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Sharp exponent of acceleration in general nonlocal equations with a weak Allee effect

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Abstract

We study a acceleration phenomenon arising in monostable integro-differential equations with a weak Allee effect. Previous works have shown its occurrence and have given correct upper bounds on the rate of expansion, but precise lower bounds were still missing. In this paper, we provide a sharp lower bound of acceleration for a large class of dispersion operators. Our results cover fractional Laplace operators and standard convolutions in a unified way. To achieve this, we construct a refined sub-solution that captures the expected dynamics of the accelerating solution. We also take advantage of a general flattening estimate for the solution, that we prove along the way and is of independent interest.

Keywords: generic nonlocal dispersion operators, fractional laplacian, convolution operator, acceleration, level lines.

1 Introduction

In this paper, we are interested in describing quantitatively propagation phenomena in the following (non-local) integro-differential equation:

$$\begin{aligned} u_t(t, x) &= \mathcal{D}[u](t, x) + f(u(t, x)) \quad \text{for } t > 0, x \in \mathbb{R}, \\ u(0, \cdot) &= u_0(x) \geq 0, \end{aligned} \tag{1.1}$$

where f is a monostable nonlinearity specified later on and

$$\mathcal{D}[u](t, x) := P.V. \left(\int_{\mathbb{R}} [u(t, y) - u(t, x)] J(x - y) dy \right).$$

Here, J is a nonnegative function satisfying the following properties.

Hypothesis 1.1. *Let $s > 0$. The kernel J is nonnegative, symmetric and such that there exists positive constants $\mathcal{J}_0, \mathcal{J}_1$ and $R_0 \geq 1$ verifying*

$$\int_{|z| \leq 1} J(z) |z|^2 dz \leq 2\mathcal{J}_1 \quad \text{and} \quad \frac{\mathcal{J}_0}{|z|^{1+2s}} \mathbb{1}_{\{|z| \geq 1\}} \geq J(z) \geq \frac{\mathcal{J}_0^{-1}}{|z|^{1+2s}} \mathbb{1}_{\{|z| \geq R_0\}}.$$

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The operator $\mathcal{D}[\cdot]$ describes the dispersion process of the individuals. Roughly, the kernel J gives the probability of a jump from a position x to a position y , so that the tails of J are of crucial importance to quantify the dynamics of the population. As a matter of fact, the parameter s will appear in rates we obtain later. One may readily notice that our hypothesis on J allows us to cover both broad types of integro-differential operators $\mathcal{D}[u]$ usually considered in the literature : the fractional laplacian $(-\Delta)^s u$ and the standard convolution operators with integrable kernels often written $J \star u - u$. This universality is one main contribution of this paper.

Without further notice, we will consider that f satisfies

Hypothesis 1.2. *The nonlinearity $f \in C^1([0, 1], \mathbb{R})$ is of the monostable type, in the sense that*

$$\begin{cases} f(0) = f(1) = 0, & f(u) > 0, & \text{for } u \in (0, 1), \\ f'(1) < 0, \\ \lim_{u \rightarrow 0} \frac{f(u)}{u^\beta} \geq r, \end{cases}$$

for some $r > 0$, $\beta > 1$.

The parameter β above describes the possibility of a weak Allee effect that the population overcomes. A biological description and discussion about the origin and relevance of such an effect may be found in a book by Courchamp *et al.* [20] but also in [7, 27, 10]. In crude terms, the Allee effect means that a too small population will not have enough strength to survive and expand. This effect is said to be weak whenever the growth rate of a very small population is eventually extremely small but still positive as opposed to a strong Allee effect leading to negative growth rates for small populations. In the sequel, and without further notice, we take $\beta > 1$ (again, yielding small growth rates for small densities). The initial data $u_0 \in C(\mathbb{R}, [0, 1])$ is such that

Hypothesis 1.3. $1 \geq u_0(x) \geq a \mathbf{1}_{(-\infty, b]}(x)$ for some $a > 0$ and $b \in \mathbb{R}$.

Existing works and earlier results

Let us review existing works relevant to position our work. Propagation phenomena in reaction diffusion and integro-differential equations has been the object of intense studies in the last decades. Starting from the work of Fisher on the propagation of an advantageous gene [29] and its analysis by Kolmogorov Petrovski and Piskunov [38] and related works by *e.g.* Aronson and Weinberger [8], the quantitative description of spreading gave birth to various mathematical tools and techniques such as travelling waves, accelerating profiles, transition fronts, among many others.

When $\beta = 1$ and f satisfies $f(s) \leq f'(0)s$, that is f is a Fisher-KPP nonlinearity, it is known that (1.1) exhibits some propagation phenomena: starting with some nonnegative nontrivial compactly supported initial data, the corresponding solution u converges to 1 locally uniformly in space when time gets large. This is referred as the *hair trigger effect* [8]. Moreover, in many cases, the convergence to 1 can be precisely characterised. Indeed, when the dispersion kernel J is exponentially bounded, travelling waves are known to exist and solutions of the Cauchy problem typically propagate at a constant speed [46, 49, 19, 24, 23, 39, 50]. On the other hand, when the dispersion kernel J has heavy tails, travelling waves do not exist and then the Cauchy problem exhibits an acceleration phenomenon [40, 50, 30]. More precisely, Garnier [30] gave the first acceleration estimates and then the first author with Garnier, Henderson and Patout [15] provided sharp level sets for convolution operators; a group around Cabré and Roquejoffre studied the fractional Fisher-KPP equation concluding to an exponential propagation behaviour [18, 17]. Related but different, since playing with the tails of the initial data, acceleration phenomena for positive solutions of a local Cauchy problem also appear in reaction diffusion equations [33]. We emphasise that in the present paper, the acceleration is only due to the structure of the dispersal operator.

When an Allee effect is introduced, the study of propagation is more subtle. Alfaro started the program with a paper about the interplay between heavy tailed initial data and Allee effect in local reaction-diffusion

equations [3]. Coville *et. al.* [24, 23, 22] have proved existence of travelling fronts when the dispersal kernel J is exponentially bounded and the Cauchy problem typically does not lead to acceleration [51]. When not, the competition between heavy tails and Allee effect leads to intense discussions. Gui and Huan discuss the existence or not of travelling waves for a fractional equation with an Allee effect in [32]. They obtain existence and thus finite speed propagation when $\frac{\beta}{2s(\beta-1)} < 1$. However, in this latter paper, no description of acceleration nor precise rate of acceleration were given in the opposite case. In the same spirit, for algebraic decaying kernels, Alfaro and Coville [4] provide the exact separation between existence and non existence of travelling waves for convolution type equations, showing also the exact separation between non acceleration and acceleration in the Cauchy problem. Before reviewing the last-to-date results on (1.1), let us also mention that acceleration phenomenon also appears in some porous medium equations [37, 47, 5, 6].

As far as (1.1) is concerned, bounds on the expansion of the level sets of u have been already obtained by the second author together with Gui and Zhao in [25] and by Alfaro in [4] showing a delicate interplay between the tails of J and the power β . Namely, they obtained an upper bound of acceleration when $\mathcal{D}[\cdot]$ is a fractional laplacian ($J \propto |\cdot|^{-(1+2s)}$) or when J is integrable with a finite first moment (this corresponds to $s > \frac{1}{2}$ here): the solution spreads at at most $t^{\frac{\beta}{2s(\beta-1)}}$ when $\frac{\beta}{2s(\beta-1)} > 1$. However, they were unable to provide a matching lower bound leaving as an open question the exact speed of the level lines. We do not state the exact exponents they get to avoid misunderstandings while reading this paper, but refer to [4, 25] where they are explicit. To have a clearer view, the next figures, Figure 1a and 1b summarize the different known behaviours for this two particular situations.

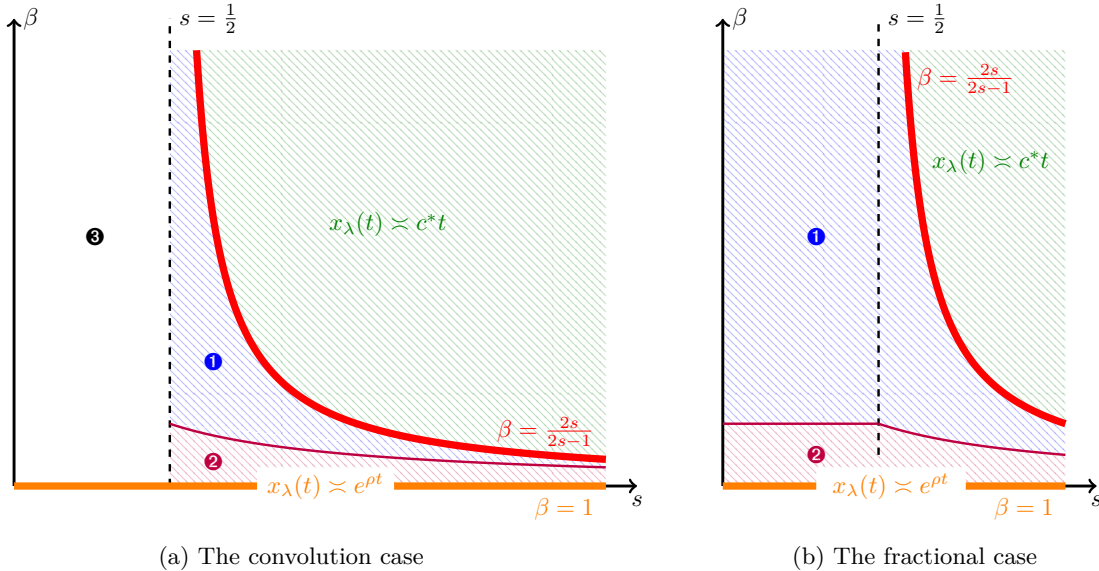


Figure 1: **For the convolution case, Figure 1a**, we have: In the green zone, the model enjoys linear propagation with existence of travelling fronts [21]: $x_\lambda(t) \asymp c^*t$. In the white zone, zone ③, no estimates are known. In the blue zone, zone ①, only an upper bounds has been derived, see [4]: $x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$. In the purple zone, zone ②, non matching lower an upper bounds has been derived, see [4]: $t^{\frac{1}{2s(\beta-1)}} \lesssim x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$. The orange zone is a zone of exponential propagation, Garnier [30], Bouin et al [15]: $x_\lambda(t) \asymp \exp(\rho t)$. **For the fractional case, Figure 1b**, we have: In the green zone, the model enjoys linear propagation with existence of travelling fronts [32, 25]: $x_\lambda(t) \asymp c^*t$. In the blue zone, zone ①, non matching upper and lower bounds have been derived, see [25]: $t^{\frac{1}{2s}} \lesssim x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$. In the purple zone, zone ②, non matching lower an upper bounds have been derived, see [25]: $t^{\frac{1}{2s(\beta-1)}} \lesssim x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$. The orange zone is a zone of exponential propagation, Roquejoffre et al. [17]: $x_\lambda(t) \asymp \exp(\rho t)$.

In a preliminary version of this work [14] (made while completing the current program), assuming that J satisfies 1.1, $s \in (0, 1)$, the present authors provide a lower bound of the acceleration of the level lines of solutions of (1.1) showing for the first time that spreading is of order $t^{\frac{\beta}{2s(\beta-1)}}$ and thus getting a sharp exponent of acceleration. This preliminary work being fully complete, we were informed by Zhang and Zlatoš that they managed to obtain in [52] similar bounds on the fractional laplacian version of the equation using a different approach that relies strongly on the properties of this operator. The present version of our work consists in presenting the full general approach initiated in [14]. We obtain the sharp estimate with the fewest possible assumptions on the measure J , in particular with the fewest restriction on s .

Statement of the result

To follow the propagation, we may define the level sets of height $\lambda \in (0, 1)$ of the solution,

$$x_\lambda(t) := \sup \{x \in \mathbb{R}, u(t, x) \geq \lambda\}.$$

Let us now state precisely our main theorem.

Theorem 1.4. *Assume that J, u_0 and f satisfy respectively Hypothesis 1.1, Hypothesis 1.3 and Hypothesis 1.2 and that*

$$\beta < 1 + \frac{1}{2s-1}.$$

Then for any $\lambda \in (0, 1)$, the level line $x_\lambda(t)$ of the solution of (1.1) accelerates with the following rate¹,

$$x_\lambda(t) \asymp_\lambda t^{\frac{\beta}{2s(\beta-1)}}.$$

To give the reader a clear panorama of the scope of our result, we have summarised our and previous contributions in Figure 2.

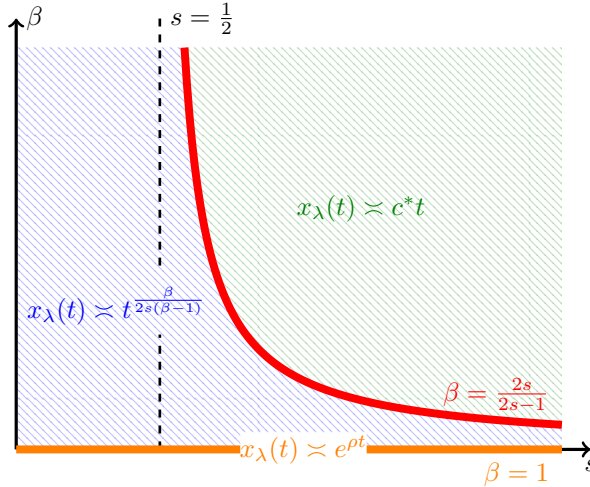


Figure 2: In the green zone, based on the previous work [4, 21, 25, 32] the model is expected to enjoy a linear propagation with existence of travelling fronts: $x_\lambda(t) \sim c^*t$. In the blue zone, we provide the sharp lower and upper bounds : $x_\lambda(t) \asymp t^{\frac{\beta}{2s(\beta-1)}}$. The orange zone is a zone of exponential propagation, by straightforward extension of the work of Bouin et al [15]: $x_\lambda(t) \asymp \exp(\rho t)$.

¹We use the notation $u \asymp_\lambda v$ for the existence of a positive constant C_λ such that $C_\lambda v \leq u \leq C_\lambda^{-1}v$.

Up to our knowledge, this is the first, sharp, unified estimate of the level sets in this generic context. As already explained above, previous papers were able to derive correct upper bounds in particular setting but failed to obtain a precise lower bound in a generic context. Note that the condition on β fits and unifies all related papers [4, 14, 25, 32, 52]. Note also that we obtain the rate of invasion for a convolution operator when $s \in (0, \frac{1}{2}]$, which remained open after [25].

The constructions made in [4, 25] to obtain upper bounds are robust and can be adapted to the range of parameters considered here for kernels satisfying (1.1). To avoid unnecessary computations, we will not duplicate here these constructions. Our contribution is thus a generic way of obtaining a lower bound that matches the already known upper bounds.

To illustrate our result, the position of the level line of height $\lambda = 0.5$ is plotted in Figure 3 as a function of time for two different values of β and several values of the fractional Laplacian exponent s . In one of the two configurations, namely for $\beta = 1.5$, the critical value of the exponent s above which there exists a travelling front is strictly greater than 1. As a consequence, the level set accelerates for any of the chosen values for s , but this acceleration clearly decreases to zero as s tends to 1, as expected from the existing results with local diffusion.

This is no more the case for $\beta = 3$, as one can observe a switching from an accelerated regime to a travel at constant speed around the critical value $s = 0.75$ (the corresponding curve is plotted with a dashed line).

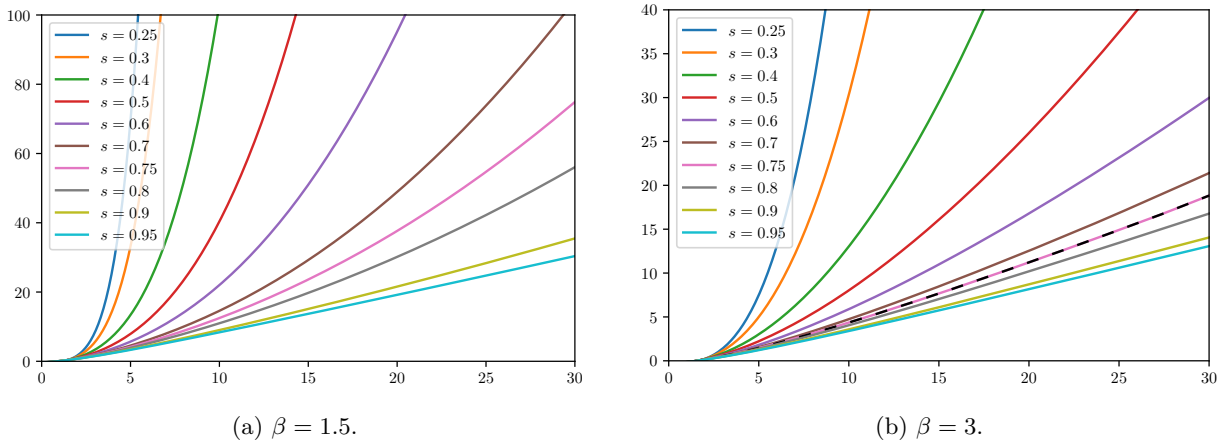


Figure 3: Position of the level line of height $\frac{1}{2}$ of numerical approximations of the solution to the problem with fractional diffusion, plotted as a function of time, for two different values of β and several values of s in $(0, 1)$.

Comments on the strategy

The first step of the proof is to study how the initial data evolves for short times, and in particular which decay at infinity is created by the dispersion with fat tails. We prove in Proposition 2.2 that at time 1, the decay of u solving (1.1) with u_0 satisfying Hypothesis 1.3 is of order x^{-2s} near $+\infty$. When $s \geq 1$, it is sufficient for the rest of the proof.

When $s \in (0, 1)$, an important aspect in our construction is to know that if u is a positive solution to (1.1) with u_0 satisfying Hypothesis 1.3 then the solution of (1.1) flattens through time:

$$\forall C > 0, \quad \exists t_C, \quad \lim_{x \rightarrow +\infty} x^{2s} u(t, x) \geq C, \quad \text{for all } t \geq t_C.$$

Figure 4 illustrates this particular behaviour, showing the deformation of the profile of the solution through time. This flattening property is more clearly seen on the right plot where solutions are shifted back to have the same value at 0. We see that there is no stabilisation of the profile and the shape of the solution changes through time, which is usually not the case when the long time behaviour is a finite speed propagation.

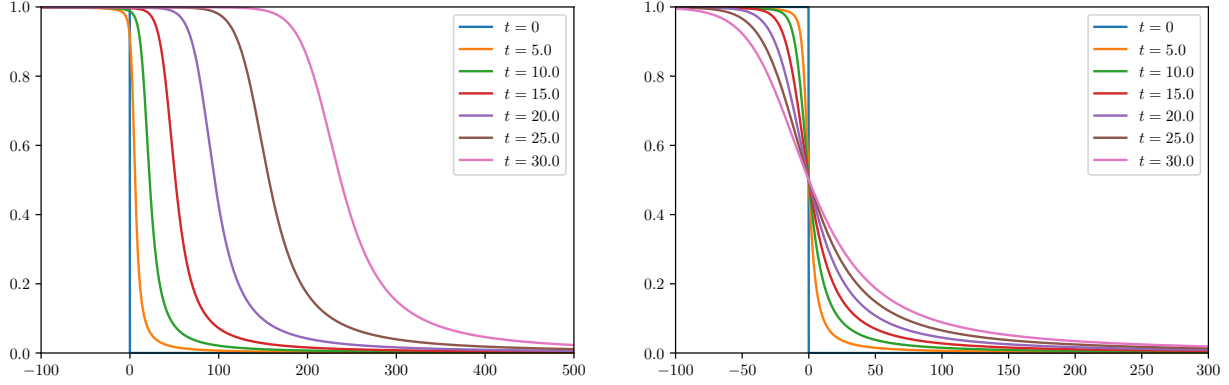


Figure 4: Numerical approximations of the solution to the problem with fractional diffusion at different times for s equal to 0.6 and β equal to 1.5. On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes. The latter exhibits more clearly the deformation of the solution.

To have a more convincing picture of this flattening effect, we may also plot the evolution over time of the best constant C such that the tail of the solution fits with $\frac{C}{x^{2s}}$ in the least square sense. After a rapid transition the graph obtained describes a linear growth of this constant C . We also refer to Figure 13 for various plots showing the adequation between u and x^{-2s} at the edge of the invasion.

