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Quentin Berger, Carsten Chong, Hubert Lacoin

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THE STOCHASTIC HEAT EQUATION WITH MULTIPLICATIVE LÉVY NOISE: EXISTENCE, MOMENTS, AND INTERMITTENCY

QUENTIN BERGER, CARSTEN CHONG, AND HUBERT LACOIN

ABSTRACT. We study the stochastic heat equation (SHE) $\partial_t u = \frac{1}{2}\Delta u + \beta u \xi$ driven by a multiplicative Lévy noise ξ with positive jumps and coupling constant $\beta > 0$, in arbitrary dimension $d \geq 1$. We prove the existence of solutions under an optimal condition if $d = 1, 2$ and a close-to-optimal condition if $d \geq 3$. Under an assumption that is general enough to include stable noises, we further prove that the solution is unique. By establishing tight moment bounds on the multiple Lévy integrals arising in the chaos decomposition of u , we further show that the solution has finite p th moments for $p > 0$ whenever the noise does. Finally, for any $p > 0$, we derive upper and lower bounds on the moment Lyapunov exponents of order p of the solution, which are asymptotically sharp in the limit as $\beta \rightarrow 0$. One of our most striking findings is that the solution to the SHE exhibits a property called *strong intermittency* (which implies moment intermittency of all orders $p > 1$ and pathwise mass concentration of the solution), for any non-trivial Lévy measure, at any disorder intensity $\beta > 0$, in any dimension $d \geq 1$. This behavior contrasts with that observed for the SHE on \mathbb{Z}^d and for the SHE on \mathbb{R}^d with Gaussian noise, for which intermittency does not occur in high dimensions for small β .

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1. INTRODUCTION

It has long been recognized that the *stochastic heat equation (SHE)*

$$(1.1) \quad \partial_t u = \frac{1}{2}\Delta u + \beta u \xi, \quad u(0, \cdot) = u_0,$$

and the *Kardar–Parisi–Zhang (KPZ) equation*

$$(1.2) \quad \partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}(\nabla h)^2 + \beta \xi, \quad h(0, \cdot) = h_0,$$

which are related through the Cole–Hopf transformation $h = \log u$ [18, 64, 68, 69], have far-reaching universality properties. In dimension $d = 1$, important progress has been made in recent years

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towards proving the *weak* and *strong KPZ universality conjectures*; see [4, 18, 47] and [5, 23, 22, 48], for instance, and [87] for a characterization of the *KPZ fixed point*, the universal limiting fluctuation field.

If $d \geq 2$, finding a non-trivial (*i.e.* non-Gaussian) notion of solution to the SHE/KPZ equation is a challenging problem, since the theories developed by [67, 70] to solve singular stochastic PDEs no longer apply. In $d = 2$, a major breakthrough has been achieved recently in [26], building on previous works by [17, 25, 65]. If $d \geq 3$ and one considers regularized versions of (1.1) and (1.2) (*e.g.*, by discretizing space or spatially mollifying the noise), it is well known that there is phase transition in β from a *weak* to a *strong disorder* regime; see [15, 39, 89]. So far, all fluctuations results in $d \geq 3$ have been obtained under weak disorder, with fluctuations belonging to the Edwards–Wilkinson/Gaussian universality class; see [41, 55, 66], for example.

For the SHE, weak/strong disorder is closely related to the phenomenon of localization of solutions and to the notion of *intermittency*, which first emerged in turbulence physics [102] and has been extensively studied for various versions of the SHE [16, 29, 31, 34, 58, 77, 82]. Intermittency roughly states that the solution exhibits large peaks on very small space scales: it is intimately related to the notion of *strong disorder*, which is in several instances shown to imply some geometric localization of the solutions (see [9, 7]); we refer to Section 3.2 for more comments. As with weak and strong disorder, these works show that as long as the noise remains short-range dependent (in space), intermittency occurs for all values of $\beta > 0$ when $d = 1, 2$ and only for β larger than some critical value when $d \geq 3$.

In this paper, we carry out a systematic study of the SHE and its intermittent behavior as $t \rightarrow \infty$ when ξ in (1.1) is no longer assumed to be *Gaussian*, which is a crucial requirement in all previously mentioned works. There are several motivations:

- (i) In [12, 99], (discrete) directed polymers with *heavy-tailed* environment are considered. It was shown in [13, 12] that, in some intermediate disorder regime, they rescale to the SHE with α -stable Lévy noise, where $\alpha \in (0, 2)$. In particular, in the same way as stable distributions arise in the generalized central limit theorem for variables with infinite variance, this suggests that the SHE with α -stable noise belongs to a new one-parameter family of KPZ universality classes. It therefore appears natural to study the properties of the SHE with α -stable noise, as the universal scaling limit of natural discrete models.
- (ii) Besides heavy-tailed noise, non-Gaussian but light-tailed noise such as (coloured) Poisson noise has also been considered before for (continuous) directed polymers; see *e.g.* [46, 49], or [40] for a review. We therefore consider here general Lévy noises, which include two central and important subclasses: α -stable noises $\alpha \in (0, 2)$ (heavy-tailed), and Poisson noises (light-tailed).
- (iii) After the KPZ equation had been introduced in [76] to model nonlinear growth phenomena, [71, Section 4.2] pointed out the need to study (1.2) with power-law noise since certain growth processes in real experiments, believed to follow the KPZ equation, exhibited scaling behaviors that could not be explained by a (white or colored) Gaussian noise.
- (iv) As we shall see, the SHE (1.1) is well posed for a large class of white noises ξ in arbitrary dimension d . So even in the absence of heavy tails, the asymptotic behavior of the SHE driven by a non-Gaussian noise may indicate to which degree the KPZ universality class in $d \geq 1$ is robust to distributional perturbations of the noise.

The most striking finding in our paper is that as soon as ξ in (1.1) is non-Gaussian, the solution to the SHE will be intermittent for all disorder intensities $\beta > 0$, in any dimension $d \geq 1$. We qualify this result as striking because when $d \geq 3$ and the noise has a finite second moment, the absence of a phase transition sharply contrasts with the results for the SHE with Gaussian noise or directed polymers in random environments (DPRE) [15, 21, 72, 82, 89]. In particular, if $d \geq 3$, our results imply that for small β , the SHE driven by a finite-variance non-Gaussian noise ξ behaves very differently from models which at first sight seem to belong to the same universality class. An

interesting open problem is therefore whether the SHE with non-Gaussian noise can be used to access the KPZ universality class in $d \geq 2$ (because it is guaranteed to be in the strong disorder regime) or whether it is an example that breaks out of KPZ universality.

The reason why a non-Gaussian noise induces intermittency/strong disorder for any β and in any dimension is subtle and is related to the fact that it is of *purely atomic* nature, but in such a way that atoms are put on random space-time points (determined by a Poisson point process) rather than on a fixed deterministic grid such as \mathbb{Z}^d , which is the standard way of discretizing (1.1). Our proofs in $d \geq 3$ show that if ξ has finite second moments and β is small, the combined effect of atoms located very close to one another becomes the dominant cause for intermittency.

Before describing our main results more precisely in Section 2, we provide a detailed introduction to Equation (1.1) with non-Gaussian noise, which aims at being as self-contained as possible.

1.1. Lévy noise with positive jumps. By [59, Chapter 4.4], a space-time white noise ξ , *i.e.* a stationary generalized random field such that for all test functions φ_1, φ_2 with $\varphi_1 \varphi_2 \equiv 0$ the variables $\xi[\varphi_1] = \int \varphi_1 d\xi$ and $\xi[\varphi_2] = \int \varphi_2 d\xi$ are independent, necessarily has an infinitely divisible distribution. Therefore, it is more commonly referred to as a *Lévy space-time white noise* and can be written as a sum of a Gaussian and a pure-jump component by the Lévy-Itô decomposition. For most parts of the paper, we assume that there is no Gaussian component and that the jumps of ξ are positive (which, as we will show, ensures positivity of the solution to the SHE). General Lévy noises, with a Gaussian part or with negative jumps, will be discussed in Section 3.3.

For a constructive definition of pure-jump Lévy noises, consider a Poisson point process ω on $\mathbb{R} \times \mathbb{R}^d \times (0, \infty)$ with intensity $\nu := dt \otimes dx \otimes \lambda(dz)$, where λ is a measure on $(0, \infty)$ that satisfies $\lambda([1, \infty)) < \infty$ and $\int_{(0,1)} z^2 \lambda(dz) < \infty$. The law of ω is denoted by \mathbb{P} . For convenience, we sometimes split ω into two processes $\omega_<$ and $\omega_>$ that correspond to the restriction of ω to $\mathbb{R} \times \mathbb{R}^d \times (0, 1)$ and $\mathbb{R} \times \mathbb{R}^d \times [1, \infty)$, respectively. We let $\mathbb{P}_<$ and $\mathbb{P}_>$ denote the associated probabilities, so that $\mathbb{P} = \mathbb{P}_< \otimes \mathbb{P}_>$. For $a \in (0, 1]$, consider the measure

$$(1.3) \quad \xi_\omega^a := \sum_{(t,x,z) \in \omega} z \mathbf{1}_{\{z \geq a\}} \delta_{(t,x)} - \kappa_a \mathcal{L},$$

where \mathcal{L} is the Lebesgue measure on $\mathbb{R} \times \mathbb{R}^d$ and $\kappa_a = \int_{[a,1)} z \lambda(dz)$. Also let

$$(1.4) \quad \xi_{\omega_<}^a := \sum_{(t,x,z) \in \omega} z \mathbf{1}_{\{z \in [a,1)\}} \delta_{(t,x)} - \kappa_a \mathcal{L}, \quad \xi_{\omega_>} := \sum_{(t,x,z) \in \omega} z \mathbf{1}_{\{z \geq 1\}} \delta_{(t,x)}.$$

Under the assumption $\int_{(0,1)} z^2 \lambda(dz) < \infty$, the measure ξ_ω^a converges a.s., in the local Sobolev space $H_{\text{loc}}^{-s}(\mathbb{R} \times \mathbb{R}^d)$ for any $s > (1+d)/2$, towards a limit $\xi_\omega \in H_{\text{loc}}^{-s}(\mathbb{R} \times \mathbb{R}^d)$ (this is a standard result, we refer to [13, Appendix A] for a proof of this exact statement). For $k \geq 1$ and $u \in \mathbb{R}$, the Sobolev space $H^u(\mathbb{R}^k)$ is the space of all Schwartz distributions $\varphi \in \mathcal{S}'(\mathbb{R}^k)$ for which

$$\|\varphi\|_{H^u(\mathbb{R}^k)} := \left(\int_{\mathbb{R}^k} (1 + |\xi|^2)^u |\hat{\varphi}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{\varphi}$ denotes the Fourier transform of φ . When $\varphi \in C_c^\infty(\mathbb{R}^k)$ (*i.e.*, φ is smooth and compactly supported), then $\hat{\varphi}(\xi) := \int_{\mathbb{R}^k} e^{i\xi \cdot x} \varphi(x) dx$. The local Sobolev space $H_{\text{loc}}^u(\mathbb{R}^k)$ is then defined by

$$H_{\text{loc}}^u(\mathbb{R}^k) := \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^k) : \forall \rho \in C_c^\infty(\mathbb{R}^k), \rho \varphi \in H^u(\mathbb{R}^k) \right\},$$

equipped with the topology induced by the family of seminorms $\{\varphi \mapsto \|\rho \varphi\|_{H^u(\mathbb{R}^k)} : \rho \in C_c^\infty(\mathbb{R}^k)\}$. The random distribution ξ_ω is called a *Lévy (space-time white) noise* with intensity or Lévy measure λ . In the simplest example, where $\lambda = \delta_1$ and ω is a unit-rate Poisson point process on \mathbb{R}^{1+d} , ξ_ω is called a standard *Poisson noise*. In the case where $\lambda(dx) = \alpha x^{-(1+\alpha)} dx$ for some $\alpha \in (0, 2)$,

ξ_ω is referred to as an α -stable noise. We refer the reader to [6, 50] for more background on Lévy noises. In the remainder, let $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ be the completed natural filtration of ξ_ω :

$$(1.5) \quad \mathcal{F}_t := \overline{\sigma(\omega \cap ([0, t] \times \mathbb{R}^d \times (0, \infty)))}.$$

Remark 1.1. Note that the Poisson process ω is measurable with respect to $\sigma(\xi_\omega)$. Hence the solution u to (1.1) that we construct in this paper (which is defined in terms of ω) is a function of the noise ξ . Our choice to index by ω instead of ξ is only for convenience because many quantities in our proofs are easier to express in terms of ω than in terms of ξ ; in particular, this is not because we need an extension of the probability space in our construction. We mention this explicitly because there are martingale constructions which are similar to the one used in this paper and that do require an extension of the probability space: this is, for instance, the case in Kahane's construction of Gaussian multiplicative chaos associated with a kernel of σ -positive type (see [74] for the seminal paper or [95, Section 2.1] for a review).

1.2. Mild solutions to SHE. A random field $(u(t, x))_{t > 0, x \in \mathbb{R}^d}$ is called a *mild solution* to the SHE (1.1) if for all $t > 0$ and $x \in \mathbb{R}^d$,

$$(1.6) \quad u(t, x) = \int_{\mathbb{R}^d} \rho(t, x - y) u_0(dy) + \beta \int_0^t \int_{\mathbb{R}^d} \rho(t - s, x - y) u(s, y) \xi(ds, dy)$$

almost surely. The statement includes the requirement that the integrals be well-defined and finite. In (1.6),

$$\rho(t, x) := (2\pi t)^{-\frac{d}{2}} e^{-\frac{\|x\|^2}{2t}}$$

is the d -dimensional heat kernel and $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d .

Remark 1.2. If $\int_{(0,1)} z \lambda(dz) = \infty$, the noise ξ does not have a locally finite total variation. In this case, the stochastic integral on the right-hand side of (1.6) cannot be defined as a path-by-path Lebesgue integral (*i.e.*, an integral with respect to a random measure defined on $[0, t] \times \mathbb{R}^d$) and must be interpreted in Itô's sense.

In [96], it was shown that (1.1) has a unique mild solution if

$$(1.7) \quad \exists q \in (1, 1 + \frac{2}{d}) : \int_{(0,\infty)} z^q \lambda(dz) < \infty.$$

While this condition covers many examples, it is not fully satisfying, in particular because it rules out α -stable noises for all $\alpha \in (0, 2)$. Subsequent papers that considered the case of α -stable noise (*e.g.*, [8, 13, 36, 37]) neither established uniqueness nor the finiteness of moments of the solution to (1.1). At this point, let us also mention [88, 91], who investigated an SHE with stable noise but with a non-Lipschitz nonlinearity.

1.3. The truncation approach to SHE. Prior to this article, the most general existence condition for the solution to the SHE (1.1) was obtained, for a Lévy noise with positive jumps, in [13], but using a (possibly) weaker notion of solution than (1.6). The approach in [13] consists in solving the equation

$$(1.8) \quad \partial_t u^a = \frac{1}{2} \Delta u^a + \beta u^a \xi_\omega^a, \quad u^a(0, \cdot) = u_0,$$

with ξ_ω^a from (1.3) and then taking the limit of u^a as $a \downarrow 0$.

Remark 1.3. In [13], it is not proved that the obtained limit satisfies (1.6). In fact, the main focus of [13] is not the SHE, but rather constructing a continuum path measure on \mathbb{R}^d : *the continuum directed polymer in Lévy noise*. This is further discussed in Section 3.1 below.

In order to review the results of [13], let us start by introducing some important notations and quantities. For $0 < s < t < \infty$, $x, y \in \mathbb{R}^d$, let $\mathfrak{X}_k(s, t) := \{(t_1, \dots, t_k) \in \mathbb{R}^k : s < t_1 < \dots < t_k < t\}$ and

$$(1.9) \quad \begin{aligned} \mathcal{X}_{s,t}^{(k)} &:= \mathfrak{X}_k(s, t) \times (0, \infty)^k, & X_{s,t}^{(k)} &:= \mathfrak{X}_k(s, t) \times (\mathbb{R}^d)^k, \\ \mathbb{X}_{s,t}^{(k)} &:= \mathfrak{X}_k(s, t) \times (\mathbb{R}^d)^k \times (0, \infty)^k. \end{aligned}$$

We drop the superscript k if $k = 1$. For $(\mathbf{t}, \mathbf{x}) \in X_{s,t}^{(k)}$, we further define

$$(1.10) \quad \rho_{s,x;t,y}(\mathbf{t}, \mathbf{x}) = \prod_{i=1}^{k+1} \rho(\Delta t_i, \Delta x_i),$$

where

$$(1.11) \quad \Delta t_i := t_i - t_{i-1} \quad \text{and} \quad \Delta x_i := x_i - x_{i-1},$$

with the convention that $t_0 := s$, $x_0 := x$ and $t_{k+1} := t$, $x_{k+1} := y$. Given $a > 0$ we let $|\xi_\omega^a|$ and $\xi_\omega^{a,+}$ denote the total variation and the positive part of ξ_ω^a (considered as a measure), that is,

$$(1.12) \quad |\xi_\omega^a| = \xi_\omega^a + 2\kappa_a \mathcal{L} \quad \text{and} \quad \xi_\omega^{a,+} = \xi_\omega^a + \kappa_a \mathcal{L}.$$

We define the *point-to-point partition function* associated with (1.8) and truncated Lévy noise ξ_ω^a as

$$(1.13) \quad \mathcal{Z}_\beta^{\omega,a}(s, x; t, y) := \rho(t - s, y - x) + \sum_{k=1}^{\infty} \beta^k \int_{X_{s,t}^{(k)}} \rho_{s,x;t,y}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^a(dt_i, dx_i),$$

under the assumption that

$$(1.14) \quad \sum_{k=0}^{\infty} \beta^k \int_{X_{s,t}^{(k)}} \rho_{s,x;t,y}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k |\xi_\omega^a|(dt_i, dx_i) < \infty.$$

It is proved in [13, Prop. 2.5 and 2.6] that (1.14) holds if and only if

$$(1.15) \quad \int_{[1,\infty)} (\log z)^{\frac{d}{2}} \lambda(dz) < \infty.$$

The case where $x = 0$ and $s = 0$ is of particular interest to us, hence we introduce the notational convention

$$(1.16) \quad \mathcal{Z}_\beta^{\omega,a}(x; t, y) := \mathcal{Z}_\beta^{\omega,a}(0, x; t, y), \quad \mathcal{Z}_\beta^{\omega,a}(t, x) := \mathcal{Z}_\beta^{\omega,a}(0, 0; t, x),$$

which applies similarly to other quantities such as $\mathfrak{X}_k(t) := \mathfrak{X}_k(0, t)$, $\mathcal{X}_t^{(k)} := \mathcal{X}_{0,t}^{(k)}$, $X_t^{(k)} := X_{0,t}^{(k)}$, $\mathbb{X}_t^{(k)} := \mathbb{X}_{0,t}^{(k)}$ and $\rho_{t,x}(\mathbf{t}, \mathbf{x}) := \rho_{0,0;t,x}(\mathbf{t}, \mathbf{x})$. When the condition (1.15) holds, $\mathcal{Z}_\beta^{\omega,a}$ is a mild solution to the SHE with noise ξ_ω^a and initial condition δ_0 . Indeed, from (1.14), the integrals are absolutely convergent, so (1.13) can be rewritten as

$$(1.17) \quad \mathcal{Z}_\beta^{\omega,a}(t, x) = \rho(t, x) + \beta \int_0^t \int_{\mathbb{R}^d} \rho(t - s, x - y) \mathcal{Z}_\beta^{\omega,a}(s, y) \xi_\omega^a(ds, dy).$$

In the same manner, $\mathcal{Z}_\beta^{\omega,a}(y; \cdot, \cdot)$ is a mild solution to the SHE with initial condition δ_y and noise ξ_ω^a , and given some uniformly bounded $u_0 : \mathbb{R}^d \rightarrow [0, \infty)$, the random field u^a defined by

$$(1.18) \quad u^a(t, x) := \int_{\mathbb{R}^d} u_0(y) \mathcal{Z}_\beta^{\omega,a}(y; t, x) dy$$

is a mild solution to the SHE with initial condition u_0 and noise ξ_ω^a ; see [13, Prop. 2.19].

Remark 1.4. Note that by translation invariance, we have

$$(1.19) \quad \begin{aligned} \left(\mathcal{Z}_\beta^{\omega,a}(s, x; t, y) \right)_{y \in \mathbb{R}^d} &\stackrel{(d)}{=} \left(\mathcal{Z}_\beta^{\omega,a}(t-s, y-x) \right)_{y \in \mathbb{R}^d} \\ &\stackrel{(d)}{=} \left(e^{-\frac{\|x-y\|^2 + \|y\|^2}{2(t-s)}} \mathcal{Z}_\beta^{\omega,a}(t-s, y) \right)_{y \in \mathbb{R}^d}. \end{aligned}$$

For this reason, we present most results only for the case $s = 0$ and $x = 0$, without loss of generality.

The next step is to investigate existence and “relevance” of the limit $\mathcal{Z}_\beta^{\omega,a}(t, x)$ as $a \downarrow 0$. If (1.15) holds, integrating the “Lebesgue part” of ξ_ω^a (see [13, Prop. 2.15] for details), we obtain the alternative expression

$$(1.20) \quad \mathcal{Z}_\beta^{\omega,a}(t, x) = e^{-\beta \kappa_a t} \left(\rho(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \right),$$

which entails the positivity of $\mathcal{Z}_\beta^{\omega,a}(t, x)$. Still under (1.15), considering the reverse filtration $\mathcal{G} := (\mathcal{G}_a)_{a \in (0,1]}$ defined by

$$(1.21) \quad \mathcal{G}_a := \sigma(\xi_\omega^a) = \sigma(\omega \cap (\mathbb{R} \times \mathbb{R}^d \times [a, \infty))),$$

one can observe that for almost every realization of ω_{\geq} , $(\mathcal{Z}_\beta^{\omega,a}(t, x))_{a \in (0,1]}$ is a non-negative càdlàg time-reversed $\mathbb{P}_{<}$ -martingale with respect to \mathcal{G} (cf. [13, Lemma 3.5]) and thus admits a limit as $a \downarrow 0$. To determine whether $\lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega,a}(t, x)$ is a good candidate for being a solution to (1.6), a first step is to determine whether this limit is degenerate or not. The answer depends on the intensity measure λ and the following result summarizes the main findings of [13, Thm. 2.7 & Prop. 2.10–2.15].

Theorem A. *Assume that λ satisfies the condition in (1.15).*

(i) *If, in addition,*

$$(1.22) \quad \begin{cases} \int_{(0,1)} z^2 \lambda(dz) < \infty & \text{if } d = 1, \\ \exists p \in (1, 1 + \frac{2}{d}) : \int_{(0,1)} z^p \lambda(dz) < \infty & \text{if } d \geq 2, \end{cases}$$

then for every $0 < s < t < \infty$ and $x, y \in \mathbb{R}^d$, we have

$$(1.23) \quad \mathcal{Z}_\beta^\omega(s, x; t, y) := \lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega,a}(s, x; t, y) > 0 \quad \mathbb{P}\text{-a.s.}$$

Moreover, the convergence holds in $L^1(\mathbb{P}_{<})$ \mathbb{P}_{\geq} -a.s., that is,

$$\lim_{a \rightarrow 0} \mathbb{E}_{<} \left[\left| \mathcal{Z}_\beta^\omega(s, x; t, y) - \mathcal{Z}_\beta^{\omega,a}(s, x; t, y) \right| \right] = 0 \quad \mathbb{P}_{\geq}\text{-a.s.}$$

(ii) *If, on the other hand, we have*

$$(1.24) \quad \begin{cases} \int_{(0,1)} z^2 \lambda(dz) = \infty & \text{if } d = 1, \\ \int_{(0,1)} z^2 |\log z| \lambda(dz) = \infty & \text{if } d = 2, \\ \int_{(0,1)} z^{1+\frac{2}{d}} \lambda(dz) = \infty & \text{if } d \geq 3, \end{cases}$$

then \mathbb{P} -a.s., we have $\lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega,a}(s, x; t, y) = 0$.

Remark 1.5. Note that the assumption $\int_{(0,1)} z^2 \lambda(dz) < \infty$ which appears as a necessary condition in (i) is required for the Lévy noise with jump intensity λ to be well defined. It is not a redundant assumption since the question of the convergence of $\mathcal{Z}_\beta^{\omega,a}$ when a tends to zero can be asked even

when λ is not a Lévy measure. The point (ii) in the above theorem shows that in that case the limit is always trivial.

One can also consider a *free-end* (or *point-to-line*) version of the partition function with truncated Lévy noise, given by

$$(1.25) \quad \mathcal{Z}_\beta^{\omega,a}(s, x; t, *) := \int_{\mathbb{R}^d} \mathcal{Z}_\beta^{\omega,a}(s, x; t, y) dy.$$

Theorem A applies to $\mathcal{Z}_\beta^{\omega,a}(s, x; t, *)$ *mutatis mutandis*. Following the convention (1.16), we use the notations $\mathcal{Z}_\beta^\omega(s, x; t, *)$, $\mathcal{Z}_\beta^\omega(x; t, *)$ and $\mathcal{Z}_\beta^\omega(t, *)$ for the limits we obtain as a tends to zero (as a consequence of the reverse martingale property). Furthermore, under the assumptions of Theorem 2.1, [13, Prop. 2.20] showed that $\lim_{a \rightarrow 0} u^a(t, x)$ exists a.s. for u^a defined in (1.18), without proving that the limit is a solution to (1.6). Since moments of the measure λ play an important role in our assumptions, we introduce

$$(1.26) \quad \mu_{a,b}(p) := \int_{[a,b)} z^p \lambda(dz),$$

for $0 \leq a \leq b \leq \infty$ and simply write $\mu := \mu_{1,\infty}(1)$.

Remark 1.6. In the case of a finite mean μ , given $t > 0$ and $x \in \mathbb{R}$, the convergence of both $\mathcal{Z}_\beta^{\omega,a}(t, x)$ and $\mathcal{Z}_\beta^{\omega,a}(t, *)$ as $a \rightarrow 0$ also holds in $L^1(\mathbb{P})$.

If $\mu < \infty$, it is convenient to consider the normalized partition functions

$$(1.27) \quad \bar{\mathcal{Z}}_\beta^{\omega,a}(t, x) := e^{-\beta\mu t} \mathcal{Z}_\beta^{\omega,a}(t, x), \quad \bar{\mathcal{Z}}_\beta^{\omega,a}(t, *) := e^{-\beta\mu t} \mathcal{Z}_\beta^{\omega,a}(t, *).$$

In the same manner, we can center the noise by setting $\bar{\xi}_\omega^a := \xi_\omega^a - \mu\mathcal{L}$, which has the effect that integrals with respect to $\bar{\xi}_\omega^a$ have mean zero. Note that

$$(1.28) \quad \bar{\mathcal{Z}}_\beta^{\omega,a}(t, x) = \rho(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \bar{\xi}_\omega^a(dt_i, dx_i)$$

and a similar identity for $\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *)$.

1.4. Intermittency and related notions. Let $(v(t, x))_{t>0, x \in \mathbb{R}^d}$ be a random field where $d \geq 0$; when $d = 0$ this means that $(v(t))_{t>0}$ is a stochastic process only indexed by t . Assuming that the *moment Lyapunov exponents*

$$(1.29) \quad \gamma(p) = \gamma(v, p) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[|v(t, x)|^p]$$

exist on $\bar{\mathbb{R}}$ for all $p \in (0, \infty)$ and are independent of $x \in \mathbb{R}^d$, we let

$$(1.30) \quad I = I(v) := \{p \geq 0 : \gamma(v, p) < \infty\},$$

with the convention $\gamma(0) = 0$. Further assuming $|\gamma(1)| < \infty$ in the following, we define the *normalized moment Lyapunov exponents* of v by

$$(1.31) \quad \bar{\gamma}(p) := \gamma(p) - p\gamma(1).$$

Clearly, $\bar{\gamma}(1) = 0$. Following the terminology of [29, 58, 102], if $p > 1$ and $|\gamma(1)| < \infty$, we say that v exhibits *(moment) intermittency of order p* if $\bar{\gamma}(p) \in (0, \infty)$. We say that v exhibits *full intermittency* if v is intermittent of all orders $p \in (1, \infty) \cap I$.

Let us also introduce some new terminology when we consider moments of order $p \in (0, 1)$: we say that v exhibits *strong intermittency* (in analogy with “very strong disorder” used in the directed polymer context, see the discussion in the next paragraph) if $\bar{\gamma}(p) \in (-\infty, 0)$ for some $p \in (0, 1)$. As $p \mapsto \bar{\gamma}(p)$ is convex on I (cf. Proposition 2.11 (i) below), strong intermittency implies $\bar{\gamma}(p) < 0$ for all $p \in (0, 1)$ but also that $\bar{\gamma}(p) > 0$ for all $p > 1$. Thus, strong intermittency implies full intermittency. As Section 3.2 below reveals, strong intermittency, plus some ergodic

properties in x , yields a *geometric characterization* of intermittency: we have a mass concentration of the paths of v at large times, characterized by the appearance of exponentially large peaks on islands covering only an exponentially small fraction of space. A more quantitative study of this phenomenon was undertaken in [78, 79], and we also refer to [57] for a similar work concerning the parabolic Anderson model on \mathbb{Z}^d .

Intermittency for the SHE. For the SHE on \mathbb{Z}^d with either a Gaussian or a finite-variance Lévy noise, the results of [1, 2, 29] and [15] show that

- if $d = 1, 2$, then strong intermittency holds for every $\beta > 0$;
- if $d \geq 3$, then strong intermittency holds if and only if $\beta > \bar{\beta}_c$ for some $\bar{\beta}_c > 0$.

For the SHE on \mathbb{R} driven by a multiplicative Gaussian white noise ξ , the analysis of intermittency has a long history: if ξ has variance 1 and $u_0 \equiv 1$, the authors of [16] derived the formula

$$(1.32) \quad \gamma(p) = \gamma(u, p) = \frac{p(p^2 - 1)}{24} \beta^4$$

for $p \in \mathbb{N}$ and proved it for $p = 2$. Later, a proof of (1.32) was given in [31] and [85] for all integers $p \in \mathbb{N}$ and all real numbers $p \geq 2$, respectively. The formula (1.32) for all $p > 0$ (and in particular, strong intermittency) was only established recently in [51, 61] using integrable probability methods. For the SHE on \mathbb{R}^d driven by a Gaussian noise that is white in time and colored in space with, say, a compactly supported correlation function, one has a similar picture to the discrete-space setting: strong intermittency always holds if $d = 1, 2$ and only for large β if $d \geq 3$; see [33, 82]. The situation may be different if the noise has long-range spatial correlation, as shown in [33, 56, 82].

For the solution to the Lévy-driven SHE, under the assumption (1.7), [38] established intermittency of order p in the following cases: for all $p \in (1, 3)$ if $d = 1$; for p close enough to (but smaller than) $1 + \frac{2}{d}$ if $d \geq 2$; for any fixed $p \in (1, 1 + \frac{2}{d})$ in any dimension $d \geq 2$ if $\beta > \beta_{p,d}$ for some $\beta_{p,d} > 0$. Whether strong intermittency (or full intermittency in dimensions $d \geq 2$) holds in the Lévy case or not, has been an open problem so far.

Very strong disorder for the directed polymer model. To complete the former discussion, let us mention some results that have been proved for the DPRE, whose partition function formally corresponds to the solution to an SHE in discrete space and time. In the context of the directed polymer model, the notion equivalent to strong intermittency is that of *very strong disorder* (see for instance [3, 28, 81]). For cultural reasons (DPRE is a statistical mechanics model), very strong disorder is a property of the *free energy*, that is, of the asymptotic behavior of $\mathbb{E}[\log Z_\beta(N, *)]$, rather than a property of the moments of order $p \in (0, 1)$ of the partition function; but this is not relevant for the present discussion. Very strong disorder has been proved to hold for directed polymers in a various settings:

- When $d = 1$ [44] and $d = 2$ [81] for any $\beta > 0$;
- When $d \geq 3$, if the environment has a power-law distribution with exponent $\alpha \in (1, 1 + \frac{2}{d}]$, for any $\beta > 0$, in [99] (this roughly corresponds to α -stable noise in the SHE context).

The method used in [81] does not rely much on the discrete nature on the model and has been adapted to prove analogous results when either space or time are continuous: Let us mention the case of the SHE on \mathbb{Z}^d with Gaussian white noise [15], the SHE on \mathbb{R}^d with a Gaussian noise which is white in time but colored in space [82, Thm. 1.2], or directed polymers in a Poisson environment [40, Rem. 3.4.3] (the partition function of which corresponds to that of the SHE with a spatially convoluted Lévy noise). In fact, the techniques can also be adapted to the SHE with Lévy noise, and the content of Section 9.2 in the present paper is strongly inspired by the proofs in [81] (for $d = 1$) and [99] (for the case of heavy-tailed noise).

On the other hand there are other situations where the directed polymer model *does not* display very strong disorder:

- When $d \geq 3$, if the environment has bounded second moment and β is small [21, 72];

- When $d \geq 3$, if the environment has a power-law distribution with exponent $\alpha > 1 + \frac{2}{d}$ and β is sufficiently small [99].

In both cases, it has been shown that the directed polymer displays *weak disorder*, a property which implies but goes beyond the absence of very strong disorder. Weak disorder implies, for instance, that the sample paths drawn from the polymer measure have diffusive behavior [45].

For this reason, the question of whether *strong intermittency* holds for the SHE with Lévy space-time white noise is particularly intriguing when $d \geq 3$ and β is small.

2. MAIN RESULTS

Our main results can be summarized as follows:

- (i) We considerably reduce the gap conditions (1.22) and (1.24) leave regarding the non-degeneracy of $\mathcal{Z}_\beta^\omega(t, x)$ if $d \geq 2$. We obtain, in Theorem 2.1 and Proposition 2.3, a necessary and sufficient condition in dimension $d = 2$ and a close-to-optimal condition in dimensions $d \geq 3$. We further prove the L^p -convergence of $\mathcal{Z}_\beta^{\omega, a}(t, x)$ to $\mathcal{Z}_\beta^\omega(t, x)$ for all $p \in (1, 1 + \frac{2}{d})$ as soon as the noise ξ_ω has a finite p th moment (*i.e.*, $\mu_{1, \infty}(p) < \infty$).
- (ii) In Theorem 2.6, we show that the limit $\mathcal{Z}_\beta^\omega(t, x)$, when non-degenerate, is indeed a mild solution to the SHE with initial condition δ_0 . Theorem 2.8 further establishes the uniqueness of solutions under conditions that are general enough to include the case of α -stable noise.
- (iii) After establishing the existence of the moment Lyapunov exponents of $\mathcal{Z}_\beta^\omega(t, *)$ and $\mathcal{Z}_\beta^\omega(t, x)$ in Proposition 2.10, we show our most striking result in Theorem 2.12: the solution to the SHE with a multiplicative Lévy noise and δ_0 -initial condition exhibits strong intermittency—and thus, in particular, full intermittency—for any non-trivial environment with finite expectation, that is, for any $\beta > 0$, in any dimension $d \geq 1$, for any non-trivial intensity measure λ with $\mu < \infty$. In Theorems 2.13 and 2.14, we further complement this result by deriving sharp asymptotic estimates on the moment Lyapunov exponents as β tends to 0.

The proof of the aforementioned results relies on two main methodological achievements:

- (i) By combining decoupling inequalities with an iterative partition of $\mathcal{X}_t^{(k)}$ (the integration domain of (t, z)), we establish, in Propositions 6.1 and 6.3, sharp moment bounds of order $p > 1$ on the multiple Lévy integrals that arise in the series representation (1.28) of $\tilde{\mathcal{Z}}_\beta^{\omega, a}(t, x)$. From these, we will then derive the upper bounds for $p > 1$ and, by convexity, the lower bounds for $p < 1$ in Theorems 2.13 and 2.14. The proof of Theorems 2.1, 2.6 and 2.8 will also rely on these moment bounds.
- (ii) By combining a change-of-measure technique with a coarse-graining approach, summarized in Lemma 9.1, we obtain moment upper bounds of order $p \in (0, 1)$ for $\mathcal{Z}_\beta^\omega(t, *)$ which are then used to show Theorem 2.12 as well as the lower bounds in Theorems 2.13 and 2.14. Moreover, we apply a simpler version of this method to prove Proposition 2.3. While the coarse-graining and change-of-measure approach is derived from that used in [81], which itself was inspired from earlier work on disordered pinning [54, 63, 97], its implementation to prove Proposition 9.5 (which is the most important part of the proof of Theorem 2.12) and Proposition 2.3 relies on important novel ideas, which we discuss in more details at the beginning of Section 9. In particular, it is crucial that atoms of the noise can occur arbitrarily close to each other, which distinguishes (2.6) from discretizations of the SHE on a lattice.

2.1. Optimal conditions for the non-degeneracy of $\mathcal{Z}_\beta^\omega(t, x)$. When $d \geq 2$, Theorem A displays a gap between the sufficient condition (1.22) and the necessary condition (1.24) for the non-degeneracy of $\mathcal{Z}_\beta^\omega(s, x; t, y)$. Our first result reduces this gap when $d \geq 3$ and identifies the

necessary and sufficient condition in dimension $d = 2$. Our new sufficient condition reads as follows:

$$(2.1) \quad \begin{cases} \int_{(0,1)} z^2 \lambda(dz) < \infty & \text{if } d = 1, \\ \int_{(0,1)} z^{1+\frac{2}{d}} |\log z| \lambda(dz) < \infty & \text{if } d \geq 2. \end{cases}$$

For later reference, we also included the case $d = 1$, which is identical to (1.22).

Theorem 2.1 (Non-degeneracy). *Let $d \geq 1$ and assume that (1.15) and (2.1) hold. Then, for any t and x , we have*

$$(2.2) \quad \mathcal{Z}_\beta^\omega(t, x) := \lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega, a}(t, x) > 0 \quad \mathbb{P}\text{-a.s.}$$

In addition, for any $p \in [1, 1 + \frac{2}{d})$, we have

$$(2.3) \quad \lim_{a \rightarrow 0} \mathbb{E}_< \left[\left| \mathcal{Z}_\beta^{\omega, a}(t, x) - \mathcal{Z}_\beta^\omega(t, x) \right|^p \right] = 0 \quad \mathbb{P}_\geq\text{-a.s.}$$

If furthermore $\mu_{1,\infty}(p) < \infty$, then

$$(2.4) \quad \lim_{a \rightarrow 0} \mathbb{E} \left[\left| \mathcal{Z}_\beta^{\omega, a}(t, x) - \mathcal{Z}_\beta^\omega(t, x) \right|^p \right] = 0.$$

The same results hold for the free-end partition function $\mathcal{Z}_\beta^\omega(t, *)$.

Remark 2.2. The restriction $p \in [1, 1 + \frac{2}{d})$ in (2.3) is simply due to the fact that for $p \geq 1 + \frac{2}{d}$, for any $a > 0$ and any non-trivial measure λ we have $\mathbb{E}[\left| \mathcal{Z}_\beta^{\omega, a}(t, x) \right|^p] = \infty$. It is easy to check that in the case of λ with bounded support the term corresponding to $k = 1$ in (1.13) has finite moments of order p if and only if $p < 1 + \frac{2}{d}$. We also refer to [38, Thm. 3.1].

Together with (1.24), this shows that (2.1) is necessary and sufficient for the non-degeneracy of $\mathcal{Z}_\beta^\omega(t, x)$ and $\mathcal{Z}_\beta^\omega(t, *)$ in dimension $d = 2$. If $d \geq 3$, the following result improves upon the necessary condition in (1.24).

Proposition 2.3 (Degeneracy). *Suppose that (1.15) holds and that $d \geq 3$. If for some $\varepsilon > 0$, we have that*

$$(2.5) \quad \int_{(0, e^{-\varepsilon})} \frac{z^{1+\frac{2}{d}} |\log z|}{(\log |\log z|)^{5+\frac{4}{d}+\varepsilon}} \lambda(dz) = \infty,$$

then for all $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega, a}(t, x) = \lim_{a \rightarrow 0} \mathcal{Z}_\beta^{\omega, a}(t, *) = 0 \quad \mathbb{P}\text{-a.s.}$$

While there is still a small gap between (2.1) and (2.5) if $d \geq 3$, we believe that (2.1) is necessary and sufficient for non-degeneracy.

Remark 2.4. There is a small abuse of language when we say that we establish an almost necessary and sufficient condition for existence of solutions to the SHE, since Proposition 2.3 does not *a priori* exclude the existence of solutions that are not given by limits of solutions with truncated noise.

