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TSO-DSOs Stable Cost Allocation for the Joint Procurement of Flexibility: A Cooperative Game Approach

Anibal Sanjab, H el ene Le Cadre, and Yuting Mou

Abstract—In this paper, a transmission-distribution systems flexibility market is introduced, in which system operators (SOs) jointly procure flexibility from different systems to meet their needs (balancing and congestion management) using a common market. This common market is, then, formulated as a cooperative game aiming at identifying a stable and efficient split of costs of the jointly procured flexibility among the participating SOs to incentivize their cooperation. The non-emptiness of the core of this game is then mathematically proven, implying the stability of the game and the naturally-arising incentive for cooperation among the SOs. Several cost allocation mechanisms are then introduced, while characterizing their mathematical properties. Numerical results focusing on an interconnected system (composed of the IEEE 14-bus transmission system and the Matpower 18-bus, 69-bus, and 141-bus distributions systems) showcase the cooperation-induced reduction in system-wide flexibility procurement costs, and identifies the varying costs borne by different SOs under various cost allocations methods.

I. INTRODUCTION

The increasing integration of distributed energy resources (DERs) and electrification of the consumer energy space (e.g., transportation and heating) pose challenges for grid operation, due to the induced uncertainty and changing load patterns. However, this new energy landscape also enables an unprecedented growing volume of invaluable flexibility¹ (from different voltage levels of the grid) thereby providing essential services (e.g., congestion management and balancing) for transmission system operators (TSOs) and distribution system operators (DSOs). In this respect, the introduction of market mechanisms for the procurement of flexibility from flexibility services providers (FSPs) has been increasingly recommended in policies [1], and has been the center of several recent works in the literature [2]–[7] and demonstration projects [8].

As FSPs could provide their flexibility as a service to different system operators (SOs), a major branch of the literature has focused on the SOs’ joint procurement (i.e. co-optimization) of flexibility [3], [6], to maximize the grid and system-level value of this flexibility. In particular, a key focus has been shed on the need for coordination between SOs to achieve joint procurement, not only for optimization of economic efficiency but also to ensure that the activated flexibility meets grid operational constraints of all the grids involved [2]–[5], [9], [10]. However, *when jointly procuring flexibility, it is paramount to decide on how the costs of this flexibility should be divided among the participating SOs*

in the most efficient and fair way, and most importantly, in a manner that incentivizes system operators to naturally collaborate and jointly procure flexibility rather than running their disjoint markets. To the best of our knowledge², no work in the literature has presented a fundamental analysis and comprehensive solution to this joint cooperation and cost allocation problem, as is the goal of the current work.

In this paper, we develop a novel cooperative game-theoretic³ approach in which 1) we develop the concept of cooperation stability between the TSO and DSOs in a joint flexibility market, 2) determine whether cooperation between the TSO and DSOs is naturally incentivized and the conditions therefore, and 3) derive several cost allocation mechanisms and analytically investigate their properties focusing on concepts such as stability, efficiency, and computational complexity.

Towards this end, we first introduce a novel flexibility market model including a TSO and multiple DSOs for jointly procuring congestion management and balancing services while explicitly accounting for grid constraints. This framework is developed by first introducing disjoint TSO and DSO level markets and joining them in a common market setting. We then formulate the joint procurement of flexibility between any subset of SOs as a cooperative (cost allocation) game and analyze its properties. We subsequently prove the concavity of this game, hence, guaranteeing the non-emptiness of its core⁴ under several pricing mechanisms. This, as a result, proves that *all SOs would be better off joining the grand coalition (i.e., the set of all SOs in a common market setting) as compared to joining any sub-coalition (i.e., a coalition of a subset of SOs, or forming disjoint markets with no cooperation).* Then, we introduce several cost allocation mechanisms to allocate the costs of the jointly procured flexibility among the different SOs (namely, *the Shapley value, normalized Banzhaf index, cost gap allocation, Lagrangian-based cost allocation, equal profit method, and proportional cost*) and investigate their stability (whether they belong to the core), allocation adequacy, and other mathematical properties such as: efficiency, symmetry, additivity, dummy player, anonymity, and computational complexity.

Our analytical conclusions and results are further corroborated using a flexibility market case analysis focusing on an interconnected test system composed of the IEEE 14-bus transmission system interconnected with three distribution

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¹Flexibility is the ability to dynamically modify consumption and generation patterns providing, as a result, a service to system operators.

²We note that the work in [3] looks at a Retailer-DSO-TSO coordination setting while focusing particularly on the Shapley value for the numerical computation of cost allocation, hence, defers in scope from the current work.

³Recent years have seen a few applications of cooperative game concepts to energy system problems, with primary focus on the residential sector and shared energy investments, such as in [11]–[13].

⁴The core is a game-theoretic concept used to assess cooperation stability, and will be formally defined in Section III.

systems – namely, the Matpower 141-bus, 69-bus and 18-bus systems. The numerical results highlight 1) the significant reduction in system cost when more SOs joint the common market grand coalition, highlighting the benefits of cooperation, 2) the essential role of higher interface flows on reaping the benefits of cooperation, and 3) the disparity that can result from different cost allocation methods, hence, providing indispensable inputs to SOs, regulators, and decision makers.

The rest of the paper is organized as follows. Section II presents the disjoint and common markets formulations. Section III introduces the cooperative game formulation. Section IV investigates the stability of the game, while Section V provides different cost allocation mechanisms and characterizes their properties. Section VI introduces the numerical results, and Section VII concludes the paper.

II. SYSTEMS AND MARKETS MODELS

Consider a transmission system (operated by a TSO) composed of a set of nodes, \mathcal{N}^T , and set of lines, \mathcal{L}^T , represented by a graph $\mathcal{G}^T(\mathcal{N}^T, \mathcal{L}^T)$. At a subset, \mathcal{N}^D , of these nodes, distribution systems (each operated by a DSO) are connected. We refer to $\mathcal{N}^D \subseteq \mathcal{N}^T$, as the set of interface nodes, containing \mathcal{N}^D nodes from each of which stems one of the N^D different distribution systems. We let a) p_n^T be the net real power injection at node $n \in \mathcal{N}^T$, b) $\mathbf{p}^{T,o}$ and $\mathbf{d}^{T,o}$ be, respectively, the vectors of anticipated base injection and load at all transmission system nodes \mathcal{N}^T , c) P_{ij}^T denote the real power flow over line $\{i, j\} \in \mathcal{L}^T$, with maximum thermal line limit $F_{ij}^{T,\max}$, d) T_{nT}^p and T_{nT}^q denote the active and reactive power transfer to the distribution system connected to node $n^T \in \mathcal{N}^D$, and e) $X_{(i,j),n}$ denote the generation shift factor of P_{ij}^T with respect to the net injection at node $n \in \mathcal{N}^T$, capturing the change in the flow over line $\{i, j\} \in \mathcal{L}^T$ due to a change in net injection at node $n \in \mathcal{N}^T$.

We denote a distribution system connected to a transmission node $n^T \in \mathcal{N}^D$ by DSO- n^T . Each DSO- n^T is composed of a radial distribution network composed of \mathcal{N}^{n^T} nodes (where node $n_0^{n^T} \in \mathcal{N}^{n^T}$ is the root node of DSO- n^T) and \mathcal{L}^{n^T} distribution lines forming a graph, $\mathcal{G}^{n^T}(\mathcal{N}^{n^T}, \mathcal{L}^{n^T})$. In each DSO- n^T , we let $A(n)$ denote the ancestor node of $n \in \mathcal{N}^{n^T} \setminus \{n_0^{n^T}\}$ and $\mathcal{K}(n)$ the set of predecessor nodes of n . For the root node $n_0^{n^T}$, $A(n_0^{n^T}) = n^T$. For each DSO- n^T , we let a) $s_n^{n^T} = p_n^{n^T} + jq_n^{n^T}$ be the net complex power injection at bus $n \in \mathcal{N}^{n^T}$, b) $\mathbf{p}^{n^T,o}$ and $\mathbf{d}^{n^T,o}$ be, respectively, the vectors of anticipated base generation and demand at all distribution system nodes \mathcal{N}^{n^T} , c) $P_{A(n)n}^{n^T}$ and $Q_{A(n)n}^{n^T}$ be, respectively, the real and reactive power flowing over the line connecting $A(n)$ and n , with a maximum apparent power flow denoted by $S_{A(n)n}^{n^T,\max}$, and d) $v_n^{n^T}$ be the magnitude squared of the voltage at node $n \in \mathcal{N}^{n^T}$, with upper and lower limits specified, respectively, by $v_n^{n^T,\max}$ and $v_n^{n^T,\min}$. Here, $P_{A(n_0^{n^T})n_0^{n^T}}^{n^T}$ and $Q_{A(n_0^{n^T})n_0^{n^T}}^{n^T}$ denote the interface flows with the transmission grid, which are equal, respectively, to T_{nT}^p and T_{nT}^q defined on the transmission side. In addition, for the line parameters, we let $r_{A(n)n}^{n^T}$ and $x_{A(n)n}^{n^T}$ be, respectively, the resistance and reactance of line $\{A(n), n\} \in \mathcal{L}^{n^T}$.

We next introduce the formulation of disjoint transmission and distribution markets, followed by the common market model, which joins the distribution and transmission-level markets.