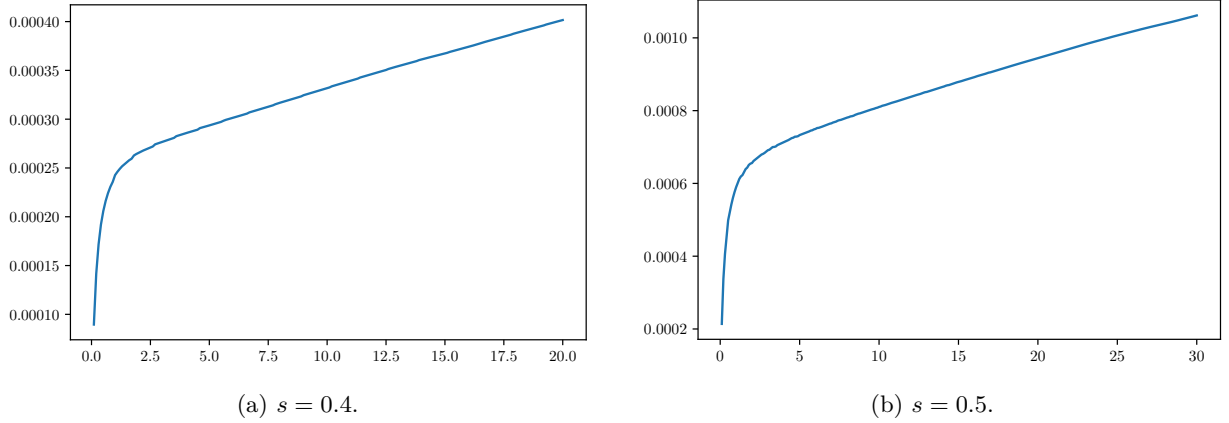


Figure 5: Evolution over time of the fitting constant for the part of the tail of the approximation solution at time $t = 1$ bounded by value 10^{-2} on the left and value 10^{-5} on the right using the function $\frac{C}{x^{2s}}$ for the solution of the problem with fractional diffusion and $\beta = 1.5$ and $s = 0.4/0.5$.

The proof relies on two ingredients. We first show an invasion property in this general context Proposition 2.4. We then combine it with a subtle construction of a sub-solution of the linear problem that mimics the expected scaling behaviour of the heat kernel. Very importantly, this flattening property is in fact true for any $s > 1$, as we show in Section 2. It is worth mentioning that the regime $s \geq 1$ being a regime for which the heat kernel is supposed to behave at large time like a Gaussian diffusion kernel, the flattening of the solution of (1.1) cannot be uniquely explained through the diffusion process and is truly a nonlinear feature. This is a clear dichotomy between the two regimes $s < 1$ and $s \geq 1$.

For particular diffusion operators such as the fractional laplacian, such flattening estimate can be obtained through time and space scaling properties of the associated heat kernel. However, although the characterisation of the heat kernel associated to the generator of a Levy process is a well known problem

in probability theory and analysis that dates back to the original work of Pólya [45] and Blumenthal and Gettoor [11] on α -stable processes, up to our knowledge characterisations of the heat kernel that may induce such flattening estimates have only been established for some specific class of Levy processes [12, 26, 31, 36] and do not exist for a generic Levy process.

Once the initial data has been properly prepared for small times, to achieve the lower bound for large times, our strategy consists in the construction of a new type of sub-solution that captures all the expected dynamics of the solution u . In particular, it turns out to be mandatory to identify several zones of space over which the behaviour of the solution u is given by one specific part of the equation. This is something new compared to previous papers. Roughly, the dynamics close to $t^{\frac{\beta}{2s(\beta-1)}}$ are due to the nonlinearity only via the related ODE, the far-field zone is purely dissipative and has the behaviour of the linearised equation, and the transition zone between the two is a subtle interplay between the two effects. This dichotomy will be detailed and illustrated in Section 5. Lastly, and related to what has just been explained, it is interesting to notice the fact that the exponent of acceleration is a function of β but not the way that the solution flattens with time: it is purely related to the rate of dispersion and will be shown numerically. See Figure 6 for a schematic view of the expected behaviour of the solution.

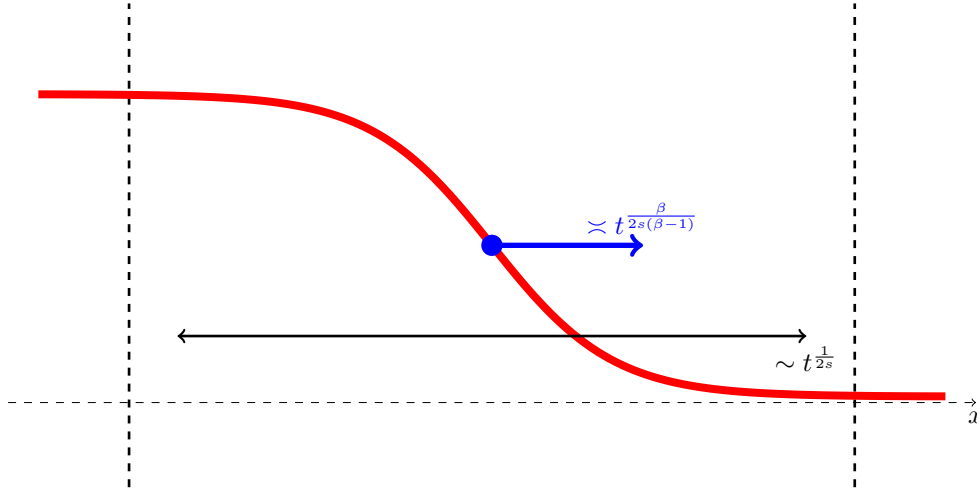


Figure 6: Schematic view of the expected behaviour of solution at a given time t .

Further comments and structure of the paper

It is worth adding that the propagation of a *compactly supported* initial data would lead to a certain amount of different considerations. In particular, the possibility of invasion is related to the size of the initial data due to the existence or not of the so-called *hair-trigger* effect. Depending on the choice of parameter, s and β , for compactly supported initial datum, it may happen that the solution get extinct at large time, which is referred as the *quenching phenomenon* [2, 53], and thus no propagation occur. We have chosen not to focus on this particular issue to concentrate on the description of the acceleration process.

It is important to have in mind that from the point of view of applications, having results at this level of generality of assumptions is of great interest. In particular in ecology, where dispersal is a fundamental process which strongly impacts the evolution species and for which our understanding is still partial [42, 43, 44]. In a sense, by giving access to the right speed of acceleration for a large class of measures, our results provide a unified view of the consequences of potential large jumps in the dispersal process.

It is worth noticing that the asymptotic behaviour obtained in Figure 5 suggests some particular behaviour of the asymptotic of the solution of the nonlinear problem u . Here for the fractional laplacian, we numerically

observe the following asymptotic behaviour: $u(t, x) \sim C_0 t / x^{2s}$ for large x . Such scaling is indeed satisfied by the sub-solution we construct to estimate from below the speed of the level set, however the super-solution used to control this speed does not enjoy it. Obtaining rigorously such asymptotic behaviour is an open question that requires a more precise description of the super-solution in the spirit of our construction. Some investigations in this direction are currently underway.

Lastly, our approach is rather robust and can be extended to more singular monostable nonlinearities such as ignition type nonlinearities. See our companion paper [13].

The rest of the paper is organised as follows. We first derive in Section 2 some estimates on the asymptotic behaviour of the solution (1.1) and prove Proposition 2.4. Then Section 3 describes in broad lines the construction of the sub-solution. We proceed in deeper calculations to achieve Theorem 1.4 in Section 5 and Section 4. We illustrate Theorem 1.4 with numerical experiments in Section 6.

2 Tails and flattening estimates.

2.1 About the tails of u at $t = 1$.

In this section, we show that starting from an Heaviside initial data, the solution immediately gets polynomial tails of order $2s$, for any positive value of s . For this, we construct a sub-solution for short times.

Let us introduce the function v defined by,

$$v(t, x) = \begin{cases} \frac{1}{\nu} & \text{for } t > 0, x \leq 0, \\ \frac{\kappa t}{x^{2s} + \kappa \nu t} & \text{for } t > 0, x > 0. \end{cases} \quad (2.2)$$

where ν and κ are positive constants to be fixed later on. Note that $v(0, \cdot) = \frac{1}{\nu} \mathbb{1}_{(-\infty, 0]}$.

Lemma 2.1. *For all ν, κ verifying $\kappa \nu \leq \frac{1}{2s\mathcal{J}_0}$ then*

$$v_t - \mathcal{D}[v] \leq \frac{\mathcal{J}_0}{2s} v \quad \text{for all } t \in (0, 1), x > R_0 + 1.$$

Proof. For $x > 0, t > 0$, compute,

$$\begin{aligned} v_t(t, x) &= \frac{\kappa x^{2s}}{(x^{2s} + \kappa \nu t)^2}, \\ v_{xx}(t, x) &= 2s v^2(t, x) \frac{x^{2s-2}}{\kappa t} \left[4s \frac{x^{2s}}{x^{2s} + \nu \kappa t} - 2s + 1 \right] = 2s v^2(t, x) \frac{x^{2s-2}}{\kappa t} \left[2s + 1 - \frac{\nu \kappa t}{x^{2s} + \nu \kappa t} \right]. \end{aligned}$$

Note that v is always convex in x for all times $t > 0$ and $\kappa \nu > 0$.

Let us now estimate $\mathcal{D}[v](t, x)$ for $t \geq 0$ and $x \geq R_0 + 1$. We have,

$$\begin{aligned}
\mathcal{D}[v](t, x) &= \int_{-\infty}^{-1} [v(t, x+z) - v(t, x)]J(z) dz \\
&\quad + \int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz + \int_1^{+\infty} [v(t, x+z) - v(t, x)]J(z) dz, \\
&\geq \int_{-\infty}^{-1} [v(t, x+z) - v(t, x)]J(z) dz + \int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz - v(t, x) \int_1^{+\infty} J(z) dz, \\
&\geq \int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz + \int_{-\infty}^{-x} [v(t, x+z) - v(t, x)]J(z) dz \\
&\quad + \int_{-x}^{-1} [v(t, x+z) - v(t, x)]J(z) dz - v(t, x) \int_1^{+\infty} J(z) dz, \\
&\geq \int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz + \left[\frac{1}{\nu} - v(t, x) \right] \int_x^{+\infty} J(z) dz - v(t, x) \int_1^{+\infty} J(z) dz, \\
&\geq \int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz + \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{\nu} - v(t, x) \right] \frac{1}{x^{2s}} - \frac{\mathcal{J}_0}{2s} v(t, x).
\end{aligned}$$

We have used that v is monotone decreasing in x for all t at the third line and Hypothesis 1.1 at the last one. The remaining integral is estimated using the regularity of v and the convexity in x , together with the symmetry of J , we can rewrite it as follows

$$\int_{-1}^1 [v(t, x+z) - v(t, x)]J(z) dz = \int_0^1 \int_0^1 \int_{-1}^1 v_{xx}(t, x + \tau\sigma z) \tau z^2 J(z) dz d\tau d\sigma \geq 0,$$

since $x \geq R_0 + 1 \geq 2$ and thus $v_{xx}(t, x + \xi) \geq 0$ for $\xi \in (-1, 1)$. We conclude,

$$\mathcal{D}[v](t, x) \geq \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{\nu} - v(t, x) \right] \frac{1}{x^{2s}} - \frac{\mathcal{J}_0}{2s} v(t, x).$$

We continue writing, for $t \in (0, 1)$ and $x \geq R_0 + 1 \geq 1$,

$$\begin{aligned}
v_t(t, x) - \mathcal{D}[v](t, x) &\leq \frac{\kappa x^{2s}}{(x^{2s} + \kappa \nu t)^2} - \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{\nu} - v(t, x) \right] \frac{1}{x^{2s}} + \frac{\mathcal{J}_0}{2s} v(t, x), \\
&\leq \frac{\kappa}{x^{2s} + \kappa \nu t} - \frac{\mathcal{J}_0^{-1}}{2s \nu} \left[1 - \frac{\kappa \nu t}{x^{2s} + \nu \kappa t} \right] \frac{1}{x^{2s}} + \frac{\mathcal{J}_0}{2s} v(t, x), \\
&= \frac{\kappa}{x^{2s} + \kappa \nu t} - \frac{\mathcal{J}_0^{-1}}{2s \nu} \frac{1}{x^{2s} + \nu \kappa t} + \frac{\mathcal{J}_0}{2s} v(t, x), \\
&= \frac{\kappa}{x^{2s} + \kappa \nu t} \left(1 - \frac{\mathcal{J}_0^{-1}}{2s \nu \kappa} \right) + \frac{\mathcal{J}_0}{2s} v(t, x) \leq \frac{\mathcal{J}_0}{2s} v(t, x),
\end{aligned}$$

when $\kappa \nu \leq \frac{1}{2s \mathcal{J}_0}$. □

Equipped with the above lemma, we obtain,

Proposition 2.2. *Let u be a solution to (1.1), with J satisfying Hypothesis 1.1. Then, there exists $D > 0$ such that,*

$$\lim_{x \rightarrow +\infty} x^{2s} u(1, x) \geq D^{2s}.$$

Proof. Observe that thanks to comparison principle and since u_0 satisfies Hypothesis 1.3 it is enough to prove this proposition for monotone initial data u_0 . In this situation, i.e. u_0 is monotone non-increasing, by a straightforward application of the comparison principle so does $x \mapsto u(t, x)$ for all times, and we have $u(t, x) \geq u(t, R_0 + 1)$ for all times $t > 0$ and $x \leq R_0 + 1$. Since $u(t, x) > 0$ for all $t > 0$ and all $x \in \mathbb{R}$, we have $\delta := \inf_{t \in [\frac{1}{2}, \frac{3}{2}]} u(t, R_0 + 1) > 0$ and thus $u(t, x) \geq \delta$ for all $t \in (\frac{1}{2}, \frac{3}{2})$, $x = R_0 + 1$.

Consider now z as above with $\nu > \frac{1}{\delta}$ and κ so that $\kappa\nu \leq \frac{1}{2s\mathcal{J}_0}$. Then for such a choice of parameters, the function $\tilde{v}(t, \cdot) := (1 - t) e^{-\frac{\mathcal{J}_0}{2s}t} z(t, \cdot)$ satisfies

$$\begin{cases} \tilde{v}_t(t, x) \leq \mathcal{D}[\tilde{v}](t, x) & \text{for } t \in (0, 1), \quad x > R_0 + 1, \\ \tilde{v}(0, \cdot) = \frac{1}{\nu} \mathbf{1}_{(-\infty, 0]}, \\ \tilde{v}(1, \cdot) = 0, & \text{for } x \geq R_0 + 1, \\ \tilde{v}(t, x) \leq \frac{1}{\nu}, & \text{for } t \in (0, 1), \quad x \leq R_0 + 1. \end{cases} \quad (2.3)$$

The function $\tilde{u} := u(\cdot + \frac{1}{2}, \cdot)$ satisfies,

$$\begin{cases} \tilde{u}_t(t, x) \geq \mathcal{D}[\tilde{u}](t, x) & \text{for } t \in (0, 1), \quad x > R_0 + 1, \\ \tilde{u}(0, \cdot) = u(\frac{1}{2}, \cdot) > \frac{1}{\nu} \mathbf{1}_{(-\infty, 0]}, \\ \tilde{u}(1, \cdot) \geq 0, & \text{for } x \geq R_0 + 1, \\ \tilde{u}(t, x) \geq \delta > \frac{1}{\nu}, & \text{for } t \in (0, 1), \quad x \leq R_0 + 1. \end{cases} \quad (2.4)$$

Using the parabolic comparison principle it follows that for all $(t, x) \in (0, 1) \times [R_0 + 1, +\infty[$, one has $u(t + \frac{1}{2}, x) \geq \tilde{v}(t, x)$ and thus

$$\lim_{x \rightarrow +\infty} x^{2s} u(1, x) \geq \lim_{x \rightarrow +\infty} x^{2s} \tilde{v}(\frac{1}{2}, x) = \frac{\kappa}{2} \left(1 - \frac{1}{2}\right) e^{-\frac{\mathcal{J}_0}{4s}} \lim_{x \rightarrow \infty} \frac{x^{2s}}{x^{2s} + \frac{1}{2}\kappa\nu} = \frac{\kappa}{4} e^{-\frac{\mathcal{J}_0}{4s}} := D^{2s}.$$

□

2.2 Flattening estimates for large times.

Here we push further our analysis of the tail of the solution of (1.1) and obtain a flattening estimate in the following sense. For any $C > 0$ there exists a positive time t_C such that $u(t, x)$ the solution of the nonlinear problem (1.1), satisfies

$$\lim_{x \rightarrow +\infty} x^{2s} u(t_C, x) \geq C.$$

More precisely, we prove the following proposition,

Proposition 2.3. *Assume J and u_0 satisfy respectively (1.1) and (1.3) and let u be a positive solution to (1.1). Then for all $C > 0$, there exists $t_C > 0$ such that $u(t_C, x)$ satisfies the following*

$$\lim_{x \rightarrow +\infty} x^{2s} u(t_C, x) \geq C.$$

Before showing this proposition, let us establish some invasion properties of the solution of (1.1). Namely,

Proposition 2.4. *Assume J, u_0 and f satisfy respectively Hypothesis 1.1, Hypothesis 1.3 and Hypothesis 1.2 then the solution to (1.1) satisfies for all $A > 0$,*

$$u(t, x) \rightarrow 1 \quad \text{uniformly in } (-\infty, A] \text{ as } t \rightarrow \infty.$$

Proof. As above, let us observe that thanks to the parabolic comparison principle and since u_0 satisfies Hypothesis 1.3 it is enough to prove this proposition for monotone initial data u_0 . Observe also that when $J \in L^1$ or \mathcal{D} is the Fractional Laplacian then the above invasion statement have already been proved in

[4, 25]. So we won't repeat this proof and only consider that J has a non-integrable singularity and \mathcal{D} is not the Fractional Laplacian. Let us also observe that since f satisfies Hypothesis 1.2 we may find r_0 small so that $f(s) \geq r_0 s^\beta (1-s)$ with $\beta > 1$ and again thanks to the parabolic comparison principle, it then is enough to prove the this invasion proposition for nonlinearity f of the form $f(s) := r_0 s^\beta (1-s)$ with $\beta > 1$. So let us assume that $f(s) := r_0 s^\beta (1-s)$ with $\beta > 1$. The idea is now to construct a subsolution to (1.1) that fills all the space. Let us observe that for any nonnegative nonlinearity f and any function v and $R \geq R_0$ we have

$$\mathcal{D}[v](x) + f(v(x)) = \int_{-\infty}^{+\infty} [v(x+h) - v(x)]J(h) dh + f(v(x)), \quad (2.5)$$

$$= \int_{|h| < R} [u(x+h) - u(x)]J(h) dh + \int_{|h| \geq R} [v(x+h) - v(x)]J(h) dh + f(v(x)), \quad (2.6)$$

$$\geq \int_{|h| < R} [v(x+h) - v(x)]J(h) dh - v(x) \int_{|h| \geq R} J(h) dh + f(v(x)), \quad (2.7)$$

$$\geq \int_{|h| < R} [v(x+h) - v(x)]J(h) dh - \frac{\mathcal{J}_0}{R^{2s}} v(x) + f(v(x)). \quad (2.8)$$

Set $f_R(s) := -\frac{\mathcal{J}_0}{R^{2s}} s + f(s)$ and let us denote by \mathcal{D}_R the diffusion operator with the kernel $J(h)\mathbf{1}_{B_R(0)}(h)$ instead of J , then from the above computations we have for any positive solution $u(t, x)$ to (1.1),

$$\partial_t u(t, x) - \mathcal{D}_R[u](t, x) - f_R(u(t, x)) \geq \partial_t u(t) - \mathcal{D}[u](t, x) + f(u(t, x)) = 0.$$

Let $0 < \theta < a := \liminf_{x \rightarrow \infty} u_0(x)$ and for $\theta < \frac{1}{2}$ let us consider $f_\theta \leq f$ a bistable function such that $f_\theta(0) = f_\theta(\theta) = f_\theta(1-\theta) = 0$, and $f_\theta(x) > 0$ for all $x \in (\theta, 1-\theta)$. Choose $\theta < \frac{a}{8}$ small, so that $1-\theta > a$ and $\int_0^{1-\theta} f_\theta(s) ds > 0$.

