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Xuelong Hu, Philippe Castagliola, Anan Tang, Jianlan Zhong. Guaranteed conditional performance of the median run length based EWMA  $X^-$  chart with unknown process parameters. Communications in Statistics - Simulation and Computation, 2021, 50 (12), pp.4280-4299. 10.1080/03610918.2019.1642485 . hal-03448266

**HAL Id: hal-03448266**

**<https://hal.science/hal-03448266>**

Submitted on 25 Nov 2021

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# Guaranteed conditional performance of the median run length based EWMA $\bar{X}$ chart with unknown process parameters

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## Abstract

Exponentially weighted moving average (EWMA) type charts are very popular and efficient in monitoring various kind of statistics. Much researches have been done on the EWMA  $\bar{X}$  chart with known process parameters. But, in practice, the process parameters used to set the control chart limits are often unknown and they need to be estimated from different Phase I samples. Moreover, because the shape of the run length distribution for the EWMA  $\bar{X}$  chart changes with the mean shift, the median

run length ( $MRL$ ) can serve as a good alternative to evaluate the performance of the EWMA  $\bar{X}$  chart. In this article, we will investigate the conditional properties of the EWMA  $\bar{X}$  chart with unknown process parameters based on the  $MRL$  metric. In order to investigate the chart's properties, the average  $MRL$  ( $AMRL$ ) and the standard deviation of  $MRL$  ( $SDMRL$ ) will be used together when the process parameters are unknown. To prevent too many lower in-control  $MRL$  values, the adjusted control limits of the  $MRL$  based EWMA chart are obtained by using a bootstrap type approach and the results show that the adjusted control limits can give a good tradeoff between the in-control and out-of-control  $MRL$  performances.

**Keywords:** EWMA  $\bar{X}$  chart; median run length; estimated parameters; average median run length; standard deviation of median run length.

## 1 Introduction

Control charts in statistical process control (SPC) are usually used to achieve process stability through the detection of the process variability. Among all types of control charts, a traditional one is the Shewhart type chart. This memoryless type chart is easy to implement and it is efficient to detect large changes in a process. But, for small or moderate changes, it is not as efficient compared to some memory type charts, like exponentially weighted moving average (EWMA) or cumulative sum (CUSUM) type charts. The EWMA type charts are popular for their efficiency in detecting small or moderate shifts in a process (Montgomery (2009)).

Since Roberts (1959) first recognized the application of the EWMA approach for monitoring processes, much researches have been conducted. Lucas and Saccucci (1990) evaluated the  $RL$  distribution of the EWMA control chart using a Markov chain method. When the

process observations are correlated, Lu and Reynolds (1999) proposed an EWMA chart to monitor the residuals from the forecast values of the correlated model. Borrer et al. (1999) investigated the properties of the EWMA  $\bar{X}$  chart when observations are non-normal, and they showed that the EWMA chart is a robust alternative when the normality assumption cannot be guaranteed. Knoth (2015) evaluated the quantile performance of EWMA control charts with variance adjusted control limits and with FIR (fast initial response) features. Maravelakis et al. (2017) investigated the Run Rules EWMA chart's properties using integral equations and the results show that the Run Rules feature can improve the performance of the EWMA chart. These researches are all focused on EWMA type charts with known process parameters.

In practice, in the implementation of control charts, the process parameters are usually unknown and they need to be estimated from in-control observations. Generally two phases (Phase I and Phase II) are applied in the implementation of control charts. In Phase I, an in-control reference sample is collected to estimate the process parameters, which are then used to set the control limits of a Phase II control chart. In Phase II, samples collected from the process are prospectively plotted on the control chart for the detection of assignable cause(s). For simplicity, a Phase II control chart's performance is usually investigated based on the assumption of known process parameters. Since different practitioners use different Phase I samples to estimate the process parameters, this causes a variability in these estimated process parameters. Thus, the control chart's performance will vary among different practitioners, causing the so called *between-practitioners variability* effect in the properties of control charts (Saleh et al. (2015b)).

Numerous researches have been conducted on EWMA type charts with unknown process parameters. Jones (2002) developed design procedures for the EWMA  $\bar{X}$  chart with

estimated parameters. Castagliola et al. (2016) investigated the properties of the EWMA median chart with estimated process parameters and they suggested dedicated chart parameters for different amount of Phase I samples. Considering the between practitioners variability, Saleh et al. (2015a) investigated the conditional performance of the EWMA  $\bar{X}$  chart and suggested to use a bootstrap approach to adjust the control chart parameters. For a comprehensive literature review involving parameter estimation, readers may refer to Jensen et al. (2006) and Psarakis et al. (2014).

A Phase II control chart's property is generally measured by the average run length ( $ARL$ ), which is defined as the average number of samples required until the chart gives an out-of-control signal. As pointed in Montgomery (2009), the shape of the run length ( $RL$ ) distribution of control charts changes for different mean shift sizes, i.e. from highly right skewed in the in-control state to nearly symmetric when the shift size in the out-of-control state is large. Therefore, the interpretation based on the  $ARL$  for highly skewed and nearly symmetric  $RL$  distributions are certainly different. For example, the EWMA  $\bar{X}$  chart with appropriate parameters can have an in-control  $ARL$  of 1000, but with a median run length ( $MRL$ ) value equals to 695, which means half of the  $RL$ s are smaller than 695. When there is one standard deviation shift in the process mean, the  $ARL$  is 11.7 and half of the  $RL$ s, i.e.  $MRL$ , are smaller than 10 (Gan (1993)).