A. Disjoint Transmission-Level Market

We consider that the TSO's anticipated base schedule $\mathbf{p}^{T,o}$ and $\mathbf{d}^{T,o}$ shows imbalance and/or line congestion which the TSO aims to solve using resources available only from the transmission level (hence, resulting in a disjoint market). These resources are represented by offers to the market from assets connected to the transmission network. Let Δp_n^{T+} and Δp_n^{T-} correspond, respectively, to the volumes of increase and reduction in generation (respectively, upward and downward flexibility) connected at the transmission bus n . In addition, let Δd_n^{T+} and Δd_n^{T-} correspond, respectively, to the volumes of reduction and increase in demand (i.e., upward and downward flexibility) at node n of the transmission network. In addition, let $c_{p_n}^{T+}$, $c_{p_n}^{T-}$, $c_{d_n}^{T+}$, and $c_{d_n}^{T-}$ represent the unit price offered by adjustable generation and loads for, respectively, Δp_n^{T+} , Δp_n^{T-} , Δd_n^{T+} , and Δd_n^{T-} . The goal of the TSO is to resolve the balancing and congestion issues at the minimum possible cost. Hence, the TSO's problem can be described as follows:

$$\min_{\Delta \mathbf{p}^T, \Delta \mathbf{d}^T} \sum_{n \in \mathcal{N}^T} (c_{p_n}^{T+} \Delta p_n^{T+} - c_{p_n}^{T-} \Delta p_n^{T-} + c_{d_n}^{T+} \Delta d_n^{T+} - c_{d_n}^{T-} \Delta d_n^{T-}), \quad (1)$$

Subject to:

$$p_n^T = p_n^{T,o} + \Delta p_n^{T+} - \Delta p_n^{T-} - d_n^{T,o} + \Delta d_n^{T+} - \Delta d_n^{T-}, \forall n \in \mathcal{N}^T, \quad (2)$$

$$P_{ij}^T = \sum_{n=1}^{N^T} p_n^T X_{(i,j),n} - \sum_{n \in \mathcal{N}^D} T_{nT}^p X_{(i,j),n}, \forall \{i, j\} \in \mathcal{L}^T, \quad (3)$$

$$p_n^T - \sum_{j.s.t. \{i,j\} \in \mathcal{L}^T} P_{ij}^T = 0 : (\lambda_n^T), \forall n \in \mathcal{N}^T \setminus \mathcal{N}^D, \quad (4)$$

$$p_n^T - T_{nT}^p - \sum_{j.s.t. \{i,j\} \in \mathcal{L}^T} P_{ij}^T = 0 : (\lambda_n^T), \forall n \in \mathcal{N}^D, \quad (5)$$

$$-F_{ij}^{T,\max} \leq P_{ij}^T \leq F_{ij}^{T,\max}, \forall \{i, j\} \in \mathcal{L}^T, \quad (6)$$

$$0 \leq \Delta p_n^{T+} \leq \Delta p_n^{T+,\max}, 0 \leq \Delta p_n^{T-} \leq \Delta p_n^{T-,\max}, \forall i \in \mathcal{N}^T, \quad (7)$$

$$0 \leq \Delta d_i^{T+} \leq \Delta d_i^{T+,\max}, 0 \leq \Delta d_i^{T-} \leq \Delta d_i^{T-,\max}, \forall i \in \mathcal{N}^T. \quad (8)$$

Here, $\Delta \mathbf{p}^T$ and $\Delta \mathbf{d}^T$ are the vectors grouping, respectively, Δp_n^{T+} and Δp_n^{T-} , and Δd_n^{T+} and Δd_n^{T-} , at all nodes $n \in \mathcal{N}^T$. (2) defines the net injection at node $n \in \mathcal{N}^T$. (3) consists of the power flow equations over all the transmission lines expressed using the generation shift factors, $X_{(i,j),n}$. (4) is the energy balance equation at node $n \in \mathcal{N}^T \setminus \mathcal{N}^D$, while (5) is the energy balance equation at interface node $n \in \mathcal{N}^D$. λ_n^T denotes the Lagrange multipliers of the nodal energy balance constraints at each node $n \in \mathcal{N}^T$. In this disjoint transmission system market formulation, T_{nT}^p is considered to be a constant (i.e. not dependent on any decision variable) to reflect the case that balancing and congestion management in this case are to be resolved using flexibility available only at the transmission system. In addition, (6) represents the congestion prevention constraints, while (7) and (8) capture the bid limits.

B. Disjoint Distribution-Level Market

The disjoint market for congestion management at the DSO- n^T level uses solely distribution grid flexibility to solve congestion issues resulting from the anticipated base generation and load profiles, $\mathbf{p}^{n^T,o}$ and $\mathbf{d}^{n^T,o}$, at all distribution system nodes \mathcal{N}^{n^T} . Let $\Delta p_n^{n^T+}$ and $\Delta p_n^{n^T-}$ correspond, respectively, to the volumes of increase and reduction in generation (corresponding to, respectively, upward and downward flexibility) connected at the DSO- n^T node n . In addition, we let $\Delta d_n^{n^T+}$ and $\Delta d_n^{n^T-}$ represent upward and downward flexibility at node n (i.e., the volumes of reduction and increase in demand at that node). In addition, similarly to the cost structure on the transmission level, we let $c_{p_n}^{n^T+}$, $c_{p_n}^{n^T-}$, $c_{d_n}^{n^T+}$, and $c_{d_n}^{n^T-}$ represent the unit prices offered by adjustable generation and loads on the distribution level for, respectively, $\Delta p_n^{n^T+}$, $\Delta p_n^{n^T-}$, $\Delta d_n^{n^T+}$, and $\Delta d_n^{n^T-}$.

As the DSO- n^T 's goal is to resolve congestion issues at minimum cost, its problem can be formulated as follows.

$$\min_{\Delta \mathbf{p}^{n^T}, \Delta \mathbf{d}^{n^T}, \mathbf{q}^{n^T}} \sum_{n \in \mathcal{N}^{n^T}} (c_{p_n}^{n^T+} \Delta p_n^{n^T+} - c_{p_n}^{n^T-} \Delta p_n^{n^T-} + c_{d_n}^{n^T+} \Delta d_n^{n^T+} - c_{d_n}^{n^T-} \Delta d_n^{n^T-}), \quad (9)$$

Subject to:

$$p_n^{n^T} = p_n^{n^T,o} + \Delta p_n^{n^T+} - \Delta p_n^{n^T-} - d_n^{n^T,o} + \Delta d_n^{n^T+} - \Delta d_n^{n^T-}, \forall n \in \mathcal{N}^{n^T}, \quad (10)$$

$$p_n^{n^T} + P_{A(n)n}^{n^T} - \sum_{k \in \mathcal{K}(n)} P_{nk}^{n^T} = 0, \forall n \in \mathcal{N}^{n^T} \setminus n_0^{n^T} : (\lambda_n^{n^T}), \quad (11)$$

$$q_n^{n^T} + Q_{A(n)n}^{n^T} - \sum_{k \in \mathcal{K}(n)} Q_{nk}^{n^T} = 0 \forall n \in \mathcal{N}^{n^T} \setminus n_0^{n^T}, \quad (12)$$

$$T_{n^T}^p - \sum_{k \in \mathcal{K}(n_0^{n^T})} P_{n_0^T k}^{n^T} = 0, : (\lambda_{n_0^T}^{n^T}), \quad (13)$$

$$T_{n^T}^q - \sum_{k \in \mathcal{K}(n_0^{n^T})} Q_{n_0^T k}^{n^T} = 0, \quad (14)$$

$$v_n^{n^T} = v_{A(n)}^{n^T} - 2r_{A(n)n}^{n^T} P_{A(n)n}^{n^T} - 2x_{A(n)n}^{n^T} Q_{A(n)n}^{n^T}, \forall n \in \mathcal{N}^{n^T} \setminus n_0^{n^T}, \quad (15)$$

$$\alpha_m P_{A(n)n}^{n^T} + \beta_m Q_{A(n)n}^{n^T} + \delta_m S_{A(n)n}^{n^T, \max} \leq 0, \forall m \in \mathcal{M}, \{A(n), n\} \in \mathcal{L}^{n^T}, \quad (16)$$

$$v_n^{n^T, \min} \leq v_n^{n^T} \leq v_n^{n^T, \max}, \forall n \in \mathcal{N}^{n^T}, \quad (17)$$

$$q_n^{n^T, \min} \leq q_n^{n^T} \leq q_n^{n^T, \max}, \forall n \in \mathcal{N}^{n^T}, \quad (18)$$

$$T_{n^T}^{q, \min} \leq T_{n^T}^q \leq T_{n^T}^{q, \max}, \quad (19)$$

$$0 \leq \Delta p_n^{n^T+} \leq \Delta p_n^{n^T+, \max}, 0 \leq \Delta p_n^{n^T-} \leq \Delta p_n^{n^T-, \max}, \forall n \in \mathcal{N}^{n^T}, \quad (20)$$

$$0 \leq \Delta d_n^{n^T+} \leq \Delta d_n^{n^T+, \max}, 0 \leq \Delta d_n^{n^T-} \leq \Delta d_n^{n^T-, \max}, \forall n \in \mathcal{N}^{n^T}. \quad (21)$$

Here, $\Delta \mathbf{p}^{n^T}$ and $\Delta \mathbf{d}^{n^T}$ are vectors grouping, respectively, $\Delta p_n^{n^T+}$ and $\Delta p_n^{n^T-}$, and $\Delta d_n^{n^T+}$ and $\Delta d_n^{n^T-}$ at each $n \in \mathcal{N}^{n^T}$. (10) returns the net generation, $p_n^{n^T}$, which depends on the base generation and load profiles and the activated flexibility at node n . The equality constraints in (11)-(15) follow directly from the *LinDistFlow* formulation [14], representing the linearized power flow equations in radial distribution

networks. In a local market mechanism (i.e. disjoint DSO- n^T level market), the interface flow $T_{n^T}^p$ is considered to be constant, i.e., not a decision variable nor a dependent variable to be optimized. Hence, this reflects that congestion management on the distribution level is to be resolved solely using flexibility provided from within the distribution system. We denote the Lagrange multiplier of constraint (11) and (13) by $\lambda_n^{n^T}$. Constraint (16) is a linearization of the complex flow limit constraint. This linearization, as proposed in [15], is a polygonal inner-approximation that transforms the feasibility region of the flow limit constraint from a circle of radius $S_{A(n)n}^{n^T, \max}$ into a polygon whose number of edges are given by the size of the approximation set \mathcal{M} . The values of α_m , β_m , and γ_m define this polygon such that all its vertices would lie on the original feasibility circle of radius $S_{A(n)n}^{n^T, \max}$ (a detailed explanation of this approximation is presented in [15]). Constraints (17) and (18) capture the limits on the nodal voltage magnitudes and reactive power injections, to ensure operational stability and the real-reactive power operational and capacity limits of load and generation, while (19) enforces a limit on the reactive power transfer with the transmission grid, $T_{n^T}^q$. Constraints (20) and (21) reflect the limits of the submitted bids.