Remark 2.5. Besides replacing (1.22) by the weaker condition (1.24) and the inclusion of L^p -convergence, Theorem 2.1 contains a third important improvement. Contrary to Theorem A, the proof of (2.3), which is sufficient to establish non-triviality of the limit, does not rely on the positivity of $\mathcal{Z}_\beta^{\omega, a}(t, x)$ and can thus be directly adapted to the case of signed noise, an observation that we further elaborate on in Section 3.3. On the other hand, our proof of Proposition 2.3 strongly relies on having only positive jumps.

2.2. Existence and uniqueness of solutions to SHE. Our next result shows that the point-to-point partition function $\mathcal{Z}_\beta^\omega(t, x)$ solves the SHE with Lévy noise ξ_ω and initial condition δ_0 .

Theorem 2.6 (Existence). *Suppose that (1.15) and (2.1) hold.*

(i) *The point-to-point partition function $\mathcal{Z}_\beta^\omega(t, x)$ is a mild solution to the SHE (1.6) with initial condition δ_0 , that is, for every $(t, x) \in (0, \infty) \times \mathbb{R}^d$, the stochastic integral below is well defined and the following identity holds a.s.:*

$$(2.6) \quad \mathcal{Z}_\beta^\omega(t, x) = \rho(t, x) + \beta \int_0^t \int_{\mathbb{R}^d} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(y, s) \xi_\omega(ds, dy).$$

(ii) *If u_0 is a locally finite signed measure on \mathbb{R}^d such that*

$$(2.7) \quad \limsup_{r \rightarrow \infty} r^{-2} \log \left(|u_0|([-r, r]^d) \right) < \frac{1}{2T}$$

for some $T > 0$, then

$$(2.8) \quad v(t, x) := \int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(y; t, x) u_0(dy)$$

is well defined and finite for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and is a solution to the SHE (1.6) with initial condition u_0 , on $[0, T] \times \mathbb{R}^d$.

Remark 2.7. The proof presented in this paper actually implies that

$$(2.9) \quad \mathcal{Z}_\beta^\omega(t, x) = \rho(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega(dt_i, dx_i),$$

where the right-hand side is well defined as a convergent sum of iterated stochastic integrals (see Section 4.3 for the corresponding framework).

Under the stronger condition (1.22), we further have uniqueness.

Theorem 2.8 (Uniqueness). *We assume that λ satisfies (1.15) and (2.1) and that u_0 and T satisfy (2.7). We let $(v(t, x))_{t \in [0, T], x \in \mathbb{R}^d}$ be the solution to the SHE defined in (2.8). Then there exists $\theta > 1$ such that for all $p \in (1, 1 + \frac{2}{d})$, $t \in (0, T]$ and $x \in \mathbb{R}^d$, we have*

$$(2.10) \quad \begin{cases} \mathbb{E}_< \left[|v(t, x)|^p \right] < \infty & \mathbb{P}_\geq\text{-a.s.}, \\ \int_{(0, t) \times \mathbb{R}^d} \rho(\theta(t-s), x)^p \mathbb{E}_< \left[|v(s, x)|^p \right] ds dx < \infty & \mathbb{P}_\geq\text{-a.s.} \end{cases}$$

Moreover, if λ satisfies (1.22), v is the unique predictable random field (up to modifications) that is a mild solution to the SHE (1.6) and satisfies (2.10).

Remark 2.9. Since Equation (1.6) only makes sense if the process ξ_ω is well defined, the assumption (2.1) present in Theorem 2.6 and 2.8 is redundant in the case $d = 1$.

2.3. Lyapunov exponents and intermittency. Having established the finiteness of moments in Theorem 2.1, we now investigate the growth rate of these moments as $t \rightarrow \infty$ and their dependence on β . Our first task is to establish the existence of the (moment) Lyapunov exponents.

Proposition 2.10. *Assume that (1.15) and (2.1) hold and let $p \in (0, 1 + \frac{2}{d})$.*

(i) *If $\mu_{1,\infty}(p) < \infty$, then the moment Lyapunov exponent*

$$(2.11) \quad \gamma_\beta(p) := \gamma(\mathcal{Z}_\beta^\omega(t, *), p) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \left[\mathcal{Z}_\beta^\omega(t, *)^p \right]$$

exists and is finite.

(ii) The exponents $\gamma_\beta(p)$ also capture the growth of the point-to-point partition function, in the sense that

$$(2.12) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\mathcal{Z}_\beta^\omega(t, 0)^p] = \gamma_\beta(p).$$

By [38, Thm. 3.1], we have $\mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^{1+2/d}] = \mathbb{E}[\mathcal{Z}_\beta^\omega(t, 0)^{1+2/d}] = \infty$ for any $t > 0$ and non-trivial λ . We also have $\mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^p] = \infty$ if $\mu_{1,\infty}(p) = \infty$. Hence, $\gamma_\beta(p)$ is well defined (possibly infinite) for every $p \geq 0$. Letting

$$(2.13) \quad p_{\max} = p_{\max}(\lambda) := \sup \left\{ p \in [0, 1 + \frac{2}{d}] : \mu_{1,\infty}(p) < \infty \right\},$$

we obtain from the above observation that the set

$$(2.14) \quad I_\lambda := I(\mathcal{Z}_\beta^\omega(t, *)) = \{p \geq 0 : \gamma_\beta(p) < \infty\}$$

is either $[0, p_{\max})$ or $[0, p_{\max}]$, the latter occurring if $p_{\max} < 1 + \frac{2}{d}$ and $\mu_{1,\infty}(p_{\max}) < \infty$.

The next result lists a few (partly classical) qualitative properties of $\gamma_\beta(p)$. If $\mu < \infty$ (in which case $1 \in I_\lambda$), we write

$$\bar{\gamma}_\beta(p) := \bar{\gamma}(\mathcal{Z}_\beta^\omega(t, *), p) = \gamma_\beta(p) - p\gamma_\beta(1) = \gamma_\beta(p) - p\mu = \gamma(\bar{\mathcal{Z}}_\beta^\omega(t, *), p).$$

By Jensen's inequality, we have $\bar{\gamma}_\beta(p) \geq 0$ for $p > 1$ and $\bar{\gamma}_\beta(p) \leq 0$ for $p < 1$.

Proposition 2.11. *Under the assumptions of Proposition 2.10, we have the following:*

- (i) The map $p \mapsto \gamma_\beta(p)$ is convex on I_λ .
- (ii) If $\mu < \infty$ and $p \in I_\lambda$, the map $\beta \mapsto \bar{\gamma}_\beta(p)$ is non-decreasing if $p \geq 1$ and non-increasing if $p \in (0, 1]$.
- (iii) If $p_{\max} > 1$, then $p \mapsto \gamma_\beta(p)/p$ is strictly increasing on $\{p \in I_\lambda : \gamma_\beta(p) - p\gamma_\beta(1) > 0\}$.

Our next goal is to understand under what conditions we have strong intermittency. Informally speaking, strong intermittency implies that the solution develops exponentially large peaks on exponentially small areas. A more formal description of the relationship between intermittency and localization is given in Section 3.2. Based on the results reviewed in Section 1.4, one may conjecture that the solution to the SHE on \mathbb{R}^d driven by a Lévy noise (that is white in time and space) exhibits a similar behavior: strong intermittency in dimensions $d = 1, 2$ and strong intermittency only for large β if $d \geq 3$. This turns out to be a fallacy: the SHE with a non-trivial Lévy noise *always* exhibits strong intermittency, irrespective of β , λ , d or p .

Theorem 2.12 (Strong intermittency). *Let $d \geq 1$. If $\lambda \neq 0$, $\mu < \infty$ and $\beta > 0$, we have $\bar{\gamma}_\beta(p) < 0$ for $p \in (0, 1)$ and $\bar{\gamma}_\beta(p) > 0$ for $p \in I_\lambda \cap (1, \infty)$.*

To shed more light on this peculiar result, we further investigate the detailed behavior of $\bar{\gamma}_\beta(p)$ in the small β limit. Let us start with the case of light-tailed noise at infinity.

Theorem 2.13 (Light-tailed noise). *Assume that (2.1) holds.*

- (i) If $d = 1$ and $\mu_{1,\infty}(2) < \infty$, then for every $p \in I_\lambda \setminus \{1\}$, there exists $C_p \in (0, \infty)$ such that for every $\beta \in (0, 1]$, we have

$$(2.15) \quad (C_p)^{-1} \beta^4 \leq |\bar{\gamma}_\beta(p)| \leq C_p \beta^4.$$

In the special case $p = 2$, we have $\bar{\gamma}_\beta(2) = \frac{1}{4} \mu_{0,\infty}(2)^2 \beta^4$.

- (ii) If $d \geq 2$ and $\mu_{1,\infty}(1 + \frac{2}{d}) < \infty$, then for every $p \in (0, 1 + \frac{2}{d}) \setminus \{1\}$,

$$(2.16) \quad \lim_{\beta \rightarrow 0} \frac{\log |\log |\bar{\gamma}_\beta(p)||}{|\log \beta|} = 1 + \frac{2}{d}.$$

If $d = 1$ and $d = 2$, these results match the asymptotics for the free energy of the directed polymer in a random environment (see [92] for $d = 1$ and [10] for $d = 2$). In dimensions $d \geq 3$, however, one has in the directed polymer setting that $\bar{\gamma}_\beta(p) = 0$ for sufficiently small β , see [21] and [99, Prop. 4.1]. Note that (2.16) implies that $|\bar{\gamma}_\beta(p)|$ vanishes very fast as β goes to zero. As a consequence, there is not a lot of margin to play with to prove that $|\bar{\gamma}_\beta(p)| > 0$ when β is small, and the proof indeed relies on a fine tuning of parameters. The behavior of $\bar{\gamma}_\beta(p)$ is different if λ displays an heavier tail at infinity. In this case, $\bar{\gamma}_\beta(p)$ has a power-law behavior close to $\beta = 0$. Let

$$(2.17) \quad \nu_p := 1 - \frac{d}{2}(p - 1).$$

Theorem 2.14 (Heavy-tailed noise). *Assume (2.1) and that $\mu < \infty$.*

(i) *If $\mu_{1,\infty}(q) < \infty$ for some $q \in (1, \min(2, 1 + \frac{2}{d}))$, then for every $p \in (0, q] \setminus \{1\}$, there exists $C_{p,q} \in (0, \infty)$ such that for every $\beta \in (0, 1]$,*

$$(2.18) \quad |\bar{\gamma}_\beta(p)| \leq C_{p,q} \beta^{\frac{q}{\nu_q}},$$

which implies

$$(2.19) \quad \liminf_{\beta \rightarrow 0} \frac{\log |\bar{\gamma}_\beta(p)|}{\log \beta} \geq \frac{q}{\nu_q}.$$

(ii) *If $\mu_{1,\infty}(q) = \infty$ for some $q \in (1, \min(2, 1 + \frac{2}{d}))$, then*

$$(2.20) \quad \limsup_{\beta \rightarrow 0} \frac{\log |\bar{\gamma}_\beta(p)|}{\log \beta} \leq \frac{q}{\nu_q}.$$

for every $p \in (0, q) \setminus \{1\}$. In particular, if

$$(2.21) \quad \lim_{z \rightarrow \infty} \frac{\log \lambda([z, \infty))}{\log z} = -\alpha$$

for some $1 < \alpha < \min(2, 1 + \frac{2}{d})$ (which includes the α -stable case), then for all $p \in (0, \alpha) \setminus \{1\}$

$$(2.22) \quad \lim_{\beta \rightarrow 0} \frac{\log |\bar{\gamma}_\beta(p)|}{\log \beta} = \frac{\alpha}{\nu_\alpha}.$$

2.4. Overview of the rest of the paper. The remainder of the paper is organized as follows. In Section 3, we discuss some extensions of our results and some related conjectures concerning geometric localization. Section 4 introduces a few important tools that are needed throughout the paper. In Section 5, we prove all basic properties concerning the moment Lyapunov exponents, that is, Propositions 2.10 and 2.11. Section 6 is devoted to proving our first main technical achievement: Propositions 6.1 and 6.3, which contain upper estimates on moments of order $p > 1$ of $\mathcal{Z}_\beta^\omega(t, x)$. In Section 7, we then use these moment bounds to derive Theorem 2.1 and the first halves of Theorems 2.13 and 2.14. The existence and uniqueness of solutions (*i.e.*, Theorems 2.6 and 2.8) are addressed in Section 8. Finally, Section 9 gathers the proofs that rely on a coarse-graining/change-of-measure approach summarized in Lemma 9.1. In particular, we show the remaining halves of Theorems 2.13 and 2.14 as well as Proposition 2.3. The Appendix contains proofs of some of the results introduced in Section 4.

3. EXTENSIONS AND CONSEQUENCES OF OUR RESULTS

3.1. Continuum directed polymer model. As mentioned in Remark 1.3, the main purpose of introducing $\mathcal{Z}_\beta^{\omega,a}(t, *)$ in [13] is not to study the SHE but rather to define a random probability measure on $C_0([0, T]) := \{\theta : [0, T] \rightarrow \mathbb{R}^d : \theta \text{ continuous and } \theta(0) = 0\}$, called the continuum directed polymer in a Lévy environment. For $T > 0$ the directed polymer in the truncated

environment ξ_ω^a is the probability measure $\mathbf{Q}_{T,\beta}^{\omega,a}$ on $C_0([0, T])$ defined via the integral of bounded Borel measurable functions as follows:

$$\mathbf{Q}_{T,\beta}^{\omega,a}(f) = \frac{e^{-\beta\kappa_a t}}{\mathcal{Z}_\beta^{\omega,a}(T, *)} \left(\mathbf{Q}[f((B_t)_{t \in [0, T]})] + \sum_{k=1}^{\infty} \beta^k \int_{X_T^{(k)}} \rho(t, \mathbf{x}, f) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \right),$$

where $\rho(t, \mathbf{x}, f) := \mathbf{Q}[f((B_t)_{t \in [0, T]}) \mid \forall i \in \llbracket k \rrbracket : B_{t_i} = x_i]$ and $\llbracket k \rrbracket := \{1, \dots, k\}$. Here, \mathbf{Q} denotes the distribution of a standard d -dimensional Brownian motion and, with some light abuse of notation, $\mathbf{Q}[\cdot \mid \forall i \in \llbracket k \rrbracket : B_{t_i} = x_i]$ is the law of the concatenation of Brownian bridges obtained by conditioning a Brownian motion on the null event $\{\forall i \in \llbracket k \rrbracket : B_{t_i} = x_i\}$. The main result in [13] is to prove that $\mathbf{Q}_{T,\beta}^{\omega,a}$ converges in distribution to a limit $\mathbf{Q}_{T,\beta}^\omega$ under the assumptions (1.15) and (1.22). By the findings of the present paper, we can replace (1.22) by (2.1) in the results of [13].

Theorem 3.1. *If the measure λ satisfies (1.15) and (2.1), then there exists a measure $\mathbf{Q}_{T,\beta}^\omega$ on $C_0([0, T])$ such that almost surely, for every bounded continuous function f in $C_0([0, T])$, we have*

$$(3.1) \quad \lim_{a \rightarrow 0} \mathbf{Q}_{T,\beta}^{\omega,a}(f) = \mathbf{Q}_{T,\beta}^\omega(f).$$

Proof. The convergence of $\mathbf{Q}_{T,\beta}^{\omega,a}(f)$ for a fixed f is a consequence of Theorem 2.1. Indeed, $\mathbf{Q}_{T,\beta}^{\omega,a}(f)$ can be written as a quotient $\mathcal{Z}_\beta^{\omega,a}(T, f) / \mathcal{Z}_\beta^{\omega,a}(T, *)$. Theorem 2.1 asserts that the denominator converges to a positive limit and the martingale argument below (1.21) entails the almost-sure convergence of $\mathcal{Z}_\beta^{\omega,a}(T, f)$. The fact that convergence holds simultaneously for all continuous bounded functions is a consequence of tightness. The proof given in [13, Section 4.8] shows that the family of non-normalized measures $\{\mathcal{Z}_\beta^{\omega,a}(T, *) \times \mathbf{Q}_{T,\beta}^{\omega,a} : a \in (0, 1]\}$ is tight, with no other assumption than (1.15). The tightness of $\{\mathbf{Q}_{T,\beta}^{\omega,a} : a \in (0, 1]\}$ then follows from the almost-sure positivity of $\mathcal{Z}_\beta^{\omega,a}(T, *)$; cf. (2.2). \square

3.2. Geometric localization of the solution to the SHE.

The case $u_0 \equiv 1$. The intermittency result given in Theorem 2.12 (especially the part concerning strong intermittency) has direct implications on the distribution of mass for the solution to (1.6) with initial condition $u_0 \equiv 1$. To illustrate this, let us present an argument from [16, §2.4], adapted to the case of non-integer moments—the argument in [16, §2.4] is about mass concentration for the square of the solution.

If $\mu < \infty$ and (2.1) holds, the solution to (1.6) with $u_0 \equiv 1$ is given by

$$U(t, x) := \int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(y; t, x) dy.$$

Let us define the normalized solution $\bar{U}(t, x) := e^{-\mu t} U(t, x)$. By symmetry, we have that $\bar{U}(t, x) \stackrel{(d)}{=} \bar{\mathcal{Z}}_\beta^\omega(t, *)$ for any fixed x . For a fixed value of t , let us assume that the field $\bar{U}(t, x)_{x \in \mathbb{R}^d}$ is ergodic (in the case of a Gaussian noise, this was proved in [32]): we then get

$$(3.2) \quad \lim_{R \rightarrow \infty} \frac{\int_{\|x\| \leq R} \bar{U}(t, x) dx}{\sigma_d R^d} = \mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, *)] = 1,$$

where $\sigma_d = \pi^{d/2} \Gamma(\frac{d}{2} + 1)$ is the volume of the unit ball in \mathbb{R}^d . On the other hand, if $p \in (0, 1)$ and $\bar{\gamma}_\beta(p) < 0$ (which is ensured by Theorem 2.12), then for any $\alpha \in (0, \frac{\bar{\gamma}_\beta(p)}{p-1})$ we have

$$(3.3) \quad \begin{aligned} \lim_{R \rightarrow \infty} \frac{\int_{\|x\| \leq R} \bar{U}(t, x) \mathbf{1}_{\{\bar{U}(t, x) \leq e^{\alpha t}\}} dx}{\sigma_d R^d} &= \mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, *) \mathbf{1}_{\{\bar{\mathcal{Z}}_\beta^\omega(t, *) \leq e^{\alpha t}\}}] \\ &\leq e^{(1-p)\alpha t} \mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, *)^p] \leq e^{[(1-p)\alpha + \bar{\gamma}_\beta(p)]t}, \end{aligned}$$

where the last step relies on super-multiplicativity, see Lemma 5.1 below. Setting $\delta := \alpha(p-1) - \bar{\gamma}_\beta(p) > 0$ and combining (3.2) and (3.3), we obtain

$$(3.4) \quad \lim_{R \rightarrow \infty} \frac{\int_{\|x\| \leq R} \bar{U}(t, x) \mathbf{1}_{\{\bar{U}(t, x) \geq e^{\alpha t}\}} dx}{\int_{\|x\| \leq R} \bar{U}(t, x) dx} \geq 1 - e^{-\delta t}.$$

Note that we also have

$$(3.5) \quad \lim_{R \rightarrow \infty} \frac{\int_{\|x\| \leq R} \mathbf{1}_{\{\bar{U}(t, x) \geq e^{\alpha t}\}} dx}{\sigma_d R^d} = \mathbb{P}(\bar{\mathcal{Z}}_\beta^\omega(t, *) \geq e^{\alpha t}) \leq e^{-\alpha t},$$

thanks to Markov's inequality. Hence, the two identities (3.4) and (3.5) show that in the large t limit, the mass of the solution concentrates on a very small portion of space.

The case $u_0 = \delta_0$. Although this is much more difficult to prove rigorously, we believe that intermittency in Theorem 2.12 also has implications on the localization of the solution $\mathcal{Z}_\beta^{\omega, a}(t, x)$ to the SHE with δ_0 initial condition. Let us consider the probability measure on \mathbb{R}^d given by

$$P_{\beta, t}^\omega(dx) = \frac{\mathcal{Z}_\beta^{\omega, a}(t, x)}{\mathcal{Z}_\beta^{\omega, a}(t, *)} dx.$$

When $\beta = 0$, $P_{\beta, t}^\omega$ is simply $\rho(t, x) dx$. In this case, for large t , its mass is roughly homogeneously spread out on the centered Euclidean ball of radius $t^{1/2}$. By contrast, if $\beta > 0$, the large time behavior of $P_{\beta, t}^\omega(dx)$ is conjectured to be different: the mass of $P_{\beta, t}^\omega$ should typically be concentrated on a set of bounded volume (the volume here should not depend on t).

Conjecture 1. Given $\varepsilon > 0$ and $\beta > 0$, there exist two constants $k = k(\varepsilon, \beta) \in \mathbb{N}$ and $R = R(\varepsilon, \beta) > 0$ such that for all $t \geq 0$,

$$\mathbb{P}\left(\exists z_1, z_2, \dots, z_k \in \mathbb{R}^d : P_{\beta, t}^\omega\left(\bigcup_{i=1}^k B(z_i, R)\right) \geq 1 - \varepsilon\right) \geq 1 - \varepsilon,$$

where $B(z, R) := \{x \in \mathbb{R}^d : \|x - z\| < R\}$.

A weaker version of the above conjecture is that localization holds in Cesàro mean, that is, for $\varepsilon > 0$ and $\beta > 0$ fixed, there exist two constants $k = k(\varepsilon, \beta) \in \mathbb{N}$ and $R = R(\varepsilon, \beta) > 0$ such that

$$(3.6) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \max_{z_1, \dots, z_k \in \mathbb{R}^d} P_{\beta, t}^\omega\left(\bigcup_{i=1}^k B(z_i, R)\right) dt \geq 1 - \varepsilon.$$

In fact, such statements have been rigorously proved for a discrete analogue of the SHE: the partition function of the discrete polymer in \mathbb{Z}^d . In this setup, the link between *very strong disorder* and localization in the Cesàro sense was first explored in [27, 43]. More recently, in [9], it was shown that for directed polymers, very strong disorder implies (3.6); the result was extended to continuous space and discrete time in [7]. The approach in [7, 9] is to prove that the measure $\frac{1}{T} \int_0^T P_{\beta, t}^\omega dt$ converges to a limiting object, using a specific topology (introduced in [90]) on the space of finite measures on \mathbb{R}^d . The corresponding localization result has also been shown for a continuum directed polymer model (and SHE) with spatially convoluted Gaussian white noise in [24], using a Gaussian multiplicative chaos approach.

While many details of the proofs presented in [7, 9] use the discrete nature of the polymer model, the general ideas the proof is based on do not seem to rely on it. For this reason, we believe that while there are technical challenges to be overcome (in particular there are moment conditions in [9] which correspond to a finite second moment for the discrete environment), these proofs could in principle be adapted to the continuum case, to show that (3.6) holds whenever $\bar{\gamma}_\beta(p) < 0$ for $p \in (0, 1)$ and hence, by Theorem 2.12, for any $\beta > 0$ as soon as $\mu < \infty$.

Remark 3.2. Note that the localization result in [9] holds for every β in dimensions 1 and 2 but only for β above a certain threshold in dimensions $d \geq 3$. This is because for the directed polymer, the analogue of our Theorem 2.12 does not hold. Quite the contrary, if $d \geq 3$ and β is small, the end-point distribution of the directed polymer satisfies the central limit theorem [21, 45, 72].

Remark 3.3. The above discussion gives a justification of Conjecture 1 in the case when $\mu < \infty$ (which implies that $\mathcal{Z}_\beta^\omega(t, *)$ has a finite expectation). As a general rule, an environment with heavier tail is expected to only increase geometric localization (cf. [98]).

Remark 3.4. When $d = 1$, known results concerning the distribution of $\mathcal{Z}_\beta^\omega(t, x)$ in the case of Gaussian white noise [5] strongly suggest that, in fact, only one big ball (that is, $k = 1$) is sufficient to capture most of the mass of $P_{\beta, t}^\omega$. For a proof of a result of this kind in the discrete polymer setup (with a special boundary condition), we refer to [42]. In dimension $d \geq 2$, whether one or finitely many balls are needed is a challenging open problem, even at the heuristic level, see [9, Section 9, (1)].

3.3. The case of more general noise. In general, a Lévy white noise ξ (on $\mathbb{R} \times \mathbb{R}^d$) is a random distribution in $\bigcap_{s > (1+d)/2} H_{\text{loc}}^{-s}(\mathbb{R} \times \mathbb{R}^d)$ that can be decomposed as follows:

$$(3.7) \quad \xi = \xi_1 - \xi_2 + \alpha_1 \xi_3 + \alpha_2 \mathcal{L}$$

where ξ_1 and ξ_2 are two independent Lévy noises with positive jumps and respective jump intensities λ_1 and λ_2 , ξ_3 is a standard Gaussian space-time white noise that is independent of ξ_1 and ξ_2 , and $\alpha_1 \geq 0$ and $\alpha_2 \in \mathbb{R}$ are constants. In our exposition so far, we have chosen to focus on the case $\xi_2 \equiv 0$ and $\alpha_1 = \alpha_2 = 0$. The constant α_2 is irrelevant for any discussion of (1.1) as it only adds a multiplicative exponential in t to the solution.

Adding a Gaussian part in dimension 1 does not pose a problem for solving (1.1); see [35]. Given that $\gamma(p)$ in (1.32) exhibits the same behavior in β as $\gamma_\beta(p)$ in (2.15), we believe that both Theorem 2.13 and 2.14 remain valid when Gaussian noise is added.

One can also consider a noise $\xi = \xi_1 - \xi_2$ that includes negative jumps. In this case, one can approximate ξ by $\xi^a = \xi_1^a - \xi_2^a$. Then the criterion (1.15) applied to both jump intensities λ_1 and λ_2 still implies the convergence of the integrals in (1.14), and many of the proofs presented in this paper can be applied verbatim. Note that when negative jumps are present in the noise, the solution is not necessarily positive. In particular, in this case, $\mathcal{Z}_\beta^\omega(t, x)$ does not correspond to the partition function of a polymer model.

Applying the proofs presented below, Theorems 2.1, 2.6 and 2.8 remain valid when signed noise is considered, and the same holds true for the upper bounds of $\mathbb{E}[|\mathcal{Z}_\beta^\omega(t, 0)|^p]$ for $p > 1$. In fact, this translates into half of the inequalities (the upper bounds) proved in Theorems 2.13 and 2.14. On the other hand, our proof of Proposition 2.10 and the upper bounds on $\mathbb{E}[|\mathcal{Z}_\beta^\omega(t, 0)|^p]$ for $p \in (0, 1)$ rely in a crucial manner on the positivity of $\mathcal{Z}_\beta^\omega(t, 0)$, and thus break down when allowing for negative jumps. Note that the upper bounds on fractional moments of the partition function are required for the other half of the inequalities (the lower bounds) in Theorems 2.13 and 2.14, and in particular, for Theorem 2.12. We believe nonetheless that intermittency occurs in the following sense.

Conjecture 2. If $\xi = \xi_1 - \xi_2$ is a signed Lévy white noise as above, with jump intensities that satisfy (2.1) and $\int_0^\infty z^{p_0} (\lambda_1 + \lambda_2)(dz)$ for some $p_0 < \min(2, 1 + \frac{2}{d})$, then

$$\gamma_\beta(p) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[|\mathcal{Z}_\beta^\omega(t, 0)|^p]$$

is well defined for $p \in [0, p_0]$ and $p \mapsto \gamma_\beta(p)/p$ is an increasing function of p on the interval $[1, p_0]$.

4. PRELIMINARY TOOLS

4.1. Size-biased measure. If $\mu < \infty$, note that $\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *)$ is non-negative by (1.20) and satisfies $\mathbb{E}[\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *)] = 1$. We can therefore define a new measure $\tilde{\mathbb{P}}_{\beta,t}^a$ for the environment by

$$(4.1) \quad \tilde{\mathbb{P}}_{\beta,t}^a(\omega \in A) := \mathbb{E}[\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *) \mathbf{1}_A],$$

which is referred to as the *size-biased measure*. If $(\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *))_{a \in (0,1)}$ is uniformly integrable, then using martingale convergence, we can also consider $\tilde{\mathbb{P}}_{\beta,t}^0$ whose density with respect to \mathbb{P} is given by $\bar{\mathcal{Z}}_\beta^\omega(t, *)$.

Lemma 3.7 in [13] gives a useful representation of the size-biased measure $\tilde{\mathbb{P}}_{\beta,t}^a$: the distribution of ω under $\tilde{\mathbb{P}}_{\beta,t}^a$ is obtained by adding to ω an independent Poisson process on the trajectory of a Brownian motion. More precisely, we let ω' be a Poisson point process on $\mathbb{R} \times (0, \infty)$ with intensity $dt \otimes \beta z \lambda(dz)$ and we denote its law by \mathbb{P}'_β . Furthermore, let $(B_t)_{t \in [0, \infty)}$ be a standard d -dimensional Brownian motion starting from 0 that is independent of both ω and ω' and denote its distribution by \mathbf{Q} . For $a \in [0, 1]$, we then define

$$\begin{aligned} \tilde{\omega}(\omega', B) &:= \{(\tau, B_\tau, \zeta) : (\tau, \zeta) \in \omega'\}, \\ \tilde{\omega}_{a,t}(\omega', B) &:= \{(\tau, B_\tau, \zeta) : (\tau, \zeta) \in \omega', \tau \in [0, t], \zeta \geq a\}, \end{aligned}$$

which corresponds to putting the Poisson process ω' on the trajectory of B , with some restrictions on τ and ζ in the second case. When $a = 0$, the dependence in a is sometimes omitted.

Lemma 4.1. *If $\mu < \infty$, then for any $\beta, t > 0$ and $a \in (0, 1]$ and for any measurable bounded function g , we have*

$$(4.2) \quad \tilde{\mathbb{E}}_{\beta,t}^a[g(\omega)] = \hat{\mathbb{E}}_\beta[\hat{g}_{a,t}(\omega, \omega', B)],$$

where $\hat{g}_{a,t}(\omega, \omega', B) := g(\omega \cup \tilde{\omega}_{a,t}(\omega', B))$ and $\hat{\mathbb{P}}_\beta := \mathbb{P} \otimes \mathbb{P}'_\beta \otimes \mathbf{Q}$. Moreover, (4.2) is also satisfied for $a = 0$ if $(\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *))_{a \in (0,1)}$ is uniformly integrable.

If $a > 0$, this was shown in [13, Lemma 3.7]. For $a = 0$, the result follows from the case $a > 0$ since $\mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, *) \mid \mathcal{G}_a] = \bar{\mathcal{Z}}_\beta^{\omega,a}(t, *)$ for any $a \in (0, 1]$, which is why (4.2) with $a = 0$ is true for any bounded \mathcal{G}_a -measurable g . The monotone class theorem yields (4.2) for general g . Note that by Theorem 2.1, (2.1) implies that $(\bar{\mathcal{Z}}_\beta^{\omega,a}(t, *))_{a \in (0,1)}$ is uniformly integrable.

Remark 4.2. In (4.2) as well as in the rest of the paper, we denote the expectation with respect to a measure \mathbb{P} , possibly with decorations, by \mathbb{E} with the same decorations. For example, we write $\tilde{\mathbb{E}}_{\beta,t}^a$, \mathbb{E}'_β and $\hat{\mathbb{E}}_\beta$ for expectation with respect to $\tilde{\mathbb{P}}_{\beta,t}^a$, \mathbb{P}'_β and $\hat{\mathbb{P}}_\beta$, respectively.

4.2. Decoupling inequalities. In the proofs below, we will make use of *moment decoupling inequalities* for multiple Poisson integrals. The results that we review in this section are based on [75], where discrete-time decoupling techniques from [52, 80] are extended to continuous time. We also refer the reader to these references for further literature (and history) on the subject. Let $N \geq 1$ be an integer, let $\mathbb{X} := (0, \infty) \times \mathbb{R}^d \times (0, \infty)$, with generic point $w := (t, x, z)$, and for concreteness let $M = M^\omega$ be the compensated Poisson random measure $M^\omega(dw) = (\delta_\omega - \nu)(dw)$ (recall that $\nu := dt \otimes dx \otimes \lambda(dz)$). Suppose that $f : \mathbb{X}^N \rightarrow \mathbb{R}$ is measurable and *tetrahedral*, that is, $f((t_1, x_1, z_1), \dots, (t_N, x_N, z_N)) = 0$ unless $t_1 < \dots < t_N$. In that case, subject to integrability conditions on f , the multiple integral

$$\int_{\mathbb{X}^N} f(\mathbf{w}) M(dw_1) \cdots M(dw_N) := \int_{\mathbb{X}} \left(\cdots \left(\int_{\mathbb{X}} f(\mathbf{w}) M(dw_1) \right) \cdots \right) M(dw_N)$$

with $\mathbf{w} = (w_1, \dots, w_N)$ can be defined as an iterated Itô integral (recall Remark 1.2). The general definition of Itô integrals used in this paper is detailed in Section 4.3. We let $(\omega_i)_{i=1}^N$ denote i.i.d. copies of our environment ω and we use the notations $\mathbb{P}^{\otimes N}$ and $\mathbb{E}^{\otimes N}$ for the associated probability and expectation, respectively. The following theorem is proved in Appendix A.

Theorem 4.3. *There exists a universal constant $C > 0$ such that for every $1 < p \leq 2$, for any integer $N \geq 1$ and any measurable and tetrahedral function $f : \mathbb{X}^N \rightarrow \mathbb{R}$ satisfying*

$$(4.3) \quad \int_{\mathbb{X}^N} |f(\mathbf{w})|^p \nu(dw_1) \cdots \nu(dw_N) < \infty,$$

the stochastic integrals below are well defined and have finite moments of order p satisfying

$$(4.4) \quad \begin{aligned} & \left(\frac{p-1}{C} \right)^N \mathbb{E}^{\otimes N} \left[\left| \int_{\mathbb{X}^N} f(\mathbf{w}) M^{\omega_1}(dw_1) \cdots M^{\omega_N}(dw_N) \right|^p \right] \\ & \leq \mathbb{E} \left[\left| \int_{\mathbb{X}^N} f(\mathbf{w}) M^\omega(dw_1) \cdots M^\omega(dw_N) \right|^p \right] \\ & \leq \left(\frac{C}{p-1} \right)^N \mathbb{E}^{\otimes N} \left[\left| \int_{\mathbb{X}^N} f(\mathbf{w}) M^{\omega_1}(dw_1) \cdots M^{\omega_N}(dw_N) \right|^p \right]. \end{aligned}$$

4.3. Stochastic integration in the absence of moments. Since $\mathcal{Z}_\beta^\omega(t, x)$ does not have any finite moments in general, the proof of Theorem 2.6 has to make use of a stochastic integration theory that does not assume existence of any moments a priori. For the reader's convenience, let us give a brief review of this L^0 -theory, which was developed by [19] in its most general form (see also [37, Appendix A] for a summary). Let $M = M^\omega$ denote either the Poisson measure δ_ω or its compensated version $\delta_\omega - \nu$.

A *predictable step process* H is of the form $H = \sum_{i=1}^r a_i \mathbf{1}_{A_i}$, where $r \in \mathbb{N}$, $a_i \in \mathbb{R}$, and for each i we have $A_i \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times (0, \infty))$ (where \mathcal{P} is the usual predictable σ -field) and $A_i \subseteq \Omega \times (0, T) \times [-N, N]^d \times (a, \infty)$ for some $T, N, a > 0$. Then, the integral of the predictable step process $H = \sum_{i=1}^r a_i \mathbf{1}_{A_i}$ is canonically defined as

$$\int_{\mathbb{X}} H(\mathbf{w}) M(d\mathbf{w}) := \sum_{i=1}^r a_i M(A_i).$$

Denoting by \mathcal{S} the collection of predictable step processes, we can extend the integral to a larger subset of predictable processes by using the metric induced by

$$(4.5) \quad \|K\|_{M,p} := \sup_{H \in \mathcal{S}, \|H\| \leq \|K\|} \left\| \int_{\mathbb{X}} H(\mathbf{w}) M(d\mathbf{w}) \right\|_{L^p},$$

defined for $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times (0, \infty))$ -measurable processes K , where we have used the notation $\|X\|_{L^p} := \mathbb{E}[|X|^p]^{1/p}$ and $\|X\|_{L^0} := \mathbb{E}[1 \wedge |X|]$. Such a process K is called *L^p -integrable with respect to M* if there exists a sequence $(H_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$ such that $\lim_{n \rightarrow \infty} \|K - H_n\|_{M,p} = 0$. The stochastic integral of K with respect to M is then defined as the L^p -limit of $\int_{\mathbb{X}} H_n(\mathbf{w}) M(d\mathbf{w})$, which exists and does not depend on the choice of $(H_n)_{n \in \mathbb{N}}$. If $p = 0$, we simply say that K is *integrable with respect to M* . According to [19, (2.8)], K is L^p -integrable with respect to M if and only if

$$(4.6) \quad \begin{cases} \|K\|_{M,p} < \infty & \text{if } p > 0, \\ \lim_{u \rightarrow 0} \|uK\|_{M,0} = 0 & \text{if } p = 0. \end{cases}$$

Furthermore, by [37, Lemma A.2],

- if $1 \leq p < \infty$ and $M^\omega = \delta_\omega - \nu$, there are $c_p > 0$ and $C_p > 0$ such that

$$(4.7) \quad c_p \|K\|_{M,p} \leq \mathbb{E} \left[\left(\int_{\mathbb{X}} K(\mathbf{w})^2 \delta_\omega(d\mathbf{w}) \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \leq C_p \|K\|_{M,p};$$

- if $0 < p \leq 1$ and $M^\omega = \delta_\omega$, then

$$\|K\|_{M,p}^p \leq \int_{\mathbb{X}} \mathbb{E}[|K(\mathbf{w})|^p] \nu(d\mathbf{w}).$$

In particular, the L^p -theory, $p \geq 1$, for $\delta_\omega - \nu$ encompasses the L^p - and L^2 -integrals considered in [96] and [100]. Even in the case when $\delta_\omega - \nu$ has a finite first moment (*i.e.*, is a martingale measure), the L^0 -theory is more general since it does not require the integrands have finite moments.

Let us end this section with a simple integrability criterion that we need in the proof of Theorem 2.6. Its proof can be found in the appendix.

Lemma 4.4. *Suppose that for \mathbb{P}_\geq -a.e. ω_\geq , the process $(\omega_<, t, x) \mapsto K(\omega_< \cup \omega_\geq, t, x)$ is $L^p(\mathbb{P}_<)$ -integrable with respect to $\xi_{\omega_<}$ for some $p > 0$. Then K is $L^0(\mathbb{P})$ -integrable with respect to $\xi_{\omega_<}$.*

4.4. A technical lemma. The following technical result will be used repeatedly in the paper. Its proof comes from a straightforward calculation and can be found in [13, Lemma A.3].

Lemma 4.5. *For any $t > 0$, $k \geq 0$ and $\zeta_1, \dots, \zeta_{k+1} > 0$,*

$$\int_{\mathfrak{X}_k(t)} \prod_{i=1}^{k+1} (\Delta t_i)^{\zeta_i-1} dt_i = t^{\sum_{i=1}^{k+1} \zeta_i - 1} \frac{\prod_{i=1}^{k+1} \Gamma(\zeta_i)}{\Gamma(\sum_{i=1}^{k+1} \zeta_i)},$$

where $\Delta t_i = t_i - t_{i-1}$, with the convention $t_0 = 0$ and $t_{k+1} = t$.

5. QUALITATIVE PROPERTIES OF LYAPUNOV EXPONENTS

In this section, we prove the statements made in Section 2.3 concerning existence and basic properties of moment Lyapunov exponents (Propositions 2.10 and 2.11), assuming that Theorem 2.1 as well as the bounds in Propositions 6.1 and 6.3 below are true. These are proved independently in Sections 6 and 7, respectively.

5.1. Existence of Lyapunov exponents. The existence of $\gamma_\beta(p)$ readily follows from sub-/supermultiplicative properties of the moments of the partition function.

