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On a Markovian game model for competitive insurance pricing

Claire Mouminoux*, Christophe Dutang†, Stéphane Loisel‡, Hansjoerg Albrecher§

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Abstract

In this paper, we extend the non-cooperative one-period game of [Dutang et al. \(2013\)](#) to model a non-life insurance market over several periods by considering the repeated (one-period) game. Using Markov chain methodology, we derive general properties of insurer portfolio sizes given a price vector. In the case of a regulated market (identical premium), we are able to obtain convergence measures of long run market shares. We also investigate the consequences of the deviation of one player from this regulated market. Finally, we provide some insights of long-term patterns of the repeated game as well as numerical illustrations of leadership and ruin probabilities.

keywords: game theory, non-cooperative game, consumers' price sensitivity, solvency constraint, Markov chains

Mathematics Subject Classification (2010): MSC 60J10, MSC 91G05, MSC 91A20

1 Introduction

In the insurance world, determining an appropriate and attractive premium is always an important issue because of the strong competition between the different insurers. This is particularly true for non-life retail insurance mass markets, such as motor insurance and household insurance, where a substantial number of insurance companies and mutuals compete. Policyholders are looking for the best deal at a low price and thus tend to migrate to firms displaying advantageous insurance premiums.

The pricing of insurance contracts is a classical research topic. In practice, insurance companies use various approaches including general principles of premium calculation, based on the expected claim expenses or its higher moments (e.g. standard deviation), credibility theory and generalized linear models (GLM), see for instance [Kaas et al. \(2008\)](#). However, in a highly competitive market that is dominated by a relatively small number of firms, insurance pricing is not limited to computing technical premiums: each insurer attempts to predict customers' behavior in order to gain market shares. The actuarial premium is thus tuned by the marketing and management departments for several reasons, such as the customers' loyalty and the market conditions.

As a consequence, game theory concepts are suitable to evaluate the strategic choices of insurers in the presence of competition. The use of game theory in actuarial science is not new. The first attempts go back to [Borch \(1962, 1974\)](#), [Bühlmann \(1980, 1984\)](#), and [Lemaire \(1984, 1991\)](#), who applied cooperative games in order to model risk transfer between insurers and/or reinsurers (see also [Aase \(1993\)](#), [Brockett & Xiaohua \(1997\)](#), [Tsanakas & Christofides \(2006\)](#), [Boonen \(2016\)](#), [Asimit et al. \(2021\)](#)). Regarding non-cooperative games, two types of models have been considered in non-life insurance markets: a) the Cournot oligopoly where insurers' strategies are based on the choice of business volume (see [Powers & Shubik \(1998\)](#), [Powers et al. \(1998\)](#)) and b) the Bertrand oligopoly where insurers set premiums (see [Polborn \(1998\)](#), [Rees et al. \(1999\)](#)).

Several single-optimization models have been proposed in the literature: in discrete-time by [Taylor \(1986, 1987\)](#) and [Pantelous & Passalidou \(2013, 2015, 2016\)](#); and in continuous-time by [Kliger &](#)

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Levikson (1998), Emms et al. (2007), Moreno-Codina & Gomez-Alvado (2008). Concerning non-cooperative games to model non-life insurance markets, in a static framework Dutang et al. (2013) and Mourdoukoutas et al. (2021) study one-period stochastic games to determine the optimal premium levels. In continuous time, Emms (2012) and Li et al. (2021) propose N -player differential games in a deterministic framework, Boonen et al. (2018) propose a N -player differential game by considering competition among each pair of insurers, based on a static concept proposed by Wu & Pantelous (2017). Asmussen et al. (2019a,b) propose two-player differential games, where the premiums result from Nash or Stackelberg equilibria.

We focus our analysis on consequences of the customers' price sensitivity (that is, customers' inertia), as well as consequences of solvency constraints on price equilibrium, market shares and the resulting profits. The model developed in Dutang et al. (2013) is the most appropriate starting point for the purpose of the present paper, since it includes a lapse model, an aggregate loss model as well as a solvency constraint function for insurers.

The consideration of a stochastic lapse behavior through a lapse probability as a function of the premiums offered by all insurers is a natural way to model consumers' inertia and to avoid the Bertrand paradox. Because of the strong regulation of the insurance market, it is essential to consider a solvency constraint in the insurer's decision-making strategies, which will affect both the premium level and the long run market patterns.

Other extensions of Dutang et al. (2013) have been proposed by Albrecher & Daily-Amir (2017), who consider a Bayesian framework in order to investigate the effects of asymmetric information as well as (Daily-Amir 2019, Chap. 4), which considers different objective functions to model market shares in health insurance studied in Daily-Amir et al. (2019). Moreover, Battulga et al. (2018) consider an m -period version of the objective function but still investigates the premium equilibrium in a static framework.

For the consideration of reinsurance aspects in this context we refer to Zeng (2010), Jin et al. (2013) who propose a zero-sum two-player dynamic reinsurance game, and Chen & Shen (2018), Chen et al. (2020) where transactions between reinsurance buyers and sellers are formulated through Stackelberg games.

In this paper we consider a repeated one-period game. We investigate convergence properties of the market using a Markov chain approach and also emphasize links of the lapse model with closed Markovian networks, see e.g. Boucherie & van Dijk (2011).

The rest of the paper is organized as follows. Section 2 recalls the one-period game based on Dutang et al. (2013). Section 3 presents general properties of the lapse and loss models given a price vector value. Section 4 focuses on properties of the repeated game in the case of a strongly regulated market, while Section 5 studies a deviation from this situation. In Section 6, we analytically investigate some asymptotic properties of the repeated game, and we present numerical computations of the leadership and the ruins probabilities, illustrating long-run pattern of the repeated game. Finally, a conclusion and perspectives for future research are given in Section 7.

2 Description of the repeated game

Consider J insurance companies competing on the price in a market of N policyholders with one-year policies, where N is considered constant across periods. In the sequel, a subscript $j \in \{1, \dots, J\}$ will always denote a player index (i.e. an insurer) whereas a subscript $i \in \{1, \dots, N\}$ denotes an insured index. Vectors will be **bolded**. The "game" for each insurer $j \in \{1, \dots, J\}$ consists in defining a premium $x_{j,t}$ at the beginning of each year t , in order to maximize the profit by selling identical policies to the insured market of size N . Let $\mathbf{x}_t = (x_{1,t}, \dots, x_{J,t}) \in \mathbb{R}^J$ be the insurers' price vector, with $x_{j,t}$ representing premium of Insurer j for year t . We denote by $\mathbf{x}_{-j,t}$ others' player action, i.e., $\mathbf{x}_{-j,t} = (x_{1,t}, \dots, x_{j-1,t}, x_{j+1,t}, \dots, x_{J,t})$ the vector \mathbf{x}_t without the j th component. In this section, we first describe the sequence of the repeated game and then explain each component in detail.

2.1 Game overview

Let us assume that the pricing is done at the beginning of the year, after an update of past year observations such as loss level, market share and capital value. In this paper we consider the Nash equilibrium (interpreted as a point at which each player, given the strategies of others players, has no incentive to change the strategy) and refrain from studying the Stackelberg equilibrium (see e.g. [Osborne & Rubinstein \(2006\)](#)). In the majority of insurance markets (particularly in retail-mass markets), it is indeed not realistic to assume that the same insurer always takes its decision before all others.

Definition 1. For a static game with J players, payoff functions O_j and constraint functions g_j , a Nash equilibrium is a vector $\mathbf{x}^* = (x_1^*, \dots, x_J^*)$, such that for all $j = 1, \dots, J$, x_j^* solves the subproblem

$$\max_{x_j \in [\underline{x}, \bar{x}]} O_j(x_j, \mathbf{x}_{-j}^*) \text{ s.t. } g_j(x_j) \geq 0.$$

Here, $O_j(x_j, \mathbf{x}_{-j}^*)$ denotes the objective function of Player j at Player j 's action x_j given the other players' equilibrium \mathbf{x}_{-j}^* , while \underline{x} and \bar{x} are exogenous finite lower and upper bounds (on the premium). $g_j(x_j) \geq 0$ is the inequality for the (solvency) constraint.

Note that the constraint function only depends on player j 's action x_j and not on the other players' strategies \mathbf{x}_{-j} (otherwise this would lead to a generalized Nash equilibrium, with very different properties, see, e.g., [Facchinei & Kanzow \(2010\)](#)).

We define the repeated game as the iteration of one-period games over T years, where $K_{j,t}$ stands for the capital of Insurer j at time t , J_t for the set of active insurers and $S_{j,t}$ for the aggregate claim amount of Insurer j .

Repeated game. At time $t = 0$, initiate $J_0 = \{1, \dots, J\}$, positive capital levels $K_{j,0}$ and (positive) portfolio sizes \mathbf{n}_0 . For period $t = 1, \dots, T$, repeat

1. The insurers among J_t maximize their objective function subject to the solvency constraint:

$$\sup_{x_{j,t} \in [\underline{x}, \bar{x}]} O_{j,t}(x_{j,t}, \mathbf{x}_{-j,t}) \text{ s.t. } g_{j,t}(x_{j,t}) \geq 0.$$

2. Once the premium equilibrium vector \mathbf{x}_t^* is determined, customers randomly lapse or renew. We get a realization $n_{j,t}^*$ of the random portfolio size $N_{j,t}(\mathbf{x}_t^*)$.
3. Aggregate claim amounts $S_{j,t}$ are randomly drawn according to the chosen loss model and the portfolio size $n_{j,t}^*$.
4. The underwriting result for Insurer j is computed by $UW_{j,t} = n_{j,t}^* \times x_{j,t}^* \times (1 - e_{j,t}) - S_{j,t}$, where $e_{j,t}$ corresponds to the rate of handling costs of Insurer j at time t .
5. The capital is updated via $K_{j,t+1} = K_{j,t} + UW_{j,t}$.
6. The set of competitors J_{t+1} is updated by removing bankrupted insurers, tiny insurers and insolvent insurers.

Since the regulator asks insurers to be solvent, insurers are removed from the market when they have a negative capital level $K_{j,t} < 0$ (bankrupted insurers). We also remove small insurers with a market share below 0.1%, $n_{j,t}^* < 0.1\%N$, assuming that in such a case an insurer will not be able to face future losses and handling costs. Indeed, such an insurer will not benefit from mutualisation concepts, essential for the insurance market and thus will decide to run-off the business. Finally, we remove players from the game when the capital is below the minimum capital requirement (MCR), whereas we keep them if capital is between MCR and solvency capital requirement (SCR). As a reminder, MCR can be defined as a percentage of the SCR computed in the solvency constraint. In general, in the non-life insurance retail market MCR is between 25% and 45% of the SCR.

For the present paper, we add the following simplifying assumptions: (i) the pricing procedure is done once a year (on January 1), (ii) all policies start at the beginning of the year, (iii) all premiums are collected on January 1, (iv) every claim is (fully) paid on December 31, (v) there is no inflation and (vi) there is no stock/bond market to invest premium.

In practice, these assumptions do not hold: (i) pricing by actuarial and marketing departments can be done more frequently, e.g. every 6 months, (ii) policies start and are renewed all over the year, (iii) premium is collected all over the year, (iv) claims are settled every day and there are reserves for incurred-but-not-reported claims, (v) there is inflation on both claims and premiums, (vi) the time between the premium payment and a possible claim payment is used to invest in stock/bond markets. The above assumptions will allow for a more tractable model and are not expected to change the premium equilibria drastically yet it will introduce more randomness on loss and premium.

Our game has four components: a lapse model, a loss model, an objective function and a solvency constraint function. We will follow again the choice made in [Dutang et al. \(2013\)](#) and briefly describe it in the sequel.

2.2 Components of the game for policyholders

2.2.1 The lapse model

Assuming Markovian behavior of policyholders, we assume that the probability to move from Insurer j to Insurer k at time t is $P(C_{i,t} = k \mid C_{i,t-1} = j) = p_{j \rightarrow k}(\mathbf{x}_t)$ with the multinomial logit assumption

$$p_{j \rightarrow k}(\mathbf{x}) = \begin{cases} \frac{1}{1 + \sum_{l \neq j} e^{f_j(\mathbf{x}_j, \mathbf{x}_l)}} & \text{if } j = k, \\ \frac{e^{f_j(\mathbf{x}_j, \mathbf{x}_k)}}{1 + \sum_{l \neq j} e^{f_j(\mathbf{x}_j, \mathbf{x}_l)}} & \text{if } j \neq k. \end{cases} \quad (1)$$

We could have chosen the multinomial probit model, but then we would have lost a tractable $p_{j \rightarrow k}$, yet the shape of the lapse function would have been very similar, see, e.g., [McFadden \(1981\)](#) for the random utility maximization setting. Here the sum is taken over the set $\{1, \dots, J\}$ and f_j is a price sensitivity function. We consider a premium ratio function and premium difference function, respectively,

$$\bar{f}_j(x, y) = \mu_j + \alpha_j \frac{x}{y} \text{ and } \tilde{f}_j(x, y) = \tilde{\mu}_j + \tilde{\alpha}_j(x - y), \quad (2)$$

where $\mu_j \in \mathbb{R}$ and $\alpha_j > 0$. Let $\mathbf{p}_{j \rightarrow}(\mathbf{x}_t) = (p_{j \rightarrow 1}(\mathbf{x}_t), \dots, p_{j \rightarrow J}(\mathbf{x}_t))$. The following assumptions on customers are made

- A1: Customer behavior is identical across the market and over time, i.e. $(C_{i,t})_t$ are identically distributed for period t per insurer but depends on \mathbf{x} .
- A2: Customers are independent, i.e. $(C_{i,t})_i$ are independent.
- A3: The customer choice of insurer at time t depends only on the previous choice at time $t - 1$ and $C_{i,t} \sim \mathcal{M}_J(1, \mathbf{p}_{j \rightarrow}(\mathbf{x}_t))$ ¹.
- A4: No customer can enter or exit the market, the total market size N is constant.

