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Sharp estimates for W -invariant Dunkl and heat kernels in the A_n case

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Abstract

In this article, we prove exact estimates for the W -invariant Dunkl kernel and heat kernel, for the root system of type A with arbitrary positive multiplicities. We apply the estimates of the W -invariant Dunkl heat kernel to compute sharp estimates for the Newton kernel and for the s -stable semigroups generated by a fractional power of the W -invariant Dunkl Laplacian.

Keywords: Dunkl kernel, root system, spherical function, heat kernel, Newton kernel, s -stable semigroup, fractional Dunkl Laplacian

MSC Classification: 33C67, 43A90, 53C35, 60J35

1 Introduction and notations

In Dunkl analysis for a root system Σ on \mathbf{R}^d , a crucial role is played by the Dunkl kernel $E_k(X, Y)$ and by the Dunkl heat kernel $p_t(X, Y)$. Finding good estimates of the kernels E_k and of the Dunkl heat kernel p_t is a challenging and important subject, developed recently in [1, 11]. In this paper we prove exact estimates of both these kernels in the W -radial rational Dunkl case, for the root system A_n with arbitrary positive multiplicities.

For a good introduction on rational Dunkl theory, the reader should consider the paper [1] or the book [9]. We provide here some details and notations on Dunkl analysis.

For every root $\alpha \in \Sigma$, let $\sigma_\alpha(X) = X - 2 \frac{\langle \alpha, X \rangle}{\langle \alpha, \alpha \rangle} \alpha$. The Weyl group W associated to the root system is generated by the reflection maps σ_α .

A function $k : \Sigma \rightarrow \mathbf{R}$ is called a multiplicity function if it is invariant under the action of W on Σ .

Let ∂_ξ be the derivative in the direction of $\xi \in \mathbf{R}^d$. The Dunkl operators indexed by ξ are then given by

$$T_\xi(k) f(X) = \partial_\xi f(X) + \sum_{\alpha \in \Sigma_+} k(\alpha) \alpha(\xi) \frac{f(X) - f(\sigma_\alpha X)}{\langle \alpha, X \rangle}.$$

The T_ξ 's, $\xi \in \mathbf{R}^d$, form a commutative family.

For fixed $Y \in \mathbf{R}^d$, the Dunkl kernel $E_k(\cdot, \cdot)$ is then the only real-analytic solution to the system

$$T_\xi(k)|_X E_k(X, Y) = \langle \xi, Y \rangle E_k(X, Y), \quad \forall \xi \in \mathbf{R}^d$$

with $E_k(0, Y) = 1$. In fact, E_k extends to a holomorphic function on $\mathbf{C}^d \times \mathbf{C}^d$.

Its W -invariant version $E_k^W(X, \lambda)$ is called a Bessel function of Dunkl type (see [9, p. 57]) or a spherical function $\psi_\lambda(X)$ of type Σ (refer to [17]). In this paper we use the latter terminology and notation. We have

$$\psi_\lambda(X) = E_k^W(X, \lambda) = \frac{1}{|W|} \sum_{w \in W} E_k(w \cdot X, \lambda)$$

and $\psi_\lambda(X)$ is the only real-analytic solution of the system

$$p(T_{\mathbf{e}_1}, \dots, T_{\mathbf{e}_d})(k)|_X \psi_\lambda(X) = p(\lambda) \psi_\lambda(X), \quad \forall \lambda \in \mathbf{R}^d$$

for every Weyl-invariant polynomial p (here $\mathbf{e}_1, \dots, \mathbf{e}_d$ represent the standard basis on \mathbf{R}^d).

Let $\omega_k(X) := \prod_{\alpha \in \Sigma^+} |\langle \alpha, X \rangle|^{2k(\alpha)}$ be the Dunkl weight function on \mathbf{R}^d . Recall that the Dunkl transform of a W -invariant function f on \mathbf{R}^d

$$\hat{f}(\lambda) := c_k^{-1} \int f(x) \psi_{-i\lambda}(X) \omega_k(X) dX, \quad \lambda \in \mathbf{R}^d,$$

plays the role of the spherical Fourier transform in W -invariant Dunkl analysis. (here the constant c_k is the Macdonald–Mehta–Selberg integral.)

In [11, Conjecture 18], we made the following conjecture on the growth of spherical functions $\psi_\lambda(x)$ of type Σ . We use the Cartan algebra notation $\mathfrak{a} = \mathbf{R}^d$. Then \mathfrak{a}^+ denotes the open positive Weyl chamber with respect to a system Σ^+ of positive roots.

Conjecture 1.1. *If $\lambda, X \in \overline{\mathfrak{a}^+}$, then*

$$\psi_\lambda(e^X) \asymp \frac{e^{\lambda(X)}}{\prod_{\alpha > 0} (1 + \alpha(X) \alpha(\lambda))^{k(\alpha)}}.$$

For the root system A_n on \mathbf{R}^d , $d \geq n$, and multiplicity $k(\alpha) = k > 0$, this becomes

$$\psi_\lambda(e^X) \asymp \frac{e^{\lambda(X)}}{\prod_{i < j \leq n+1} (1 + (x_i - x_j) (\lambda_i - \lambda_j))^k}, \quad \lambda, X \in \overline{\mathfrak{a}^+} \quad (1.1)$$

(the underlying constants here only depend on k).

The notation $f \asymp g$ in a domain D means that there exists $C_1 > 0$ and $C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ with C_1 and C_2 independent of $x \in D$. Recall that for the root system A_n on \mathbf{R}^{n+1} , the positive Weyl chamber is defined by $\mathfrak{a}^+ = \{X \in \mathbf{R}^{n+1} \mid x_1 > x_2 > \dots > x_{n+1}\}$.

Remark 1.2. *This conjecture includes the cases of the symmetric spaces of noncompact type $\mathbf{POS}_1(n, \mathbf{F})$, the positive definite matrices of determinant 1 over \mathbf{F} where $\mathbf{F} = \mathbf{R}$ (the real numbers with $k = 1/2$), $\mathbf{F} = \mathbf{C}$ (the complex numbers with $k = 1$), $\mathbf{F} = \mathbf{H}$ (the quaternion numbers with $k = 2$ or $\mathbf{F} = \mathbf{O}$ (the Octonions with $k = 4$) when $n = 3$. In [11], we proved the conjecture for the root system A_n in the complex case $k = 1$.*

The main tool of the proof of Conjecture 1.1 for root systems A_n is the following iterative formula for the spherical functions of type A , proven in [17]. Here we do not assume that the elements of the Lie algebra have trace 0. Here the Cartan subalgebra \mathfrak{a} for the root system A_n is isomorphic to \mathbf{R}^{n+1} . For $\lambda \in \mathfrak{a} = \mathbf{R}^{n+1}$ and $X \in \mathfrak{a}^+$, we have

$$\begin{aligned} \psi_\lambda(e^X) &= e^{\lambda(X)} \text{ if } n = 1 \text{ and} \\ \psi_\lambda(e^X) &= \frac{\Gamma(k(n+1))}{(\Gamma(k))^{n+1}} e^{\lambda_{n+1} \sum_{r=1}^{n+1} x_r} \pi(X)^{1-2k} \int_{x_{n+1}}^{x_n} \dots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \end{aligned} \quad (1.2)$$

$$\left[\prod_{i=1}^n \left(\prod_{j=1}^i (x_j - y_i) \prod_{j=i+1}^{n+1} (y_i - x_j) \right) \right]^{k-1} \prod_{i < j \leq n} (y_i - y_j) dy_1 \cdots dy_n$$

where $\lambda_0(U) = \sum_{r=1}^n (\lambda_r - \lambda_{n+1}) u_k$ and $\pi(X) = \prod_{i < j \leq n+1} (x_i - x_j)$.