C. Common Market Model

In the common market, the TSO can readily use resources offered from within the different distribution systems connected to its transmission network as well as resources connected to its transmission network to perform balancing and congestion management. Concurrently, the DSOs can use resources offered from their distribution networks for congestion management. In the common market, flexibility resources (i.e. submitted bids) are accessible to all participating SOs and the market is jointly cleared, in a collaborative manner, to optimally meet the needs of all the SOs while abiding by the operational limits (i.e. constraints) of all the grids involved. The proposed common market formulation, incorporating the TSO and the \mathcal{N}^D DSOs, combines the disjoint market models as follows:

$$\min_{\Delta \mathbf{p}, \Delta \mathbf{d}, \mathbf{q}} \left[\sum_{n \in \mathcal{N}^T} (c_{p_n}^{n^T+} \Delta p_n^{n^T+} - c_{p_n}^{n^T-} \Delta p_n^{n^T-} + c_{d_n}^{n^T+} \Delta d_n^{n^T+} - c_{d_n}^{n^T-} \Delta d_n^{n^T-}) + \sum_{n^T \in \mathcal{N}^D} \sum_{n \in \mathcal{N}^{n^T}} (c_{p_n}^{n^T+} \Delta p_n^{n^T+} - c_{p_n}^{n^T-} \Delta p_n^{n^T-} + c_{d_n}^{n^T+} \Delta d_n^{n^T+} - c_{d_n}^{n^T-} \Delta d_n^{n^T-}) \right], \quad (22)$$

$$\text{Subject to:} \quad (2)-(8), (10)-(21) \forall n^T \in \mathcal{N}^D, \quad (23)$$

$$\text{and } T_{n^T}^{p, \min} \leq T_{n^T}^p \leq T_{n^T}^{p, \max} \forall n^T \in \mathcal{N}^D. \quad (24)$$

$\Delta \mathbf{p}$ and $\Delta \mathbf{d}$ are the vectors of generation and demand flexibility variables (i.e., respectively, $\Delta \mathbf{p}^T$ and $\Delta \mathbf{p}^{n^T}$ for all $n^T \in \mathcal{N}^D$, and $\Delta \mathbf{d}^T$ and $\Delta \mathbf{d}^{n^T}$ for all $n^T \in \mathcal{N}^D$). In addition, \mathbf{q} is the vector of \mathbf{q}^{n^T} for all $n^T \in \mathcal{N}^D$. The common market formulation sums the objective functions of the disjoint markets and uses all the operational and bid level constraints from these markets. An additional constraint in the common market is (24). In this regard, in contrast to the

disjoint markets, the common market model treats $T_{n^T}^p$ as a dependent variable (rather than a constant as in the disjoint markets) with limits shown in (24). This enables the interaction between the previously disjoint markets to jointly procure the flexibility needed by all SOs.

Remark II.1. We note that the common market formulation in (22)–(24) can be readily adapted to reflect a sub-common market setting joining any subset of the SOs. This is readily achieved by replacing \mathcal{N}^D by any defined subset of DSOs in (22)–(24), and by optionally including or excluding the TSO-level market from the formulation.

D. Compact Formulation

Due to its linearity, and for ease of notation, the common market model can be readily expressed using a compact linear programming formulation. This is carried out next, which will be useful in defining the cooperative game framework.

We introduce \mathbf{x}_0 as the column vector that contains the TSO's decision variables: $\mathbf{x}_0 \triangleq \text{col}(\Delta \mathbf{p}^{T+}, \Delta \mathbf{p}^{T-}, \Delta \mathbf{d}^{T+}, \Delta \mathbf{d}^{T-})$. The TSO's feasibility set \mathcal{X}_0 is defined based on the constraints that depend only on the TSO's decision variables. Each DSO- n^T 's decision variable column vector, denoted by \mathbf{x}_{n^T} , contains $\mathbf{x}_{n^T} \triangleq \text{col}(\Delta \mathbf{p}^{n^T+}, \Delta \mathbf{p}^{n^T-}, \Delta \mathbf{d}^{n^T+}, \Delta \mathbf{d}^{n^T-}, \mathbf{q}^{n^T})$. We define \mathcal{X}_{n^T} as the feasibility set of DSO- n^T . The set is made of constraints that depend only on the DSO's decision variables. We let $\mathbf{x} \in \mathcal{X}_0 \times \prod_{n^T} \mathcal{X}_{n^T}$ be the vector that contains the joint decisions of the TSO and DSOs. Dependent variables, which can be expressed as functions of the TSO and DSOs' decision variables, are concatenated in a column vector \mathbf{z} . Precisely, we set: $\mathbf{z} \triangleq \text{col}(\mathbf{p}^T, \mathbf{P}^T, (\mathbf{p}^{n^T})_{n^T \in \mathcal{N}^D}, (\mathbf{P}^{n^T})_{n^T \in \mathcal{N}^D}, (\mathbf{Q}^{n^T})_{n^T \in \mathcal{N}^D}, (\mathbf{v}^{n^T})_{n^T \in \mathcal{N}^D}, \mathbf{T}^p, \mathbf{T}^q)$. Constants such as $\mathbf{p}^{T,o}$, $\mathbf{d}^{T,o}$, $(\mathbf{p}^{n^T,o})_{n^T \in \mathcal{N}^D}$, $(\mathbf{d}^{n^T,o})_{n^T \in \mathcal{N}^D}$, $\Delta \mathbf{p}^{T+, \max}$, $\Delta \mathbf{p}^{T-, \max}$, $\Delta \mathbf{d}^{T+, \max}$, $\Delta \mathbf{d}^{T-, \max}$, $(\Delta \mathbf{p}^{n^T+, \max})_{n^T \in \mathcal{N}^D}$, $(\Delta \mathbf{p}^{n^T-, \max})_{n^T \in \mathcal{N}^D}$, $(\mathbf{v}^{n^T, \max})_{n^T \in \mathcal{N}^D}$, $(\mathbf{v}^{n^T, \min})_{n^T \in \mathcal{N}^D}$, $(\mathbf{q}^{n^T, \max})_{n^T \in \mathcal{N}^D}$, $(\mathbf{q}^{n^T, \min})_{n^T \in \mathcal{N}^D}$, \mathbf{T}^q, \max , \mathbf{T}^q, \min , \mathbf{T}^p, \max , \mathbf{T}^p, \min , $\Xi((i, j), n) = X_{(i, j), n}$, $(\mathbf{r}^{n^T}, \mathbf{x}^{n^T})_{n^T \in \mathcal{N}^D}$, α, β, δ , $\mathbf{F}^{T, \max}$, and $(\mathbf{S}^{n^T, \max})_{n^T \in \mathcal{N}^D}$, are taken as input parameters. We also define the TSO's cost vector $\mathbf{c}_0 \triangleq (\mathbf{c}_p^{T+}, -\mathbf{c}_p^{T-}, \mathbf{c}_d^{T+}, -\mathbf{c}_d^{T-})$ and DSO- n^T 's cost vector $\mathbf{c}_{n^T} \triangleq (\mathbf{c}_p^{n^T+}, -\mathbf{c}_p^{n^T-}, \mathbf{c}_d^{n^T+}, -\mathbf{c}_d^{n^T-}, \mathbf{0})$.

Based on this compact notation, the TSO's objective function can be written as $\Phi_0(\mathbf{x}_0) \triangleq \mathbf{c}_0^t \mathbf{x}_0$, while the objective function of each DSO- n^T is given by $\Phi_n(\mathbf{x}_{n^T}) \triangleq \mathbf{c}_{n^T}^t \mathbf{x}_{n^T}$, where \cdot^t is the transpose operator. The social cost of the common market, introduced in (22), is defined as the sum of the TSO and DSOs' objective functions, i.e., $\Phi(\mathbf{x}) \triangleq \Phi_0(\mathbf{x}_0) + \sum_{n^T \in \mathcal{N}^T} \Phi_{n^T}(\mathbf{x}_{n^T})$. To write the common market model (22)–(24) in a compact matrix form, a change of notations is needed. For that purpose, we order the DSOs in ascending order from 1 to $N = |\mathcal{N}^D|$ based on the values of their initial n^T . In addition, we denote $\mathbf{x} \triangleq (\mathbf{x}_0, (\mathbf{x}_n)_n)$ as the concatenation of the TSO and DSOs' decision variables.

As the common market model in (22)–(24) is linear, it can readily be presented as a standard compact linear program (LP), using our defined vector notation, as follows:

$$\text{(LP)} \quad \min_{\mathbf{x}, \mathbf{z}} \Phi(\mathbf{x}), \quad (25a)$$

$$\text{s.t. } \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} \leq \mathbf{d}, \quad (\lambda) \quad (25b)$$

$$\mathbf{x}_0 \in \mathcal{X}_0, \quad (25c)$$

$$\mathbf{x}_n \in \mathcal{X}_n, \forall n \in \mathcal{N}, \quad (25d)$$

$$\mathbf{z} \in \mathcal{Z}. \quad (25e)$$

The steps towards this compact formulation are presented in the Appendix. Note that (25b) can be equivalently written as $(\mathbf{A}_0 \mathbf{x}_0 + \mathbf{B}_0 \mathbf{z}_0) + \sum_n (\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{z}_n) \leq \mathbf{d}$ to differentiate between the TSO and the different DSO n 's variables. We let λ be the dual variable of (25b).