Lemma 5.1. *Under assumption (1.15), for every $a \in (0, 1]$, $s, t > 0$ and $p \in [0, 1 + \frac{2}{a})$, we have*

$$(5.1) \quad \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(t+s, *)^p] \leq \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(s, *)^p] \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(t, *)^p] \quad \text{if } p \geq 1,$$

$$(5.2) \quad \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(t+s, *)^p] \geq \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(s, *)^p] \mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(t, *)^p] \quad \text{if } p \leq 1.$$

Proof. The statement being trivial if $\mu_{1,\infty}(p) = \infty$, we can assume $\mu_{1,\infty}(p) < \infty$ for the remainder of the proof. Let $P_{\beta,t}^{\omega,a}$ denote the (random) probability measure on \mathbb{R}^d whose density with respect to the Lebesgue measure is given by $\mathcal{Z}_\beta^{\omega,a}(t, x) / \mathcal{Z}_\beta^{\omega,a}(t, *)$ and X denote an \mathbb{R}^d -valued random variable with distribution $P_{\beta,t}^{\omega,a}$. Recalling the definition (1.25) of $\mathcal{Z}_\beta^{\omega,a}(s, x; t, *)$ and using the Markov property for Brownian motion, we have

$$\begin{aligned} \mathcal{Z}_\beta^{\omega,a}(s+t, *) &= \int_{\mathbb{R}^d} \mathcal{Z}_\beta^{\omega,a}(s, x) \mathcal{Z}_\beta^{\omega,a}(s, x; s+t, *) dx \\ &= \mathcal{Z}_\beta^{\omega,a}(s, *) \int_{\mathbb{R}^d} \mathcal{Z}_\beta^{\omega,a}(s, x; s+t, *) P_{\beta,s}^{\omega,a}(dx) \\ &= \mathcal{Z}_\beta^{\omega,a}(s, *) E_{\beta,s}^{\omega,a} [\mathcal{Z}_\beta^{\omega,a}(s, X; s+t, *)]. \end{aligned}$$

For $p \geq 1$, we get by Jensen's inequality that

$$\mathcal{Z}_\beta^{\omega,a}(s+t,*)^p \leq \mathcal{Z}_\beta^{\omega,a}(s,*)^p E_{\beta,s}^{\omega,a} \left[\mathcal{Z}_\beta^{\omega,a}(s,X;s+t,*)^p \right],$$

and the inequality goes in the other direction if $p \in (0,1)$. Now, recalling (1.5) and (1.19), we have

$$\begin{aligned} & \mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(s,*)^p E_{\beta,s}^{\omega,a} \left[\mathcal{Z}_\beta^{\omega,a}(s,X;s+t,*)^p \right] \mid \mathcal{F}_s \right] \\ &= \mathcal{Z}_\beta^{\omega,a}(s,*)^p E_{\beta,s}^{\omega,a} \left[\mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(s,X;s+t,*)^p \mid \mathcal{F}_s \right] \right] \\ &= \mathcal{Z}_\beta^{\omega,a}(s,*)^p \mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(t,*)^p \right]. \end{aligned}$$

Altogether, we have proven that for $p \geq 1$,

$$\mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(s+t,*)^p \mid \mathcal{F}_s \right] \leq \mathcal{Z}_\beta^{\omega,a}(s,*)^p \mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(t,*)^p \right],$$

which yields (5.1). Repeating the same computation with reversed inequalities when $p \in (0,1)$, we obtain (5.2). \square

Proof of Proposition 2.10. As a consequence of (2.4) in Theorem 2.1, we have $\mathbb{E}[\mathcal{Z}_\beta^{\omega,a}(t,*)^p] \rightarrow \mathbb{E}[\mathcal{Z}_\beta^\omega(t,*)^p]$ as $a \rightarrow 0$. Hence, the previous lemma remains valid for $\mathcal{Z}_\beta^\omega(t,*)$ instead of $\mathcal{Z}_\beta^{\omega,a}(t,*)$. An immediate consequence is that the limit in (2.11) exists by the continuous version of Fekete's subadditive (or superadditive) lemma. It remains to check that $\gamma_\beta(p)$ is finite when (2.1) is satisfied and $\mu_{1,\infty}(p) < \infty$. When $p \in [1, 1 + \frac{2}{d})$, $\gamma_\beta(p) < \infty$ follows from

$$\begin{aligned} (5.3) \quad \gamma_\beta(p) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \left[\mathcal{Z}_\beta^\omega(n,*)^p \right] \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \left[\mathcal{Z}_\beta^\omega(1,*)^p \right]^n \\ &= \log \mathbb{E} \left[\mathcal{Z}_\beta^\omega(1,*)^p \right], \end{aligned}$$

and the finiteness of $\mathbb{E}[\mathcal{Z}_\beta^\omega(1,*)^p]$, see Theorem 2.1.

For $p \in (0,1)$, we prove $\gamma_\beta(p) > -\infty$ and $\gamma_\beta(p) < \infty$ separately. For the first part, let $\gamma_\beta^<(p)$ be the moment Lyapunov exponents that one obtains after replacing ω by $\omega_<$. Thus, by an obvious comparison and convexity (see Proposition 2.11, applied to $\lambda_<(dz) = \mathbf{1}_{(0,1)}(z) \lambda(dz)$), it follows that

$$\gamma_\beta(p) \geq \gamma_\beta^<(p) \geq \frac{\gamma_\beta^<(1) - (1-\theta)\gamma_\beta^<(1+\frac{1}{d})}{\theta} > -\infty,$$

where $\theta \in (0,1)$ is such that $1 = \theta p + (1-\theta)(1+\frac{1}{d})$. Note that $\gamma_\beta^<(1) = 0$ and that $\gamma_\beta^<(1+\frac{1}{d}) < \infty$ thanks to (5.3).

In order to show that $\gamma_\beta(p) < \infty$ when $p \in (0,\infty)$ and $\mu_{1,\infty}(p) < \infty$, we use Jensen's inequality for $\mathbb{E}_<$ so that for every $a \in [0,1)$,

$$\mathbb{E} \left[\mathcal{Z}_\beta^{\omega,a}(t,*)^p \right] \leq \mathbb{E} \left[\mathcal{Z}_\beta^{\omega \geq}(t,*)^p \right].$$

To bound the right-hand side, we use the following simple inequality, which will be used extensively in the remainder of the paper: given $\kappa \in (0,1)$ and any countable collection of non-negative numbers $(a_i)_{i \in I}$, we have

$$(5.4) \quad \left(\sum_{i \in I} a_i \right)^\kappa \leq \sum_{i \in I} a_i^\kappa.$$

We will refer to (5.4), which can be proved by induction (using the fact that $(a+b)^\kappa \leq a^\kappa + b^\kappa$), as the *subadditivity property*; note that it obviously extends to stochastic integrals with respect to discrete measures.

Moreover, note that we have

$$(5.5) \quad \rho(t,x)^p = t^{\nu_p-1} \vartheta(p) \rho\left(\frac{t}{p}, x\right),$$

where $\nu_p = 1 - \frac{d}{2}(p-1)$ was defined in (2.17) and $\vartheta(p) := (2\pi)^{\nu_p-1}p^{-\frac{d}{2}}$. Thus, using (5.4) for the first step, (5.5) and the fact that ρ is a density for the second step, and Lemma 4.5 for the last step, we derive

$$\begin{aligned} \mathbb{E}[\mathcal{Z}_\beta^{\omega \geq}(t, *)^p] &\leq \sum_{k=0}^{\infty} (\beta^p \mu_{1,\infty}(p))^k \int_{X_t^{(k)}} \prod_{i=1}^k \rho(\Delta t_i, \Delta x_i)^p dt_i dx_i \\ &= \sum_{k=0}^{\infty} (\beta^p \vartheta(p) \mu_{1,\infty}(p))^k \int_{\mathfrak{X}_k(t)} \prod_{i=1}^k (\Delta t_i)^{\nu_p-1} dt_i \\ &= \sum_{k=0}^{\infty} \frac{(\beta^p \vartheta(p) \mu_{1,\infty}(p) \Gamma(\nu_p) t^{\nu_p})^k}{\Gamma(\nu_p k + 1)}. \end{aligned}$$

Using Stirling's formula for the gamma function (or the estimate (7.13) below), one can check that the right-hand side grows exponentially in t : as a consequence, we have $\gamma_\beta(p) < \infty$. This proves part (i) of the proposition.

For part (ii), we are going to prove the upper and lower bounds in (2.12) separately. Using Jensen's inequality for the first inequality and the translation invariance (1.19) for the last identity, we have, for $p > 1$,

$$\begin{aligned} \mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^p] &\leq \int_{\mathbb{R}^d} \rho(t, x) \mathbb{E}[(\rho(t, x)^{-1} \mathcal{Z}_\beta^\omega(t, x))^p] dx \\ (5.6) \quad &= \mathbb{E}[(\rho(t, 0)^{-1} \mathcal{Z}_\beta^\omega(t, 0))^p]. \end{aligned}$$

Since $\rho(t, 0)^{-1}$ is of order $t^{d/2}$, after taking logarithm, dividing by t and taking the limit as $t \rightarrow \infty$, we obtain

$$(5.7) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\mathcal{Z}_\beta^\omega(t, 0)^p] \geq \gamma_\beta(p).$$

The same proof yields for $p \in (0, 1)$ that

$$(5.8) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\mathcal{Z}_\beta^\omega(t, 0)^p] \leq \gamma_\beta(p).$$

For the complementary bound, we first consider the case $p > 1$ and define the probability measure $\bar{P}_{\beta,t}^\omega$ with density $\mathcal{Z}_\beta^\omega(t, x) \rho(1, x) / \int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) dx$. Then we have

$$\begin{aligned} \mathcal{Z}_\beta^\omega(t+1, 0) &= \int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) \mathcal{Z}_\beta^\omega(t, x; t+1, 0) \rho(1, x)^{-1} dx \\ &= \left(\int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) dx \right) \bar{E}_{\beta,t}^\omega [\mathcal{Z}_\beta^\omega(t, X; t+1, 0) \rho(1, X)^{-1}]. \end{aligned}$$

Proceeding as in the proof of Lemma 5.1 and using the translation invariance (1.19), we obtain for $p > 1$

$$(5.9) \quad \mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \leq \mathbb{E} \left[\left(\int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) dx \right)^p \right] \mathbb{E}[\mathcal{Z}_\beta^\omega(1, 0)^p \rho(1, 0)^{-p}].$$

In particular, we have

$$\mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \leq (2\pi)^{-\frac{pd}{2}} \mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^p] \mathbb{E}[\mathcal{Z}_\beta^\omega(1, 0)^p \rho(1, 0)^{-p}],$$

which implies, for $p > 1$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \leq \gamma_\beta(p).$$

When $p \in (0, 1)$, repeating the proof of (5.9), but using concavity instead of convexity, we obtain

$$(5.10) \quad \mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \geq \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) dx\right)^p\right] \mathbb{E}[\mathcal{Z}_\beta^\omega(1, 0)^p \rho(1, 0)^{-p}].$$

Now setting $\mathcal{C}_z := z + [0, 1)^d$, we have

$$(5.11) \quad \mathcal{Z}_\beta^\omega(t, *) = \sum_{z \in \mathbb{Z}^d} \int_{\mathcal{C}_z} \mathcal{Z}_\beta^\omega(t, x) dx \leq \sum_{z \in \mathbb{Z}^d} \left(\max_{x \in \mathcal{C}_z} \rho(t, x)\right) Y_z(t)$$

with $Y_z(t) := \int_{\mathcal{C}_z} \mathcal{Z}_\beta^\omega(t, x) \rho(t, x)^{-1} dx$.

Note that the variables Y_z are identically distributed by (1.19), which together with (5.4) implies

$$(5.12) \quad \mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^p] \leq \left(\sum_{z \in \mathbb{Z}^d} \max_{x \in \mathcal{C}_z} \rho(t, x)^p\right) \mathbb{E}[Y_0(t)^p] \leq C t^{\frac{d}{2}(1-p)} \mathbb{E}[Y_0(t)^p]$$

for $t > 1$. Moreover, also for $t > 1$,

$$\int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(t, x) \rho(1, x) dx \geq \left(\min_{x \in \mathcal{C}_0} \rho(1, x) \rho(t, x)\right) Y_0(t) \geq (2\pi)^{-d} t^{-\frac{d}{2}} Y_0(t).$$

Therefore, together with (5.10) and (5.12), we obtain that

$$\mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \geq C' t^{-\frac{d}{2}} \mathbb{E}[\mathcal{Z}_\beta^\omega(t, *)^p].$$

This allows us to conclude that, for $p \in (0, 1)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\mathcal{Z}_\beta^\omega(t+1, 0)^p] \geq \gamma_\beta(p). \quad \square$$

5.2. Monotonicity and convexity properties. To show the monotonicity in β of the Lyapunov exponents, we prove a more general result linking monotonicity and convexity. Its proof is inspired by an analogous result proved in the discrete setup [45, Lemma 3.3]. Recall the definition of the renormalized partition function (1.27).

Lemma 5.2. *Assume that $\mu_{1,\infty}(q) < \infty$ for some $q \in [1, 1 + \frac{2}{d})$ and let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a convex function that satisfies*

$$\sup_{u \geq 1} |\varphi(u)| u^{-q} < \infty.$$

Then $\beta \mapsto \mathbb{E}[\varphi(\bar{\mathcal{Z}}_\beta^{\omega,a}(t, 0))]$ is a non-decreasing function, for any $a > 0$.

Proof of Proposition 2.11. As a consequence of Lemma 5.2, if $\mu_{1,\infty}(p) < \infty$ for some $p \in [1, 1 + \frac{2}{d})$, then $\beta \mapsto \mathbb{E}[\bar{\mathcal{Z}}_\beta^{\omega,a}(t, 0)^p]$ is non-decreasing if $p > 1$ and non-increasing if $p \in (0, 1)$ (apply Lemma 5.2 to the convex function $x \mapsto -x^p$). Letting $a \rightarrow 0$, we obtain the same results for $\mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, 0)^p]$ because of (2.4). Since $\bar{\gamma}_\beta(p) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, 0)^p]$, this completes the proof of (ii) in Proposition 2.11. Both (i) and (iii) are standard: The convexity of $p \mapsto \gamma_\beta(p)$ follows from that of $p \mapsto \log \mathbb{E}[X^p]$, valid for an arbitrary non-negative random variable X by Hölder's inequality; (iii) is a direct consequence of convexity since for $1 < p < p'$ we have $\bar{\gamma}_\beta(p) \leq \frac{p-1}{p'-1} \bar{\gamma}_\beta(p')$, recalling that $\bar{\gamma}_\beta(1) = 0$ by definition. \square

Proof of Lemma 5.2. Since φ is convex, there exists a and b such that $\varphi(u) + au + b \geq 0$ for every $u \geq 0$. Hence, replacing $\varphi(u)$ by $\varphi(u) + au + b$, we can assume that φ is non-negative, since $\mathbb{E}[\bar{\mathcal{Z}}_\beta^{\omega,a}(t, 0)] = \rho(t, 0)$ does not depend on β . We may also assume without loss of generality that φ is differentiable on $[0, \infty)$ and that φ' is bounded. Indeed, if this is not the case, we can find a sequence $\varphi_n \uparrow \varphi$ that has these properties and then use monotone convergence.

Recalling (1.12) and setting $\bar{\kappa}_a = \kappa_a + \mu$, we have, similarly to (1.20),

$$\bar{Z}_\beta^{\omega,a}(t, 0) = e^{-\beta \bar{\kappa}_a t} \sum_{k=0}^{\infty} \beta^k \int_{X_t^{(k)}} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i).$$

Taking the derivative with respect to β , we obtain, after an index shift for the first term,

$$(5.13) \quad \partial_\beta \bar{Z}_\beta^{\omega,a}(t, 0) = e^{-\beta \bar{\kappa}_a t} \sum_{k=0}^{\infty} \beta^k \left((k+1) \int_{X_t^{(k+1)}} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^{k+1} \xi_\omega^{a,+}(dt_i, dx_i) \right. \\ \left. - \bar{\kappa}_a t \int_{X_t^{(k)}} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \right).$$

The first term in the sum can be viewed as the sum of $k+1$ integrals where the variables of integration are $(t_1, \dots, t_i, s, t_{i+1}, \dots, t_k) \in \mathfrak{X}_{k+1}(t)$ and $(x_1, \dots, x_i, y, x_{i+1}, \dots, x_k)$ for $i = 0, \dots, k$. Recombining the terms, we arrive at the identity (recall the convention $t_0 := 0$ and $t_{k+1} := t$)

$$(k+1) \int_{X_t^{(k+1)}} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \\ = \int_{X_t^{(k)}} \left(\int_{X_t} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \rho(s, y | \mathbf{t}, \mathbf{x}) \xi_\omega^{a,+}(ds, dy) \right) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i),$$

where

$$\rho(s, y | \mathbf{t}, \mathbf{x}) := \frac{\rho(s - t_{i-1}, y - x_{i-1}) \rho(t_i - s, x_i - y)}{\rho(t_i - t_{i-1}, x_i - x_{i-1})} \quad \text{if } s \in (t_{i-1}, t_i)$$

and $\rho(t_i, y | \mathbf{t}, \mathbf{x}) := 0$ for all $i = 1, \dots, k$. As $\int_{\mathbb{R}^d} \rho(s, y | \mathbf{t}, \mathbf{x}) dy = 1$, the second term in (5.13) corresponds to a centering of $\xi_\omega^{a,+}(ds, dy)$, so

$$(5.14) \quad \partial_\beta \bar{Z}_\beta^{\omega,a}(t, 0) = e^{-\beta \bar{\kappa}_a t} \sum_{k=0}^{\infty} \beta^k \int_{X_t^{(k)}} \left(\int_{X_t} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \rho(s, y | \mathbf{t}, \mathbf{x}) \bar{\xi}_\omega^a(ds, dy) \right) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i).$$

Note that we have

$$|\partial_\beta \bar{Z}_\beta^{\omega,a}(t, 0)| \leq e^{-\beta \bar{\kappa}_a t} \sum_{k=0}^{\infty} \beta^k \int_{X_t^{(k)}} \left(\int_{X_t} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \rho(s, y | \mathbf{t}, \mathbf{x}) |\bar{\xi}_\omega^a|(ds, dy) \right) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i),$$

where $|\bar{\xi}_\omega^a| = \bar{\xi}_\omega^a + 2\bar{\kappa}_a \mathcal{L}$. Since we have reduced to the case where φ' is bounded, the expression above implies (cf. [13, Prop. 2.5]) for any $\beta_0 > 0$

$$(5.15) \quad \mathbb{E} \left[\sup_{\beta \in [0, \beta_0]} \left| \partial_\beta \bar{Z}_\beta^{\omega,a}(t, 0) \varphi'(\bar{Z}_\beta^{\omega,a}(t, 0)) \right| \right] < \infty.$$

This allows to interchange derivative and expectation, so by (5.14),

$$(5.16) \quad \partial_\beta \mathbb{E}[\varphi(\bar{Z}_\beta^{\omega,a}(t, 0))] = \mathbb{E}[\partial_\beta \bar{Z}_\beta^{\omega,a}(t, 0) \varphi'(\bar{Z}_\beta^{\omega,a}(t, 0))] \\ = e^{-\beta \bar{\kappa}_a t} \sum_{k=0}^{\infty} \beta^k \mathbb{E} \left[\int_{X_t^{(k)}} U_k^\omega(\mathbf{t}, \mathbf{x}) \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \right],$$

where we have set

$$U_k^\omega(\mathbf{t}, \mathbf{x}) := \left(\int_{[0, t) \setminus \{t_i\}_{i=1}^k \times \mathbb{R}^d} \rho(s, y | \mathbf{t}, \mathbf{x}) \bar{\xi}_\omega^a(ds, dy) \right) \varphi'(\bar{Z}_\beta^{\omega,a}(t, 0)).$$

Given $(\mathbf{t}, \mathbf{x}, \mathbf{z}) \in \mathfrak{X}_k(t) \times (\mathbb{R}^d)^k \times (0, \infty)^k$, we let $\omega \cup (\mathbf{t}, \mathbf{x}, \mathbf{z})$ be the point process obtained by adding the k points $(t_i, x_i, z_i)_{1 \leq i \leq k}$ to ω . By Mecke's multivariate equation (see [84, Thm. 4.4]), we have

$$\begin{aligned}
 (5.17) \quad & \mathbb{E} \left[\int_{X_t^{(k)}} U_k^\omega(\mathbf{t}, \mathbf{x}) \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_\omega^{a,+}(dt_i, dx_i) \right] \\
 &= \mathbb{E} \left[\int_{\mathbb{X}_t^{(k)}} U_k^\omega(\mathbf{t}, \mathbf{x}) \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k z_i \mathbf{1}_{[a,\infty)}(z_i) \delta_\omega(dw_i) \right] \\
 &= \int_{\mathbb{X}_t^{(k)}} \mathbb{E} \left[U_k^{\omega \cup (\mathbf{t}, \mathbf{x}, \mathbf{z})}(\mathbf{t}, \mathbf{x}) \right] \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k z_i \mathbf{1}_{[a,\infty)}(z_i) dt_i dx_i \lambda(dz_i).
 \end{aligned}$$

If $k, \mathbf{t}, \mathbf{x}$ and \mathbf{z} are fixed, then $\omega \mapsto \varphi'(\bar{\mathcal{Z}}_\beta^{\omega \cup (\mathbf{t}, \mathbf{x}, \mathbf{z}), a}(t, 0))$ and $\omega \mapsto \int_{[(0,t) \setminus \{t_i\}_{i=1}^k] \times \mathbb{R}^d} \rho(s, y | \mathbf{t}, \mathbf{x}) \bar{\xi}_\omega^a(ds, dy)$ are non-decreasing for the inclusion order. Thus we can apply the FKG inequality for Poisson point processes (see [73, Lemma 2.1]) and obtain that

$$\mathbb{E} \left[U_k^{\omega \cup (\mathbf{t}, \mathbf{x}, \mathbf{z})}(\mathbf{t}, \mathbf{x}) \right] \geq \mathbb{E} \left[\int_{[(0,t) \setminus \{t_i\}_{i=1}^k] \times \mathbb{R}^d} \rho(s, y | \mathbf{t}, \mathbf{x}) \bar{\xi}_\omega^a(ds, dy) \right] \mathbb{E} \left[\varphi'(\bar{\mathcal{Z}}_\beta^{\omega \cup (\mathbf{t}, \mathbf{x}, \mathbf{z}), a}(t, 0)) \right] = 0.$$

Together with (5.16) and (5.17), we obtain that $\partial_\beta \mathbb{E}[\varphi(\bar{\mathcal{Z}}_\beta^{\omega, a}(t, 0))] \geq 0$. \square

6. MOMENTS OF ORDER $p > 1$

The goal of this section is to prove Propositions 6.1 and 6.3 below, which form the core of all moment upper bounds for $p > 1$ in this paper.

6.1. The statements. Let $\bar{\mathcal{Z}}_\beta^{\omega, 0}(t, x) := \bar{\mathcal{Z}}_\beta^\omega(t, x)$.

6.1.1. The case of dimension $d = 1$. We start with $d = 1$, where the statement is easier to state (and easier to prove). Recall that $\nu_p := 1 - \frac{d}{2}(p - 1)$.

Proposition 6.1. *Assume that $d = 1$ and that (2.1) holds.*

(i) *If $\mu_{1,\infty}(2) < \infty$, then for any $a \in [0, 1]$,*

$$\begin{aligned}
 (6.1) \quad & \mathbb{E} \left[\rho(t, x)^{-2} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^2 \right] = \sum_{k=0}^{\infty} \left(\frac{\beta^2 \mu_{a,\infty}(2) \sqrt{t}}{2} \right)^k \frac{\sqrt{\pi}}{\Gamma((k+1)/2)} \\
 &= 1 + \beta^2 \mu_{a,\infty}(2) \sqrt{\pi t} \exp \left(\frac{1}{4} \beta^4 \mu_{a,\infty}(2)^2 t \right) \Phi \left(\beta^2 \mu_{a,\infty}(2) \sqrt{\frac{t}{2}} \right),
 \end{aligned}$$

where Φ is the standard normal distribution function.

(ii) *There exists a constant $C \in (0, \infty)$ such that if $\mu_{1,\infty}(p) < \infty$ for some $p \in (1, 2)$, then for any value of $\eta \in (0, 1]$, $a \in [0, 1]$ and $t, \beta > 0$,*

$$(6.2) \quad \mathbb{E} \left[\rho(t, x)^{-p} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^p \right]^{\frac{1}{p}} \leq \sqrt{2} \Gamma(\nu_p)^{\frac{1}{p}} \sum_{k_1, k_2=0}^{\infty} \left(\frac{C\beta}{p-1} \right)^{k_1+k_2} \left(\mu_{0,\eta}(2) t^{\frac{1}{2}} \right)^{\frac{k_1}{2}} \frac{(\mu_{\eta,\infty}(p) t^{\nu_p})^{\frac{k_2}{p}}}{\Gamma(\nu_p(k_2+1))^{\frac{1}{p}}}.$$

(iii) *There exists a constant $C \in (0, \infty)$ such that if $\mu_{1,\infty}(p) < \infty$ for some $p \in (2, 3)$, then for any value of $a \in [0, 1]$ and $t, \beta > 0$,*

$$\begin{aligned}
 (6.3) \quad & \mathbb{E} \left[\rho(t, x)^{-p} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^p \right]^{\frac{1}{p}} \leq \pi^{\frac{1}{4}} \Gamma(\nu_p)^{\frac{1}{p}} \sum_{k_2=0}^{\infty} \frac{(C\beta \mu_{0,\infty}(2)^{\frac{1}{2}} \Gamma(\nu_p)^{\frac{1}{p}} t^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}}} \\
 & \times \left(\sum_{\ell=0}^{\infty} \frac{(C'\beta \mu_{0,\infty}(p)^{\frac{1}{p}} t^{\frac{\nu_p}{p}} \Gamma(\nu_p)^{\frac{1}{p}})^{\ell}}{\Gamma(\nu_p(\ell+1))^{\frac{1}{p}}} \right)^{k_2+1}.
 \end{aligned}$$

Remark 6.2. The formula (6.1) is rather straightforward to prove. Starting from (1.28) and using that $\bar{\xi}_\omega^a$ is centered, the second moment is given by

$$\mathbb{E}\left[\rho(t, x)^{-2} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^2\right] = 1 + \sum_{k=1}^{\infty} (\beta^2 \mu_{a, \infty}(2))^k \int_{X_t^{(k)}} \rho(t, x)^{-2} \prod_{i=1}^{k+1} \rho(\Delta t_i, \Delta x_i)^2 dt_i dx_i,$$

with the notation of (1.11). Integrating over x_1, \dots, x_k , we get

$$\mathbb{E}\left[\rho(t, x)^{-2} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^2\right] = 1 + \sum_{k=1}^{\infty} (\beta^2 \mu_{a, \infty}(2))^k (2\sqrt{\pi t}) \int_{\mathfrak{X}_k(t)} \prod_{i=1}^{k+1} \frac{1}{2\sqrt{\pi \Delta t_i}} dt_i,$$

which gives the first equality in (6.1) by Lemma 4.5. The second equality follows from the formula $\sum_{k=0}^{\infty} x^k / \Gamma((k+1)/2) = \pi^{-1/2} + 2xe^{x^2} \Phi(\sqrt{2}x)$. Note that this formula coincides with the one we obtain if $\bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)$ in (6.1) is replaced by the solution to the SHE (1.1) with a space-time *Gaussian* noise with variance $\mu_{a, \infty}(2)$ (and $u_0 = \delta_0$); cf. [31, Eq. (2.31)].

6.1.2. The case of dimension $d \geq 2$. For the statement with $d \geq 2$, we need to introduce a few auxiliary quantities. With the usual convention $t_0 := 0$ and $t_{k+1} := t$ and $\Delta t_i := t_i - t_{i-1}$, we define $\Lambda(0, t, p) := 1$ and

$$(6.4) \quad \Lambda(k, t, p) := t^{1-\nu_p} \int_{\mathfrak{X}_k(t)} (\Delta t_{k+1})^{\nu_p-1} \prod_{i=1}^k G_p(\Delta t_i) dt_i$$

for $k \geq 1$, where

$$(6.5) \quad G_p(s) := \begin{cases} s^{\frac{1}{3}\nu_p-1} & \text{if } t \leq 1, \\ s^{\nu_p-1} & \text{if } t \geq 1. \end{cases}$$

Also, we let

$$(6.6) \quad \begin{aligned} \zeta_1(\eta, p, t) &:= \left(8 \mu_{0, \eta}^{\log}(1 + \tfrac{2}{d})(1 + \log_+ t)\right)^{\frac{p}{1+2/d}}, \\ \zeta_2(\eta, p) &:= \mu_{0, \eta}^{\log}(1 + \tfrac{2}{d}) + \mu_{\eta, \infty}(p), \end{aligned}$$

where we have set $\log_+ t := \log(t \vee 1)$ and

$$\mu_{0, \eta}^{\log}(1 + \tfrac{2}{d}) := \int_{(0, \eta)} z^{1+\frac{2}{d}} (3|\log z| + 1) \lambda(dz).$$

Note that $\zeta_1(\eta, p, t)$ and $\zeta_2(\eta, p)$ are finite if (2.1) holds and $\mu_{1, \infty}(p) < \infty$.

Proposition 6.3. *Assume that $d \geq 2$ and that (2.1) holds. There exists a constant $C \in (0, \infty)$ that only depends on d such that if $\mu_{1, \infty}(p) < \infty$ for $p \in (1, 1 + \frac{2}{d})$, then for any value of $\eta \in (0, 1]$, $a \in [0, 1]$ and $t, \beta > 0$,*

$$(6.7) \quad \mathbb{E}\left[\rho(t, x)^{-p} \bar{\mathcal{Z}}_\beta^{\omega, a}(t, x)^p\right]^{\frac{1}{p}} \leq \sum_{k_1, k_2=0}^{\infty} \left(\frac{C\beta}{p-1}\right)^{k_1+k_2} \zeta_1(\eta, p, t)^{\frac{k_1}{p}} \zeta_2(\eta, p)^{\frac{k_2}{p}} \Lambda(k_2, t, p)^{\frac{1}{p}}.$$

Remark 6.4. In our proofs of Propositions 6.1 and 6.3, we keep track of the dependence in p . The reason for this is that in the proof of Theorem 2.13—more precisely in the proof of (2.16)—we need to apply (6.7) for p that depends of β . The only important point we need to make sure is that our estimates remain uniform for p in an interval around $1 + \frac{2}{d}$.

6.1.3. *Finiteness of the moments.* An almost immediate consequence of the proposition is the following uniform moment bound.

Corollary 6.5. *If (2.1) holds and $\mu_{1,\infty}(p) < \infty$ for some $p \in (1, 1 + \frac{2}{d})$, then for any β and T there exists $C(\beta, p, T)$ such that*

$$\sup_{a \in [0,1]} \sup_{(t,x) \in (0,T] \times \mathbb{R}^d} \mathbb{E} \left[\left(\rho(t,x)^{-1} \bar{Z}_\beta^{\omega,a}(t,x) \right)^p \right]^{\frac{1}{p}} < C(\beta, p, T).$$

Proof. First let us note that the bounds in Propositions 6.1 and 6.3 already are uniform in a and x , so we only need to check uniformity in t .

Let us start with the case $d = 1$. The statement for $p = 2$ is obvious from (6.1). If $p \in (1, 2)$, by reorganizing (6.2), we obtain that for every $t \in [0, T]$,

$$\mathbb{E} \left[\rho(t,x)^{-p} \bar{Z}_\beta^{\omega,a}(t,x)^p \right]^{\frac{1}{p}} \leq C' \sum_{k_1=0}^{\infty} \left(C_p \beta \mu_{0,\eta}(2)^{\frac{1}{2}} T^{\frac{1}{4}} \right)^{k_1} \sum_{k_2=0}^{\infty} \frac{\left(C_p \beta \mu_{\eta,\infty}(p)^{\frac{1}{p}} T^{\frac{\nu_p}{p}} \right)^{k_2} \Gamma(\nu_p)^{\frac{1}{p}}}{\Gamma(\nu_p(k_2+1))^{\frac{1}{p}}},$$

where $C_p = C/(p-1)$. The first sum is finite if one chooses η such that $C\beta\mu_{0,\eta}(2)^{1/2}T^{1/4} < \frac{1}{2}$ and the second one is always finite since $\Gamma(\nu_p(k_2+1))$ grows super-exponentially. If $p \in (2, 3)$, we have from (6.3) that for every $t \in [0, T]$,

$$\mathbb{E} \left[\rho(t,x)^{-p} \bar{Z}_\beta^{\omega,a}(t,x)^p \right]^{\frac{1}{p}} \leq C_p \sum_{k_2=0}^{\infty} \frac{(C_p \beta T^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}}} \left(\sum_{\ell=0}^{\infty} \frac{(C'_p \beta T^{\frac{\nu_p}{p}})^{\ell}}{\Gamma(\nu_p(\ell+1))} \right)^{k_2+1}.$$

Since $\Gamma(\nu_p(\ell+1))$ and $\Gamma(\frac{k_2+1}{2})$ grow super-exponentially, the sums are finite.

When $d \geq 2$, since ζ_1 is monotone in t , we have, in the same manner,

$$(6.8) \quad \mathbb{E} \left[\rho(t,x)^{-p} \bar{Z}_\beta^{\omega,a}(t,x)^p \right]^{\frac{1}{p}} \leq \left(\sum_{k_1=0}^{\infty} \left(C_p \beta \zeta_1(\eta, p, T)^{\frac{1}{p}} \right)^{k_1} \right) \left(\sum_{k_2=0}^{\infty} \left(C_p \beta \zeta_2(\eta, p)^{\frac{1}{p}} \right)^{k_2} \Lambda(k_2, t, p)^{\frac{1}{p}} \right)$$

for $t \leq T$. The first sum is finite provided that η is chosen sufficiently small. Considering the second term, we have

$$\begin{aligned} \Lambda(k_2, t, p) &\leq t^{1-\nu_p} (t \vee 1)^{\frac{2}{3}\nu_p k_2} \int_{\mathfrak{X}_{k_2}(t)} (\Delta t_{k_2+1})^{\nu_p-1} \prod_{i=1}^{k_2} (\Delta t_i)^{\frac{1}{3}\nu_p-1} dt_i \\ &= (t \vee 1)^{\frac{2}{3}\nu_p k_2} t^{\frac{1}{3}\nu_p k_2} \frac{\Gamma(\frac{1}{3}\nu_p)^{k_2} \Gamma(\nu_p)}{\Gamma(\frac{1}{3}\nu_p(k_2+3))}, \end{aligned}$$

where we used Lemma 4.5 for the last identity. As a result, assuming that $T \geq 1$, we obtain that for every $t \in [0, T]$,

$$\left(C_p \beta \zeta_2(\eta, p)^{\frac{1}{p}} \right)^{k_2} \Lambda(k_2, t, p)^{\frac{1}{p}} \leq \frac{(C_p \beta \zeta_2(\eta, p)^{\frac{1}{p}} \Gamma(\frac{1}{3}\nu_p)^{\frac{1}{p}} T^{\frac{\nu_p}{p}})^{k_2} \Gamma(\nu_p)^{\frac{1}{p}}}{\Gamma(\frac{1}{3}\nu_p(k_2+3))^{\frac{1}{p}}},$$

and since $\Gamma(\nu_p[(k_2/3)+1])$ grows super-exponentially, the sum over k_2 in (6.8) is finite. \square

6.2. Bounding moments in the chaos expansion: the first term. From now on, we focus on the case $x = 0$, which yields no loss of generality by (1.19). Also, in both Propositions 6.1 and 6.3 above, the case $a = 0$ can be deduced from the case $a > 0$ using Fatou's lemma: in the following, we can always assume that $a > 0$. Starting from the chaos decomposition (1.28), we can use Minkowski's inequality to get

$$(6.9) \quad \mathbb{E} \left[\rho(t, 0)^{-p} \bar{Z}_\beta^{\omega,a}(t, 0)^p \right]^{\frac{1}{p}} \leq \sum_{k=0}^{\infty} \beta^k \mathbb{E} [|W_{a,k}(t)|^p]^{\frac{1}{p}},$$

where $W_{a,0}(t) := 1$ and

$$(6.10) \quad W_{a,k}(t) := \int_{X_t^{(k)}} \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})}{\rho(t, 0)} \prod_{i=1}^k \bar{\xi}_\omega^a(dt_i, dx_i) \quad \text{for } k \geq 1.$$

The estimates for $\mathbb{E}[|W_{a,k}(t)|^p]$ are intricate, so let us spend some time on the case $k = 1$ to illustrate the intuition behind our proof. For simplicity, let us consider the expansion of the free-end partition function. Because the integrals in (6.9) are martingales in t for the filtration (1.5), we can apply the Burkholder–Davis–Gundy (BDG) inequality and obtain

$$(6.11) \quad \mathbb{E} \left[\left(\int_{(0,t) \times \mathbb{R}^d} \rho(s, x) \bar{\xi}_\omega^a(ds, dx) \right)^p \right] \leq C_p \mathbb{E} \left[\left(\int_{(0,t) \times \mathbb{R}^d \times [a, \infty)} (\rho(s, x)z)^2 \delta_\omega(dw) \right)^{\frac{p}{2}} \right],$$

recalling that $w = (t, x, z)$. Since we are tracking the dependence in p , it is worth noting that it is possible to take $C_p = (4p)^p \leq 64$ if $p \leq 2$; see [53, Chapter VII, Theorem 92]. In order to bound the right-hand side uniformly in a , we replace $[a, \infty)$ by $(0, \infty)$. Let us further restrict ourselves to the case $d \geq 3$ for simplicity (note that in particular $p < 2$). By Jensen's inequality and the subadditivity property (5.4), we have, for $\theta \in [p, 2]$,

$$(6.12) \quad \begin{aligned} \mathbb{E} \left[\left(\int_{\mathbb{X}_t} \rho(s, x)^2 z^2 \delta_\omega(dw) \right)^{\frac{p}{2}} \right] &\leq \mathbb{E} \left[\left(\int_{\mathbb{X}_t} \rho(s, x)^2 z^2 \delta_\omega(dw) \right)^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} \\ &\leq \left(\int_{\mathbb{X}_t} \rho(s, x)^\theta z^\theta ds dx \lambda(dz) \right)^{\frac{p}{\theta}} = \vartheta(\theta)^{\frac{p}{\theta}} \left(\int_{\mathcal{X}_t} s^{\nu_\theta-1} z^\theta ds \lambda(dz) \right)^{\frac{p}{\theta}}, \end{aligned}$$

recalling also (5.5) and $\nu_\theta = 1 - \frac{d}{2}(\theta - 1)$ for the last line. On the right-hand side, we see that when θ increases, the integrability in z around 0 improves but the one in s worsens, forcing us to choose $\nu_\theta > 0$ (i.e., $\theta < 1 + \frac{2}{d}$) to obtain a finite integral. As a consequence, we need to assume $\mu_{0,\infty}(\theta) < \infty$ for some $\theta \in [p, 1 + \frac{2}{d})$. If this holds, then, in fact, the same estimate can be applied iteratively in order to obtain bounds for any of the multiple integrals in (6.9). Let us remark that this is essentially the assumption (and the method) used in [96] to obtain existence, uniqueness and moments for the solution to (2.6).