Now since $f_R \rightarrow f$ as $R \rightarrow \infty$, we may find R_θ such that $f_\theta \leq f_{R_\theta}$ and so we have for $R \geq R_\theta$

$$\partial_t u(t, x) - \mathcal{D}_R[u](t, x) - f_\theta(u(t, x)) \geq 0. \quad (2.9)$$

Let us smoothly extend f_θ outside $[0, 1-\theta]$ as follow:

$$\begin{cases} f'_\theta(0)s & \text{when } s < 0 \\ f_\theta(s) & \text{when } 0 \leq s \leq 1-\theta \\ f'_\theta(1-\theta)(s-1+\theta) & \text{when } 1-\theta < s \end{cases}$$

and for convenience let us denote f_θ this extension. Let us now consider the following problem

$$\partial_t v(t, x) - \mathcal{D}_R[v](t, x) - f_\theta(v(t, x)) = 0. \quad (2.10)$$

Observe that from (2.9), u is a supersolution to (2.10). Let us now construct a adequate subsolution to (2.10). Thanks to [1], we know that the problem (2.10) admits a unique monotone travelling wave solution $(\varphi_\theta, c_\theta)$ connecting 0 to $1-\theta$ which is smooth since J has a non integrable singularity. That is $(\varphi_\theta, c_\theta)$ is a smooth solution to

$$\begin{aligned} c_\theta \varphi'_\theta(z) + \mathcal{D}_R[\varphi_\theta](z) + f_\theta(\varphi_\theta(z)) &= 0 \quad \text{for all } z \in \mathbb{R} \\ \lim_{z \rightarrow -\infty} \varphi_\theta(z) &= 1-\theta, \quad \lim_{z \rightarrow +\infty} \varphi_\theta(z) = 0. \end{aligned}$$

Observe that by definition of f_θ we must have $c_\theta > 0$ since the sign of the speed in such context is given by the sign of $\int_0^{1-\theta} f_\theta(s) ds$. Let us now normalize φ_θ by $\varphi_\theta(0) = \theta$ and set

$$w_{\varepsilon, \kappa, L}(t, x) := \varphi_\theta \left(x - c_\theta t + \kappa(1 - e^{-\varepsilon t}) + L \right) - \left(1 - \frac{a}{2} \right) e^{-\varepsilon t},$$

with ε, κ, L as free parameters to be fixed later on. Observe that at $t = 0$, we have

$$w_{\varepsilon, \kappa, L}(0, x) \leq \frac{a}{2} - \theta < a \quad \text{for all } \kappa, L, \varepsilon, x \quad (2.11)$$

$$w_{\varepsilon, \kappa, L}(0, x) \leq \frac{a}{2} + \theta - 1 < 0 \quad \text{for all } \kappa > 0, L > 0, \varepsilon > 0, x > 0. \quad (2.12)$$

As a consequence, since φ_θ is monotone for ε and κ fixed we may always find L_0 such that $u_0 \geq w_{\varepsilon, \kappa, L_0}(t, x)$. Let us now show that for the right choice of parameters, ε, κ , the function $w_{\varepsilon, \kappa, L_0}(t, x)$ is a sub-solution to (2.10).

Claim 2.5. *There exists, ε, κ such that for all L , $w_{\varepsilon, \kappa, L}$ is a sub-solution to (2.10).*

Let us postpone the proof of the claim for the moment, then having this result at hand by using the parabolic comparison principle we then deduce that $u(t, x) \geq w_{\varepsilon, \kappa, L_0}$ and thus for all $A \in \mathbb{R}$, $\lim_{t \rightarrow \infty} u(t, x) \geq 1 - \theta$ in $(-\infty, A]$. θ being arbitrary small the latter argument then implies that $u(t, x) \rightarrow 1$ locally uniformly in $(-\infty, A]$ as $t \rightarrow +\infty$ and since $u(t, x)$ is monotone non increasing in x , the convergence is then uniform. \square

To complete the above proof, let us established the claim.

Proof of the Claim. Compute $\partial_t w_{\varepsilon, \kappa, L}$, then we have

$$\partial_t w_{\varepsilon, \kappa, L} = (-c + \kappa \varepsilon e^{-\varepsilon t}) \varphi' (x - c_\theta t + \kappa(1 - e^{-\varepsilon t}) + L) + \varepsilon \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}$$

Set $\xi(t, x) := x - c_\theta t + \kappa(1 - e^{-\varepsilon t}) + L$, then by using the equation satisfied by φ_θ we have

$$\begin{aligned} \partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &= \kappa \varepsilon e^{-\varepsilon t} \varphi'(\xi(t, x)) + \varepsilon \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \\ &\quad + f_\theta(\varphi_\theta(\xi(t, x))) - f_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \end{aligned} \quad (2.13)$$

Choose $0 < \delta_0 < \frac{a}{8}$ such that $f_\theta(s)$ satisfies

$$\begin{aligned} f_\theta(s) &\leq \frac{f'_\theta(0)}{2} s \quad \text{for } s \in (0, \delta_0) \\ f_\theta(s) &\leq \frac{f'_\theta(1-\theta)}{2} (s - 1 + \theta) \quad \text{for } s \in (1 - \theta - 4\delta_0, 1 - \theta) \end{aligned}$$

Inspired by the construction made in [16], let $\delta < \delta_0$ and choose now $A(\delta) \gg 1$ such that $\varphi_\theta(z) \leq \delta$ if $z \geq A$ and $\varphi_\theta(z) \geq 1 - \theta - \delta$ for $z \leq -A$. We now treat the three situations $\xi(t, x) > A$, $\xi(t, x) < -A$ and $|\xi(t, x)| < A$ separately.

The case $\xi(t, x) > A$: When $\xi(t, x) > A$, we have two possibilities, either $\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} > 0$ or $\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \leq 0$. In the latter case, we have $f_\theta(\varphi_\theta(\xi(t, x))) \leq \frac{f'_\theta(0)}{2} \varphi_\theta(\xi(t, x))$ and

$$f_\theta(\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}) = f'_\theta(0) \left[\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \right]$$

So, since $\varphi'_\theta < 0$, we have

$$\begin{aligned} \partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &\leq \varepsilon \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} + \frac{f'_\theta(0)}{2} \varphi_\theta(\xi(t, x)) - f'_\theta(0) \left[\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \right], \\ &\leq \left[\varepsilon + \frac{f'_\theta(0)}{2} \right] \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} - \frac{f'_\theta(0)}{2} \left[\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \right], \\ &\leq 0, \end{aligned}$$

as soon as $\varepsilon \leq -\frac{f'_\theta(0)}{2}$. In the other situation, we have $\delta \geq \varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \geq 0$ and therefore

$$f_\theta(\varphi_\theta(\xi(t, x))) - f_\theta\left(\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}\right) \leq \frac{f'_\theta(0)}{2} \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}.$$

As above, we conclude that

$$\begin{aligned} \partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &\leq \left[\varepsilon + \frac{f'_\theta(0)}{2}\right] \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \\ &\leq 0, \end{aligned}$$

as soon as $\varepsilon \leq -\frac{f'_\theta(0)}{2}$.

The case $\xi(t, x) < -A$: Let us now assume that $\xi(t, x) < -A$. We then have $\varphi_\theta(\xi(t, x)) \geq 1 - \theta - \delta$ and $f(\varphi_\theta(\xi(t, x))) \leq -\frac{f'_\theta(1-\theta)}{2}\delta$. If $\left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \leq 3\delta_0$ then $\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \leq 1 - \theta - 4\delta_0$ and therefore

$$f_\theta(\varphi_\theta(\xi(t, x))) - f_\theta\left(\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}\right) \leq \frac{f'_\theta(1)}{2} \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}.$$

Thus we have

$$\begin{aligned} \partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &\leq \left[\varepsilon + \frac{f'_\theta(1)}{2}\right] \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \\ &\leq 0, \end{aligned}$$

provided $\varepsilon \leq -\frac{f'_\theta(1)}{2}$.

In the other situation, we have $\left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \geq 3\delta_0$ and

$$\frac{3a}{8} - \delta \leq \frac{a}{2} - \theta - \delta \leq \varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \leq 1 - \theta - 3\delta_0.$$

Since $\delta \leq \frac{a}{8}$, we can ensure that

$$f_\theta(\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}) \geq m_0 := \min_{s \in [\frac{a}{4}, 1-\theta-3\delta_0]} f_\theta(s).$$

As a consequence, we have

$$\begin{aligned} \partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &\leq \varepsilon \left(1 - \frac{a}{2}\right) - \frac{f'_\theta(1-\theta)}{2} \delta - m_0 \\ &\leq 0, \end{aligned}$$

provided ε and δ are chosen small enough says $\varepsilon \leq \frac{m_0}{2-a}$ and $\delta \leq \frac{m_0}{f'_\theta(1-\theta)}$.

The case $|\xi(t, x)| < A$: Lastly, let us assume that $|\xi| < A$. On that region $\varphi'_\theta(z) < 0$ and therefore

$$\varphi'_\theta(\varphi(\xi(t, x))) \leq -\nu_0 := \sup_{z \in [-A, A]} \varphi'_\theta(z) < 0.$$

Recall that f_θ is a Lipschitz function, so we also have

$$f_\theta(\varphi_\theta(\xi(t, x))) - f_\theta\left(\varphi_\theta(\xi(t, x)) - \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}\right) \leq \|f'_\theta\| \left(1 - \frac{a}{2}\right) e^{-\varepsilon t}.$$

Thus, we end up with

$$\begin{aligned}\partial_t w_{\varepsilon, \kappa, L} - \mathcal{D}_R[w_{\varepsilon, \kappa, L}] - f_\theta(w_{\varepsilon, \kappa, L}) &\leq -\kappa \varepsilon e^{-\varepsilon t} \nu_0 + (\varepsilon + \|f'_\theta\|) \left(1 - \frac{a}{2}\right) e^{-\varepsilon t} \\ &\leq \left[-\kappa \varepsilon \nu_0 + (\varepsilon + \|f'_\theta\|) \left(1 - \frac{a}{2}\right)\right] e^{-\varepsilon t} \\ &\leq 0\end{aligned}$$

provided κ is chosen large enough, says $\kappa \geq \frac{(\varepsilon + \|f'_\theta\|)(2-a)}{2\varepsilon \nu_0}$. □

Remark 2.6. *The above proof do not need any specific form of the nonlinearity f , we only need that f is a monostable nonlinearity, in the sense that $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$. As a consequence, the Proposition then holds true for any monostable f . In addition with some minor adaptation in the way the bistable function is construct, the proof will also stand for ignition type nonlinearity f .*

Let us now obtain the flattening result and prove Proposition 2.3.

Proof of Proposition 2.3. As in the proof of Proposition 2.2, we will construct an adequate subsolution. To this end, let v be the parametric function defined in the proof of Proposition 2.2 where we have set $\nu = 2$ and $\kappa = \frac{\mathcal{J}_0^{-1}}{8s}$

$$v(t, x) = \begin{cases} \frac{1}{2} & \text{for } t > 0, x \leq 0, \\ \frac{\kappa t}{x^{2s} + \kappa 2t} & \text{for } t > 0, x > 0. \end{cases} \quad (2.14)$$

Let us now refine our estimate of $\mathcal{D}[v]$. Let $R > 1$ to be chosen then as in the proof of Proposition 2.2 for $t > 0$ and $x \geq R_0 + R$, we have

$$\begin{aligned}\mathcal{D}[v](t, x) &= \int_{-\infty}^{-R} [v(t, x+z) - v(t, x)] J(z) dz \\ &\quad + \int_{-R}^R [v(t, x+z) - v(t, x)] J(z) dz + \int_R^{+\infty} [v(t, x+z) - v(t, x)] J(z) dz, \\ &\geq \int_{-\infty}^{-R} [v(t, x+z) - v(t, x)] J(z) dz + \int_{-R}^R [v(t, x+z) - v(t, x)] J(z) dz - v(t, x) \int_R^{+\infty} J(z) dz, \\ &\geq \int_{-R}^R [v(t, x+z) - v(t, x)] J(z) dz + \int_{-\infty}^{-x} [v(t, x+z) - v(t, x)] J(z) dz \\ &\quad + \int_{-x}^{-R} [v(t, x+z) - v(t, x)] J(z) dz - v(t, x) \int_R^{+\infty} J(z) dz, \\ &\geq \int_{-R}^R [v(t, x+z) - v(t, x)] J(z) dz + \left[\frac{1}{2} - v(t, x)\right] \int_x^{+\infty} J(z) dz - v(t, x) \int_R^{+\infty} J(z) dz, \\ &\geq \int_{-R}^R [v(t, x+z) - v(t, x)] J(z) dz + \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{2} - v(t, x)\right] \frac{1}{x^{2s}} - \frac{\mathcal{J}_0}{2sR^{2s}} v(t, x).\end{aligned}$$

As well the remaining integral is estimated similarly using the regularity of v and the convexity in x , together with the symmetry of J , and thus for $x > R_0 + R$,

$$\mathcal{D}[v](t, x) \geq \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{2} - v(t, x)\right] \frac{1}{x^{2s}} - \frac{\mathcal{J}_0}{2sR^{2s}} v(t, x).$$

All together, we then have for $t > 0$ and $x \geq R_0 + R$,

$$\begin{aligned}
v_t(t, x) - \mathcal{D}[v](t, x) &\leq \frac{\kappa x^{2s}}{(x^{2s} + \kappa \nu t)^2} - \frac{\mathcal{J}_0^{-1}}{2s} \left[\frac{1}{2} - v(t, x) \right] \frac{1}{x^{2s}} + \frac{\mathcal{J}_0}{2s R^{2s}} v(t, x), \\
&\leq \frac{1}{x^{2s} + 2\kappa t} \left(-\frac{\mathcal{J}_0^{-1}}{8s} + \frac{\mathcal{J}_0 \kappa t}{2s R^{2s}} \right).
\end{aligned}$$

For any $C > 0$, let us now define $t^* := \frac{2C}{\kappa}$ and choose R large enough says $R \geq R_C := (8C\mathcal{J}_0^2)^{\frac{1}{2s}}$. From the above computation we then have for all $t \in (0, t^*)$ and $x \geq R_0 + R_C$

$$v_t(t, x) - \mathcal{D}[v](t, x) \leq 0 \quad \text{for all } t \in (0, t^*), x \geq R_0 + R_C. \quad (2.15)$$

Equipped with this subsolution, let us now conclude our proof. By using the invasion property, Proposition 2.4, there exists t_C such that for all $t \geq t_C$ we have

$$u(t, x) \geq \frac{3}{4} \quad \text{for all } t > 0, x \leq R_0 + R_C.$$

So the function $\tilde{u}(t, x) := u(t + t_C, x)$ then satisfies

$$\begin{aligned}
\tilde{u}_t(t, x) - \mathcal{D}[\tilde{u}](t, x) &\geq 0 \quad \text{for all } t \in (0, t^*), x \in \mathbb{R}, \\
\tilde{u}(t, x) &\geq v(t, x) \quad \text{for all } t \in [0, t^*], x \leq R_0 + R_C.
\end{aligned}$$

Using the comparison principle it follows that for all $(t, x) \in (0, t^*) \times [R_0 + R_C, +\infty[$, one has $\tilde{u}(t, x) \geq v(t, x)$ and thus

$$\lim_{x \rightarrow +\infty} x^{2s} u(t_C + t^*/2, x) \geq \lim_{x \rightarrow +\infty} x^{2s} v(t^*/2, x) = C.$$

□

3 Strategy for the construction of sub-solutions.

As previously mentioned, our main strategy is to construct a sub-solution to (1.1) that will mimic some expected behaviours. As observed in the previous section, since f satisfies Hypothesis 1.2, we have for r_0 small, $f(s) \geq r_0 s^\beta (1 - s)$. Consequently, we only need to construct a sub-solution for equation (1.1) with f having this specific form. Let us also observe that by scaling in space in time the solution as well the measure J , i.e. $v(t, x) := u\left(\frac{t}{r_0}, \frac{x}{r_0}\right)$ and taking $J\left(\frac{z}{r_0}\right) dz$, we can reduce our construction to finding a sub-solution to the following problem:

$$\partial_t v(t, x) = \mathcal{D}_{r_0}[v](t, x) + v^\beta(t, x)(1 - v(t, x)). \quad (3.16)$$

where \mathcal{D}_{r_0} denote the operator \mathcal{D} with the rescaled measure $J\left(\frac{z}{r_0}\right) dz$. In the sequel, to keep tractable notations, we drop the subscript of the diffusion operator.

3.1 The form of the sub-solutions.

We are looking for a sub-solution \underline{u} to (3.16) that satisfies everywhere

$$\underline{u}_t \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon)\underline{u}^\beta \quad \text{and} \quad \underline{u} \leq \varepsilon, \quad (3.17)$$

for some $\varepsilon \in (0, 1)$. Indeed, this would give, if $\underline{u}(0, \cdot) \leq u(t', \cdot)$, for some $t' > 0$,

$$\underline{u}_t \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon)\underline{u}^\beta \leq \mathcal{D}[\underline{u}] + (1 - \underline{u})\underline{u}^\beta$$

and thus \underline{u} is a subsolution to u . We construct an at least of class \mathcal{C}^2 function \underline{u} piecewise,

$$\begin{aligned}\underline{u} &:= \varepsilon, & \text{on } \{x \leq X(t)\}, \\ \underline{u} &:= \phi, & \text{else,}\end{aligned}$$

with $\phi(t, X(t)) = \varepsilon$. The point $X(t)$ is unknown at that stage. We expect ϕ to solve an ODE of the form $n' = n^\beta$ near $x = X(t)$ and to look like a solution of the standard fractional Laplace equation with Heaviside initial data at the far edge. A natural candidate would be given by

$$w(t, x) := \left[\left(\frac{\kappa t}{x^{2s}} \right)^{1-\beta} - \gamma(\beta-1)t \right]^{-\frac{1}{\beta-1}}. \quad (3.18)$$

Note that this function is well defined for $t \geq 1$ and $x > X_0 := \kappa^{\frac{1}{2s}} (\gamma(\beta-1)t)^{\frac{\beta}{(\beta-1)2s}}$ and has visually the structure of a solution to the ODE $n' = n^\beta$. The expected decay in space of a solution of the standard fractional Laplace equation with Heaviside initial data being at least of order tx^{-2s} , such a w would have the good asymptotics. Let us define $X(t)$ such that $w(t, X(t)) = \varepsilon$, that is

$$X(t) = (\kappa t)^{\frac{1}{2s}} [\varepsilon^{1-\beta} + \gamma(\beta-1)t]^{\frac{1}{2s(\beta-1)}}. \quad (3.19)$$

The positive constants κ and γ are for the moment free parameters to be chosen later on. One may observe that $X(t)$ moves with the speed that we expect in Theorem 1.4. However, taking ϕ as this w would not lead to a \mathcal{C}^2 function at $x = X(t)$. To remedy this issue, we complete our construction by taking ϕ such that

$$\underline{u}(t, x) := \begin{cases} \varepsilon & \text{for all } x \leq X(t), \\ 3 \left(1 - \frac{w(t, x)}{\varepsilon} + \frac{w^2(t, x)}{3\varepsilon^2} \right) w(t, x) & \text{for all } x > X(t), \end{cases} \quad (3.20)$$

for $t > 1$.

Start by observing that \underline{u} satisfies (3.17) if and only if,

$$0 \leq \mathcal{D}[\underline{u}] + \varepsilon^\beta(1 - \varepsilon), \quad x \leq X(t), \quad (3.21)$$

$$\phi_t \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon)\phi^\beta, \quad \text{else} \quad (3.22)$$

As a consequence, the main work is to derive good estimates for $\mathcal{D}[\underline{u}]$ in both regions $x \leq X(t)$ and $x \geq X(t)$. The estimate in the first region will be rather direct to get and will rely mostly on the fact that \underline{u} is constant there together with the tails of J . In the latter region, things are more intricate. We have to split it into three zones, as depicted on Figure 7 below, each one been the stage of one specific character of the model and thus demanding a specific way to estimate $\mathcal{D}[\underline{u}]$.