Constraints (25c) and (25d) can be analytically written as $\Psi_0(\mathbf{x}_0) \leq \mathbf{0}$ and $\Psi_n(\mathbf{x}_n) \leq \mathbf{0}, \forall n \in \mathcal{N}$ respectively. We denote μ_0 and (μ_n) as the associated dual variables.

The Lagrangian function associated with the optimization problem (25) is defined as follows:

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \mathbf{z}, \lambda, \mu) &\triangleq \Phi(\mathbf{x}) + \lambda^t (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} - \mathbf{d}) + \mu_0^t \Psi_0(\mathbf{x}_0) \\ &\quad + \sum_n \mu_n^t \Psi_n(\mathbf{x}_n). \end{aligned} \quad (26)$$

In addition, we introduce the relaxed formulation of (25) where the coupling constraint (25b) is replaced by a penalty in the objective:

$$\max_{\lambda \geq 0} \min_{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z}} \left(\Phi(\mathbf{x}) + \lambda^t (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} - \mathbf{d}) \right), \quad (27a)$$

$$\text{s.t. } \mathbf{x}_0 \in \mathcal{X}_0, \quad (27b)$$

$$\mathbf{x}_n \in \mathcal{X}_n, \forall n \in \mathcal{N}, \quad (27c)$$

$$\mathbf{z} \in \mathcal{Z}. \quad (27d)$$

The dual function of problem (25) is $g(\lambda, \mu) \triangleq \min_{\mathbf{x}, \mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \lambda, \mu)$ where $\mathcal{L}(\cdot)$ is defined in (26). The associated dual problem is $\max_{\lambda \geq 0, \mu \geq 0} g(\lambda, \mu)$. The duality gap is the non-negative number $\Phi(\mathbf{x}^*) - g(\lambda^*, \mu^*)$ where \mathbf{x}^* is solution to the primal problem (25) and λ^*, μ^* are solutions to the dual problem. Under constraint qualification (e.g. Slater's condition) for (25), strong duality implies:

$$\min_{\mathbf{x}, \mathbf{z}} \max_{\lambda \geq 0} \mathcal{L}(\mathbf{x}, \mathbf{z}, \lambda) = \max_{\lambda \geq 0} \min_{\mathbf{x}, \mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \lambda). \quad (28)$$

Since $\Psi_0(\cdot), \Psi_n(\cdot), \forall n \in \mathcal{N}$ and (25b), (25e) are all defined through affine functions, Slater's condition is not strictly required and can be replaced with non-strict inequalities in the (LP) feasibility set.

III. COST GAME DEFINITION

We consider a game that is populated by the non-empty set \mathcal{N} of the N DSOs and the TSO. We will refer to them as the players of the game. A coalition is a subset of $\mathcal{N}^G \triangleq \mathcal{N} \cup \{\text{TSO}\}$. The grand coalition is the set \mathcal{N}^G of all SOs (players), with cardinality N^G . We cast this game as a characteristic function cost game, G , formally defined as follows.

Definition 1 (Characteristic Function Cost Game [16]). A characteristic function cost game G is given by a pair (\mathcal{N}^G, v) , where $v : 2^{\mathcal{N}^G} \rightarrow \mathbb{R}$ is a characteristic function,

which maps each coalition $C \subseteq \mathcal{N}^G$ to a real number $v(C)$. The real number $v(C)$ denotes the value of the coalition C .

The common market in (25), and its special cases for a singleton TSO or DSO market (or any sub-common market containing any subset of the SOs as discussed in Remark II.1), corresponds to a characteristic function v given by:

$$v(C) = \sum_{n \in C} c_n^t x_n^* \text{ where } C \subseteq \mathcal{N}^G, \quad (29)$$

and $(x_n^*)_{n \in C}$ is the optimum of the optimization problem

$$\min_{(x_n)_{n \in C}, (z_n)_{n \in C}} \sum_{n \in C} \Phi_n(x_n), \quad (30a)$$

$$\text{s.t. } \mathbf{Ax} + \mathbf{Bz} \leq \mathbf{d}, \quad (30b)$$

$$x_n \in \mathcal{X}_n, z_n \in \mathcal{Z}_n, \forall n \in C. \quad (30c)$$

As the coalitional value $v(C)$ (i.e., the total cost of the common and sub-common markets) can be divided amongst the members of C in any way that the members of C choose, G is classified as a transferable utility (TU) game [16].

An outcome of the characteristic function game, G , consists of two parts: i) a partition of the SOs (i.e. the players) into coalitions, known as the coalition structure; and ii) a cost vector, distributing the value of each coalition among its members. These concepts are formally defined next.

Definition 2 (Coalition Structure). *Given our game $G = (\mathcal{N}^G, v)$, a coalition structure over \mathcal{N}^G is a collection of non-empty subsets $CS = \{C^1, \dots, C^s\}$ such that $\cup_{j=1}^s C^j = \mathcal{N}^G$ and $C^i \cap C^j = \emptyset, \forall i, j \in \{1, \dots, s\}$ such that $i \neq j$.*

A vector $\mathbf{y} = (y_1, \dots, y_{N^G}) \in \mathbb{R}^{N^G}$ is a cost allocation vector for a coalition structure $CS = \{C^1, \dots, C^s\}$ over \mathcal{N}^G if: $y_n \geq 0, \forall n \in \mathcal{N}^G, \sum_{n \in C^{s'}} y_n \geq v(C^{s'})$ for any $s' \in \{1, \dots, s\}$ (feasibility condition). The efficiency of \mathbf{y} is defined as follows.

Definition 3 (Efficiency). *A cost allocation vector \mathbf{y} is efficient if all the coalition cost is distributed amongst coalition members, i.e., $\sum_{n \in C^{s'}} y_n = v(C^{s'}), \forall s' \in \{1, \dots, s\}$.*

The space of all coalition structures will be denoted \mathcal{CS} . An outcome of G is, hence, a pair (CS, \mathbf{y}) . For a cost allocation vector \mathbf{y} , we let $y(C) \triangleq \sum_{n \in C} y_n$ denote the total cost allocation of a coalition $C \subseteq \mathcal{N}^G$ under \mathbf{y} . By extension, the social cost of the coalition structure CS will be denoted $v(CS) \triangleq \sum_{C \in CS} v(C)$.

For the derivations that ensue, we now recall classical definitions of two subclasses of coalitional games that will be useful thereafter: submodular games and concave games.

Definition 4 (Submodularity [16]). *A characteristic function v is said to be submodular if it satisfies*

$$v(C \cup C') + v(C \cap C') \leq v(C) + v(C'),$$

for every pair of coalitions $C, C' \subseteq \mathcal{N}^G$. A game with a submodular characteristic function is said to be concave.

Concave games have an intuitive characterization in terms of players' marginal contributions: in a concave game, a player (i.e. an SO) is more useful (decreasing the group cost) when it joins a bigger coalition, as formally defined next.

Definition 5 (Concavity [16]). *A characteristic function game G is concave if and only if for every pair of coalitions C, C' such that $C \subset C'$ and every player $n \in \mathcal{N}^G \setminus C'$ it holds that $v(C \cup \{n\}) - v(C) \leq v(C' \cup \{n\}) - v(C')$.*

To define whether a common market can naturally arise, the stability of the cooperation between the SOs in a common market must be defined and verified. A stable coalition is a coalition from which no SO has an incentive to deviate. A stable grand coalition, is the coalition including the TSO and all DSOs and which is stable. Consider an outcome (CS, \mathbf{y}) of the cost game G . If $y(C) > v(C)$ for some $C \subseteq \mathcal{N}^G$, the SOs in C could do better by abandoning the coalition structure CS and forming a coalition of their own. Thus, in this case, the outcome (CS, \mathbf{y}) is unstable. The set of stable outcomes, i.e., outcomes where no subset of SOs (players) has an incentive to deviate, is called the core of G .

Definition 6. *The core $\mathcal{C}(G)$ of the characteristic function game $G = (\mathcal{N}^G, v)$ is the set of all efficient outcomes (CS, \mathbf{y}) such that $y(C) \leq v(C), \forall C \subseteq \mathcal{N}^G$. The core of our cost game G can formally be defined as follows: $\mathcal{C}(G) \triangleq \{\mathbf{y} \in \mathbb{R}^{N^G} | y(\mathcal{N}^G) = v(\mathcal{N}^G) \text{ and } y(C) \leq v(C), \forall C \subseteq \mathcal{N}^G\}$.*

The constraints imposed on $\mathcal{C}(G)$ ensure that no TSO or DSO has an incentive to leave the grand coalition (of all SOs in a common market) and form any subcoalition (a sub-common market as defined in Remark II.1, encompassing any subset of SOs including the singleton coalitions/disjoint markets).

We next prove and analyze the stability of the grand coalition in our game and, hence, the naturally arising common benefit of the SOs to cooperate in a common flexibility market.

IV. STABILITY OF COST GAME G

For the common market to naturally arise and be sustained, collaboration among SOs should be naturally beneficial to all of them. This is achieved if their collaboration is stable, i.e., when the core of our formulated TSO-DSO game G , defined in Definition 6, is non-empty.

We next prove that the core of G is non-empty and hence the TSO and DSOs have a natural incentive to collaborate and form a common market. To this end, we first begin by proving that it is beneficial for any DSO to collaborate with the TSO than forming a disjoint market.