In fact, the random choice $(C_{i,t})_t$ of Policyholder i is governed by a (discrete-time) Markov chain with transition matrix

$$P_{\rightarrow}(\mathbf{x}_t) \triangleq \begin{pmatrix} p_{1 \rightarrow 1}(\mathbf{x}_t) & \dots & p_{1 \rightarrow J}(\mathbf{x}_t) \\ & \ddots & \\ p_{J \rightarrow 1}(\mathbf{x}_t) & \dots & p_{J \rightarrow J}(\mathbf{x}_t) \end{pmatrix}.$$

In general, the Markov chain is time-inhomogeneous, as P_{\rightarrow} depends on the price vector \mathbf{x}_t which evolves over time. Note that using Markov chains to study customer behavior was also done in [Marker \(1998\)](#), where a special case is considered: the duopolistic situation of one insurer versus the market.

From an insurer point of view, we are interested in the number of policyholders at time t . Let $n_{j,t-1}$ be the portfolio size at the previous period $t - 1$. We define $\bar{\mathbf{C}}_{j,t} = (\bar{C}_{j,1,t}, \dots, \bar{C}_{j,J,t})$ as the random assignment of customers of Insurer j at time t , where $\bar{C}_{j,k,t}$ denotes the (random) number of policyholders moving from Insurer j to Insurer k .

¹ $\mathcal{M}_d(n, \mathbf{p})$ denotes the multinomial distribution of dimension d with trial number parameter n and event probability parameter $\mathbf{p} \in [0, 1]^d$.

Based on A1, A2, A3, policyholders of Insurer j will choose insurers according to a multinomial distribution $\bar{\mathbf{C}}_{j,t} \sim \mathcal{M}_J(n_{j,t-1}, \mathbf{p}_{j \rightarrow}(\mathbf{x}_t))$ given the portfolio size $n_{j,t-1}$. Let \mathbf{N}_t be the vector of (random) portfolio sizes at time t . They are obtained by summing the (independent) choices of each Insurers' customers

$$\mathbf{N}_t = \sum_{j=1}^J \bar{\mathbf{C}}_{j,t} = \begin{pmatrix} \bar{C}_{1,1,t} \\ \vdots \\ \bar{C}_{1,J,t} \end{pmatrix} + \cdots + \begin{pmatrix} \bar{C}_{J,1,t} \\ \vdots \\ \bar{C}_{J,J,t} \end{pmatrix} = \begin{pmatrix} N_{1,t} \\ \vdots \\ N_{J,t} \end{pmatrix}.$$

We denote by $\mathbf{n}_t = (n_{1,t}, \dots, n_{J,t})$ the realizations of the random vector.

For Insurer j , his portfolio size $N_{j,t}$ is a sum of independent variables. It is important to note that the insurers' portfolio sizes are not independent, since the total market size remains constant (A4).

Finally, the process $(\mathbf{N}_t)_t$ is a J -dimensional discrete-time Markov process (cf. Proposition 3 later). The process $(\mathbf{N}_t)_t$ takes values in the set of portfolio sizes

$$\mathcal{S}_{ms} = \left\{ \mathbf{n} \in \mathbb{N}^J, \sum_{j=1}^J n_j = N \right\}, \quad (3)$$

which has $\binom{N+J-1}{N}$ elements, see e.g. Breuer & Baum (2005).

The lapse/renewal process of policyholders at each point in time can be seen as a closed Markovian network of discrete-time queues with batch services (see e.g. (Boucherie & van Dijk 2011, Chap. 6)), for which the service time corresponds to the number of years a policyholder stays with the same insurer. In that context, $p_{j \rightarrow k}$ are called routing probabilities, an insurer is a server and a policyholder is a customer.

2.2.2 The loss model

Let $Y_{i,t}$ be the aggregate loss of policy i during the period t . We assume

- A5: There is no adverse selection, i.e. $Y_{i,t}$ are independent and identically distributed (i.i.d.) random variables, $\forall i = 1, \dots, N$.
- A6: Catastrophic events are excluded and Y_i follows a frequency – average severity loss model

$$Y_{i,t} = \sum_{l=1}^{M_{i,t}} Z_{i,l,t}, \quad (4)$$

where the claim number $M_{i,t}$ is independent of the claim severity $Z_{i,l,t}$.

- A7: The insurance business is short-tailed, i.e. the loss Y_i is paid in total on December 31 of each year.

Assumption A5 allows us to simplify the simulation process because the i.i.d. assumption implies that individual losses $Y_{i,t}$ do not need to be simulated. The aggregate claim amount for Insurer j is

$$S_{j,t}(\mathbf{x}_t) = \sum_{i=1}^{N_{j,t}(\mathbf{x}_t)} Y_{i,t}, \quad (5)$$

where $N_{j,t}(\mathbf{x}_t)$ is the portfolio size of Insurer j given the price vector \mathbf{x}_t .

In the sequel, we assume that the claim frequency distribution belongs to the $(a, b, 0)$ class (cf. Klugman et al. (2012)). Table 1 displays the three well known members of this class, where the last column gives the scale parameter of the distribution, which will play an important role below. Using the $(a, b, 0)$ class assumption, Proposition 5 derives the distribution of $S_{j,t}(\mathbf{x}_t)$ in a general setting and Proposition 8 derives the distribution of $S_{j,t}(\mathbf{x}_t)$ for an invariant measure μ . Obtaining a compound distribution for $S_{j,t}(\mathbf{x}_t)$ also allows to easily carry out numerical illustrations, as we do not need to simulate insurance losses by customer but by insurer.

Regarding the claim severity distribution, we assume a light-tailed distribution so that the moment-generating function exists for some positive arguments. Assumption A6 can be summarized as

Distribution	Symbol	a	b	m.p.f. at 0	scale param.
Poisson	$\mathcal{P}(\lambda)$	0	λ	$\exp(-\lambda)$	λ
binomial	$\mathcal{B}(r, q)$	$-\frac{q}{1-q}$	$\frac{q(r+1)}{1-q}$	$(1-q)^r$	r
negative binomial	$\mathcal{NB}(r, q)$	$1-q$	$(r-1)(1-q)$	q^r	r

Table 1: The $(a, b, 0)$ class

- $M_{j,t} \stackrel{\text{i.i.d.}}{\sim} M$ with M in the $(a, b, 0)$ class,
- $(Z_{j,l,t})_l \stackrel{\text{i.i.d.}}{\sim} Z$ with $G_Z^M(\alpha) := E(e^{\alpha Z}) < \infty$ for some $\alpha > 0$.

2.3 Components of the game for insurers

2.3.1 Objective function

Insurers have a history of past premium levels $x_{j,t}^*$, gross written premium $\text{GWP}_{j,t}$, portfolio size $n_{j,t}$ and capital $K_{j,t}$ at the beginning of year t . Let d be the past number of years considered for which economic variables (e.g. market premium) are computed and available for making decisions.

We define the estimated mean of overall costs including handling costs and claims of Insurer j as

$$\pi_{j,t} = \omega_j \bar{a}_{j,t-1} + (1 - \omega_j) \bar{m}_{t-1}, \quad (6)$$

where $\omega_j \in [0, 1]$ is the credibility factor of Insurer j and the average market premium is determined, available e.g. via rating agencies or through insurer associations, as

$$\bar{m}_{t-1} = \frac{1}{d} \sum_{u=1}^d m_{t-u}, \quad \text{with } m_{t-u} = \frac{\sum_{j \in J_t} \text{GWP}_{j,t-u} \times x_{j,t-u}^*}{\sum_{j \in J_t} \text{GWP}_{j,t-u}},$$

which is the mean of last d market premiums. Insurer j computes its actuarially based premium as the empirical mean of individual loss averages

$$\bar{a}_{j,t-1} = \frac{1}{1 - e_{j,t}} \frac{1}{d} \sum_{u=1}^d \frac{S_{j,t-u}}{n_{j,t-u}},$$

where $S_{j,t}$ denotes the observed aggregate loss of Insurer j during year t and $e_{j,t}$ denotes the expense rate as a percentage of gross written premium. Note that the claim amount is not adjusted against large claims (i.e. $y_{i,t}$ are not capped). We choose the demand function as

$$D_{j,t}(\mathbf{x}_t) = \frac{n_{j,t-1}}{N} \left(1 - \beta_j \left(\frac{x_{j,t}}{m_j(\mathbf{x}_t)} - 1 \right) \right), \quad (7)$$

where $\beta_j > 0$ is the elasticity parameter and $m_j(\mathbf{x})$ is a market premium proxy. Indeed, we assume that the insurance product is a normal product where price elasticity of consumers is negative. In this form, $D_j(\mathbf{x}_t)$ approximates the expected market share $E(N_{j,t}(\mathbf{x}_t))/N$ presented in Section 2.2.1.

The market proxy used in Equation (7) is the mean of other competitors' premium

$$m_{j,t}(\mathbf{x}) = \frac{1}{J_t - 1} \sum_{k \in J_t \setminus \{j\}} x_k. \quad (8)$$

The market proxy aims to assess other insurers' premiums.

Consequently, Insurer j typically does not target the cheapest, the most expensive or the leader. From a mathematical point of view, we would lose the continuity of the demand function if chose

the cheapest premium $\min_{k \neq j} x_k$. Furthermore, the term $\frac{x_j}{m_j(\mathbf{x})} - 1$ in the demand function is closely related to the average of the relative premium differences since

$$\frac{1}{J_t - 1} \sum_{k \in J_t \setminus \{j\}} \left(\frac{x_k}{x_j} - 1 \right) = \frac{1}{x_j} \frac{1}{J_t - 1} \sum_{k \in J_t \setminus \{j\}} x_k - \frac{J_t - 1}{J_t - 1} = \frac{m_{j,t}(\mathbf{x})}{x_j} - 1.$$

Consider now an alternative market proxy value, where firms do not attribute the same weight to each competitor, and we assume here that firms put more importance on the biggest competitors' price. In the following, we analyze the following market proxy (a weighted mean of other competitors' prices)

$$m_{j,t}(\mathbf{x}, \mathbf{n}) = \frac{1}{N - n_j} \sum_{k \in J_t \setminus \{j\}} n_k x_k. \quad (9)$$

Assuming a positive price elasticity, we state the objective function defined as the product of a demand function and an expected profit per policy representing a company-wide expected profit

$$O_{j,t}(\mathbf{x}_t) = \frac{n_{j,t}}{N} \left(1 - \beta_j \left(\frac{x_{j,t}}{m_{j,t}(\mathbf{x}_t, \mathbf{n}_t)} - 1 \right) \right) (x_{j,t} - \pi_{j,t}), \quad (10)$$

where $\pi_{j,t}$ is the break-even premium j in (6) and $m_{j,t}(\mathbf{x}_t, \mathbf{n}_t)$ is either (8) or (9).

2.3.2 Constraint function

Based on the Solvency II framework, we choose the tractable solvency constraint function

$$g_{j,t}^1(x_{j,t}) = \frac{K_{j,t-1} + n_{j,t-1}(x_{j,t} - \pi_{j,t})}{k_{995} \sigma(Y) \sqrt{n_{j,t-1}}} - 1, \quad (11)$$

where $k_{995} = 3$ is the solvency coefficient approximating the 99.5% quantile level, see [Dutang et al. \(2013\)](#) for details. Two constraints functions g_j^2 and g_j^3 are considered to ensure $x_{j,t} \in [\underline{x}, \bar{x}]$.

When the capital level of an insurer is between the MCR and SCR, the regulator will completely monitor it. If the capital drops below the MCR, then the regulator will demand the company to run-off their business (cf. Section 2.1).