3.2 Facts and formulas on X and w .

First, from direct computations we have:

$$\underline{u}_t = \underline{u}_x = \underline{u}_{xx} = 0 \quad \text{for all } t > 0, x < X(t) \quad (3.23)$$

$$\underline{u}_t = 3w_t \left(1 - \frac{w}{\varepsilon} \right)^2 \quad \text{for all } t > 1, x > X(t) \quad (3.24)$$

$$\underline{u}_x = 3w_x \left(1 - \frac{w}{\varepsilon} \right)^2 \quad \text{for all } t > 1, x > X(t) \quad (3.25)$$

$$\underline{u}_{xx}(t, x) = 3 \left(1 - \frac{w}{\varepsilon} \right) \left[w_{xx} \left(1 - \frac{w}{\varepsilon} \right) - \frac{2w_x^2}{\varepsilon} \right] \quad \text{for all } t > 1, x > X(t) \quad (3.26)$$

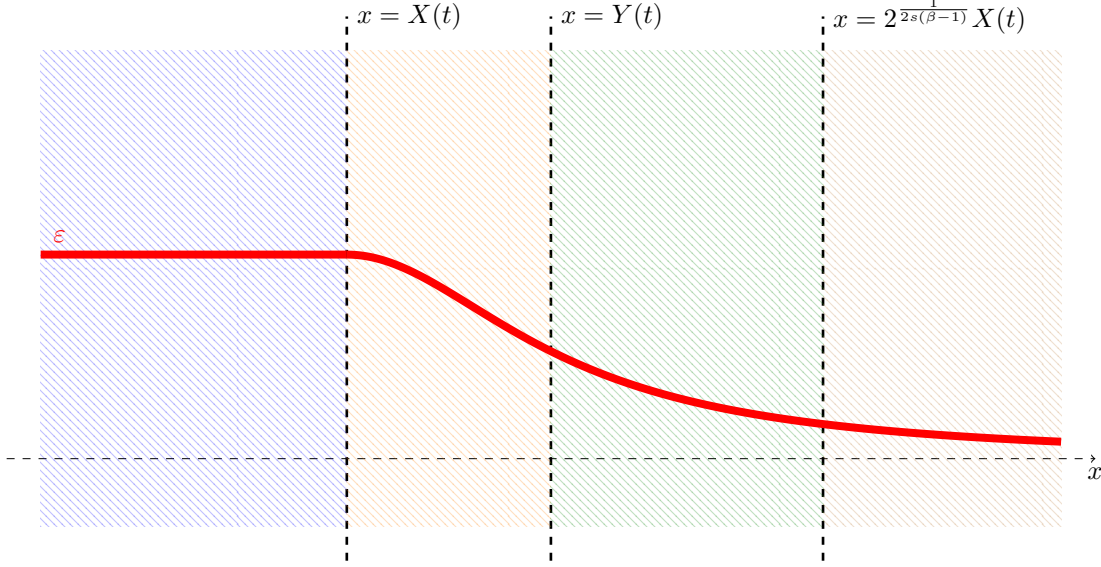


Figure 7: Schematic view of the sub-solution at a given time t . Several zones have to be considered. The exact expression of $Y(t)$ will appear naturally later. The blue zone is where \underline{u} is constant, making computations easier. In the orange zone, the fact that \underline{u} looks like a solution to an ODE $n' = n^\beta$ is crucial. In the brown (far-field) zone, the decay imitating a fractional Laplace equation gives the right behaviour. The green zone is subtle and needs a mixture between both surrounding zones.

Note crucially that \underline{u} is then at a \mathcal{C}^2 function in x and \mathcal{C}^1 in t . For convenience, let us denote

$$\Phi(t, x) := \frac{\kappa t}{x^{2s}} \quad U := \frac{w}{\Phi}.$$

We will need repeatedly the following information on derivatives of w at any point (t, x) where w is defined.

$$w_t = w^\beta \left(\gamma + \frac{\Phi_t}{\Phi^\beta} \right) \quad (3.27)$$

$$w_x = w^\beta \frac{\Phi_x}{\Phi^\beta} = -2s w^\beta \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} = U^\beta \Phi_x \quad (3.28)$$

$$w_{xx} = \beta w^{\beta-1} w_x \frac{\Phi_x}{\Phi^\beta} + w^\beta \frac{\Phi_{xx} \Phi^\beta - |\Phi_x|^2 \beta \Phi^{\beta-1}}{\Phi^{2\beta}} = [\Phi_{xx} + \beta |\Phi_x|^2 \Phi^{-1} (U^{\beta-1} - 1)] U^\beta \quad (3.29)$$

Since $U \geq 1$, we deduce from the latter that w is convex. In addition, by rewriting \underline{u}_{xx} in terms of U and Φ let us observe that

$$\begin{aligned} \underline{u}_{xx} &= 3 \left(1 - \frac{w}{\epsilon} \right) \left[|\Phi_x|^2 \Phi^{-1} U^\beta \left[\left(1 + \frac{1}{2s} \right) + \beta (U^{\beta-1} - 1) \right] \left(1 - \frac{w}{\epsilon} \right) - \frac{2}{\epsilon} U^{2\beta} |\Phi_x|^2 \right], \\ &= 3 \left(1 - \frac{w}{\epsilon} \right) |\Phi_x|^2 U^{2\beta} \left[\Phi^{-1} U^{-\beta} \left[\left(1 + \frac{1}{2s} \right) + \beta (U^{\beta-1} - 1) \right] \left(1 - \frac{w}{\epsilon} \right) - \frac{2}{\epsilon} \right], \\ &= 3\epsilon \left(1 - \frac{w}{\epsilon} \right) |\Phi_x|^2 U^{2\beta} \left[\left[\left(1 + \frac{1}{2s} - \beta \right) U^{1-\beta} + \beta \right] (\epsilon w^{-1} - 1) - 2 \right], \end{aligned}$$

and so $\underline{u}(t, x)$ is convex, i.e. $\underline{u}_{xx}(t, x) \geq 0$, for x such that

$$\left[\left(1 + \frac{1}{2s} - \beta \right) U^{1-\beta} + \beta \right] (\epsilon w^{-1} - 1) \geq 2.$$

Lemma 3.1. We have $\underline{u}_{xx}(t, x) \geq 0$ as soon as $w(t, x) \leq \frac{\varepsilon}{\delta_c}$, where

$$\delta_c := 1 + \frac{2}{\min(\beta, 1 + \frac{1}{2s})}.$$

Proof. Recall first that $U \geq 1$, so that $0 \leq U^{1-\beta} \leq 1$. Assume first that $1 + \frac{1}{2s} - \beta \geq 0$, then in this situation, if $w \leq \frac{\varepsilon}{1 + \frac{2}{\beta}}$, from the above inequality we have

$$\left[\left(1 + \frac{1}{2s} - \beta \right) U^{1-\beta} + \beta \right] (\varepsilon w^{-1} - 1) \geq \beta (\varepsilon w^{-1} - 1) \geq 2$$

and so $\underline{u}_{xx}(t, x) \geq 0$ if $w \leq \frac{\varepsilon}{1 + \frac{2}{\beta}}$. When $1 + \frac{1}{2s} - \beta \leq 0$, if $w \leq \frac{\varepsilon}{1 + \frac{2}{1 + \frac{1}{2s}}}$,

$$\left[\left(1 + \frac{1}{2s} - \beta \right) U^{1-\beta} + \beta \right] (\varepsilon w^{-1} - 1) \geq \left(1 + \frac{1}{2s} \right) (\varepsilon w^{-1} - 1) \geq 2.$$

□

Proposition 3.2. For all $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$,

$$w(t, x) \leq \frac{2^{\frac{1}{\beta-1}} \kappa t}{x^{2s}}.$$

Proof. By using (3.18), the definition of w , since we have

$$w(t, x) = \frac{\kappa t}{x^{2s}} \left(1 - \frac{\gamma(\beta-1)t^\beta \kappa^{\beta-1}}{x^{2s(\beta-1)}} \right)^{-\frac{1}{\beta-1}}$$

As a consequence, for all $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$, by using the definition of $X(t)$, (3.19), we have the following estimate for w ,

$$\begin{aligned} w(t, x) &\leq \frac{\kappa t}{x^{2s}} \left(1 - \frac{\gamma(\beta-1)t^\beta \kappa^{\beta-1}}{2X(t)^{2s(\beta-1)}} \right)^{-\frac{1}{\beta-1}} \\ &\leq \frac{\kappa t}{x^{2s}} \left(1 - \frac{\gamma(\beta-1)t^\beta \kappa^{\beta-1}}{2(\kappa t)^{\beta-1} [\varepsilon^{1-\beta} + \gamma(\beta-1)t]} \right)^{-\frac{1}{\beta-1}} \\ &\leq \frac{\kappa t}{x^{2s}} \left(1 - \frac{\gamma(\beta-1)t}{2[\varepsilon^{1-\beta} + \gamma(\beta-1)t]} \right)^{-\frac{1}{\beta-1}} \leq \frac{2^{\frac{1}{\beta-1}} \kappa t}{x^{2s}}. \end{aligned}$$

□

Last, let us observe that for all $t \geq 1$, $X(t)$ satisfies the following:

$$\frac{t}{X(t)} \leq \left(\frac{1}{\kappa^{\frac{1}{2s}} (\gamma(\beta-1))^{\frac{1}{2s(\beta-1)}}} \right) t^{1 - \frac{\beta}{2s(\beta-1)}} \quad \text{so that} \quad \lim_{t \rightarrow \infty} \frac{t}{X(t)} = 0, \quad (3.30)$$

$$\frac{\kappa t}{X^{2s}(t)} \leq \frac{\varepsilon}{(1 + \varepsilon^{\beta-1} \gamma(\beta-1)t)^{\frac{1}{\beta-1}}} \quad \text{so that} \quad \lim_{t \rightarrow \infty} \frac{\kappa t}{X^{2s}(t)} = 0. \quad (3.31)$$

From the above estimates we can also derive the following useful limits

$$\lim_{t \rightarrow \infty} \frac{t \ln t}{X(t)} = 0, \quad (3.32)$$

$$\lim_{t \rightarrow \infty} w_x(t, X(t)) = 0. \quad (3.33)$$

4 Proof of Theorem 1.4 when $s \geq 1$.

4.1 Choice of parameters and consequences.

Let us define $t_\varepsilon := \frac{\sigma}{\varepsilon}$ for some $\sigma > 0$ and let show that for the right choice of $\varepsilon, \kappa, \sigma$ and γ the function \underline{u} defined in (3.20) is indeed a subsolution to (3.17) for all $t \geq t_\varepsilon$. In the rest of the section, let us fix

$$\kappa := \frac{D^{2s}}{2} \sigma^{-1} \varepsilon, \quad \gamma := \frac{\varepsilon^{2-\beta}}{\beta-1},$$

with D the positive constant given in Proposition 2.2. Let us also define $X_c(t) < Y(t)$ so that $w(t, X_c(t)) = \frac{\varepsilon}{\delta_c}$ and $w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$, this is

$$Y(t) = (\kappa t)^{\frac{1}{2s}} \left[(2\delta_c)^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}}, \quad X_c(t) = (\kappa t)^{\frac{1}{2s}} \left[\delta_c^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}}.$$

Consequently,

$$\begin{aligned} Y(t) - X_c(t) &= (\kappa t)^{\frac{1}{2s}} \left(\left[(2\delta_c)^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}} - \left[\delta_c^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}} \right) \\ &\geq (\kappa t)^{\frac{1}{2s}} \frac{2^{\beta-1} - 1}{2s(\beta-1)} \delta_c^{\beta-1} \varepsilon^{1-\beta} \left[\zeta_{s,\beta} \delta_c^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}-1} \end{aligned}$$

where

$$\zeta_{s,\beta} = \begin{cases} 1, & \text{if } 2s(\beta-1) < 1, \\ 2^{\beta-1}, & \text{if } 2s(\beta-1) > 1. \end{cases}$$

The latter being increasing in t in both configurations, we obtain,

$$\begin{aligned} Y(t) - X_c(t) &\geq (\kappa t_\varepsilon)^{\frac{1}{2s}} \frac{2^{\beta-1} - 1}{2s(\beta-1)} \delta_c^{\beta-1} \varepsilon^{1-\beta} \left[\zeta_{s,\beta} \delta_c^{\beta-1} \varepsilon^{1-\beta} + \gamma(\beta-1)t_\varepsilon \right]^{\frac{1}{2s(\beta-1)}-1} \\ &\geq D \frac{2^{\beta-1} - 1}{2s(\beta-1)} \delta_c^{\beta-1} \varepsilon^{1-\beta} \varepsilon^{(1-\beta)\left(\frac{1}{2s(\beta-1)}-1\right)} \left[\zeta_{s,\beta} \delta_c^{\beta-1} + \sigma \right]^{\frac{1}{2s(\beta-1)}-1} \\ &\geq D \frac{2^{\beta-1} - 1}{2s(\beta-1)} \delta_c^{\beta-1} \varepsilon^{-\frac{1}{2s}} \left[\zeta_{s,\beta} \delta_c^{\beta-1} + \sigma \right]^{\frac{1}{2s(\beta-1)}-1} := \mathcal{C}_1 \varepsilon^{-\frac{1}{2s}}. \end{aligned}$$

We end this section with a useful computation for further use. Since w is decreasing and convex we have

$$|w_x(t, X(t))|^2 \leq |w_x(t_\varepsilon, X(t_\varepsilon))|^2 := \frac{4s^2 [1 + \sigma]^{2 - \frac{2}{(\beta-1)2s}}}{D^2} \varepsilon^{2(1-\beta+\frac{1}{2s})} \varepsilon^{2\beta}. \quad (4.34)$$

4.2 Estimating $\mathcal{D}[\underline{u}]$ when $x \leq X(t)$.

On this region, by definition of \underline{u} , we have

$$\mathcal{D}[\underline{u}](t, x) = \int_{y \geq X(t)} [\underline{u}(t, y) - \varepsilon] J(x - y) dy.$$

This section aims at showing (3.21). For the convenience of the reader, we shall state this is the following

Proposition 4.1. *For all σ there exists $\varepsilon_0(\sigma)$ such that for $t \geq t_\varepsilon$ and $\varepsilon \leq \varepsilon_0(\sigma)$ we have*

$$\mathcal{D}[\underline{u}](t, x) + \frac{\varepsilon^\beta}{2} (1 - \varepsilon) \geq 0 \quad \text{for all } x \leq X(t).$$

Proof. We split the interval $(-\infty, X(t)]$ into two sub-intervals $(-\infty, X(t) - B]$ and $(X(t) - B, X(t)]$ with $B > 1$ to be chosen later, and thus estimate $\mathcal{D}[\underline{u}]$ on both subsets.

When $x \leq X(t) - B$: On this subset, then a short computation gives

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &= \int_{y \geq X(t)} \frac{\underline{u}(t, y) - \varepsilon}{|x - y|^{1+2s}} J(x - y) |x - y|^{1+2s} dy \geq -\varepsilon \mathcal{J}_0 \int_{y \geq X(t)} \frac{dy}{(y - x)^{1+2s}} \\ &\geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{(X(t) - x)^{2s}} \geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{B^{2s}}. \end{aligned}$$

When $X(t) - B < x \leq X(t)$: On this subset, by making the change of variable $z = y - x$, since $B > 1$ and $\underline{u}(t, x) = \varepsilon$, a short computation gives

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &= \int_{z \geq X(t) - x + B} [\underline{u}(t, x + z) - \varepsilon] J(z) dz + \int_{X(t) - x + B \geq z \geq X(t) - x} [\underline{u}(t, x + z) - \varepsilon] J(z) dz \\ &\geq -\varepsilon \mathcal{J}_0 \int_{z \geq X(t) - x + B} \frac{dz}{z^{1+2s}} + \int_{X(t) - x + B \geq z \geq X(t) - x} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz \\ &\geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{(X(t) + B - x)^{2s}} + \int_{X(t) - x + B \geq z \geq X(t) - x} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dy \\ &\geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{B^{2s}} + \int_{X(t) - x + B \geq z \geq X(t) - x} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dy. \end{aligned}$$

By using the Taylor formula we have

$$\underline{u}(t, x + z) - \underline{u}(t, x) = z \int_0^1 \underline{u}_x(t, x + \tau z) d\tau$$

and thus we can estimate the last integral by

$$I := \int_{X(t) - x}^{X(t) - x + B} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz = \int_{X(t) - x}^{X(t) - x + B} \int_0^1 \underline{u}_x(t, x + \tau z) z J(z) dz d\tau.$$

Since \underline{u}_x is a \mathcal{C}^1 function, we can reapply a Taylor formula and the last integral may be rewritten as follows

$$I = \int_0^1 \int_0^1 \int_{X(t) - x}^{B + X(t) - x} \underline{u}_{xx}(t, x + \tau \sigma z) J(z) \tau z^2 dz d\tau d\sigma.$$

We have $\underline{u}_{xx}(t, x) = 0$ for $x \leq X(t)$, see (3.23), so I reduces to

$$I = \int_0^1 \int_0^1 \int_0^{B + X(t) - x} \underline{u}_{xx}(t, x + \tau \sigma z) J(z) \tau z^2 dz d\tau d\sigma.$$

Using the definition of \underline{u}_{xx} , (3.26), for $x \geq X(t)$ and the convexity of w , we get

$$\begin{aligned} I &\geq -\frac{6}{\varepsilon} w_x(t, X(t))^2 \int_0^1 \int_0^1 \int_0^{2B} J(z) \tau z^2 dz d\tau d\sigma, \\ &\geq -\frac{6}{\varepsilon} w_x(t, X(t))^2 \left(\int_0^1 \int_0^1 \int_0^1 J(z) \tau z^2 d\tau d\sigma dz + \int_1^{2B} \int_0^1 \int_0^1 J(z) \frac{|z|^{1+2s}}{|z|^{1+2s}} \tau z^2 d\sigma d\tau dz \right), \\ &\geq -\frac{3}{\varepsilon} w_x(t, X(t))^2 \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right), \end{aligned}$$

where we have used Hypothesis 1.1 to estimate the integrals. As a consequence, we obtain the following estimate:

$$\mathcal{D}[\underline{u}] \geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{B^{2s}} - \frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right) w_x(t, X(t))^2. \quad (4.35)$$

We are now ready to choose $B := \left(\frac{2\mathcal{J}_0}{s\varepsilon^{\beta-1}(1-\varepsilon)} + 1 \right)^{\frac{1}{2s}}$ above. This implies then in both cases

$$\begin{aligned} \mathcal{D}[\underline{u}] &\geq -\frac{\varepsilon^\beta(1-\varepsilon)}{4} - \frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right) w_x(t, X(t))^2, \\ &\geq -\frac{\varepsilon^\beta(1-\varepsilon)}{4} - \frac{3}{\varepsilon} (\mathcal{J}_1 + \mathcal{J}_0 \ln(2B)) w_x(t, X(t))^2, \end{aligned}$$

which for all $t \geq t_\varepsilon$ using (4.34) leads to

$$\mathcal{D}[\underline{u}] \geq - \left[\frac{1-\varepsilon}{4} + \frac{12s^2 [1+\sigma]^{2-\frac{2}{(\beta-1)2s}}}{D^2} (\mathcal{J}_1 + \mathcal{J}_0 \ln(2B)) \varepsilon^{1-\beta+\frac{1}{s}} \right] \varepsilon^\beta.$$

Rewrite $1-\beta+\frac{1}{s} = \frac{2-\beta+\beta-2s(\beta-1)}{2s}$. Since $s \geq 1$, we have $\beta < 2$ and thus $1-\beta+\frac{1}{s} > 0$. As a consequence, $\varepsilon^{\frac{2-\beta}{2s}} (\mathcal{J}_1 + \mathcal{J}_0 \ln(2B)) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we may find an explicit $\varepsilon_0(\sigma)$ such that for all $\varepsilon \leq \varepsilon_0$ and for all $t \geq t_\varepsilon$,

$$\mathcal{D}[\underline{u}] + \frac{\varepsilon^\beta}{2}(1-\varepsilon) \geq 0.$$

□

4.3 Estimate of $\mathcal{D}[\underline{u}]$ on $x > X(t)$.

As exposed earlier and shown in Figure 7, we shall estimate $\mathcal{D}[\underline{u}]$ in three separate intervals

$$[X(t), Y(t)], \quad [Y(t), 2^{\frac{1}{2s(\beta-1)}} X(t)], \quad [2^{\frac{1}{2s(\beta-1)}} X(t), +\infty).$$

The exact expression of $Y(t)$ is explicit and is such that $w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$. Note that by definition $Y(t) > X_c(t)$ for all t and that $Y(t) \geq X(t) + R_0$ for $t \geq t_\varepsilon$, when ε is small enough.