Proposition 1. *Any DSO $n \in \mathcal{N}$ prefers cooperating with the TSO than remaining alone.*

Proof. It is profitable for any DSO $n \in \mathcal{N}$ to cooperate with the TSO if and only if:

$$v(\{\text{TSO} \cup n\}) \leq v(\{n\}) + v(\{\text{TSO}\}). \quad (31)$$

Considering the relaxed version of (25) which we proved to be equivalent to (25) under weak Slater's condition, the inequality (31) is equivalent to the following one:

$$\begin{aligned} & \max_{\lambda} \min_{x_0, z_0, x_n, z_n} \left[c_0^t x_0 + c_n^t x_n + \lambda^t (\mathbf{Ax} + \mathbf{Bz} - \mathbf{d}) \right] \\ & \leq \max_{\lambda} \min_{x_0, z_0} \left[c_0^t x_0 + \lambda^t (\mathbf{Ax} + \mathbf{Bz} - \mathbf{d}) \right] \\ & \quad + \max_{\lambda} \min_{x_n, z_n} \left[c_n^t x_n + \lambda^t (\mathbf{Ax} + \mathbf{Bz} - \mathbf{d}) \right], \quad (32) \end{aligned}$$

where $\mathbf{x}_0 \in \mathcal{X}_0, \mathbf{x}_n \in \mathcal{X}_n, \mathbf{z}_0 \in \mathcal{Z}_0, \mathbf{z}_n \in \mathcal{Z}_n$. Since (32) holds by definition, we can conclude that any DSO $n \in \mathcal{N}$ has an incentive to cooperate with the TSO. \square

We next prove that not only would any DSO benefit from collaborating with the TSO, but also any DSO has an incentive to be part of the grand coalition made of all of the \mathcal{N} DSOs and the TSO. We start by proving an intermediate result that characterizes the cost game G .

Proposition 2. *The characteristic function game $G = (\mathcal{N}^G, v)$ is concave.*

Proof. Let $C \subseteq \mathcal{N}^G$. We compute $v(C \cup \{n\}) - v(C) = \mathbf{c}_n^t \mathbf{x}_n^*$ where \mathbf{x}_n^* solves (30), and observe that $v(C' \cup \{n\}) - v(C') = v(C \cup \{n\}) - v(C)$ for any coalition $C' \subset C$. The concavity property introduced in Definition 5 holds. \square

Corollary IV.1. *The characteristic function cost game G is submodular.*

Proof. This results follows directly from Proposition 2. Indeed, every concave game is necessarily submodular [16]. \square

This, hence, allows us to prove the non-emptiness of the core, as shown next.

Theorem 1. *The core of the characteristic function cost game G is non-empty and the grand coalition of G is stable.*

Proof. Consider the following allocation linear program:

$$\max_{\mathbf{y}} \quad \sum_{n \in \mathcal{N}^G} y_n, \quad (33a)$$

$$s.t. \quad \sum_{n \in C} y_n \leq v(C), \forall C \subseteq \mathcal{N}^G. \quad (33b)$$

It is quite obvious that $\mathcal{C}(G) \neq \emptyset$ if and only if the optimum value of the linear program (33) is equal to $v(\mathcal{N}^G)$, in which case any optimal solution to (33) lies in $\mathcal{C}(G)$. Taking the linear program dual to (33), an equivalent condition for $\mathcal{C}(G) \neq \emptyset$ can be obtained based on the concept of balanced sets. A collection \mathcal{B} of nonempty subsets of \mathcal{N}^G is balanced if $\sum_{C \in \mathcal{B}} \gamma_C v(C) \leq v(\mathcal{N})$ holds for every balanced collection \mathcal{B} with weights $(\gamma_C)_{C \in \mathcal{B}}$. A game has a non-empty core if and only if it is balanced [17].

The cost game G being submodular from Corollary IV.1, (33) is equivalent to:

$$\max_{\mathbf{y}} \quad \sum_{n \in \mathcal{N}^G} y_n, \quad (34a)$$

$$s.t. \quad \sum_{n \in \mathcal{N}^G} y_n = \sum_{n \in \mathcal{N}^G} \Phi_n(\mathbf{x}_n^*), \quad (34b)$$

$$\mathbf{x}^* = \arg \min_{\mathbf{x}, \mathbf{z}} \sum_{n \in \mathcal{N}^G} \Phi_n(\mathbf{x}_n), \quad (34c)$$

$$s.t. \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} \leq \mathbf{d}, \quad (34d)$$

$$\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z}. \quad (34e)$$

By construction, the Shapley⁵ value of the cost game G is solution to (34) and as such, belongs to the core of the game [16], [18]. This implies that the core of G is never empty. \square

⁵The Shapley value is introduced in detail and analyzed in Section V-A.

Theorem 1, hence, proves that it is beneficial for all DSOs and TSO to cooperate in a common market. In the derivation, c_n was used in the cost function of the players, which would reflect a cost-based market clearing or a pay-as-bid mechanism. However, Theorem 1 can be readily extended beyond pay-as-bid for other pricing schemes, such as nodal pricing (i.e., locational marginal pricing) as shown next.

Proposition 3. *The core of the cost game G is non-empty under nodal pricing.*

Proof. Under nodal pricing, the cost game G coalitional value becomes $v(C) = \sum_{n \in C} \lambda_n^t \mathbf{x}_n^*$ where $C \subseteq \mathcal{N}^G$, and $(\mathbf{x}_n^*)_{n \in C}$ is the optimum of (30) where $\Phi_n(\mathbf{x}_n)$ is replaced with $\lambda_n^t \mathbf{x}_n$.

Using the same reasoning as in the proof of Proposition 2, it is straightforward to check that cost game G is concave under both pricing schemes, rendering the core of the cost game G non-empty under nodal pricing. \square

After proving the stability of the cooperation of the SOs in the common market, we next present several cost allocation methods – based on which the total cost of flexibility procurement in the common market can be split – and analytically characterize their properties.

V. ALLOCATION MECHANISMS

We introduce several cost allocation mechanisms and study their properties for our TSO-DSOs cooperative game, based on efficiency, stability, and fairness criteria, which measure how well each SO allocated cost reflects its contribution to the total cost. The properties for evaluating a cost allocation mechanism, $\Phi(G)$, are defined as follows:

- (i) **efficiency**: $\sum_n \Phi_n(G) = v(\mathcal{N}^G)$.
- (ii) **dummy player**: if a player n is a dummy in G , i.e., $v(C \cup \{n\}) = v(C), \forall C \subseteq \mathcal{N}^G \setminus \{n\}$, then $\Phi_n(G) = 0$.
- (iii) **symmetry** (equal treatment of equals): if n and n' are equivalent in G , in the sense that $v(C \cup \{n\}) = v(C \cup \{n'\}), \forall C \subseteq \mathcal{N}^G$, then $\Phi_n(G) = \Phi_{n'}(G)$.
- (iv) **additivity**: $\Phi_n(G + \tilde{G}) = \Phi_n(G) + \Phi_n(\tilde{G}), \forall n \in \mathcal{N}^G$.
- (v) **stability**: the cost allocation $\Phi(G)$ belongs to the core.
- (vi) **anonymity**: players' relabeling does not affect their cost allocation. If n and n' are two players, and game \tilde{G} is identical to G except for exchanging the roles of n and n' , then $\Phi_n(G) = \Phi_{n'}(\tilde{G})$. Note that (vi) implies (iii).

The studied cost allocation mechanisms are defined next.

A. Shapley Value (SV)

The Shapley value is a solution concept typically formulated with respect to the grand coalition: it defines a way of distributing the value $v(\mathcal{N}^G)$ that could be obtained by the grand coalition. The SV is based on the intuition that the cost allocated to each agent (in our case to each SO) should be *proportional to her contribution* to the grand coalition. Define $\mathcal{P}_{\mathcal{N}^G}$ as the set of permutations, e.g., one-to-one mappings from \mathcal{N}^G to itself. We introduce $S_p(n)$ as the set of all the predecessors of n in $p \in \mathcal{P}_{\mathcal{N}^G}$, i.e., $S_p(n) \triangleq \{n' \in \mathcal{N}^G | p(n') < p(n)\}$ where $<$ denotes the predecessor relationship. The SV of SO n is denoted

$SV_n(G)$ and is given by $SV_n(G) \triangleq \frac{1}{N^G} \sum_{p \in \mathcal{P}_{N^G}} \Delta_p(n)$ where $\Delta_p(n) \triangleq v(S_p(n) \cup \{n\}) - v(S_p(n))$ measures the marginal contribution of n with respect to a permutation p . It can equivalently be written under the extended form: $SV_n(G) = \sum_{C \subseteq \mathcal{N}^G \setminus \{n\}} \frac{|C|!(N^G - |C| - 1)!}{N^G!} [v(C \cup \{n\}) - v(C)]$.

For each permutation (ordering) of the SOs, each SO is imputed a cost based on how much the SO contributes to the coalition formed by its predecessors in this permutation. The allocated cost is averaged over all possible permutations to guarantee the symmetry of the allocation.

By construction, $SV(G)$ meets properties (i)-(iv) [18]. In fact, the SV is the *only* cost allocation method that has the four properties (i)-(iv) simultaneously. The anonymity property (vi) is also met by the SV, meaning that the SV does not discriminate between the SOs solely on the basis of their indices [19]. Finally, following the proof of Theorem 1, property (v) holds true for our cost game G . The main challenge of the SV is in its computational complexity (NP - complete). The complexity of the SV also hinders the interpretation of its fairness.

B. Normalized Banzhaf Index ($B^\#$)

Like the SV, the Banzhaf index $B(G)$ measures the agents' expected marginal contributions; however, instead of averaging over all permutations of players, it averages over all coalitions in the game. The Banzhaf index of an SO n is denoted $B_n(G)$ and is given by $B_n(G) \triangleq \frac{1}{2^{N^G - 1}} \sum_{C \subseteq \mathcal{N}^G \setminus \{n\}} [v(C \cup \{n\}) - v(C)]$. $B(G)$ meets properties (ii)-(iv) [16]. Similarly to the SV, it meets also the anonymity property (vi). Because it lacks the efficiency property (i), $B(G)$ is not in the core. To meet the efficiency property (i), a rescaled version of the Banzhaf index has been proposed, called the normalized Banzhaf index: $B_n^\#(G) \triangleq \frac{B_n(G)}{\sum_{n' \in \mathcal{N}^G} B_{n'}(G)}$. The normalized Banzhaf index meets properties (i)-(iii); however it loses (iv) [16]. Similarly to Banzhaf index, it meets property (vi). We next prove that the normalized Banzhaf index leads to a stable cost allocation.