3 Properties of insurer portfolio sizes given a price vector

3.1 Properties of the lapse model

In this first subsection, we study the theoretical properties of the lapse model at both policyholder level and insurer level given a series of (bounded) price vector \mathbf{x}_t . We derive a proposition for the process $(C_{i,t})_t$ of Policyholder i as well as a proposition for the process $(\bar{C}_{j,t})_t$ and $(N_{j,t})_t$ of Insurer j . All proofs are postponed to Appendix A.1.

For a single Policyholder i , the following result shows that the choice sequence $(C_{i,t})_t$ follows a time-inhomogeneous Markov chain.

Proposition 1. *The choice $(C_{i,t})_t$ of Customer i at time t is a time-inhomo-geneous Markov chain with transition matrix $P_{\rightarrow}^{(t)} = P_{\rightarrow}(\mathbf{x}_1) \times \dots \times P_{\rightarrow}(\mathbf{x}_t)$. The Markov chain $(C_{i,t})_t$ has an invariant measure.*

The choice vector $(C_{1,t}, \dots, C_{N,t})_t$ of all customers at time t is a time-inhomogeneous Markov chain with transition matrix $P_{\rightarrow}(\mathbf{x}_t) \otimes^n$ (the n -times Kronecker product of the matrix $P_{\rightarrow}(\mathbf{x}_t)$).

Hence, the distribution of policyholders for Insurer j can be obtained.

Proposition 2. *For all $t \in \mathbb{N}^*$, $\bar{C}_{j,t} \sim \mathcal{M}_J(n_{j,0}, \tilde{\mathbf{p}}_j)$ given $N_{j,0} = n_{j,0}$ where $\tilde{\mathbf{p}}_j$ is the j th row of the matrix $P_{\rightarrow}^{(t)}$. In particular, $\bar{C}_{j,t} \sim \mathcal{M}_J(n_{j,t-1}, \mathbf{p}_{j \rightarrow}(\mathbf{x}_t))$ given $N_{j,t-1} = n_{j,t-1}$.*

Following Propositions 1 and 2, we show that insurer portfolio sizes $(\mathbf{N}_t)_t$ constitute a (multidimensional) Markov chain in the set of all possible portfolio sizes \mathcal{S}_{ms} defined in (3). We also characterize the conditional distribution $\mathbf{N}_t \mid \mathbf{N}_{t-1} = \mathbf{n}$ which in the limit is a multinomial distribution:

Proposition 3. *The insurer portfolio size vector $(\mathbf{N}_t)_t$ is a time-inhomogeneous Markov chain with state space \mathcal{S}_{ms} . The probability generating function of $\mathbf{N}_t \mid \mathbf{N}_{t-1} = \mathbf{n}$ is given by*

$$G_{\mathbf{N}_t \mid \mathbf{N}_{t-1} = \mathbf{n}}^P(\mathbf{z}) = (\mathbf{z}^T \mathbf{p}_{1 \rightarrow}(\mathbf{x}_t))^{n_1} \times \cdots \times (\mathbf{z}^T \mathbf{p}_{J \rightarrow}(\mathbf{x}_t))^{n_J},$$

where $\mathbf{z} \in \mathbb{R}^J$ and T denotes the matrix transpose.

Let μ be the invariant measure of $(C_{i,t})_t$. $\mathbf{N}_t \mid \mathbf{N}_0 = \mathbf{n}$ tends to a multinomial distribution $\mathcal{M}_J(N, \mu)$ and the invariant measure of $(\mathbf{N}_t)_t$ is the vector with all probabilities of that multinomial distribution $\mathcal{M}_J(N, \mu)$.

Remark 1. *Note that $\mathbf{N}_t \mid \mathbf{N}_{t-1} = \mathbf{n}$ is a sum of multinomial random variables, and not a multinomial random variable (unless the rows of $P_{\rightarrow}(\mathbf{x}_t)$ are identical). The probability mass function of $\mathbf{N}_t \mid \mathbf{N}_{t-1} = \mathbf{n}$ is given in Appendix A.*

Next, we establish the conditional distribution of a particular portfolio size $N_{j,t} = m_j \mid \mathbf{N}_{t-1} = \mathbf{n}$ of Insurer j . It is difficult to derive other general properties of the distribution of a sum of multinomial or binomial variables with different probabilities $p_{i \rightarrow j}$, except when the size parameters n_j are reasonably large, in which case the normal approximation is appropriate.

Proposition 4. *The distribution of $N_{j,t} = m_j \mid \mathbf{N}_{t-1} = \mathbf{n}$ has probability generating function*

$$G_{N_{j,t} \mid \mathbf{N}_{t-1} = \mathbf{n}}^P(z) = \prod_{k=1}^J (1 - p_{k \rightarrow j}(\mathbf{x}_t) + p_{k \rightarrow j}(\mathbf{x}_t)z)^{n_k}. \quad (12)$$

That is, $N_{j,t} = m_j \mid \mathbf{N}_{t-1} = \mathbf{n}$ is a sum of binomial random variables.

Remark 2. *The probability mass function of $N_{j,t} = m_j \mid \mathbf{N}_{t-1} = \mathbf{n}$ is given in Appendix A. In particular, $E(N_{j,t} \mid \mathbf{N}_{t-1} = \mathbf{n}) = n_j \times p_{j \rightarrow j}(\mathbf{x}_t) + \sum_{l \neq j} n_l \times p_{l \rightarrow j}(\mathbf{x}_t)$.*

3.2 Properties of aggregate loss distributions

We now focus on the loss distribution of a given insurer.

Proposition 5. *Under Assumptions A5-A7, the moment generating function of the aggregate claim amount per insurer $S_{j,t}$ at period t is given by*

$$G_{S_{j,t}}^M(z) = G_{N_{j,t}}^P \left(G_{\widetilde{M}_{j,t}}^P (G_Z^M(z)) \right),$$

where G^P stands for the **probability** generating function and G^M for the **moment** generating function. In particular, the insurer's aggregate claim amount $S_{j,t}(\mathbf{x}_t)$ given that $N_{j,t} = n_{j,t}$ is a compound distribution of the same kind as the individual loss amount $Y_{i,t}$

$$S_{j,t}(\mathbf{x}_t) = \sum_{l=1}^{\widetilde{M}_{j,t}} Z_{i,l},$$

where $(Z_{i,l})_{i,l}$ are i.i.d. claim amounts and $\widetilde{M}_{j,t}$ is given in Table 2.

In particular, the expectation is $E(S_{j,t}) = E(N_{i,t})E(M_{i,t})E(Z)$. A general formula for higher order moments is given in Appendix A.2.

Distribution	$M_{i,t}$	$\widetilde{M}_{j,t}$
Poisson	$\mathcal{P}(\lambda)$	$\mathcal{P}(\lambda n_{j,t})$
binomial	$\mathcal{B}(r, q)$	$\mathcal{B}(n_{j,t}r, q)$
negative binomial	$\mathcal{NB}(r, q)$	$\mathcal{NB}(n_{j,t}r, q)$

Table 2: Correspondence for compound distributions

4 Properties for a one-price market

Insurance price regulation has important consequences on competition, capital and thus on insurers' solvability (Klein et al. (2002)). Using a convergence measure, we investigate the long-run market share distribution among insurers benefiting from different consumer demand patterns. We first consider a special case for which tractable convergence measures can be derived, namely when there is no competition between insurers or when there is a strict regulation. That is, for all $t > 0$, $\mathbf{x}_t = (x, \dots, x)$, or the regulator sets the price from one year to another, i.e. $\mathbf{x}_1 = (x, \dots, x)$, $\mathbf{x}_2 = (y, \dots, y)$, $\mathbf{x}_3 = (z, \dots, z)$, etc. Since premiums are the same for all insurers, the price sensitivity functions remain the same ($\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j$ and $\tilde{f}_j(x_j, x_l) = \tilde{\mu}_j$ from (2)) and the premium level does not have consequences on our lapse model.

In this section, we therefore omit time dependence on t . $p_{j \rightarrow k}$ then simplifies to

$$p_{j \rightarrow k}(\mathbf{x}) = \begin{cases} \frac{1}{1+(J-1)e^{f_j}} & \text{if } j = k, \\ \frac{e^{f_j}}{1+(J-1)e^{f_j}} & \text{if } j \neq k, \end{cases} = \begin{cases} p_{j \rightarrow j} & \text{if } j = k, \\ p_{j \neq} & \text{if } j \neq k, \end{cases} \quad (13)$$

Note that this expression is only a function of j (and not of k). Since $\sum_k p_{j \rightarrow k} = 1$, we get $p_{j \neq} = (1 - p_{j \rightarrow j}) / (J - 1)$.

The two following results are directly derived from Propositions 3 and 4.

Proposition 6. $\mathbf{N}_t | \mathbf{N}_{t-1} = \mathbf{n}$ has the following transition probabilities

$$P(\mathbf{N}_t = \mathbf{m} | \mathbf{N}_{t-1} = \mathbf{n}) = \sum_{\substack{0 \leq c_{11}, \dots, c_{1J} \leq N, \\ \text{s.t. } \sum_l c_{1l} = n_1}} \dots \sum_{\substack{0 \leq c_{J1}, \dots, c_{JJ} \leq N, \\ \text{s.t. } \sum_l c_{Jl} = n_J}} \prod_{\substack{j=1, \\ \sum_k c_{kj} = m_j}}^J \frac{n_j!}{c_{j1}! \dots c_{jJ}!} (p_{j \rightarrow j})^{c_{jj}} (p_{j \neq})^{n_j - c_{jj}}, \quad (14)$$

where $p_{j \rightarrow j}$ and $p_{j \neq}$ are given in (13).

Proposition 7. Assuming identical lapse probabilities $p_{j \rightarrow j} = p_{=}$ and $p_{j \neq} = p_{\neq}$, $N_{j,t} | \mathbf{N}_{t-1} = \mathbf{n}$ is a sum of two binomially distributed random variables $\mathcal{B}(n_j, p_{j \rightarrow j})$ and $\mathcal{B}(n - n_j, \frac{1 - p_{j \rightarrow j}}{J - 1})$. In particular, given $\mathbf{N}_{t-1} = \mathbf{n}$, the probability generating function is

$$P(N_{j,t} = m_j | \mathbf{N}_{t-1} = \mathbf{n}) = \sum_{k=0^+}^{n_j^-} \binom{n_j}{k} (p_{=})^k (1 - p_{=})^{n_j - k} \binom{n - n_j}{m_j - k} (p_{\neq})^{m_j - k} (1 - p_{\neq})^{n - n_j - m_j + k}, \quad (15)$$

where $0^+ = \max(0, (m_j - (n - n_j)))$ and $n_j^- = \min(n_j, m_j)$.

In this particular setting, a tractable expression of the invariant measure is available:

Theorem 1. The choice $(C_{i,t})_t$ of Customer i at time t is a time-homogeneous Markov chain when $\mathbf{x}_t = \mathbf{x}$. In particular, $P_{\rightarrow}^{(t)} = (P_{\rightarrow}(\mathbf{x}))^t$. There exists a unique invariant measure μ for $(C_{i,t})_t$ given by

$$\mu = \left(\frac{c_1^\Pi}{c_1^\Pi + \dots + c_J^\Pi}, \dots, \frac{c_J^\Pi}{c_1^\Pi + \dots + c_J^\Pi} \right) \text{ with } c_i^\Pi = \prod_{j=1, j \neq i}^J p_{j \neq i}. \quad (16)$$

If in addition the choice probabilities $p_{j \rightarrow k}$ are identical for all insurers, then $\mu = (1/J, \dots, 1/J)$.

The portfolio size vector $(\mathbf{N}_t)_t$ at time t is a time-homogeneous Markov chain with state space \mathcal{S}_{ms} for which the invariant measure is the vector with all probabilities of that multinomial distribution $\mathcal{M}_J(N, \mu)$.

Remark 3. Similarly to Proposition 1, the choice vector $(C_{1,t}, \dots, C_{N,t})_t$ is a time-homogeneous Markov chain. The invariant measure is obtained by applying N times the Kronecker product of μ defined in Equation (16). If, in addition, the choice probability $p_{j \rightarrow k}$ are identical across insurers, then $\mu^{\otimes N} = (1/J^N, \dots, 1/J^N)$.

Remark 4. Similarly to Proposition 2, for all $t \in \mathbb{N}^*$, $\bar{\mathbf{C}}_{j,t} \sim \mathcal{M}_J(n_{j,0}, \tilde{\mathbf{p}}_j)$ given $N_{j,0} = n_{j,0}$ where $\tilde{\mathbf{p}}_j$ is the j th row of the matrix $(P_{\rightarrow})^t$. In particular, $\bar{\mathbf{C}}_{j,t} \sim \mathcal{M}_J(n_{j,t-1}, \mathbf{p}_{j \rightarrow}(\mathbf{x}))$ given $N_{j,t-1} = n_{j,t-1}$.