4.3.1 The region $X(t) \leq x \leq Y(t)$.

We start this with an estimate

Lemma 4.2. *For all $B > 1$,*

$$\mathcal{D}[\underline{u}](t, x) \geq -\frac{\mathcal{J}_0\varepsilon}{sB^{2s}} - \frac{6}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^B z^{1-2s} dz \right) (w_x(t, X(t)))^2.$$

Proof. By definition of \underline{u} , for any $\delta \geq R_0$ we have, using Hypothesis 1.1,

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &= \int_{x+z \leq X(t)-\delta} \frac{\varepsilon - \underline{u}(t, x)}{|z|^{1+2s}} J(z) |z|^{1+2s} dz + \int_{x+z \geq X(t)-\delta} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz, \\ &\geq \int_{x+z \geq X(t)-\delta} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz, \end{aligned}$$

which leads to

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &= \int_{x+z \geq X(t)-\delta, |z| \leq B} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \\ &\quad + \int_{x+z \geq X(t)-\delta, |z| \geq B} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz. \end{aligned} \quad (4.36)$$

The second integral in the right hand side of the above expression is the easiest. Since \underline{u} is positive and J satisfies (1.1) we then have for $B > 1$,

$$\int_{x+z \geq X(t)-\delta, |z| \geq B} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \geq -\underline{u}(t, x) \mathcal{J}_0 \int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}}.$$

When $X(t) - \delta \leq x - B$, a short computation shows that

$$\begin{aligned} \int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}} &= \int_{X(t)-x-\delta \leq z \leq -B} \frac{dz}{|z|^{1+2s}} + \int_{z \geq B} \frac{dz}{|z|^{1+2s}} \\ &= \int_{X(t)-x-\delta \leq z \leq -B} \frac{dz}{z^{1+2s}} + \int_{z \geq B} \frac{dz}{z^{1+2s}} \\ &= \frac{1}{2sB^{2s}} - \frac{1}{2s(x+\delta-X(t))^{2s}} + \frac{1}{2sB^{2s}}. \end{aligned}$$

On the other hand if $X(t) - \delta \geq x - B$ then

$$\int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}} = \int_{z \geq B} \frac{dz}{|z|^{1+2s}} = \int_{z \geq B} \frac{dz}{z^{1+2s}} = \frac{1}{2sB^{2s}}.$$

In each situation we then have

$$\int_{x+z \geq X(t)-\delta, |z| \geq B} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \geq -\frac{\underline{u}(t, x) \mathcal{J}_0}{sB^{2s}} \geq -\frac{\mathcal{J}_0 \varepsilon}{sB^{2s}}.$$

Let us now estimate the first integral of the right hand side of the inequality (4.36), that is, let us estimate

$$I := \int_{x+z \geq X(t)-\delta, |z| \leq B} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz.$$

Following the same steps as for the proof of Proposition 4.1, since $\underline{u}(t, x)$ is \mathcal{C}^1 in x we have, for all $t \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned} I &= \int_{x+z \geq X(t)-\delta, |z| \leq B} \int_0^1 \int_0^1 \underline{u}_{xx}(t, x + \sigma \tau z) \tau z^2 J(z) d\tau d\sigma dz, \\ &\geq \min_{-B < \xi < B} \underline{u}_{xx}(t, x + \xi) \left(\int_{|z| \leq B} \int_0^1 \int_0^1 \tau z^2 J(z) d\tau d\sigma dz \right) \\ &\geq \min_{-B < \xi < B} \underline{u}_{xx}(t, x + \xi) \left(\int_{|z| \leq 1} \int_0^1 \int_0^1 \tau z^2 J(z) d\tau d\sigma dz + \int_{1 \leq |z| \leq B} \int_0^1 \int_0^1 \tau z^2 J(z) d\tau d\sigma dz \right). \end{aligned}$$

By using (3.23) and (3.26) and the convexity of w , we deduce that

$$\begin{aligned} I &\geq -\frac{6}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^B z^{1-2s} dz \right) \sup_{\substack{-B < \xi < B, \\ x+\xi > X(t)}} w_x(t, x + \xi)^2, \\ &\geq -\frac{6}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^B z^{1-2s} dz \right) w_x(t, X(t))^2, \end{aligned}$$

the latter since $w(t, x)$ is convex. Collecting all previous steps, we get the following estimate for all $B > 1$,

$$\mathcal{D}[\underline{u}](t, x) \geq -\frac{\mathcal{J}_0 \varepsilon}{sB^{2s}} - \frac{6}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^B z^{1-2s} dz \right) (w_x(t, X(t)))^2.$$

with ends the proof of the lemma. \square

With this lemma at hand, we claim that

Proposition 4.3. *For all σ there exists $\varepsilon_1(\sigma)$ such that for $t \geq t_\varepsilon$ and $\varepsilon \leq \varepsilon_1$ we have, on $[X(t), Y(t)]$,*

$$\mathcal{D}[\underline{u}] + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta \geq 0.$$

Proof. Specify in the previous lemma $B = \nu\varepsilon^{\frac{1-\beta}{2s}}$ with $\nu > 1$ to be chosen later on. Note that $B > 1$ since $\nu, \beta > 1$ and $\varepsilon \leq 1$. With this B we get from the above inequality that

$$\mathcal{D}[\underline{u}](t, x) \geq -\frac{\mathcal{J}_0\varepsilon^\beta}{s\nu^{2s}} - \frac{6}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{\nu\varepsilon^{\frac{1-\beta}{2s}}} z^{1-2s} dz \right) (w_x(t, X(t)))^2.$$

Observe that the definition of \underline{u} gives $3w(t, x) \geq \underline{u}(t, x) \geq w(t, x)$ for all $x \geq X(t)$. Since for $x \leq Y(t)$ we have $w(t, x) \geq \frac{\varepsilon}{2\delta_c}$, we get $\underline{u}(t, x) \geq \frac{\varepsilon}{2\delta_c}$. As a consequence,

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) + \frac{(1-\varepsilon)}{2}\underline{u}^\beta(t, x) &\geq \varepsilon^\beta \frac{1-\varepsilon}{2(2\delta_c)^\beta} - \frac{\mathcal{J}_0\varepsilon^\beta}{s\nu^{2s}} - \frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{\nu\varepsilon^{\frac{1-\beta}{2s}}} z^{1-2s} dz \right) (w_x(t, X(t)))^2, \\ &\geq \varepsilon^\beta \frac{1-\varepsilon}{4(2\delta_c)^\beta} - \frac{6}{\varepsilon} w_x(t, X(t))^2 \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{\nu\varepsilon^{\frac{1-\beta}{2s}}} z^{1-2s} dz \right) \end{aligned}$$

where we have chosen $\nu := \sup \left\{ \left(\frac{4(2\delta_c)^\beta \mathcal{J}_0}{s(1-\varepsilon)} \right)^{\frac{1}{2s}}; 1 \right\}$. We may now reproduce the argument used in the proof of Proposition 4.1 and find a explicit ε_1 so that for all $\varepsilon \leq \varepsilon_1$, $t \geq t_\varepsilon$,

$$\mathcal{D}[\underline{u}](t, x) + \frac{1-\varepsilon}{2}\underline{u}^\beta(t, x) \geq 0.$$

□

4.3.2 A preliminary estimate in the range $x \geq Y(t)$.

In this zone, \underline{u} is convex since $Y(t) \geq X_c(t)$.

Lemma 4.4. *For any time $t > 1$, any $x \geq Y(t)$, and $B > R_0$ such that $x - B \geq X_c(t)$,*

$$\mathcal{D}[\underline{u}] \geq \frac{\varepsilon - \underline{u}}{2s\mathcal{J}_0x^{2s}} - \frac{\mathcal{C}_0}{B^{2s-1}} \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} w^\beta. \quad (4.37)$$

Proof. Let us go back to the definition of $\mathcal{D}[\underline{u}](t, x)$, that we split into three parts:

$$\mathcal{D}[\underline{u}](t, x) = \int_{-\infty}^{-B} [u(t, x+z) - \underline{u}(t, x)] J(z) dz + \int_{-B}^B [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz + \int_B^\infty [u(t, x+z) - \underline{u}(t, x)] J(z) dz.$$

To obtain an estimate of the second integral, we actually follow the same steps as several times previously to obtain via Taylor expansion,

$$\int_{-B}^B [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz = \int_{-B}^B \int_0^1 \int_0^1 \underline{u}_{xx}(t, x + \tau\sigma z) \tau z^2 J(z) d\tau d\sigma dz \geq 0,$$

since \underline{u} is convex in the zone of integration. By using again a Taylor formula, the last integral may be rewritten as

$$\int_B^\infty [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz = \int_B^\infty \int_0^1 \underline{u}_x(t, x + \tau z) z J(z) d\tau dz.$$

Observe that since $x \geq Y(t)$ and w is convex, (3.25) implies

$$\underline{u}_x(t, x + \tau z) = 3w_x(t, x + \tau z) \left(1 - \frac{w(t, x + \tau z)}{\varepsilon}\right)^2 \geq 3 \left(1 - \frac{1}{2\delta_c}\right)^2 w_x(t, x).$$

It follows that

$$\begin{aligned} \int_B^\infty [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz &\geq 3 \left(1 - \frac{1}{2\delta_c}\right)^2 \left(\int_B^\infty z J(z) dz\right) w_x(t, x) \\ &\geq 3 \left(1 - \frac{1}{2\delta_c}\right)^2 \frac{\mathcal{J}_0}{(2s-1)B^{2s-1}} w_x(t, x). \end{aligned}$$

using Hypothesis 1.1. Finally, since $X(t) - x \leq X(t) - X_c(t) - B \leq -B$, the first integral can be estimated as follows:

$$\begin{aligned} \int_{-\infty}^{-B} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz &\geq \mathcal{J}_0^{-1} \int_{-\infty}^{X(t)-x} \frac{\underline{u}(t, x + z) - \underline{u}(t, x)}{|z|^{1+2s}} dz + \int_{X(t)-x}^{-B} [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz, \\ &\geq \frac{\mathcal{J}_0^{-1}}{2s} \frac{\varepsilon - \underline{u}(t, x)}{(x - X(t))^{2s}}, \end{aligned}$$

taking advantage of the fact that \underline{u} is decreasing. Collecting all previous steps and recalling the expression of w_x set in (3.28) gives the lemma. \square

4.3.3 The region $Y(t) < x < 2^{\frac{1}{2s(\beta-1)}} X(t)$.

The previous lemma at hand, let us now estimate $\mathcal{D}[\underline{u}]$ when $x \geq Y(t)$.

Proposition 4.5. *For all σ there exists $\varepsilon_2(\sigma)$ such that for $t \geq t_\varepsilon$ and $\varepsilon \leq \varepsilon_2$ we have*

$$\mathcal{D}[\underline{u}] + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta \geq 0 \quad \text{for all } Y(t) < x < 2^{\frac{1}{2s(\beta-1)}} X(t).$$

Proof. Let us recall that $Y(t)$ is such that $w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$ and consider $x \geq Y(t)$. For such x , as long as B is chosen such that $x - B \geq X_c(t)$, we get by Lemma 4.4,

$$\mathcal{D}[\underline{u}] \geq -\frac{\mathcal{C}_0}{B^{2s-1}} \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} w^\beta.$$

The rest of the proof will explain the choice of B . Since $X(t) < x < 2^{\frac{1}{2s(\beta-1)}} X(t)$, we have directly

$$\begin{aligned} \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} &\leq \frac{2X(t)^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} = 2(\kappa t)^{-\frac{1}{2s}} [\varepsilon^{1-\beta} + \gamma(\beta-1)t]^{1-\frac{1}{2s(\beta-1)}} \\ &\leq 2D^{-1} [1 + \sigma]^{1-\frac{1}{2s(\beta-1)}} \varepsilon^{(\beta-1)(\frac{1}{2s(\beta-1)}-1)} \end{aligned}$$

Consequently,

$$\mathcal{D}[\underline{u}] + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta \geq \left[\frac{1}{2}(1 - \varepsilon) - 2\frac{\mathcal{C}_0}{D} [1 + \sigma]^{1-\frac{1}{2s(\beta-1)}} \frac{\varepsilon^{-(\beta-1)(1-\frac{1}{2s(\beta-1)})}}{B^{2s-1}} \right] w^\beta.$$

Observe that if the last bracket is positive, the proof is finished. This is where the choice of B is crucial. Define

$$B = \left[\frac{4\mathcal{C}_0}{(1 - \varepsilon)D} \right]^{\frac{1}{2s-1}} (1 + \sigma)^{\frac{1}{2s-1} - \frac{1}{2s(2s-1)(\beta-1)}} \varepsilon^{-\frac{\beta-1}{2s-1}(1-\frac{1}{2s(\beta-1)})+}.$$

Once this choice is made, since $\varepsilon \leq 1$, the proof is finished. Recall that the limitation in the choice of B is due to the fact that we need to ensure that for all $t \geq t_\varepsilon$ and $x \geq Y(t)$, $x - X_c(t) - B \geq 0$. Since $Y(t) - X_c(t) \geq C_1 \varepsilon^{-\frac{1}{2s}}$, this is satisfied as long as $B \leq C_1 \varepsilon^{-\frac{1}{2s}}$. Since $\frac{1}{2s} - \frac{\beta-1}{2s-1} \left(1 - \frac{1}{2s(\beta-1)}\right) = \frac{2-\beta}{2s-1} > 0$, one may observe that the condition is, somewhat miraculously, satisfied by taking ε small, after any wanted choice of σ . \square

4.3.4 The region $x > 2^{\frac{1}{2s(\beta-1)}} X(t)$

Let us now obtain an estimate for the last region, $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$. In this region we claim

Proposition 4.6. *There exists σ_{LR} such that for all $\sigma \geq \sigma_{LR}$, we have for $t \geq t_\varepsilon$ and $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$*

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{4\mathcal{J}_0 s x^{2s}}.$$

with $\tau := \frac{3}{2\delta_c} \left(1 - \frac{1}{2\delta_c} + \frac{1}{3} \frac{1}{(2\delta_c)^2}\right)$.

Proof. For a given x , we define $B = \frac{x}{K}$ with $K = \frac{2^{\frac{1}{2s(\beta-1)}+1}}{2^{\frac{1}{2s(\beta-1)}} - 1}$. Note that by definition of K , we have then $2^{\frac{1}{2s(\beta-1)}} \left(1 - \frac{1}{K}\right) = \frac{2^{\frac{1}{2s(\beta-1)}+1}}{2} := C_2$. As a consequence, $x - B \geq C_2 X(t)$. A straightforward computation shows that $C_2 X(t) \geq X_c(t)$ for all $t \geq t_\varepsilon$ as soon as

$$\frac{1+\sigma}{\delta_c^{\beta-1} + \sigma} \geq C_2^{-2s(\beta-1)}.$$

Since $C_2 > 1$ and $\lim_{u \rightarrow \infty} \frac{1+u}{\delta_c^{\beta-1} + u} = 1$, the above inequality is always true for large σ , says $\sigma \geq \sigma_1$. We may thus apply Lemma 4.4 to get

$$\mathcal{D}[\underline{u}] \geq \frac{\varepsilon - \underline{u}(t, x)}{2s\mathcal{J}_0 x^{2s}} - \frac{C_0}{B^{2s-1}} \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} w^\beta,$$

Let us recall that $Y(t)$ is such that $w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$ and so $\underline{u}(t, Y(t)) = \frac{3\varepsilon}{2\delta_c} \left(1 - \frac{1}{2\delta_c} + \frac{1}{3} \frac{1}{(2\delta_c)^2}\right) := \tau$. We can easily check that since $\frac{1}{2\delta_c} < 1$, we have $\tau < 1$. Therefore, we have

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &\geq \frac{\varepsilon(1-\tau)}{2\mathcal{J}_0 s x^{2s}} - \frac{C_0}{B^{2s-1}} w^\beta(t, x) \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}}, \\ &\geq \frac{\varepsilon(1-\tau)}{2\mathcal{J}_0 s x^{2s}} - \frac{C_0 2^{\frac{\beta}{\beta-1}}}{B^{2s-1}} \frac{1}{x^{2s}} \frac{\kappa t}{x} = \left[\frac{1-\tau}{2\mathcal{J}_0 s} - C_0 2^{\frac{\beta}{\beta-1}} K^{2s-1} \frac{\kappa t}{\varepsilon x^{2s}} \right] \frac{\varepsilon}{x^{2s}}, \\ &\geq \left[\frac{1-\tau}{2\mathcal{J}_0 s} - 2C_0 K^{2s-1} \frac{\kappa t}{\varepsilon X(t)^{2s}} \right] \frac{\varepsilon}{x^{2s}} = \left[\frac{1-\tau}{2\mathcal{J}_0 s} - \frac{2C_0 K^{2s-1}}{[1+\varepsilon t]^{\frac{1}{\beta-1}}} \right] \frac{\varepsilon}{x^{2s}}. \end{aligned}$$

We have used Proposition 3.2 to estimate w after the first line. We then get for $t \geq t_\varepsilon$,

$$\mathcal{D}[\underline{u}](t, x) \geq \left[\frac{1-\tau}{2\mathcal{J}_0 s} - \frac{2C_0 K^{2s-1}}{[1+\sigma]^{\frac{1}{\beta-1}}} \right] \frac{\varepsilon}{x^{2s}} \geq \frac{\varepsilon(1-\tau)}{4\mathcal{J}_0 s x^{2s}},$$

choosing σ large enough. \square

4.4 Tuning the parameters σ and ε .

In this last part of the proof, we choose our parameters σ and ε in order that for $t > t_\varepsilon$, \underline{u} is indeed a sub-solution to (3.16). Recall that \underline{u} is a subsolution if and only if (3.21) and (3.22) hold simultaneously. Since (3.21) holds unconditionally for $t \geq t_\varepsilon$ and $\varepsilon \leq \varepsilon_0$, the only thing left to check is that (3.22) holds for a suitable choice of σ .