Clearly, $\mu_{0,\infty}(\theta) < \infty$ does not hold, for any $\theta > 0$, if ξ_ω is an α -stable noise with $\alpha \in (1, 2)$; recall that we assume $\mu_{1,\infty}(p) < \infty$ for some $p > 1$. In that case, at least when $k = 1$, it is easy to do better than (6.12). The key point is to first separate $z \geq 1$ and $z < 1$ and then, for $z < 1$, further whether $z \leq s^{d/2}$ or $z > s^{d/2}$. For $z \geq 1$, we can simply apply (6.12) with $\theta = p$, that is,

$$\mathbb{E} \left[\left(\int_{\mathbb{X}_t} \rho(s, x)^2 z^2 \mathbf{1}_{\{z \geq 1\}} \delta_\omega(dw) \right)^{\frac{p}{2}} \right] \leq \vartheta(p) \mu_{1,\infty}(p) \nu_p^{-1} t^{\nu_p}.$$

For $z \leq 1$, we first consider the contribution coming from $z \leq s^{d/2}$. By Jensen's inequality, we can take the exponent $\frac{p}{2}$ outside the expectation on the right-hand side of (6.11), which leads to the bound

$$\mathbb{E} \left[\int_{\mathbb{X}_t} \mathbf{1}_{\{z \leq s^{d/2} \wedge 1\}} \rho(s, x)^2 z^2 \delta_\omega(dw) \right]^{\frac{p}{2}} = \left(\int_{\mathbb{X}_t} \mathbf{1}_{\{z \leq s^{d/2} \wedge 1\}} \rho(s, x)^2 z^2 ds dx \lambda(dz) \right)^{\frac{p}{2}}.$$

Since $d \geq 3$, we have, integrating first with respect to x (recall (5.5)), then with respect to s and finally with respect to z ,

$$\int_{\mathbb{X}_t} \mathbf{1}_{\{z \leq s^{d/2} \wedge 1\}} \rho(s, x)^2 z^2 ds dx \lambda(dz) = \int_{\mathcal{X}_t} \frac{z^2 \mathbf{1}_{\{z \leq s^{d/2} \wedge 1\}}}{(4\pi s)^{\frac{d}{2}}} ds \lambda(dz) \leq \frac{1}{(\frac{d}{2} - 1)(4\pi)^{\frac{d}{2}}} \mu_{0,1}(1 + \frac{2}{d}).$$

For the contribution to the integral coming from $z > s^{d/2}$, applying (6.12) with $\theta = p$, we get that it is smaller than

$$\vartheta(p) \int_{\mathcal{X}_t} \mathbf{1}_{\{z > s^{d/2}, z < 1\}} s^{\nu_p - 1} z^p ds \lambda(dz) = \frac{\vartheta(p)}{\nu_p} \mu_{0,1} \left(1 + \frac{2}{d}\right),$$

where we have first integrated with respect to s using $\nu_p = 1 - \frac{d}{2}(p-1) > 0$ for $p < 1 + \frac{2}{d}$, and then with respect to z . Altogether, we have shown:

Lemma 6.6. *In dimension $d \geq 3$, there exists a constant C (which may depend on d but not on p) such that for all $p \in (1, 1 + \frac{2}{d})$,*

$$\mathbb{E} \left[\left(\int_{X_t} \rho(s, x) \bar{\xi}_{\omega}^a(ds, dx) \right)^p \right] \leq \frac{C}{\nu_p} \left[\mu_{0,1} \left(1 + \frac{2}{d}\right)^{\frac{p}{2}} + \mu_{0,1} \left(1 + \frac{2}{d}\right) + \mu_{1,\infty}(p) t^{\nu_p} \right].$$

In other words, the single integral has a finite p th moment for some $p \in (1, 1 + \frac{2}{d})$ if the intensity measure satisfies $\mu_{0,1}(1 + \frac{2}{d}) + \mu_{1,\infty}(p) < \infty$. In fact, this condition is necessary and sufficient by [94, Thm. 3.3], so Lemma 6.6 is optimal for single integrals.

6.3. Decoupling and partitioning: Key tools in proving Propositions 6.1 and 6.3.

6.3.1. Decoupling. When $k \geq 2$, there is no direct analogue of (6.11) since we cannot apply the BDG inequality for the k -fold iterated integral. The first step of our proof is to use the decoupling inequalities from Section 4.2 in order to obtain, instead of $W_{a,k}(t)$, multiple Poisson integrals with respect to k independent copies of the original noise. An important advantage of the decoupled integral is that we can change the order of integration without losing the martingale property. Using Theorem 4.3 with the tetrahedral function $f(\mathbf{t}, \mathbf{x}, \mathbf{z}) := \rho(t, 0)^{-1} \rho_{t,0}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k z_i \mathbf{1}_{\{z_i \geq a\}}$ (the reader can check that (4.3) is satisfied whenever $\mu_{1,\infty}(p) < \infty$), we have

$$(6.13) \quad \mathbb{E}[|W_{a,k}(t)|^p] \leq \left(\frac{C}{p-1} \right)^k \mathbb{E}^{\otimes k}[|V_{a,k}(t)|^p],$$

where

$$(6.14) \quad V_{a,k}(t) := \int_{X_t^{(k)}} \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})}{\rho(t, 0)} \prod_{i=1}^k \bar{\xi}_{\omega_i}^a(dt_i, dx_i)$$

and ω_i , for $i = 1, \dots, k$, are i.i.d. copies of ω . With a small abuse of notation, we write \mathbb{P} , $\mathbb{P}^{\otimes k}$, \mathbb{E} and $\mathbb{E}^{\otimes k}$ in the remainder of the proof.

6.3.2. Partitioning and integrating over space. As in Section 6.2, we want to estimate the p th moment of the right-hand side of (6.14) using a combination of the BDG inequality, Jensen's inequality and subadditivity and then integrate over space. Because we want to use an intermediate exponent as in (6.12) that depends on the value of \mathbf{t} and \mathbf{z} , a first task is to decompose $V_{a,k}(t)$ by considering a (non-random) partition \mathfrak{P}_k of the parameter space $\mathcal{X}_t^{(k)}$. For each element $\mathcal{P} \in \mathfrak{P}_k$ of our partition, we will further determine a partition $J_1(\mathcal{P}) \cup J_2(\mathcal{P})$ of $\llbracket k \rrbracket$ and first integrate with respect to (t_i, z_i) with $i \in J_1(\mathcal{P})$ and then with respect to (t_i, z_i) with $i \in J_2(\mathcal{P})$. We let k_1 and k_2 denote the respective cardinalities of $J_1 = J_1(\mathcal{P})$ and $J_2 = J_2(\mathcal{P})$.

When $J \subseteq \llbracket k \rrbracket$, we use the following notation:

$$(6.15) \quad \Delta_J t_i := t_i - t_{i(J,-1)} \quad \text{and} \quad \Delta_J x_i := x_i - x_{i(J,-1)}$$

where $x_0 = x_{k+1} = 0$, $t_0 = 0$ and $t_{k+1} = t$ as usual and where for $i \in J \cup \{k+1\}$ we denote the predecessor of i in $J \cup \{0, k+1\}$ by $i(J, -1)$. As a result of our partitioning procedure, we obtain the following estimate:

Lemma 6.7. For $p \in (1, 2 \wedge (1 + \frac{2}{d}))$ and $\theta \in (p, 2]$, we have for all $a \in (0, 1]$,

$$(6.16) \quad \mathbb{E}[|V_{a,k}(t)|^p]^{\frac{1}{p}} \leq C^k \sum_{\mathcal{P} \in \mathfrak{P}_k} t^{\frac{1-\nu_p}{p}} \left(\int_{\mathbb{X}_t^{(k_2)}} \left(\int_{\mathbb{X}_t^{(k_1)}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \prod_{i=1}^{k+1} (\Delta t_i)^{\nu_\theta-1} \prod_{i \in J_1} z_i^\theta dt_i \lambda(dz_i) \right)^{\frac{p}{\theta}} \right. \\ \left. \times \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{(\nu_p-1) + \frac{p}{\theta}(1-\nu_\theta)} \prod_{i \in J_2} z_i^p dt_i \lambda(dz_i) \right)^{\frac{1}{p}}.$$

Proof. For $\mathcal{P} \in \mathfrak{P}_k$, let us define

$$(6.17) \quad V_{a,k}(t, \mathcal{P}) := \int_{\mathbb{X}_t^{(k)}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})}{\rho(t, 0)} \prod_{i=1}^k z_i \mathbf{1}_{\{z_i \geq a\}} (\delta_{\omega_i} - \nu)(dw_i).$$

Then, by (6.14) and Minkowski's inequality, we have that

$$\mathbb{E}[|V_{a,k}(t)|^p]^{\frac{1}{p}} \leq \sum_{\mathcal{P} \in \mathfrak{P}_k} \mathbb{E}[|V_{a,k}(t, \mathcal{P})|^p]^{\frac{1}{p}}.$$

Since the ω_i 's are independent, we can, similarly to (A.4) in the appendix, permute the integrals in (6.14) and integrate with respect to the indices in J_1 first and J_2 afterwards. By the BDG inequality, subadditivity (recall $p \leq 2$) and Jensen's inequality (applied in the fashion as in (6.12)), we obtain

$$(6.18) \quad \mathbb{E}[|V_{a,k}(t, \mathcal{P})|^p] \leq C^{k_2} \int_{\mathbb{X}_t^{(k_2)}} \mathbb{E} \left[\left| \int_{\mathbb{X}_t^{(k_1)}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})}{\rho(t, 0)} \prod_{i \in J_1} z_i \mathbf{1}_{\{z_i \geq a\}} (\delta_{\omega_i} - \nu)(dw_i) \right|^p \right] \\ \times \prod_{i \in J_2} z_i^p \mathbf{1}_{\{z_i \geq a\}} dt_i dx_i \lambda(dz_i) \\ \leq C^k \int_{\mathbb{X}_t^{(k_2)}} \left(\int_{\mathbb{X}_t^{(k_1)}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})^\theta}{\rho(t, 0)^\theta} \prod_{i \in J_1} z_i^\theta dt_i dx_i \lambda(dz_i) \right)^{\frac{p}{\theta}} \prod_{i \in J_2} z_i^p dt_i dx_i \lambda(dz_i).$$

Remark 6.8. In the above integrals and for the remainder of the proof, with a small abuse of notation, the coordinates of elements of $\mathbb{X}_t^{(k_1)}$ are indexed by J_1 instead of $\llbracket k_1 \rrbracket$ (and similarly for J_2). With this convention, $\mathbb{X}_t^{(k)}$ is a strict subset of $\mathbb{X}_t^{(k_1)} \times \mathbb{X}_t^{(k_2)}$, but this is not a problem since the indicator function $\mathbf{1}_{\mathcal{P}}$ restricts the integral to a subset of $\mathbb{X}_t^{(k)}$.

The next step is to carry out integration with respect to x_1, \dots, x_k explicitly. First of all, notice that for any $m \geq 1$ and any $p > 1$, we have

$$(6.19) \quad \int_{(\mathbb{R}^d)^m} \rho_{s,y}(\mathbf{t}, \mathbf{x})^p \prod_{i=1}^m dx_i = \vartheta(p)^m \rho(s, y)^p s^{1-\nu_p} \prod_{i=1}^{m+1} (\Delta t_i)^{\nu_p-1}$$

for every $\mathbf{t} \in \mathfrak{X}_m(s)$. This can be proved by induction on m after checking the case $m = 1$ by hand, using (5.5); we leave the details to the reader.

We then successively apply (6.19) with $p = \theta$ to the segments of J_1 (i.e., maximal sets of consecutive indices in J_1). Recalling (6.15), we get

$$(6.20) \quad \int_{(\mathbb{R}^d)^{k_1}} \rho_{t,0}(\mathbf{t}, \mathbf{x})^\theta \prod_{i \in J_1} dx_i = \vartheta(\theta)^{k_1} \prod_{i=1}^{k+1} (\Delta t_i)^{\nu_\theta-1} \prod_{j \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_j)^{1-\nu_\theta} \rho(\Delta_{J_2} t_j, \Delta_{J_2} x_j)^\theta.$$

Applying (6.19) in the case $y = 0$, we also get

$$(6.21) \quad \int_{(\mathbb{R}^d)^{k_2}} \prod_{j \in J_2 \cup \{k+1\}} \rho(\Delta_{J_2} t_j, \Delta_{J_2} x_j)^p \prod_{i \in J_2} dx_i = \vartheta(p)^{k_2} \rho(t, 0)^p t^{1-\nu_p} \prod_{j \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_j)^{\nu_p-1}.$$

We conclude the proof by inserting (6.20) and (6.21) in (6.18). \square

6.4. The proof of Proposition 6.1. The second moment was computed in Remark 6.2. In order to prove (6.2), we only need to separate small and large values of z . We consider a partition \mathfrak{P}_k indexed by the subsets $J \subseteq \llbracket k \rrbracket$ and we define

$$(6.22) \quad \mathcal{P}(J) := \{(\mathbf{t}, \mathbf{z}) \in \mathcal{X}_t^{(k)} : z_i < \eta \text{ for all } i \in J \text{ and } z_i \geq \eta \text{ for all } i \in \llbracket k \rrbracket \setminus J\}$$

as well as $J_1 := J$ and $J_2 := \llbracket k \rrbracket \setminus J$. Applying Lemma 6.7 with $\theta = 2$ and noting that $\nu_2 = \frac{1}{2}$ and $\nu_p = 1 - \frac{1}{2}(p-1)$ in dimension $d = 1$, we obtain

$$(6.23) \quad \mathbb{E} \left[|V_{a,k}(t)|^p \right]^{\frac{1}{p}} \leq C^k \sum_{J \subseteq \llbracket k \rrbracket} t^{\frac{1-\nu_p}{p}} \mu_{0,\eta}(2)^{\frac{k_1}{2}} \mu_{\eta,\infty}(p)^{\frac{k_2}{p}} \\ \times \left(\int_{(0,t)^{k_2}} \left(\int_{(0,t)^{k_1}} \mathbf{1}_{\{t_1 < \dots < t_k\}} \prod_{i=1}^{k+1} (\Delta t_i)^{-\frac{1}{2}} \prod_{i \in J_1} dt_i \right)^{\frac{p}{2}} \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\frac{2-p}{4}} \prod_{i \in J_2} dt_i \right)^{\frac{1}{p}}.$$

Integrating successively over the segments of J_1 and writing $\Delta_{J_2}(i) = i - i(J, -1)$ for the distance between $i \in J_2 \cup \{k+1\}$ and the previous index in J_2 (recall (6.15)), we get from Lemma 4.5 that

$$(6.24) \quad \int_{(0,t)^{k_1}} \mathbf{1}_{\mathfrak{x}_k(t)}(\mathbf{t}) \prod_{i=1}^{k+1} (\Delta t_i)^{-\frac{1}{2}} \prod_{i \in J_1} dt_i = \mathbf{1}_{\mathfrak{x}_{k_2}(t)}((t_i)_{i \in J_2}) \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\frac{1}{2} \Delta_{J_2}(i)-1} \frac{\Gamma(\frac{1}{2} \Delta_{J_2}(i))}{\Gamma(\frac{1}{2} \Delta_{J_2}(i))}.$$

Now, if we bound $(\Delta_{J_2} t_i)^{(\Delta_{J_2}(i)-1)/2} \leq t^{(\Delta_{J_2}(i)-1)/2}$ and $\Gamma(\frac{1}{2} \ell) \geq \frac{1}{2} \sqrt{\pi}$ for any $\ell \geq 1$, we get that

$$\int_{(0,t)^{k_1}} \mathbf{1}_{\mathfrak{x}_k(t)}(\mathbf{t}) \prod_{i=1}^{k+1} (\Delta t_i)^{-\frac{1}{2}} \prod_{i \in J_1} dt_i \leq \mathbf{1}_{\mathfrak{x}_{k_2}(t)}((t_i)_{i \in J_2}) 2^{k_2+1} (\pi t)^{\frac{1}{2} k_1} \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{-\frac{1}{2}}.$$

Going back to (6.23) and taking the factor $[(\Delta_{J_2} t_i)^{-1/2}]^{p/2}$ over to the outer integral, we compute

$$(6.25) \quad \int_{(0,t)^{k_2}} \mathbf{1}_{\mathfrak{x}_{k_2}(t)}((t_i)_{i \in J_2}) \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\frac{1-p}{2}} \prod_{i \in J_2} dt_i = t^{\nu_p k_2 + \nu_p - 1} \frac{\Gamma(\nu_p)^{k_2+1}}{\Gamma(\nu_p(k_2+1))},$$

thanks again to Lemma 4.5. We therefore conclude that

$$(6.26) \quad \mathbb{E} \left[|V_{a,k}(t)|^p \right]^{\frac{1}{p}} \leq \sum_{J \subseteq \llbracket k \rrbracket} \frac{(\sqrt{\pi} t \mu_{0,\eta}(2))^{\frac{k_1}{2}} (2^{\frac{p}{2}} \mu_{\eta,\infty}(p) t^{\nu_p} \Gamma(\nu_p))^{\frac{k_2}{p}} \sqrt{2} \Gamma(\nu_p)^{\frac{1}{p}}}{\Gamma(\nu_p(k_2+1))^{\frac{1}{p}}} \\ \leq C^k \max_{k_1+k_2=k} \frac{(\sqrt{\pi} t \mu_{0,\eta}(2))^{\frac{k_1}{2}} (2^{\frac{p}{2}} \mu_{\eta,\infty}(p) t^{\nu_p} \Gamma(\nu_p))^{\frac{k_2}{p}} \Gamma(\nu_p)^{\frac{1}{p}}}{\Gamma(\nu_p(k_2+1))^{\frac{1}{p}}}.$$

Absorbing $\pi^{k_1/4}$ and $(2^{p/2} \Gamma(\nu_p))^{k_2/p}$, which is uniformly bounded in $p \in [1, 2]$, into the constant C and replacing the maximum by a sum, we derive (6.2) from (6.9), (6.13) and (6.26).

If $p \in (2, 3)$, the subadditivity argument in (6.18) does not apply. Instead, we shall use a variant of the BDG estimate (6.11) that only contains the intensity measure $\nu := dt \otimes dx \otimes \lambda(dz)$ instead of δ_ω . For any $p \geq 2$, there exists a constant $C'_p \in (0, \infty)$ such that for all $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times (0, \infty))$ -measurable process $K = K(\omega, w) = K(\omega, t, x, z)$,

$$(6.27) \quad \mathbb{E} \left[\left| \int_{\mathbb{X}} K(w) (\delta_\omega - \nu)(dw) \right|^p \right] \leq C'_p \left\{ \mathbb{E} \left[\left(\int_{\mathbb{X}} K(w)^2 \nu(dw) \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\int_{\mathbb{X}} |K(w)|^p \nu(dw) \right] \right\},$$

where $C'_p = (2^{p-1} p(p-1)(p/(p-1))^p)^{p/2} \vee 2(p/(p-1))^p (2^p + 1 + p) \leq 729$ when $p \in [2, 3]$; see [93, Theorem 1 (b)] (and its proof for the value of C'_p). We also refer to [86] for a survey of various

versions and proofs (and names) of this inequality. Let us define $\widetilde{W}_{a,0}(t, x) := \rho(t, x)$ and

$$\widetilde{W}_{a,k}(t, x) := \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \bar{\xi}_\omega^a(dt_i, dx_i) \quad \text{for } k \geq 1,$$

so that in particular $W_{a,k}(t) = \widetilde{W}_{a,k}(t, 0)/\rho(t, 0)$. With this definition, applying (6.27), we have

$$\begin{aligned} \mathbb{E} \left[|\widetilde{W}_{a,k}(t, x)|^p \right]^{\frac{1}{p}} &= \mathbb{E} \left[\left| \int_{X_t} \rho(t - t_k, 0 - x_k) \widetilde{W}_{a,k-1}(t_k, x_k) \bar{\xi}_\omega^a(dt_k, dx_k) \right|^p \right]^{\frac{1}{p}} \\ &\leq C' \left\{ \mathbb{E} \left[\left(\mu_{0,\infty}(2) \int_{X_t} (\rho(t - t_k, x_k) \widetilde{W}_{a,k-1}(t_k, x_k))^2 dt_k dx_k \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \right. \\ &\quad \left. + \left(\mu_{0,\infty}(p) \int_{X_t} \rho(t - t_k, x_k)^p \mathbb{E}[|\widetilde{W}_{a,k-1}(t_k, x_k)|^p] dt_k dx_k \right)^{\frac{1}{p}} \right\}. \end{aligned}$$

By Minkowski's integral inequality, this is further bounded by we get

$$\begin{aligned} C' \left\{ \mu_{0,\infty}(2)^{\frac{1}{2}} \left(\int_{X_t} \rho(t - t_k, x_k)^2 \mathbb{E}[|\widetilde{W}_{a,k-1}(t_k, x_k)|^p]^{\frac{2}{p}} dt_k dx_k \right)^{\frac{1}{2}} \right. \\ \left. + \mu_{0,\infty}(p)^{\frac{1}{p}} \left(\int_{X_t} \rho(t - t_k, x_k)^p \mathbb{E}[|\widetilde{W}_{a,k-1}(t_k, x_k)|^p] dt_k dx_k \right)^{\frac{1}{p}} \right\}. \end{aligned}$$

Repeating this estimate and recalling (6.9), we get

$$(6.28) \quad \mathbb{E} \left[\bar{\mathcal{Z}}_\beta^{\omega,a}(t, 0)^p \right]^{\frac{1}{p}} \leq \rho(t, 0) + \sum_{k=1}^{\infty} (C' \beta)^k \sum_{\boldsymbol{\theta} \in \{2, p\}^k} \mu_{0,\infty}(p)^{\frac{k_1}{p}} \mu_{0,\infty}(2)^{\frac{k_2}{2}} \|\rho_{t,0}(\mathbf{t}, \mathbf{x})\|_{\boldsymbol{\theta}, t},$$

where k_1 is the number of p 's and k_2 the number of 2's in $\boldsymbol{\theta}$, and where we have defined

$$\|f\|_{\boldsymbol{\theta}, t} := \left(\int_{X_t} \left(\cdots \left(\int_{X_t} f(\mathbf{t}, \mathbf{x})^{\theta_1} dt_1 dx_1 \right)^{\frac{\theta_2}{\theta_1}} \cdots \right)^{\frac{\theta_k}{\theta_{k-1}}} dt_k dx_k \right)^{\frac{1}{\theta_k}}$$

for $f : ((0, \infty) \times \mathbb{R}^d)^k \rightarrow [0, \infty)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in [1, \infty)^k$.

Now in order to conclude, we want to replace $\|\rho_{t,0}(\mathbf{t}, \mathbf{x})\|_{\boldsymbol{\theta}, t}$ by an integral which is analogous to that found in the on the right-hand side of (6.18). To do so, we use the following identity, valid for any positive function f and any measures μ_1 and μ_2 :

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} f(w_1, w_2)^2 \mu_2(dw_2) \right)^{\frac{p}{2}} \mu_1(dw_1) \right)^{\frac{1}{p}} \leq \left(\int_{\Omega_2} \left(\int_{\Omega_1} f(w_1, w_2)^p \mu_1(dw_1) \right)^{\frac{2}{p}} \mu_2(dw_2) \right)^{\frac{1}{2}}.$$

This inequality is a special case of [80, Lemma 3.3.1]. Now, letting J_1 denote the set of indices for which $\theta_i = p$ and J_2 those for which $\theta_i = 2$, we apply the above inequality iteratively to take all the integrals with respect to J_1 inside. We obtain (recall Remark 6.8)

$$\|\rho_{t,0}(\mathbf{t}, \mathbf{x})\|_{\boldsymbol{\theta}, t} \leq \left(\int_{X_t^{(k_2)}} \left(\int_{X_t^{(k_1)}} \mathbf{1}_{\mathbf{x}_k(t)}(\mathbf{t}) \rho_{t,0}(\mathbf{t}, \mathbf{x})^p \prod_{i \in J_1} dt_i dx_i \right)^{\frac{2}{p}} \prod_{i \in J_2} dt_i dx_i \right)^{\frac{1}{2}}.$$

Now we can first integrate with respect to the x_i 's using (6.20) and (6.21) (with p instead of θ and 2 instead of p). Recalling that $\nu_2 = \frac{1}{2}$, we obtain

$$\|\rho_{t,0}(\mathbf{t}, \mathbf{x})\|_{\theta,t} \leq \vartheta(p)^{\frac{k_1}{p}} \vartheta(2)^{\frac{k_2}{2}} \rho(t,0) t^{\frac{1}{4}} \times \left(\int_{(0,t)^{k_2}} \left(\int_{(0,1)^{k_1}} \mathbf{1}_{\mathbf{x}_k(t)}(\mathbf{t}) \prod_{i=1}^{k+1} (\Delta t_i)^{\nu_p-1} \prod_{i \in J_1} dt_i \right)^{\frac{2}{p}} \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\frac{2}{p}(1-\nu_p)-\frac{1}{2}} \prod_{i \in J_2} dt_i \right)^{\frac{1}{2}}.$$

Using Lemma 4.5 to integrate first with respect to $(t_i)_{i \in J_1}$, we get as in (6.24)

$$(6.29) \quad \int_{(0,1)^{k_1}} \mathbf{1}_{\mathbf{x}_k(t)}(\mathbf{t}) \prod_{i=1}^{k+1} (\Delta t_i)^{\nu_p-1} \prod_{i \in J_1} dt_i = \mathbf{1}_{\mathbf{x}_{k_2}(t)}((t_i)_{i \in J_2}) \prod_{i \in J_2 \cup \{k+1\}} \frac{(\Delta_{J_2} t_i)^{\nu_p \Delta_{J_2}(i)-1} \Gamma(\nu_p)^{\Delta_{J_2}(i)}}{\Gamma(\nu_p \Delta_{J_2}(i))} \\ \leq \mathbf{1}_{\mathbf{x}_{k_2}(t)}((t_i)_{i \in J_2}) \Gamma(\nu_p)^{k+1} t^{\nu_p k_1} \frac{\prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\nu_p-1}}{\prod_{i \in J_2 \cup \{k+1\}} \Gamma(\nu_p \Delta_{J_2}(i))},$$

because $(\Delta_{J_2} t_i)^{\nu_p \Delta_{J_2}(i)-1} \leq t^{\nu_p(\Delta_{J_2}(i)-1)} (\Delta_{J_2} t_i)^{\nu_p-1}$. Pulling $(\Delta_{J_2} t_i)^{\nu_p-1}$ to the outer integral, we evaluate

$$\int_{(0,t)^{k_2}} \mathbf{1}_{\mathbf{x}_{k_2}(t)}((t_i)_{i \in J_2}) \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{-\frac{1}{2}} \prod_{i \in J_2} dt_i = t^{\frac{k_2}{2}-\frac{1}{2}} \frac{\Gamma(\frac{1}{2})^{k_2+1}}{\Gamma(\frac{k_2+1}{2})},$$

using once more Lemma 4.5. Altogether, bounding $\vartheta(p), \vartheta(2) \leq 1$, we obtain

$$\|\rho_{t,0}(\mathbf{t}, \mathbf{x})\|_{\theta,t} \leq \rho(t,0) t^{\frac{\nu_p}{p} k_1 + \frac{1}{4} k_2} \frac{\Gamma(\nu_p)^{\frac{1}{p}(k+1)} \pi^{\frac{1}{4}(k_2+1)}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}} \prod_{i \in J_2 \cup \{k+1\}} \Gamma(\nu_p \Delta_{J_2}(i))^{\frac{1}{p}}}.$$

Going back to (6.28), bounding $\vartheta(p), \vartheta(2) \leq 1$ and expressing everything in terms of J_2 and $k_2 = |J_2|$, we get that $\mathbb{E}[\rho(t,0)^{-p} \tilde{\mathcal{Z}}_{\beta}^{\omega,a}(t,0)^p]^{1/p}$ is bounded by

$$\pi^{\frac{1}{4}} \Gamma(\nu_p)^{\frac{1}{p}} \sum_{k=0}^{\infty} \sum_{J_2 \subseteq \llbracket k \rrbracket} \frac{(C' \beta \mu_{0,\infty}(p)^{\frac{1}{p}} t^{\frac{\nu_p}{p}} \Gamma(\nu_p)^{\frac{1}{p}})^{k_1} (C' \beta \mu_{0,\infty}(2)^{\frac{1}{2}} \Gamma(\nu_p)^{\frac{1}{p}} \pi^{\frac{1}{4}} t^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}} \prod_{i \in J_2 \cup \{k+1\}} \Gamma(\nu_p \Delta_{J_2}(i))^{\frac{1}{p}}} \\ = \pi^{\frac{1}{4}} \Gamma(\nu_p)^{\frac{1}{p}} \sum_{k=0}^{\infty} \sum_{k_2=0}^k \frac{(C' \beta \mu_{0,\infty}(2)^{\frac{1}{2}} \Gamma(\nu_p)^{\frac{1}{p}} \pi^{\frac{1}{4}} t^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}}} \\ \times \sum_{\substack{\ell_1, \dots, \ell_{k_2+1} \geq 1 \\ \ell_1 + \dots + \ell_{k_2+1} = k+1}} \prod_{i=1}^{k_2+1} \frac{(C' \beta \mu_{0,\infty}(p)^{\frac{1}{p}} t^{\frac{\nu_p}{p}} \Gamma(\nu_p)^{\frac{1}{p}})^{\ell_i-1}}{\Gamma(\nu_p \ell_i)^{\frac{1}{p}}},$$

where we have used a change of variable and the fact that $\ell_1 + \dots + \ell_{k_2+1} = k+1$ implies that $k_1 = \sum_{i=1}^{k_2+1} (\ell_i - 1)$. Exchanging the first two sums and factorizing the last one yields (6.3). \square

6.5. Intermezzo: $d \geq 2$ under stronger moment conditions. The method we have used for $d = 1$ can further be used to prove boundedness of moments under the more restrictive assumption given in (1.22). We illustrate this by Proposition 6.9 below whose proof can be achieved by replicating that of (6.2). This quantitative estimate is sufficient to prove the upper bound parts of Theorem 2.14 and similar ideas could in principle be used to prove Theorem 2.6 under the more restrictive assumption (1.22). The more involved method used in the proof of Proposition 6.3 below, however, is necessary to bridge the gap between the conditions (1.22) and (1.24); recall that the latter is optimal in dimension 2 and very close to optimal when $d \geq 3$.

Proposition 6.9. *There exists a constant $C \in (0, \infty)$ that only depends on d such that if $d \geq 2$, $\mu_{0,1}(q) < \infty$ and $\mu_{1,\infty}(p) < \infty$ for some $1 < p \leq q < 1 + \frac{2}{d}$, we have, for all $a \in [0, 1]$, $\eta \in (0, 1]$ and $\beta, t > 0$,*

$$\mathbb{E} \left[\rho(t, x)^{-p} \bar{Z}_\beta^{\omega, a}(t, x)^p \right]^{\frac{1}{p}} \leq \sum_{k_1, k_2=0}^{\infty} \left(\frac{C\Gamma(\nu_q)\beta}{p-1} \right)^{k_1+k_2} (\mu_{0,\eta}(q)t^{\nu_q})^{\frac{k_1}{q}} \frac{(\Gamma(\nu_p)\mu_{\eta,\infty}(p)t^{\nu_p})^{\frac{k_2}{p}} \Gamma(\nu_p)^{\frac{1}{p}}}{\Gamma(\nu_p(k_2+1))^{\frac{1}{p}}}.$$

Proof. Let us give a very short guideline for the proof: we use the same partition \mathfrak{P}_k as in (6.22) and apply Lemma 6.7 with $\theta = q$. The proof is identical to that of Proposition 6.1 (ii) except that (6.24) has to be replaced by (6.29), which together with the bound $\Gamma(\nu_q\Delta_{J_2}(i)) \geq \frac{1}{2}$ gives

$$\int_{(0,t)^{k_1}} \mathbf{1}_{\mathfrak{X}_k(t)}(\mathbf{t}) \prod_{i=1}^{k+1} (\Delta t_i)^{\nu_q-1} \prod_{i \in J_1} dt_i \leq \mathbf{1}_{\mathfrak{X}_{k_2}(t)}((t_i)_{i \in J_2}) 2^{k_2+1} \Gamma(\nu_q)^k t^{\nu_q k_1} \prod_{i \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_i)^{\nu_q-1}. \quad \square$$

6.6. The proof of Proposition 6.3. We use the same idea as for the proof of Proposition 6.1. We will consider a partition \mathfrak{P}_k (defined below) of the parameter space $\mathcal{X}_t^{(k)}$: each element $\mathcal{P} \in \mathfrak{P}_k$ induces a partition of $\llbracket k \rrbracket$ into two sets J_1 and J_2 and we are going to use Lemma 6.7 with $\theta = 1 + \frac{2}{d}$ (and hence $\nu_\theta = 0$). The main technical part of this section will then be to bound

$$(6.30) \quad U(\mathcal{P}, t) := t^{1-\nu_p} \int_{\mathcal{X}_t^{(k_2)}} \left(\int_{\mathcal{X}_t^{(k_1)}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \prod_{i=1}^{k+1} (\Delta t_i)^{-1} \prod_{i \in J_1} z_i^{1+\frac{2}{d}} dt_i \lambda(dz_i) \right)^{\frac{p}{1+2/d}} \\ \times \prod_{j \in J_2 \cup \{k+1\}} (\Delta_{J_2} t_j)^{\frac{p}{1+2/d} + \nu_p - 1} \prod_{i \in J_2} z_i^p dt_i \lambda(dz_i).$$

The following is the main technical estimate of this section.

Proposition 6.10. *Recall (6.4) and (6.6). With the choice \mathfrak{P}_k defined in Section 6.6.1 below (see in particular (6.35)), we have for every $\mathcal{P} \in \mathfrak{P}_k$, $\eta \in (0, 1]$ and $t > 0$ that*

$$(6.31) \quad U(\mathcal{P}, t) \leq \zeta_1(\eta, p, t)^{k_1} \zeta_2(\eta, p)^{k_2} \Lambda(k_2, t, p),$$

where $k_1 := |J_1|$ and $k_2 := |J_2|$.

Proposition 6.3 follows immediately from Proposition 6.10.

Proof of Proposition 6.3. By Lemma 6.7, (6.13) and (6.14), we have that

$$(6.32) \quad \mathbb{E}[W_{a,k}(t)^p]^{\frac{1}{p}} \leq \left(\frac{C}{p-1} \right)^k |\mathfrak{P}_k| \max_{\mathcal{P} \in \mathfrak{P}_k} U(\mathcal{P}, t)^{\frac{1}{p}} \\ \leq 2 \left(\frac{9C}{p-1} \right)^k \max_{k_1+k_2=k} \left(\zeta_1(\eta, p, t)^{k_1} \zeta_2(\eta, p)^{k_2} \Lambda(k_2, t, p) \right)^{\frac{1}{p}},$$

since $|\mathfrak{P}_k| \leq 2 \times 9^k$ (see below). Replacing max by a sum yields (6.7). \square

6.6.1. Constructing the partition \mathfrak{P}_k . The construction of \mathfrak{P}_k when $d \geq 2$ is considerably more involved, as it no longer suffices to only differentiate between $z_i \geq \eta$ and $z_i < \eta$ as we did for $d = 1$. Recall that in Section 6.2, in order to obtain optimal bounds, we had to split the integral according to how z_i and $\Delta t_i^{d/2}$ compare to each other. A problem arises when trying to generalize this method to $k \geq 2$: On the right-hand side of (6.30), the value of Δt_i plays the same role as Δt_{i+1} , which is why we cannot break down the k -fold integral into 2^k parts as we did in the proof of Proposition 6.1.

We now describe our solution to this problem. To give the idea behind our partition and its link to the bound (6.31), recall from (6.30) that we first integrate with respect to the variables with indices in J_1 and then with respect to those with indices in J_2 . Loosely speaking, indices in

J_1 correspond to values of z_i that are small compared to $(\Delta t_i)^{d/6}$ and $(\Delta t_{i+1})^{d/6}$; this gives rise to a factor $\zeta_1(\eta, p, t)^{k_1}$ that can be explained by the calculations in (6.38)–(6.39) below. Note that after integrating with respect to such an index, one may have to update the set of parameters since for the next step one has to compare z_i with time increments formed between the remaining variables. These may differ from Δt_i and Δt_{i+1} , which explains why the partition has to be defined iteratively.

Each element of our partition $\mathcal{P} \in \mathfrak{P}_k$ is encoded by a finite sequence $(L^j, I_-^j, I_+^j, D^j)_{j=1}^m$ of partitions of $\llbracket k+1 \rrbracket$; the length m of the sequence is a variable. With some abuse of notation, we identify \mathcal{P} with this sequence. We also use the notation $I^j := I_-^j \cup I_+^j$. Not every sequence is admissible, so let us present the rules for constructing the set of admissible sequences:

- First we partition $\llbracket k+1 \rrbracket$ into three sets L^1 , I_+^1 , and I_-^1 , imposing that $k+1 \in I^1$, and we define $D^1 := \emptyset$.
- The procedure is then iterative. Assume that $m \geq j$ and that one has constructed the sets of the first j steps. We let $i(+1, j)$ denote the successor of i and $i(-1, j)$ denote the predecessor of i in $L^j \cup I_-^j \cup I_+^j$:

$$(6.33) \quad \begin{aligned} i(+1, j) &:= \min\{\ell \in L^j \cup I_-^j \cup I_+^j : \ell \geq i+1\}, \\ i(-1, j) &:= \max\{\ell \in L^j \cup I_-^j \cup I_+^j : \ell \leq i-1\}, \end{aligned}$$

with the convention $\max \emptyset = 0$; we only use the notation $i(+1, j)$ if the set over which the minimum is taken is non-empty. Unless

$$(6.34) \quad L_\times^j := \{i \in \llbracket k \rrbracket : i \in L^j \text{ and } i(+1, j) \in I_-^j\} = \emptyset,$$

we need to add extra terms corresponding to $j+1$ to our sequence, which will be described in the next point. This procedure is repeated until at some stage j , (6.34) is satisfied. We then define $m = j$ and our sequence is complete.

- When $L_\times^j \neq \emptyset$, the next sets $(L^{j+1}, I_-^{j+1}, I_+^{j+1}, D^{j+1})$ are only partially determined by (L^j, I_-^j, I_+^j, D^j) : while we prescribe the choice $D^{j+1} := D^j \cup L_\times^j$, there is some liberty for choosing L^{j+1} , I_-^{j+1} and I_+^{j+1} . We set

$$I_\times^j := \{i(+1, j) : i \in L_\times^j\}$$

and consider an arbitrary partition $\{I_-^{(j+1)}, I_+^{(j+1)}, L^{(j+1)}\}$ of I_\times^j subject to only one constraint: if $k+1 \in I_\times^j$, then $k+1 \in I_-^{(j+1)} \cup I_+^{(j+1)}$ (this is to guarantee that $k+1 \in I^j$ for all j). We then define

$$\begin{aligned} L^{j+1} &:= (L^j \setminus L_\times^j) \cup L^{(j+1)}, & I_-^{j+1} &:= (I_-^j \setminus I_\times^j) \cup I_-^{(j+1)}, \\ I_+^{j+1} &:= I_+^j \cup I_+^{(j+1)}. \end{aligned}$$

Given an element in \mathfrak{P}_k , we further define

$$J_1 := D^m \cup \{i \in L^m : i(+1, m) \in L^m\} \quad \text{and} \quad J_2 := \llbracket k \rrbracket \setminus J_1.$$

An example of an admissible sequence of partitions as well as the associated sets J_1 and J_2 are shown in Figure 1.

Let us now determine, or rather bound from above, the cardinality of \mathfrak{P}_k , that is, the number of admissible sequences that can be constructed according to the above rules. Note that we have 2×3^k possibilities for choosing L^1 , I_+^1 and I_-^1 . Afterwards, at each step, one has to assign one out of three labels $I_-^{(j+1)}$, $I_+^{(j+1)}$ or $L^{(j+1)}$ to each element in I_\times^j . Since $|I_\times^j| = |L_\times^j|$ and the L_\times^j 's are disjoint by construction, there are at most k choices to be made, hence at most 3^k possibilities. We therefore have $|\mathfrak{P}_k| \leq 2 \times 9^k$.

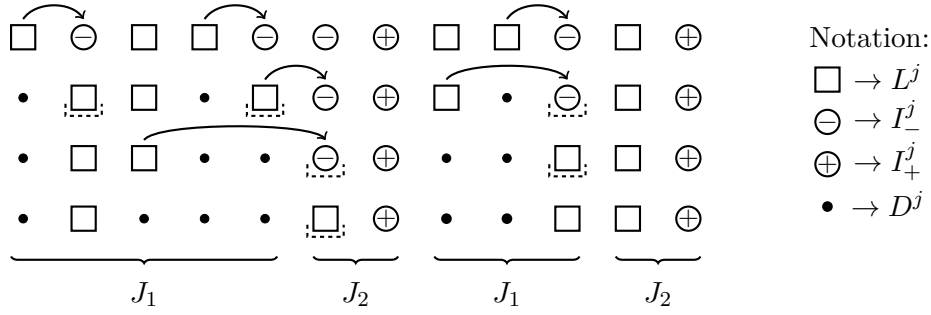


FIGURE 1. Example of an admissible sequence of partitions of $\llbracket k+1 \rrbracket$ with $k = 11$. The successive partitions $(L^j, I_-^j, I_+^j, D^j)_{1 \leq j \leq m}$ (with $m = 4$) are represented from top to bottom. Each curved arrow represents an element of I_-^j and points from an index in L_-^j to an index in I_-^j . The symbol \dots represents indices that can be chosen to belong to $L^{(j+1)}$ (\square), $I_-^{(j+1)}$ (\ominus) or $I_+^{(j+1)}$ (\oplus). In the proof of Proposition 6.10 we first integrate with respect to variables with indices in J_1 , proceeding in the order $L_-^1, L_-^2, \dots, L_-^{m-1}, \{i \in L^m : i(+1, m) \in L^m\}$, and afterwards integrate with respect to variables with indices in J_2 .