Remark 1.3. Formula (1.2) concerns the action of the root system A_n on \mathbf{R}^{n+1} . If we assume $\sum_{k=1}^{n+1} x_k = 0 = \sum_{k=1}^{n+1} \lambda_k$, we have then the action of the root system A_n on \mathbf{R}^n . We can also consider the action of A_n on any \mathbf{R}^m with $m \geq n$ by deciding on which $n+1$ entries x_k , the roots act. These considerations do not affect the results of this article.

The Dunkl heat kernel $p_t(X, Y)$ is given as

$$p_t(X, Y) = \frac{1}{2^{\gamma+d/2} C_k} t^{-\frac{d}{2}-\gamma} e^{-\frac{|X|^2-|Y|^2}{4t}} E_k \left(X, \frac{Y}{2t} \right), \quad (1.3)$$

where $\gamma = \sum_{\alpha > 0} k(\alpha)$. Establishing estimates of the Dunkl heat kernel is equivalent to estimating the Dunkl kernel as demonstrated by equation (1.3).

In [14, Lemma 4.5], it is shown that

$$\int_{\mathbf{R}^d} p_t(X, Y) \omega_k(Y) dY = 1$$

$$\Delta_k|_X p_t(X, Y) = \frac{\partial}{\partial t} p_t(X, Y)$$

where the Dunkl Laplacian Δ_k equals

$$\begin{aligned} \Delta_k f(X) &= \sum_{i=1}^d T_{\mathbf{e}_i}^2 f(X) \\ &= \Delta f(X) + 2 \sum_{\alpha \in \Sigma^+} k(\alpha) \left[\frac{\langle \alpha, \nabla f(X) \rangle}{\langle \alpha, X \rangle} - \frac{f(X) - f(\sigma_\alpha X)}{\langle \alpha, X \rangle^2} \right]. \end{aligned}$$

Here Δ and ∇ denote the regular Laplacian and gradient.

The formula (1.3) remains true for the W -invariant kernels p_t^W and E^W and translates in a similar relationship between the spherical function ψ_λ and the heat kernel $p_t^W(X, Y)$:

$$p_t^W(X, Y) = \frac{1}{2^{\gamma+d/2} C_k} t^{-\frac{d}{2}-\gamma} e^{-\frac{|X|^2-|Y|^2}{4t}} \psi_X \left(\frac{Y}{2t} \right). \quad (1.4)$$

In Section 2, we prove the Conjecture 1.1 for the root system A_n , with an arbitrary multiplicity $k > 0$, i.e. we prove the formula (1.1) providing exact estimates for the spherical functions $\psi_\lambda(X)$ in the two variables X, λ when λ is real.

In Section 3.1, we apply the sharp estimates (1.1) of the spherical functions $\psi_\lambda(X)$ to the W -invariant Dunkl heat kernel $p_t(X, Y)$ for the root system A_n , with an arbitrary multiplicity $k > 0$. In the Theorem 3.1, we obtain sharp estimates of $p_t(X, Y)$ in three variables t, X, Y .

Next, in Sections 3.2 and 3.3, we apply the Theorem 3.1 to the W -invariant Dunkl Newton kernel and to the W -invariant s -stable semigroups, respectively. In all cases, we obtain sharp estimates.

2 Proof of the Conjecture in the case A_n .

We will assume from now on that $X \in \mathfrak{a}^+$ and $\lambda \in \overline{\mathfrak{a}^+}$.

Notation 2.1. We will write $f(x) \lesssim g(x)$ ($f(x) \gtrsim g(x)$) for $x \in D$ if there exists a constant $C > 0$ independent of x such that $f(x) \leq C g(x)$ ($f(x) \geq C g(x)$) for all $x \in D$. We will use the notation $M_k = (x_k + x_{k+1})/2$.

Remark 2.2. Suppose $i < j$. We will use repeatedly the fact that the functions

$$\frac{x}{1 + (\lambda_i - \lambda_j)x} \quad \text{and} \quad \frac{x}{(1 + (\lambda_i - \lambda_j)x)^k}, \quad k \leq 1$$

are increasing functions of x . A feature of our proofs will be the distinction between the cases $0 < k \leq 1$ and $k > 1$.

Proposition 2.3. Conjecture 1.1 is equivalent to

$$I^{(n)} \asymp \frac{\pi(X)^{2k-1}}{\prod_{i < j \leq n+1} ((1 + (\lambda_i - \lambda_j)(x_i - x_j))^k)} \quad (2.1)$$

where

$$I^{(n)} = \int_{x_{n+1}}^{x_n} \dots \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \left(\prod_{i < j \leq n} (x_i - y_j) \prod_{i < j \leq n+1} (y_i - x_j) \right)^{k-1} \prod_{i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \dots dy_n.$$

Proof. The integral $I^{(n)}$ corresponds to a constant multiple of $e^{-\lambda(X)} \pi(X)^{2k-1} \psi_\lambda(e^X)$ in which we have replaced $\psi_{\lambda_0}(e^Y)$ in (1.2) by its asymptotic expression conjectured in (1.1). \square

We start by two technical results.

Lemma 2.4. For $k > 0$ and $x \geq 0$, we have

$$\int_0^x u^{k-1} e^{-u} du \asymp \left(\frac{x}{1+x} \right)^k.$$

Proof. The result is clearly true if $0 \leq x < 1$ (use $e^{-1} \leq e^{-x} \leq 1$ and integrate). If $x \geq 1$ then

$$\int_0^1 u^{k-1} e^{-u} du \leq \int_0^x u^{k-1} e^{-u} du < \int_0^\infty u^{k-1} e^{-u} du$$

and the result follows. \square

Proposition 2.5. Assume that $\gamma = x_n - x_{n+1}$ is the largest positive root and let

$$I_1 = \int_{M_n}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \left(\prod_{i \leq j \leq n} (x_i - y_j) \prod_{i < j \leq n+1} (y_i - x_j) \right)^{k-1} \\ \prod_{i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \cdots dy_n.$$

Then $I_1 \asymp I^{(n)}$.