Proposition 4. *The normalized Banzhaf index $B^\#(G)$ meets the stability property (v) for the cost game G .*

Proof. By construction, the normalized Banzhaf index is efficient. We need to check the non-deviation property, i.e., that $\sum_{n \in C} B_n^\#(G) \leq v(C), \forall C \subseteq \mathcal{N}^G$. We notice that the normalized Banzhaf index is a convex combination of elements that are constructed in the proof of Th.2.27 in [16] to exhibit elements from the core of G . Since these elements belong to the core by construction, the normalized Banzhaf index is itself a convex combination of these elements, and the core of the cost game G can be shown to be a convex set. Hence, the normalized Banzhaf index meets the non-deviation property, which implies that it belongs to the core of G . \square

Similarly to the SV, both versions of Banzhaf index share exponential computational complexity rates, implying challenges for implementations on a large scale and no simple interpretation of fairness.

C. Cost Gap Allocation Method (CGA)

This method coincides with the τ -value, introduced in [20]. Similarly to the SV, we define $\Delta = (\Delta(n))_n$ as the marginal cost vector. Its n -th coordinate is the separable cost of SO n , i.e., $\Delta(n) \triangleq v(\mathcal{N}^G) - v(\mathcal{N}^G \setminus \{n\}), \forall n \in \mathcal{N}^G$. For each coalition $C \subseteq \mathcal{N}^G$, we define the cost gap of C by:

$$\begin{cases} g(C) \triangleq v(C) - \sum_{n \in C} \Delta(n) & \text{if } C \neq \emptyset, \\ g(\emptyset) \triangleq 0. \end{cases}$$

The map $g : 2^{\mathcal{N}^G} \rightarrow \mathbb{R}$ is the cost gap function of game G . Note that $g(\mathcal{N}^G)$ is equal to the non separable cost in G . In general, we consider $g \geq 0$. We define the weight vector \mathbf{w} such that: $w_n \triangleq \min_{\{C | n \in C\}} g(C), \forall n \in \mathcal{N}^G$. For any characteristic function v such that $g(C) \geq 0, \forall C \subseteq \mathcal{N}^G$ and $\sum_{n \in \mathcal{N}^G} w_n \geq g(\mathcal{N}^G)$, the cost gap allocation method assigns the cost allocation:

$$\mathbf{y} \triangleq \begin{cases} \Delta & \text{if } g(\mathcal{N}^G) = 0, \\ \Delta + g(\mathcal{N}^G)(\sum_{n \in \mathcal{N}^G} w_n)^{-1} \mathbf{w} & \text{if } g(\mathcal{N}^G) > 0. \end{cases}$$

The cost gap allocation CGA(G) meets the efficiency (i), dummy player (ii), anonymity (vi) and, therefore, symmetry (iii) properties [20]. CGA(G) is stable for $N^G < 4$, but can lead to unstable outcomes for $4 \geq N^G$ [20]. On the other hand, CGA(G) gives rise to an exact analytical expression, and therefore to a simple interpretation of fairness.

D. Lagrangian Based Allocation (L)

This method is an extension of the classical shadow price based cost split [21], with which it coincides when weak duality holds for the cost game G . In (25), we get the dual λ for the coupling constraint (25b), and duals $\mu_0, (\mu_n)_n$ for the individual constraints (25c), (25d). When solving (25) for the grand coalition, we get $v(\mathcal{N}^G)$. The optimal dual solution has the property that:

$$v(\mathcal{N}^G) = \lambda^t (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} - \mathbf{d}) + \mu_0^t \Psi_0(\mathbf{x}_0) + \sum_n \mu_n^t \Psi_n(\mathbf{x}_n). \quad (35)$$

Since our problem is linear, each SO's contribution can be found by computing its contribution to the dual objective function value [21]. This cost allocation is efficient (i) under weak duality [22]. Stability (v) is also achieved since efficiency (i) holds and the individual rationality condition is met by definition of the cost allocation and weak duality. From (35), dummy player (ii) and additivity (iv) hold. However, symmetry (iii) and anonymity (vi) do not hold in general.

Lagrangian based cost allocation $L(G)$ requires the computation of the grand coalition optimal value (e.g., solving a linear optimization program) and all the dual variables associated with the constraints in (25). Making use of a solver, $L(G)$ implementation is simple and leads to a rather intuitive interpretation of fairness.

E. Equal Profit Method (EPM)

The motivation behind this cost allocation is to propose a method which is aimed at finding a stable cost allocation, such that the maximum difference in pairwise relative savings is minimized. We call this method, the Equal Profit Method (EPM) [21]. The relative savings of SO n is, then, computed as $\frac{v(\{n\}) - y_n}{v(\{n\})} = 1 - \frac{y_n}{v(\{n\})}$. When a cost allocation is stable,

$v(\{n\}) \geq y_n$. This allocation is obtained using the solution of a linear optimization problem:

$$\min \quad f, \quad (36a)$$

$$\text{s.t.} \quad f \geq \frac{y_n}{v(\{n\})} - \frac{y_{n'}}{v(\{n'\})}, \forall n, n', \quad (36b)$$

$$\sum_{n' \in C} y_{n'} \leq v(C), \forall C \subseteq \mathcal{N}^G, \quad (36c)$$

$$\sum_{n' \in \mathcal{N}^G} y_{n'} = v(\mathcal{N}^G), y_n \geq 0, \forall n. \quad (36d)$$

EPM belongs to the core of the cost game G by construction, as (36c), (36d) define efficiency and individual rationality⁶ respectively. This means that it meets the efficiency (i) and stability (v) properties. The symmetry property (iii) is naturally met from (36b) at the optimum. However, additivity (iv), anonymity (vi) and dummy player (ii) are not met in general. EPM requires the optimal solution of a linear constrained optimization problem. Making use of a solver, its implementation is simple and leads to an intuitive interpretation of fairness.

F. Proportional Cost Allocation (PCA)

A straightforward allocation is to distribute the total cost of the grand coalition (i.e., the total cost of the common market), $v(\mathcal{N}^G)$, among the SOs according to how much flexibility is used by each of them. This is expressed by $y_n = w_n \cdot v(\mathcal{N}^G)$, where w_n is equal to SO n 's share of the total activated flexibility, and $w_n \geq 0, \forall n \in \mathcal{N}^G, \sum_{n' \in \mathcal{N}^G} w_{n'} = 1$, or, alternatively in case stand-alone costs are used, it is equal to $\frac{v(\{n\})}{\sum_{n' \in \mathcal{N}^G} v(\{n'\})}$ [21].

The proportional cost allocation method is easy to understand, implement, and compute. Besides, it is stable if and only if $\sum_{n \in C} w_n \leq \frac{v(C)}{v(\mathcal{N}^G)}, \forall C \subseteq \mathcal{N}^G$. In case stand-alone costs are used (we refer to this specifically as PCA), the proportional allocation is stable, meeting (v). The efficiency property (i) is always met due to the normalization of the weights. The symmetry property (iii) also holds, but neither anonymity (vi) nor dummy player (ii) hold.

As PCA gives rise to an exact analytical expression, it yields a simple, intuitive interpretation of fairness.

A comparison of the properties of the different proposed cost allocation methods is provided in Table I.

VI. NUMERICAL RESULTS

For the numerical results, we consider an interconnected system composed of the IEEE 14-bus transmission network connected to three distribution networks – namely, the Matpower 18-bus, 69-bus, and 141-bus systems. All network parameters are based on the corresponding cases in Matpower [23]. We add base demand to the buses and adapt the capacity limit of lines, so that the initial system state (without any flexibility activation) show anticipated congestions and system imbalance. Flexibility bids for both upward and downward flexibility are created over the different nodes.

⁶A cost allocation vector y for a coalition structure CS satisfies the individual rationality property if $y_n \leq v(\{n\}), \forall n \in \mathcal{N}^G$, i.e., each SO weakly prefers being in the coalition structure to being on his own.

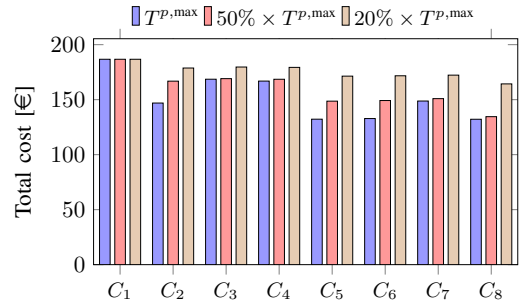


Fig. 1: Total cost reduction when different coalitions are formed in three cases of interface flow limit, where: $C_1 = \{\{TN\}, \{DN_{18}\}, \{DN_{69}\}, \{DN_{141}\}\}$, $C_2 = \{\{TN, DN_{18}\}, \{DN_{69}\}, \{DN_{141}\}\}$, $C_3 = \{\{TN, DN_{141}\}, \{DN_{18}\}, \{DN_{69}\}\}$, $C_4 = \{\{TN, DN_{69}\}, \{DN_{18}\}, \{DN_{141}\}\}$, $C_5 = \{\{TN, DN_{18}, DN_{69}\}, \{DN_{141}\}\}$, $C_6 = \{\{TN, DN_{18}, DN_{141}\}, \{DN_{69}\}\}$, $C_7 = \{\{TN, DN_{69}, DN_{141}\}, \{DN_{18}\}\}$, $C_8 = \{\{TN, DN_{18}, DN_{69}, DN_{141}\}\}$.