Remark 5. Theorem 1 provides an invariant measure for the lapse model without any solvency consideration. If Insurer k is bankrupted at time t , $k \notin J_{t+1}, J_{t+2}, \dots$, then invariant measures can be derived by putting $p_{k \rightarrow k} = 0$ and $p_{k \neq} = 1/(J-1)$.

In other words, for a constant price vector \mathbf{x} and a large t , the distribution of \mathbf{N}_t is independent of \mathbf{N}_0 , converges in distribution to $\mathcal{M}_J(N, \mu)$ and for $\mathbf{n} \in \mathcal{S}_{ms}$, the probability mass function is

$$P(\mathbf{N}_t = \mathbf{n}) = \frac{N!}{n_1! \dots n_J!} \mu_1^{n_1} \dots \mu_J^{n_J}, \quad (17)$$

with μ defined in (16).

Remark 6. The last equation is in line with the so-called product form solution of a closed Markovian network of queues with batch services. In fact, Theorem 2 of Henderson et al. (1990) show a similar form to (4), where the invariance condition $\mu^T = \mu^T P_{\rightarrow}$ is called the balance equation.

In the special case of identical choice probabilities $p_{j \rightarrow k}$ leading to $\mu_j = 1/J$, we simply obtain

$$P(\mathbf{N}_t = \mathbf{n}) = \frac{N!}{n_1! \dots n_J!} \left(\frac{1}{J} \right)^N, \quad E(N_{i,t}) = \frac{N}{J}, \quad \text{Var}(N_{i,t}) = \frac{N(J-1)}{J^2}.$$

In the current setting, a tractable expression of the survival function of insurer loss $S_{j,t}$ is given in the following proposition.

Proposition 8. Consider the invariant measure μ given in (16). The survival function of the aggregate claim amount is given by

$$P(S_{j,t} > s) = \sum_{0 \leq m \leq N} \binom{N}{m} (\mu_j)^m (1 - \mu_j)^{N-m} \cdot \sum_{0 \leq k} P(\tilde{M}_{j,t} = k | N_{j,t} = m) P\left(\sum_{l=1}^k Z_l > s\right),$$

where the distribution of the total claim number $\tilde{M}_{j,t}$ is given in Table 2.

Remark 7. The expression of the survival function of the aggregate claim amount can be further expanded in some classic cases:

- if Z follows a gamma distribution $\mathcal{G}(\mu_1, \sigma_1)$, then $\sum_{l=1}^k Z_l$ follows a gamma distribution $\mathcal{G}(k\mu_1, \sigma_1)$,
- if Z follows an inverse Gaussian distribution $\mathcal{IG}(\mu_1, \sigma_1)$, then $\sum_{l=1}^k Z_l$ follows an inverse Gaussian distribution $\mathcal{IG}(k\mu_1, k^2\sigma_1)$.

For other distributions, say a lognormal distribution, a large number of approximations exist, see Asmussen et al. (2016) and the references therein.

5 Properties when one insurer deviates from the one-price policy

We consider the case of a deviation from a regulated price or a market-accepted level by one insurer, yet the other competitors remain at the same level. That is, we study $\mathbf{x}_t = (x, \rho x, \dots, \rho x)$ with $\rho > 0$ a fixed parameter. Proofs are postponed in Appendix C.

Again, the dependence on time t is omitted. In that case using (2), we have three possible exponents, see Table 3.

	price ratio	price difference	notation
$l \in \{1, \dots, J\}$	$\bar{f}_1(x_1, x_l) = \mu_1 + \frac{\alpha_1}{\rho}$	$\tilde{f}_1(x_1, x_l) = \mu_1 + \alpha_1(1 - \rho)x$	f_1
$l \neq 1$	$\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j$	$\tilde{f}_j(x_j, x_l) = \mu_j$	f_j
$l = 1$	$\bar{f}_j(x_j, x_1) = \mu_j + \alpha_j \rho$	$\tilde{f}_j(x_j, x_1) = \mu_j + \alpha_j(\rho - 1)x$	$f_{j,\rho}$

Table 3: Price sensitivity functions

Therefore, $p_{j \rightarrow k}$ is only a function of j and not of k , and simplifies to

$$p_{1 \rightarrow j}(\mathbf{x}) = \begin{cases} \frac{1}{1+(J-1)e^{f_1}} & \text{if } j = 1, \\ \frac{e^{f_1}}{1+(J-1)e^{f_1}} & \text{if } j \neq 1, \end{cases} = \begin{cases} p_{1 \rightarrow 1} & \text{if } j = 1, \\ p_{1 \neq} & \text{if } j \neq 1, \end{cases} \quad (18)$$

$$p_{j \rightarrow l}(\mathbf{x}) = \begin{cases} \frac{1}{1+(J-2)e^{f_j} + e^{f_{j,\rho}}} & \text{if } j = l, \\ \frac{e^{f_j}}{1+(J-2)e^{f_j} + e^{f_{j,\rho}}} & \text{if } j \neq l \neq 1, \\ \frac{e^{f_{j,\rho}}}{1+(J-2)e^{f_j} + e^{f_{j,\rho}}} & \text{if } j \neq 1, l = 1, \end{cases} = \begin{cases} p_{j \rightarrow j} & \text{if } j = k, \\ p_{j \neq} & \text{if } j \neq l \neq 1, \\ p_{j \rightarrow 1} & \text{if } j \neq 1, l = 1, \end{cases} \quad (19)$$

Now we are able to derive the invariant measure for the case $\mathbf{x}_t = (x, \rho x, \dots, \rho x)$:

Theorem 2. *The choice $(C_{i,t})_t$ of Customer i at time t is a time-homogeneous Markov chain. There exists a unique invariant measure μ for $(C_{i,t})_t$ given by*

$$\begin{aligned} \mu_1 &= \frac{d_{-1}^{\Pi} - \sum_{j=2}^J d_{-1,-j}^{\Pi} p_{j \neq}}{d_{-1}^{\Pi} + \sum_{j=2}^J d_{-1,-j}^{\Pi} (p_{1 \neq} - p_{j \neq})}, \\ \mu_j &= \frac{d_{-1,-j}^{\Pi} p_{1 \neq}}{d_{-1}^{\Pi} + \sum_{j=2}^J d_{-1,-j}^{\Pi} (p_{1 \neq} - p_{j \neq})}, \quad j = 2, \dots, J. \end{aligned} \quad (20)$$

with $d_l = (J - 1)p_{l \neq} + p_{l \rightarrow 1}$ and

$$d_{-1,-j}^{\Pi} = \prod_{l=2, l \neq j}^J d_l, \quad d_{-1}^{\Pi} = \prod_{l=2}^J d_l.$$

The portfolio sizes $(\mathbf{N}_t)_t$ at time t is a time-homogeneous Markov chain with state space \mathcal{S}_{ms} for which the invariant measure is the vector with all probabilities of that multinomial distribution $\mathcal{M}_J(N, \mu)$.

Remark 8. *In the special case of identical insurers, the invariant measure becomes $\mu_1 = \frac{p_{2 \rightarrow 1}}{p_{2 \rightarrow 1} + (J-1)p_{1 \neq}}, \mu_j = \frac{p_{1 \neq}}{p_{2 \rightarrow 1} + (J-1)p_{1 \neq}}$ for $j = 2, \dots, J$.*

Remark 9. *Again, $(\mathbf{N}_t)_t$ converges in distribution to a multinomial distribution and an analogous expression to Equation (17) is available with μ from (20).*

Remark 10. *Note that the case where $\mathbf{x}_t = (x_1, \dots, x_J)$ without assuming that some insurers propose the same price is very complex and should be solved numerically. We cannot use the simplification of transition probability $p_{j \rightarrow k}$ in this case, yet the transition matrix P_{\rightarrow} will be a circulant matrix.*

Remark 11. When $\mathbf{x} = (x, \rho x, \dots, \rho x)$, we can also have an explicit expression of the asymptotic distribution of $N_{j,t}$ which will lead to an expression of the survival function of the aggregate loss similar to Proposition 8.

Let us analyze the case $\rho > 1$, i.e. Insurer 1 is the cheapest insurer. From Table 3, we deduce that $f_j < f_{j,\rho}$. Therefore, we can order the transition probabilities

$$e^{f_j} < e^{f_{j,\rho}} \Rightarrow \forall l, j \neq 1, p_{j \neq l} = p_{j \rightarrow l} < p_{j \rightarrow 1}.$$

In order to easily compare transition probabilities, we further assume that insurers are identical, with lapse parameters $\mu_j = \mu_1$ and $\alpha_j = \alpha_1$. So $f_1 < f_2 < f_{2,\rho}$ yields

$$\begin{cases} e^{f_{2,\rho}} > e^{f_1} \\ e^{f_1 + f_{2,\rho}} > e^{f_1 + f_2} \end{cases} \Rightarrow \frac{e^{f_{2,\rho}}}{1 + (J-2)e^{f_2} + e^{f_{2,\rho}}} > \frac{e^{f_1}}{1 + (J-1)e^{f_1}}.$$

Since the invariant measure simplifies, we have $\mu_1 > \mu_j$ for $j = 2, \dots, J$.

We now study the stochastic ordering of the empirical average loss of insurers, see (Shaked & Shanthikumar 2007, Chapter 3) for details on the convex order \leq_{cx} .

Proposition 9. *If Insurer 1 is the cheapest insurer with $\rho > 1$, then the loss average by policy of Insurer 1, at any time t , is stochastically smaller than the one of the others in the following sense:*

$$\frac{1}{N_{1,t}(\tilde{\mathbf{x}})} \sum_{i=1}^{N_{1,t}(\tilde{\mathbf{x}})} Y_i \leq_{cx} \frac{1}{N_{k,t}(\tilde{\mathbf{x}})} \sum_{i=1}^{N_{k,t}(\tilde{\mathbf{x}})} Y_i, \quad \forall k \neq 1,$$

where $\tilde{\mathbf{x}} = (x, \rho x, \dots, \rho x)$.

Remark 12. *In particular, the convex order implies that*

$$\text{Var} \left(\frac{1}{N_{1,t}(\tilde{\mathbf{x}})} \sum_{i=1}^{N_{1,t}(\tilde{\mathbf{x}})} Y_i \right) \leq \text{Var} \left(\frac{1}{N_{k,t}(\tilde{\mathbf{x}})} \sum_{i=1}^{N_{k,t}(\tilde{\mathbf{x}})} Y_i \right).$$

Proposition 9 is in line with results from Wang et al. (2010), where they find in a dynamic model that larger firms experience less premium variation than smaller firms. Indeed, since premium equilibrium is highly correlated to loss history, we can reasonably expect that, in the long run, the firm proposing the lowest premium benefits from the largest market share. The loss average is therefore less volatile, allowing insurers to be less constrained by solvency regulation and potential loss shocks and leading, in a dynamic pattern, to a more stable premium. We analyze below the effect on the underwriting result by policy using the increasing convex order \leq_{icx} , see (Shaked & Shanthikumar 2007, Chapter 4).

Proposition 10. *If in addition to Insurer 1 being the cheapest, for all $k \neq 1$, $x_1(1 - e_1) \leq x_k(1 - e_k)$, then the underwriting result by policy is ordered $UW_{1,t} \leq_{icx} UW_{k,t}$, where $UW_{j,t}$ is the random variable*

$$UW_{j,t} = x_j(1 - e_j) - \frac{1}{N_{j,t}(\mathbf{x})} \sum_{i=1}^{N_{j,t}(\mathbf{x})} Y_i,$$

where $\tilde{\mathbf{x}} = (x, \rho x, \dots, \rho x)$.

Remark 13. *If Insurer 1 is the most expensive insurer with $0 < \rho < 1$, we obtain the reverse situation for Propositions 9 and 10: Insurer 1's empirical average loss and underwriting loss per policy will be stochastically larger.*

6 Properties of the repeated game

In this section, we investigate some long-run properties of the repeated game after ensuring the existence and uniqueness of the premium equilibrium. All proofs are postponed to Appendix D.

6.1 Asymptotic properties of the repeated game

We first show that the premium equilibrium of the repeated game admits a unique premium equilibrium.

Proposition 11. *If there are at least two non-bankrupted insurers at time t , the repeated game with objective function (10) and solvency constraint (11) admits a unique (Nash) premium equilibrium \mathbf{x}^* . If in addition no constraint function is active, the premium equilibrium solves a linear system of equations $M_1 M_2 \mathbf{x}^* = v$ with*

$$M_1 = \begin{pmatrix} 2b_1/w_1 & -a_1 & \dots & & \\ -a_2 & 2b_2/w_2 & -a_2 & \dots & \\ & & \ddots & & \\ & & & \dots & -a_J & 2b_J/w_J \end{pmatrix}, M_2 = \begin{pmatrix} w_1 & 0 & \dots \\ & \ddots & \\ \dots & 0 & w_J \end{pmatrix}, v = \begin{pmatrix} b_1 c_1 \\ \vdots \\ b_J c_J \end{pmatrix},$$

where $w_j = 1$, $b_j = \beta_j(J - 1)$ for arithmetic market proxy (8), and $w_j = n_j$, $b_j = \beta_j(N - n_j)$ for weighted market proxy (9).