Next, let us explain how each sequence in \mathfrak{P}_k is associated to a subset $\mathcal{P} \subseteq \mathcal{X}_t^{(k)}$. With the convention $(L^{(1)}, I_-^{(1)}, I_+^{(1)}) := (L^1, I_-^1, I_+^1)$, we define

$$(6.35) \quad \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) := \prod_{j=1}^m \left(\prod_{i \in I_-^{(j)}} \mathbf{1}_{\{z_i \geq (\Delta_j t_i)^{d/6 \wedge \eta}, z_{i(-1,j)} < (\Delta_j t_i)^{d/6 \wedge \eta}\}} \right. \\ \left. \times \prod_{i \in I_+^{(j)}} \mathbf{1}_{\{z_i \wedge z_{i(-1,j)} \geq (\Delta_j t_i)^{d/6 \wedge \eta}\}} \prod_{i \in L^{(j)}} \mathbf{1}_{\{z_i < (\Delta_j t_i)^{d/6 \wedge \eta}\}} \right),$$

where $\Delta_j t_i := t_i - t_{i(-1,j)}$ (recall (6.33)) and $z_{k+1} := \infty$ by convention.

It is easy to verify that (6.35) induces a partition of $\mathcal{X}_t^{(k)}$ indexed by \mathfrak{P}_k : Indeed, for any fixed $(\mathbf{t}, \mathbf{z}) \in \mathcal{X}_t^{(k)}$, the values of z_i and Δt_i uniquely determine $(L^1, I_-^1, I_+^1) = (L^{(1)}, I_-^{(1)}, I_+^{(1)})$. Assuming that (L^j, I_-^j, I_+^j) has been identified up to some $j \geq 1$, we can check whether (6.34) is satisfied. If so, we set $m = j$ and we are done. Otherwise, we again use the values of z_i and Δt_i to find $(L^{(j+1)}, I_-^{(j+1)}, I_+^{(j+1)})$, from which we can then determine $(L^{j+1}, I_-^{j+1}, I_+^{j+1})$. Therefore, every (\mathbf{t}, \mathbf{z}) is contained in some $\mathcal{P} \in \mathfrak{P}_k$. The uniqueness of \mathcal{P} is straightforward.

6.6.2. Proof of Proposition 6.10. Recall the formula (6.30) of $U(\mathcal{P}, t)$. The proof is divided into two parts: first we integrate with respect to variables with indices in J_1 and then with respect to those with indices in J_2 .

Step 1. Integrating with respect to indices in J_1 . Our first task is to prove that, roughly speaking, the integral with respect to each z_i and t_i with indices $i \in J_1$ yields at most a factor $\zeta_1(\eta, p, t)^{k_1}$. More precisely, we prove

$$(6.36) \quad U(\mathcal{P}, t) \leq \zeta_1(\eta, p, t)^{k_1} U_2(\mathcal{P}, t),$$

where

$$U_2(\mathcal{P}, t) := t^{1-\nu_p} \int_{\mathcal{X}_t^{(k_2)}} \mathbf{1}_{\mathcal{P}'} (\Delta_{J_2} t_{k+1})^{\nu_p-1} \prod_{i \in J_2} (\Delta_{J_2} t_i)^{\nu_p-1} z_i^p (3|\log z_i| + 1)^{\mathbf{1}_{\{z_i < \eta\}}} dt_i \lambda(dz_i)$$

and

$$(6.37) \quad \mathbf{1}_{\mathcal{P}'} := \prod_{i \in I_-^{(m)}} \mathbf{1}_{\{z_i \geq (\Delta_{J_2} t_i)^{d/6} \wedge \eta\}} \prod_{i \in I_+^{(m)}} \mathbf{1}_{\{z_i \wedge z_{i(-1, J_2)} \geq (\Delta_{J_2} t_i)^{d/6} \wedge \eta\}} \prod_{i \in L^{(m)} \cap J_2} \mathbf{1}_{\{z_i < (\Delta_{J_2} t_i)^{d/6} \wedge \eta\}}$$

represents constraints on the values of $\Delta_{J_2} t_i$ and z_i that are inherited from \mathcal{P} for $i \in J_2$. Note that compared to (6.35), we only have $j = m$ and we have replaced $\Delta_m t_i$ by $\Delta_{J_2} t_i$. This makes no difference when $i \in I_-^{(m)}$ or $i \in I_+^{(m)}$ since $i(-1, m) \in J_2$ in both cases; for $i \in L^{(m)}$, the constraint is implied by \mathcal{P} since $\Delta_{J_2} t_i \geq \Delta_m t_i$.

First, let us reduce to the case where $m = 0$, that is, $L_\times^1 = \emptyset$, : the idea is to integrate with respect to variables with indices in L_\times^1 , then with respect to those with indices in L_\times^2 , etc. One may refer to Figure 1 to understand how this procedure goes on. We start with integration with respect to z_i and t_i when $i \in L_\times^1$. In this case, we have $z_i < [\Delta t_i \wedge \Delta t_{i+1}]^{d/6} \wedge \eta$. To treat these indices, note that by symmetry,

$$(6.38) \quad \int_0^t s^{-1}(t-s)^{-1} \mathbf{1}_{\{z < [s \wedge (t-s)]^{d/6}\}} ds \leq 2 \int_0^{t/2} s^{-1}(t-s)^{-1} \mathbf{1}_{\{z < s^{d/6}\}} ds \leq 4t^{-1} \log(\tfrac{1}{2} t z^{-\frac{6}{d}}),$$

where for the last bound we used that $t-s \geq t/2$. From now on, we assume $\eta \leq 1$. Then, if $z \in (0, \eta)$, using also that $6/d \leq 3$, we have

$$\int_0^t s^{-1}(t-s)^{-1} \mathbf{1}_{\{z < [s \wedge (t-s)]^{d/6}\}} ds \leq 8t^{-1}(3|\log z| + 1)(1 + \log_+ t).$$

Recalling the definition (6.6) of $\zeta_1(\eta, p, t)$, we end up with

$$(6.39) \quad \int_0^t \int_{(0, \eta)} s^{-1}(t-s)^{-1} \mathbf{1}_{\{z < [s \wedge t-s]^{d/6}\}} z^{1+\frac{2}{d}} ds \lambda(dz) \leq t^{-1} \zeta_1(\eta, p, t)^{\frac{1+2/d}{p}}.$$

So if $i \in L_\times^1$, using the trivial fact that $\Delta^2 t_i := t_{i+1} - t_{i-1} \leq t$, we have

$$(6.40) \quad \int_{t_{i-1}}^{t_{i+1}} \int_{(0, \infty)} (\Delta t_i \Delta t_{i+1})^{-1} z_i^{1+\frac{2}{d}} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) dt_i \lambda(dz_i) \leq \zeta_1(\eta, p, t)^{\frac{1+2/d}{p}} (\Delta^2 t_i)^{-1} \mathbf{1}_{\mathcal{P}^{(i)}}(\mathbf{t}^{(i)}, \mathbf{z}^{(i)}),$$

where $\mathbf{t}^{(i)}$ and $\mathbf{z}^{(i)}$ denote the vectors \mathbf{t} and \mathbf{z} with the i th coordinate omitted and $\mathcal{P}^{(i)}$ is obtained from \mathcal{P} by ignoring all the constraints that involve either z_i or t_i . Thus, at the cost of a multiplicative factor $\zeta_1(\eta, p, t)^{(1+2/d)/p}$ one can, for each $i \in L_\times^1$, simplify the integrals with respect to $\lambda(dz_i) dt_i$. After relabeling the indices, we can then iterate this process, making use of the iterative construction of \mathcal{P} . As a result, we are left to prove (6.36) in the case where $L_\times^1 = \emptyset$. Since $m = 1$ in this case, we drop the superscript 1 from the notation.

Under the assumption $L_\times^1 = \emptyset$, we have $J_1 = \{i \in L : i+1 \in L\}$. We may treat different segments of J_1 separately. Let us assume, for instance, that $\llbracket i, i+\ell-1 \rrbracket \subseteq J_1$, where $\llbracket i, j \rrbracket := \{i, i+1, \dots, j\}$. In the same spirit as (6.38)–(6.40), we have that

$$\begin{aligned} & \int_{t_{i-1}}^{t_{i+1}} (\Delta t_i \Delta t_{i+1})^{-1} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) dt_i \\ & \leq \frac{8(1 + \log_+ t)}{\Delta^2 t_i} (1 + 3|\log(z_i \wedge z_{i+1})|) \mathbf{1}_{\mathcal{P}^{(i)}}(\mathbf{t}^{(i)}, \mathbf{z}^{(i)}) \mathbf{1}_{\{z_i \vee z_{i+1} < (\Delta^2 t_i)^{6/d} \wedge \eta\}}. \end{aligned}$$

Iterating this and noticing that $|\log(z_i \wedge z_{i+1})| = |\log z_i| \vee |\log z_{i+1}|$ since $z_i, z_{i+1} < \eta \leq 1$, we get

$$\int_{\mathfrak{X}_\ell(t_{i-1}, t_{i+\ell})} \prod_{j=0}^{\ell} (\Delta t_{i+j})^{-1} \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) dt_i \cdots dt_{i+\ell-1} \leq \frac{(8(1 + \log_+ t))^\ell}{t_{i+\ell} - t_{i-1}} \times \frac{\prod_{j=0}^{\ell} (3|\log z_{i+j}| + 1)}{\min_{j \in \llbracket \ell \rrbracket} (3|\log z_{i+j}| + 1)}.$$

The second denominator can be ignored since it is larger than 1. If one then integrates with respect to $z_i, \dots, z_{i+\ell-1}$ (recall (6.37) and the remark afterwards), one obtains

$$\begin{aligned} \int_{\mathfrak{X}_\ell(t_{i-1}, t_{i+\ell}) \times (0, \infty)^\ell} \left(\prod_{j=0}^{\ell-1} (\Delta t_{i+j})^{-1} \right) \mathbf{1}_{\mathcal{P}}(\mathbf{t}, \mathbf{z}) \prod_{j=0}^{\ell-1} z_{i+j}^{1+\frac{2}{d}} dt_{i+j} \lambda(dz_{i+j}) \\ \leq \mathbf{1}_{\mathcal{P}'} \frac{\zeta_1(\eta, p, t)^{\frac{1+2/d}{p}\ell}}{t_{i+\ell} - t_{i-1}} (3|\log z_{i+\ell}| + 1). \end{aligned}$$

This completes integration with respect to (t_i, z_i) for $i \in J_1$ (note that $i + \ell - 1 \in J_1$ implies $i + \ell \in L$ and hence $z_{i+\ell} < \eta$). Recalling (6.30) and bounding $(3|\log z| + 1)^{p/(1+2/d)} \leq 3|\log z| + 1$, we obtain (6.36).

Step 1. Integrating with respect to indices in J_2 . To complete the proof, we need to show that

$$(6.41) \quad U_2(\mathcal{P}, t) \leq \zeta_2(\eta, p)^{k_2} \Lambda(k_2, t, p).$$

To simplify notation, let us assume that $J_2 = \llbracket k \rrbracket$ and hence $k_2 = k$ (again, this simply amounts to relabeling the vertices). Recalling the definition (6.6) of $\zeta_2(\eta, p)$, the bound (6.41) follows once we have shown that if $(\mathbf{t}, \mathbf{z}) \in \mathcal{P}'$, then

$$(6.42) \quad (\Delta t_{k+1})^{\nu_p-1} \prod_{i=1}^k (\Delta t_i)^{\nu_p-1} z_i^p \leq \mathbf{G}_p(\mathbf{t}) \prod_{i=1}^k \left(z_i^p \mathbf{1}_{\{z_i \geq \eta\}} + z_i^{1+\frac{2}{d}} \mathbf{1}_{\{z_i < \eta\}} \right),$$

where, recalling the definition (6.5) of $G_p(s)$, we set

$$\mathbf{G}_p(\mathbf{t}) := \begin{cases} (\Delta t_{k+1})^{\nu_p-1} \prod_{i \in \llbracket k \rrbracket} G_p(\Delta t_i) & \text{if } k \notin L, \\ (\Delta t_k)^{\nu_p-1} \prod_{i \in \llbracket k+1 \rrbracket \setminus \{k\}} G_p(\Delta t_i) & \text{if } k \in L. \end{cases}$$

Note that in both cases, $t^{1-\nu_p}$ times the integral of $\mathbf{G}_p(\mathbf{t})$ over the simplex $\mathfrak{X}_k(t)$ is equal to $\Lambda(k, t, p)$ as defined in (6.4) (in the second case one simply needs to exchange the role of Δt_k and Δt_{k+1}).

The reason why (6.42) is needed is that, on the left-hand side of (6.42), the exponent p makes z_i^p *a priori* not integrable near zero. But since $\nu_p - 1 > -1$, there is some margin for the integrability of $(\Delta t_i)^{\nu_p-1}$: the idea is to use the constraints in \mathcal{P}' to “transfer” a part of the exponent of Δt_i onto that of z_i or z_{i-1} or both.

To this end, we shall use the equivalence

$$(6.43) \quad (\Delta t)^{\frac{d}{6}} \leq z \iff (\Delta t)^{\frac{\nu_p}{3}} \leq z^{1+\frac{2}{d}-p}$$

in two cases: (i) if $i \in I_- \cup I_+$ and $z_i < \eta$, with $z = z_i$ and $\Delta t = \Delta t_i$; (ii) if $i \in L$ (hence $z_i < \eta$), with $z = z_i$ and $\Delta t = \Delta t_{i+1}$. The reader can check that for $(\mathbf{t}, \mathbf{z}) \in \mathcal{P}'$, the left-hand side of (6.43) is satisfied in these two cases. Notice now that if $i \in L$, then necessarily we have $i+1 \in I_+$: indeed, $i+1$ cannot be in I_- since $L_\times^1 = \emptyset$ and it cannot be in L , either, since $i \in J_2$ by assumption. Therefore, we obtain

$$(6.44) \quad \begin{aligned} (\Delta t_{k+1})^{\nu_p-1} \prod_{i=1}^k (\Delta t_i)^{\nu_p-1} z_i^p &\leq \prod_{i \in I_-} (\Delta t_i)^{\nu_p-1-\frac{\nu_p}{3}} \mathbf{1}_{\{z_i < \eta\}} \prod_{i \in I_+} (\Delta t_i)^{\nu_p-1-\frac{\nu_p}{3}} (\mathbf{1}_{\{z_i < \eta\}} + \mathbf{1}_{\{i-1 \in L\}}) \\ &\quad \times \prod_{i \in L} (\Delta t_i)^{\nu_p-1} \prod_{i=1}^k \left(z_i^p \mathbf{1}_{\{z_i \geq \eta\}} + z_i^{1+\frac{2}{d}} \mathbf{1}_{\{z_i < \eta\}} \right). \end{aligned}$$

The bound (6.42) now follows from the fact that

$$\prod_{i \in I_-} (\Delta t_i)^{\nu_p-1-\frac{\nu_p}{3}} \mathbf{1}_{\{z_i < \eta\}} \prod_{i \in I_+} (\Delta t_i)^{\nu_p-1-\frac{\nu_p}{3}} (\mathbf{1}_{\{z_i < \eta\}} + \mathbf{1}_{\{i-1 \in L\}}) \prod_{i \in L} (\Delta t_i)^{\nu_p-1}$$

is bounded by $\mathbf{G}_p(\mathbf{t})$, which is a consequence of the following three observations concerning the exponent of Δt_i in the last display:

- It is larger than or equal to $\frac{\nu_p}{3} - 1$ (which we use if $\Delta t_i \leq 1$).
- It is equal to $\nu_p - 1$ if $\Delta t_i \geq 1 > \eta$ (as can be checked from (6.37)).
- It is equal to $\nu_p - 1$ if $i = k$ and $k \in L$ or if $i = k + 1$ and $k \notin L$ (this follows from the convention $z_{k+1} = \infty$).

This finishes the proof of (6.41), which together with (6.36) shows (6.31). \square

7. PROOF OF THEOREM 2.1 AND UPPER BOUNDS IN THEOREMS 2.13–2.14

7.1. Proof of Theorem 2.1. First let us note that the results for the free-end partition function follows from those for the point-to-point version and Jensen's inequality; see (5.6). The L^p -convergence (2.4) for $p > 1$ is a direct consequence of Corollary 6.5 and the fact that $(\mathcal{Z}_\beta^{\omega,a}(t, x))_{a \in (0,1]}$ is a time-reversed martingale for the filtration \mathcal{G} defined in (1.21). The convergence in L^1 when $\mu < \infty$ cannot be obtained in this manner but can be recovered by a truncation argument. More precisely, we consider the partition function $\mathcal{Z}_\beta^{\omega,[a,b]}(t, x)$ obtained by replacing ω by $\omega \cap (\mathbb{R} \times \mathbb{R}^d \times [0, b))$ (see [13, Equation (2.40) and below]) and reproduce the argument used in [13, Prop. 4.6 and 4.7] but with a moment of order $p \in (1, 1 + \frac{2}{d}) \cap (1, 2]$ instead of 2.

Let us move to the proof of (2.3). Recall that $\omega_<$ and ω_\geq denote the set of points in ω with third coordinate in $(0, 1)$ and $[1, \infty)$, respectively. Consider the reduced partition function

$$\mathcal{Z}_\beta^{\omega_<,a}(t, x) = \rho(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_{\omega_<}^a(dt_i, dx_i).$$

It is easy to obtain that

$$(7.1) \quad \mathcal{Z}_\beta^{\omega,a}(t, x) = \mathcal{Z}_\beta^{\omega_<,a}(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \mathcal{Z}_{\beta,t,x}^{\omega_<,a}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_{\omega_\geq}^a(dt_i, dx_i),$$

where $\mathcal{Z}_{\beta,t,x}^{\omega_<,a}(\mathbf{t}, \mathbf{x}) := \prod_{i=1}^{k+1} \mathcal{Z}_\beta^{\omega_<,a}(t_{i-1}, x_{i-1}; t_i, x_i)$ and $x_0 := 0$, $x_{k+1} := x$, $t_0 := 0$, $t_{k+1} := t$. As a consequence, using Minkowski's inequality, the identity (1.19) and Corollary 6.5, we have for any $p \in (1, 1 + \frac{2}{d})$,

$$\mathbb{E}_< \left[\mathcal{Z}_\beta^{\omega,a}(t, x)^p \right]^{\frac{1}{p}} \leq C(\beta, p, t) \rho(t, x) + \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \prod_{i=1}^{k+1} \mathbb{E}_< \left[\mathcal{Z}_\beta^{\omega_<,a}(\Delta t_i, \Delta x_i)^p \right]^{\frac{1}{p}} \prod_{i=1}^k \xi_{\omega_\geq}^a(dt_i, dx_i).$$

Using Corollary 6.5, we have $\mathbb{E}_< [\mathcal{Z}_\beta^{\omega_<,a}(\Delta t_i, \Delta x_i)^p]^{1/p} \leq C(\beta, p, t) \rho(\Delta t_i, \Delta t_i)$ for every $a \in (0, 1)$. Setting $\beta' := \beta C(\beta, p, t)$, we conclude that

$$(7.2) \quad \mathbb{E}_< \left[\mathcal{Z}_\beta^{\omega,a}(t, x)^p \right]^{\frac{1}{p}} \leq C(\beta, p, t) \mathcal{Z}_{\beta'}^{\omega_\geq}(t, x) < \infty$$

for every $a \in (0, 1)$. As $\mathcal{Z}_\beta^{\omega,a}(t, x)$ is a reversed martingale in a under $\mathbb{P}_<$, this shows (2.3). For the proof of the positivity statement in (2.2), we refer to [13, Section 4.7]. \square

7.2. Proof of Theorem 2.13 (first half: upper bounds). In this section, we prove all upper bounds on $|\bar{\gamma}_\beta(p)|$. By convexity and the fact that $\bar{\gamma}_\beta(1) = 0$, we have for $q \in (0, 1)$ and $p \in (1, 1 + \frac{2}{d})$ that

$$(7.3) \quad \frac{q-1}{p-1} \bar{\gamma}_\beta(p) \leq \bar{\gamma}_\beta(q) \leq 0.$$

Hence it is sufficient to prove only an upper bound on $\bar{\gamma}_\beta(p)$ for $p \in (1, 1 + \frac{2}{d})$. Furthermore, by Proposition 2.10 (ii), we can consider the point-to-point partition function in our computations.

7.2.1. *Dimension $d = 1$.* If $d = 1$ and $p = 2$, the first part of Proposition 6.1 immediately yields

$$\bar{\gamma}_\beta(2) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \left[\rho(t, x)^{-2} \bar{\mathcal{Z}}_\beta^\omega(t, x)^2 \right] = \frac{1}{4} \beta^4 \mu_{0, \infty}(2)^2.$$

The result for other values of $p \in (1, 2)$ follows by convexity, which gives $0 \leq \bar{\gamma}_\beta(p) \leq (p-1)\bar{\gamma}_\beta(2)$.

If $p \in (2, 3)$, we use the bound (6.3) for some $t = t(\beta)$. Indeed, by sub-additivity of $\log \mathbb{E}[\bar{\mathcal{Z}}_\beta^\omega(t, *)^p]$ (see Lemma 5.1), we get that

$$\bar{\gamma}_\beta(p) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \left[\bar{\mathcal{Z}}_\beta^\omega(t, *)^p \right] = \inf_{t > 0} \frac{1}{t} \log \mathbb{E} \left[\bar{\mathcal{Z}}_\beta^\omega(t, *)^p \right].$$

Using (5.6), we therefore get that for any $p > 1$ and any $t > 0$,

$$(7.4) \quad \bar{\gamma}_\beta(p) \leq \frac{1}{t} \log \mathbb{E} \left[\rho(t, 0)^{-p} \bar{\mathcal{Z}}_\beta^\omega(t, 0)^p \right].$$

If $t = t(\beta) = c_p \beta^{-4}$ with a sufficiently small constant c_p , (6.3) implies that

$$\begin{aligned} \mathbb{E} \left[\rho(t, 0)^{-p} \bar{\mathcal{Z}}_\beta^\omega(t, 0)^p \right] &\leq C_p \sum_{k_2=0}^{\infty} \frac{(C_p \beta t^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}}} \left(\sum_{\ell=0}^{\infty} \frac{(C'_p \beta t^{\frac{\nu_p}{p}})^{\ell}}{\Gamma(\nu_p(\ell+1))^{\frac{1}{p}}} \right)^{k_2+1} \\ &\leq 2C_p \sum_{k_2=0}^{\infty} \frac{(2C_p \beta t^{\frac{1}{4}})^{k_2}}{\Gamma(\frac{k_2+1}{2})^{\frac{1}{2}}} \leq 4C_p. \end{aligned}$$

Indeed, since $\frac{1}{p}\nu_p < \frac{1}{4}$ for $p \in (2, 3)$, we can choose c_p sufficiently small so that both the internal sum over ℓ and the sum over k_2 in the second line are bounded by 2. Altogether, thanks to (7.4), we get $\bar{\gamma}_\beta(p) \leq C_p'' \beta^4$ with $C_p'' = c_p^{-1} \log 4C_p$.

7.2.2. *Dimension $d \geq 2$.* If $d \geq 2$, we are going to use Proposition 6.3 with $\eta = 1$ and drop the dependence in η in the notation. Our first task is to replace the functions $\zeta_1(p, t)$, $\zeta_2(p)$ and $\Lambda(k_2, t, p)$ in (6.7) by something more tractable. First, we let C_1 and C_2 be constants, which only depend on the measure λ and are chosen to be larger than 1, such that for every $p \in [1 + \frac{1}{d}, 1 + \frac{2}{d})$ and $t \geq e$, we have

$$(7.5) \quad \left(\frac{C}{p-1} \right)^k \zeta_1(p, t)^{\frac{k_1}{p}} \zeta_2(p)^{\frac{k_2}{p}} \leq \left(C_1 (\log t)^{\frac{1}{1+2/d}} \right)^{k_1} C_2^{k_2},$$

where C is the constant in (6.7). For $\Lambda(k_2, t, p)$, we use the following lemma, which we prove afterwards.

Lemma 7.1. *For any $k \geq 0$, $p \in (1, 1 + \frac{2}{d})$, $t \geq 1$ and $\alpha \in (0, 1)$,*

$$\Lambda(k, t, p) \leq (k+1)^2 (4\nu_p^{-1} \alpha^{-\nu_p})^k e^{\alpha t}.$$

Using Proposition 6.3, (7.5) and Lemma 7.1 (and the bound $(k+1)^{2/q} \leq (k+1)^2$), we have for any $\alpha > 0$ and $q \in (1, 1 + \frac{2}{d})$

$$(7.6) \quad \begin{aligned} &\mathbb{E} \left[\rho(t, 0)^{-q} \bar{\mathcal{Z}}_\beta^\omega(t, 0)^q \right]^{\frac{1}{q}} \\ &\leq e^{\frac{\alpha}{q} t} \sum_{k_1=0}^{\infty} (C_1 \beta (\log t)^{\frac{1}{1+2/d}})^{k_1} \sum_{k_2=0}^{\infty} (k_2+1)^2 \left(4^{\frac{1}{q}} C_2 \beta \nu_q^{-\frac{1}{q}} \alpha^{-\frac{\nu_q}{q}} \right)^{k_2}. \end{aligned}$$

Now, for sufficiently small β we choose $q \in (1 + \frac{1}{d}, 1 + \frac{2}{d})$, $\alpha \in (0, 1)$, $t \geq e$ in the following way (the dependence in β may sometimes be omitted in the computations below, for readability):

$$(7.7) \quad \begin{aligned} q(\beta) &:= 1 + \frac{2}{d} - \frac{8e}{d} (2C_2 \beta)^{1+\frac{2}{d}}, & \alpha(\beta) &:= (4(2C_2 \beta)^q \nu_q^{-1})^{\frac{1}{\nu_q}}, \\ t(\beta) &:= e^{(2C_1 \beta)^{-(1-\frac{2}{d})}}. \end{aligned}$$

Note that with this choice we have $\nu_q = 4e(2C_2\beta)^{1+\frac{2}{d}}$ and thus

$$(7.8) \quad \alpha = (2C_2\beta)^{-\frac{2}{d}} \exp(-(4e(2C_2\beta)^{1+\frac{2}{d}})^{-1}),$$

and

$$(7.9) \quad C_1\beta(\log t)^{\frac{1}{1+2/d}} = \frac{1}{2} \quad \text{and} \quad 4^{\frac{1}{q}}C_2\beta\nu_q^{-\frac{1}{q}}\alpha^{-\frac{\nu_q}{q}} = \frac{1}{2},$$

so that the sums in k_1 and k_2 in (7.6) are respectively equal to 2 and 12. We end up with

$$(7.10) \quad \mathbb{E}[\rho(t, 0)^{-q} \bar{Z}_\beta^\omega(t, 0)^q]^{\frac{1}{q}} \leq 24 e^{\frac{\alpha}{q}t}.$$

For a fixed $p \in (1, 1 + \frac{2}{d})$, we can now deduce the upper bound in (2.16) from (7.10). We consider β sufficiently small so that $q = q(\beta) \geq p$. Using first (7.4) and then Jensen's inequality, we get

$$(7.11) \quad \begin{aligned} \bar{\gamma}_\beta(p) &\leq \frac{1}{t} \log \mathbb{E}[\rho(t, 0)^{-p} \bar{Z}_\beta^\omega(t, 0)^p] \leq \frac{p}{t} \log \mathbb{E}[\rho(t, 0)^{-q} \bar{Z}_\beta^\omega(t, 0)^q]^{\frac{1}{q}} \\ &\leq \frac{p \log 24}{t} + \frac{p\alpha}{q}. \end{aligned}$$

Using (7.8) and replacing t by its value (7.7) yield the upper bound in (2.16) since $p/q \leq 1$. Let us finally comment on our choice (7.7). The parameters α and t have been chosen so that (7.9), hence (7.10), holds. The value of $q = q(\beta)$ is chosen so that α , seen as a function of q for fixed β , is minimized. This completes the proof of the upper bounds in (2.15) and (2.16). \square

The lower bounds in (2.15) and (2.16) will be shown after Proposition 9.5.

Proof of Lemma 7.1. Recall the definitions (6.4) and (6.5) of $\Lambda(k, t, p)$ and $G_p(s)$. For fixed $j \in \llbracket k \rrbracket$, using a symmetry argument, we have that

$$\begin{aligned} \int_{\mathfrak{X}_k(t)} \mathbf{1}_{\{\Delta t_{k+1} = \max_{i \in \llbracket k+1 \rrbracket} \Delta t_i\}} (\Delta t_{k+1})^{\nu_p-1} \prod_{i=1}^k G_p(\Delta t_i) dt_i \\ = \int_{\mathfrak{X}_k(t)} \mathbf{1}_{\{\Delta t_j = \max_{i \in \llbracket k+1 \rrbracket} \Delta t_i\}} (\Delta t_j)^{\nu_p-1} \prod_{i \in \llbracket k+1 \rrbracket \setminus \{j\}} G_p(\Delta t_i) dt_i \\ \geq \int_{\mathfrak{X}_k(t)} \mathbf{1}_{\{\Delta t_j = \max_{i \in \llbracket k+1 \rrbracket} \Delta t_i\}} (\Delta t_{k+1})^{\nu_p-1} \prod_{i \in \llbracket k \rrbracket} G_p(\Delta t_i) dt_i, \end{aligned}$$

where we have used that $(\Delta t_{k+1})^{\nu_p-1} G_p(\Delta t_j) \leq (\Delta t_j)^{\nu_p-1} G_p(\Delta t_{k+1})$ if $\Delta t_{k+1} \leq \Delta t_j$ for the second step. As a consequence, again by a symmetry argument, we have

$$\Lambda(k, t, p) \leq (k+1) t^{1-\nu_p} \int_{\mathfrak{X}_k(t)} \mathbf{1}_{\{\Delta t_{k+1} = \max_{i \in \llbracket k+1 \rrbracket} \Delta t_i\}} (\Delta t_{k+1})^{\nu_p-1} \prod_{i=1}^k G_p(\Delta t_i) dt_i.$$

Together with the inequality $\max_{i \in \llbracket k+1 \rrbracket} \Delta t_i \geq t/(k+1)$, this implies

$$\Lambda(k, t, p) \leq (k+1)^{2-\nu_p} \int_{\mathfrak{X}_k(t)} \prod_{i=1}^k G_p(\Delta t_i) dt_i.$$

We can conclude our proof by observing that

$$\int_{\mathfrak{X}_k(t)} \prod_{i=1}^k G_p(\Delta t_i) dt_i \leq e^{\alpha t} \left(\int_0^\infty G_p(s) e^{-\alpha s} ds \right)^k$$

and

$$\int_0^\infty G_p(s) e^{-\alpha s} ds \leq \int_0^1 s^{\frac{1}{3}\nu_p-1} ds + \int_0^\infty s^{\nu_p-1} e^{-\alpha s} ds = 3\nu_p^{-1} + \Gamma(\nu_p) \alpha^{-\nu_p},$$

which is less than $4\nu_p^{-1}\alpha^{-\nu_p}$ because $\nu_p \in (0, 1)$ and $\alpha \in (0, 1)$. \square

7.3. Proof of Theorem 2.14 (first half: upper bounds). Again, by (7.3) it suffices to prove (2.18) for $p > 1$. Moreover, (2.19) follows immediately from (2.18). If $d \geq 2$, the arguments from Section 7.2 remain valid if we let q be equal to the value corresponding to our assumption $\mu_{1,\infty}(q) < \infty$ (we consider C_1 and C_2 such that (7.5) is valid for this value of q) but retain the choice in (7.7) for α and t . The conclusion (7.11) is still valid for any $p \in (1, q]$, and because $\alpha(\beta) = C_3\beta^{q/\nu_q}$, (2.18) follows.

If $d = 1$, it suffices by convexity to treat the case when $p = q$. By (6.2) (with $\eta = 1$) and our assumptions, there are constants C_1 and C_2 , which may depend on q , such that

$$(7.12) \quad \mathbb{E} \left[\rho(t, x)^{-q} \bar{\mathcal{Z}}_\beta^\omega(t, x)^q \right]^{\frac{1}{q}} \leq \sum_{k_1=0}^{\infty} (C_1 \beta t^{\frac{1}{4}})^{k_1} \sum_{k_2=0}^{\infty} \frac{(C_2 \beta t^{\frac{\nu_q}{q}})^{k_2}}{\Gamma(\nu_q(k_2 + 1))^{\frac{1}{q}}}.$$

Let us choose $t = t(\beta) := (2C_1)^4 \beta^{-4}$ so that the first sum is equal to 2.

For the second sum, consider the three-parameter function $F_{\alpha,\delta}^{(\gamma)}(x) = \sum_{m=0}^{\infty} x^m / \Gamma(\alpha m + \delta)^\gamma$. A precise asymptotic of $F_{\alpha,\delta}^{(\gamma)}$ is given in [60, Thm. 1], from which one gets that there is $K_{\alpha,\delta}^{(\gamma)} > 0$ such that for all $x \geq 0$,

$$(7.13) \quad F_{\alpha,\delta}^{(\gamma)}(x) \leq K_{\alpha,\delta}^{(\gamma)} e^{2\gamma x^{\frac{1}{\alpha\gamma}}}.$$

Thanks to this estimate, the second sum in (7.12) is bounded by $C_3(1 + \beta^{c_q}) \exp(\frac{1}{q}(C_2\beta)^{q/\nu_q}t)$ with $c_q = (q-1)(\frac{1}{\nu_q} - \frac{4}{q}) < 0$. By (7.4), we obtain

$$\bar{\gamma}_\beta(q) \leq \frac{q \log(2C_3(1 + \beta^{c_q}))}{t(\beta)} + C_2 \beta^{\frac{q}{\nu_q}} \leq C_4 \beta^4 \log(1 + \beta^{-1}) + C_2 \beta^{\frac{q}{\nu_q}}.$$

Recalling that $q/\nu_q < 4$ if $p < 2$, this gives the upper bound in (2.18). Finally, the lower bound in (2.22) is a direct consequence of (2.19) since $\lim_{z \rightarrow \infty} \log \lambda([z, \infty)) / \log z = \alpha$ implies $\mu_{1,\infty}(q) < \infty$ for all $q < \alpha$. \square

The second half of Theorem 2.14 will be shown after Proposition 9.5.

8. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE SHE

This section is devoted to the proof of Theorems 2.6 and 2.8. The existence part, that is, Theorem 2.6 is detailed in the Sections 8.1–8.4. Theorem 2.8, is addressed afterwards: in Section 8.5, we verify that the solution defined in Theorem 2.6 satisfies the condition (2.10), while uniqueness is shown in Section 8.6.

8.1. Overview of the proof of Theorem 2.6. For the ease of exposition, we first present the case where the initial condition u_0 is the Dirac function δ_0 . By (1.17), we have

$$(8.1) \quad \begin{aligned} \mathcal{Z}_\beta^{\omega,a}(t, x) &= \rho(t, x) + \beta \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega,a}(s, y) \xi_{\omega_{<}}^a(ds, dy) \\ &\quad + \beta \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega,a}(s, y) \xi_{\omega_{\geq}}^a(ds, dy). \end{aligned}$$

We want to let a tend to 0 and conclude that

$$(8.2) \quad \begin{aligned} \mathcal{Z}_\beta^\omega(t, x) &= \rho(t, x) + \beta \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y) \xi_{\omega_{<}}(ds, dy) \\ &\quad + \beta \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y) \xi_{\omega_{\geq}}(ds, dy). \end{aligned}$$

We have to prove that the two integrals in the on the right-hand side are well defined in Itô's and Lebesgue's sense, respectively (cf. Remark 1.2), that they are finite and that they are the limit of the integrals displayed in (8.1). For this last point, setting

$$\begin{aligned} I_1^a(t, x) &:= \int_{X_t} \rho(t-s, x-y) (\mathcal{Z}_\beta^{\omega, a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)) \xi_{\omega_<}^a(ds, dy), \\ I_2^a(t, x) &:= \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y) \xi_{\omega_<}^{[0, a)}(ds, dy), \\ I_3^a(t, x) &:= \int_{X_t} \rho(t-s, x-y) (\mathcal{Z}_\beta^{\omega, a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)) \xi_{\omega_\geq}(ds, dy), \end{aligned}$$

with $\xi_\omega^{[a, b)} := \xi_\omega^a - \xi_\omega^b$ (and $\xi_\omega^0 := \xi_\omega$), we need to prove that for all (t, x) and $j = 1, 2, 3$, the following convergence holds in probability:

$$(8.3) \quad \lim_{a \rightarrow 0} I_j^a(t, x) = 0.$$

The finiteness and convergence of these integrals are proved in Section 8.2 and Section 8.3, respectively. The case of general initial condition is discussed in Section 8.4.

8.2. Finiteness of stochastic integrals. We first introduce a technical statement that is a direct consequence of Corollary 6.5.

Lemma 8.1. *Assuming (1.15) and (2.1), for all $p \in (1, 1 + \frac{2}{d})$, $T > 0$, $x \in \mathbb{R}^d$ and $\beta > 0$, there exist positive constants $\beta' = \beta'(\beta, p, T)$ and $C = C(\beta, p, T)$ such that for every $t \in [0, T]$,*

$$\mathbb{E}_< \left[\sup_{a \in (0, 1)} \mathcal{Z}_\beta^{\omega, a}(t, x)^p \right]^{\frac{1}{p}} \leq C \mathcal{Z}_{\beta'}^{\omega_\geq}(t, x) \quad \mathbb{P}_\geq\text{-a.s.}$$

Proof. By Doob's maximal inequality, it suffices to bound $\mathbb{E}_< [\mathcal{Z}_\beta^{\omega, a}(t, x)^p]^{1/p}$ uniformly in a , t and x . With the conventions that $t_{k+1} := t$, $x_{k+1} := x$ and the term corresponding to $k = 0$ is equal to $\mathcal{Z}_\beta^{\omega_<, a}(t, x)$, we have that

$$(8.4) \quad \mathcal{Z}_\beta^{\omega, a}(t, x) = \sum_{k=0}^{\infty} \beta^k \int_{X_t^{(k)}} \prod_{i=1}^{k+1} \mathcal{Z}_\beta^{\omega_<, a}(t_{i-1}, x_{i-1}; t_i, x_i) \prod_{j=1}^k \xi_{\omega_\geq}(dt_j, dx_j).$$

Averaging with respect to $\omega_<$ and using the Minkowski inequality, translation invariance and Corollary 6.5 (we let $C_0 = C_0(\beta, p, T)$ denote the constant obtained from Corollary 6.5), we can bound $\mathbb{E}_< [\mathcal{Z}_\beta^{\omega, a}(t, x)^p]^{1/p}$ by

$$\begin{aligned} & \sum_{k=0}^{\infty} \beta^k \int_{X_t^{(k)}} \prod_{i=1}^{k+1} \mathbb{E}_< \left[\mathcal{Z}_\beta^{\omega_<, a}(\Delta t_i, \Delta x_i)^p \right]^{\frac{1}{p}} \prod_{j=1}^k \xi_{\omega_\geq}(dt_j, dx_j) \\ & \leq C_0 \sum_{k=0}^{\infty} [C_0 \beta]^k \int_{X_t^{(k)}} \rho_{t, x}(\mathbf{t}, \mathbf{x}) \prod_{j=1}^k \xi_{\omega_\geq}(dt_j, dx_j) = C_0 \mathcal{Z}_{C_0 \beta}^{\omega_\geq}(t, x). \quad \square \end{aligned}$$

Remark 8.2. In this section and the next, the dependence in β, p and T of the constants does not have a major importance: most of the time, they are omitted in computations.

We now proceed to showing that the integrals in (8.2) are well defined and finite. The integral I_3^a is a Lebesgue integral. Because of (2.2), it clearly suffices to show that

$$(8.5) \quad \int_{X_t} \rho(t-s, x-y) \sup_{a \in (0, 1)} \mathcal{Z}_\beta^{\omega, a}(s, y) \xi_{\omega_\geq}(ds, dy) < \infty.$$

In fact, we are going to show that the $L^p(\mathbb{P}_<)$ -norm of the quantity in (8.5) is finite \mathbb{P}_\geq -a.s. By Minkowski's integral inequality and Lemma 8.1, we have

$$\begin{aligned}
 (8.6) \quad \mathbb{E}_< \left[\left(\int_{X_t} \rho(t-s, x-y) \sup_{a \in (0,1)} \mathcal{Z}_\beta^{\omega,a}(s,y) \xi_{\omega_\geq}(ds, dy) \right)^p \right]^{\frac{1}{p}} \\
 \leq \int_{X_t} \rho(t-s, x-y) \mathbb{E}_< \left[\sup_{a \in (0,1)} \mathcal{Z}_\beta^{\omega,a}(s,y)^p \right]^{\frac{1}{p}} \xi_{\omega_\geq}(ds, dy) \\
 \leq C \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_{\beta'}^{\omega_\geq}(s,y) \xi_{\omega_\geq}(ds, dy).
 \end{aligned}$$

By (8.1), applied with $a = 1$, the integral in the last line is equal to $(\mathcal{Z}_{\beta'}^{\omega_\geq}(t, x) - \rho(t, x))/\beta'$, which is finite and therefore proves (8.5).