The submitted bids are drawn from a uniform distribution in the range of $[10, 15]$ €/MWh for downward bids and $[50, 55]$ €/MWh for upward offers, not to induce biases stemming from the submitted bids/offers.

We first showcase the benefit of cooperation in reducing the total system costs. Fig. 1 shows the incremental benefit of cooperation when adding additional DSOs up to the grand coalition as well as the impact that the interface flow limits have on the cooperation benefit. We consider first each SO alone (i.e. disjoint markets), then incrementally add one DSO to the coalition and compute the total costs. The coalition formed are denoted C_1 to C_8 (defined in Fig. 1), where C_1 represents the case of disjoint markets (i.e., singleton SOs) and C_8 the case of the grand coalition forming a common market. The process is repeated for three different levels of maximum interface flow limits, where the $50\% \times T^{p,\max}$ and $20\% \times T^{p,\max}$ cases reduce the interface flow maximum limit for each DSO to, respectively, 20% and 50% of its original value, $T^{p,\max}$. As can be seen from Fig. 1, adding more DSOs in the cooperation further reduces the total costs. For example, by comparing C_1, C_2, C_5 and C_8 , we can observe the way the sequential addition of DN_{18}, DN_{69} , and DN_{141} to the TSO (TN) led to a significant decrease in total system costs for all values of interface flow limits (C_8 achieves 29%, 28%, and 12% reduction with respect to C_1 for, respectively, $T^{p,\max}, 50\% \times T^{p,\max}$, and $20\% \times T^{p,\max}$). This also highlights the effect of allowing a higher level of interface power exchange on achieving a more efficient procurement of flexibility, as out of the three interface flow limits, the $20\% \times T^{p,\max}$ case achieves the least amount of savings.

Indeed, Fig. 1 further showcases the way in which a tighter interface flow limit reduces the benefits introduced by cooperation, by showcasing the increased total cost under each coalition for the three limits $T^{p,\max}, 50\% \times T^{p,\max}$, and $20\% \times T^{p,\max}$. However, the impacts of the interface flow limits on different DSOs vary significantly. By comparing bars of the same color in C_2, C_3 and C_4 , we can see that C_2 shows the most severe variation in costs for a tighter interface flow limit. This reflects the fact that DN_{18} is affected more significantly, as in this case study, DN_{18} has the largest

TABLE I: Properties of the Cost Allocation Methods for cost game G .

Properties	SV	$B^\#$	CGA	L	EPM	PCA
Efficiency	✓		✓	✓	✓	✓
Dummy player	✓	✓	✓	✓		
Symmetry	✓	✓	✓		✓	✓
Additivity	✓	✓		✓		✓
Stability	✓		✓ if $N^G < 4$	✓	✓	✓
Anonymity	✓	✓	✓			
Complexity	NP-complete	NP-complete	exact	$O(N^G)$	$O(N^G)$	exact
Simplicity			✓	✓	✓	✓

TABLE II: Total costs when identical DSOs located at different buses cooperate with the TSO.

Coalition	Total Costs [€]
$\{\{TN\}, \{DN_A\}, \{DN_B\}, \{DN_C\}\}$	43.15
$\{\{TN, DN_A\}, \{DN_B\}, \{DN_C\}\}$	39.81
$\{\{TN, DN_B\}, \{DN_A\}, \{DN_C\}\}$	38.74
$\{\{TN, DN_C\}, \{DN_A\}, \{DN_B\}\}$	40.4

contribution to the total cost reduction in the common market. Hence, when its flow with the TSO is more limited, the benefit from cooperation significantly decreases.

In addition to the flexibility bids (prices and quantities) submitted from a given distribution network, the location of the distribution network itself within the system (i.e., the node in the transmission system to which the distribution network is connected) also plays a key role in its contribution to the cost reduction of the coalition. For example, we consider a case in which we place three identical 18-bus distribution networks including the same set of bids (denoted DN_A , DN_B , and DN_C) at different buses of the transmission network and evaluate the total costs when each DSO cooperates with the TSO. The numerical results are shown in Table II. Even though the distribution systems and the set of submitted bids from those systems are the same, the resulting cooperation-induced reduction in system costs are different – as shown Table II – due to the location of transmission system congestions. As shown Table II, the total cost is reduced by 10.2% when the TSO cooperates with DN_B (third row in Table II), but this saving drops to only 6.4% when the TSO cooperates with DN_C (fourth row in Table II).

We next investigate the resulting cost for each SO under the different cost allocation methods. The stacked bars in Fig. 2 showcase the cost allocated to each system operator via the different mechanisms presented in Section V and summarized in Table I. Fig. 2 highlights the varying proportions of the total cooperation cost to be taken up by each SO under different cost allocation methods. For example, in the results of Fig. 2, considering the nominal $T^{p,\max}$ case, the Shapley value leads all DSOs to bear a higher portion of the total cost as compared to, e.g., the PCA method. The results show that the SV, in this numerical setting, induces a shift in total cost from the TSO to the DSOs as compared to the PCA. Under this setting, the CGA is more favorable to the TSO as compared to the SV, while the PCA is less favorable, which is the opposite case to that of the DSOs. In addition, the EMP, for example, can help reducing the cost allocated to DN_{18} , who, in general, bears

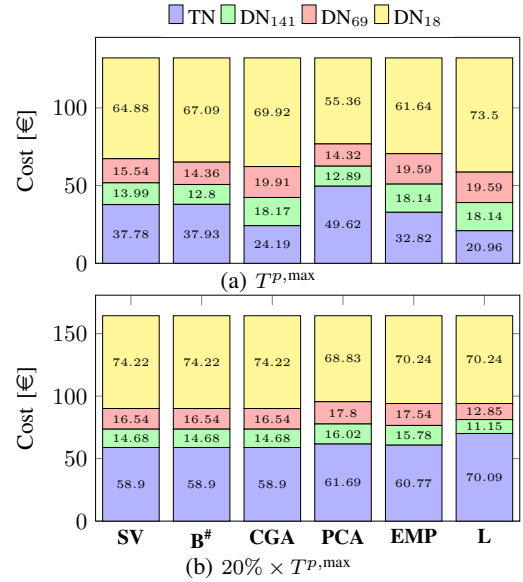


Fig. 2: SOs' costs under different cost allocation methods and interface flow limit.

the highest cost in all the allocation methods. However, the CGA and EPM do not cover some of the essential properties for adequate cost allocation (as shown in Table I).

The interface flow limits have also a direct effect on the disparity of the costs borne by each SO. For example, as shown in Fig. 2, the $20\% \times T^{p,\max}$ limit increases the cost borne by the TSO for all cost allocation methods, while this limit helps reducing the cost borne by DN_{18} under the Lagrangian-based allocation method. In general, Fig. 2 shows that the tighter the interface flow limits the lower the difference in costs under the different cost allocation methods as cooperation becomes more limited and, hence, less consequential.

The presented numerical results are specific to our numerical case analysis. However, they serve to highlight the fact that different cost allocations and interface flow limits can lead to disparity in the cost borne by each SO. Hence, choosing an adequate allocation scheme (meeting key properties in Table I) while reducing this disparity, are essential to achieving beneficial, stable, efficient, and fair cooperation among SOs for the joint procurement of flexibility resources.

VII. CONCLUSION

In this paper, we have introduced a common market model for the joint TSO-DSOs procurement of flexibility. We have then developed a cooperative game approach to analyze the cooperation potential of SOs in the common market. In this

regard, we have proven the stability of their cooperation, implying that cooperation in this common market can naturally arise without external intervention. In addition, we have analyzed several possible cost allocation mechanisms, to split the cost of the jointly procured flexibility among the participating SOs in a stable and adequate manner, while analyzing the properties of each of those methods. Our numerical results have further highlighted the benefits of this cooperation. In addition, the results have shown the paramount effect of the interface flow limits on the benefits of cooperation, highlighting the need for further investments to improve power exchange capabilities among the different grids. The numerical results also highlighted the disparity that can be introduced by different cost allocation methods, where some methods lead to a shift in the costs borne by different SOs for the same market.