It is important to note that the linear system deduced in the previous proposition is **not equivalent** to the original Nash equilibrium since it relies on the strong assumptions of non-active constraint functions, see Appendix D. Nevertheless, it helps to understand the effect of some parameters on the premium equilibrium in that particular case. Similarly to Proposition 2.2 of Dutang et al. (2013), Table 4 presents the sensitivity analysis of parameters on the premium equilibrium.

z	$x_j^* = \underline{x}, \bar{x}$	x_j^* solv. constr.	no act. constraint	
	$z \mapsto x_j^*(z)$	$z \mapsto x_j^*(z)$	$z \mapsto x_j^*(z)$ if (8)	$z \mapsto x_j^*(z)$ if (9)
π_j	—	\nearrow	\nearrow	\nearrow
β_j	—	—	\searrow	\searrow
n_j	—	unknown	—	\nearrow if $\pi_j > 2$, \searrow oth.
K_j	—	\searrow	—	—
$\sigma(Y)$	—	\nearrow	—	—

Table 4: Sensitivity analysis of premium x_j^*

Let us define the ruin and the leadership probabilities of Insurer j .

Definition 2 (Ruin probability). *The ruin probability of Insurer j at time t is*

$$\psi_j(k_j, n_j) = P(K_{j,t} < 0 \cup N_{j,t} \leq 0.01\%N \mid K_{j,0} = k, N_{j,0} = n_j).$$

Definition 3 (Leadership probability). *The leadership probability of Insurer j at time t is*

$$\phi_j(k_j, n_j) = P(\forall k \neq j, N_{j,t} x_{j,t}^* \geq N_{k,t} x_{k,t}^* \mid K_{j,0} = k, N_{j,0} = n_j).$$

The following proposition gives the asymptotic value of the leadership probability under the special case of the one-price vector of Section 4.

Proposition 12. *If $\mathbf{x}_t = \mathbf{x}$, then the leadership probability for large t is independent of initial conditions and given by*

$$\phi_j(k_j, n_j) = \sum_{\mathbf{n} \in \mathcal{S}_{ms}} \frac{N!}{n_1! \dots n_J!} \mu_1^{n_1} \dots \mu_J^{n_J} \mathbb{1}_{\forall k \neq j, n_j x_j \geq n_k x_k},$$

where μ is either (16) or (20).

Finally, one can show that the repeated game will necessarily end at some (large) time t .

Proposition 13. *For the repeated insurance game, the probability that there are at least two non-bankrupt insurers at time t decreases geometrically as t increases, that is*

$$P(\text{Card}(J_t) > 1) < (1 - \bar{\xi})^t.$$

where $\bar{\xi}$ is a positive probability minoring the one-period ruin probability of survived insurers.

6.2 Numerical illustrations of a three-insurer market

We now provide some numerical computations of the ruin and leadership probabilities for the game repeated over 10 periods.

We consider a game where three insurers fight for a market of $N = 500$ policyholders. Insurer 1 is the leader with more than one half of the market, whereas Insurer 2 is the challenger with 30% of policyholders and Insurer 3 the outsider with the last 10% of policyholders with objective and constraint functions defined in Equations (10) and (11) with parameters given in Table 9 in Appendix D. Policyholders face the loss model of Table 8 and follow a new multinomial logit model with parameters given in Table 7 in Appendix D.

In Table 5, with market proxy (8), we observe that the leadership of Insurer 1 is greater than those of his competitors, yet there is a decreasing trend over time. The longer the repeated game is, the lower the probability is that Insurer 1 is the leader. In Table 5, the ruin probability is also computed: having a lower initial solvency ratio, Insurer 3 is more exposed to the insolvency risk (13% against 1% for Insurer 2 and 0% for Insurer 1).

Period	Ruin probabilities			Leadership probabilities		
	Insurer 1	Insurer 2	Insurer 3	Insurer 1	Insurer 2	Insurer 3
1	0.00	0.01	0.05	1.00	0.00	0.00
2	0.00	0.01	0.09	0.94	0.00	0.00
3	0.00	0.01	0.10	0.86	0.02	0.03
4	0.00	0.01	0.12	0.73	0.12	0.06
5	0.00	0.01	0.12	0.59	0.16	0.14
6	0.00	0.01	0.13	0.52	0.18	0.16
7	0.00	0.01	0.13	0.45	0.23	0.18
8	0.00	0.01	0.13	0.42	0.25	0.18
9	0.00	0.01	0.13	0.38	0.25	0.21
10	0.00	0.01	0.13	0.44	0.24	0.15

Table 5: Empirical probabilities of ruin and leadership over 100 runs, 3-player game with market proxy (8)

Looking at Table 6, we observe similarly to Table 5 that Insurer 1 is the leader and is never ruined. In contrast, Insurer 3 gets bankrupted more often as time evolves (from 2% to 14%) and has little chance to become leader (around 10%). Finally Insurer 2 remains the best competitor of Insurer 1 with a significant probability of leadership.

Period	Ruin probabilities			Leadership probabilities		
	Insurer 1	Insurer 2	Insurer 3	Insurer 1	Insurer 2	Insurer 3
1	0.00	0.01	0.02	1.00	0.00	0.00
2	0.00	0.01	0.04	0.98	0.00	0.00
3	0.00	0.01	0.05	0.88	0.04	0.03
4	0.00	0.01	0.04	0.65	0.15	0.09
5	0.00	0.01	0.05	0.62	0.17	0.10
6	0.00	0.01	0.07	0.60	0.18	0.11
7	0.00	0.01	0.09	0.57	0.21	0.10
8	0.00	0.01	0.10	0.59	0.20	0.09
9	0.00	0.01	0.11	0.55	0.20	0.13
10	0.00	0.01	0.14	0.56	0.23	0.08

Table 6: Empirical probabilities of ruin and leadership over 100 runs, 3-player game with weighted market proxy (9)

7 Conclusion

The present paper proposes a repeated game based on the static game of [Dutang et al. \(2013\)](#). We first analyze a situation where insurers do not compete on price and derive the asymptotic distribution of insurer portfolio size and insurer aggregate loss. This situation could appear when the insurance market is strictly regulated or where there is a collusion between insurers. Secondly, we show that deviating from this situation by offering a lower premium leads to significant advantages in terms of market shares, leadership probability and loss volatility. We also provide some insights on the effect of competition in the repeated game.

The current work can be extended in many directions: considering a dividend rule when the solvency ratio exceeds a certain threshold, adding a cost of capital in the objective function, and taking into account adverse selection and moral hazard of policyholders, to name a few examples. Also, it will be more realistic to take into account the asymmetry of information. For example, assuming the insurer only observes its own claim history and receives a market signal through the level of the market premium (price).

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A Proofs of Section 3

We prove below Propositions [1](#), [2](#), [3](#), [5](#) with standard probabilistic arguments. We recall that bolded notation are reserved for vectors, G^P and G^M stand for the probability and the moment generating functions, \otimes the Kronecker product.

A.1 Properties of the lapse model

Proof of Prop. 1. By A3, $(C_{i,t})_t$ is a Markov chain with transition matrix $P_{\rightarrow}(\mathbf{x}_t)$ defined as

$$P_{\rightarrow}(\mathbf{x}_t) = \begin{pmatrix} p_{1 \rightarrow 1}(\mathbf{x}_t) & \dots & p_{1 \rightarrow J}(\mathbf{x}_t) \\ \dots & \dots & \dots \\ p_{J \rightarrow 1}(\mathbf{x}_t) & \dots & p_{J \rightarrow J}(\mathbf{x}_t) \end{pmatrix}.$$

In fact, P_{\rightarrow} is a matrix function. By Proposition [14](#), the transition matrix has no null terms. It is immediate that the transition from $C_{i,0}$ to $C_{i,t}$ is the multiplication of the t matrices $P_{\rightarrow}(\mathbf{x}_1), \dots, P_{\rightarrow}(\mathbf{x}_t)$. On the finite state space $\{1, \dots, J\}$, the Markov chain is both irreducible and aperiodic using Proposition [14](#) in Appendix [B.1](#). By (Norris 1997, p. 41), the process $(C_{1,t}, C_{2,t})_t$ is still a Markov chain on the space $E^2 = \{1, \dots, J\}^2$ with transition matrix $P_{\rightarrow}(\mathbf{x}_t) \otimes P_{\rightarrow}(\mathbf{x}_t)$. Iterating $N - 1$ more times leads to the result. \square

Proof of Prop. 2. Let \mathcal{N}_j be the set of customers of Insurer j at time 0. That is $\forall i \in \mathcal{N}_j, C_{i,0} = j$. As $(C_{i,t})_t$ is a Markov chain, the transition from Insurer j to Insurer k is governed by the j th row \tilde{p}_j of the matrix $P_{\rightarrow}^{(t)} = P_{\rightarrow}(\mathbf{x}_1) \times \dots \times P_{\rightarrow}(\mathbf{x}_t)$, $\tilde{p}_j = (P_{\rightarrow,j,1}^{(t)}, \dots, P_{\rightarrow,j,J}^{(t)})$. Thus $\forall i \in \mathcal{N}_j, C_{i,t} \mid C_{i,0} = j \sim \mathcal{M}_J(1, \tilde{\mathbf{p}}_j)$. By A1 and A2, those policyholders of Insurer j will choose insurers according to a multinomial distribution $\overline{\mathbf{C}}_{j,t} \sim \mathcal{M}_J(n_{j,t-1}, \tilde{\mathbf{p}}_j)$ given $N_{j,0} = n_{j,0}$. From period $t - 1$ to period t , the transition matrix simplifies to $P_{\rightarrow}(\mathbf{x}_t)$ and $\tilde{\mathbf{p}}_j = \mathbf{p}_{j \rightarrow}(\mathbf{x}_t)$. \square

Proof of Prop. 3. The portfolio sizes vector is the sum of choice vectors $\mathbf{N}_t = \overline{\mathbf{C}}_{1,t} + \dots + \overline{\mathbf{C}}_{J,t}$. By A2, Proposition [2](#) and given $\mathbf{N}_{t-1} = \mathbf{n}$, $(\overline{\mathbf{C}}_{j,t})_j$ are independent multinomial vectors with parameters $\mathcal{M}_J(n_j, \mathbf{p}_{j \rightarrow}(\mathbf{x}_t))$ for $j = 1, \dots, J$. Therefore, \mathbf{N}_t (obtained by summing over j) has a known distribution given $\mathbf{N}_{t-1} = \mathbf{n}$. Since $(\mathbf{N}_t)_t$ is a discrete-time process taking values in \mathcal{S}_{ms} , $(\mathbf{N}_t)_t$ is a Markov chain. By recurrence, the number of elements of \mathcal{S}_{ms} is $\text{Card}(\mathcal{S}_{ms}) = \binom{N+J-1}{N}$. The transition matrix of size $\text{Card}(\mathcal{S}_{ms}) \times \text{Card}(\mathcal{S}_{ms})$ has a complex expression $\overline{P}_t = (P(\mathbf{N}_t = \mathbf{m} \mid \mathbf{N}_{t-1} = \mathbf{n}))_{\mathbf{n}, \mathbf{m}}$ where $\mathbf{n}, \mathbf{m} \in \mathcal{S}_{ms}$ and

$$\begin{aligned} & P(\mathbf{N}_t = \mathbf{m} \mid \mathbf{N}_{t-1} = \mathbf{n}) \\ = & \sum_{\substack{0 \leq c_{11}, \dots, c_{1J} \leq N, \\ \text{s.t. } \sum_l c_{1l} = n_1}} \dots \sum_{\substack{0 \leq c_{J1}, \dots, c_{JJ} \leq N, \\ \text{s.t. } \sum_l c_{Jl} = n_J}} \prod_{j=1}^J \frac{n_j!}{c_{j1}! \dots c_{jJ}!} (p_{j \rightarrow 1}(\mathbf{x}_t))^{c_{j1}} \dots (p_{j \rightarrow i}(\mathbf{x}_t))^{c_{jJ}}. \end{aligned}$$