Let us now deal with I_1^a and I_2^a . We show that for \mathbb{P}_\geq -a.e. realization of ω_\geq , the process $(\omega_<, s, y) \mapsto \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega_< \cup \omega_\geq}(s, y)$ is $L^p(\mathbb{P}_<)$ -integrable with respect to $\xi_{\omega_<}$, in the sense described in Section 4.3. Thanks to Lemma 4.4, this implies that $(\omega, s, y) \mapsto \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y)$ is $L^0(\mathbb{P})$ -integrable with respect to $\xi_{\omega_<}$. To this end, according to (4.6) and (4.7), we only need to prove that for some $p \in (1, 1 + \frac{2}{d})$, we have

$$(8.7) \quad \mathbb{E}_< \left[\left(\int_{\mathbb{X}_t} \rho(t-s, x-y)^2 \mathcal{Z}_\beta^\omega(s, y)^2 z^2 \delta_{\omega_<}(dw) \right)^{\frac{p}{2}} \right] < \infty.$$

Using Fatou's Lemma for the inner integral, we can replace $\mathcal{Z}_\beta^\omega(s, y)$ by $\mathcal{Z}_\beta^{\omega,a}(s, y)$ and add a restriction $\mathbf{1}_{\{z \geq a\}}$ provided that the bound we prove is uniform in a . Using the BDG inequality in the reverse direction, we obtain

$$\begin{aligned}
 (8.8) \quad \mathbb{E}_< \left[\left(\int_{\mathbb{X}_t} \rho(t-s, x-y)^2 \mathcal{Z}_\beta^{\omega,a}(s, y)^2 z^2 \mathbf{1}_{\{z \geq a\}} \delta_{\omega_<}(dw) \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
 \leq C \mathbb{E}_< \left[\left| \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega,a}(s, y) \xi_{\omega_<}^a(ds, dy) \right|^p \right]^{\frac{1}{p}}.
 \end{aligned}$$

By (8.1), β times the last expectation is bounded by

$$(8.9) \quad \rho(t, x) + \mathbb{E}_< \left[\mathcal{Z}_\beta^{\omega,a}(t, x)^p \right]^{\frac{1}{p}} + \beta \mathbb{E}_< \left[\left(\int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega,a}(s, y) \xi_{\omega_\geq}(ds, dy) \right)^p \right]^{\frac{1}{p}}.$$

We conclude by observing that all terms are uniformly bounded in a : we use Lemma 8.1 for the second one, while the third one has been controlled in the previous paragraph (see (8.6)). \square

8.3. Convergence of stochastic integrals. We give the proof of (8.3) in this section. The convergence of I_3^a follows from (8.5) and dominated convergence. The convergence of I_2^a follows from dominated convergence for stochastic integrals. The main piece of work consists in proving the convergence of I_1^a . We define

$$\begin{aligned}
 J_1^{a,b}(t, x) &:= \int_{X_t} \rho(t-s, x-y) (\mathcal{Z}_\beta^{\omega,a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)) \xi_{\omega_<}^{[a,b]}(ds, dy), \\
 J_2^{a,b}(t, x) &:= \int_{X_t} \rho(t-s, x-y) (\mathcal{Z}_\beta^{\omega,a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)) \xi_{\omega_<}^b(ds, dy),
 \end{aligned}$$

and we prove that the following holds in \mathbb{P}_\geq -probability:

$$\lim_{b \rightarrow 0} \sup_{a \in (0,1)} \mathbb{E}_< [|J_1^{a,b}(t, x)|] = 0 \quad \text{and} \quad \lim_{a \rightarrow 0} \mathbb{E}_< [|J_2^{a,b}(t, x)|] = 0.$$

Convergence of $J_2^{a,b}(t, x)$. By Jensen's inequality, it suffices to prove that $\mathbb{E}_<[|J_2^{a,b}(t, x)|^p] \rightarrow 0$ for some $p \in (1, \min(2, 1 + \frac{2}{d}))$. By the BDG inequality, we can further reduce this to proving

$$(8.10) \quad \lim_{a \rightarrow 0} \mathbb{E}_< \left[\left(\int_{X_t \times [b, 1]} \rho(t-s, x-y)^2 (\mathcal{Z}_\beta^{\omega, a}(s, y) - \mathcal{Z}_\beta^\omega(s, y))^2 z^2 \delta_{\omega_<}(\mathrm{d}w) \right)^{\frac{p}{2}} \right] = 0.$$

By subadditivity (5.4), for $a < b$ this quantity is smaller than

$$\begin{aligned} \mathbb{E}_< \left[\int_{X_t \times [b, 1]} \rho(t-s, x-y)^p |\mathcal{Z}_\beta^{\omega, a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)|^p z^p \delta_{\omega_<}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \right] \\ = \mu_{b,1}(p) \int_{X_t} \rho(t-s, x-y)^p \mathbb{E}_< \left[|\mathcal{Z}_\beta^{\omega, a}(s, y) - \mathcal{Z}_\beta^\omega(s, y)|^p \right] \mathrm{d}s \mathrm{d}y. \end{aligned}$$

As the integrand tends to zero thanks to (2.3), by dominated convergence we will get that the last integral converges to 0 for \mathbb{P}_\geq -a.s. if we show that for \mathbb{P}_\geq -a.e. realization of ω_\geq ,

$$(8.11) \quad \int_{X_t} \rho(t-s, x-y)^p \mathbb{E}_< \left[\sup_{a \in (0, 1)} \mathcal{Z}_\beta^{\omega, a}(s, y)^p \right] \mathrm{d}s \mathrm{d}y < \infty.$$

To this end, we use Lemma 8.1 to bound the left-hand side of (8.11) by a constant times

$$(8.12) \quad \begin{aligned} & \int_{X_t} \rho(t-s, x-y)^p \mathcal{Z}_{\beta'}^{\omega_\geq}(s, y)^p \mathrm{d}s \mathrm{d}y \\ & \leq \left(\sum_{k=0}^{\infty} (\beta')^k \int_{X_t^{(k)}} \left(\int_{(t_k, t) \times \mathbb{R}^d} \rho(t-s, x-y)^p \rho(s-t_k, y-x_k)^p \mathrm{d}s \mathrm{d}y \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \prod_{i=1}^k \rho(\Delta t_i, \Delta x_i) \xi_{\omega_\geq}(\mathrm{d}t_i, \mathrm{d}x_i) \right)^p, \end{aligned}$$

where the last inequality follows from Minkowski's inequality (for the L^p -norm of the measure $\rho(t-s, x-y)^p \mathrm{d}s \mathrm{d}y$). Recalling (5.5) and using Lemma 4.5, one has

$$(8.13) \quad \begin{aligned} \int_{(t_k, t) \times \mathbb{R}^d} \rho(t-s, x-y)^p \rho(s-t_k, y-x_k)^p \mathrm{d}s \mathrm{d}y &= \vartheta(p)^2 \frac{\Gamma(\nu_p)^2}{\Gamma(2\nu_p)} (t-t_k)^{2\nu_p-1} \rho\left(\frac{1}{p}(t-t_k), x-x_k\right) \\ &= \vartheta(p) \frac{\Gamma(\nu_p)^2}{\Gamma(2\nu_p)} (t-t_k)^{\nu_p} \rho(t-t_k, x-x_k)^p. \end{aligned}$$

Thus, the right-hand side of (8.12) is bounded by $\vartheta(p)\Gamma(\nu_p)^2\Gamma(2\nu_p)^{-1}t^{\nu_p}$ times

$$(8.14) \quad \left(\sum_{k=0}^{\infty} (\beta')^k \int_{\mathfrak{X}_k(t) \times (\mathbb{R}^d)^k} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_{\omega_\geq}(\mathrm{d}t_i, \mathrm{d}x_i) \right)^p = \mathcal{Z}_{\beta'}^{\omega_\geq}(t, x)^p,$$

which is finite. This concludes the proof of (8.10). \square

Convergence of $J_1^{a,b}(t, x)$. Using Jensen's inequality, we need to show that for some $p \in (1, 1 + \frac{2}{d})$,

$$(8.15) \quad \begin{aligned} \lim_{b \rightarrow 0} \sup_{a \in (0, 1)} \mathbb{E}_< \left[\left| \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^{\omega, a}(s, y) \xi_{\omega_<}^{[a, b]}(\mathrm{d}s, \mathrm{d}y) \right|^p \right] &= 0, \\ \lim_{b \rightarrow 0} \sup_{a \in (0, 1)} \mathbb{E}_< \left[\left| \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y) \xi_{\omega_<}^{[a, b]}(\mathrm{d}s, \mathrm{d}y) \right|^p \right] &= 0. \end{aligned}$$

For the second line, we apply the BDG inequality and realize that

$$\begin{aligned} \sup_{a \in (0,1)} \mathbb{E}_< \left[\left| \int_{X_t} \rho(t-s, x-y) \mathcal{Z}_\beta^\omega(s, y) \xi_{\omega_<}^{[a,b]}(ds, dy) \right|^p \right] \\ \leq C \mathbb{E}_< \left[\left(\int_{X_t \times [0,b]} \rho(t-s, x-y)^2 \mathcal{Z}_\beta^\omega(s, y)^2 z^2 \delta_{\omega_<}(dw) \right)^{\frac{p}{2}} \right]. \end{aligned}$$

By (8.7), this converges to 0 as $b \rightarrow 0$ by dominated convergence. Concerning the first line in (8.15), we rely on the proof of Propositions 6.1 and 6.3 but with a small variation. Let us define a modified partition function by

$$\mathcal{Y}_\beta^{\omega, a, b}(s, x; t, y) := \sum_{k=1}^{\infty} \beta^{k-1} \int_{X_{s,t}^{(k)}} \rho_{s,x;t,y}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^{k-1} \xi_\omega^a(dt_i, dx_i) \xi_\omega^{[a,b]}(dt_k, dx_k)$$

and $\mathcal{Y}_\beta^{\omega, a, b}(t, x) := \mathcal{Y}_\beta^{\omega, a, b}(0, 0; t, x)$. Then, the $\mathbb{E}_<[(\cdot)]$ -term in the first line of (8.15) is simply $\mathbb{E}_<[|\mathcal{Y}_\beta^{\omega, a, b}(t, x)|^p]$. In analogy with (7.1),

$$\begin{aligned} \mathcal{Y}_\beta^{\omega, a, b}(t, x) &= \mathcal{Y}_\beta^{\omega_<, a, b}(t, x) \\ &+ \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \mathcal{Y}_\beta^{\omega_<, a, b}(t_k, x_k; t, x) \prod_{i=1}^k \mathcal{Z}_\beta^{\omega_<, a}(t_{i-1}, x_{i-1}; t_i, x_i) \xi_{\omega_\geq}(dt_i, dx_i). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \mathbb{E}_<[|\mathcal{Y}_\beta^{\omega, a, b}(t, x)|^p]^{\frac{1}{p}} &\leq \mathbb{E}_<[\mathcal{Y}_\beta^{\omega_<, a, b}(t, x)^p]^{\frac{1}{p}} \\ &+ \sum_{k=1}^{\infty} \beta^k \int_{X_t^{(k)}} \mathbb{E}_<[|\mathcal{Y}_\beta^{\omega_<, a, b}(\Delta t_{k+1}, \Delta x_{k+1})|^p]^{\frac{1}{p}} \prod_{i=1}^k \mathbb{E}_<[\mathcal{Z}_\beta^{\omega_<, a}(\Delta t_i, \Delta x_i)^p]^{\frac{1}{p}} \xi_{\omega_\geq}(dt_i, dx_i). \end{aligned}$$

We then need the following estimate, which comes from a small adaptation of the proofs of Propositions 6.1 and 6.3 and Corollary 6.5. The proof of this lemma is postponed to the end of the section.

Lemma 8.3. *Assume that $\lambda([1, \infty)) = 0$. Given $\beta, T > 0$, and $p \in (1, 1 + \frac{2}{d})$ there exists a function $\delta(b) = \delta_{\beta, p, T}(b)$ such that $\lim_{b \rightarrow 0} \delta(b) = 0$ and*

$$\sup_{t \in (0, T]} \sup_{x \in \mathbb{R}^d} \sup_{a \in (0, 1]} \mathbb{E} \left[\rho(t, x)^{-p} |\mathcal{Y}_\beta^{\omega, a, b}(t, x)|^p \right]^{\frac{1}{p}} \leq \delta(b).$$

By Corollary 6.5 and Lemma 8.3, we obtain $C_0 = C_0(\beta, p, T)$ such that

$$\begin{aligned} \mathbb{E}_<[|\mathcal{Y}_\beta^{\omega, a, b}(t, x)|^p]^{\frac{1}{p}} &\leq \delta(b) \sum_{k=0}^{\infty} (C_0 \beta)^k \int_{X_t^{(k)}} \rho_{t,x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k \xi_{\omega_\geq}(dt_i, dx_i) \\ &= \delta(b) \mathcal{Z}_{\beta'}^{\omega_\geq}(t, x), \end{aligned}$$

with $\beta' := C_0 \beta$. Since $\mathcal{Z}_{\beta'}^{\omega_\geq}(t, x)$ is \mathbb{P}_\geq -a.s. finite, this shows the first line of (8.15) and the proof of the convergence to zero of $J_1^{a, b}(t, x)$ is complete. \square

Proof of Lemma 8.3. The proof is considerably easier for $d = 1$, so we only provide details in the case $d \geq 2$. We follow exactly the plan of the proof of Proposition 6.3 up to Equation (6.14).

Because of the restriction on the last iteration of the integral over the noise, the quantity we need to bound is not $\mathbb{E}[|V_{a,k}(t)^p|]$ but rather $\mathbb{E}[|V_{a,k}^b(t)|^p]$ where

$$V_{a,k}^b(t) := \int_{\mathbb{X}_t^{(k)}} \frac{\rho_{t,0}(\mathbf{t}, \mathbf{x})}{\rho(t, 0)} \mathbf{1}_{\{z_k < b\}} \prod_{i=1}^k z_i \mathbf{1}_{\{z_i \geq a\}} (\delta_{\omega_i} - \nu)(d\omega_i).$$

To conclude, we only need to slightly improve the bounds in further computations. The reader can check that the proof of Lemma 6.7 yields the inequality (6.16) for $V_k^b(t)$ with an additional $\mathbf{1}_{\{z_k < b\}}$ in each of the integrals on the right-hand side. Let $\mathcal{U}(\mathcal{P}, t, b)$ denote the corresponding integrals in the case $\theta = 1 + \frac{2}{d}$, cf. (6.30). Adapting the proof of Proposition 6.10, we want to prove that

$$(8.16) \quad \mathcal{U}(\mathcal{P}, t, b) \leq \varepsilon(b, \eta) \zeta_1(\eta, p, t)^{k_1} \zeta_2(\eta, p)^{k_2} \Lambda(k_2, t, p)$$

for some $\varepsilon(b, \eta)$ that converges to 0 as $b \rightarrow 0$ for any fixed $\eta \in (0, 1)$. Because of (8.16), the factor $\varepsilon(b, \eta)$ appears on the right-hand side of (6.32) so that, provided that η has been chosen sufficiently small, we obtain that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $a \in (0, b)$,

$$\mathbb{E}[\rho(t, x)^{-p} \mathcal{Y}_\beta^{\omega, a, b}(t, x)^p] \leq \beta^{-1} \varepsilon(b, \eta) C(\beta, p, T)$$

with the constant $C(\beta, p, T)$ of Corollary 6.5. This is exactly the desired result. Note that the factor β^{-1} comes from the fact that in the definition of $\mathcal{Y}_\beta^{\omega, a, b}(t, x)$, we have β^{k-1} instead of β^k .

To prove (8.16), we follow the proof of Proposition 6.10 and realize that we can improve the upper bound by a factor of $(\mu_{0,b}^{\log}(1 + \frac{2}{d}) / \mu_{0,\eta}^{\log}(1 + \frac{2}{d}))^{p/(1+2/d)}$ if $k \in J_1$ and a factor of $\mu_{0,b}^{\log}(1 + \frac{2}{d}) / \zeta_2(\eta, p)$ if $k \in J_2$, so (8.16) is valid for

$$\varepsilon(b, \eta) := \left(\frac{\mu_{0,b}^{\log}(1 + \frac{2}{d})}{\mu_{0,\eta}^{\log}(1 + \frac{2}{d})} \right)^{\frac{p}{1+2/d}} \vee \frac{\mu_{0,b}^{\log}(1 + \frac{2}{d})}{\zeta_2(\eta, p)}. \quad \square$$

8.4. General initial condition. Using translation invariance, we have already shown that the point-to-point partition function $\mathcal{Z}_\beta^\omega(y; t, x)$ is a solution to (1.6) with $u_0 = \delta_y$, for all $y \in \mathbb{R}^d$. The fact that v as defined in (2.8) is a mild solution to the SHE with initial condition u_0 follows immediately by integrating (2.6) on both sides with respect to u_0 , provided that integrals can be permuted in the following manner:

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_{(0,t) \times \mathbb{R}^d} \rho(t-s, x-y') \mathcal{Z}_\beta^\omega(y; s, y') \xi_\omega(ds, dy') \right) u_0(dy) \\ = \int_{(0,t) \times \mathbb{R}^d} \rho(t-s, x-y') \left(\int_{\mathbb{R}^d} \mathcal{Z}_\beta^\omega(y; s, y') u_0(dy) \right) \xi_\omega(ds, dy'). \end{aligned}$$

Writing $\xi_\omega = \xi_{\omega_{<}} + \xi_{\omega_{\geq}}$, we can use the ordinary Fubini theorem for the $\xi_{\omega_{\geq}}$ -integral and a stochastic version of Fubini's theorem [37, Thm. A.3] for the integration with respect to $\xi_{\omega_{<}}$. Both cases require some integrability properties, which we are going to show by recycling the estimates from Section 8.2.

Concerning the integral with respect to $\xi_{\omega_{\geq}}$, we simply need to check integrability in the Lebesgue sense, that is,

$$\int_{\mathbb{R}^d} \int_{X_t} \rho(t-s, x-y') \mathcal{Z}_\beta^\omega(y; s, y') \xi_{\omega_{\geq}}(ds, dy') |u_0|(dy) < \infty \quad \mathbb{P}_{\geq}\text{-a.s.}$$

By Minkowski's integral inequality and Lemma 8.1, for $p \in (1, 1 + \frac{2}{d})$, the $L^p(\mathbb{P}_<)$ -norm of the left-hand side is bounded by a constant times

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{X_t} \rho(t-s, x-y') \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; s, y') \xi_{\omega_{\geq}}(ds, dy') |u_0|(dy) \\ = (\beta')^{-1} \left(\int_{\mathbb{R}^d} \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; t, x) |u_0|(dy) - \int_{\mathbb{R}^d} \rho(t, x-y) |u_0|(dy) \right), \end{aligned}$$

where the equality follows from (8.1) with $a = 1$. The second integral on the right-hand side is finite as a consequence of our assumption (2.7). The first one is finite by [13, Prop. 2.21].

For the integral with respect to $\xi_{\omega_{<}}$, according to [37, Thm. A.3], changing the order of integration is permitted if

$$\int_{\mathbb{R}^d} \mathbb{E}_{<} \left[\left(\int_{\mathbb{X}_t} \rho(t-s, x-y')^2 \mathcal{Z}_{\beta}^{\omega}(y; s, y')^2 z^2 \delta_{\omega_{<}}(ds, dy', dz) \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} |u_0|(dy)$$

is finite. Using (8.8) and (8.9), together with Lemma 8.1 and (8.6), we can bound the quantity above by a constant times

$$\int_{\mathbb{R}^d} \left(\rho(t, x-y) + \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; t, x) + \int_{X_t} \rho(t-s, x-y') \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; s, y') \xi_{\omega_{\geq}}(ds, dy') \right) |u_0|(dy).$$

The integrability of each of the three terms has been shown above. \square

8.5. Integrability property of the solution. Let us show that the solution v defined in Equation (2.8) satisfies the integrability condition (2.10). The important part is to show that whenever u_0 satisfies (2.7) for a given $T > 0$, then for any $p \in (1, 1 + \frac{2}{d})$, we have

$$(8.17) \quad \int_{X_T} \rho(\theta(T-t), x)^p \mathbb{E}_{<} \left[\left| \int_{\mathbb{R}^d} \mathcal{Z}_{\beta}^{\omega}(y; t, x) u_0(dy) \right|^p \right] dt dx < \infty.$$

In our proof of (8.17), we will check along the way that the expectation term, which equals $\mathbb{E}_{<}[|v(t, x)|^p]$, is finite for any t and x , proving the first line of (2.10). By (2.7), we can find $\varepsilon > 0$ such that

$$(8.18) \quad 2T \limsup_{r \rightarrow \infty} r^{-2} \log \left(|u_0|([-r, r]^d) \right) < 1 - 2\varepsilon.$$

We also let $\theta := (1 - \varepsilon)^{-1}$. In the following proof, the parameters $p \in (1, 1 + \frac{2}{d})$, T , β and λ and ε are all fixed, and the constants appearing in the inequalities may depend on them.

Using Minkowski's inequality first and Lemma 8.1 afterwards, we have for any $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}_{<} \left[\left| \int_{\mathbb{R}^d} \mathcal{Z}_{\beta}^{\omega}(y; t, x) u_0(dy) \right|^p \right]^{\frac{1}{p}} &\leq \int_{\mathbb{R}^d} \mathbb{E}_{<} \left[\mathcal{Z}_{\beta}^{\omega}(y; t, x)^p \right]^{\frac{1}{p}} |u_0|(dy) \\ &\leq C \int_{\mathbb{R}^d} \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; t, x) |u_0|(dy). \end{aligned}$$

The a.s. finiteness of the right-hand side for fixed x and t is a consequence of [13, Prop. 2.19]. Using the inequality above, to prove (8.17), we now need to show that \mathbb{P}_{\geq} -a.s.,

$$(8.19) \quad \int_{X_T} \rho(\theta(T-t), x)^p \left(\int_{\mathbb{R}^d} \mathcal{Z}_{\beta'}^{\omega_{\geq}}(y; t, x) |u_0|(dy) \right)^p dt dx < \infty.$$

We do so by comparing $\mathcal{Z}_{\beta'}^{\omega \geq}(y; t, x)$ with a variable with finite p th moment. For a small $\varepsilon > 0$, we introduce the quantity

$$(8.20) \quad \mathcal{T} := \sup_{k \geq 1} \sup_{y \in \mathbb{R}^d} \sup_{\substack{(t_i, x_i, z_i)_{i=1}^k \subseteq \omega \geq: \\ 0 < t_1 < \dots < t_k < T}} \left\{ \sum_{i=1}^k \log z_i - \varepsilon \sum_{i=2}^k \frac{\|x_i - x_{i-1}\|^2}{2(t_i - t_{i-1})} - \varepsilon \left(\frac{\|y\|^2}{2T} + \frac{\|x_1 - y\|^2}{2t_1} \right) \right\}.$$

Note that optimizing over y leads to the simpler expression

$$(8.21) \quad \mathcal{T} = \sup_{k \geq 1} \sup_{\substack{(t_i, x_i, z_i)_{i=1}^k \subseteq \omega \geq: \\ 0 < t_1 < \dots < t_k < T}} \left\{ \sum_{i=1}^k \log z_i - \varepsilon \sum_{i=2}^k \frac{\|x_i - x_{i-1}\|^2}{2(t_i - t_{i-1})} - \frac{\varepsilon \|x_1\|^2}{2(T + t_1)} \right\}.$$

The following lemma is an easy consequence of [13, Lemma 4.17].

Lemma 8.4. *For any $T > 0$ and $\varepsilon > 0$, \mathcal{T} is finite \mathbb{P}_{\geq} -a.s.*

Proof. By [13, Lemma 4.17], we have (with the convention $x_0 = 0$ and $t_0 = 0$)

$$\sup_{k \geq 1} \sup_{\substack{(t_i, x_i, z_i)_{i=1}^k \subseteq \omega \geq: \\ 0 < t_1 < \dots < t_k < 2T}} \sum_{i=1}^k \left(\log z_i - \varepsilon \frac{\|x_i - x_{i-1}\|^2}{2(t_i - t_{i-1})} \right) < \infty.$$

Taking a further restriction (namely $t_1 > T$) and shifting time by $-T$, we obtain by translation invariance that

$$\sup_{k \geq 1} \sup_{\substack{(t_i, x_i, z_i)_{i=1}^k \subseteq \omega \geq: \\ 0 < t_1 < \dots < t_k < \bar{T}}} \left\{ \sum_{i=1}^k (\log z_i) - \varepsilon \sum_{i=2}^k \frac{\|x_i - x_{i-1}\|^2}{2(t_i - t_{i-1})} - \varepsilon \frac{\|x_1\|^2}{2(t_1 + T)} \right\} < \infty. \quad \square$$

By Lemma 8.4, we have for every $x, y \in \mathbb{R}^d$ and $t \in [0, T]$ that

$$(8.22) \quad \mathcal{Z}_{\beta'}^{\omega \geq}(y; t, x) \leq e^{\mathcal{T} + \frac{\varepsilon \|y\|^2}{2T}} \sum_{k=0}^{\infty} (\beta')^k \int_{\mathbb{X}_t^{(k)}} \rho_{y; t, x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^{k+1} e^{\varepsilon \frac{\|\Delta x_i\|^2}{2\Delta t_i}} \prod_{j=1}^k \delta_{\omega_{\geq}}(dw_j)$$

with the convention that $x_0 = y$. Because

$$(8.23) \quad \rho(t, x) e^{\eta \frac{\|x\|^2}{2t}} = \theta_{\eta}^{\frac{d}{2}} \rho(\theta_{\eta} t, x) \quad \text{with} \quad \theta_{\eta} = (1 - \eta)^{-1}$$

and $\theta = (1 - \varepsilon)^{-1}$, we have $\mathcal{Z}_{\beta'}^{\omega \geq}(y; t, x) \leq e^{\mathcal{T} + \frac{\varepsilon \|y\|^2}{2T}} \widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x)$, where

$$(8.24) \quad \widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x) := \sum_{k=0}^{\infty} (\theta^{\frac{d}{2}} \beta')^k \int_{\mathbb{X}_t^{(k)}} \rho_{y; \theta t, x}(\theta \mathbf{t}, \mathbf{x}) \prod_{j=1}^k \delta_{\omega_{\geq}}(dw_j).$$

Setting $\hat{u}_0(dy) := e^{\frac{\varepsilon \|y\|^2}{2T}} |u_0|(dy)$, we can reduce (8.19) to showing

$$(8.25) \quad \int_{X_T} \rho(\theta(T - t), x)^p \left(\int_{\mathbb{R}^d} \widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x) \hat{u}_0(dy) \right)^p dt dx < \infty.$$

We take the expectation and apply Minkowski's inequality twice to get

$$\begin{aligned}
(8.26) \quad & \mathbb{E} \left[\int_{X_T} \rho(\theta(T-t), y)^p \left(\int_{\mathbb{R}^d} \widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x) \hat{u}_0(dy) \right)^p dt dy \right] \\
& \leq \int_{X_T} \rho(\theta(T-t), y)^p \left(\int_{\mathbb{R}^d} \mathbb{E} \left[\widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x)^p \right]^{\frac{1}{p}} \hat{u}_0(dy) \right)^p dt dx \\
& \leq \left(\int_{\mathbb{R}^d} \left(\int_{X_T} \rho(\theta(T-s), y)^p \mathbb{E} \left[\widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x)^p \right] dt dx \right)^{\frac{1}{p}} \hat{u}_0(dy) \right)^p.
\end{aligned}$$

Observe now that $\widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x)$ is a (non-normalized) point-to-point partition function corresponding to a Poisson environment with intensity measure $\lambda([1, \infty))\delta_{\theta^{d/2}\beta'}$ and a time-rescaling by θ . Therefore, by Corollary 6.5,

$$(8.27) \quad \mathbb{E}[\widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(y; t, x)^p] \leq C\rho(\theta t, x - y)^p,$$

for a constant C that depends on all parameters but not on y , t and x . Hence, the $\frac{1}{p}$ th power of (8.26) is bounded by a constant times

$$\int_{\mathbb{R}^d} \left(\int_{X_T} \rho(\theta(T-t), x)^p \rho(\theta t, x - y)^p dt dx \right)^{\frac{1}{p}} \hat{u}_0(dy) \leq C' \int_{\mathbb{R}^d} e^{-\frac{\|y\|^2}{2\theta T}} \hat{u}_0(dy),$$

where the inner integral has been computed as in (8.13). The above integral is finite thanks to our choice of ε and θ in (8.18). Thus, (8.25) holds and the proof is complete. \square

8.6. Uniqueness of solutions to the SHE. We assume in this section that λ satisfies (1.22) and prove uniqueness among solutions satisfying (2.10). By the fact that the condition (2.10) is stable under linear combinations, it is sufficient to show that any solution to (1.6) with $u_0 \equiv 0$ is equal to zero. Consider v that satisfies (2.10) and is such that for every $t \in (0, T]$ and $x \in \mathbb{R}^d$,

$$v(t, x) = \beta \int_0^t \int_{\mathbb{R}^d} \rho(t-s, x-y) v(s, y) \xi_\omega(ds, dy).$$

Applying this identity to $v(s, y)$ in the integrand and repeating this, we obtain that for any $k \geq 1$,

$$(8.28) \quad v(t, x) = \beta^k \int_{\mathcal{X}_t^{(k)}} \rho_{t_1, x_1; t, x}(\mathbf{t}^{(1)}, \mathbf{x}^{(1)}) v(t_1, x_1) \prod_{i=1}^k \xi_\omega(dt_i, dx_i),$$

where $\mathbf{t}^{(1)}$ and $\mathbf{x}^{(1)}$ are obtained from \mathbf{t} and \mathbf{x} , respectively, by removing the first component. Fixing $x \in \mathbb{R}^d$ and $t \in (0, T)$, we are going to prove that $v(t, x) = 0$ by showing that the right-hand side in (8.28) is summable in k . More precisely, we set

$$\mathcal{U}(t, x) := \sum_{k=0}^{\infty} \beta^k \int_{\mathcal{X}_t^{(k)}} \rho_{t_1, x_1; t, x}(\mathbf{t}^{(1)}, \mathbf{x}^{(1)}) v(t_1, x_1) \prod_{i=1}^k \xi_\omega(dt_i, dx_i),$$

the term for $k=0$ being simply $v(t, x)$. We fix $p \in (1, 1 + \frac{2}{d})$ for which $\mu_{0,1}(p) < \infty$ (when $d=1$ we fix $p=2$) and we are going to prove that

$$(8.29) \quad \mathbb{E}_{<} \left[|\mathcal{U}(t, x)|^p \right] < \infty \quad \mathbb{P}_{\geq}\text{-a.s.}$$

Separating $\omega_{<}$ and ω_{\geq} in ξ_ω , we have that

$$(8.30) \quad \mathcal{U}(t, x) = \sum_{k=0}^{\infty} \beta^k \int_{\mathcal{X}_t^{(k)}} \mathcal{V}(t_1, x_1) \prod_{i=2}^{k+1} \mathcal{Z}_{\beta}^{\omega_{<}}(t_{i-1}, x_{i-1}; t_i, x_i) \prod_{i=1}^k \xi_{\omega_{\geq}}(dt_i, dx_i),$$

where

$$(8.31) \quad \mathcal{V}(t, x) := \sum_{k=0}^{\infty} \beta^k \int_{\mathcal{X}_t^{(k)}} \rho_{t_1, x_1; t, x}(\mathbf{t}^{(1)}, \mathbf{x}^{(1)}) v(t_1, x_1) \prod_{i=1}^k \xi_{\omega_{<}}(dt_i, dx_i)$$

and the terms for $k = 0$ in (8.30) and (8.31) are $\mathcal{V}(t, x)$ and $v(t, x)$ respectively. By Corollary 6.5 and translation invariance, there exists $\beta' > 0$ such that $\mathbb{E}_{<}[|\mathcal{U}(t, x)|^p]^{1/p}$ is bounded by

$$(8.32) \quad \sum_{k=0}^{\infty} (\beta')^k \int_{\mathcal{X}_t^{(k)}} \mathbb{E}_{<}[|\mathcal{V}(t_1, x_1)|^p]^{\frac{1}{p}} \prod_{i=2}^{k+1} \rho(\Delta t_i, \Delta x_i) \prod_{i=1}^k \xi_{\omega_{\geq}}(dt_i, dx_i).$$

Using the triangle inequality and iterating the BDG inequality and the subadditivity property (5.4) for each term (as done in (A.1), for example), we can bound $\mathbb{E}_{<}[|\mathcal{V}(t, x)|^p]^{1/p}$ by

$$\sum_{k=0}^{\infty} (C_p \beta \mu_{0,1}(p)^{\frac{1}{p}})^k \left(\int_{\mathcal{X}_t^{(k)}} \rho_{t_1, x_1; t, x}(\mathbf{t}^{(1)}, \mathbf{x}^{(1)})^p \mathbb{E}_{<}[|v(t_1, x_1)|^p] \prod_{i=1}^k dt_i dx_i \right)^{\frac{1}{p}}.$$

Then, applying (6.19) and Lemma 4.5 with the trivial inequality $t - t_1 \leq t$, we get that

$$(8.33) \quad \mathbb{E}_{<}[|\mathcal{V}(t, x)|^p]^{\frac{1}{p}} \leq \mathbb{E}_{<}[|v(t, x)|^p]^{\frac{1}{p}} + \sum_{k=1}^{\infty} \frac{[C_p \beta (\mu_{0,1}(p) \Gamma(\nu_p) t^{\nu_p})^{\frac{1}{p}}]^k}{\Gamma(k \nu_p)^{\frac{1}{p}}} \times \left(\int_{X_t} \rho(t - t_1, x - x_1)^p \mathbb{E}_{<}[|v(t_1, x_1)|^p] dt_1 dx_1 \right)^{\frac{1}{p}}.$$

Since the sum in k is finite, the two conditions in our assumption (2.10) implies the \mathbb{P}_{\geq} -a.s. finiteness of the right-hand side of (8.33).

Coming back to the series in (8.32), we apply a similar trick to (8.22) and replace the z_i 's coming from the atoms of ω_{\geq} by one. More precisely, we set

$$(8.34) \quad \mathcal{T}(t, x) := \sup_{k \geq 1} \sup_{\substack{(t_i, x_i, z_i)_{i=1}^k \subseteq \omega_{\geq}: \\ 0 < t_1 < \dots < t_k < t}} \left[\sum_{i=1}^k (\log z_i) - \eta \sum_{i=2}^{k+1} \frac{\|x_i - x_{i-1}\|^2}{2\Delta t_i} \right],$$

with the convention that $t_{k+1} = t$ and $x_{k+1} = x$. The value of $\eta > 0$ in (8.34) is chosen such that $\theta = (1 - \eta \frac{3p-1}{2})^{-1}$, with the same θ as in the assumption (2.10). By the translation invariance and symmetry properties of ω_{\geq} , [13, Lemma 4.17] guarantees that $\mathcal{T}(t, x) < \infty$, \mathbb{P}_{\geq} -a.s. Defining

$$(8.35) \quad \widehat{\mathcal{Z}}_{\beta'}^{\omega_{\geq}}(s, y; t, x) := \sum_{k=0}^{\infty} (\beta')^k \int_{\mathbb{X}_t^{(k)}} \rho_{s, y; t, x}(\mathbf{t}, \mathbf{x}) \prod_{i=1}^{k+1} e^{\eta \frac{\|\Delta x_i\|^2}{2\Delta t_i}} \prod_{j=1}^k \delta_{\omega}(dt_j, dx_j dz_j),$$

we find that the series in (8.32) starting from $k = 1$ is smaller than the following integral (we have replaced t_1, x_1, z_1 by s, y, z):

$$(8.36) \quad \beta' e^{\mathcal{T}(t, x)} \int_{(0, t) \times \mathbb{R}^d \times [1, \infty)} \widehat{\mathcal{Z}}_{\beta'}^{\omega_{\geq}}(s, y; t, x) \mathbb{E}_{<}[|\mathcal{V}(s, y)|^p]^{\frac{1}{p}} \delta_{\omega_{\geq}}(ds, dy, dz).$$

Note that the integral in (8.36) is a sum and that all terms are finite almost surely by (8.33) and (8.27). In order to show that the integral in (8.36) is finite, we proceed in several steps. The first one is to replace $\widehat{\mathcal{Z}}_{\beta'}^{\omega_{\geq}}(s, y; t, x)$ by $\rho(t - s, x - y) e^{\eta \|x - y\|^2 / (t - s)}$. This is the purpose of the following lemma whose short proof is postponed to the end of the section.

Lemma 8.5. *Let*

$$\Theta(t, x) := \left\{ (s, y) \in (0, t) \times \mathbb{R}^d : \widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(s, y; t, x) \geq \rho(t-s, x-y) e^{\eta \frac{\|x-y\|^2}{(t-s)}} \right\},$$

$$\bar{\omega}_{\geq} := \left\{ (s, y) \in (0, t) \times \mathbb{R}^d : \exists z \geq 1, (s, y, z) \in \omega \right\}.$$

Then we have $\#(\Theta(t, x) \cap \bar{\omega}_{\geq}) < \infty$ \mathbb{P}_{\geq} -a.s.

As a consequence of Lemma 8.5, the integral in (8.36) is finite if

$$(8.37) \quad \int_{\mathbb{X}_t} \mathbb{E}_{<}[|\mathcal{V}(s, y)|^p]^{\frac{1}{p}} \rho(t-s, x-y) e^{\eta \frac{\|x-y\|^2}{(t-s)}} \delta_{\omega_{\geq}}(ds, dy, dz) < \infty.$$

By [84, Prop. 12.1],

(8.37) holds if we can show that

$$(8.38) \quad \int_{X_t} \mathbb{E}_{<}[|\mathcal{V}(s, y)|^p]^{\frac{1}{p}} \rho(t-s, x-y) e^{\eta \frac{\|x-y\|^2}{(t-s)}} ds dy < \infty.$$

To this end, we use the bound (8.33) for $\mathbb{E}_{<}[|\mathcal{V}_{\beta}^{\omega}(s, y)|^p]^{1/p}$ and check that the integrals corresponding to each of the two summands on the right-hand side of (8.33) are finite. In the first case, we must show that

$$(8.39) \quad \int_{X_t} \mathbb{E}_{<}[|v(s, y)|^p]^{\frac{1}{p}} \rho(t-s, x-y) e^{\eta \frac{\|x-y\|^2}{(t-s)}} ds dy < \infty.$$

Applying Jensen's inequality for the finite measure $e^{-\eta \|x-y\|^2/2(t-s)} ds dy$, we can bound the quantity above by

$$(8.40) \quad \left(\int_{X_t} \mathbb{E}_{<}[|v(s, y)|^p] \rho(t-s, x-y)^p e^{\eta(3p-1) \frac{\|x-y\|^2}{2(t-s)}} ds dy \right)^{\frac{1}{p}}.$$

Recalling (8.23) and the relation $\theta = (1 - \eta \frac{3p-1}{2})^{-1}$, the finiteness of (8.40) follows from our assumption (2.10).