REFERENCES

- [1] Distribution Systems Working Group, "CEER paper on DSO procedures of procurement of flexibility," *CEER Ref: C19-DS-55-05*, 2020.
- [2] H. Le Cadre, I. Mezghani, and A. Papavasiliou, "A game-theoretic analysis of transmission-distribution system operator coordination," *European Journal of Operational Research*, vol. 274, no. 1, pp. 317–339, 2019.
- [3] A. Vicente-Pastor, J. Nieto-Martin, D. W. Bunn, and A. Laur, "Evaluation of flexibility markets for retailer–DSO–TSO coordination," *IEEE Transactions on Power Systems*, vol. 34, no. 3, pp. 2003–2012, 2019.
- [4] A. Papavasiliou and I. Mezghani, "Coordination schemes for the integration of transmission and distribution system operations," in *Power Systems Computation Conference (PSCC)*, 2018, pp. 1–7.
- [5] A. Sanjab, Y. Mou, A. Virag, and K. Kessels, "A linear model for distributed flexibility markets and DLMPs: A comparison with the SOCP formulation," in *CIREC*, 2021, arxiv.org/abs/2111.02328.
- [6] A. Roos, "Designing a joint market for procurement of transmission and distribution system services from demand flexibility," *Renewable Energy Focus*, vol. 21, pp. 16–24, 2017.
- [7] A. Hermann, J. Kazempour, S. Huang, and J. Østergaard, "Congestion management in distribution networks with asymmetric block offers," *IEEE Transactions on Power Systems*, vol. 34, no. 6, pp. 4382–4392, 2019.
- [8] T. Schittekatte and L. Meeus, "Flexibility markets: Q&A with project pioneers," *Utilities Policy*, vol. 63, p. 101017, 2020.
- [9] H. Gerard, E. I. Rivero Puento, and D. Six, "Coordination between transmission and distribution system operators in the electricity sector: A conceptual framework," *Utilities Policy*, vol. 50, pp. 40–48, 2018.
- [10] P. Sheikhamadi, S. Bahramara, A. Mazza, G. Chicco, and J. P. Catalão, "Bi-level optimization model for the coordination between transmission and distribution systems interacting with local energy markets," *International Journal of Electrical Power & Energy Systems*, vol. 124, p. 106392, 2021.
- [11] M. Hupez, J.-F. Toubeau, Z. De Grève, and F. Vallée, "A new cooperative framework for a fair and cost-optimal allocation of resources within a low voltage electricity community," *IEEE Transactions on Smart Grid*, vol. 12, no. 3, pp. 2201–2211, 2021.
- [12] V. Robu, M. Vinyals, A. Rogers, and N. R. Jennings, "Efficient buyer groups with prediction-of-use electricity tariffs," *IEEE Transactions on Smart Grid*, vol. 9, no. 5, pp. 4468–4479, 2017.
- [13] A. Sanjab and W. Saad, *Power System Analysis: Competitive Markets, Demand Management, and Security*. Cham: Springer International Publishing, 2017, pp. 1–38.
- [14] M. Baran and F. F. Wu, "Optimal sizing of capacitors placed on a radial distribution system," *IEEE Transactions on Power Delivery*, vol. 4, no. 1, pp. 735–743, 1989.
- [15] S. Wang, S. Chen, L. Ge, and L. Wu, "Distributed generation hosting capacity evaluation for distribution systems considering the robust optimal operation of OLTC and SVC," *IEEE Transactions on Sustainable Energy*, vol. 7, no. 3, pp. 1111–1123, 2016.
- [16] G. Chalkiadakis, E. Elkind, and M. Wooldridge, "Computational aspects of cooperative game theory," *Synthesis Lectures on Artificial Intelligence and Machine Learning*, vol. 5, no. 6, pp. 1–168, 2011.
- [17] O. N. Bondareva, "Some applications of linear programming methods to the theory of cooperative games," *Problemy kibernetiki*, vol. 10, no. 119, p. 139, 1963.
- [18] L. S. Shapley, "Cores of convex games," *International journal of game theory*, vol. 1, no. 1, pp. 11–26, 1971.
- [19] S. Hart, "Shapley value," in *Game Theory*. Springer, 1989, pp. 210–216.
- [20] S. H. Tijs and T. S. Driessen, "Game theory and cost allocation problems," *Management science*, vol. 32, no. 8, pp. 1015–1028, 1986.

- [21] M. Frisk, M. Göthe-Lundgren, K. Jörnsten, and M. Rönnqvist, "Cost allocation in collaborative forest transportation," *European Journal of Operational Research*, vol. 205, no. 2, pp. 448–458, 2010.
- [22] G. Owen, "On the core of linear production games," *Mathematical Programming*, no. 9, pp. 358–370, 1975.
- [23] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Transactions on power systems*, vol. 26, no. 1, pp. 12–19, 2010.

APPENDIX

COMPACT LP FORMULATION OF THE COMMON MARKET

We aim at expressing (22)-(24) in the compact linear matrix form in (25). To that purpose, we introduce $\mathbf{M}_{adj}^{n^T} \in \text{Mat}(\mathcal{N}^{n^T}, \mathcal{N}^{n^T})$ as the adjacency matrix of graph $\mathcal{G}^{n^T}(\mathcal{N}^{n^T}, \mathcal{L}^{n^T})$, where $\mathbf{M}_{adj}^{n^T}(i, j) = 1$ if there exists a link connecting node $i \in \mathcal{N}^{n^T}$ to node $j \in \mathcal{N}^{n^T}$; 0 otherwise. Similarly, we define $\mathbf{M}_{adj}^T \in \text{Mat}(\mathcal{N}^T, \mathcal{N}^T)$ as the adjacency matrix of $\mathcal{G}^T(\mathcal{N}^T, \mathcal{L}^T)$, where $\mathbf{M}_{adj}^T(i, j) = 1$ if there exists a link connecting node $i \in \mathcal{N}^T$ to node $j \in \mathcal{N}^T$; 0 otherwise. In addition, we consider the generation shift factor (GSF) matrix $\Xi \in \text{Mat}(\mathcal{L}^T, \mathcal{N}^T)$ such that $\Xi((i, j), n) = X_{(i,j),n}$ is the GSF of line $(i, j) \in \mathcal{L}^T$ and node $n \in \mathcal{N}^T$. Moreover, for notation simplicity, we introduce vector $\mathbf{1}_{n^T}$ of size $1 \times \mathcal{N}^T$, with $\mathbf{1}_{n^T}(n^T) = 1$; 0 otherwise. Similarly, for any $i \in \mathcal{N}^{n^T}$, we introduce vector $\mathbf{1}_i^{n^T}$ of size $1 \times \mathcal{N}^{n^T}$, with $\mathbf{1}_i^{n^T}(i) = 1$; 0 otherwise. The square matrix with vector \mathbf{x} on its main diagonal is denoted $\text{diag}(\mathbf{x})$. $\mathbf{M}(n, :)$ denotes the n -th row of matrix \mathbf{M} , while $\mathbf{M}(:, n)$ denotes its n -th column.

We start with the *transmission side*. It is clear that (2) is linear in \mathbf{x}_0 and \mathbf{z} . It can be written under the compact form: $(\mathbf{1}_n - \mathbf{1}_n \mathbf{1}_n - \mathbf{1}_n) \mathbf{x}_0 - (\mathbf{0} \mathbf{1}_n \mathbf{0}) \mathbf{z} = \mathbf{d}_n^{T,o} - \mathbf{p}_n^{T,o}, \forall n \in \mathcal{N}^T$. Then, (3) is linear in \mathbf{z} leading to the compact linear formulation: $-\mathbf{p}^{T,t} \Xi(l, :)^t + \mathbf{P}^T(i, j) + \mathbf{T}^p \Xi(l, :)^t = 0, \forall l = \{i, j\} \in \mathcal{L}^T$. Equation (4) applies to nodes on the transmission network, excluding nodes at the interface with the distribution networks. Equation (4) is also linear in \mathbf{z} , taking the form: $\mathbf{p}_n^T - \mathbf{M}_{adj}^T(n, :)(\mathbf{P}^T(n, :))^t = 0, \forall n \in \mathcal{N}^T \setminus \mathcal{N}^D$. Equation (5) focuses on nodes at the interface between the transmission and the distribution networks. It takes the compact form below, which is again linear in \mathbf{z} : $\mathbf{p}_n^T - \mathbf{T}_n^p - \mathbf{M}_{adj}^T(n, :)(\mathbf{P}^T(n, :))^t = 0, \forall n \in \mathcal{N}^D$. Finally, (6)–(8) provides lower and upper-bounds on the variables $\mathbf{P}^T(i, j)$ in \mathbf{z} and on \mathbf{x}_0 , respectively.

At the *distribution side*, we consider any $\text{DSO-}n^T \in \mathcal{N}^{n^T}$. (10) is linear in \mathbf{x}_{n^T} and \mathbf{z} : $(-\mathbf{1}_n^{n^T} \mathbf{1}_n^{n^T} - \mathbf{1}_n^{n^T} \mathbf{1}_n^{n^T} \mathbf{0}) \mathbf{x}_{n^T} + (\mathbf{0} \mathbf{1}_n^{n^T} \mathbf{0}) \mathbf{z} = \mathbf{1}_n^{n^T} (\mathbf{p}^{n^T,o} - \mathbf{d}^{n^T,o}), \forall n \in \mathcal{N}^{n^T}$. (11) is linear only in \mathbf{z} , and can be written as follows: $\mathbf{p}_n^{n^T} + \mathbf{1}_n^{n^T} \mathbf{P}^{n^T}(n, :) - \mathbf{M}_{adj}^{n^T}(n, :)(\mathbf{P}^{n^T}(n, :))^t = \mathbf{0}$. (12) is linear in \mathbf{x}_{n^T} and \mathbf{z} : $\mathbf{q}_n^{n^T} + \mathbf{1}_n^{n^T} \mathbf{Q}^{n^T}(n, :) - \mathbf{M}_{adj}^{n^T}(n, :)(\mathbf{Q}^{n^T}(n, :))^t = \mathbf{0}$, (13)-(14) are linear in \mathbf{z} , giving rise to: $\mathbf{T}_n^{p,q} - \mathbf{M}_{adj}^{n^T}(n_0^{n^T}, :)(\mathbf{P}^{n^T}(n_0^{n^T}, :))^t = 0$ and $\mathbf{T}_n^{q,q} - \mathbf{M}_{adj}^{n^T}(n_0^{n^T}, :)(\mathbf{Q}^{n^T}(n_0^{n^T}, :))^t = 0$. For (15), we get: $v_n^{n^T} - v_{A(n)}^{n^T} + 2 \cdot \mathbf{M}_{adj}^{n^T}(n, :)(\text{diag}(\mathbf{r}^{n^T}))(\mathbf{P}^{n^T}(n, :))^t - \text{diag}(\mathbf{x}^{n^T})(\mathbf{Q}^{n^T}(n, :))^t = 0 \forall n \in \mathcal{N}^{n^T} \setminus n_0^{n^T}$. (16) can be expressed as a function of \mathbf{z} : $\alpha_m \mathbf{P}^{n^T}(i, j) + \beta_m \mathbf{Q}^{n^T}(i, j) + \delta_m \mathbf{S}^{n^T} \max(i, j) \leq 0, \forall m \in \mathcal{M} \forall \{i, j\} \in \mathcal{L}^{n^T}$. Finally, (17)-(21) impose lower and upper-bounds on \mathbf{x}^{n^T} and \mathbf{z} components.