The probability $P(\bar{\mathbf{C}}_{j,t} = \mathbf{c}_j | N_{j,t-1} = n_j)$ depends on the price vector \mathbf{x}_t , and therefore is time dependent. By A2, the probability generating function of $\mathbf{N}_t | \mathbf{N}_{t-1} = \mathbf{n}$ is in constrast simpler

$$G_{\mathbf{N}_t | \mathbf{N}_{t-1} = \mathbf{n}}^P(\mathbf{z}) = G_{\mathbf{C}_{1,t}}^P(\mathbf{z}) \times \cdots \times G_{\mathbf{C}_{J,t}}^P(\mathbf{z}) = (\mathbf{z}^T p_{1 \rightarrow}(\mathbf{x}_t))^{n_1} \times \cdots \times (\mathbf{z}^T p_{J \rightarrow}(\mathbf{x}_t))^{n_J},$$

where $z \in \mathbb{R}^J$, T denotes the matrix transpose and $G^P(\cdot)$ denotes the probability generating function. Using Proposition 2, we have $\bar{\mathbf{C}}_{j,t} | \mathbf{N}_{j,0} = \mathbf{n}$ follows a multinomial distribution with parameters $\mathcal{M}_J(n_j, \tilde{\mathbf{p}}_j)$. By similar arguments, $G_{\mathbf{N}_t | \mathbf{N}_0 = \mathbf{n}}^P(\mathbf{z}) = (P_{\rightarrow}^{(t)} \times \mathbf{z})^{\mathbf{n}}$. If μ is the invariant measure of $(C_{i,t})_t$, then

$$P_{\rightarrow}^{(t)} \xrightarrow{t \rightarrow +\infty} \begin{pmatrix} \mu \\ \dots \\ \mu \end{pmatrix} \Rightarrow P_{\rightarrow}^{(t)} \times \mathbf{z} = \begin{pmatrix} \mu^T \mathbf{z} \\ \dots \\ \mu^T \mathbf{z} \end{pmatrix} \Rightarrow G_{\mathbf{N}_t | \mathbf{N}_{t-1} = \mathbf{n}}^P(\mathbf{z}) = (\mu^T \mathbf{z})^{\sum_i n_i}.$$

In other words, the probability generating function of $\mathbf{N}_t | \mathbf{N}_0 = \mathbf{n}$ is the p.g.f. of a multinomial distribution. Since we obtain a limiting distribution for the Markov chain $(\mathbf{N}_t)_t$ is also its invariant measure, see, e.g., (Norris 1997, p. 33). \square

Proof of Prop. 4. By A4, the probability generating function of the sum constituting $N_{j,t}$ is the product of generating function of each binomially distributed random variables

$$G_{N_{j,t} | \mathbf{N}_{t-1} = \mathbf{n}}^P(\mathbf{z}) = \prod_{k=1}^J (1 - p_{k \rightarrow j}(\mathbf{x}_t) + p_{k \rightarrow j}(\mathbf{x}_t)z)^{n_k}.$$

Differentiating with respect to z , we get

$$\begin{aligned} & G_{N_{j,t} | \mathbf{N}_{t-1} = \mathbf{n}}^P{}'(z) \\ &= \sum_{k=1}^J n_k p_{k \rightarrow j}(\mathbf{x}_t) (1 - p_{k \rightarrow j}(\mathbf{x}_t) + p_{k \rightarrow j}(\mathbf{x}_t)z)^{n_k - 1} \prod_{l \neq k} (1 - p_{l \rightarrow j}(\mathbf{x}_t) + p_{l \rightarrow j}(\mathbf{x}_t)z)^{n_l}. \end{aligned}$$

Taking $z = 1$ leads to the result. The mass probability function of the portfolio size $N_{j,t}$ is given by

$$P(N_{j,t} = m_j | \mathbf{N}_{t-1} = \mathbf{n}) = \sum_{\substack{0 \leq c_1, \dots, c_J \leq n \\ \text{s.t. } \sum_k c_k = m_j}} \prod_{l=1}^J \binom{n_l}{c_j} (p_{l \rightarrow j}(\mathbf{x}_t))^{c_j} (1 - p_{l \rightarrow j}(\mathbf{x}_t))^{n_l - c_j}.$$

\square

A.2 Properties of the loss model

Proof of Prop. 5. Using assumptions A5, A6, A7, the moment generating function of $S_{j,t}$ given that $N_{j,t} = n_j$ using (5) is

$$\forall u, G_{S_{j,t} | N_{j,t} = n_j}^M(u) = E(e^{uS_{j,t}} | N_{j,t} = n_j) = E\left(\prod_{i=1}^{n_j} e^{uY_{i,t}}\right) = \left(G_{Y_{i,t}}^M(u)\right)^{n_j}.$$

Since $Y_{i,t}$ is a compound distribution by (4), we get $G_{Y_{i,t}}^M(u) = G_{M_{i,t}}^P(G_Z^M(u))$. As the claim frequency belongs to the $(a, b, 0)$ class, the resulting distribution for total claim number can be easily derived, see Table 1. Again using assumptions A5, A6, A7, we have

$$\forall u, G_{S_{j,t}}^M(u) = E(E(e^{uS_{j,t}} | N_{j,t})) = E\left(\left(G_{Y_{i,t}}^M(u)\right)^{N_{j,t}}\right) = G_{N_{i,t}}^P(G_{M_{i,t}}^P(G_Z^M(u))).$$

Using Lemma 1, with $f = G_{N_{i,t}}^P$, $g = G_{M_{i,t}}^P$ and $h = G_Z^M$, we set $x = 0$ in order to compute moments so that $h^{(j)}(0) = (G_Z^M)^{(j)}(0) = E(Z^j)$, $h(0) = 1$, $g^{(l)}(1) = E(M_{i,t} \dots (M_{i,t} - l + 1))$, $f^m(1) = E(N_{i,t} \dots (N_{i,t} - m + 1))$. So we have

$$\frac{d^n G_{S_{j,t}}^M(0)}{dx^n} = \sum_{m_j \in \mathbb{N}} \frac{n! E(N_{i,t} \dots (N_{i,t} - m_j + 1))}{m_1! \dots m_n!} \prod_{j=1}^n \left(\sum_{l_j \in \mathbb{N}} \frac{E(M_{i,t} \dots (M_{i,t} - l_j + 1))}{l_1! \dots l_j!} \right)^{m_j} \prod_{k=1}^j \left(\frac{E(Z^k)}{k!} \right)^{l_k m_j},$$

with $m_j = m_1 + \dots + m_n$ and $l_j = l_1 + \dots + l_j$ where the multiple summation is a finite set of integers, see Lemma 1. In particular, $n = 1$ leads to $m_1 = 1$, $l_1 = 1$. Hence $E(S_{j,t}) = \frac{E(N_{i,t})}{1!} \left(\frac{E(M_{i,t})}{1!} \right)^1 \left(\frac{E(Z)}{1!} \right)^1 = E(N_{i,t})E(M_{i,t})E(Z)$. \square

Using recursively the Faà di Bruno formula, we obtain the following lemma.

Lemma 1. *Assuming f , g and h are n th-time differentiable, we have*

$$\frac{d^n f(g(h(x)))}{dx^n} = \sum_{m_j \in \mathbb{N}} \frac{n! f^{(m_1 + \dots + m_n)}(g(h(x)))}{m_1! m_2! \dots m_n!} \prod_{j=1}^n \left(\sum_{l_j \in \mathbb{N}} \frac{g^{(l_1 + \dots + l_j)}(h(x))}{l_1! \dots l_j!} \right)^{m_j} \prod_{k=1}^j \left(\frac{h^{(k)}(x)}{k!} \right)^{l_k m_j},$$

where the multiple summation is over integers

$$\left\{ m_j \in \mathbb{N}, \sum_{j=1}^n j m_j = n \right\}, \left\{ l_k \in \mathbb{N}, \sum_{k=1}^j k l_k = j \right\}.$$

B Proofs of Section 4

B.1 Properties of the transition probability

Proposition 14. *Transition probability $p_{l \rightarrow j}(\mathbf{x})$ is a strictly decreasing function of x_j given \mathbf{x}_{-j} and verifies $0 < p_{l \rightarrow j}(\mathbf{x}) < 1$.*

Proof. The expression of $p_{j \rightarrow k}$ can be rewritten as

$$p_{j \rightarrow k}(\mathbf{x}) = p_{j \rightarrow j}(\mathbf{x}) \left(\delta_{jk} + (1 - \delta_{jk}) e^{f_j(x_j, x_k)} \right), p_{j \rightarrow j}(\mathbf{x}) = \frac{1}{1 + \sum_{l \neq j} e^{f_j(x_j, x_l)}},$$

with δ_{ij} denoting the Kronecker delta where the summation is over $l \in \{1, \dots, J\} \setminus \{j\}$ and f_j is the price function. The price function f_j goes from $(t, u) \in \mathbb{R}^2 \mapsto f_j(t, u) \in \mathbb{R}$. Partial derivatives are denoted by $\frac{\partial f_j(t, u)}{\partial t} = f'_{j1}(t, u)$ and $\frac{\partial f_j(t, u)}{\partial u} = f'_{j2}(t, u)$. Derivatives of higher order use the same notation principle. The $p_{j \rightarrow k}(\mathbf{x})$ function has the good property to be infinitely differentiable. Since we have

$$\frac{\partial}{\partial x_i} \sum_{l \neq j} e^{f_j(x_j, x_l)} = \delta_{ji} \sum_{l \neq j} f'_{j1}(x_j, x_l) e^{f_j(x_j, x_l)} + (1 - \delta_{ji}) f'_{j2}(x_j, x_l) e^{f_j(x_j, x_l)},$$

we deduce

$$\frac{\partial p_{j \rightarrow j}(x)}{\partial x_i} = - \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l(x) \right) p_{i \rightarrow j}(x) \delta_{ij} - f'_{j2}(x_j, x_l) p_{j \rightarrow i}(x) p_{j \rightarrow j}(x) (1 - \delta_{ij}).$$

Furthermore,

$$\frac{\partial}{\partial x_i} \left(\delta_{jk} + (1 - \delta_{jk}) e^{f_j(x_j, x_k)} \right) = (1 - \delta_{jk}) \left(\delta_{ik} f'_{j2}(x_j, x_k) e^{f_j(x_j, x_k)} + \delta_{ij} f'_{j1}(x_j, x_k) e^{f_j(x_j, x_k)} \right).$$

Hence, we get

$$\begin{aligned} \frac{\partial p_{j \rightarrow k}(x)}{\partial x_i} &= -\delta_{ij} \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) p_{j \rightarrow l}(x) \right) p_{j \rightarrow k}(x) - (1 - \delta_{ij}) f'_{j2}(x_j, x_i) p_{j \rightarrow i}(x) p_{j \rightarrow k}(x) \\ &\quad + (1 - \delta_{jk}) \left[\delta_{ij} f'_{j1}(x_j, x_k) p_{j \rightarrow k}(x) + \delta_{ik} f'_{j2}(x_j, x_k) p_{j \rightarrow k}(x) \right]. \end{aligned}$$

Let ϕ_l be the family function $x_j \mapsto p_{l \rightarrow j}(\mathbf{x})$ for $l = 1, \dots, J$. ϕ_j has the following derivative

$$\phi'_j(x_j) = - \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) p_{j \rightarrow l}(\mathbf{x}) \right) p_{j \rightarrow j}(\mathbf{x}).$$

Since for the two considered price function, we have $f'_{j1}(x_j, x_l) = \alpha_j/x_l > 0$ and $\tilde{f}'_{j1}(x_j, x_l) = \tilde{\alpha}_j > 0$, then the function ϕ_j is strictly decreasing. For $l \neq j$, the function ϕ_l has the following derivative $\phi'_l(x_j) = f'_{j2}(x_l, x_j) p_{l \rightarrow j}(\mathbf{x})(1 - p_{l \rightarrow j}(\mathbf{x}))$. Again, for the two considered price function, we have $f'_{j2}(x_j, x_l) = -\alpha_j x_j/x_l^2 < 0$ and $\tilde{f}'_{j2}(x_j, x_l) = -\tilde{\alpha}_j < 0$. So, the function ϕ_l is strictly decreasing.