Proof. Let $I_2 = I^{(n)} - I_1$. In I_1 and I_2 , consider only the corresponding integral in y_n , calling the resulting expressions \tilde{I}_1 and \tilde{I}_2 . Observing that $y_n - x_{n+1} \asymp \gamma$ for $y_n \in [x_n, M_n]$, we have

$$\tilde{I}_1 \asymp \gamma^{k-1} \int_{M_n}^{x_n} e^{-(\lambda_n - \lambda_{n+1})(x_n - y_n)} \left(\prod_{i \leq n} (x_i - y_n) \right)^{k-1} \\ \prod_{i < n} \frac{y_i - y_n}{(1 + (\lambda_i - \lambda_n)(y_i - y_n))^k} dy_n.$$

If $0 < k \leq 1$ then,

$$\tilde{I}_1 \gtrsim e^{-(\lambda_n - \lambda_{n+1})\gamma/2} \gamma^{k-1} \int_{M_n}^{x_n} (x_1 - y_n)^{k-1} \prod_{i=2}^n (x_i - y_n)^{k-1} \prod_{i < n} \frac{x_{i+1} - y_n}{(1 + (\lambda_i - \lambda_n)\gamma)^k} dy_n \\ \gtrsim \gamma^{k-1} e^{-(\lambda_n - \lambda_{n+1})\gamma/2} \int_{M_n}^{x_n} \prod_{i < n} \frac{(x_{i+1} - y_n)^k}{(1 + (\lambda_i - \lambda_n)\gamma)^k} (x_1 - y_n)^{k-1} dy_n \\ \gtrsim \gamma^{k-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k} \int_{M_n}^{x_n} (x_n - y_n)^{(n-1)k} (x_1 - y_n)^{k-1} dy_n \\ \gtrsim \gamma^{(n+1)k-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k}.$$

Indeed, if $n > 1$ then $(x_1 - y_n)^{k-1} \gtrsim \gamma^{k-1}$ and the rest can easily be integrated. If $n = 1$, then

$$\int_{M_n}^{x_n} (x_n - y_n)^{(n-1)k} (x_1 - y_n)^{k-1} dy_n = \int_{M_n}^{x_n} (x_n - y_n)^{k-1} dy_n \asymp \gamma^k.$$

If $k > 1$, we have

$$\tilde{I}_1 \gtrsim \gamma^{k-1} e^{-(\lambda_n - \lambda_{n+1})\gamma/2} \int_{M_n}^{x_n} \prod_{i=1}^{n-1} (x_i - y_n)^{k-1} \prod_{i < n} \frac{y_i - y_n}{(1 + (\lambda_i - \lambda_n)(y_i - y_n))^k} \\ (x_n - y_n)^{k-1} dy_n \\ \gtrsim \gamma^{k-1} e^{-(\lambda_n - \lambda_{n+1})\gamma/2} \int_{M_n}^{x_n} \left(\prod_{i < n} \frac{y_i - y_n}{1 + (\lambda_i - \lambda_n)(y_i - y_n)} \right)^k (x_n - y_n)^{k-1} dy_n$$

$$\begin{aligned}
&\gtrsim \gamma^{k-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k} \int_{M_n} \prod_{i < n} (y_i - y_n)^k (x_n - y_n)^{k-1} dy_n \\
&\gtrsim \gamma^{k-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k} \int_{M_n} (x_n - y_n)^{nk-1} dy_n \\
&\asymp \gamma^{(n+1)k-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k}.
\end{aligned}$$

On the other hand for $k > 0$, observing that $x_i - y_n \asymp \gamma$ and $y_i - y_n \asymp \gamma$ for $y_n \in [M_n, x_{n+1}]$, we have

$$\begin{aligned}
\tilde{I}_2 &\asymp \gamma^{n(k-1)} \prod_{i < n} \frac{\gamma}{(1 + (\lambda_i - \lambda_n)\gamma)^k} \int_{x_{n+1}}^{M_n} e^{-(\lambda_n - \lambda_{n+1})(x_n - y_n)} (y_n - x_{n+1})^{k-1} dy_n \\
&\lesssim \gamma^{n(k-1)+n-1} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k} \int_{x_{n+1}}^{M_n} (y_n - x_{n+1})^{k-1} dy_n \\
&\asymp \gamma^{n(k-1)+n-1+k} \frac{e^{-(\lambda_n - \lambda_{n+1})\gamma/2}}{\prod_{i < n} (1 + (\lambda_i - \lambda_n)\gamma)^k} \lesssim \tilde{I}_1
\end{aligned}$$

which allows us to conclude. \square

Theorem 2.6. *Conjecture 1.1 holds for the root system A_n , $n \geq 1$ with root multiplicity $k > 0$.*

Proof. The result is proven using induction. Using Proposition 2.5 with $n = 1$, we have, using $u = (\lambda_1 - \lambda_2)(x_1 - y_1)$ and Lemma 2.4,

$$\begin{aligned}
I^{(1)} &\asymp I_1 = \int_{M_1}^{x_1} e^{-(\lambda_1 - \lambda_2)(x_1 - y_1)} (x_1 - y_1)^{k-1} (y_1 - x_2)^{k-1} dy_1 \\
&\asymp (x_1 - x_2)^{k-1} \int_{M_1}^{x_1} e^{-(\lambda_1 - \lambda_2)(x_1 - y_1)} (x_1 - y_1)^{k-1} dy_1 \\
&= (x_1 - x_2)^{k-1} (\lambda_1 - \lambda_2)^{-k} \int_0^{(\lambda_1 - \lambda_2)(x_1 - x_2)/2} e^{-u} u^{k-1} du \\
&\asymp (x_1 - x_2)^{k-1} \left(\frac{(x_1 - x_2)/2}{1 + (\lambda_1 - \lambda_2)(x_1 - x_2)/2} \right)^k
\end{aligned}$$

which proves the formula (2.1) in the case $n = 1$.

Assume that the result holds for the root systems A_1, A_2, \dots, A_{n-1} . We will use Proposition 2.3 and will proceed by assuming, in turn for each $m < n$, that $\alpha_m = x_m - x_{m+1}$ is the largest root. We will discuss the case $m = n$ at the end.

We will proceed as follows. As in the proof of Proposition 2.5, we will divide the integral in two parts I_1 and I_2 , show that I_1 has the desired asymptotics and that $I_2 \lesssim I_1$.