Recently, the only use of the  $ARL$  to measure the performance of control charts has been criticized by many researchers, to name a few, see Gan (1993), Gan (1994), Jones et al. (2004), Chakraborti (2007) and Tang et al. (2018). All these researches pointed out the disadvantage of the interpretation based only on the  $ARL$ , because the shape of the  $RL$  distribution changes with the mean shift. On the other hand, the  $MRL$  does not have this interpretation problem. For this advantage, some researchers recommended the usage of the

$MRL$  as an efficiency measure for control charts. Gan (1993, 1994) have investigated the properties of the EWMA and CUSUM  $\bar{X}$  charts based on  $MRL$ . Khoo et al. (2012, 2011) have also investigated the properties of the univariate and multivariate synthetic charts based on the  $MRL$ . For other more researches on the design of  $MRL$  based charts, readers may refer to Teoh et al. (2016a) for variable sample size chart, Tang et al. (2018) for the adaptive EWMA chart, and Lee and Khoo (2017) for the the synthetic  $np$  attribute chart. All the researches based on the  $MRL$ , referenced above, assume that the chart parameters are all known. For the unknown process parameters case, the properties of  $MRL$  based control charts with estimated parameters have also been investigated by some researchers, see Chakraborti (2007), Teoh et al. (2015), Teoh et al. (2016b) and You et al. (2016). These researches only focused on the *marginal*  $MRL$  performance of control charts with estimated parameters. When the process parameters are estimated from different Phase I samples, the  $MRL$  based charts will also have the “between-practitioners variability” problem.

Considering the between-practitioners variability, on one hand, much researches have been conducted on the  $ARL$  based control charts for different type charts, see Saleh et al. (2015b), Saleh et al. (2015a), Faraz et al. (2015), Hu and Castagliola (2017), Keefe et al. (2015), Aly et al. (2015), Aly et al. (2016), Geodhart et al. (2017a) and Geodhart et al. (2017b). These researches recommended to use the average  $ARL$  ( $AARL$ ), combined with the standard deviation of  $ARL$  ( $SDARL$ ) to give a more complete picture of control charts’ properties. On the other hand, for the  $MRL$  based control charts, it is also important to investigate the between-practitioners variability.

Although numerous researches have been done on EWMA type charts, as far as we know, considering the between-practitioners variability, no research has been conducted on the  $MRL$  based EWMA  $\bar{X}$  chart with unknown process parameters. Since the between-

practitioners variability causes a large variation in the in-control and out-of-control performances of control charts, this suggests us that the *MRL* based EWMA  $\bar{X}$  chart also needs to be investigated based on the average (*AMRL*) and the standard deviation (*SDMRL*) of the *MRL*.

The rest of this paper is organized as follows: First, Section 2 presents the run length properties of the EWMA  $\bar{X}$  chart with unknown process parameters using the Markov chain approach. Based on the average, the standard deviation, and several percentiles of the *MRL* distribution, we investigate in Section 3 the *MRL* performance of the EWMA  $\bar{X}$  chart with unknown process parameters. In Section 4, the bootstrap adjustment of the control limits and the corresponding in-control and out-of-control performances of the EWMA  $\bar{X}$  chart are discussed. Finally, some conclusions and recommendations are made in Section 5.

In addition, an online supplement is provided to discuss the following: (i) a brief background of the *MRL* based EWMA  $\bar{X}$  chart; (ii) an example introduced in Montgomery (2009) is given to illustrate the use of the *MRL* based EWMA  $\bar{X}$  chart.

## 2 The run length properties of the EWMA $\bar{X}$ chart with unknown process parameters

### 2.1 Model standardization

Suppose that the subgroups  $\{Y_{i,1}, Y_{i,2}, \dots, Y_{i,n}\}$  of size  $n$ , at sampling time  $i$  are consecutively collected. We assume that there is independence within and between these subgroups and,  $Y$  follows a normal distribution  $Y \sim N(\mu_0 + \delta\sigma_0, \sigma_0)$ , where  $\mu_0$  and  $\sigma_0$  are the in-control

mean and standard deviation, respectively, and  $\delta$  is the magnitude of the standardized mean shift. The process is considered to be in-control when  $\delta = 0$ . Otherwise ( $\delta \neq 0$ ), the process is out-of-control. The EWMA chart statistic  $Z'_i$ , at sampling time  $i$ , is defined as

$$Z'_i = \lambda \bar{Y}_i + (1 - \lambda) Z'_{i-1} \quad (1)$$

where  $\bar{Y}_i = \frac{1}{n} \sum_{j=1}^n Y_{i,j}$  is the sample mean of subgroup  $i$  and  $\lambda$  ( $0 < \lambda \leq 1$ ) is a smoothing constant. The initial value  $Z'_0$  is usually set to be the target value of a process, i.e.  $Z'_0 = \mu_0$ .

When the process parameters  $\mu_0$  and  $\sigma_0$  are known, the lower control limit ( $LCL'$ ) and the upper control limit ( $UCL'$ ) of the EWMA  $\bar{X}$  chart are equal to,

$$LCL' = \mu_0 - K \sqrt{\frac{\lambda}{2 - \lambda}} \frac{\sigma_0}{\sqrt{n}}, \quad (2)$$

$$UCL' = \mu_0 + K \sqrt{\frac{\lambda}{2 - \lambda}} \frac{\sigma_0}{\sqrt{n}}. \quad (3)$$

where  $K$  is a constant chosen to satisfy the desired in-control performance of the control chart. The EWMA  $\bar{X}$  chart signals when the statistic  $Z'_i$  fall outside the control limits in (2) and (3).