Futhermore, the function ϕ_l decreases from 1 to 0 such that $\phi_l(x_j) \rightarrow 1$ (resp. $\phi_l(x_j) \rightarrow 0$) when $x_j \rightarrow -\infty$ (resp. $x_j \rightarrow \infty$). When $i \neq j$, functions $x_i \mapsto p_{i \rightarrow j}(x)$ are also strictly increasing. Let $\underline{\mathbf{x}}^j = (\underline{x}, \dots, \underline{x}, \bar{x}, \underline{x}, \dots, \underline{x})$ and $\bar{\mathbf{x}}^j = (\bar{x}, \dots, \bar{x}, \underline{x}, \bar{x}, \dots, \bar{x})$. We have

$$\forall \mathbf{x} \in [\underline{x}, \bar{x}]^J, 0 < p_{i \rightarrow j}(\bar{\mathbf{x}}^j) < p_{i \rightarrow j}(\mathbf{x}) < p_{i \rightarrow j}(\underline{\mathbf{x}}^j) < 1.$$

□

B.2 Properties of a constant regulated price vector

Proof of Prop. 6. Using Proposition 3 and when $\mathbf{x} = (x, \dots, x)$ using (13), we have

$$P_{\rightarrow} = \begin{pmatrix} p_{1 \rightarrow 1} & \frac{1-p_{1 \rightarrow 1}}{J-1} & \dots & \dots & \frac{1-p_{1 \rightarrow 1}}{J-1} \\ & \ddots & & & \\ \dots & \frac{1-p_{j \rightarrow j}}{J-1} & p_{j \rightarrow j} & \frac{1-p_{j \rightarrow j}}{J-1} & \dots \\ & & & \ddots & \\ \frac{1-p_{J \rightarrow J}}{J-1} & \dots & \dots & \frac{1-p_{J \rightarrow J}}{J-1} & p_{J \rightarrow J} \end{pmatrix} \in \mathbb{R}^{J \times J}. \quad (21)$$

We deduce that the probabilities appearing in the proof of Proposition 3 simplify to

$$\frac{n_j!}{c_{j1}! \dots c_{jJ}!} (p_{j \rightarrow j})^{c_{jj}} (p_{j \neq})^{c_{j1}} \dots (p_{j \neq})^{c_{jJ}} = \frac{n_j!}{c_{j1}! \dots c_{jJ}!} (p_{j \rightarrow j})^{c_{jj}} (p_{j \neq})^{n_j - c_{jj}}.$$

We get the desired result by summing over appropriate indexes. □

Proof of Prop. 7. For a constant price vector, identical players ($p_{j \rightarrow j} = p_{=}$ and $p_{j \neq} = p_{\neq}$) and Proposition 4, the probability generating function is

$$\begin{aligned} G_{N_{j,t} | \mathbf{N}_{t-1} = \mathbf{n}}^P(z) &= (1 - p_{j \rightarrow j} + p_{j \rightarrow j} z)^{n_j} \prod_{l \neq j} (1 - p_{\neq} + p_{\neq} z)^{n_l} \\ &= (1 - p_{j \rightarrow j} + p_{j \rightarrow j} z)^{n_j} (1 - p_{\neq} + p_{\neq} z)^{N - n_j}. \end{aligned}$$

$N_{j,t} | \mathbf{N}_{t-1} = \mathbf{n}$ is a sum of two binomially distributed random variables $\mathcal{B}(n_j, p_{j \rightarrow j})$ and $\mathcal{B}(n - n_j, p_{\neq})$. □

Proof of Th. 1. When $\mathbf{x}_t = \mathbf{x}$, $(C_{i,t})_t$ is a Markov chain with a transition matrix $P_{\rightarrow}(\mathbf{x}_t) = P_{\rightarrow}$. When $\mathbf{x} = (x, \dots, x)$ (13) leads to (21). Since the number of state is finite (J) and the Markov chain is irreducible by Prop. 14, there exists a unique invariant measure μ , see e.g. Norris (1997).

Let us consider the general matrix $M \in \mathbb{R}^{J \times J}$ with general term $M_{i,j} = a_i(1 - \delta_{ij}) + (1 - (J-1)a_i)\delta_{ij}$. Note that the rows of M equal 1 and M has only two different terms by row. The reversibility conditions for a measure μ are

$$\begin{cases} \mu_1 M_{1,2} = \mu_2 M_{2,1} \\ \dots \\ \mu_1 M_{1,J} = \mu_J M_{J,1} \end{cases} \Leftrightarrow \begin{cases} \mu_1 a_1 / a_2 = \mu_2 \\ \dots \\ \mu_1 a_1 / a_J = \mu_J. \end{cases}$$

Let $a_{-i}^{\Pi} = \prod_{j=1, j \neq i}^J a_j$. Using $\mu_1 + \dots + \mu_J = 1$, we get by multiplying both sides by a_{-i}^{Π}

$$\mu_1 + \sum_{i>2} \mu_i \frac{a_1}{a_i} = 1 \Leftrightarrow \mu_1 = \frac{a_{-1}^{\Pi}}{\sum_{i=1}^J a_{-i}^{\Pi}} \Leftrightarrow \mu = \left(\frac{a_{-1}^{\Pi}}{\sum_{i=1}^J a_{-i}^{\Pi}}, \dots, \frac{a_{-J}^{\Pi}}{\sum_{i=1}^J a_{-i}^{\Pi}} \right).$$

The measure μ above is in detailed balance with M and also an invariant measure for M . Setting $a_j = p_{j \neq}$ leads to the desired result. In the special case where $p_{j \rightarrow j}$ are identical across insurers, $p_{j \neq} = p_{\neq}$ is constant. Hence for all $j = 1, \dots, J$

$$a_{-i}^{\Pi} = \prod_{j \neq i} p_{j \neq} = (p_{\neq})^{J-1} \Rightarrow \mu_i = \frac{(p_{\neq})^{J-1}}{\sum_{l=1}^J (p_{\neq})^{J-1}} = \frac{(p_{\neq})^{J-1}}{J \times (p_{\neq})^{J-1}} = \frac{1}{J}.$$

□

Proof of Prop. 8. Using (4) and Proposition 1, for large t , we have

$$P(\mathbf{N}_t = \mathbf{n}) = \frac{N!}{(c_1^{\Pi} + \dots + c_J^{\Pi})^N} \prod_{j=1}^J \frac{(c_j^{\Pi})^{n_j}}{n_j!}.$$

The asymptotic marginal distribution is binomial $\mathcal{B}(N, \mu_j)$ with $P(N_{j,t} = m) = \binom{N}{m} (\mu_j)^m (1 - \mu_j)^{N-m}$. This leads to the desired result for the aggregate claim amount. □

C Proofs of Section 5

Proof of Th. 2. Since $\sum_k p_{j \rightarrow k} = 1$, we get $p_{j \neq} = (1 - p_{j \rightarrow j} - p_{j \rightarrow 1}) / (J - 2)$, and $p_{1 \neq} = \frac{1 - p_{1 \rightarrow 1}}{J - 1}$. We use the following notation

$$P_{\rightarrow} = \begin{pmatrix} p_{1 \rightarrow 1} & p_{1 \neq} & \dots & & \\ p_{2 \rightarrow 1} & p_{2 \rightarrow 2} & p_{2 \neq} & \dots & \\ p_{3 \rightarrow 1} & p_{3 \neq} & p_{3 \rightarrow 3} & p_{3 \neq} & \dots \\ & & & \ddots & \\ p_{J \rightarrow 1} & p_{J \neq} & \dots & p_{J \neq} & p_{J \rightarrow J} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & \dots & & \\ b_2 & a_2 & c_2 & \dots & \\ b_3 & c_3 & a_3 & c_3 & \dots \\ & & & \ddots & \\ b_J & c_J & \dots & c_J & a_J \end{pmatrix}. \quad (22)$$

A first series of equation for the invariant measure is obtained from $\mu' = \mu' P_{\rightarrow}$. Ignoring the first equation and subtracting the second equation from all others, we get

$$\begin{cases} \mu_2 = b_1 \mu_1 + a_2 \mu_2 + \dots + c_J \mu_J \\ \mu_3 - \mu_2 = (c_2 - a_2) \mu_2 + (a_3 - c_3) \mu_3 \\ \vdots \\ \mu_J - \mu_2 = (c_2 - a_2) \mu_2 + (a_J - c_J) \mu_J \end{cases} \Leftrightarrow \begin{cases} -b_1 \mu_1 = -\mu_2 + a_2 \mu_2 + c_3 \mu_3 + \dots + c_J \mu_J \\ \mu_3(1 - a_3 + c_3) = (c_2 - a_2 + 1) \mu_2 \\ \vdots \\ \mu_J(1 - a_J + c_J) = (c_2 - a_2 + 1) \mu_2. \end{cases}$$

The $J - 2$ equations give $\mu_j = \mu_2 \frac{c_2 - a_2 + 1}{c_j - a_j + 1}, j > 2$. Recalling that any row of M sums up to 1, $a_i + b_i + c_i(J - 2) = 1$ for $i \neq 1$, we have

$$\forall j = 3, \dots, J, a_j = 1 - c_j(J - 2) - b_j \Rightarrow c_j - a_j + 1 = (J - 1)c_j + b_j =: d_j.$$

For $j = 3, \dots, J$, $\mu_j = \mu_2 \frac{d_2}{d_j}$. The first equation for μ_1 becomes

$$b_1 \mu_1 = \mu_2 - a_2 \mu_2 - \sum_{j=3}^J c_j \mu_2 \frac{d_2}{d_j} \Leftrightarrow \mu_1 = \mu_2 \frac{b_2}{b_1} + \mu_2 \frac{d_2}{b_1} \sum_{j=3}^J \left(\frac{c_2}{d_2} - \frac{c_j}{d_j} \right).$$

Using the condition $\sum_i \mu_i = 1$ yields to $\mu_2 = \left(1 + \frac{b_2}{b_1} + \frac{d_2}{b_1} \sum_{j=3}^J \left(\frac{c_2}{d_2} - \frac{c_j}{d_j} + \frac{b_1}{d_j} \right) \right)^{-1}$. Reintroducing the product notation $d_{-1,-j}^{\Pi} = \prod_{l \neq 1,j} d_l$ yields the following reformulation

$$\mu_j = \frac{d_{-1,-j}^{\Pi} b_1}{d_{-1}^{\Pi} + \sum_{j=2}^J d_{-1,-j}^{\Pi} (b_1 - c_j)}, j > 2, \mu_1 = \frac{d_{-1}^{\Pi} + \sum_{j=2}^J d_{-1,-j}^{\Pi} (-c_j)}{d_{-1}^{\Pi} + \sum_{j=2}^J d_{-1,-j}^{\Pi} (b_1 - c_j)}.$$

Let us go back to the original transition matrix (22) with $a_1 = p_{1 \rightarrow 1}$, $b_1 = p_{1 \neq} = \frac{1 - p_{1 \rightarrow 1}}{J - 1}$, $\forall j > 1$, $a_j = p_{j \rightarrow j}$, $b_j = p_{j \rightarrow 1}$, $c_j = p_{j \neq}$. With $d_j = (J - 1)c_j + b_j = (J - 1)p_{j \neq} + p_{j \rightarrow 1}$, we obtain the desired result. In the special case of identical insurers, we have $\forall j \neq 1, p_{j \neq} = p_{2 \neq}$ and $p_{j \rightarrow 1} = p_{2 \rightarrow 1}$. With $d_j = d_2 \Rightarrow d_{-1,-j}^{\Pi} = d_2^{J-2} \Rightarrow d_{-1}^{\Pi} = d_2^{J-1}$, we obtain the desired result. \square

Proof of Prop. 9. Consider Insurer j is the cheapest, i.e. $x_j < x_k$ for all $k \neq j$. $p_{k \rightarrow j}(x) > p_{k \rightarrow l}(x)$ for $l \neq j$ given the initial portfolio sizes n_j 's are constant, since the change probability $p_{k \rightarrow j}$ (for $k \neq j$) is a decreasing function (see Appendix B.1). Using the stochastic order (\leq_{st}), the convex order (\leq_{cx}), the majorization order (\leq_m), see (Shaked & Shanthikumar 2007, resp. Chap. 1, Chap. 3, p. 2), we can show a stochastic order of the portfolio size by applying the convolution property of the stochastic order J times: $N_k(\mathbf{x}) \leq_{st} N_j(\mathbf{x}), \forall k \neq j$.