By using (3.24) and (3.27), (3.22) holds if particular

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon) \underline{u}^\beta - \gamma w^\beta, \quad x > X(t),$$

Set $\varepsilon^* := \inf\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, where $\varepsilon_0, \varepsilon_1$ and ε_2 are respectively determined by Proposition 4.1, Proposition 4.3, and Proposition 4.5. To make our choice, let us decompose the set $[X(t), +\infty) = I_1 \cup I_2$ into two subsets defined as follows

$$I_1 := [X(t), 2^{\frac{1}{2s(\beta-1)}} X(t)], \quad I_2 := [2^{\frac{1}{2s(\beta-1)}} X(t), +\infty).$$

On the first interval, we have

Lemma 4.7. *There exists ε_4 such that for all $\varepsilon \leq \varepsilon_4$ and all $\sigma \geq 1$ one has, for $t \geq t_\varepsilon$,*

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon) \underline{u}^\beta - (\beta - 1)^{-1} \varepsilon^{2-\beta} w^\beta, \quad \text{for all } x \in I_1.$$

Proof. By definition of Φ , we have, at (t, x) ,

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = \frac{3}{t} \frac{x^{2s(\beta-1)}}{(\kappa t)^{\beta-1}} w^\beta.$$

By exploiting the definition of $X(t)$ it follows that for $x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$,

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = \frac{3}{t} \frac{x^{2s(\beta-1)}}{(\kappa t)^{\beta-1}} w^\beta \leq \left[\frac{6}{t} \left(\frac{1}{\varepsilon} \right)^{\beta-1} + 6\varepsilon^{2-\beta} \right] w^\beta.$$

So for $t \geq t_\varepsilon$, since $\sigma \geq 1$ we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq 6\varepsilon^{2-\beta} [1 + \sigma^{-1}] w^\beta \leq 12\varepsilon^{2-\beta} w^\beta.$$

From the above, we see that for $\varepsilon' := \left(\frac{1}{96}\right)^{\frac{1}{2-\beta}}$ we have for all $\varepsilon \leq \varepsilon'$

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \frac{(1 - \varepsilon)}{4} w^\beta.$$

Recall that by Proposition 4.3 and Proposition 4.5, we have for all $x \in I_1$ and $t \geq t_\varepsilon$, and $\varepsilon \leq \varepsilon^*$

$$\mathcal{D}[\underline{u}] + (1 - \varepsilon) \underline{u}^\beta - \gamma w^\beta \geq \left(\frac{1 - \varepsilon}{2} - \frac{\varepsilon^{2-\beta}}{\beta - 1} \right) w^\beta,$$

since $\underline{u} \geq w$ for all $x \geq X(t)$. We then end our proof by taking $\varepsilon \leq \varepsilon_4 := \inf \left\{ \varepsilon', \left(\frac{(1-\varepsilon)(\beta-1)}{4} \right)^{\frac{1}{2-\beta}} \right\}$. \square

Finally, let us check what happens on I_2 ,

Lemma 4.8. *There exists $\sigma \geq 1$ such that for all $\varepsilon \leq \varepsilon_4$, one has for all $t \geq t_\varepsilon$,*

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \mathcal{D}[\underline{u}] + (1 - \varepsilon) \underline{u}^\beta - (\beta - 1)^{-1} \varepsilon^{2-\beta} w^\beta, \quad \text{for all } x \in I_2.$$

Proof. As in the above proof, by definition of Φ we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = 3\kappa \frac{x^{2s(\beta-1)}}{(\kappa t)^\beta} w^\beta.$$

By Proposition 3.2, we have for $x \in I_2$ and $t \geq t_\varepsilon$,

$$w^\beta(t, x) \leq 2^{\frac{\beta}{\beta-1}} \frac{(\kappa t)^\beta}{x^{2s\beta}}.$$

Therefore, we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq 3\kappa \frac{x^{2s(\beta-1)}}{(\kappa t)^\beta} 2^{\frac{\beta}{\beta-1}} \frac{(\kappa t)^\beta}{x^{2s\beta}} = 2^{\frac{\beta}{\beta-1}} \frac{3D^{2s}\sigma^{-1}\varepsilon}{x^{2s}}.$$

Observe that for $\sigma \geq \sup \left\{ 1, D^{2s} 2^{\frac{\beta}{\beta-1}} \frac{12\mathcal{J}_0 s}{1-\tau} \right\}$ we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \frac{\varepsilon(1-\tau)}{4\mathcal{J}_0 s x^{2s}}.$$

Now recall that by Proposition 4.6, we have for all $x \in I_2$, $\varepsilon \leq \varepsilon^*$ and $t \geq t_\varepsilon$,

$$\mathcal{D}[\underline{u}] + (1-\varepsilon)\underline{u}^\beta - (\beta-1)^{-1}\varepsilon^{2-\beta}w^\beta \geq (1-\varepsilon - (\beta-1)^{-1}\varepsilon^{2-\beta})w^\beta + \frac{\varepsilon(1-\tau)}{4\mathcal{J}_0 s x^{2s}},$$

since $\underline{u} \geq w$ for all $x \geq X(t)$. The claim is then proved by taking $\varepsilon \leq \varepsilon_4$. \square

4.5 The final argument

From the above, section, we may find ε small so that $\underline{u}(t, x)$ is a subsolution to (3.16) for all $t \geq t_\varepsilon$. Having now this sub-solution $\underline{u}(t, x)$ at hand, to conclude the proof of Theorem, we only need to check that for some R^* and T we have $u(T, x + R^*) \geq \underline{u}(t_\varepsilon, x)$. Indeed, if so then by the parabolic comparison principle, we will then have $u(t+1, x + R^*) \geq \underline{u}(t_\varepsilon + t, x)$ for all t and the level set

$$E_\varepsilon(t) := \{x \in \mathbb{R} | u(t+1, x + R^*) \geq \varepsilon\} \supset (\infty, X(t_\varepsilon + t)].$$

So let us find the adequate R^* and T . To do so, let us first recall that by Proposition 2.2 we have

$$\lim_{x \rightarrow +\infty} x^{2s} u(1, x) = D^{2s}.$$

On the other hand by definition of \underline{u} a quick computation shows that

$$\lim_{x \rightarrow +\infty} x^{2s} \underline{u}(t_\varepsilon, x) = \frac{D^{2s}}{2}.$$

Therefore there exists $R_1 > 0$ such that for all $x \geq R_1$, $u(1, x) \geq \underline{u}(t_\varepsilon, x)$ and in particular we have $u(1, x - R_1) \geq \underline{u}(t_\varepsilon, x)$ for all $x \geq R_1$ since $u(1, x)$ is monotone non increasing. To conclude, we just need to ensure that $\liminf_{x \rightarrow \infty} u(1, x) > \varepsilon$. Indeed, if so, then there exist $R_2 > 0$ such that for all $x < -R_2$ we have $u(1, x) > \underline{u}(t_\varepsilon, x)$ and thus we conclude that $u(1, x - R_1 - R_2) \geq \underline{u}(t_\varepsilon, x)$ since by monotonicity of $u(1, x)$ we have $u(1, x - R_1 - R_2) \geq \varepsilon \geq \underline{u}(t_\varepsilon, x)$ for $x \in (\infty, R_1]$ and $u(1, x - R_1 - R_2) \geq u(1, x - R_1) \geq \underline{u}(t_\varepsilon, x)$ for all $x \geq R_1$.

To prove that $\liminf_{x \rightarrow \infty} u(1, x) > \varepsilon$, we just need to observe that by a straightforward application of the comparison principle, we have $u(t, x) \geq p(t, x)$ where $p(t, x)$ is the solution of the linear problem

$$\begin{aligned} p_t &= \mathcal{D}[p] \quad \text{for } t > 0, x \in \mathbb{R}, \\ p(0, \cdot) &= a \mathbb{1}_{(-\infty, b]}. \end{aligned} \tag{4.38}$$

By denoting by $G(t, x)$ the Green function associated to the above linear equation, that is the solution defined by

$$\begin{aligned} G_t &= \mathcal{D}[G] \quad \text{for } t > 0, x \in \mathbb{R}, \\ G(0, \cdot) &= \delta_{x=0}, \end{aligned} \tag{4.39}$$

the solution p is then given by

$$p(t, x) = aG(t, \cdot) \star \mathbb{1}_{(-\infty, b]}(\cdot)(x) = a \int_{x-b}^{+\infty} G(t, y) dy,$$

$$\text{and thus } \liminf_{x \rightarrow -\infty} u(1, x) \geq \liminf_{x \rightarrow -\infty} p(1, x) = a \lim_{x \rightarrow -\infty} \int_{x-b}^{+\infty} G(1, y) dy = a.$$

Having the lower bound at hand, to obtain the lower bound for all level lines, we can argue as in the proof in [4, 25] using the adequate invasion property, namely Proposition 2.4.

5 Proof of Theorem 1.4 when $s < 1$.

In this section, we prove Theorem 1.4 when $s < 1$. In such a situation, the above construction based on a fine control of the time $t_\varepsilon = \frac{\sigma}{\varepsilon}$, is inadequate for a large set of parameters (s, β) , especially when $\beta \geq 2$. In particular, in this latter case, the constraint imposed on the form of t_ε would make the proof fail. To cover all the possible situations new ideas have then to be developed. When $s < 1$, the diffusion process plays a much important role by inducing by himself a flattening of the solution. So, having this in mind, our idea is to exploit the flattening properties of the solution to (3.16) to remove the constraint imposed on t_ε in the above construction hoping that we can find a time t^* after which $\underline{u}(t, x)$ is a sub-solution. By doing so, we get more flexibility on the construction but at the expense of a clear understanding of the time after which the true acceleration regime starts.

Let us now show that for the right choice of ε, κ and γ the function \underline{u} is indeed a subsolution to (3.17) for all $t \geq t^*$ for some t^* .

5.1 Estimating $\mathcal{D}[\underline{u}]$ when $x \leq X(t)$.

On this region, by definition of \underline{u} , we have

$$\mathcal{D}[\underline{u}](t, x) = \int_{y \geq X(t)} [\underline{u}(t, y) - \varepsilon] J(x - y) dy.$$

This section aims at showing (3.21). For the convenience of the reader, we shall state this is the following

Proposition 5.1. *For all $\varepsilon \leq \frac{1}{2}, \gamma$ and κ there exists $t_0(\varepsilon, \kappa, \gamma, \beta, s)$ such that for all $t \geq t_0$*

$$\mathcal{D}[\underline{u}](t, x) + \frac{\varepsilon^\beta}{2}(1 - \varepsilon) \geq 0 \quad \text{for all } x \leq X(t).$$

Proof. The starting point is the same as for Proposition 4.1 so we shall now reproduce the exact same proof, but start right away from (4.35), that is

$$\mathcal{D}[\underline{u}] \geq -\frac{\varepsilon \mathcal{J}_0}{2s} \frac{1}{B^{2s}} - \frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right) w_x(t, X(t))^2.$$

As in the proof of Proposition 4.1 by choosing $B := \left(\frac{2\mathcal{J}_0}{s\varepsilon^{\beta-1}(1-\varepsilon)} + 1 \right)^{\frac{1}{2s}}$ we then get

$$\mathcal{D}[\underline{u}] \geq -\frac{\varepsilon^\beta(1-\varepsilon)}{4} - \frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right) (w_x(t, X(t)))^2.$$

Since by (3.33), $(w_x(t, X(t)))^2 \rightarrow 0$ as $t \rightarrow +\infty$ we may then find a t_0 so that for all $t \geq t_0$

$$\frac{3}{\varepsilon} \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{2B} z^{1-2s} dz \right) (w_x(t, X(t)))^2 \leq \frac{\varepsilon^\beta (1-\varepsilon)}{4},$$

and thus for $t \geq t_0$ we then achieve

$$\mathcal{D}[\underline{u}] + \frac{\varepsilon^\beta}{2} (1-\varepsilon) \geq 0.$$

□

5.2 Estimate of $\mathcal{D}[\underline{u}]$ on $x > X(t)$.

As exposed earlier and shown in Figure 7, we shall again estimate $\mathcal{D}[\underline{u}]$ in the three separate intervals

$$[X(t), Y(t)], \quad [Y(t), 2^{\frac{1}{2s(\beta-1)}} X(t)], \quad [2^{\frac{1}{2s(\beta-1)}} X(t), +\infty).$$

where we recall that $Y(t) > X_c(t)$ for all t is such that $w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$.

Note that for all $\varepsilon, \gamma, \kappa, s, \beta$ we may find $t^\# > 0$ such that $Y(t) \geq X_c(t) + R_0$ for $t \geq t^\#$.

5.2.1 The region $X(t) \leq x \leq Y(t)$.

In this region, owing to Lemma 4.2, we claim that

Proposition 5.2. *For all $\varepsilon < \frac{1}{2}, \kappa$ and γ there exists t_1 such that*

$$\mathcal{D}[\underline{u}] \geq -\frac{1}{2}(1-\varepsilon)\underline{u}^\beta \quad \text{for all } t \geq t_1, \quad X(t) < x < Y(t).$$

Proof. The proof follows essentially the same step as the proof of Proposition 4.3. Namely, by using Lemma 4.2 with $B := \nu \varepsilon^{\frac{1-\beta}{2s}}$ where $\nu := \sup \left\{ \left(\frac{4(2\delta_c)^\beta \mathcal{J}_0}{s(1-\varepsilon)} \right)^{\frac{1}{2s}}; 1 \right\}$ and the fact that in the zone $x \leq Y(t)$, we have $\underline{u}(t, x) \geq \underline{u}(t, Y(t)) \geq w(t, Y(t)) = \frac{\varepsilon}{2\delta_c}$, one has

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1-\varepsilon)\underline{u}^\beta(t, x) \geq \frac{1}{4}(1-\varepsilon) \frac{\varepsilon^\beta}{(2\delta_c)^\beta} - \frac{6}{\varepsilon} (w_x(t, X(t)))^2 \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{\nu \varepsilon^{\frac{1-\beta}{2s}}} z^{1-2s} dz \right).$$

From there we can argue as in the proof of Proposition 5.1 using that $\lim_{t \rightarrow +\infty} (w_x(t, X(t)))^2 = 0$ and find a t_1 such that for all $t \geq t_1$

$$\frac{6}{\varepsilon} (w_x(t, X(t)))^2 \left(\mathcal{J}_1 + \mathcal{J}_0 \int_1^{\nu \varepsilon^{\frac{1-\beta}{2s}}} z^{1-2s} dz \right) \leq \frac{1}{4}(1-\varepsilon) \frac{\varepsilon^\beta}{(2\delta_c)^\beta},$$

enforcing

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1-\varepsilon)\underline{u}^\beta(t, x) \geq 0.$$

□

5.2.2 A preliminary estimate in the range $x \geq Y(t)$.

In this region the estimate of Lemma 4.4 does not hold for all $s < 1$ and we need to derive a new generic estimate of $\mathcal{D}[\underline{u}]$. We then start by deriving an estimate of $\mathcal{D}[\underline{u}](t, x)$ only valid in the range $x \geq Y(t)$.

Lemma 5.3. *For any time $t > t^\#$, $B > 1$ and $x \geq Y(t)$,*

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{2s\mathcal{J}_0x^{2s}} - \frac{\mathcal{J}_0\varepsilon\tau}{2sB^{2s}} + 3\mathcal{J}_0 \left(\int_1^B z^{-2s} dz \right) w_x(t, x). \quad (5.40)$$

with $\tau := \frac{3}{2\delta_c} \left(1 - \frac{1}{2\delta_c} + \frac{1}{3(2\delta_c)^2} \right)$.

Proof. Let us go back to the definition of $\mathcal{D}[\underline{u}](t, x)$ that we split into three parts:

$$\mathcal{D}[\underline{u}](t, x) = \int_{-\infty}^{-1} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz + \int_{-1}^1 [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz + \int_1^\infty [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz.$$

Since for $t \geq t^\#$, $x \geq Y(t) \geq X_c(t) + R_0 \geq X(t) + R_0$ and \underline{u} is decreasing, the first integral can be estimated as follows:

$$\begin{aligned} & \int_{-\infty}^{-1} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \\ & \geq \mathcal{J}_0^{-1} \int_{-\infty}^{X(t)-x} \frac{\underline{u}(t, x+z) - \underline{u}(t, x)}{|z|^{1+2s}} dz + \int_{X(t)-x}^{-1} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \\ & \geq \frac{\mathcal{J}_0^{-1}}{2s} \frac{\varepsilon - \underline{u}(t, x)}{(x - X(t))^{2s}}. \end{aligned} \quad (5.41)$$

To obtain an estimate of the second integral, we actually follow the same steps as several times previously to obtain via Taylor expansion,

$$\begin{aligned} \int_{-1}^1 [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz &= \int_{-1}^1 \int_0^1 \int_0^1 \underline{u}_{xx}(t, x + \tau\sigma z) \tau z^2 J(z) d\tau d\sigma dz \\ &\geq \mathcal{J}_1 \min_{-1 < \xi < 1} \underline{u}_{xx}(t, x + \xi) \geq 0, \end{aligned} \quad (5.42)$$

since $x - 1 \geq Y(t) - 1 \geq X_c(t) + R_0 - 1 \geq X_c(t)$ so that \underline{u} is convex there.

Finally, let us estimate the last integral and for $B > 1$ let us split it up into two parts. Namely,

$$\begin{aligned} I &:= \int_1^{+\infty} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \\ &= \int_1^B [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz + \int_B^{+\infty} [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz. \end{aligned}$$

Since \underline{u} is positive we have

$$\int_B^\infty [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz \geq -\frac{\mathcal{J}_0 \underline{u}(t, x)}{2sB^{2s}}. \quad (5.43)$$

By using again a Taylor formula, the last integral rewrites

$$\int_1^B [\underline{u}(t, x+z) - \underline{u}(t, x)] J(z) dz = \int_1^B \int_0^1 \underline{u}_x(t, x + \tau z) z J(z) d\tau dz.$$

Observe that by definition of \underline{u}_x , (3.25), for all $\tau z \geq 0$ we have

$$\underline{u}_x(t, x + \tau z) \geq 3w_x(t, x + \tau z) \geq 3w_x(t, x),$$

since w is convex.

It then follows that

$$\int_1^B [\underline{u}(t, x + z) - \underline{u}(t, x)] J(z) dz \geq 3 \left(\int_1^B z J(z) dz \right) w_x(t, x) \geq 3\mathcal{J}_0 \left(\int_1^B z^{-2s} dz \right) w_x(t, x) \quad (5.44)$$

using Hypothesis 1.1. Collecting (5.41), (5.42), (5.43), (5.44), we find for $x \geq Y(t)$, and $t \geq t^\#$

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon - \underline{u}(t, x)}{2s\mathcal{J}_0 x^{2s}} - \frac{\mathcal{J}_0 \underline{u}(t, x)}{2sB^{2s}} + 3\mathcal{J}_0 \left(\int_1^B z^{-2s} dz \right) w_x(t, x).$$

The Lemma is then proved by observing that for $x \geq Y(t)$, $\underline{u}(t, x) \leq \underline{u}(t, Y(t)) = \varepsilon\tau$. \square

5.2.3 The region $Y(t) < x < 2^{\frac{1}{2s(\beta-1)}} X(t)$.

The previous lemma at hand, let us now estimate $\mathcal{D}[\underline{u}](t, x)$ when $x \geq Y(t)$.

Proposition 5.4. *For any $0 < \varepsilon \leq \frac{1}{2}$ and any $\gamma, \kappa > 0$, there exists $t_2 > 0$ such that for all $t \geq t_2$,*

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq 0 \quad \text{for all } Y(t) < x < 2^{\frac{1}{2s(\beta-1)}} X(t).$$

Proof. First let us observe that since $Y(t) \rightarrow +\infty$ we may find $t' > t^\#$ such that for all $t \geq t'$

$$\left(\frac{2\tau\mathcal{J}_0^2}{1-\tau} \right)^{\frac{1}{2s}} Y(t) > 1.$$

Fix $B := \left(\frac{2\tau\mathcal{J}_0^2}{1-\tau} \right)^{\frac{1}{2s}} x$, then from Lemma 5.3 and by using the definition of $w_x(t, x)$, (3.28), we deduce that for $t \geq t'$ and $x \geq Y(t)$

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{4s\mathcal{J}_0 x^{2s}} - 6s\mathcal{J}_0 w^\beta(t, x) \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} \left(\int_1^B z^{-2s} dz \right).$$

Therefore, since $\underline{u}(t, x) \geq w(t, x)$ we get

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - 6s\mathcal{J}_0 \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} \left(\int_1^B z^{-2s} dz \right) \right].$$

Set $\mathcal{C}_3 := \left(\frac{2\tau\mathcal{J}_0^2}{1-\tau} \right)^{\frac{1}{2s}}$ and let us now treat the three cases $\frac{1}{2} < s < 1$, $s = \frac{1}{2}$ and $s < \frac{1}{2}$ separately.

Case $\frac{1}{2} < s < 1$: In this situation, the above integral is bounded from above by $\frac{1}{2s-1}$ and we have

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - \frac{6s\mathcal{J}_0}{2s-1} \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} \right].$$

Since $X(t) \leq x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$,

$$\frac{tx^{2s(\beta-1)-1}}{\kappa^{\beta-1}t^\beta} \leq \frac{2X(t)^{2s(\beta-1)-1}}{\kappa^{\beta-1}t^{\beta-1}}.$$

which by using the definition of $X(t)$, (3.19), enforces

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - \frac{12s\mathcal{J}_0}{2s-1} \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t]}{X(t)} \right].$$

Using that $\frac{1}{X(t)} \rightarrow 0$ and $\frac{t}{X(t)} \rightarrow 0$, (3.30), we may find t_2 so that for all $t \geq t_2$

$$\frac{1}{2}(1 - \varepsilon) \geq \frac{12s\mathcal{J}_0}{2s-1} \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t]}{X(t)}.$$

Case $s = \frac{1}{2}$: In this situation, the above integral is bounded from above by $\ln(B)$ and as above since $X(t) \leq x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$ we have

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - 12s\mathcal{J}_0 \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t] \ln(2^{\frac{1}{2s(\beta-1)}} \mathcal{C}_3 X(t))}{X(t)} \right].$$

Using the definition of $X(t)$, (3.19), we have $\ln(X(t)) \lesssim \ln(t)$ and thus since by (3.32), $\frac{t \ln(t)}{X(t)} \rightarrow 0$, we may find t_2 so that for all $t \geq t_2$

$$\frac{1}{2}(1 - \varepsilon) \geq -12s\mathcal{J}_0 \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t] \ln(2^{\frac{1}{2s(\beta-1)}} \mathcal{C}_3 X(t))}{X(t)}.$$

Case $0 < s < \frac{1}{2}$: In this last situation, the integral is bounded from above by $\frac{\mathcal{C}_3^{1-2s} x^{1-2s}}{1-2s}$ and therefore

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - \frac{6s\mathcal{J}_0\mathcal{C}_3^{1-2s}}{1-2s} \frac{x^{2s(\beta-1)-2s}}{(\kappa t)^{\beta-1}} \right].$$

which using that $X(t) \leq x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$ enforces

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq w^\beta(t, x) \left[\frac{1}{2}(1 - \varepsilon) - \frac{12s\mathcal{J}_0\mathcal{C}_3^{1-2s}}{1-2s} \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t]}{X^{2s}(t)} \right].$$

Again using the definition of $X(t)$, (3.19) and since by (3.31), $\frac{t}{X^{2s}(t)} \rightarrow 0$, we may find t_2 so that for all $t \geq t_2$

$$\frac{1}{2}(1 - \varepsilon) \geq \frac{12s\mathcal{J}_0\mathcal{C}_3^{1-2s}}{1-2s} \frac{[\varepsilon^{1-\beta} + \gamma(\beta-1)t]}{X^{2s}(t)}.$$

In each situation, we then find t_2 such that for all $t \geq t_2$ and $Y(t) \leq x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$

$$\mathcal{D}[\underline{u}](t, x) + \frac{1}{2}(1 - \varepsilon)\underline{u}^\beta(t, x) \geq 0.$$

□

5.2.4 The region $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$

Let us now obtain an estimate for the last region. In this region we claim

Proposition 5.5. *For all $\varepsilon \leq \frac{1}{2}, \gamma$ and κ there exists t_3 such that for all $t \geq t_3$*

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{8s\mathcal{J}_0 x^{2s}} \quad \text{for all } x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$$

with $\tau := \frac{3}{2\delta_c} \left(1 - \frac{1}{2\delta_c} + \frac{1}{3(2\delta_c)^2} \right)$.