In order to investigate the properties of the EWMA  $\bar{X}$  chart, this article adopts the standardized form of the EWMA statistic (see Jones (2002) and Saleh et al. (2015a)):

$$Z_i = \lambda W_i + (1 - \lambda) Z_{i-1} \quad (4)$$

with

$$W_i = \frac{\bar{Y}_i - \mu_0}{\sigma_0 / \sqrt{n}}. \quad (5)$$



and the initial value of the EWMA statistic  $W_0$  is 0.

When the process parameters  $\mu_0$  and  $\sigma_0$  are unknown, they need to be estimated from a Phase I data set, composed of  $i = 1, 2, \dots, m$  subgroups  $\{X_{i,1}, \dots, X_{i,n}\}$  of size  $n$ , where  $X$  is assumed to be a  $N(\mu_0, \sigma_0)$  random variable. A natural estimator  $\hat{\mu}_0$  of  $\mu_0$  is

$$\hat{\mu}_0 = \frac{1}{m} \sum_{i=1}^m \bar{X}_i, \quad (6)$$

and a pooled estimator  $\hat{\sigma}_0$  for  $\sigma_0$  (Jensen et al. (2006)) is,

$$\hat{\sigma}_0 = \sqrt{\frac{1}{m(n-1)} \sum_{i=1}^m \sum_{j=1}^n (X_{i,j} - \bar{X}_i)^2}, \quad (7)$$

where  $\bar{X}_i = \frac{1}{n} \sum_{j=1}^n X_{i,j}$  is the  $i$ th sample mean.

By substituting  $\mu_0$  and  $\sigma_0$  with the corresponding estimators in (6) and (7), the standardized form of the EWMA statistic with estimated parameters is,

$$\widehat{Z}_i = \lambda \widehat{W}_i + (1 - \lambda) \widehat{Z}_{i-1} \quad (8)$$

with

$$\widehat{W}_i = \frac{\bar{Y}_i - \hat{\mu}_0}{\hat{\sigma}_0 / \sqrt{n}} \quad (9)$$

After some manipulations, we can re-rewrite (9) as follows:

$$\widehat{W}_i = (W_i - U) \frac{\sqrt{n}}{V} \quad (10)$$

where  $W_i = \frac{\bar{Y}_i - \mu_0}{\sigma_0 / \sqrt{n}}$ ,  $U = (\hat{\mu}_0 - \mu_0) \frac{\sqrt{n}}{\sigma_0}$  and  $V = \frac{\hat{\sigma}_0}{\sigma_0} \sqrt{n}$ .

Under the standardized case, the control limits of the EWMA  $\bar{X}$  chart become,

$$LCL = -K\sqrt{\frac{\lambda}{2-\lambda}}, \quad (11)$$

$$UCL = +K\sqrt{\frac{\lambda}{2-\lambda}}. \quad (12)$$

It must be noted that these control limits remain the same no matter if the chart parameters  $\mu_0$  and  $\sigma_0$  are known or estimated.

## 2.2 The Markov chain approach

Generally, three approaches can be used to obtain the run length distribution of the EWMA type charts: (i) Integral equations; (ii) Markov chain approach; (iii) Monte-Carlo simulations. Following some researches done on the EWMA type charts (see You et al. (2016), Castagliola et al. (2016) and Saleh et al. (2015a)), this paper mainly focused on the Markov chain approach. Using the Markov chain approach introduced in Brook and Evans (1972), the run length properties of the EWMA  $\bar{X}$  chart can be obtained by dividing the interval  $[LCL, UCL]$  into  $2m + 1$  subintervals  $(H_j - \Delta, H_j + \Delta)$ ,  $j = \{-m, \dots, 0, \dots, m\}$ , each of width  $2\Delta = \frac{UCL-LCL}{2m+1}$ . Let  $H_j = \frac{LCL+UCL}{2} + 2j\Delta$  represents the midpoint of the  $j$ th interval. By dividing the interval  $[LCL, UCL]$  into a large number of subintervals, the statistic  $\hat{Z}_i$  is said to be equal to  $H_j$  at time  $i$ , if  $H_j - \Delta < \hat{Z}_i < H_j + \Delta$ ,  $j = -m, \dots, 0, \dots, m$ . Otherwise, if the statics falls outside the control limits, the Markov chain reached the absorbing state. The transient states are referred to be as in-control states, and the absorbing state is referred to be as the out-of-control state.

The probability  $p_{j,k}$  that the control statistic goes from state  $j$  to state  $k$  in one step is equal to,

$$p_{j,k} = P(\widehat{Z}_i \in (H_k - \Delta, H_k + \Delta) | \widehat{Z}_{i-1} = H_j), \quad (13)$$

Replacing  $\widehat{Z}_i = \lambda \widehat{W}_i + (1 - \lambda) \widehat{Z}_{i-1}$  yields,

$$\begin{aligned} p_{j,k} &= P\left(\frac{H_k - \Delta - (1 - \lambda)H_j}{\lambda} \leq \widehat{W}_i \leq \frac{H_k + \Delta - (1 - \lambda)H_j}{\lambda}\right), \\ &= P\left(\frac{H_k - \Delta - (1 - \lambda)H_j}{\lambda} \frac{V}{\sqrt{n}} + U \leq W_i \leq \frac{H_k + \Delta - (1 - \lambda)H_j}{\lambda} \frac{V}{\sqrt{n}} + U\right), \end{aligned} \quad (14)$$

Using the fact that  $W_i \sim N(\delta\sqrt{n}, 1)$ , we can obtain

$$\begin{aligned} p_{j,k} &= F_N\left(\frac{H_k + \Delta - (1 - \lambda)H_j}{\lambda} \frac{V}{\sqrt{n}} + U - \delta\sqrt{n}\right) \\ &\quad - F_N\left(\frac{H_k - \Delta - (1 - \lambda)H_j}{\lambda} \frac{V}{\sqrt{n}} + U - \delta\sqrt{n}\right), \end{aligned} \quad (15)$$

where  $F_N(\cdot)$  is the c.d.f. (cumulative distribution function) of the standard normal  $N(0, 1)$  distribution.