Let us consider the empirical loss average of an insurer with portfolio size n $\bar{A}(n) = \frac{1}{n} \sum_{i=1}^n Y_i$, where Y_i denotes the total claim amount for Policy i . For $n < \tilde{n}$, we define two policy numbers $\mathbf{a}_n, \mathbf{b}_{\tilde{n}} \in \mathbb{R}^{\tilde{n}}$ as

$$\mathbf{b}_{\tilde{n}} = \left(\frac{1}{\tilde{n}}, \dots, \frac{1}{\tilde{n}} \right) \text{ and } \mathbf{a}_n = \left(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{\text{size } n}, \underbrace{0, \dots, 0}_{\text{size } \tilde{n} - n} \right).$$

Since $\mathbf{b}_{\tilde{n}} \leq_m \mathbf{a}_n$ and $(Y_i)_i$'s are i.i.d. random variables, we have

$$\sum_i b_{\tilde{n},i} Y_i \leq_{cx} \sum_i a_{n,i} Y_i \Leftrightarrow \sum_{i=1}^{\tilde{n}} \frac{1}{\tilde{n}} Y_i \leq_{cx} \sum_{i=1}^n \frac{1}{n} Y_i \Leftrightarrow \bar{A}(\tilde{n}) \leq_{cx} \bar{A}(n).$$

Using Theorem 3.A.23 of Shaked & Shanthikumar (2007), except that for all ϕ convex, $E(\phi(\bar{A}(n)))$ is a decreasing function (rather an increasing function) of n and $N_k(\mathbf{x}) \leq_{st} N_j(\mathbf{x})$, we can show $\bar{A}(N_j(\mathbf{x})) \leq_{cx} \bar{A}(N_k(\mathbf{x}))$. \square

Proof of Prop. 10. Using Prop. 9, we have $\sum_{i=1}^{\tilde{n}} \frac{1}{\tilde{n}} Y_i \leq_{cx} \sum_{i=1}^n \frac{1}{n} Y_i$. For all increasing convex functions ϕ , the function $x \mapsto \phi(x + a)$ is still increasing and convex. Thus for all random variables X, Y such that $X \leq_{icx} Y$ and real numbers $a, b, a \leq b$, we have

$$E(\phi(X + a)) \leq E(\phi(X + b)) \leq E(\phi(Y + b)) \Leftrightarrow a + X \leq_{icx} b + Y.$$

Consider Insurer j is the cheapest and $x_j(1 - e_j) \leq x_k(1 - e_k)$ and using the fact that $X \leq_{cx} Y$ is equivalent to $-X \leq_{cx} -Y$, we have $uw_j(\mathbf{x}, \tilde{n}) \leq_{icx} uw_k(\mathbf{x}, n), \forall k \neq j$. Using Theorem 3.A.23 of Shaked & Shanthikumar (2007), except that for all ϕ convex, $E(\phi(uw_j(\mathbf{x}, n)))$ is a decreasing function of n and $N_k(\mathbf{x}) \leq_{st} N_j(\mathbf{x})$, we can show $UW_j = uw_j(\mathbf{x}, N_j(\mathbf{x})) \leq_{icx} uw_k(\mathbf{x}, N_k(\mathbf{x})) = UW_k$. \square

D Proofs of Section 6

Proof of Prop. 11. Assuming $\#J_t \geq 2$, since the strategy set is $[\underline{x}, \bar{x}]^{J_t}$, it guarantees the market proxy $m_j = m_j(\mathbf{x})$ or $m_j = m_j(\mathbf{x}, \mathbf{n})$ to be positive. Given $\mathbf{x}_{-j,t}$, the function $x_{j,t} \mapsto O_j(\mathbf{x}_t)$ is a quadratic (hence concave) function. Given that the constraint functions (11) are linear, by Theorem 1 of Rosen (1965), the existence of a premium equilibrium at time t is guaranteed. The proof of uniqueness is exactly the same as in Dutang et al. (2013). Omitting the time index t , consider a generic objective function

$$O_j(\mathbf{x}) = \left(a_j - b_j \frac{x_j}{m_j(\mathbf{x})} \right) (x_j - c_j), \text{ with } m_j(\mathbf{x}) = \sum_{i \neq j} w_i x_i,$$

with known weights $w_i > 0$ and positive constant $a_j, b_j, c_j > 0$. In the case of no active constraint functions. Similarly to Dutang et al. (2013), if \mathbf{x}^* is a Nash equilibrium, \mathbf{x}^* must verify

$$\forall j, \nabla_{x_j} O_j(\mathbf{x}^*) = 0 \Leftrightarrow M_1 M_2 \mathbf{x} = v,$$

with

$$M_1 = \begin{pmatrix} 2b_1/w_1 & -a_1 & \dots & & \\ -a_2 & 2b_2/w_2 & -a_2 & \dots & \\ & & \ddots & & \\ & & & -a_J & 2b_J/w_J \end{pmatrix}, M_2 = \begin{pmatrix} w_1 & 0 & \dots \\ & \ddots & \\ \dots & 0 & w_J \end{pmatrix}, v = \begin{pmatrix} b_1 c_1 \\ \vdots \\ b_J c_J \end{pmatrix}.$$

NB: When the market proxy is the arithmetic mean (8), we set $w_j = 1$ and $b_j = \beta_j(J-1)$. When the market proxy is the weighted mean (9), we set $w_j = n_j$ and $b_j = \beta_j(N-n_j)$. \square

Remark 14. *Getting a linear system for the premium equilibrium, we are looking for a necessary and sufficient condition. The linear system for the premium equilibrium has a solution when the determinant of M is non null. Since $\det(M) = \det(M_1) \det(M_2)$, M_2 being diagonal yields a positive determinant $\det(M_2) = \prod_i w_i > 0$ for the weights considered. So $\det(M_0) = 0$ is equivalent to $\det(M_1) = 0$.*

Using Lemma 2, we have with $a_j = 1 + \beta_j$ and $b_j = \beta_j$

$$\det(M_1) = \frac{2b_1}{w_1} \prod_{k \neq 1} (a_k + 2b_k/w_k) - \sum_{j=2}^J a_j \prod_{k \neq j} (a_k + 2b_k/w_k).$$

Using $\tilde{\beta}_j = 1 + \beta_j + 2\beta_j w_{-j}^\Sigma / w_j$, $w_{-j}^\Sigma = \sum_{k \neq j} w_k$, $\tilde{\beta}_{-j}^\Pi = \prod_{k \neq j} \tilde{\beta}_k$,

$$\det(M) = 0 \Leftrightarrow 2\beta_1 w_{-1}^\Sigma \tilde{\beta}_1 = w_1 \sum_{j=2}^J (1 + \beta_j) \tilde{\beta}_j.$$

There are many solutions to this equation, but there is a unique solution when $w_j = w$ is constant and β_j 's are all identical. Say $\beta_j = \beta$ leading to $\tilde{\beta}_j = 1 + \beta + 2\beta(J-1)$. Hence $\det(M) = 0$ yields to $\beta = 1$

A sufficient condition for the linear system to have a solution is M to be diagonally dominant. That is $\forall j = 1, \dots, J$,

$$|2b_j| > \sum_{k \neq j} | -a_j w_k | \Leftrightarrow 2b_j > a_j \sum_{k \neq j} w_k \Leftrightarrow 2b_j > a_j w_{-j}^\Sigma$$

In the case of objective function (10), we choose $a_j = 1 + \beta_j$, $b_j = \beta_j w_{-j}^\Sigma$. So the sufficient condition is $2\beta_j w_{-j}^\Sigma > (1 + \beta_j) w_{-j}^\Sigma \Leftrightarrow \beta_j > 1$. In that case, one can check that the determinant $\det(M_1)$ is strictly positive. This fact was also seen in Dutang et al. (2013).

Lemma 2. Consider the following multi-diagonal matrix for $n \geq 2$

$$M_n = \begin{pmatrix} u_1 & v_1 & \dots & & \\ v_2 & u_2 & v_2 & \dots & \\ & & \ddots & & \\ \dots & v_{n-1} & u_{n-1} & v_{n-1} & \\ & \dots & v_n & u_n & \end{pmatrix}.$$

With $w_1 = u_1$ and $w_j = v_j, \forall j = 2, \dots, n$, the determinant is given by $\det(M_n) = (-1)^{n+1} \sum_{j=1}^n w_j \prod_{k \neq j} (v_k - u_k)$.

Proof. If $\det(M_n) = (-1)^{n+1} \sum_{j=1}^n w_j \prod_{k \neq j} (v_k - u_k)$, then

$$\begin{aligned} \det(M_{n+1}) &= (u_{n+1} - v_{n+1}) \det(M_n) + (-1)^{n+1} (v_1 - u_1) \dots (v_n - u_n) v_{n+1} \\ &= (u_{n+1} - v_{n+1}) (-1)^{n+1} \sum_{j=1}^n w_j \prod_{k \neq j} (v_k - u_k) + (-1)^{n+1} (v_1 - u_1) \dots (v_n - u_n) v_{n+1} \\ &= (-1)^{n+2} \sum_{j=1}^{n+1} w_j \prod_{k \neq j} (v_k - u_k). \end{aligned}$$

For $n = 2$, $\det(M_2) = u_1 u_2 - v_1 v_2 = (u_2 - v_2) u_1 + (-1)^{2+1} (v_1 - u_1) v_2$. □

Proof of Prop. 13. Using Proposition 14, we have

$$0 < p_{i \rightarrow j}(\underline{\mathbf{x}}^{j-}) < p_{i \rightarrow j}(\mathbf{x}) < p_{i \rightarrow j}(\bar{\mathbf{x}}_j^-) < 1,$$

for all $\mathbf{x} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]^J$. Taking supremum and infimum on player j , we get $0 < \underline{p}_j = \inf_j p_{i \rightarrow j}(\underline{\mathbf{x}}^{j-})$ and $\sup_j p_{i \rightarrow j}(\bar{\mathbf{x}}_j^-) = \bar{p}_j < 1$. Using the proof of Th. 4 in Appendix A, we have

$$\begin{aligned} &P(N_{j,t}(\mathbf{x}) = m_j | N_{j,t-1} > 0, \text{Card}(J_{t-1}) > 1) \\ &= \sum_{\substack{\tilde{m}_1, \dots, \tilde{m}_{J_{t-1}} \geq 0 \\ \text{s.t. } \sum_l \tilde{m}_l = m_j}} \prod_{l \in J_{t-1}} \binom{n_{l,t-1}}{\tilde{m}_l} p_{i \rightarrow j}(\mathbf{x})^{\tilde{m}_l} (1 - p_{i \rightarrow j}(\mathbf{x}))^{n_{l,t-1} - \tilde{m}_l} \\ &> \sum_{\substack{\tilde{m}_1, \dots, \tilde{m}_{J_{t-1}} \geq 0 \\ \text{s.t. } \sum_l \tilde{m}_l = m_j}} \prod_{l \in J_{t-1}} \binom{n_{l,t-1}}{\tilde{m}_l} \underline{p}_l^{\tilde{m}_l} (1 - \bar{p}_l)^{n_{l,t-1} - \tilde{m}_l} = \xi > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &P(\text{Card}(J_t) = 0 | \text{Card}(J_{t-1}) > 1) \\ &\geq P\left(\forall j \in J_{t-1}, N_{j,t}(\mathbf{x}) > 0, K_{j,t-1} + N_{j,t}(\mathbf{x}) x_{j,t}^* (1 - e_j) < \sum_{i=1}^{N_{j,t}(\mathbf{x})} Y_i | \text{Card}(J_{t-1}) > 1\right) \\ &\geq \sum_{m_j=1}^N P_t(N_{j,t}(\mathbf{x}) = m_j | \text{Card}(J_{t-1}) > 1) P\left(K_{j,t-1} + m_j x_{j,t}^* (1 - e_j) < \sum_{i=1}^{m_j} Y_i\right) \\ &> \sum_{m_j=1}^N \xi P\left(K_{j,t-1} + m_j x_{j,t}^* (1 - e_j) < \sum_{i=1}^{m_j} Y_i\right) = \bar{\xi} > 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} &P(\text{Card}(J_t) > 1 | \text{Card}(J_{t-1}) > 1) \\ &= 1 - P(\text{Card}(J_t) = 0 | \text{Card}(J_{t-1}) > 1) - P(\text{Card}(J_t) = 1 | \text{Card}(J_{t-1}) > 1) \\ &\leq 1 - P(\text{Card}(J_t) = 0 | \text{Card}(J_{t-1}) > 1) \\ &< 1 - \bar{\xi} < 1. \end{aligned}$$

By successive conditioning, we get

$$P(\text{Card}(J_t) > 1) = P(\text{Card}(J_0) > 1) \prod_{s=1}^t P(\text{Card}(J_s) > 1 | \text{Card}(J_{s-1}) > 1) < (1 - \bar{\xi})^t.$$

So, the probability $P(\text{Card}(J_t) > 1)$ decreases geometrically as t increases. \square

Proof of Prop. 12. In a regulated market ($x_{j,t}^* = x_j$), using Theorems 1 and 2, the asymptotic distribution of \mathbf{N}_t is a multinomial distribution. Hence the leadership probability is independent of initial condition $\phi_j(k_j, n_j) = \phi_j$. Using the probability mass function leads to the desired result. \square

Policy of	Lapse MLN PD model					
	f_j	μ_j	α_j	$1 - p_{j \rightarrow j}((1, 1, 1))$	$1 - p_{j \rightarrow j}((1, 1.05, 1.05))$	
Insurer 1	\hat{f}_j	-2.890372	7.401976	10%		15%
Insurer 2	\hat{f}_j	-2.508437	5.844477	14%		19%
Insurer 3	\tilde{f}_j	-2.209495	4.928581	18%		23%

Table 7: Parameters of the lapse model

	Loss frequency		Loss severity		
	dist.	λ	dist.	μ_1	σ_1
Policyholder (Expec. & Var.)	Poisson	0.1	lognormal	1.931616	0.8613578
	$(E(M) = Var(M) = 0.1)$		$(E(Z) = 10, Var(Z) = 110)$		

Table 8: Parameters of the loss model PLN

	Objective and constraint param.						
	β_j	ω_j	e_j	$\pi_{j,0}$	$n_{j,0}$	$K_{j,0}$	solv. ratio
Insurer 1	3	1	30%	1.28494	300	317.49	133%.
Insurer 2	3	1	30%	1.28612	150	224.49	133%
Insurer 3	4	1	40%	1.48396	50	97.21	100%
	\underline{x}	\bar{x}	N	$E(Y)$	$Var(Y)$	k_{95}	d
Market	1	3	500	1	4.5826	3	3

Table 9: Parameters of insurers