Proof. We follow the same steps as for the proof of Proposition 5.4 but with some adaptations. Set $B, t' > t^\#$ as in the proof of Proposition 5.4, and observe that from the definition of $X(t)$, (3.19), by a straightforward computation, we see that there exists $t'' > 0$ so that for all $t \geq t''$ we have $2^{\frac{1}{2s(\beta-1)}} X(t) > Y(t)$.

So from Lemma 5.3 we have, for $t \geq \sup\{t', t''\}$ and $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{4s\mathcal{J}_0 x^{2s}} - 6s\mathcal{J}_0 w^\beta(t, x) \frac{x^{2s(\beta-1)-1}}{(\kappa t)^{\beta-1}} \left(\int_1^B z^{-2s} dz \right).$$

By using Proposition 3.2, we have

$$w^\beta(t, x) \leq 2^{\frac{\beta}{\beta-1}} \frac{(\kappa t)^\beta}{x^{2s\beta}}$$

and therefore we get

$$\begin{aligned} \mathcal{D}[\underline{u}](t, x) &\geq \frac{\varepsilon(1-\tau)}{4s\mathcal{J}_0 x^{2s}} - \frac{6s\mathcal{J}_0 2^{\frac{\beta}{\beta-1}} \kappa t}{x^{2s}} \frac{1}{x} \left(\int_1^B z^{-2s} dz \right), \\ &\geq \frac{1}{x^{2s}} \cdot \left[\frac{\varepsilon(1-\tau)}{4s\mathcal{J}_0} - 6s\mathcal{J}_0 2^{\frac{\beta}{\beta-1}} \frac{\kappa t}{x} \left(\int_1^B z^{-2s} dz \right) \right]. \end{aligned}$$

By considering separately the three cases $\frac{1}{2} < s < 1$, $s = \frac{1}{2}$, $0 < s < \frac{1}{2}$ and reproducing the argument used in the proof of Proposition 5.4 we may find t_3 such that for all $t \geq t_3$ and $x \geq 2^{\frac{1}{2s(\beta-1)}} X(t)$,

$$\mathcal{D}[\underline{u}](t, x) \geq \frac{\varepsilon(1-\tau)}{8s\mathcal{J}_0 x^{2s}}.$$

□

5.3 Tuning the parameters κ and γ

In this last part of the proof, we choose our parameters γ and κ in order that for some $t^* > 0$, \underline{u} is indeed a sub-solution to (3.16) for $t \geq t^*$

Recall that \underline{u} is a subsolution if and only if (3.21) and (3.22) hold simultaneously. Since (3.21) holds unconditionally for t sufficiently large, the only thing left to check is that (3.22) holds for a suitable choice of γ and κ .

By using (3.24) and (3.27), (3.22) holds if particular

$$3 \frac{\Phi_t(t, x)}{\Phi^\beta(t, x)} w^\beta(t, x) \leq \mathcal{D}[\underline{u}](t, x) + (1-\varepsilon)\underline{u}^\beta(t, x) - \gamma w^\beta, \quad x > X(t),$$

Set $t^* := \sup\{t_0, t_1, t_2, t_3\}$, where t_0, t_1, t_2 and t_3 are respectively determined by Proposition 5.1, Proposition 5.2, Proposition 5.4 and Proposition 5.5. To make our choice, let us decompose the set $[X(t), +\infty) = I_1 \cup I_2$ into two subsets defined as follows

$$I_1 := [X(t), 2^{\frac{1}{2s(\beta-1)}} X(t)], \quad I_2 := [2^{\frac{1}{2s(\beta-1)}} X(t), +\infty).$$

On the first interval, we have

Lemma 5.6. *For all $\varepsilon < \frac{1}{2}$, there exists γ^* such that for all κ and $\gamma \leq \gamma^*$, one has, for $t \geq \sup\{\frac{48}{\varepsilon^{\beta-1}(1-\varepsilon)}, t^*\}$,*

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \mathcal{D}[\underline{u}] + (1-\varepsilon)\underline{u}^\beta - \gamma w^\beta, \quad \text{for all } x \in I_1.$$

Proof. By definition of Φ , we have, at (t, x) ,

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = \frac{3}{t} \frac{x^{2s(\beta-1)}}{(\kappa t)^{\beta-1}} w^\beta.$$

By exploiting the definition of $X(t)$ it follows that for $x \leq 2^{\frac{1}{2s(\beta-1)}} X(t)$,

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = \frac{3}{t} \frac{x^{2s(\beta-1)}}{(\kappa t)^{\beta-1}} w^\beta \leq \left[\frac{6}{t} \left(\frac{1}{\varepsilon} \right)^{\beta-1} + 6\gamma(\beta-1) \right] w^\beta.$$

Let $\gamma_0 := \frac{1-\varepsilon}{48(\beta-1)}$, then for all $\gamma \leq \gamma_0$ we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \left[\frac{6}{t} \left(\frac{1}{\varepsilon} \right)^{\beta-1} + \frac{1-\varepsilon}{8} \right] w^\beta,$$

which for t large, say $t \geq \frac{48\varepsilon^{1-\beta}}{1-\varepsilon}$, gives $3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \frac{1-\varepsilon}{4} w^\beta$.

Recall that by Proposition 5.2 and Proposition 5.4, we have for all $x \in I_1$ and $t \geq t^*$,

$$\mathcal{D}[\underline{u}] + (1-\varepsilon)\underline{u}^\beta - \gamma w^\beta \geq \left(\frac{1-\varepsilon}{2} - \gamma \right) w^\beta,$$

since $\underline{u} \geq w$ for all $x \geq X(t)$. We then end our proof by taking $\gamma^* := \inf\{\gamma_0, \frac{1-\varepsilon}{4}\}$ and $t \geq \sup\{\frac{48\varepsilon^{1-\beta}}{1-\varepsilon}, t^*\}$. \square

Finally, let us check what happens on I_2 ,

Lemma 5.7. *For all $\varepsilon \leq \frac{1}{2}$, there exists κ^* such that for all $\gamma \leq \gamma^*$ and $\kappa \leq \kappa^*$, one has for all $t \geq t^*$,*

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq \mathcal{D}[\underline{u}] + (1-\varepsilon)\underline{u}^\beta - \gamma w^\beta, \quad \text{for all } x \in I_2.$$

Proof. As in the above proof, by definition of Φ we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta = 3\kappa \frac{x^{2s(\beta-1)}}{(\kappa t)^\beta} w^\beta.$$

By Proposition 3.2, we have for $x \in I_2$,

$$w^\beta(t, x) \leq 2^{\frac{\beta}{\beta-1}} \frac{(\kappa t)^\beta}{x^{2s\beta}},$$

therefore, we have

$$3 \frac{\Phi_t}{\Phi^\beta} w^\beta \leq 3\kappa \frac{x^{2s(\beta-1)}}{(\kappa t)^\beta} 2^{\frac{\beta}{\beta-1}} \frac{(\kappa t)^\beta}{x^{2s\beta}} = 3\kappa 2^{\frac{\beta}{\beta-1}} \frac{1}{x^{2s}}.$$

Now recall that by Proposition 5.5, we have for all $x \in I_2$ and $t \geq t^*$,

$$\mathcal{D}[\underline{u}] + (1-\varepsilon)\underline{u}^\beta - \gamma w^\beta \geq \left(\frac{1-\varepsilon}{2} - \gamma \right) w^\beta + \frac{\varepsilon(1-\tau)}{8\mathcal{J}_0 s x^{2s}},$$

since $\underline{u} \geq w$ for all $x \geq X(t)$. The Lemma is then proved by taking $\gamma \leq \gamma_0$ and $\kappa \leq \kappa^* := \frac{\varepsilon(1-\tau)}{24 \cdot 2^{\frac{\beta}{\beta-1}} \mathcal{J}_0 s}$. \square

5.4 The final argument

From the above, for all fixed $\varepsilon \leq \frac{1}{2}$ there exists κ^*, γ^* and t^* such that $\underline{u}(t, x)$ is a subsolution to (3.16) for all $t \geq t^*$. As in Section 4 to conclude the proof, we need to check that for some T we have $u(T, x) \geq \underline{u}(t^*, x)$ for all $x \in \mathbb{R}$.

To do so, let us observe that $\underline{u}(t^*, x) \leq 3w(t^*, x)$ and by using the definition of w , we have

$$\lim_{x \rightarrow +\infty} x^{2s} \underline{u}(t^*, x) \leq 3\kappa^* t^*.$$

Now thanks to Proposition 2.3, there exists $t_{3\kappa^* t^*}$ such that for all $t \geq t_{3\kappa^* t^*}$

$$\lim_{x \rightarrow +\infty} x^{2s} u(t, x) \geq 3\kappa^* t^*.$$

Therefore, we achieve $\underline{u}(t^*, x) \leq u(t_{3\kappa^* t^*}, x)$ for $x \gg 1$ says, $x > x_0$, and moreover thanks to the monotone behaviour of u , we have for all $t \geq t_{3\kappa^* t^*}$, $\underline{u}(t^*, x) \leq u(t, x)$ for $x > x_0$. On the other hand, by Proposition 2.4 $u(t, x) \rightarrow 1$ uniformly in $(-\infty, x_0]$ and since $\underline{u} \leq \frac{1}{2} < 1$ we can find \hat{t} so that $\underline{u}(t^*, x) \leq \frac{1}{2} < u(\hat{t}, x)$ for all $x \leq x_0$. Thus, by taking $T \geq \sup\{t_{3\kappa^* t^*}, \hat{t}\}$ we then achieve $\underline{u}(t^*, x) \leq u(T, x)$ for all $x \in \mathbb{R}$.

The proof of Theorem 1.4 is then complete for all $\varepsilon \leq \frac{1}{2}$. To obtain the speed of the level line for $\varepsilon \geq \frac{1}{2}$, again we can reproduce the proof used in [4, 25] using the adequate invasion property, namely Proposition 2.4.

6 Numerical experiments

In this Section, we provide numerical experiments complementing/illustrating the quantitative findings reported in the present work.

To compute approximations to the solution of the Cauchy problem (3.16), the PDE is first discretised in space using a quadrature rule-based finite difference method on a uniform Cartesian grid, and then integrated in time using an implicit-explicit (IMEX) scheme. To do so, one needs to set the problem on a bounded domain, which is achieved by truncating the real line to a bounded interval and imposing an *exterior* boundary condition.

Contrary to the nonlocal diffusion operator, for which the assumptions on the kernel J eliminate the possibility of a singularity, the integral fractional Laplacian features the singular non-integral integrand, and proper care is needed when discretising this operator. A common approach to dealing with this difficulty is to split the singular integral into a sum of an isolated contribution from the singular part with another having a smooth integrand and on which standard quadrature rules can be employed. Such a strategy has been used to solve both nonlocal (see [48]) and fractional (see [34, 28, 41]) diffusion models. In the present work, we followed the splitting approach introduced in [28]. It consists in writing the singular integral representation of the fractional Laplacian as a weighted integral of a weaker singular function by introducing a splitting parameter, namely

$$(-\Delta)^s u(x) = C_{1,s} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^\gamma} |x - y|^{\gamma-1-2s} dy \text{ for } s \text{ in } (0, 1),$$

where γ is a real number appropriately chosen in $(2s, 2)$. The discretisation of the fractional Laplacian on the one-dimensional bounded domain $\Omega = (0, L)$, $L > 0$, with the extended Dirichlet boundary condition $u = g$ in $\mathbb{R} \setminus \Omega$ then works as follows. A uniform Cartesian grid $\{x_j = jh \mid j \in \mathbb{Z}\}$, with $h = \frac{L}{N}$ for some integer N , is used. The fractional operator, evaluated at a given gridpoint x_j in Ω , is then decomposed into two parts

$$(-\Delta)^s u(x_j) = -C_{1,s} \left(\int_0^L \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^{1+2s}} dz + \int_L^{+\infty} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^{1+2s}} dz \right). \quad (6.45)$$

The first integral in the decomposition being singular, the splitting is used. Denoting $z_k = kh$, for k in $\{0, \dots, N\}$, one writes

$$\begin{aligned} \int_0^L \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^{1+2s}} dz &= \int_0^L \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^\gamma} |z|^{\gamma-1-2s} dz \\ &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^\gamma} z^{\gamma-1-2s} dz. \end{aligned}$$

For k in $\{2, \dots, N\}$, the integrals in the above sum are regular and approximated by the weighted trapezoidal rule, that is

$$\begin{aligned} \int_{z_{k-1}}^{z_k} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^\gamma} z^{\gamma-1-2s} dz &\approx \\ \frac{1}{2(\gamma-2s)} \left(\frac{u(x_j - z_{k-1}) - 2u(x_j) + u(x_j + z_{k-1}))}{z_{k-1}^\gamma} + \frac{u(x_j - z_k) - 2u(x_j) + u(x_j + z_k)}{z_k^\gamma} \right) &(z_k^{\gamma-2s} - z_{k-1}^{\gamma-2s}). \end{aligned}$$

For $k = 1$, assuming that the solution u is smooth enough (of class \mathcal{C}^2 for instance), the integral can also be formally approximated by the weighted trapezoidal rule,

$$\int_{z_0}^{z_1} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^\gamma} z^{\gamma-1-2s} dz \approx \frac{h^{\gamma-2s}}{2(\gamma-2s)} \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1}))}{h^\gamma},$$

using that

$$\lim_{z \rightarrow 0} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^\gamma} \approx \lim_{z \rightarrow 0} z^{2-\gamma} u''(z) = 0.$$

Note that an optimal convergence rate for this scheme is obtained for $\gamma = 1 + s$ (see the discussion in [28]).

Next, for any z larger than L , $x_j \pm z$ belongs to $\mathbb{R} \setminus \Omega$ and thus the value of $u(x_j \pm z)$ is given by the extended Dirichlet boundary condition. The second integral in (6.45) then reduces to

$$\int_L^{+\infty} \frac{u(x_j - z) - 2u(x_j) + u(x_j + z)}{z^{1+2s}} dz = -\frac{1}{sL^{2s}} u(x_j) + \int_L^{+\infty} \frac{g(x_j - z) + g(x_j + z)}{z^{1+2s}} dz,$$

and may be computed explicitly depending on the extended boundary datum g .

The discrete linear system resulting from this quadrature-based finite difference method is Toeplitz if a uniform grid is used and its solution can be advantageously tackled by the Levinson recursion, for a cost of $O(N^2)$ arithmetic operations. So-called superfast algorithms, which use the fast Fourier transform or other trigonometric transformations, also exist but may lack numerical stability in practice.

A forward-backward Euler (1, 1, 1) IMEX scheme (see [9]) is then applied to the semi-discretized equation, the diffusion term in the equation being treated implicitly (by the backward Euler method) and the nonlinear reaction term being dealt with explicitly (by the forward Euler method). Knowing that the solution tends to 1 at $-\infty$ and 0 at $+\infty$, we used Dirichlet boundary conditions with constant data taking these values where appropriate.

To cope with the algebraic decay of solutions and their spreading over a given period of time, which is necessary in order to observe the setting of a travelling or accelerated front, we implemented a very crude adaptation mechanism of the domain size along the iteration. At each time step, a criterion decides if the discretisation grid is to be expanded on each side or not, according the measured spreading of the numerical approximation at the current time and a given tolerance. This allows for discretisation points to be added to the grid (the space step being fixed one and for all at the beginning) over the course of the computation, which results in an ever increasing cost for each new iteration. The maximum number of added points at each step is a fixed parameter in the code, and, to complete the values of the approximation at these points, the boundary conditions are used, that is the value 1 on the left side of the grid, and the value 0 on the right

one. This results in using extremely large computational domains as the simulation progresses, and thus an ever increasing computational effort. Such crude approach nevertheless allowed to qualitatively confirm a number of theoretical results established in the present paper, but it showed its limitations in experiments in which smaller values of the fractional exponent were used, the computational domain being too small (with the parameters chosen for the computations) to correctly account for the spreading of the solution. As a consequence, the influence of the Dirichlet boundary conditions is felt and the asymptotic behaviour of the approximation is affected.

A more refined, but also more biased, way of both adding points and completing the approximated solution (or more generally of replacing the approximation by a Dirichlet problem by a problem set on the whole real line, see Section 4.4 in [35]) would be to follow some ansatz based on existing results for the asymptotic behaviour of solutions at infinity to construct an approximation of the solution outside of the computational domain, see for instance Theorem 1.3 in [32] for a generalised Fisher–KPP model (or Corollary 3.9 in [35]).

The method presented above was implemented using the standard NumPy and SciPy Python libraries and the `scipy.linalg.solve_toeplitz` routine to solve the Toeplitz linear system. In all the computations below, we used a timestep of length 0.01 and started from a bounded computational domain equal to $[-1000, 1000]$, discretised with 10001 points, that is a stepsize equal to 0.2 in space. The maximum number of points that could be added to each side of the domain at each iteration was 150.

Let us now show and describe our numerical results. We recover the expected dynamics of the invasion solution with respect to the Allee effect parameter β . Namely, after a rather short transition period, the dynamics stabilise in a regime where the level set of the solution evolves with order $t^{\frac{\beta}{2s(\beta-1)}}$. A first evidence is given in Figure 8, where we plot in semi-log scale the position of the level line $\frac{1}{2}$ with respect to time. Except when $\beta = 1$ where the dynamics is known to be different (an exponential in time expansion, yielding a linear in time curve in semi-log scale), the shape of position of the level lines are somehow identical i.e. the $\log(x_\lambda(t)) \sim C(s, \beta) \log(t)$. We also observe that $C(s, \beta)$ becomes constant in β as β grows, as expected.

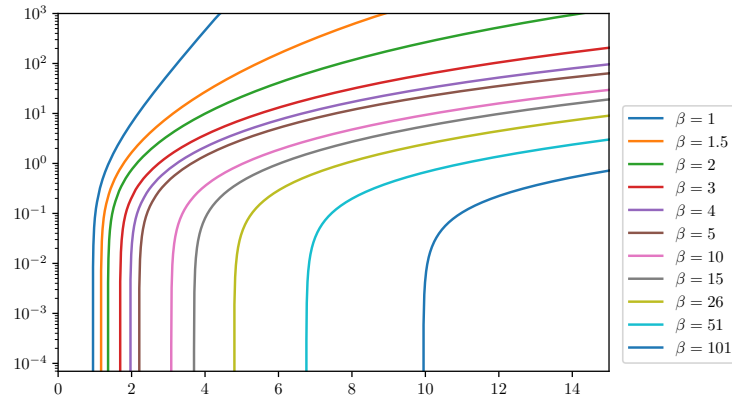


Figure 8: Logarithm of the position of the level line of height $\frac{1}{2}$ of numerical approximations of the solution to the problem with fractional diffusion, plotted as a function of time, for different values of β and s equal to $\frac{1}{2}$.