Let  $\mathbf{Q}$  be the  $(2m + 1, 2m + 1)$  matrix of probabilities  $p_{j,k}$  of moving from one transient

state to another, i.e.

$$\mathbf{Q} = \begin{pmatrix} p_{-m,-m} & \cdots & p_{-m,-1} & p_{-m,0} & p_{-m,1} & \cdots & p_{-m,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-1,-m} & \cdots & p_{-1,-1} & p_{-1,0} & p_{-1,1} & \cdots & p_{-1,m} \\ p_{0,-m} & \cdots & p_{0,-1} & p_{0,0} & p_{0,1} & \cdots & p_{0,m} \\ p_{1,-m} & \cdots & p_{1,-1} & p_{1,0} & p_{1,1} & \cdots & p_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m,-m} & \cdots & p_{m,-1} & p_{m,0} & p_{m,1} & \cdots & p_{m,m} \end{pmatrix}, \quad (16)$$

The run length distribution of the EWMA  $\bar{X}$  chart is determined by the transition probability matrix  $\mathbf{Q}$  and the initial probability vector  $\mathbf{q}$ , which can be represented by a  $(2m+1, 1)$  vector  $\mathbf{q} = (q_{-m}, \dots, q_{-1}, q_0, q_1, \dots, q_m)$ , where

$$q_j = \begin{cases} 0 & \text{if } Z_0 \notin (H_j - \Delta, H_j + \Delta) \\ 1 & \text{if } Z_0 \in (H_j - \Delta, H_j + \Delta) \end{cases} \quad (17)$$

Since the run length ( $RL$ ) of the EWMA  $\bar{X}$  chart is a Discrete PHase-type (DPH) random variable of parameters  $(\mathbf{Q}, \mathbf{q})$  (see Neuts (1981) or Latouche and Ramaswami (1999)), the p.m.f. (probability mass function)  $f_{RL}(\ell|\mathbf{Q}, \mathbf{q})$  and the c.d.f.  $F_{RL}(\ell|\mathbf{Q}, \mathbf{q})$  of the  $RL$  defined for  $\ell = \{1, 2, 3, \dots\}$  can easily be obtained as follows:

$$f_{RL}(\ell|\mathbf{Q}, \mathbf{q}) = \mathbf{q}^\top \mathbf{Q}^{\ell-1} \mathbf{r}, \quad (18)$$

$$F_{RL}(\ell|\mathbf{Q}, \mathbf{q}) = 1 - \mathbf{q}^\top \mathbf{Q}^\ell \mathbf{1}. \quad (19)$$

where  $\mathbf{r} = 1 - \mathbf{Q}\mathbf{1}$  (i.e. row probabilities must sum to 1) with  $\mathbf{1} = (1, 1, \dots, 1)^T$ .

Moreover, the  $100\xi$  ( $0 < \xi < 1$ ) percentage point  $RL_\xi$  of the run length distribution is determined as follows (Gan (1994)):

$$F_{RL}(RL_\xi - 1) \leq \xi \quad \text{and} \quad F_{RL}(RL_\xi) > \xi. \quad (20)$$

By setting  $\xi = 0.5$ , we can easily obtain the conditional  $MRL$  ( $\widehat{MRL}$ ) of the EWMA  $\bar{X}$  chart.

It is important to note that, the  $\widehat{MRL}$  computed using (20) is conditioned on  $U$  and  $V$ . From the results in Zhang et al. (2011), we can note that the random variable  $U$  follows a normal  $(0, \frac{1}{\sqrt{m}})$  distribution with the p.d.f.  $f_U(u|m) = f_N\left(u \left| 0, \frac{1}{\sqrt{m}} \right.\right)$ , where  $f_N(\cdot)$  is the p.d.f. of a normal distribution and, the random variable  $V$  follows a gamma distribution with the p.d.f.  $f_V(v|m, n) = 2vf_\gamma\left(v^2 \left| \frac{m(n-1)}{2}, \frac{2n}{m(n-1)} \right.\right)$ , where  $f_\gamma(\cdot)$  is the p.d.f. of the gamma distribution with parameters  $\frac{m(n-1)}{2}$  and  $\frac{2n}{m(n-1)}$ .

Considering random variables  $U$  and  $V$ , the unconditional  $MRL$ , denoted as  $AMRL$ , of the EWMA  $\bar{X}$  chart with unknown process parameters can be obtained as follows:

$$AMRL = E(\widehat{MRL}) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \widehat{MRL} \times f_U(u|m)f_V(v|m, n)dvdu, \quad (21)$$

and the standard deviation of  $\widehat{MRL}$ , denoted as  $SDMRL$ , is

$$SDMRL = \sqrt{E(\widehat{MRL}^2) - (E(\widehat{MRL}))^2}, \quad (22)$$

where

$$E(\widehat{MRL}^2) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \widehat{MRL}^2 \times f_U(u|m)f_V(v|m,n)dvdu. \quad (23)$$

Since the integrals in (21) and (23) are difficult to compute directly, the Gauss–Legendre quadrature is used in our article to obtain an approximation of these integrals.