For the second summand on the right-hand side of (8.33), we have to show that

$$\int_{X_t} e^{\eta \frac{\|x-y\|^2}{(t-s)}} \rho(t-s, x-y) \left(\int_{X_s} \rho(s-r, y-v)^p \mathbb{E}_{<}[|v(r, v)|^p] dr dv \right)^{\frac{1}{p}} ds dy$$

is finite. Using the same trick as in (8.40), we see that it is sufficient to prove the finiteness of

$$\int_{X_t} \int_{X_s} \left[e^{-\eta(3p-1) \frac{\|x-y\|^2}{2(t-s)}} \rho(t-s, x-y)^p \right] \rho(s-r, y-v)^p \mathbb{E}_{<}[|v(r, v)|^p] dr dv ds dy.$$

Now, by (8.23) and thanks to our choice of η , we can replace the $[\cdots]$ -term by a constant times $\rho(\theta(t-s), x-y)^p$ and bound $\rho(s-r, y-v)^p$ by a constant times $\rho(\theta(s-r), y-v)^p$. Then, using (5.5) as in (8.13), we have

$$\int_{X_{r,t}} \rho(\theta(t-s), x-y)^p \rho(\theta(s-r), y-v)^p ds dy \leq C'_{\eta,p} (t-r)^{\nu_p} \rho(\theta(t-r), x-v)^p,$$

so (2.10) completes the proof. \square

Proof of Lemma 8.5. By Markov's inequality and (8.27), we have for any given $s, y \in (0, t) \times \mathbb{R}^d$,

$$\mathbb{P}_{\geq} \left[\widehat{\mathcal{Z}}_{\beta'}^{\omega \geq}(s, y; t, x) \geq \rho(t-s, x-y) e^{\eta \frac{\|x-y\|^2}{(t-s)}} \right] \leq C \left(\frac{\rho(\frac{t-s}{1-\eta}, x-y)}{\rho(t-s, x-y)} e^{-\eta \frac{\|x-y\|^2}{(t-s)}} \right)^p = C' e^{-\eta p \frac{\|x-y\|^2}{2(t-s)}},$$

using (8.23) for the last inequality. Hence, we have

$$\begin{aligned} \mathbb{E}_{\geq} \left[\#(\Theta(t, x) \cap \bar{\omega}_{\geq}) \right] &\leq C' \int_{(0, t) \times \mathbb{R}^d \times [1, \infty)} e^{-\eta p \frac{\|x-y\|^2}{2(t-s)}} ds dy \lambda(dz) \\ &= C_p \lambda([1, \infty)) \int_{(0, t)} (t-s)^{\frac{d}{2}} ds < \infty. \end{aligned} \quad \square$$

9. MOMENTS OF ORDER $p < 1$ AND LOWER BOUNDS IN THEOREMS 2.13–2.14

This section is dedicated to estimating fractional moments of $\mathcal{Z}_{\beta}^{\omega}(t, *)$. More precisely, we focus on $\mathbb{E}[\mathcal{Z}_{\beta}^{\omega}(t, *)^{1/2}]$ to simplify notation, but the method would be equally efficient to directly estimate $\mathbb{E}[\mathcal{Z}_{\beta}^{\omega}(t, *)^p]$ for $p \in (0, 1)$. The estimates obtained in this section allow us to complete the proof of Theorems 2.13 and 2.14, since by convexity and the fact that $\bar{\gamma}_{\beta}(1) = 0$ and $\bar{\gamma}_{\beta}(0) = 0$, we have

$$(9.1) \quad \begin{cases} \bar{\gamma}_{\beta}(p) \leq 2p\bar{\gamma}_{\beta}(\frac{1}{2}) & \text{for } p \in (0, \frac{1}{2}), \\ \bar{\gamma}_{\beta}(p) \leq 2(1-p)\bar{\gamma}_{\beta}(\frac{1}{2}) & \text{for } p \in (\frac{1}{2}, 1), \\ \bar{\gamma}_{\beta}(p) \geq -2(p-1)\bar{\gamma}_{\beta}(\frac{1}{2}) & \text{for } p > 1. \end{cases}$$

We also prove Proposition 2.3 in Section 9.4, partially using ideas developed in Sections 9.1–9.3.

In Section 9.1, we introduce a method that combines a coarse-graining and a change-of-measure argument, which allows to obtain upper bounds that decay exponentially in t (this is crucial for the proof of Theorems 2.13 and 2.14). This approach originates in the study of the discrete pinning model [54, 63]: in its refined form, which first appeared in [97], it can be used in a continuum setup such as the SHE. Note that the method found many implementations in the last decade, such as for disordered pinning [11, 62], directed polymers and variants [10, 15, 81, 82], the random walk pinning model [14, 20], large deviations of random walks in a random environment [101], self-avoiding walks in a random environment [83], and anomalous path detection in a random environment [30].

The achievement of the method (presented in Lemma 9.1) is to reduce the problem of estimating moments to that of showing that the original measure \mathbb{P} significantly differs (in total variation) from an alternative measure where the environment has been modified along a Brownian trajectory (the size-biased measure). This last statement is proved in Sections 9.2 (for $d = 1$ or heavy-tailed noises) and 9.3 (for $d \geq 2$ and light-tailed noises). While Lemma 9.1 is quite general and can be adapted to various settings, the proof that the original and the tilted measures differ in total variation heavily depends on context.

The proof presented in Section 9.2 for $d = 1$ relies on ideas found in [81, Section 3]: in essence, we discriminate between the original and the tilted environments by looking at the average in a box of length T and width \sqrt{T} . The proof for heavy-tailed environments, also found in Section 9.2, relies on a similar idea to that found in [99, Section 2].

On the other hand, the approach taken in Section 9.3 and Section 9.4 are completely new and have no discrete analogue. We show that what makes the tilted measure different from the original one is the presence of clusters of points that are very close to one another. It requires a fine analysis to identify exactly what the characteristics of these clusters are.

9.1. A general coarse-graining lemma. For notational simplicity, we set $\mathcal{Z}_{\beta, t}^{\omega} := \bar{\mathcal{Z}}_{\beta}^{\omega}(t, *)$ in the remainder of this section. Our coarse-graining approach reduces the problem of bounding fractional moments of $\mathcal{Z}_{\beta, t}^{\omega}$ to identifying a single event that is unlikely under the original measure \mathbb{P} but becomes likely under the size-biased measure $\tilde{\mathbb{P}}_{\beta, t}^0$ (see Section 4.1).

We say that a measurable event \mathcal{A} for the point process ω has *time range* $[0, T]$ if $\mathbf{1}_{\mathcal{A}}(\omega)$ is a function of $\omega \cap ([0, T] \times \mathbb{R}^d \times (0, \infty))$. Moreover, defining $\hat{\omega} = \hat{\omega}(\omega, \omega', B) := \omega \cup \tilde{\omega}(\omega', B)$, we will

use, for a generic real-valued function f defined on the set of environments, the abbreviation

$$(9.2) \quad \widehat{f}(\omega, \omega', B) := f \circ \widehat{\omega}(\omega, \omega', B).$$

Also, we write \mathbf{Q}_x for the law of a d -dimensional standard Brownian motion starting from x .

Lemma 9.1. *Assume that $\mu < \infty$. There exists a constant $K \in (0, \infty)$, which only depends on the dimension d , with the following property: For any $\beta > 0$ and $T > 0$, if there exists an event \mathcal{A} with time range $[0, T)$ satisfying both*

$$(H1) \quad \mathbb{P}(\omega \in \mathcal{A}) \leq e^{-K},$$

$$(H2) \quad \max_{x \in [0, \sqrt{T})^d} \widehat{\mathbb{P}}_{\beta, x}(\widehat{\omega} \notin \mathcal{A}) \leq e^{-K},$$

where $\widehat{\mathbb{P}}_{\beta, x} = \mathbb{P} \otimes \mathbb{P}'_{\beta} \otimes \mathbf{Q}_x$, then for all $m \geq 1$ we have

$$(9.3) \quad \mathbb{E}[(\mathcal{Z}_{\beta, mT}^{\omega})^{\frac{1}{2}}] \leq 2^{-m},$$

and as a consequence $\bar{\gamma}_{\beta}(\frac{1}{2}) \leq -(\log 2)T^{-1}$.

Below, we are going to apply Lemma 9.1 to events of the type

$$(9.4) \quad \mathcal{A} := \{\omega : f(\omega) - \mathbb{E}[f(\omega)] > h\} \quad \text{or} \quad \mathcal{A} := \{\omega : f(\omega) > h\}$$

where f is a function that only depends on $\omega \cap ([0, T) \cap \mathbb{R}^d \times (0, \infty))$ and $h \in \mathbb{R}$ is a threshold to be chosen appropriately. More precisely, we consider functions f of the form (recall that $w = (t, x, z)$)

$$(9.5) \quad f(\omega) = \int_{\mathbb{X}_T^{(k)}} \Upsilon(t, x, z) \prod_{i=1}^k \delta_{\omega}(dw_i),$$

where k is a positive integer and Υ is a non-negative function on $\mathbb{X}_T^{(k)}$ (recall the notation (1.9)). Given f as above, we define

$$(9.6) \quad f'(\omega', B) = \int_{\mathcal{X}_T^{(k)}} \Upsilon(t, (B_{t_i})_{i=1}^k, z) \prod_{i=1}^k \delta_{\omega'}(dt_i, dz_i).$$

In other words, f' is the total contribution of the additional terms in \widehat{f} that are obtained by considering (only) the atoms on the added “environment spine” (ω', B) (recall (9.2)). Then, for f of the form (9.5), since Υ is non-negative, we have

$$(9.7) \quad \widehat{f}(\omega, \omega', B) \geq f(\omega) + f'(\omega', B).$$

In this setting, verifying (H1) and (H2) boils down to first and second moment computations for f and f' .

Corollary 9.2. *Assume that f is as in (9.5) and satisfies one of the two following conditions:*

(C1) *For all $x \in [0, \sqrt{T})^d$, we have that*

$$\mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)]^2 \geq 4e^K \left\{ \text{Var}(f(\omega)) + \text{Var}_{\mathbb{P}'_{\beta} \otimes \mathbf{Q}_x}(f'(\omega', B)) \right\};$$

(C2) *The function Υ is integer-valued, we have $\mathbb{E}[f(\omega)] \leq e^{-K}$ and*

$$\forall x \in [0, \sqrt{T})^d : \quad \mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)]^2 \geq e^K \text{Var}_{\mathbb{P}'_{\beta} \otimes \mathbf{Q}_x}(f'(\omega', B)).$$

In both cases, also assume that the variances on the right-hand side are finite. Then (H1) and (H2) are satisfied and $\gamma_{\beta}(\frac{1}{2}) \leq -(\log 2)T^{-1}$.

Proof. If (C1) is satisfied, define

$$m := \mathbb{E}[f(\omega)] \quad \text{and} \quad \sigma := \sqrt{\text{Var}(f(\omega)) + \text{Var}_{\mathbb{P}'_\beta \otimes \mathbf{Q}_x}(f'(\omega', B))}$$

and consider the event $\mathcal{A} := \{\omega : f(\omega) > m + e^{\frac{K}{2}}\sigma\}$. We can apply Chebyshev's inequality, using that $\sigma^2 \geq \text{Var}(f(\omega))$, to check the validity of (H1). For (H2), we observe that as a consequence of (9.7), we have

$$\{\widehat{\omega} \notin \mathcal{A}\} \subseteq \{f(\omega) + f'(\omega', B) \leq m + e^{\frac{K}{2}}\sigma\}.$$

Using Chebyshev's inequality, we deduce that

$$\widehat{\mathbb{P}}_{\beta,x}(\widehat{\omega} \notin \mathcal{A}) \leq \frac{\text{Var}(f(\omega)) + \text{Var}_{\mathbb{P}'_\beta \otimes \mathbf{Q}_x}(f'(\omega', B))}{(e^{\frac{K}{2}}\sigma - \mathbb{E}'_\beta \otimes \mathbf{Q}_x[f'(\omega', B)])^2} \leq e^{-K}.$$

If (C2) is satisfied, we set $\mathcal{A} := \{\omega : f(\omega) \geq 1\} = \{\omega : f(\omega) > 0\}$. We then deduce (H1) from Markov's inequality applied to $f(\omega)$ and (H2) from Chebychev's inequality applied to $f'(\omega', B)$. \square

Remark 9.3. We will use the assumption (C1) for $d = 1$ in the light-tailed case and the assumption (C2) in all heavy-tailed cases and for $d \geq 2$. These two assumptions correspond to two different ways of distinguishing between the original measure and the size-biased one. In Assumption (C1), f is increased under the size-biased measure by an amount f' that, on average, exceeds the typical fluctuation (measured as $\text{Var}(f(\omega))$) under the original measure. Assumption (C2) implies that there are patterns in ω that appear with large probability under the size-biased measure and are most likely absent under the original one.

Proof of Lemma 9.1. Let us assume that T and \mathcal{A} satisfy (H1) and (H2). We divide \mathbb{R}^d into cubes of side length \sqrt{T} : let

$$\mathcal{C}_j = \mathcal{C}_j^{(T)} := j\sqrt{T} + [0, \sqrt{T})^d, \quad j \in \mathbb{Z}^d$$

and define, for $\mathbf{j} = (j_1, \dots, j_m) \in (\mathbb{Z}^d)^m$, the function $\chi_{\mathbf{j}} : C([0, mT]) \rightarrow \mathbb{R}$,

$$\chi_{\mathbf{j}}(\varphi) := \mathbf{1}\{\forall i \in \llbracket m \rrbracket : \varphi(iT) \in \mathcal{C}_{j_i}\}.$$

In other words, $\chi_{\mathbf{j}}(\varphi)$ is the indicator that the function φ passes through a prescribed sequence of cubes of diameter \sqrt{T} at times iT , where $i \in \llbracket m \rrbracket$. If $\varphi = (B_t)_{t \in [0, mT]}$ is a Brownian motion, the unique value of \mathbf{j} such that $\chi_{\mathbf{j}}(B) = 1$ corresponds to a coarse-grained trajectory (or skeleton) of B on time scale T . Now, recalling that we have set $\mathcal{Z}_{\beta,t}^\omega = \bar{\mathcal{Z}}_\beta^\omega(t, *)$ and recalling the notation from Section 4.1, we have

$$(9.8) \quad \mathcal{Z}_{\beta,mT}^\omega = \int_{(\mathbb{R}^d)^m} \prod_{i=1}^M \bar{\mathcal{Z}}_\beta^\omega((i-1)T, x_{i-1}; iT, x_i) dx_i.$$

This can be checked directly from the definition if $\mathcal{Z}_{\beta,mT}^\omega$ and $\bar{\mathcal{Z}}_\beta^\omega((i-1)T, x_{i-1}; iT, x_i)$ are replaced by $\bar{\mathcal{Z}}_\beta^{\omega,a}(mT, *)$ and $\bar{\mathcal{Z}}_\beta^{\omega,a}((i-1)T, x_{i-1}; iT, x_i)$, respectively. The case $a = 0$ then follows from the L^1 -convergence in Theorem 2.1. We therefore get $\mathcal{Z}_{\beta,mT}^\omega = \sum_{\mathbf{j} \in (\mathbb{Z}^d)^m} \mathcal{Z}_{\beta,mT}^\omega(\chi_{\mathbf{j}})$, where

$$\mathcal{Z}_{\beta,mT}^\omega(\chi_{\mathbf{j}}) := \int_{(\mathbb{R}^d)^m} \prod_{i=1}^M \bar{\mathcal{Z}}_\beta^\omega((i-1)T, x_{i-1}; iT, x_i) \mathbf{1}_{\{x_i \in \mathcal{C}_{j_i}\}} dx_i.$$

As a consequence, by subadditivity (5.4), we have

$$\mathbb{E}\left[(\mathcal{Z}_{mT}^\omega)^{\frac{1}{2}}\right] \leq \sum_{\mathbf{j} \in (\mathbb{Z}^d)^m} \mathbb{E}\left[\mathcal{Z}_{mT}^\omega(\chi_{\mathbf{j}})^{\frac{1}{2}}\right].$$

Our aim is to obtain a good bound on $\mathbb{E}[(\mathcal{Z}_{\beta,mT}^\omega(\chi_j))^{1/2}]$. This is where we introduce a change-of-measure procedure. We use the Cauchy–Schwarz inequality as follows:

$$(9.9) \quad \mathbb{E}\left[(\mathcal{Z}_{\beta,mT}^\omega(\chi_j))^{\frac{1}{2}}\right]^2 \leq \mathbb{E}\left[G_j(\omega)^{-1}\right] \mathbb{E}\left[G_j(\omega) \mathcal{Z}_{\beta,mT}^\omega(\chi_j)\right],$$

for some non-negative function $G_j(\omega)$ that penalizes the ω 's that contribute most to $\mathbb{E}[\mathcal{Z}_{\beta,mT}^\omega(\chi_j)]$. This procedure is referred to as a change of measure since $\mathbb{E}[G_j(\omega) \mathcal{Z}_{\beta,mT}^\omega(\chi_j)]$ corresponds to the expectation of $\mathcal{Z}_{\beta,mT}^\omega(\chi_j)$ under a new measure whose density with respect to the original one is given by $G_j(\omega)$. We now construct G_j with a product structure in order to gain a constant factor per coarse-grained step of the trajectory. For $K > 0$, a fixed constant to be determined below, we define (with $j_0 := 0$)

$$(9.10) \quad \begin{aligned} g(\omega) &:= \exp(-K \mathbf{1}_{\mathcal{A}}(\omega)), & g_{i,j}(\omega) &:= g \circ \theta_{((i-1)T, j\sqrt{T})}(\omega), \\ G_j(\omega) &:= \prod_{i=1}^m g_{i,j_{i-1}}(\omega), \end{aligned}$$

where $\theta_{(t,x)}$ is the space-time shift operator acting on point collections in $\mathbb{R} \times \mathbb{R}^d \times (0, \infty)$ (i.e., $\theta_{(t,x)}(\omega) = \{(s-t, y-x, z) : (s, y, z) \in \omega\}$). Because \mathcal{A} has time range T , for given j , the variables $(g_{i,j_{i-1}}(\omega))_{i=1}^m$ are independent and identically distributed under \mathbb{P} , so that, by (H1),

$$\mathbb{E}[G_j(\omega)^{-1}] = (1 + (e^K - 1)\mathbb{P}(\mathcal{A}))^m \leq 2^m.$$

We then derive from (9.9) that

$$(9.11) \quad \mathbb{E}\left[(\mathcal{Z}_{\beta,mT}^\omega)^{\frac{1}{2}}\right] \leq 2^{\frac{m}{2}} \sum_{j \in (\mathbb{Z}^d)^m} \sqrt{\mathbb{E}[G_j(\omega) \mathcal{Z}_{\beta,mT}^\omega(\chi_j)]}.$$

Using the size-biased representation of Lemma 4.1 and recalling the notation (9.2), we obtain that

$$\mathbb{E}[G_j(\omega) \mathcal{Z}_{\beta,mT}^\omega(\chi_j)] = \mathbb{E} \otimes \mathbb{E}'_\beta \otimes \mathbf{Q}[\widehat{G}_j(\omega, \omega', B) \chi_j(B)].$$

Letting $B_t^{(i)} := B_{(i-1)T+t} - j_{i-1}\sqrt{T}$, using the stationarity of ω and ω' and the fact that \mathcal{A} has time-range T , we have

$$\begin{aligned} \mathbb{E} \otimes \mathbb{E}'_\beta \otimes \mathbf{Q}[\widehat{G}_j(\omega, \omega', B) \chi_j(B)] &= \widehat{\mathbb{E}}_\beta \left[\prod_{i=1}^m \widehat{g}_{i,j_{i-1}}(\omega, \omega', B^{(i)}) \mathbf{1}_{\{B_T^{(i)} \in \mathcal{C}_{j_i-j_{i-1}}\}} \right] \\ &= \mathbf{Q} \left[\prod_{i=1}^m \mathbb{E} \otimes \mathbb{E}'_\beta [\widehat{g}_{i,j_{i-1}}(\omega, \omega', B^{(i)}) \mathbf{1}_{\{B_T^{(i)} \in \mathcal{C}_{j_i-j_{i-1}}\}}] \right] \\ &= \mathbf{Q} \left[\prod_{i=1}^m \mathbb{E} \otimes \mathbb{E}'_\beta [\widehat{g}(\omega, \omega', B^{(i)}) \mathbf{1}_{\{B_T^{(i)} \in \mathcal{C}_{j_i-j_{i-1}}\}}] \right]. \end{aligned}$$

Conditionally on $(B_t)_{t \in [0, (i-1)T]}$, $(B_t^{(i)})_{t \geq 0}$ is a Brownian motion starting from $B_{(i-1)T} - j_{i-1}\sqrt{T}$, which belongs to \mathcal{C}_0 . Thus, using the Brownian motion's Markov property iteratively (starting with $i = m$), we obtain that

$$\widehat{\mathbb{E}}_\beta [\widehat{G}_j(\omega, \omega', B) \chi_j(B)] \leq \prod_{i=1}^m \max_{x \in \mathcal{C}_0} \widehat{\mathbb{E}}_{\beta,x} [\widehat{g}(\omega, \omega', B) \mathbf{1}_{\{B_T \in \mathcal{C}_{j_i-j_{i-1}}\}}].$$

Reindexing the sums yields

$$(9.12) \quad \sum_{j \in (\mathbb{Z}^d)^m} \sqrt{\mathbb{E}[G_j(\omega) \mathcal{Z}_{\beta,mT}^\omega(\chi_j)]} \leq \left(\sum_{j \in \mathbb{Z}^d} \max_{x \in \mathcal{C}_0} \sqrt{\widehat{\mathbb{E}}_{\beta,x} [\widehat{g}(\omega, \omega', B) \mathbf{1}_{\{B_T \in \mathcal{C}_j\}}]} \right)^m.$$

Hence, it remains to show that the sum on the right-hand side is small. Given an integer N , since $g \leq 1$, we have

$$(9.13) \quad \begin{aligned} & \sum_{j \in \mathbb{Z}^d} \max_{x \in \mathcal{C}_0} \sqrt{\widehat{\mathbb{E}}_{\beta,x} [\widehat{g}(\omega, \omega', B) \mathbf{1}_{\{B_T \in \mathcal{C}_j\}}]} \\ & \leq \sum_{\|j\|_\infty \geq N} \max_{x \in \mathcal{C}_0} \sqrt{\mathbf{Q}_x[B_T \in \mathcal{C}_j]} + (2N)^d \max_{x \in \mathcal{C}_0} \sqrt{\widehat{\mathbb{E}}_{\beta,x} [\widehat{g}(\omega, \omega', B)]}. \end{aligned}$$

We then choose N to be the smallest integer such that the first sum is smaller than $(4\sqrt{2})^{-1}$; note it depends on d but not on T . For the second term, recalling the definition (9.10) of g and using assumption (H2) for the last inequality, we have

$$(9.14) \quad \widehat{\mathbb{E}}_{\beta,x}[\widehat{g}(\omega, \omega', B)] = e^{-K} + (1 - e^{-K}) \widehat{\mathbb{P}}_{\beta,x}(\widehat{\omega} \notin \mathcal{A}) \leq 2e^{-K}.$$

Finally, combining (9.11)–(9.14), we obtain that

$$\mathbb{E}[(\mathcal{Z}_{\beta,mT}^\omega)^{\frac{1}{2}}] \leq 2^{\frac{m}{2}} \left(\frac{1}{4\sqrt{2}} + (2N)^d \sqrt{2e^{-K}} \right)^m = 2^{-m},$$

where the last inequality holds by choosing $K = 2[(3+d)\log 2 + d\log N]$. \square

9.2. One-body estimates. We now present a simple choice for f that yields the desired bounds on the Lyapunov exponents both in dimension $d = 1$ and in any dimension $d \geq 2$ for heavy-tailed environments. This gives the second half of Theorem 2.13 (i) and of Theorem 2.14, that is, the lower bounds on $|\gamma_\beta(p)|$. More precisely, we prove the following.

Proposition 9.4. *Assume that $\mu < \infty$.*

(i) *In dimension $d = 1$, we have*

$$(9.15) \quad \limsup_{\beta \rightarrow 0+} \beta^{-4} \bar{\gamma}_\beta(\tfrac{1}{2}) < 0.$$

(ii) *In any dimension $d \geq 1$, if $\liminf_{A \rightarrow \infty} A^\alpha \lambda([A, \infty)) > 0$ for some $\alpha < 1 + \frac{2}{d}$, then*

$$(9.16) \quad \limsup_{\beta \rightarrow 0+} \beta^{-\frac{\alpha}{\nu_\alpha}} \bar{\gamma}_\beta(\tfrac{1}{2}) < 0.$$

If we only assume $\limsup_{A \rightarrow \infty} A^\alpha \lambda([A, \infty)) > 0$, then

$$(9.17) \quad \liminf_{\beta \rightarrow 0+} \beta^{-\frac{\alpha}{\nu_\alpha}} \bar{\gamma}_\beta(\tfrac{1}{2}) < 0.$$

Proof. The proofs of (i) and (ii) slightly differ but have the same starting point, that is, the same function f (to which we will apply Corollary 9.2), up to a choice of parameter.

Preparing the setup. We start by fixing $R > 1$ such that

$$(9.18) \quad \left((2\pi)^{-\frac{1}{2}} \int_{[-R, R-1]} e^{-\frac{u^2}{2}} du \right)^d \geq 1 - \frac{1}{64} e^{-K}.$$

where K is the constant from Lemma 9.1. Given $T > 0$ and $0 < a < b \leq \infty$, we then consider the functional

$$(9.19) \quad f(\omega) := \int_{\mathbb{X}_T^{(1)}} \mathbf{1}_{\{\|x\|_\infty \leq R\sqrt{T}, z \in [a, b]\}} \delta_\omega(dt, dx, dz),$$

which is Poisson distributed with

$$(9.20) \quad \mathbb{E}[f(\omega)] = \text{Var}(f(\omega)) = \lambda([a, b])(2R\sqrt{T})^d T.$$

Using the formalism of the previous subsection, we further have

$$f'(\omega', B) = \int_{(0, T) \times (0, \infty)} \mathbf{1}_{\{\|B_t\|_\infty \leq R\sqrt{T}, z \in [a, b]\}} \delta_{\omega'}(dt, dz).$$

To estimate the expectation and variance of f' , we define

$$\bar{f}(\omega') := \int_{(0,T) \times (0,\infty)} \mathbf{1}_{\{z \in [a,b]\}} \delta_{\omega'}(dt, dz).$$

Note that we have $f'(\omega', B) \leq \bar{f}(\omega')$ and, recalling that ω' has intensity measure $\beta z dt \lambda(dz)$, also

$$(9.21) \quad \mathbb{E}'_{\beta}[\bar{f}(\omega')] = \beta \mu_{a,b}(1)T, \quad \mathbb{E}'_{\beta}[\bar{f}(\omega')^2] = \beta \mu_{a,b}(1)T + (\beta \mu_{a,b}(1)T)^2.$$

The condition (9.18) guarantees that for any $x \in [0, \sqrt{T}]^d$ and $t \in (0, T)$

$$\mathbf{Q}_x(\|B_t\|_{\infty} \leq R\sqrt{T}) \geq 1 - \frac{1}{64} e^{-K},$$

so that we have

$$(9.22) \quad \mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)] \geq (1 - \frac{1}{64} e^{-K}) \beta \mu_{a,b}(1)T \geq \frac{1}{\sqrt{2}} \beta \mu_{a,b}(1)T.$$

We also have, combining (9.21) and the first inequality of (9.22),

$$(9.23) \quad \begin{aligned} \text{Var}_{\mathbb{P}'_{\beta} \otimes \mathbf{Q}_x}(f'(\omega', B)) &\leq \mathbb{E}'_{\beta}[\bar{f}(\omega')^2] - \mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)]^2 \\ &\leq \beta \mu_{a,b}(1)T + \frac{1}{32} e^{-K} (\beta \mu_{a,b}(1)T)^2. \end{aligned}$$

We are now ready to apply these estimates to specific cases.

Proof of (i). We first choose a and b such that $\lambda([a, b]) > 0$ (in a way that does not depend on T or β) and then we let

$$T = T(\beta) := \frac{2^{10} R^2 e^{2K} \lambda([a, b])^2}{\beta^4 \mu_{a,b}(1)^4}.$$

With this choice and for small β , the second term on the right-hand side of (9.23) dominates: using (9.22) for the second inequality, we get

$$\text{Var}_{\mathbb{P}'_{\beta} \otimes \mathbf{Q}_x}(f'(\omega', B)) \leq \frac{e^{-K}}{16} (\beta \mu_{a,b}(1)T)^2 \leq \frac{e^{-K}}{8} \mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)]^2.$$

The reader can check that with our definition of T , we have (recall (9.20))

$$\text{Var}(f(\omega)) = 2R\lambda([a, b])T^{\frac{3}{2}} = \frac{e^{-K}}{16} (\beta \mu_{a,b}(1)T)^2 \leq \frac{e^{-K}}{8} \mathbb{E}'_{\beta} \otimes \mathbf{Q}_x[f'(\omega', B)]^2.$$

Therefore, condition (C1) in Corollary 9.2 is indeed satisfied for small β , and we thus have

$$\limsup_{\beta \rightarrow 0+} \beta^{-4} \bar{\gamma}_{\beta}(\frac{1}{2}) \leq -\frac{(\log 2) \mu_{a,b}(1)^4}{2^{10} R^2 e^{2K} \lambda([a, b])^2}.$$

Proof of (ii). Let $\ell > 0$ denote the limit superior/inferior of $A^{-\alpha} \lambda([A, \infty))$. We choose

$$(9.24) \quad A = A(\beta) := M \beta^{-\frac{d+2}{2-d(\alpha-1)}},$$

where $M = M(K, R, \ell)$ is a large constant that depends on K , R and ℓ . For the remainder of the proof, we assume that β is such that we have, for the value of A specified above,

$$(9.25) \quad \lambda([A, \infty)) \geq \frac{1}{2} \ell A^{-\alpha}.$$

(We assume that β is sufficiently small if the assumption is about the limit inferior, and we take β along a well-chosen subsequence tending to 0 if the assumption is about the limit superior.) We are going to use Corollary 9.2 for the function (9.19), with the choices $a = A$, $b = \infty$ and

$$(9.26) \quad T := \left[e^{-K} (2R)^{-d} \lambda([A, \infty))^{-1} \right]^{\frac{2}{2+d}}.$$

We have $T \leq [e^{-K} (2R)^{-d} \frac{2}{\ell} M^{\alpha}]^{2/(2+d)} \beta^{-\alpha/\nu_{\alpha}}$ by our choice of A and our tail assumption. Therefore, once we show that (C2) is satisfied, Corollary 9.2 yields the desired bound on $\bar{\gamma}_{\beta}(\frac{1}{2})$.

For the expectation of f , by (9.20), our choice for T directly yields

$$\mathbb{E}[f(\omega)] = (2R)^d \lambda([A, \infty)) T^{1+\frac{d}{2}} = e^{-K}.$$

To check the condition on the variance of f' , in view of (9.22) and (9.23), it is sufficient to have $\beta \mu_{A,\infty}(1)T \geq 4e^K$. To this end, we bound $\mu_{A,\infty}(1) \geq A\lambda([A, \infty))$ and use the definition (9.26) of T to get that $\beta \mu_{A,\infty}(1)T \geq C(K, R)\beta A\lambda([A, \infty))^{d/(2+d)}$. By (9.25) and the definition of A in (9.24), the latter can be made large by choosing $M = M(K, R, \ell)$ large. \square

9.3. Multi-body estimates. The aim of this section is to prove:

Proposition 9.5. *Suppose that $d \geq 2$ and $\mu < \infty$. Then*

$$(9.27) \quad \limsup_{\beta \rightarrow 0} \frac{\log|\log|\bar{\gamma}_\beta(\frac{1}{2})||}{|\log \beta|} \leq 1 + \frac{2}{d}.$$

With Propositions 9.4 and 9.5 at hand, we can complete the proof of Theorems 2.13 and 2.14. Recall that the first half of the proof was given in Sections 7.2 and 7.3, respectively.

Proof of Theorems 2.13 and 2.14 (second half). By (9.1), the lower bounds in (2.15) and (2.16) directly follow from (9.15) and (9.27). Next, (2.20) follows from (9.17) since $\mu_{1,\infty}(q) = \infty$ implies that $\limsup_{A \rightarrow \infty} A^\alpha \lambda([A, \infty)) > 0$ for every $\alpha > q$. Finally, the upper bound in (2.22) follows from (2.20) because (2.21) implies $\mu_{1,\infty}(q) = \infty$ for all $q > \alpha$. \square

Proof of Proposition 9.5. In Section 9.2, counting the number of atoms in ω with size in a certain range was sufficient to determine the correct order of magnitude (in β) of $\bar{\gamma}_\beta(\frac{1}{2})$ (and hence of $\bar{\gamma}_\beta(p)$ for other values of p). If $d \geq 2$ and if λ is light-tailed, in order to obtain optimal estimates, we must consider certain *clusters* of atoms. Furthermore, the size of these point configurations, which we denote by k , must be taken large. We shall obtain our result (9.27) by proving that for any fixed $k \geq 3$, we have

$$\limsup_{\beta \rightarrow 0} \frac{\log|\log|\bar{\gamma}_\beta(\frac{1}{2})||}{|\log \beta|} \leq (1 + \frac{2}{d})\frac{k}{k-2}.$$

This will be achieved again by using Corollary 9.2. We need to introduce a few parameters for the functional f used in this proof. First, we fix $0 < a < b < \infty$ with $\lambda([a, b)) > 0$ and two parameters $0 < \kappa < \kappa'$ satisfying

$$(9.28) \quad \kappa \in (\frac{1}{k}, \frac{1}{k-1}) \quad \text{and} \quad (k-2)\kappa' + \kappa < 1.$$

Furthermore, we let $R = R(k, d)$ be such that

$$(9.29) \quad \left(\frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2}R}^{\sqrt{2}(R-1)} e^{-\frac{u^2}{2}} du \right)^d \left(\frac{1}{\sqrt{2\pi}} \int_{-R}^R e^{-\frac{u^2}{2}} du \right)^{d(k-1)} \geq 1 - \frac{e^{-K}}{8},$$

where K is the constant from Lemma 9.1. With this choice of R , we have, for any $T > 0$ and $\mathbf{t} \in \mathfrak{X}_k(T)$ such that $t_1 \leq \frac{T}{2}$, that

$$(9.30) \quad \inf_{x \in [0, \sqrt{T}]^d} \mathbf{Q}_x \left(\|B_{t_1}\|_\infty \leq R\sqrt{T}, \forall i \in \llbracket 2, k \rrbracket : \|\Delta B_{t_i}\|_\infty \leq R\sqrt{\Delta t_i} \right) \geq 1 - \frac{e^{-K}}{8}.$$

Finally, for some small $\varepsilon \in (0, 1)$ (independent of β and to be chosen later), we set

$$(9.31) \quad T := \exp \left(\varepsilon^{-\frac{4+d}{4(k-2)}} \beta^{-(1+\frac{2}{d})\frac{k}{k-2}} \right).$$

Writing $v_I := \prod_{i \in I} v_i$ for $I \subseteq \llbracket k \rrbracket$, we consider the multi-body functional

$$(9.32) \quad f(\omega) = \int_{\mathbb{X}_T^{(k)}} \mathbf{1}_{\{\forall i \in \llbracket k \rrbracket: z_i \in [a, b]\}} \mathbf{1}_{\{\|x_1\|_\infty \leq R\sqrt{T}, \forall i \in \llbracket 2, k \rrbracket: \|\Delta x_i\|_\infty \leq R\sqrt{\Delta t_i}\}} \\ \times \mathbf{1}_{\{t_1 \leq \frac{T}{2}, \forall i \in \llbracket 2, k-1 \rrbracket: \Delta t_i \in [T^{-\kappa'}, T^{-\kappa}], \Delta t_{\llbracket 2, k \rrbracket} \leq \varepsilon \beta^{2k/d} T^{-1}\}} \prod_{i=1}^k \delta_\omega(dw_i).$$

This function f counts clusters of atoms that are so close to each other (in time) that the benefit of visiting all of them, which is $\prod_{i=2}^k z_i \rho(\Delta t_i, \Delta x_i)$ or roughly $(\Delta t_{\llbracket 2, k \rrbracket})^{-d/2}$, outweighs the cost of visiting the first atom of the group, which is of order $T^{-d/2}$ in the bulk of the box. Limiting the possible range for Δt_i is convenient in the computations and our particular choice for this limitation is largely *ad hoc*. If

$$(9.33) \quad \varepsilon \beta^{\frac{2k}{d}} T^{-1+(k-2)\kappa'} \leq T^{-\kappa},$$

then the conditions on the t_i 's in the first line of (9.32) imply $\Delta t_k \leq T^{-\kappa}$. With our choice (9.31) for T and (9.28), the condition (9.33) is satisfied for any $\beta < 1$.

Let us now check that condition (C2) in Corollary 9.2 is satisfied. We start with the condition on $\mathbb{E}[f(\omega)]$. Integrating over the x_i 's and t_1 , making the change of variable $s_i = \Delta t_{i+1}$ and using that we have $\Delta t_k \leq T^{-\kappa}$, we get

$$(9.34) \quad \mathbb{E}[f(\omega)] = \frac{1}{2} T^{1+\frac{d}{2}} \left[(2R)^d \lambda([a, b]) \right]^k \int_{[T^{-\kappa'}, T^{-\kappa}]^{k-2} \times [0, T^{-\kappa}]} \mathbf{1}_{\{s_{\llbracket k-1 \rrbracket} \leq \varepsilon \beta^{2k/d} T^{-1}\}} \prod_{i=1}^{k-1} s_i^{\frac{d}{2}} ds_i.$$

Next, we integrate with respect to s_{k-1} and use (9.33) to obtain

$$(9.35) \quad T^{1+\frac{d}{2}} \int_{[T^{-\kappa'}, T^{-\kappa}]^{k-2} \times [0, T^{-\kappa}]} \mathbf{1}_{\{s_{k-1} \leq \varepsilon \beta^{2k/d} T^{-1}/s_{\llbracket k-2 \rrbracket}\}} \prod_{i=1}^{k-1} s_i^{\frac{d}{2}} ds_i \\ = \varepsilon^{1+\frac{d}{2}} \beta^{(1+\frac{2}{d})k} (1 + \frac{d}{2})^{-1} \int_{[T^{-\kappa'}, T^{-\kappa}]^{k-2}} \prod_{i=1}^{k-2} s_i^{-1} ds_i \\ = \varepsilon^{1+\frac{d}{2}} \beta^{(1+\frac{2}{d})k} (d+2)^{-1} [(\kappa' - \kappa) \log T]^{k-2}.$$

Therefore, by (9.31), there exists a constant $C = C(R, \lambda, a, b, \kappa, \kappa', k, d)$ such that $\mathbb{E}[f(\omega)] = C \varepsilon^{1+d/2} \beta^{(1+2/d)k} (\log T)^{k-2} = C \varepsilon^{d/4}$. Hence, choosing $\varepsilon \leq (C e^K)^{-4/d}$ fulfills the assumption on $\mathbb{E}[f(\omega)]$ in (C2); let us stress that ε must satisfy a second condition which will be specified later in the proof.