We illustrate in Figure 9, Figure 10 and Figure 11 the different behaviour that can be observed when s varies. For $\beta = 1.5$, Figure 9, we can see that acceleration always occur for all the values considered ($s = 0.3, 0.5, 0.7$). For $\beta = 3$, depicted in Figure 10 and Figure 11, the picture is more complex. In accordance with the theory we observe a transition from an accelerated invasion when $s \leq 0.7$ to a constant speed invasion when $s = 0.8$, the transition being captured for the value $s = 0.75$. As expected, we can also see that for both Allee effect considered, acceleration always occurs when $s < \frac{1}{2}$.

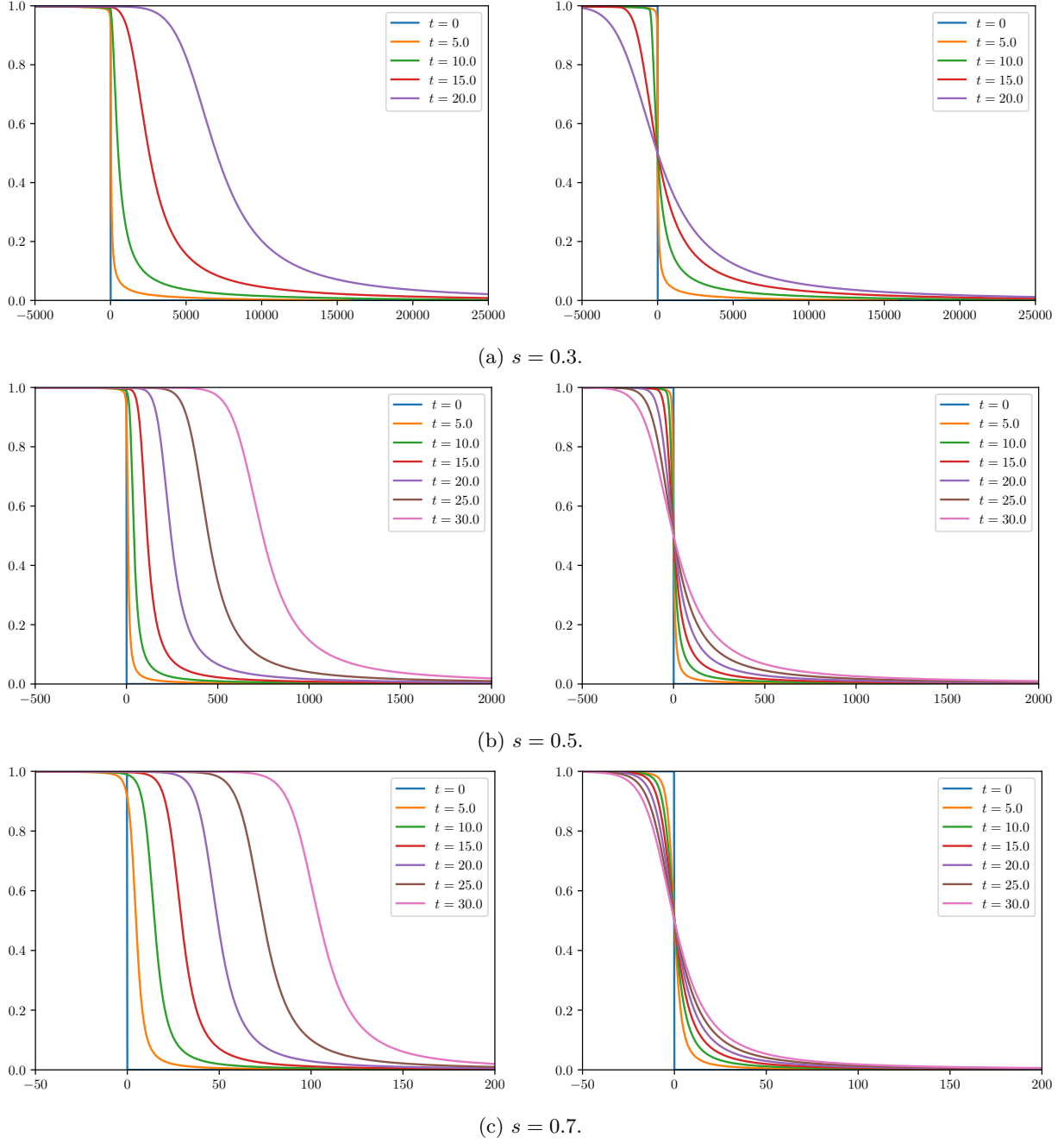
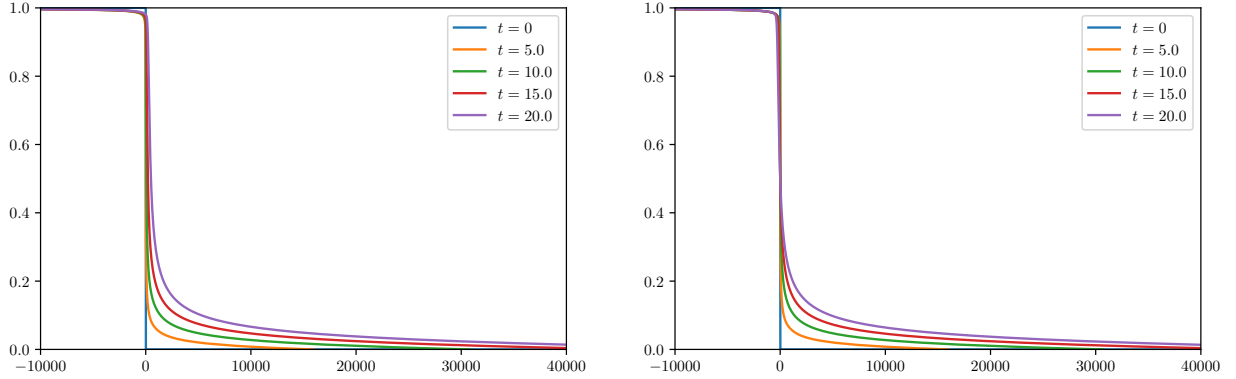


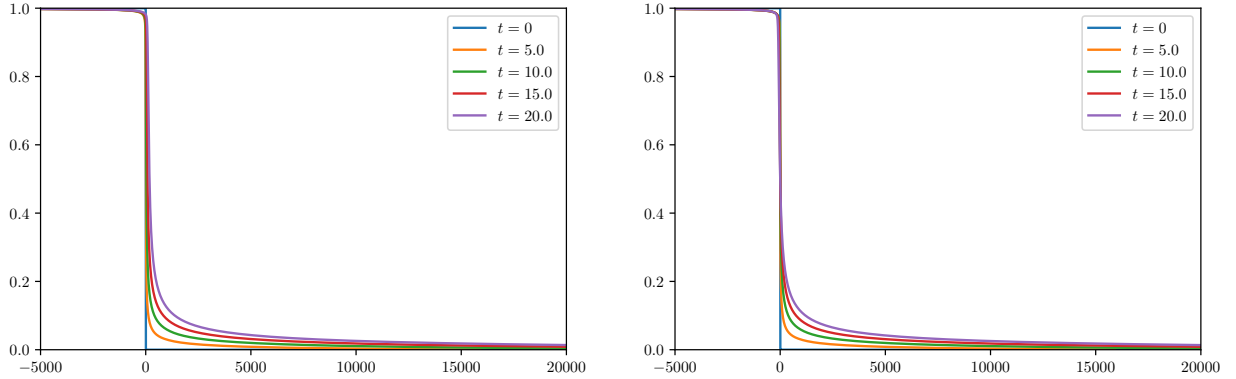
Figure 9: Numerical approximations of the solution to the problem with fractional diffusion at different times for $\beta = 1.5$ and different values of s . On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes.

Beware crucially that the space range of values where profiles are plotted varies drastically between plots: the flattening effect is larger and larger as s gets smaller and smaller. There is for example a factor 20 between the range of x used for $s = 0.3$ and for $s = 0.5$. This could lead to a possible misinterpretation of the numerical results if one is not careful enough while taking notice of the figures. To cope with this problem, we have plot in Figure 12, the shifted profile of the level-set at a given fixed time and on a given

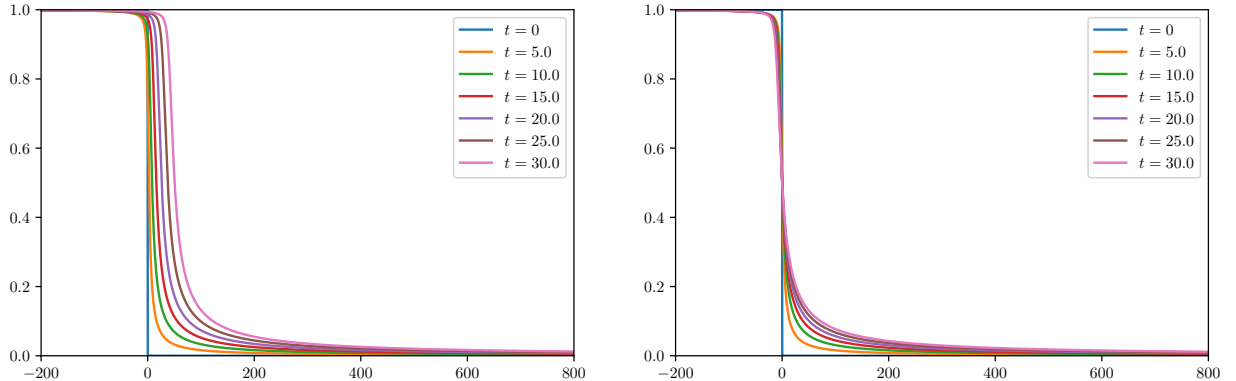
portion of space. By doing so, we see more easily the transition at $s = 0.75$: for $s \geq 0.75$ the profiles become rapidly very similar and front-like, whereas when $s < 0.75$ we observe a noticeable deformation.



(a) $s = 0.25$.

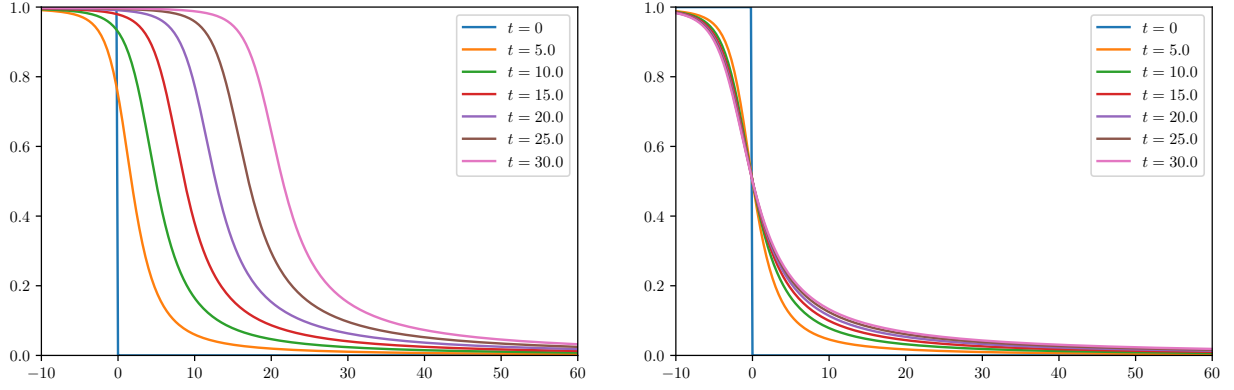


(b) $s = 0.3$.

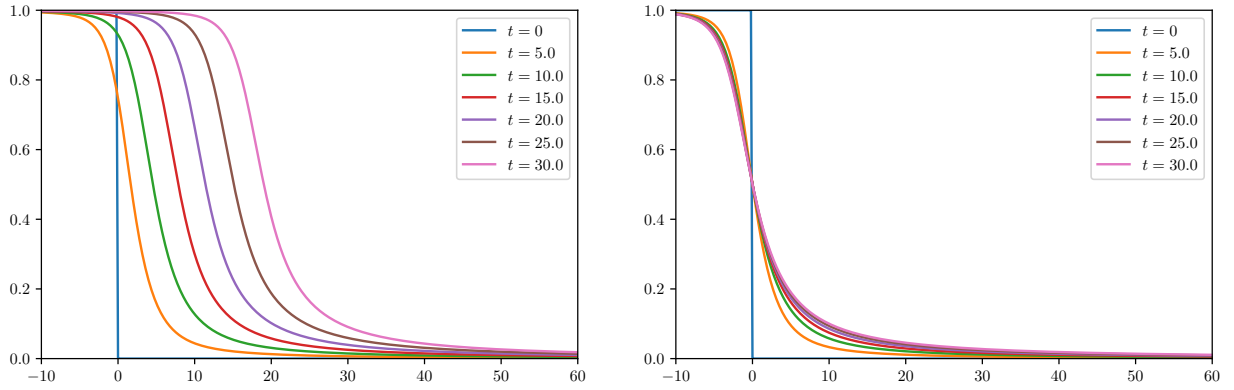


(c) $s = 0.5$.

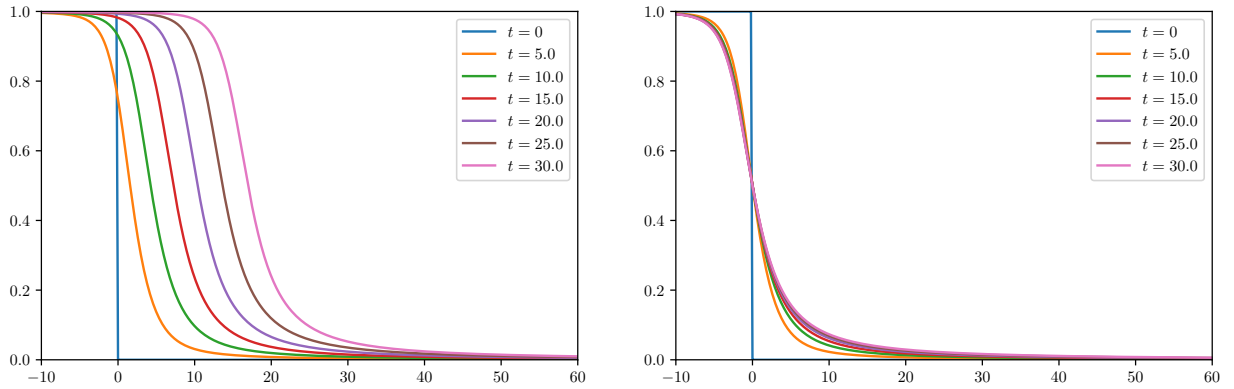
Figure 10: Numerical approximations of the solution to the problem with fractional diffusion at different times for $\beta = 3$ and different values of the fractional Laplacian exponent s . On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes.



(a) $s = 0.7$.



(b) $s = 0.75$.



(c) $s = 0.8$.

Figure 11: Numerical approximations of the solution to the problem with fractional diffusion at different times for $\beta = 3$ and different values of s . On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes.

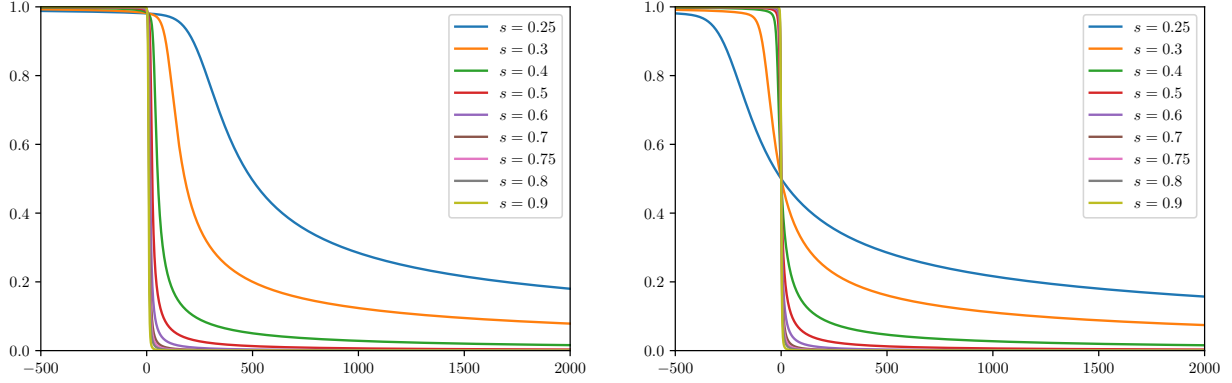


Figure 12: Numerical approximations of the solution to the problem with fractional diffusion at time $t = 20$ for $\beta = 3$ and different values of s . On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes.

Last, we illustrate in Figure 13, the expected asymptotic behaviour of the solution, showing that the numerical solution at the front edge behaves like $\frac{C}{x^{2s}}$. The fitting was achieved with the least squares method in the `scipy.optimize.curve_fit` routine. Observe that for the smallest value of s , the effect of the right boundary is huge and somehow ruins a little bit the adequation for large x . However, the routine manages to maintain a good fit for intermediate values of x , which is rather satisfactory.

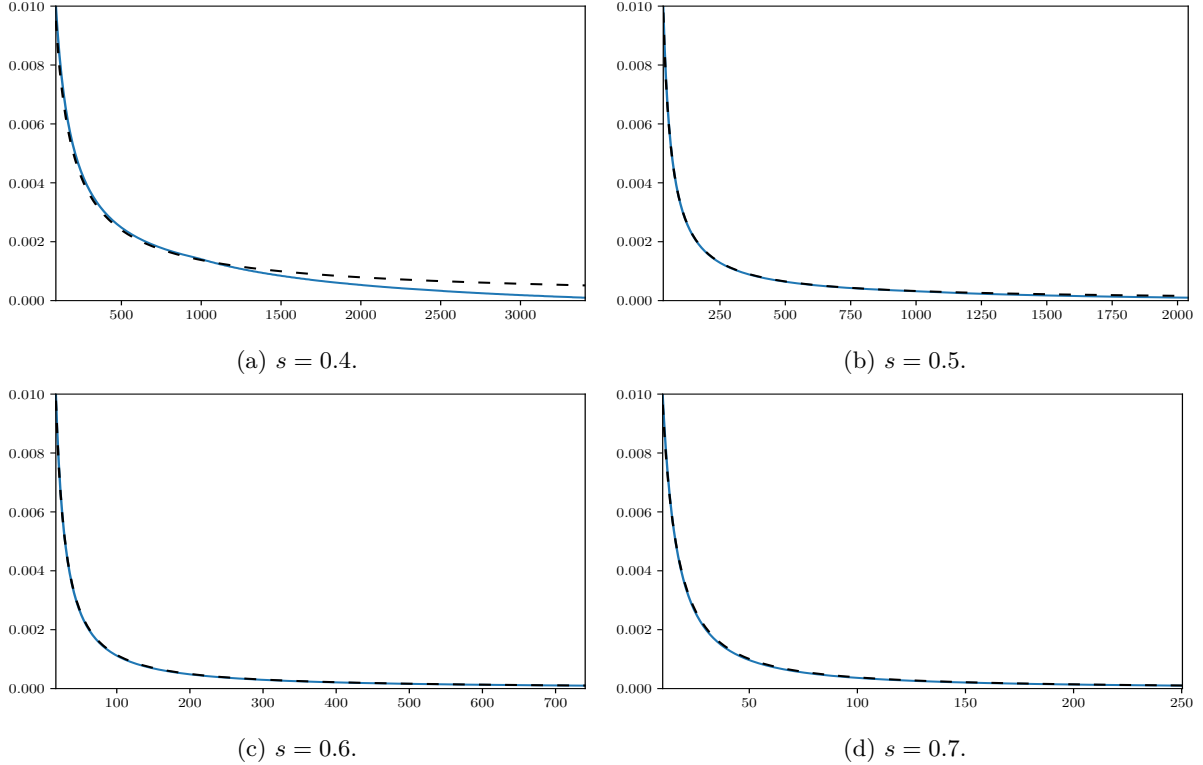


Figure 13: Fitting of the part of the tail of the approximation solution at time $t = 1$ bounded by value 10^{-2} on the left and value 10^{-4} on the right using the function $\frac{C}{x^{2s}}$ for $\beta = 1.5$ and different values of s . The blue line is the solution u and the black dashed line is the best function $\frac{C}{x^{2s}}$.

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