### 3 The $MRL$ performance of the EWMA $\bar{X}$ chart with unknown process parameters

$MRL$  based control charts with unknown process parameters are often investigated solely based on the unconditional  $MRL$  properties (see Teoh et al. (2015), You et al. (2016) and Teoh et al. (2016b)). This approach does not unfortunately reflect the sampling variation or the *between-practitioners variability* in control charts and unfortunately, it does neither account for the variability in the conditional  $MRL$  values. If this variability is large, the conditional  $MRL$  value of the EWMA  $\bar{X}$  chart corresponding to a specific practitioner can be far away from the desired in-control  $MRL$  (denoted as  $MRL_0$ ).

To illustrate this variability in the conditional in-control  $MRL$  performances of the EWMA  $\bar{X}$  chart, Table 1 displays the  $AMRL$ ,  $SDMRL$  and several percentiles of the in-control  $MRL$  values of the EWMA  $\bar{X}$  chart for different combinations of  $\lambda \in \{0.1, 0.2, 0.5, 1.0\}$  and  $m$  varying from 20 to 500, when  $MRL_0 = 200$  and  $n = 5$ . The parameters  $(\lambda, K)$  used in Table 1 are selected to produce the desired  $MRL_0$  for the *known* parameters case and, the in-control  $MRL$  values of the EWMA  $\bar{X}$  chart have been obtained by simulation using

$10^5$  replications. Because the  $MRL$  is an integer, for each value of  $\lambda$ , there are generally several values of  $K$  satisfying the desired  $MRL_0$ . Without loss of generality, in this article, we always choose the smallest  $K$  that satisfies the desired  $MRL_0$ .

From Table 1, it can be noted that for small numbers ( $m \leq 100$ ) of Phase I samples, the in-control  $MRL$  values are quite variable. For example, when  $(\lambda, K) = (0.1, 2.5986)$  and  $m = 50$ , 50% of the in-control  $MRL$  values are smaller than 133 and 10% of the in-control  $MRL$  values are larger than 236. It can be concluded that many  $MRL$  values are dispersed from the desired  $MRL_0 = 200$ . Even with a larger number of sample size ( $m = 500$ ), there is also a large variation in the in-control  $MRL$  values. For example, considering the same example above but with  $m = 500$ , there are still 50% of the in-control  $MRL$  values smaller than 187 and 10% of the in-control  $MRL$  values larger than 218. This fact shows that, with a realistic amount of Phase I samples, the conditional in-control  $MRL$  values of the EWMA  $\bar{X}$  chart can be still widely dispersed. Moreover, in order to emphasize this variation intuitively, Figure 1 also shows the histograms of the in-control  $MRL$  values based on  $10^5$  simulated EWMA  $\bar{X}$  charts for  $m \in \{50, 100, 200, 400\}$  when  $(\lambda, K) = (0.1, 2.5986)$ ,  $n = 5$  and  $MRL_0 = 200$ . Percentiles of the conditional  $MRL$  values as well as the  $SDMRL$  value are presented in the upper part of Table 1. For example, in Figure 1 (a), when  $m = 50$  Phase I samples are used to estimate the process parameters, 5% of the in-control  $MRL$  values are smaller than 47, 5% of the in-control  $MRL$  values are larger than 274 and, the  $SDARL$  value equals to 71.63. In Figure 1 (d), even with  $m = 400$  Phase I samples, a larger variation of nearly  $SDMRL = 28.29$  still exists.

For other combinations of  $(\lambda, K)$  in Table 1, we can observe a similar trend in the variation of the in-control  $MRL$  distributions. For simplicity, these histograms are not presented here. Moreover, with the increase in the number  $m$  of Phase I samples, most of the percentiles

of the  $MRL$  values increase and, the extreme upper percentiles and the  $SDMRL$  decrease. For example, when  $(\lambda, K) = (0.1, 2.5986)$ , if we increase the number  $m$  of Phase I samples from 50 to 400, the 10% and 50% percentiles of the  $MRL$  values increase from 60 up to 148 and from 133 up to 184, respectively. And the 90% percentiles decreases from 236 up to 219, and the  $SDMRL$  value decreases from 71.63 up to 28.29. It can also be noted that, for fixed value of  $m$ , with the increase in  $\lambda$ , the corresponding  $SDMRL$  values of the EWMA  $\bar{X}$  chart increases. For example, when  $m = 50$ , the  $SDMRL$  value increases from 71.63 up to 108.09 when  $\lambda$  increases from 0.1 up to 1. Because the Shewhart  $\bar{X}$  chart is a specific case of the EWMA  $\bar{X}$  chart with  $\lambda = 1$ , this fact shows that the between-practitioners variability in the  $MRL$  performance of the Shewhart  $\bar{X}$  chart is larger than the one of the EWMA  $\bar{X}$  chart.

(Please Insert Table 1 Here)

(Please Insert Figure 1 Here)

The simulation results in Table 1 and Figure 1 show that, with a realistic amount of Phase I samples, a larger variation exists in the in-control  $MRL$  values of the EWMA  $\bar{X}$  chart, especially for large values of  $\lambda$ . This variation is caused by the between-practitioners variability. To account for the between-practitioners variability, the  $SDARL$  is suggested as a measure for the  $ARL$  based control charts. A direct measure of this variability for the  $MRL$  based EWMA  $\bar{X}$  chart is the standard deviation of the  $MRL$  ( $SDMRL$ ).