Assume now that $\alpha_m = x_m - x_{m+1}$, $1 \leq m \leq n-1$, is the largest root. Noting that $x_i - y_j \asymp \alpha_m$, $i \leq m$, $m < j \leq n$, $y_i - x_j \asymp \alpha_m$, $i \leq m$, $j \geq m+2$, $y_i - y_j \asymp \alpha_m$, $i \leq m$, $m < j \leq n$, for $y_m \in [M_m, x_m]$, we have

$$\begin{aligned}
I_1 &= \int_{x_{n+1}}^{x_n} \cdots \int_{M_m}^{x_m} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \\
&\quad \left(\prod_{i < j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1} \\
&\quad \left(\prod_{m < i < j \leq n} (x_i - y_j) \prod_{m < i < j \leq n+1} (y_i - x_j) \right)^{k-1} \\
&\quad \left(\prod_{i \leq m < j \leq n} (x_i - y_j) \prod_{\substack{i \leq m, \\ m+1 < j \leq n+1}} (y_i - x_j) \right)^{k-1} \\
&\quad \prod_{i < j \leq m} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \prod_{m < i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \\
&\quad \prod_{i \leq m < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \dots dy_n \\
&\asymp \alpha_m^{2m(n-m)(k-1)} \prod_{i \leq m < j \leq n} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_j)\alpha_m)^k} \\
&\quad \int_{x_{n+1}}^{x_n} \cdots \int_{M_m}^{x_m} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \\
&\quad \left(\prod_{i < j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1} \\
&\quad \left(\prod_{m < i < j \leq n} (x_i - y_j) \prod_{m < i < j \leq n+1} (y_i - x_j) \right)^{k-1} \\
&\quad \prod_{i < j \leq m} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \prod_{m < i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \dots dy_n \\
&= \frac{\alpha_m^{2m(n-m)(k-1) + m(n-m)}}{\prod_{i \leq m < j \leq n} (1 + (\lambda_i - \lambda_j)\alpha_m)^k} \int_{M_m}^{x_m} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^m (\lambda_i - \lambda_{n+1})(x_i - y_i)} \\
&\quad \left(\prod_{i < j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1} \\
&\quad \prod_{i < j \leq m} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \dots dy_m
\end{aligned}$$

$$\begin{aligned}
& \int_{x_{n+1}}^{x_n} \cdots \int_{x_{m+2}}^{x_{m+1}} e^{-\sum_{i=m+1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \\
& \left(\prod_{m < i < j \leq n} (x_i - y_j) \prod_{m < i < j \leq n+1} (y_i - x_j) \right)^{k-1} \\
& \prod_{m < i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_{m+1} \cdots dy_n \\
& \asymp \frac{\alpha_m^{m(n-m)(2k-1)}}{\prod_{i \leq m < j \leq n} (1 + (\lambda_i - \lambda_j) \alpha_m)^k} \\
& \frac{\prod_{i < j \leq m} (x_i - x_j)^{2k-1}}{\prod_{i < j \leq m} (1 + (\lambda_i - \lambda_j)(x_i - x_j))^k \prod_{i \leq m} (1 + (\lambda_i - \lambda_{n+1})(x_i - x_{m+1}))^k} \\
& \prod_{m < i < j \leq n+1} \frac{(x_i - x_j)^{2k-1}}{(1 + (\lambda_i - \lambda_j)(x_i - x_j))^k}
\end{aligned}$$

which has the desired asymptotics (we used Proposition 2.5 and the induction hypothesis on A_m and on A_{n-m}).

It remains to show that $I_2 = I^{(n)} - I_1 \lesssim I_1$. As in the proof of Proposition 2.5, it suffices to show that $\tilde{I}_1 \gtrsim \tilde{I}_2$ where \tilde{I}_1 (respectively \tilde{I}_2) is the portion of I_1 (I_2) integrated with respect to y_m .

Now, since $y_m - x_j \asymp \alpha_m$, $m < j \leq n+1$, and $y_m - y_j \asymp \alpha_m$, $m < j \leq n$, when $y_m \in [M_m, x_m]$, we have

$$\begin{aligned}
\tilde{I}_1 & \asymp \frac{\alpha_m^{(n+1-m)(k-1)+n-m}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \int_{M_m}^{x_m} e^{-(\lambda_m - \lambda_{n+1})(x_m - y_m)} \prod_{i \leq m} (x_i - y_m)^{k-1} \\
& \prod_{i < m} \frac{y_i - y_m}{(1 + (\lambda_i - \lambda_m)(y_i - y_m))^k} dy_m.
\end{aligned}$$

If $k > 1$ then

$$\begin{aligned}
\tilde{I}_1 & \gtrsim \frac{\alpha_m^{(n+1-m)(k-1)+n-m}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \int_{M_m}^{x_m} \prod_{i \leq m} (x_m - y_m)^{k-1} \\
& \prod_{i < m} \frac{x_m - y_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} dy_m \\
& \gtrsim \frac{\alpha_m^{(n+1-m)(k-1)+n-m}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \frac{e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2}}{\prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k} \\
& \int_{M_m}^{x_m} (x_m - y_m)^{m(k-1)+m-1} dy_m \\
& = \frac{\alpha_m^{(n+1)k-1}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \frac{e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2}}{\prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k}.
\end{aligned}$$

If $0 < k \leq 1$ then

$$\begin{aligned}
\tilde{I}_1 &\gtrsim \frac{\alpha_m^{(n+1-m)(k-1)+n-m}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \int_{M_m}^{x_m} \prod_{i < m} \alpha_m^{k-1} (x_m - y_m)^{k-1} \\
&\quad \prod_{i < m} \frac{x_m - y_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} dy_m \\
&= \frac{\alpha_m^{(n+1-m)(k-1)+n-m+(m-1)(k-1)}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \frac{e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2}}{\prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k} \\
&\quad \int_{M_m}^{x_m} (x_m - y_m)^{m-1+k-1} dy_m \\
&\asymp \frac{\alpha_m^{(n+1-m)(k-1)+n-m+(m-1)(k-1)}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \frac{e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2}}{\prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k} \alpha_m^{m-1+k} \\
&= \frac{\alpha_m^{(n+1)k-1}}{\prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} \frac{e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2}}{\prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k}.
\end{aligned}$$

On the other hand, since $x_i - y_m \asymp \alpha_m$, $i \leq m$, and $y_i - y_m \asymp \alpha_m$, $i < m$, when $y_m \in [x_{m+1}, M_m]$,

$$\begin{aligned}
\tilde{I}_2 &\asymp \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} \int_{x_{m+1}}^{M_m} e^{-(\lambda_m - \lambda_{n+1})(x_m - y_m)} \\
&\quad \prod_{m < j \leq n+1} (y_m - x_j)^{k-1} \prod_{m < j \leq n} \frac{y_m - y_j}{(1 + (\lambda_m - \lambda_j)(y_m - y_j))^k} dy_m.
\end{aligned}$$

If $k > 1$ then

$$\begin{aligned}
\tilde{I}_2 &\lesssim \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \int_{x_{m+1}}^{M_m} \prod_{m < j \leq n} (y_m - y_j)^{k-1} \\
&\quad \prod_{m < j \leq n} \frac{y_m - y_j}{(1 + (\lambda_m - \lambda_j)(y_m - y_j))^k} (y_m - x_{n+1})^{k-1} dy_m \\
&= \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \\
&\quad \int_{x_{m+1}}^{M_m} \prod_{m < j \leq n} \left(\frac{y_m - y_j}{1 + (\lambda_m - \lambda_j)(y_m - y_j)} \right)^k (y_m - x_{n+1})^{k-1} dy_m \\
&\lesssim \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \alpha_m^k \\
&\quad \prod_{m < j \leq n} \left(\frac{\alpha_m}{1 + (\lambda_m - \lambda_j) \alpha_m} \right)^k \lesssim \tilde{I}_1.
\end{aligned}$$