Let us now estimate the expectation and variance of $f'(\omega', B)$. We consider an alternative function without restrictions on B :

$$(9.36) \quad \bar{f}(\omega') := \int_{([0, T] \times [a, b])^k} \mathbf{1}_{\{t_1 \leq \frac{T}{2}, \forall i \in \llbracket 2, k-1 \rrbracket: \Delta t_i \in [T^{-\kappa'}, T^{-\kappa}]\}} \times \mathbf{1}_{\{\Delta t_{\llbracket 2, k \rrbracket} \leq \varepsilon \beta^{2k/d} T^{-1}\}} \prod_{i=1}^k \delta_{\omega'}(dt_i, dz_i)$$

By (9.30), similarly to (9.22) and (9.23), we have for any $x \in [0, \sqrt{T}]^d$ that

$$(9.37) \quad \mathbb{E}'_\beta \otimes \mathbf{Q}_x[f'(\omega', B)] \geq (1 - \frac{1}{8} e^{-K}) \mathbb{E}'_\beta[\bar{f}(\omega')] \geq \frac{1}{\sqrt{2}} \mathbb{E}'_\beta[\bar{f}(\omega')], \\ \text{Var}_{\mathbb{P}'_\beta \otimes \mathbf{Q}_x}(f'(\omega', B)) \leq \text{Var}_{\mathbb{P}'_\beta}(\bar{f}(\omega')) + \frac{1}{4} e^{-K} \mathbb{E}'_\beta[\bar{f}(\omega')]^2.$$

We can thus conclude that the variance bound in (C2) is satisfied if

$$(9.38) \quad \text{Var}_{\mathbb{P}'_\beta}(\bar{f}(\omega')) \leq \frac{1}{4} e^{-K} \mathbb{E}'_\beta[\bar{f}(\omega')]^2.$$

Proceeding as in (9.34)–(9.35) and recalling that ω' has intensity measure $\beta z dt \lambda(dz)$, we have

$$\begin{aligned}
 \mathbb{E}'_\beta[\bar{f}(\omega')] &= (\beta\mu_{a,b}(1))^k \frac{T}{2} \int_{[T-\kappa', T-\kappa]^{k-2} \times [0, T-\kappa]} \mathbf{1}_{\{s_{\llbracket k-1 \rrbracket} \leq \varepsilon \beta^{2k/d} T^{-1}\}} \prod_{i=1}^{k-1} ds_i \\
 (9.39) \quad &= \frac{\varepsilon}{2} \mu_{a,b}(1)^k \beta^{(1+\frac{2}{d})k} \int_{[T-\kappa', T-\kappa]^{k-2}} \prod_{i=1}^{k-2} s_i^{-1} ds_i \\
 &= C' \varepsilon \beta^{(1+\frac{2}{d})k} (\log T)^{k-2} = C' \varepsilon^{-\frac{d}{4}},
 \end{aligned}$$

with $C' = C'(\lambda, a, b, \kappa, \kappa', k, d)$, and where we have used the value (9.31) of T for the last identity. In order to compute the second moment of $f(\omega')$, we expand the integral in (9.36) by writing $\prod_{i=1}^k \delta_{\omega'}(dt_i, dz_i)$ as

$$(9.40) \quad \sum_{I \subseteq \llbracket k \rrbracket} \prod_{i \in I} \left(\delta_{\omega'}(dt_i, dz_i) - \beta z_i dt_i \lambda(dz_i) \right) \prod_{j \in I^c} \beta z_j dt_j \lambda(dz_j),$$

where $I^c := \llbracket k \rrbracket \setminus I$. We let f_I denote the integral corresponding to the term I . Since $f_\emptyset = \mathbb{E}'_\beta[\bar{f}(\omega')]$, we have

$$(9.41) \quad \text{Var}_{\mathbb{P}'_\beta}(\bar{f}(\omega')) = \sum_{I, J \subseteq \llbracket k \rrbracket: I, J \neq \emptyset} \mathbb{E}'_\beta[f_I(\omega') f_J(\omega')].$$

The terms in the sum above are equal to zero if I and J do not have the same cardinality. If $|I| = |J|$, then $\mathbb{E}'_\beta[f_I(\omega') f_J(\omega')]$ is obtained by matching the values of $(t_i)_{i \in I}$ to that of $(t_i)_{i \in J}$ in the integral before averaging. More precisely, we have

$$\mathbb{E}'_\beta[f_I(\omega') f_J(\omega')] = (\beta\mu_{a,b}(1))^{2k-|I|} \int_{[0, T]^{2k-|I|}} \left(\mathbf{1}_{\mathcal{Q}} \left((t_i^{(1)})_{i=1}^k \right) \prod_{j \in I^c} dt_j^{(1)} \right) \left(\mathbf{1}_{\mathcal{Q}} \left((t_i^{(2)})_{i=1}^k \right) \prod_{j \in J^c} dt_j^{(2)} \right) \prod_{i \in I} dt_i,$$

where \mathcal{Q} is the subset of $[0, T]^k$ induced by the indicator function in (9.36), and the vectors $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ are completed by setting

$$(9.42) \quad t_i^{(1)} = t_{\sigma(i)}^{(2)} = t_i \quad \text{for } i \in I,$$

with σ being the unique increasing bijection from I to J . In particular, we have

$$(9.43) \quad \mathbb{E}'_\beta[f_{\llbracket k \rrbracket}(\omega')^2] = \mathbb{E}'_\beta[\bar{f}(\omega')].$$

When $|I| = |J| = \ell \in \llbracket k-1 \rrbracket$, we are going to show that

$$(9.44) \quad \int_{[0, T]^{2k-|I|}} \left(\mathbf{1}_{\mathcal{Q}} \left((t_i^{(1)})_{i=1}^k \right) \prod_{j \in I^c} dt_j^{(1)} \right) \left(\mathbf{1}_{\mathcal{Q}} \left((t_i^{(2)})_{i=1}^k \right) \prod_{j \in J^c} dt_j^{(2)} \right) \prod_{i \in I} dt_i \leq T^{1-(2k-\ell-1)\kappa}.$$

So combining (9.43) with (9.44), we obtain that

$$\text{Var}_{\mathbb{P}'_\beta}(\bar{f}(\omega')) \leq \mathbb{E}'_\beta[\bar{f}(\omega')] + \sum_{\ell=1}^{k-1} \binom{k}{\ell}^2 (\beta\mu_{a,b}(1))^{2k-\ell} T^{1-(2k-\ell-1)\kappa},$$

which, for small β , is bounded by $\mathbb{E}'_\beta[\bar{f}(\omega')] + 1$ since $1 - (2k - \ell - 1)\kappa \leq 0$ thanks to (9.28). For this reason, (9.38) is satisfied if $\mathbb{E}'_\beta[\bar{f}(\omega')] \geq 8e^K$, which according to (9.39) holds true provided that ε is chosen small enough.

To complete the proof, let us show how (9.44) is obtained: the key is to integrate in the correct order the different variables. We construct a graph $G_{I,J}$ with $2k - \ell$ vertices where each vertex is identified with one of the variables $(t_i)_{i \in I}$, $(t_i^{(1)})_{i \in I^c}$, $(t_i^{(2)})_{i \in J^c}$. We start with two (initially disconnected) paths of k vertices labeled $(t_i^{(1)})_{i=1}^k$ and $(t_i^{(2)})_{i=1}^k$, respectively, the edges being $\{t_i^{(r)}, t_{i+1}^{(r)}\}$ for $i \in \llbracket k-1 \rrbracket$. Then we glue the two paths together by identifying the vertices $(t_i^{(1)})_{i \in I}$

with $(t_i^{(2)})_{i \in J}$ according to (9.42); any double edge that might have been created is replaced by a single edge. As $I \neq \emptyset$, the graph we obtain is connected. Now we consider an enumeration $v_1, v_2, v_3, \dots, v_{2k-\ell}$ of the vertices of $G_{I,J}$ such that for every $j \in \llbracket 2k - \ell - 1 \rrbracket$, the subgraph $G_{I,J}^{(j)}$ of $G_{I,J}$ induced by $v_{j+1}, \dots, v_{2k-\ell}$ is connected. One way of finding such an enumeration is, for example, to construct a spanning tree $T_{I,J}$ of $G_{I,J}$ and then, after v_1, \dots, v_j have been determined for some $j \in \{0, \dots, 2k - \ell - 1\}$, to take any leaf of $T_{I,J} \setminus \{v_1, \dots, v_j\}$ as v_{j+1} . We refer to Figure 2 for one example of $G_{I,J}$ together with a permitted and a forbidden enumeration according to the rule we just introduced.

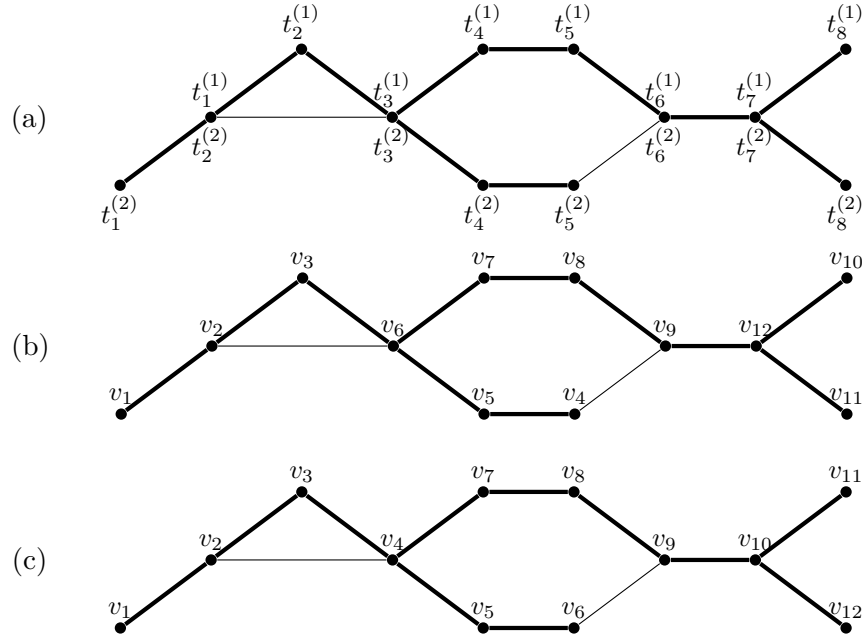


FIGURE 2. (a) The graph $G_{I,J}$ with $k = 8$, $I = \{1, 3, 6, 7\}$ and $J = \{2, 3, 6, 7\}$ and a spanning tree $T_{I,J}$ (thick edges); (b) a permitted enumeration of $G_{I,J}$ based on $T_{I,J}$; (c) a forbidden enumeration of $G_{I,J}$ based on $T_{I,J}$. Integration in the forbidden order only yields a suboptimal bound: once v_1, \dots, v_4 have been integrated (*i.e.*, removed from the graph), we obtain two disconnected components, so if integration is continued, v_6 becomes isolated and contributes a factor T when integrated (similarly, both v_{11} and v_{12} become isolated when v_1, \dots, v_{10} have been integrated: each of them contributes a factor of T).

Having fixed such an enumeration, in order to obtain (9.44), we integrate $v_1, \dots, v_{2k-\ell}$ in this order. Note that if $v_j = t_i^{(r)}$ for some $j \in \llbracket 2k - \ell - 1 \rrbracket$, $r \in \{1, 2\}$ and $i \in I^c$, then, by construction,

$$\{t_{i-1}^{(r)}, t_{i+1}^{(r)}\} \cap \{v_{j+1}, \dots, v_{2k-\ell}\} \neq \emptyset;$$

indeed, if both $t_{i-1}^{(r)}$ and $t_{i+1}^{(r)}$ had been picked before, then $v_j = t_i^{(r)}$ would be an isolated vertex in $G_{I,J}^{(j-1)}$ because $i \in I^c$. In the same manner, if $v_j = t_i$ for some $j \in \llbracket 2k - \ell - 1 \rrbracket$ and $i \in I$, then

$$\{t_{i-1}^{(1)}, t_{i+1}^{(1)}, t_{\sigma(i)-1}^{(2)}, t_{\sigma(i)+1}^{(2)}\} \cap \{v_{j+1}, \dots, v_{2k-\ell}\} \neq \emptyset.$$

As a result, for each $j \in \llbracket 2k - \ell - 1 \rrbracket$, the range when we integrate with respect to the variable v_j and the values of the variables $v_{j+1}, \dots, v_{2k-\ell}$ are fixed is included in an interval of length $T^{-\kappa}$: Indeed, by (9.33), the indicator function $\mathbf{1}_Q$ in the integral implies that $\Delta t_i^{(r)} \leq T^{-\kappa}$ for all $i \in \llbracket 2, k \rrbracket$ and $r \in \{1, 2\}$. We no longer have constraints only for the last variable $v_{2k-\ell}$, which yields a factor of T . \square

9.4. Proof of Proposition 2.3. It suffices to consider the free-end partition function. Since $(\mathcal{Z}_\beta^{\omega,a}(t, x))_{a \in (0,1]} \stackrel{(d)}{=} (e^{-\|x\|^2/(2t)} \mathcal{Z}_\beta^{\omega,a}(t, 0))_{a \in (0,1]}$, the a.s. positivity of the limit when $a \rightarrow 0$ does not depend on x . Hence the result in the point-to-point case follows from the free-end case by Fatou's Lemma. Although this is not essential to the reasoning, it is practical for our computations to assume that $\int_{(0,1)} z^{1+2/d} \lambda(dz) < \infty$. If this assumption is violated, then the result has already been proved in [13, Prop. 2.10]. We also assume that $\mu < \infty$: the general case follows from a truncation procedure, see the end of Section 5 in [13]. For notational simplicity, we actually assume that $\mu = 0$ (i.e., that λ has support in $(0, 1)$).

While the proof does not rely on Lemma 9.1, it partially builds on the same idea, which is to show that the size-biased measure transforms an atypical event under the original probability measure into a typical one. Let us summarize this in a lemma that is similar in spirit to Corollary 9.2. Recall (1.21) and the notation $\mathcal{Z}_{\beta,t}^{\omega,a} := \tilde{\mathcal{Z}}_\beta^{\omega,a}(t, *)$.

Lemma 9.6. *Let us assume that there exists a sequence of integer-valued functions $(f_n)_{n \in \mathbb{N}}$ of the form (9.5) (with Υ integer-valued and with T replaced by t) with the following properties:*

- (i) *For every n , f_n is \mathcal{G}_a -measurable for some $a = a_n > 0$.*
- (ii) *Using the notation (9.6), we have that*

$$(9.45) \quad \lim_{n \rightarrow \infty} \mathbb{E}[f_n(\omega)] = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\text{Var}_{\mathbb{P}'_\beta \otimes \mathbf{Q}}(f'_n(\omega', B))}{\mathbb{E}'_\beta \otimes \mathbf{Q}[f'_n(\omega', B)]^2} = 0.$$

Then we have

$$\lim_{a \rightarrow 0} \mathbb{E}[(\mathcal{Z}_{\beta,t}^{\omega,a})^{\frac{1}{2}}] = 0.$$

Proof. Consider $\varepsilon > 0$ and let n be sufficiently large such that both quantities in (9.45) are smaller than ε . Furthermore, let $a = a_n$ be such that f_n is \mathcal{G}_a -measurable. Defining $K := \log \varepsilon$ and $G(\omega) := \exp(-K \mathbf{1}_{\{f_n(\omega) \geq 1\}})$, we have by the Cauchy–Schwarz inequality,

$$\mathbb{E}[(\mathcal{Z}_{\beta,t}^{\omega,a})^{\frac{1}{2}}] \leq \mathbb{E}[G(\omega)^{-1}]^{\frac{1}{2}} \mathbb{E}[G(\omega) \mathcal{Z}_{\beta,t}^{\omega,a}]^{\frac{1}{2}}.$$

Clearly, by Markov's inequality,

$$\mathbb{E}[G(\omega)^{-1}] \leq 1 + e^K \mathbb{P}(f_n(\omega) \geq 1) \leq 2.$$

Using Lemma 4.1 and the fact that f_n is \mathcal{G}_a -measurable, we further have (recall the notation (9.2))

$$\mathbb{E}[G(\omega) \mathcal{Z}_{\beta,t}^{\omega,a}] = \mathbb{E} \otimes \mathbb{E}'_\beta \otimes \mathbf{Q}[\hat{G}_{a,t}(\omega, \omega', B)].$$

As f_n is integer-valued, the inequality (9.7) implies that

$$\hat{G}_{a,t}(\omega, \omega', B) \leq e^{-K} + \mathbf{1}_{\{\hat{f}_n(\omega, \omega', B) = 0\}} \leq e^{-K} + \mathbf{1}_{\{f'(\omega', B) = 0\}},$$

which, by an application of the Chebychev inequality, results in

$$\mathbb{E}_\beta[\hat{G}_{a,t}(\omega, \omega', B)] \leq e^{-K} + \mathbb{P}'_\beta \otimes \mathbf{Q}(f'_n(\omega', B) \leq 0) \leq e^{-K} + \varepsilon = 2\varepsilon.$$

Because $a \mapsto (\mathcal{Z}_{\beta,t}^{\omega,a})^{1/2}$ is a supermartingale, we obtain that

$$\limsup_{a' \rightarrow 0} \mathbb{E}[(\mathcal{Z}_{\beta,t}^{\omega,a'})^{\frac{1}{2}}] \leq \mathbb{E}[(\mathcal{Z}_{\beta,t}^{\omega,a})^{\frac{1}{2}}] \leq 2\sqrt{\varepsilon},$$

which finishes the proof since ε is arbitrary. \square

Before we specify the sequence of functions f_n , we need to introduce a few parameters. We let $k_n := 1 + \lfloor n^{1/3} \rfloor$ and we set for $n \geq 1$

$$M_n := \prod_{i=1}^n \frac{k_{i-1}(k_{i-1} + \frac{1}{2})}{(k_{i-1} + 1)(k_{i-1} - \frac{1}{2})} \quad \text{and} \quad b_n := e^{-M_n}.$$

Let us stress that $M_{n+1}/M_n = 1 + (2k_n^2)^{-1} + O(k_n^{-3})$, so $\log M_n \sim \frac{3}{2}n^{1/3} \sim \frac{3}{2}k_n$ as $n \rightarrow \infty$. On the interval $[b_{n+1}, b_n)$, $|\log z|/(\log|\log z|)^{5+4/d+\varepsilon}$ is of order $M_n(\log M_n)^{-(5+4/d+\varepsilon)}$, so the assumption (2.5) is equivalent to

$$(9.46) \quad \sum_{n=1}^{\infty} \frac{M_n}{(\log M_n)^{5+4/d+\varepsilon}} \int_{[b_{n+1}, b_n)} z^{1+\frac{2}{d}} \lambda(dz) = \infty.$$

Then, using that $\log M_n \sim cn^{1/3}$ and in particular that $M_n^{1/k_n} \leq C$ for some constant $C > 0$, we get that

$$(9.47) \quad \limsup_{n \rightarrow \infty} k_n^{-2(1+\frac{2}{d})-\frac{\varepsilon}{2}} M_n^{1-\frac{1}{k_n}} \int_{[b_{n+1}, b_n)} z^{1+\frac{2}{d}} \lambda(dz) = \infty,$$

because $k_n^{-3-\frac{\varepsilon}{2}}$ is summable. Considering a subsequence if necessary, we assume from now on, without loss of generality, that (9.47) holds with a limit instead of a lim sup. We set

$$(9.48) \quad V_n := M_n^{1-\frac{1}{k_n}} \int_{[b_{n+1}, b_n)} z^{1+\frac{2}{d}} \lambda(dz),$$

so that $\lim_{n \rightarrow +\infty} k_n^{-2(1+\frac{2}{d})-\frac{\varepsilon}{2}} V_n = \infty$. We let $\delta = \delta_\varepsilon$ be sufficiently small so that

$$(9.49) \quad \frac{d-2\delta}{d+2} \left(2(1+\frac{2}{d}) + \frac{\varepsilon}{2} \right) > 2,$$

and we set

$$(9.50) \quad R_n := V_n^{\delta/d} \quad \text{and} \quad \eta_n := 1 - \left[(2\pi)^{-\frac{1}{2}} \int_{[-R_n, R_n]} e^{-\frac{u^2}{2}} du \right]^{dk_n}.$$

One can easily check that $\lim_{n \rightarrow \infty} R_n = \infty$ and also $\lim_{n \rightarrow \infty} \eta_n = 0$ (using that $V_n \geq k_n^{2(1+2/d)}$).

We are now ready to introduce f_n . As in the proof of Proposition 9.5, we want to count clusters of k_n atoms that are beneficial to visit; note that here we have $k_n \rightarrow \infty$. More precisely, assuming without loss of generality that $t = 1$, we define

$$(9.51) \quad f_n(\omega) := \int_{\mathbb{X}_1^{(k_n)}} \mathbf{1}_{\{(\Delta t_{\llbracket k_n \rrbracket})^{-d/2} z_{\llbracket k_n \rrbracket} \geq A_n, \forall i \in \llbracket 2, k_n \rrbracket: \Delta t_i \in [b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}]\}} \\ \times \mathbf{1}_{\{\forall i \in \llbracket k_n \rrbracket: \|\Delta x_i\|_\infty \leq R_n \sqrt{\Delta t_i}, z_i \in [b_{n+1}, b_n)\}} \prod_{i=1}^{k_n} \delta_\omega(dw_i),$$

where we have used the shorthand notation

$$(9.52) \quad A_n := V_n^{\frac{d}{d+2}(1+\delta)k_n}, \quad \theta_n := \frac{2k_n+1}{2k_n-1} \quad \text{and} \quad \theta'_n := \frac{k_n}{k_n-1}.$$

Note that f_n is $\mathcal{G}_{b_{n+1}}$ -measurable. The restriction on the range for x_i and the requirement that $(\Delta t_{\llbracket k_n \rrbracket})^{-d/2} z_{\llbracket k_n \rrbracket}$ be large are important features of f . The remaining constraints are *ad hoc* and mainly serve the purpose to lighten the computations. If z_i and t_i satisfy the constraints above, then recalling the definition of θ'_n , we have

$$(9.53) \quad t_1^{\frac{d}{2}} \leq A_n^{-1} (\Delta t_{\llbracket 2, k_n \rrbracket})^{-\frac{d}{2}} z_{\llbracket k_n \rrbracket} \leq A_n^{-1} \exp(M_n[(k_n-1)\theta'_n - k_n]) = A_n^{-1}.$$

Since $A_n \rightarrow \infty$ (recall that $V_n \rightarrow \infty$) and $k_n b_n^{2\theta_n/d} \leq k_n \exp(-\frac{2}{d}M_n) \rightarrow 0$, we have $A_n^{-1} + (k_n - 1)b_n^{2\theta_n/d} \leq 1$ for sufficiently large n , and hence

$$\left\{ (\Delta t_{\llbracket k_n \rrbracket})^{-\frac{d}{2}} z_{\llbracket k_n \rrbracket} \geq A_n, \forall i \in \llbracket 2, k_n \rrbracket: \Delta t_i \in [b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}] \right\} \implies t_{k_n} \leq 1.$$

Consequently, integrating first with respect to the x_i 's, making a change of variables $s_i = \Delta t_i$ and then integrating only with respect to s_{k_n} , we have

$$\begin{aligned}
 \mathbb{E}[f_n(\omega)] &= (2R_n)^{dk_n} \int_{[b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}]^{k_n-1} \times [0,1] \times [b_{n+1}, b_n]^{k_n}} \mathbf{1}_{\{(s_{\llbracket k_n \rrbracket})^{d/2} \leq A_n^{-1} z_{\llbracket k_n \rrbracket}\}} \prod_{i=1}^{k_n} s_i^{\frac{d}{2}} ds_i \lambda(dz_i) \\
 (9.54) \quad &= \frac{2}{d+2} (2R_n)^{dk_n} A_n^{-(1+\frac{2}{d})} \int_{[b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}]^{k_n-1} \times [b_{n+1}, b_n]^{k_n}} \prod_{j=1}^{k_n-1} s_j^{-1} ds_j \prod_{i=1}^{k_n} z_i^{1+\frac{2}{d}} \lambda(dz_i) \\
 &= \frac{2}{d+2} 2^{dk_n} V_n^{-k_n} \left(\frac{2(\theta'_n - \theta_n)}{d} M_n \right)^{k_n-1} \left(\int_{[b_{n+1}, b_n]} z^{1+\frac{2}{d}} \lambda(dz) \right)^{k_n},
 \end{aligned}$$

where we have used the definition of R_n and A_n in the last line. Using the definition of V_n , together with the fact that $\theta'_n - \theta_n \leq \frac{1}{2(k_n-1)^2}$, we get that

$$\mathbb{E}[f_n(\omega)] \leq \frac{2^{1+d}}{d+2} \left(\frac{2^d}{d(k_n-1)^2} \right)^{k_n-1}.$$

Since $k_n \rightarrow \infty$, this shows the first condition in (9.45).

To check the second condition in (9.45), we compute the expectation and variance of

$$\bar{f}_n(\omega') := \int_{\mathfrak{X}_{k_n}(1) \times [b_{n+1}, b_n]^{k_n}} \mathbf{1}_{\{(\Delta t_{\llbracket k_n \rrbracket})^{-d/2} z_{\llbracket k_n \rrbracket} \geq A\}} \mathbf{1}_{\{\forall i \in \llbracket 2, k_n \rrbracket: \Delta t_i \in [b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}]\}} \prod_{i=1}^{k_n} \delta_{\omega'}(dt_i, dz_i).$$

Recalling the notation (9.50) of η_n , we have $\mathbf{Q}(\forall i \in \llbracket k_n \rrbracket: \|\Delta B_{t_i}\|_\infty \leq R\sqrt{\Delta t_i}) = 1 - \eta_n$. Hence, repeating the computations that led to (9.37) in the previous section, we obtain

$$\begin{aligned}
 \mathbb{E}'_\beta \otimes \mathbf{Q}[f'_n(\omega', B)] &= (1 - \eta_n) \mathbb{E}'_\beta[\bar{f}_n(\omega')], \\
 \text{Var}_{\mathbb{P}'_\beta \otimes \mathbf{Q}}[f'_n(\omega', B)] &\leq \text{Var}_{\mathbb{P}'_\beta}(\bar{f}_n(\omega')) + 2\eta_n \mathbb{E}'_\beta[f'_n(\omega')]^2.
 \end{aligned}$$

Since η_n tends to zero, the reader can check that the second condition in (9.45) follows from the following claims:

$$\begin{aligned}
 (9.56) \quad &\lim_{n \rightarrow \infty} \mathbb{E}'_\beta[\bar{f}_n(\omega')] = \infty, \\
 &\text{Var}_{\mathbb{P}'_\beta}(\bar{f}_n(\omega')) \leq 2\mathbb{E}'_\beta[\bar{f}_n(\omega')] \quad \text{for sufficiently large } n.
 \end{aligned}$$

The first property in (9.56) follows from direct calculation. We have

$$\begin{aligned}
 \mathbb{E}'_\beta[\bar{f}_n(\omega')] &= \int_{[b_n^{2\theta'_n/d}, b_n^{2\theta_n/d}]^{k_n-1} \times [0,1] \times [b_{n+1}, b_n]^{k_n}} \mathbf{1}_{\{(s_{\llbracket k_n \rrbracket})^{d/2} \leq A_n^{-1} z_{\llbracket k_n \rrbracket}\}} \prod_{i=1}^{k_n} ds_i \beta z_i \lambda(dz_i) \\
 (9.57) \quad &= \beta^{k_n} A_n^{-\frac{2}{d}} \left(\frac{2(\theta'_n - \theta_n)}{d} M_n \right)^{k_n-1} \left(\int_{[b_{n+1}, b_n]} z^{1+\frac{2}{d}} \lambda(dz) \right)^{k_n} \\
 &= \frac{d}{2(\theta'_n - \theta_n)} \left(\frac{2\beta}{d} (\theta'_n - \theta_n) V_n^{\frac{d-2\delta}{d+2}} \right)^{k_n},
 \end{aligned}$$

where the computation is similar to (9.54). Now, since $\theta'_n - \theta_n \sim \frac{1}{2}k_n^{-2}$ and $k_n^{-2(1+2/d)-\varepsilon/2} V_n \rightarrow \infty$, (9.49) implies that $(\theta'_n - \theta_n) V_n^{(d-2\delta)/(d+2)} \rightarrow \infty$, which leads to the first part of (9.56).

As in the previous section, to compute the second moment of $\bar{f}_n(\omega')$, we can use (9.40) to write $\text{Var}_{\mathbb{P}'_\beta}(\bar{f}_n(\omega'))$ in the form (9.41). As seen before, only sets with $|I| = |J|$ contribute, and the term corresponding to $I = J = \llbracket k_n \rrbracket$ is simply $\mathbb{E}'_\beta[\bar{f}_n(\omega')]$. If $|I| = |J| = \ell \in \llbracket k_n - 1 \rrbracket$, then

$\mathbb{E}'_\beta[f_I(\omega')f_J(\omega')]$ equals

$$\begin{aligned} & \beta^{2k_n-\ell} \int_{((0,1) \times [b_{n+1}, b_n])^{2k_n-\ell}} \mathbf{1}_{\mathcal{R}}\left((t_i^{(1)})_{i=1}^{k_n}, (z_i^{(1)})_{i=1}^{k_n}\right) \prod_{j \in I^c} z_i^{(1)} dt_j^{(1)} \lambda(dz_i^{(1)}) \\ & \quad \times \mathbf{1}_{\mathcal{R}}\left((t_i^{(2)})_{i=1}^{k_n}, (z_i^{(2)})_{i=1}^{k_n}\right) \prod_{j \in J^c} z_i^{(2)} dt_j^{(2)} \lambda(dz_i^{(2)}) \prod_{i \in I} z_i dt_i \lambda(dz_i), \end{aligned}$$

where \mathcal{R} is the subset of $((0,1) \times [b_{n+1}, b_n])^{k_n}$ induced by the indicator function in (9.55) and for $i \in I$, we used the notation

$$t_i^{(1)} = t_{\sigma(i)}^{(2)} = t_i \quad \text{and} \quad z_i^{(1)} = z_{\sigma(i)}^{(2)} = z_i.$$

Next, we relax the constraints imposed by \mathcal{R} by only keeping $\Delta t_i^{(r)} \leq b_n^{2\theta_n/d}$. Then we proceed as in the proof of (9.44): by choosing an optimal order, we obtain that integration over each time variable produces a factor of at most $b_n^{2\theta_n/d}$, except for the last integration, which produces a factor of one. Thus,

$$\begin{aligned} \mathbb{E}'_\beta[f_I(\omega')f_J(\omega')] & \leq \beta^{2k_n-\ell} b_n^{\frac{2\theta_n}{d}(2k_n-\ell-1)} \left(\int_{[b_{n+1}, b_n]} z \lambda(dz) \right)^{2k_n-\ell} \\ & \leq \beta^{2k_n-\ell} e^{\frac{2}{d}[(2k_n-\ell)M_{n+1}-(2k_n-\ell-1)\theta_n M_n]} \mu_{0,1} \left(1 + \frac{2}{d}\right)^{2k_n-\ell}, \end{aligned}$$

where in the second line we simply used the fact that

$$\int_{[b_{n+1}, b_n]} z \lambda(dz) \leq (b_{n+1})^{-\frac{2}{d}} \mu_{0,1} \left(1 + \frac{2}{d}\right),$$

together with the definition of b_n and b_{n+1} . Recalling the definition of θ_n and of M_n , we have $M_{n+1} = \frac{k_n}{k_n+1} M_n \theta_n$, so that

$$(2k_n - \ell)M_{n+1} - (2k_n - \ell - 1)\theta_n M_n = M_n \theta_n k_n \left[\frac{2k_n - \ell}{k_n + 1} - \frac{2k_n - \ell - 1}{k_n} \right] \leq 0$$

for any $\ell \leq k_n - 1$. We therefore end up with

$$\begin{aligned} \text{Var}_{\mathbb{P}'_\beta}(\bar{f}_n(\omega')) & \leq \mathbb{E}'_\beta[\bar{f}_n(\omega')] + \sum_{\ell=1}^{k_n-1} \binom{k_n}{\ell}^2 \beta^{2k_n-\ell} \mu_{0,1} \left(1 + \frac{2}{d}\right)^{2k_n-\ell} \\ & \leq \mathbb{E}'_\beta[\bar{f}_n(\omega')] + C'(d, \lambda, \beta)^{k_n}. \end{aligned}$$

Recalling (9.57), we have that for any constant $C > 0$ the expectation $\mathbb{E}'_\beta[\bar{f}_n(\omega')]$ is larger than C^{k_n} for sufficiently large n . This proves the second claim in (9.56) and concludes the proof. \square

APPENDIX A. SOME TECHNICAL RESULTS

A.1. Proof of Theorem 4.3. The result is contained in [75], but only implicitly, so we give a short proof. Let us first check that (4.3) ensures that all integrals in (4.4) are well defined and finite. Applying the BDG inequality and using the subadditivity (5.4) of the function $x \mapsto x^{p/2}$ for $x > 0$ iteratively, we deduce that

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\mathbb{X}^N} f(\mathbf{w}) \prod_{j=1}^N M^\omega(dw_j) \right|^p \right] & \leq C_p \mathbb{E} \left[\left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}^{N-1}} f(\mathbf{w}) \prod_{j=1}^{N-1} M^\omega(dw_j) \right)^2 \delta_\omega(dw_N) \right)^{\frac{p}{2}} \right] \\ (A.1) \quad & \leq C_p \int_{\mathbb{X}} \mathbb{E} \left[\left| \int_{\mathbb{X}^{N-1}} f(\mathbf{w}) \prod_{j=1}^N M^\omega(dw_j) \right|^p \right] \nu(dw_N) \\ & \leq \dots \leq C_p^N \int_{\mathbb{X}^N} |f(\mathbf{w})|^p \nu(dw_1) \dots \nu(dw_N) < \infty. \end{aligned}$$

These inequalities remain unchanged if ω is replaced by $\omega_1, \dots, \omega_N$ and \mathbb{E} is replaced by $\mathbb{E}^{\otimes N}$.

Now we move to the proof of (4.4), for which we shall prove the second inequality. To obtain the reverse inequality it is sufficient follow the same proof and observe that all estimates are two-sided (*i.e.*, one can always substitute “ \geq ” for “ \leq ” if one also replaces $1/C$ by C). We now consider a state space on which ω is jointly defined with our i.i.d. copies $\omega_1, \dots, \omega_N$ (and ω is independent of $(\omega_1, \dots, \omega_N)$). We let $\bar{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}^{\otimes N}$ denote the associated probability. In analogy with (1.5), we consider a filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ on this state space defined by $\bar{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^{(1)} \otimes \dots \otimes \mathcal{F}_t^{(N)}$, where $\mathcal{F}_t^{(i)} := \sigma(\omega \cap ([0, t] \cap \mathbb{R}^d \times (0, \infty)))$ for $i = 1, \dots, N$. Our result follows from applying the following inequality, valid for $i = 1, \dots, N$, iteratively:

$$(A.2) \quad \bar{\mathbb{E}} \left[\int_{\mathbb{X}^N} f(\mathbf{w}) \prod_{j=1}^i M^\omega(dw_j) \prod_{j=i+1}^N M^{\omega_j}(dw_j) \right] \leq \frac{C}{1-p} \bar{\mathbb{E}} \left[\int_{\mathbb{X}^N} f(\mathbf{w}) \prod_{j=1}^{i-1} M^\omega(dw_j) \prod_{j=i}^N M^{\omega_j}(dw_j) \right].$$

Let us first spend some time on the first step $i = N$. By elementary properties of Itô integrals, $X_t := \int_{\mathbb{X}^N} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_N) \prod_{j=1}^N M^\omega(dw_j)$ and $Y_t := \int_{\mathbb{X}^N} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_N) (\prod_{j=1}^{N-1} M^\omega(dw_j)) M^{\omega_N}(dw_N)$ define $(\bar{\mathcal{F}}_t)_{t \geq 0}$ -martingales, with $[X]_t = \int_{\mathbb{X}} (\int_{\mathbb{X}^{N-1}} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_N) \prod_{j=1}^{N-1} M^\omega(dw_j))^2 \delta_{\omega_N}(dw_N)$ and $[Y]_t = \int_{\mathbb{X}} (\int_{\mathbb{X}^{N-1}} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_N) \prod_{j=1}^{N-1} M^\omega(dw_j))^2 \delta_{\omega_N}(dw_N)$ being the corresponding quadratic variation processes. Since ω and ω_N are Poisson random measures with the same $(\bar{\mathcal{F}}_t)_{t \geq 0}$ -intensity measures (namely $dt \otimes dx \otimes \lambda(dz)$), the jump measures associated to $[X]$ and $[Y]$ have the same predictable compensator in $(\bar{\mathcal{F}}_t)_{t \geq 0}$. As a result, X and Y are *weakly tangential* martingales in the sense of [75]. Thus, by [75, Thm. 4.1] and Doob’s inequality, there is $C_p > 0$ such that

$$(A.3) \quad \bar{\mathbb{E}}[|X_\infty|^p] \leq \bar{\mathbb{E}} \left[\sup_{t \geq 0} |X_t|^p \right] \leq C_p \bar{\mathbb{E}} \left[\sup_{t \geq 0} |Y_t|^p \right] \leq C_p \left(\frac{p}{p-1} \right)^p \bar{\mathbb{E}}[|Y_\infty|^p].$$

The reader can check that for the implicit constant C_p from [75, Thm. 4.2], one may take $C_p := (24p^2)^p [2 \times 3^{p/2} (2^{p/2+1} + 14 \times 3^{p/2} (28 \times 3^{p/2})^{p/2})]$. The expression in brackets comes from the last equation in [75, p. 38] (with $\varphi(x) = |x|^{p/2}$ and $c = (14 \times 3^{p/2})^{-1}$), while the computation in [75, p. 39], combined with the BDG inequality as in [53, Ch. VII, Thm. 92], entails an additional factor of $(24p^2)^p$. Most importantly, C_p is bounded uniformly in $p \in (1, 2]$, so that there exists a universal constant C such that for $p \in (1, 2]$, $\bar{\mathbb{E}}[|X_\infty|^p] \leq C(p-1)^{-1} \bar{\mathbb{E}}[|Y_\infty|^p]$, which finishes the proof of (A.2) for $i = N$.

In order to iterate and prove (A.2) for $i \leq N-1$, we wish to interchange $M^\omega(dw_1) \dots M^\omega(dw_i)$ with $M^{\omega_{i+1}}(dw_{i+1}) \dots M^{\omega_N}(dw_N)$ as follows:

$$(A.4) \quad \int_{\mathbb{X}^N} f(\mathbf{w}) \prod_{j=1}^i M^\omega(dw_i) \prod_{j=i+1}^N M^{\omega_j}(dw_j) = \int_{\mathbb{X}^i} \left(\int_{\mathbb{X}^{N-i}} f(\mathbf{w}) \prod_{j=i+1}^N M^{\omega_j}(dw_j) \right) \prod_{j=1}^i M^\omega(dw_i).$$

Even though the integral on the r.h.s. is anticipative when considering the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$, we can recover an integral in Itô’s sense by constructing the inner integrals $\int_{\mathbb{X}^{N-i}} f(\mathbf{w}) \prod_{j=i+1}^N M^{\omega_j}(dw_j)$ using the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ and the outer integrals using the filtration $\bar{\mathcal{F}}_t^{(i)}$ defined by $\bar{\mathcal{F}}_t^{(i)} := \mathcal{F}_t \otimes \bigotimes_{j=1}^i \mathcal{F}_t^{(j)} \otimes \bigotimes_{j=i+1}^N \mathcal{F}_\infty^{(j)}$. Note that the inner integrals are $\bar{\mathcal{F}}_0^{(i)}$ -measurable. With this convention, we can justify (A.4) as follows: It certainly holds if f is a step function, that is, if f only assumes finitely many values (in that case, the integrals are simply finite sums). For general f , take a sequence of step functions $(f_n)_{n \in \mathbb{N}}$ such that $|f_n| \leq |f|$ for all n and $f_n \rightarrow f$ pointwise as $n \rightarrow \infty$. Equation (A.4) holds for f_n , and arguing similarly to (A.1) and using dominated convergence, we deduce (A.4). The fact that we are able to interpret the latter integral in Itô’s sense is crucial for the BDG inequality, which was needed in (A.1), to apply. Once (A.4) is

established, we can prove (A.2) by considering the weakly tangential martingales (for the filtration $(\bar{\mathcal{F}}_t^{(i)})_{t \geq 0}$)

$$X_t^{(i)} := \int_{\mathbb{X}^i} \left(\int_{\mathbb{X}^{N-i}} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_i) \prod_{j=i+1}^N M^{\omega_j}(dw_j) \right) \prod_{j=1}^i M^{\omega_j}(dw_j)$$

$$Y_t^{(i)} := \int_{\mathbb{X}^i} \left(\int_{\mathbb{X}^{N-i}} f(\mathbf{w}) \mathbf{1}_{(0,t]}(t_i) \prod_{j=i+1}^N M^{\omega_j}(dw_j) \right) \prod_{j=1}^{i-1} M^{\omega_j}(dw_j) M^{\omega_i}(dw_i),$$

applying (A.3) and re-arranging the integrals similarly to (A.4).

A.2. Proof of Lemma 4.4. For $A, u, \varepsilon > 0$ and $H \in \mathcal{S}$ that satisfies $|H| \leq u|K|$, we have that

$$\mathbb{P} \left(\left| \int_{\mathbb{X}} H(\omega, t, x) \xi_{\omega <} (dt, dx) \right| > \varepsilon \right) \leq \mathbb{P}_{\geq} (Au \|K\|_{\xi_{\omega <}, p; \mathbb{P}_{<}} > \varepsilon)$$

$$+ \mathbb{P} \left(\left| \int_{\mathbb{X}} H(\omega, t, x) \xi_{\omega <} (dt, dx) \right| > uA \|K\|_{\xi_{\omega <}, p; \mathbb{P}_{<}} \right).$$

The second probability is bounded by

$$\mathbb{E}_{\geq} \left[\frac{1}{u^p A^p \|K\|_{\xi_{\omega <}, p; \mathbb{P}_{<}}^p} \mathbb{E}_{<} \left[\left| \int_{\mathbb{X}} H(\omega, t, x) \xi_{\omega <} (dt, dx) \right|^p \right] \right] \leq A^{-p}.$$

Therefore, using dominated convergence for the first term, we get

$$\lim_{u \rightarrow 0} \sup_{H \in \mathcal{S}, |H| \leq u|K|} \mathbb{P} \left(\left| \int_{\mathbb{X}} H(\omega, t, x) \xi_{\omega <} (dt, dx) \right| > \varepsilon \right) \leq \lim_{u \rightarrow 0} \mathbb{E}_{\geq} [\mathbf{1}_{\{Au \|K\|_{\xi_{\omega <}, p; \mathbb{P}_{<}} > \varepsilon\}}] + A^{-p} \leq A^{-p}.$$

Sending $A \rightarrow \infty$ shows that the left-hand side is 0, which is equivalent to $\|uK\|_{\xi_{<}, 0} \rightarrow 0$ as $u \rightarrow 0$. By (4.6), K is integrable with respect to $\xi_{\omega <}$. \square

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DMA, ÉCOLE NORMALE SUPÉRIEURE, UNIVERSITÉ PSL, 75005 PARIS, FRANCE.

Email address: `quentin.berger@sorbonne-universite.fr`

DEPARTMENT OF STATISTICS, COLUMBIA UNIVERSITY, 1255 AMSTERDAM AVENUE, NEW YORK, NY 10027, USA

Email address: `carsten.chong@columbia.edu`

IMPA, INSTITUTO DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA 110, RIO DE JANEIRO CEP-22460-320, BRAZIL

Email address: `lacoin@impa.br`