In order to investigate the properties of the  $MRL$  based EWMA  $\bar{X}$  chart with unknown process parameters, we do not have to only focus on the  $AMRL$  performance of the EWMA  $\bar{X}$  chart, but we also need to focus on its  $SDMRL$  performance. Some researchers have considered  $SDARL$  values about 10% of the desired  $ARL_0$  as being small enough (see Zhang et al. (2014) and Saleh et al. (2015a)). Similar to these researches conducted on the  $ARL$



based control charts, for the  $MRL$  based EWMA  $\bar{X}$  chart, we consider the  $SDMRL$  values about 10% of the desired  $MRL_0$  as small enough in this article. Under this case, the specific practitioner may have his(her) own control chart's  $MRL$  value close to the desired  $MRL_0$ .

Now we are wondering “How many Phase I samples are needed for the EWMA  $\bar{X}$  chart to have an  $AMRL$  or an actual  $MRL$  value close to the desired  $MRL_0$ ?” To answer this question, Tables 2 to 4 present the  $AMRL$  and  $SDMRL$  values for different combinations of  $(\lambda, K)$ , the desired  $MRL_0 \in \{100, 200, 500\}$  and  $m$  ranging from 20 to 5000 when the sample size  $n = 5$ . The smaller the  $SDMRL$ , the smaller the variability in the conditional  $MRL$  values and, at the same time, an actual  $MRL$  corresponding to a specific practitioner could be closer to the desired  $MRL_0$ . Similar to Saleh et al. (2015a), several values of  $\lambda \in \{0.1, 0.2, 0.5, 1.0\}$  are chosen and the parameter combinations  $(\lambda, K)$  are chosen to satisfy the desired  $MRL_0$  of the EWMA  $\bar{X}$  chart with *known* process parameters. The  $SDMRL$  values that are about 10% of the desired  $MRL_0$  are bolded in each table. Moreover, Table 5 also presents the relative difference between the in-control  $AMRL$  and the desired  $MRL_0$ . Here, the relative difference is defined as  $\Delta = \frac{|AMRL - MRL_0|}{MRL_0} \times 100\%$ .

Some conclusions can be made as follows:

- Based on the  $AMRL$  metric, it can be noted from Table 5 that, when  $MRL_0 = 100$  and  $\lambda = 0.1$ , if a practitioner wants the relative difference  $\Delta$  smaller than 10%, it requires about  $m \simeq 250$  Phase I samples to estimate the process parameters and it requires about  $m \simeq 600$  Phase I samples to have a  $\Delta$  smaller than 5% (see column 2). With the increase in the value of  $\lambda$ , the required number of Phase I samples to have a  $\Delta$  smaller than the predetermined percentage decreases. For example, if  $\lambda$  increases from 0.1 to 1, the required number of Phase I samples decreases from 600 to 20 to have

a  $\Delta$  smaller than 5% (see columns 2 and 5). Moreover, for fixed value of  $\lambda$ , with the increase in the desired  $MRL_0$ , the required number of Phase I samples to have a  $\Delta$  smaller than the predetermined percentage increases. For example, when  $\lambda = 0.1$ , the number of Phase I samples increases from 600 up to 1100 to have a  $\Delta$  smaller than 5% when  $MRL_0$  increases from 100 up to 500 (see columns 2 and 10).

- Based on the  $SDMRL$  metric, it can be noted from Table 2 that: (1) About  $m \simeq 500$  Phase I samples are needed for the EWMA  $\bar{X}$  chart to have an in-control  $SDMRL$  to be about 10% of the desired  $MRL_0$  (see the bolded entry in column 3) when  $MRL_0 = 100$ ,  $n = 5$  and  $\lambda = 0.1$ ; (2) For fixed values of  $m$ , with the increase in  $\lambda$ , the  $SDMRL$  value also increases. For example, when  $m = 100$ , if  $\lambda$  increases from 0.1 up to 1, the  $SDMRL$  values increases from 22.54 up to 29.94. This fact also shows that the  $SDMRL$  values of the Shewhart  $\bar{X}$  chart is larger than the one of the EWMA  $\bar{X}$  chart for fixed value of  $m$ .
- Through the results of Tables 2 to 5, it can be noted that, when  $MRL_0 = 100$  and  $\lambda = 0.1$ ,  $m \simeq 250$  and  $m \simeq 600$  Phase I samples are needed to have  $\Delta$  smaller than 10% and 5%, respectively (see Table 5). While considering the  $SDMRL$  values in Table 2, it requires about  $m \simeq 500$  Phase I samples to have a  $SDMRL$  value about 10% of the desired  $MRL_0 = 100$  ( $SDMRL = 9.72$ ) and about  $m \simeq 2000$  Phase I samples to have a  $SDMRL$  value about 5% of the desired  $MRL_0 = 100$  ( $SDMRL = 4.37$ ). We can note that much more Phase I samples are needed based on the  $SDMRL$  metric compared to the  $AMRL$  metric. For larger values of  $\lambda$ , in order to have  $\Delta$  or  $SDMRL$  values smaller than the predetermined percentage, the required number of Phase I samples needed for the  $SDMRL$  metric are far more larger than the one for the  $AMRL$  metric. For example, in Tables 2 and 5, when  $\lambda = 0.5$ , it requires about  $m \simeq 150$  Phase I samples (see column 4 in Table 5) to have a  $\Delta$  smaller than 5%.