If $0 < k \leq 1$ then

$$\tilde{I}_2 \lesssim \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1}) \alpha_m / 2} \int_{x_{m+1}}^{M_m} \prod_{m+1 < j \leq n+1} (y_m - x_j)^{k-1}$$

$$\begin{aligned}
& \prod_{m < j \leq n} \frac{y_m - x_{j+1}}{(1 + (\lambda_m - \lambda_j)(y_m - x_{j+1}))^k} (y_m - x_{m+1})^{k-1} dy_m \\
&= \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m)\alpha_m)^k} \\
& \quad e^{-(\lambda_m - \lambda_{n+1})\alpha_m/2} \int_{x_{m+1}}^{M_m} \left(\prod_{m < j \leq n} \frac{y_m - x_{j+1}}{1 + (\lambda_m - \lambda_j)(y_m - x_{j+1})} \right)^k \\
& \quad (y_m - x_{m+1})^{k-1} dy_m \\
&\lesssim \alpha_m^{m(k-1)} \prod_{i < m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m)\alpha_m)^k} e^{-(\lambda_m - \lambda_{n+1})\alpha_m/2} \alpha_m^k \\
& \quad \left(\prod_{m < j \leq n} \frac{\alpha_m}{1 + (\lambda_m - \lambda_j)\alpha_m} \right)^k \lesssim \tilde{I}_1.
\end{aligned}$$

By the structure of the root system A_n , the case α_n maximal is equivalent to the case α_1 maximal. Indeed, in formula (1.2), one does not assume that $\lambda \in \overline{\mathfrak{a}^+}$. We also know that $\psi_\lambda(e^X)$ is invariant under permutation of its λ argument. Hence one can re-write (1.2) by exchanging λ_1 and λ_{n+1} ,

$$\begin{aligned}
\psi_\lambda(e^X) &= e^{\lambda(X)} \text{ if } n = 1 \text{ and} \\
\psi_\lambda(e^X) &= \frac{\Gamma(k(n+1))}{(\Gamma(k))^{n+1}} e^{\lambda_1 \sum_{r=1}^{n+1} x_r} \left(\prod_{i < j \leq n+1} (x_i - x_j) \right)^{1-2k} \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} \psi_{\tilde{\lambda}_0}(e^Y) \\
& \quad \left[\prod_{i=1}^n \left(\prod_{j=1}^i (x_j - y_i) \prod_{j=i+1}^{n+1} (y_i - x_j) \right) \right]^{k-1} \prod_{i < j \leq n} (y_i - y_j) dy_1 \cdots dy_n
\end{aligned}$$

where $\tilde{\lambda}_0(U) = \sum_{r=2}^{n+1} (\lambda_r - \lambda_1) u_r$. We used the fact that

$$\psi_{[\lambda_{n+1}-\lambda_1, \lambda_2-\lambda_1, \dots, \lambda_n-\lambda_1]}(e^Y) = \psi_{[\lambda_2-\lambda_1, \dots, \lambda_n-\lambda_1, \lambda_{n+1}-\lambda_1]}(e^Y).$$

Conjecture 1.1 is equivalent to

$$J^{(n)} \asymp \frac{\pi(X)^{2k-1}}{\prod_{i < j \leq n+1} ((1 + (\lambda_i - \lambda_j)(x_i - x_j))^k)}$$

where

$$\begin{aligned}
J^{(n)} &= \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_1 - \lambda_{i+1})(y_i - x_{i+1})} \left(\prod_{i < j \leq n} (x_i - y_j) \prod_{i < j \leq n+1} (y_i - x_j) \right)^{k-1} \\
& \quad \prod_{i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_{i+1} - \lambda_{j+1})(y_i - y_j))^k} dy_1 \cdots dy_n.
\end{aligned}$$

The term $J^{(n)}$ corresponds to a constant multiple of $e^{-\lambda(X)} \pi(X)^{2k-1} \psi_\lambda(e^X)$ in which we have replaced $\psi_{\lambda_0}(e^Y)$ in (1.2) by its asymptotic expression conjectured in (1.1). One then proves the case α_n maximal as one proves the case α_1 maximal.

This concludes the proof of the estimate (1.1) for $X \in \mathfrak{a}^+$ (recall that the formula (1.2) holds for $X \in \mathfrak{a}^+$). The estimates that we find for $\psi_\lambda(e^X)$ extend to $X \in \overline{\mathfrak{a}^+}$ by continuity. \square

3 Applications

3.1 Estimates of the W -invariant Dunkl Heat Kernel

The following theorem establishes, for root systems A_n and for any multiplicity $k > 0$, the estimates of the W -invariant Dunkl Heat Kernel conjectured in the Conjecture 18 of [11].

Theorem 3.1. *For the root systems of type A , we have for $X, Y \in \overline{\mathfrak{a}^+}$*

$$p_t^W(X, Y) \asymp \frac{t^{-d/2} e^{-|X-Y|^2/(4t)}}{\prod_{\alpha>0} (t + \alpha(X) \alpha(Y))^k}.$$

Proof. Consider the relation (1.4)

$$p_t^W(X, Y) = t^{-d/2-\gamma} e^{-|X-Y|^2/(4t)} \psi_X(Y/(2t))$$

with $\gamma = \sum_{\alpha>0} k(\alpha) = k|\Sigma^+|$. From Theorem 2.6, we have

$$\begin{aligned} p_t^W(X, Y) &\asymp t^{-d/2-\gamma} e^{-(|X|^2+|Y|^2)/(4t)} \frac{e^{\langle X, Y/(2t) \rangle}}{\prod_{\alpha>0} (1 + \alpha(X) \alpha(Y/(2t)))^k} \\ &= t^{-d/2-\gamma+k|\Sigma^+|} e^{-(|X|^2+|Y|^2)/(4t)} \frac{e^{\langle X, Y/(2t) \rangle}}{\prod_{\alpha>0} (2t + \alpha(X) \alpha(Y))^k} \\ &\asymp \frac{t^{-d/2} e^{-|X-Y|^2/(4t)}}{\prod_{\alpha>0} (t + \alpha(X) \alpha(Y))^k}. \end{aligned}$$

\square

3.2 Estimates of the W -invariant Dunkl Newton Kernel

The W -invariant Dunkl Newton kernel $N^W(X, Y)$ is the kernel of the inverse operator of the Dunkl Laplacian Δ^W . It is the fundamental solution of the problem $\Delta^W u = f$ where f is given and $|u(x)| \rightarrow 0$ as $x \rightarrow \infty$. It is defined by

$$N^W(X, Y) = \int_0^\infty p_t^W(X, Y) dt,$$

where $p_t^W(X, Y)$ is the heat kernel of Δ^W .

In [10], we stated the following conjecture for the Weyl invariant Newton kernel for $d \geq 3$ and proved it for complex root systems.

Conjecture 3.2. For $X, Y \in \overline{\mathfrak{a}^+}$ and $d \geq 3$, we have

$$N^W(X, Y) \asymp \frac{1}{|X - Y|^{d-2} \prod_{\alpha \in \Sigma^+} |X - \sigma_\alpha Y|^{2k(\alpha)}}.$$

In this section, we prove the conjecture in the case of root systems of type A and prove a similar result in the case $d = 2$.