This number increases up to 3000 to have a  $SDMRL$  value about 5% of the desired  $MRL_0 = 100$  (see column 7 in Table 2). Moreover, for fixed values of  $m$  and  $\lambda$ , it can also be noted that the  $SDMRL$  value increases with the increase in the desired  $MRL_0$ . For example, when  $m = 200$  and  $\lambda = 0.1$ , the  $SDMRL$  value increases from 16.19 up to 127.48 when the  $MRL_0$  value increases from 100 to 500 (see columns 3 in Tables 2 and 4). This fact shows that if practitioners want a larger in-control  $MRL_0$  for the EWMA  $\bar{X}$  chart with unknown process parameters, they should be more cautious about a large variability in the actual in-control  $MRL$  values.

(Please Insert Tables 2 to 5 Here)

Based on the results presented above,  $m \simeq 200$  Phase I samples are large enough to have  $\Delta$  values smaller than 10%, except for the  $\lambda = 0.1$  case. Even for the  $\lambda = 0.1$  case, at most  $m \simeq 400$  Phase I samples can ensure that the  $\Delta$  value is smaller than 10% (see column 10 of Table 5). While at least  $m \simeq 500$  Phase I samples are needed for the EWMA  $\bar{X}$  chart to obtain an in-control  $SDMRL$  value about 10% of the desired  $MRL_0$  (see column 3 in Table 2). However in practice, this large number of Phase I samples is sometimes hard and too costly to collect. So with a realistic amount of Phase I samples, the practitioner has no confidence that the in-control  $MRL$  value of the specific EWMA  $\bar{X}$  chart will be close to the desired  $MRL_0$ . As presented in Table 1, if  $\lambda = 0.1$ ,  $n = 5$  and the desired  $MRL_0 = 200$ , there are still 50% of the in-control  $MRL$  values smaller than 187 with  $m = 500$  Phase I samples. For this reason, in the next section, the control limits of the EWMA  $\bar{X}$  chart are adjusted to avoid too many small in-control  $MRL$  values.

## 4 Adjustments of the control limits of the $MRL$ based EWMA $\bar{X}$ chart

With a realistic number of Phase I samples, the variability in the in-control  $MRL$  values of the EWMA  $\bar{X}$  chart is large. The EWMA  $\bar{X}$  charts corresponding to many practitioners may have  $MRL$  values smaller than the desired  $MRL_0$ . This will cause a larger false alarm than the one desired in the implementation of the  $MRL$  based EWMA  $\bar{X}$  chart. Therefore, there is the need of having an alternative technique for controlling the in-control performances of the  $MRL$  based EWMA  $\bar{X}$  chart.

In order to avoid many too small in-control  $MRL$  values for the EWMA  $\bar{X}$  chart, a bootstrap type approach introduced in Gandy and Kvaloy (2013) is used here to adjust the control limits to guarantee the conditional in-control  $MRL$  values to be larger than the desired  $MRL_0$  with a high fixed probability. Recently, some researchers have applied this approach to design  $ARL$  based control charts (see Saleh et al. (2015a), Faraz et al. (2015), Hu and Castagliola (2017), Aly et al. (2015), Saleh et al. (2016), Faraz et al. (2017)). The bootstrap approach was shown to balance the in-control and out-of-control performances of control charts well. In this article, it will also be adopted to design the  $MRL$  based EWMA  $\bar{X}$  chart with unknown process parameters. The steps for obtaining the adjusted control limits of the  $MRL$  based EWMA  $\bar{X}$  chart are summarized as follows:

- (1) Generate a Phase I data set  $X_{i,j}$  composed of  $i = 1, 2, \dots, m$  samples  $\{X_{i,1}, X_{i,2}, \dots, X_{i,n}\}$  of size  $n$  from the (assumed) true in-control distribution  $P = P(\mu_0, \sigma_0)$  of  $X_{i,j}$ , and compute the estimated parameters  $\hat{\mu}_0$  and  $\hat{\sigma}_0$  as in (6) and (7).
- (2) Set the value of  $\lambda$  and compute the quantity  $K(\hat{P}, \hat{\theta})$  that produces a desired in-control  $MRL_0$ , where  $\hat{P} = P(\hat{\mu}_0, \hat{\sigma}_0)$  is the estimated in-control distribution and  $\hat{\theta} = (\hat{\mu}_0, \hat{\sigma}_0)$

is the estimated process parameters used to set the EWMA statistic in its standardized form.

- (3) Generate  $m$  samples each of size  $n$  from  $\hat{P} = P(\hat{\mu}_0, \hat{\sigma}_0)$  and compute the estimated process parameters  $\hat{\theta}_b^* = (\hat{\mu}_b^*, \hat{\sigma}_b^*)$ .
- (4) Find the quantity  $K(\hat{P}, \hat{\theta}_b^*)$  that produces the desired  $MRL_0$ , where  $\hat{\theta}_b^*$  is used to set the EWMA statistic in its standardized form.
- (5) Repeat Steps (3) to (4)  $B$  times and get the adjusted control limit  $K_a$  as the  $\alpha^*$  ordered value of these control limits, where  $\alpha^*$  is the specified probability that the practitioner expected the in-control  $MRL$  values larger than the desired  $MRL_0$ .
- (6) Set up the EWMA  $\bar{X}$  chart with the initial Phase I data set  $X_{i,j}$  and the adjusted control limit  $K_a$ . Compute the in-control and out-of-control  $MRL$  values for the EWMA  $\bar{X}$  chart with Phase II data from the (assumed) in-control distribution.