The next three lemmas will be useful to derive sharp estimates for the Newton kernel.

Lemma 3.3. Suppose $k > 0$, $a \geq 0$, $b_i \geq 0$, $a + b_i > 0$, $i = 1, \dots, m$ and $N > km - 1$. Then

$$J := \int_0^\infty \frac{u^N e^{-u} du}{\prod_{i=1}^m (a + b_i u)^k} \asymp \frac{1}{\prod_{i=1}^m (a + b_i)^k}.$$

Proof. Note that $a + b_i u \leq (a + b_i)u$ whenever $u \geq 1$. Therefore, we have

$$J \geq \int_1^\infty \frac{u^N e^{-u} du}{\prod_{i=1}^m (a + b_i u)^k} \geq \frac{1}{\prod_{i=1}^m (a + b_i)^k} \int_1^\infty u^{N-mk} e^{-u} du.$$

Let Λ be the (possibly empty) set of indices where $a \leq b_i$.

$$\begin{aligned} J &\leq \frac{1}{\left(\prod_{i \notin \Lambda} a\right)^k} \int_0^\infty \frac{u^N e^{-u} du}{\prod_{i \in \Lambda} (b_i u)^k} = \frac{1}{\left(\prod_{i \notin \Lambda} a\right)^k \left(\prod_{i \in \Lambda} b_i\right)^k} \int_0^\infty u^{N-|\Lambda|k} e^{-u} du \\ &\lesssim \frac{1}{\prod_{i=1}^k (a + b_i)^k} \max_{0 \leq m_0 \leq m} \int_0^\infty u^{N-m_0k} e^{-u} du \end{aligned}$$

(we understand an empty product to be equal to 1). □

Lemma 3.4. Suppose $k > 0$, $a > 0$ and $b \geq 0$. Then

$$J := \int_0^\infty \frac{u^{k-1} e^{-u} du}{(a + bu)^k} \asymp \frac{\ln(2 + b/a)}{(a + b)^k}.$$

Proof. If $0 \leq b \leq a$ then

$$\int_0^\infty \frac{u^{k-1} e^{-u} du}{(a + au)^k} \leq J \leq \int_0^\infty \frac{u^{k-1} e^{-u} du}{a^k}$$

and the result holds. We now assume $a \leq b$. We then have

$$\begin{aligned} J &\asymp \int_0^{a/b} \frac{u^{k-1}}{a^k} du + \int_{a/b}^1 \frac{u^{k-1} du}{(bu)^k} + \int_1^\infty \frac{u^{k-1} e^{-u} du}{(bu)^k} \\ &\asymp \frac{1}{b^k} + \frac{1}{b^k} \int_{a/b}^1 u^{-1} du + \frac{1}{b^k} \int_1^\infty u^{-1} e^{-u} du \\ &\asymp \frac{1}{b^k} + \frac{1}{b^k} \ln(b/a) + \frac{1}{b^k} \asymp \frac{\ln(2 + b/a)}{b^k} \end{aligned}$$

which concludes the proof. □

Lemma 3.5. *Suppose $k > 0$, $a \geq 0$ and $0 \leq b_1 \leq b_2 \leq b_3$ then*

$$J := \int_0^\infty \frac{u^{3k-1} e^{-u} du}{(a + b_1 u)^k (a + b_2 u)^k (a + b_3 u)^k} \asymp \frac{\ln(2 + b_1/a)}{(a + b_1)^k (a + b_2)^k (a + b_3)^k}.$$

Proof. If $b_1 \leq a$ then

$$\int_1^\infty \frac{u^{3k-1} e^{-u} du}{(a + a u)^k ((a + b_2) u)^k ((a + b_3) u)^k} \leq J \leq \int_0^\infty \frac{u^{3k-1} e^{-u} du}{a^k (a + b_2 u)^k (a + b_3 u)^k}$$

and the result follows in this case using Lemma 3.3 for the upper bound.

If $a \leq b_1 \leq b_2 \leq b_3$ then

$$J \gtrsim \int_{a/b_1}^2 \frac{u^{3k-1} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} \asymp \frac{\ln(2 b_1/a)}{(b_1 b_2 b_3)^k}$$

while

$$\begin{aligned} J &\lesssim \int_0^{a/b_1} \frac{u^{3k-1} du}{a^k (b_2 u)^k (b_3 u)^k} + \int_{a/b_1}^2 \frac{u^{3k-1} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} + \int_2^\infty \frac{u^{3k-1} e^{-u} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} \\ &\asymp \frac{1}{b_1^k b_2^k b_3^k} + \frac{\ln(2 b_1/a)}{b_1^k b_2^k b_3^k} + \frac{1}{b_1^k b_2^k b_3^k} \end{aligned}$$

and the result follows in this case. □

Theorem 3.6. *For the root system A_n and $d \geq 3$, we have for $X, Y \in \overline{\mathfrak{a}^+}$*

$$N^W(X, Y) \asymp \frac{|X - Y|^{2-d}}{\prod_{\alpha > 0} |X - \sigma_\alpha Y|^{2k}}.$$

Proof. We have, using Theorem 3.1 and the change of variables $u = |X - Y|^2/(4t)$

$$\begin{aligned} N^W(X, Y) &= \int_0^\infty p_t^W(X, Y) dt \asymp \int_0^\infty \frac{t^{-d/2} e^{-|X-Y|^2/(4t)} dt}{\prod_{\alpha > 0} (t + \alpha(X) \alpha(Y))^k} \\ &\asymp |X - Y|^{2-d} \int_0^\infty \frac{u^{d/2-2} e^{-u} du}{\prod_{\alpha > 0} (|X - Y|^2/(4u) + \alpha(X) \alpha(Y))^k} \\ &\asymp |X - Y|^{2-d} \int_0^\infty \frac{u^{k|\Sigma^+|+d/2-2} e^{-u} du}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y) u)^k} \\ &\asymp |X - Y|^{2-d} \frac{1}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k} \asymp \frac{|X - Y|^{2-d}}{\prod_{\alpha > 0} |X - \sigma_\alpha Y|^{2k}} \end{aligned}$$

(we have used Lemma 3.3 and the fact that $|X - \sigma_\alpha Y|^2 = |X - Y|^2 + 2\alpha(X) \alpha(Y)$). □

Proposition 3.7. *If $d = 2$, the Newton kernel in the A_1 case satisfies*

$$N^W(X, Y) \asymp \frac{\ln\left(1 + \frac{|X - \sigma_\alpha Y|^2}{|X - Y|^2}\right)}{|X - \sigma_\alpha Y|^{2k}} \quad X, Y \in \overline{\mathfrak{a}^+}.$$

Here, it is important to recall that for $X, Y \in \overline{\mathfrak{a}^+}$, we have $|X - \sigma_\alpha Y| \geq |X - Y|$ and therefore, the numerator of the last expression is at least $\ln 2$ for $X \neq Y$. This remark also applies to the estimate in Proposition 3.8.

Proof. With computations similar as in the case $d \geq 3$, using Lemma 3.4,

$$\begin{aligned} N^W(X, Y) &= \int_0^\infty \frac{t^{-1} e^{-|X-Y|^2/(4t)} dt}{(t + \alpha(X) \alpha(Y))^k} \asymp \int_0^\infty \frac{u^{k-1} e^{-u} du}{(|X - Y|^2 + \alpha(X) \alpha(Y) u)^k} \\ &\asymp \frac{\ln(2 + \alpha(X) \alpha(Y)/|X - Y|^2)}{(|X - Y|^2 + \alpha(X) \alpha(Y))^k} = \frac{\ln\left(\frac{2|X-Y|^2 + \alpha(X)\alpha(Y)}{|X-Y|^2}\right)}{(|X - Y|^2 + \alpha(X) \alpha(Y))^k} \\ &= \frac{\ln\left(\frac{3}{2} + \frac{1}{2} \frac{|X - \sigma_\alpha Y|^2}{|X - Y|^2}\right)}{(|X - Y|^2 + \alpha(X) \alpha(Y))^k} \asymp \frac{\ln\left(1 + \frac{|X - \sigma_\alpha Y|^2}{|X - Y|^2}\right)}{|X - \sigma_\alpha Y|^{2k}}. \end{aligned}$$

□

Proposition 3.8. *If $d = 2$, the Newton kernel in the A_2 case satisfies*

$$N^W(X, Y) \asymp \frac{\ln\left(1 + \frac{|X - \sigma_\omega Y|^{2k}}{|X - Y|^2}\right)}{|X - \sigma_\alpha Y|^{2k} |X - \sigma_\beta Y|^{2k} |X - \sigma_{\alpha+\beta} Y|^{2k}}, \quad X, Y \in \overline{\mathfrak{a}^+},$$

where ω gives the minimum of $|X - \sigma_\omega Y|$ for $\omega \in \{\alpha, \beta\}$.

Proof. With computations similar as in the case $d \geq 3$, using Lemma 3.5,

$$\begin{aligned} N^W(X, Y) &\asymp \int_0^\infty \frac{t^{-1} e^{-|X-Y|^2/(4t)} dt}{(t + \alpha(X) \alpha(Y))^k (t + \beta(X) \beta(Y))^k (t + (\alpha + \beta)(X) (\alpha + \beta)(Y))^k} \\ &\asymp \int_0^\infty \frac{u^{3k-1} e^{-u} du}{\prod_{\eta \in \{\alpha, \beta, \alpha+\beta\}} (|X - Y|^2 + \eta(X) \eta(Y) u)^k} \\ &\asymp \frac{\ln(2 + \omega(X) \omega(Y)/|X - Y|^2)}{\prod_{\eta \in \{\alpha, \beta, \alpha+\beta\}} (|X - Y|^2 + \eta(X) \eta(Y))^k} \\ &\asymp \frac{\ln\left(1 + \frac{|X - \sigma_\omega Y|^2}{|X - Y|^2}\right)}{|X - \sigma_\alpha Y|^{2k} |X - \sigma_\beta Y|^{2k} |X - \sigma_{\alpha+\beta} Y|^{2k}}. \end{aligned}$$

where ω gives the minimum of $|X - \sigma_\omega Y|$ for $\omega \in \{\alpha, \beta\}$. □

Remark 3.9. *In the Dunkl analysis, an important role is played by the intertwining operator V_k , defined as a unique linear isomorphism on the space of polynomial functions on \mathbf{R}^d which intertwines the Dunkl operators with the usual partial derivatives:*

$$T_\xi V_k = V_k \partial_\xi \quad \text{for all } \xi \in \mathbf{R}^d$$

and is normalized by $V_k(1) = 1$.

The following general formula for the Dunkl Newton kernel $N_k(x, y)$ involving the intertwining operator V_k was proven in [6]:

$$N_k(X, Y) = \frac{2^{2\gamma} ((d-2)/2)_\gamma}{|W| (d-2) w_d \pi(\rho)} V_k \left((|Y|^2 - 2 \langle X, \cdot \rangle + |X|^2)^{(2-d-2\gamma)/2} \right) (Y)$$

(we are using a slightly different normalization of the operator V_k than [9, 14], see [10] for details.) Little is known explicitly on the intertwining operator. Theorem 3.6 and the formula $N^W(X, \cdot, Y) = \frac{1}{|W|} \sum_{w \in W} N_k(wX, Y)$ imply the following explicit asymptotic formula.

Corollary 3.10. For the root system A_n and $d \geq 3$, we have for $X, Y \in \overline{\mathfrak{a}^+}$

$$V_k \left((|Y|^2 - \frac{2}{|W|} \sum_{w \in W} \langle wX, \cdot \rangle + |X|^2)^{(2-d-2\gamma)/2} \right) (Y) \asymp \frac{|X - Y|^{2-d}}{\prod_{\alpha > 0} |X - \sigma_\alpha Y|^{2k}}.$$

3.3 Heat semigroups for fractional powers of Δ_k^W

Let $s \in (0, 2)$. The fractional powers $(-\Delta_k^W)^{s/2}$ of the W -invariant Dunkl Laplacian are the infinitesimal generators of important semigroups $(h_t^W(X, Y))_{t \geq 0}$, called *W-invariant Dunkl s-stable semigroups*.

Fractional powers of the Dunkl Laplacian and related semigroups and processes were considered for $s = 1$ in [15, p.75], [16, Section 5] and for any $s \in (0, 2)$ in [5, 13]. Stable semigroups on Riemannian symmetric spaces of non-compact type were studied in [7, 12].

Like the heat semigroup $p_t^W(X, Y)$, the densities $h_t^W(X, Y)$ are to be considered with respect to the Dunkl weight function $\omega_k(Y)$ on \mathbf{R}^d . We have

$$h_t^W(X, Y) = \int_0^\infty p_u^W(X, Y) \eta_t(u) du$$

where $\eta_t(u)$ is the density of the $s/2$ -stable subordinator, i.e. of a positive Lévy process $(Y_t)_{t > 0}$ with the Laplace transform $\mathbf{E}(\exp(zY_t)) = \exp(-tz^{s/2})$, $z > 0$ (see [2] for more details).

Denote by $(h_t^{\mathbf{R}^d}(X, Y))_{t \geq 0}$ the s -stable rotationally invariant semigroup on \mathbf{R}^d , with generator $(-\Delta)^{s/2}$. It is known ([3]) that

$$h_t^{\mathbf{R}^d}(X, Y) \asymp \min \left\{ \frac{1}{t^{d/s}}, \frac{t}{|X - Y|^{d+s}} \right\} \asymp \frac{t}{(t^{2/s} + |X - Y|^2)^{(d+s)/2}}. \quad (3.1)$$

Remark 3.11. It is useful to note that $\min \{t^{-d/s}, t|X - Y|^{-(d+s)}\} = t^{-d/s}$ if and only if $t^{2/s} \geq |X - Y|^2$.

Theorem 3.12. Consider the W -invariant Dunkl Laplacian in the A_n case with multiplicity $k > 0$. Then for $X, Y \in \overline{\mathfrak{a}^+}$,

$$h_t^W(X, Y) \asymp \frac{h_t^{\mathbf{R}^d}(X, Y)}{\prod_{\alpha > 0} (t^{2/s} + |X - \sigma_\alpha Y|^2)^k} \asymp \frac{h_t^{\mathbf{R}^d}(X, Y)}{\prod_{\alpha > 0} (t^{2/s} + |X - Y|^2 + \alpha(X)\alpha(Y))^k}.$$

Proof. The proof is inspired by the proof of [4, Theorem 3.1] providing estimates of stable semi-groups on fractals.

Given Remark 3.11, it will make sense to consider the cases $t^{2/s} \geq |X - Y|^2$ and $t^{2/s} \leq |X - Y|^2$ separately. In the proof, m will denote the number of positive roots.

We start by showing that our estimate is an upper bound with an appropriate constant. In [4, (14), page 168], it is shown that the subordinator density $\eta_t(u)$ satisfies

$$\eta_t(u) \leq C t u^{-1-s/2} e^{-t u^{-s/2}}. \quad (3.2)$$

Hence, using our estimates of the W -invariant Dunkl heat kernel in Theorem 3.1 and the change of variable $u = |X - Y|^2/(4w)$, we have

$$\begin{aligned} h_t^W(X, Y) &\lesssim t \int_0^\infty \frac{u^{-d/2} e^{-|X-Y|^2/(4u)} u^{-1-s/2} e^{-t u^{-s/2}} du}{\prod_{\alpha>0} (u + \alpha(X) \alpha(Y))^k} \\ &\lesssim t |X - Y|^{-(d+s)} \int_0^\infty \frac{w^{(d+s)/2-1} e^{-w} dw}{\prod_{\alpha>0} (|X - Y|^2/(4w) + \alpha(X) \alpha(Y))^k} \\ &\asymp t |X - Y|^{-(d+s)} \int_0^\infty \frac{w^{(d+s)/2+km-1} e^{-w} dw}{\prod_{\alpha>0} (|X - Y|^2 + \alpha(X) \alpha(Y) w)^k} \\ &\asymp \frac{t |X - Y|^{-(d+s)}}{\prod_{\alpha>0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k} \end{aligned}$$

with an application of Lemma 3.3 to get the last equivalence. This proves the upper bound in the case $|X - Y|^2 \geq t^{2/s}$.

We use Theorem 3.1 and the inequality (3.2) again with the change of variable $u = t^{2/s} w^{-2/s}$. Let Λ be the set of $\alpha > 0$ such that $t^{2/s} \leq \alpha(X) \alpha(Y)$ with m' the number of elements in Λ . We have

$$\begin{aligned} h_t^W(X, Y) &\lesssim t \int_0^\infty \frac{u^{-d/2} u^{-1-s/2} e^{-t u^{-s/2}} du}{\prod_{\alpha>0} (u + \alpha(X) \alpha(Y))^k} \\ &\asymp t^{-d/s} \int_0^\infty \frac{w^{d/s} e^{-w} dw}{\prod_{\alpha>0} (t^{2/s} w^{-2/s} + \alpha(X) \alpha(Y))^k} \\ &\asymp t^{-d/s} \int_0^\infty \frac{w^{d/s+2km/s} e^{-w} dw}{\prod_{\alpha>0} (t^{2/s} + \alpha(X) \alpha(Y) w^{2/s})^k} \\ &\lesssim t^{-d/s} \frac{1}{(\prod_{\alpha \notin \Lambda} t^{2/s})^k} \int_0^\infty \frac{w^{d/s+2km/s} e^{-w} dw}{\prod_{\alpha \in \Lambda} (\alpha(X) \alpha(Y) w^{2/s})^k} \\ &\asymp \frac{t^{-d/s}}{\prod_{\alpha>0} (t^{2/s} + \alpha(X) \alpha(Y))^k} \int_0^\infty w^{d/s+2k(m-m')/s} e^{-w} dw. \end{aligned}$$

This proves the upper bound in the case $|X - Y|^2 \leq t^{2/s}$.

Now we will justify the lower bound. Recall (see [4, (9,10), page 167] or [12, (9), page 89]) that for $u \geq t^{2/s}$, we have

$$\eta_t(u) \asymp t u^{-1-s/2}. \quad (3.3)$$

If $|X - Y|^2 \geq t^{2/s}$ then using formula (3.3), Theorem 3.1 and the change of variable $u = |X - Y|^2/(4w)$, we have

$$\begin{aligned}
h_t^W(X, Y) &\gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-d/2} e^{-|X-Y|^2/(4u)} u^{-1-s/2} du}{\prod_{\alpha>0} (u + \alpha(X) \alpha(Y))^k} \\
&\gtrsim t |X - Y|^{-(d+s)} \int_0^{|X-Y|^2/(4t^{2/s})} \frac{w^{(d+s)/2-1} e^{-w} dw}{\prod_{\alpha>0} (|X - Y|^2/(4w) + \alpha(X) \alpha(Y))^k} \\
&\gtrsim t |X - Y|^{-(d+s)} \int_{1/8}^{1/4} \frac{w^{(d+s)/2-1} e^{-w} dw}{\prod_{\alpha>0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k}
\end{aligned}$$

which proves the lower bound in that case.

Now assume $|X - Y|^2 \leq t^{2/s}$. We use formula (3.3) and Theorem 3.1. Then, since $-1/4 \leq -|X - Y|^2/(4t^{2/s}) \leq -|X - Y|^2/(4u) \leq 0$ for $u \geq t^{2/s}$, using the change of variable $u = t^{2/s} w$, we have

$$\begin{aligned}
h_t^W(X, Y) &\gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-d/2} e^{-|X-Y|^2/(4u)} u^{-1-s/2} du}{\prod_{\alpha>0} (u + \alpha(X) \alpha(Y))^k} \\
&\gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-(d+s)/2-1} du}{\prod_{\alpha>0} (u + \alpha(X) \alpha(Y))^k} \\
&\gtrsim t^{-d/s} \int_1^{\infty} \frac{w^{-(d+s)/2-1} dw}{\prod_{\alpha>0} (t^{2/s} w + \alpha(X) \alpha(Y))^k} \\
&\gtrsim t^{-d/s} \int_1^{\infty} \frac{w^{-(d+s)/2-1} dw}{w^{km} \prod_{\alpha>0} (t^{2/s} + \alpha(X) \alpha(Y))^k} \\
&\asymp \frac{t^{-d/s}}{\prod_{\alpha>0} (t^{2/s} + \alpha(X) \alpha(Y))^k}
\end{aligned}$$

which proves the lower bound in that case. □

Remark 3.13. *The upper estimate in Theorem 3.12 may be deduced from our estimates of the W -invariant Dunkl heat kernel in Theorem 3.1 and from [8, Corollary 3.8 p. 11].*

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