In the previous steps, we consider  $m = 50$  Phase I samples, each with size  $n = 5$  as realistic and they are generated from the true in-control distribution  $P$ , which is assumed to be a standard normal distribution. The desired in-control  $MRL_0$  is set to be 200. It is also expected that the in-control  $MRL$  values will be larger than the desired  $MRL_0$  with a specified probability  $\alpha^* = 0.9$ . By repeating 1000 times the steps (1)-(6), it is expected that 90% of the conditional in-control  $MRL$  values for the EWMA  $\bar{X}$  chart will be larger than the desired  $MRL_0 = 200$ . As a comparison, by setting  $\lambda = 1$  in the EWMA  $\bar{X}$  chart, the conditional performances for the Shewhart  $\bar{X}$  chart with unknown process parameters are also simulated with the same parameter settings. Programs written in Matlab are used for the simulations in this paper.

Figures 2 and 3 show the boxplots of these in-control and out-of-control  $MRL$  values for the EWMA and the Shewhart  $\bar{X}$  charts with the adjusted and unadjusted limits. The control limits are denoted as “Adjusted Limits” when they are adjusted with the bootstrap approach. Otherwise, they are denoted as “Unadjusted Limits”.

(Please Insert Figures 2 to 3 Here)

When the process is in-control, it can be noted from Figures 2 that the “Adjusted Limits” can guarantee about 90% of the in-control  $MRL$  values for the EWMA and Shewhart  $\bar{X}$  charts larger than the desired  $MRL_0 = 200$ . While for the “Unadjusted Limits” cases, about 50% of the EWMA  $\bar{X}$  charts have in-control  $MRL$  values smaller than 134 and about 90% of the EWMA  $\bar{X}$  charts have in-control  $MRL$  values only larger than 229. This fact shows that many of the “Unadjusted Limits” EWMA  $\bar{X}$  charts have a higher false alarm rate than the ones with the “Adjusted Limits”. Moreover, in the case of adjusted limit, it is noted that most of the in-control  $MRL$  values for the EWMA  $\bar{X}$  charts are larger than the ones for the Shewhart  $\bar{X}$  charts.

When the process is out-of-control, suppose there is a shift  $\delta \in \{0.2, 0.4, 0.6, 0.8\}$  in the process mean. The out-of-control  $MRL$  values for the EWMA and Shewhart  $\bar{X}$  charts with the “Adjusted Limits” and “Unadjusted Limits” are presented in Figure 3. It can be noted that for small shift size ( $\delta \leq 0.2$ ), the out-of-control  $MRL$  values for the EWMA  $\bar{X}$  charts with “Adjusted Limits” are obviously larger than the ones for the EWMA  $\bar{X}$  charts with “Unadjusted Limits”. For example, in Figure 3 (a), when  $\delta = 0.2$ , we have  $MRL = 47$  for the EWMA  $\bar{X}$  chart with “Adjusted Limits” which is larger than  $MRL = 24$  for the EWMA  $\bar{X}$  chart with “Unadjusted Limits”. For moderate to large shifts, the difference between the out-of-control  $MRL$  values for the EWMA  $\bar{X}$  charts with adjusted or unadjusted limits are

not so obvious. For example, in Figure 3 (c), when  $\delta = 0.6$ , the median of the out-of-control *MRL* values for the EWMA  $\bar{X}$  charts are about 8 and 6 for the “Adjusted Limits” and “Unadjusted Limits” cases, respectively. Moreover, most of the out-of-control *MRL* values for the Shewhart  $\bar{X}$  charts are larger than the ones for the EWMA  $\bar{X}$  charts, no matter if the limits are adjusted or not, which shows the advantage of the *MRL* based EWMA  $\bar{X}$  chart over the *MRL* based Shewhart  $\bar{X}$  chart for the detection of process mean shifts.

## 5 Conclusions and recommendations

The researches on the EWMA  $\bar{X}$  chart with unknown process parameters done before are mainly focused on the chart’s unconditional performance, thus ignoring the variability between practitioners. Considering this issue, in this article, the conditional performance of the *MRL* based EWMA  $\bar{X}$  chart with unknown process parameters is investigated. The *MRL* metric is used as the conditional measure for the highly skewed run length distributions of the EWMA  $\bar{X}$  chart. Some tables and figures have been presented to assess the between-practitioners variability for the *MRL* based EWMA  $\bar{X}$  chart. General results show that the number of Phase I samples to reduce the between-practitioners variability of the EWMA  $\bar{X}$  chart up to an acceptable level is far larger than the one needed for Phase I samples based on the unconditional *MRL* performances. With a realistic number of Phase I samples, the specific practitioner has no confidence of his/her own EWMA  $\bar{X}$  chart’s *MRL* performance. Thus, the bootstrap approach is recommended to adjust the control limits of the *MRL* base EWMA  $\bar{X}$  chart and the results show that the adjusted control limits can actually balance the in-control and out-of-control *MRL* performances well, except for some small mean shifts. As this work is done for the univariate case, future works can be extended to the multivariate control charts.

# Acknowledgements

The authors would like to acknowledge the helpful suggestions of the anonymous reviewers and the editor. This research are partially supported by National Natural Science Foundation of China No. 71802110 and 71801049, Natural Science Foundation of JiangSu Province No. BK20170894, Humanity and Social Science Foundation of Ministry of Education of China No. 17YJC630043 and 17YJC630231.

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Table 1: The distribution of the in-control  $MRL$  of the EWMA  $\bar{X}$  chart for different values of  $(\lambda, K)$  and  $m$  when  $MRL_0 = 200$  and  $n = 5$

$m$	5th	10th	20th	30th	40th	50th	60th	70th	80th	90th	95th	$AMRL$	$SDMRL$
$(\lambda, K) = (0.1, 2.5986)$													
20	24	31	44	58	74	92	113	139	175	238	307	119.36	102.00
30	32	42	58	75	92	111	131	155	186	238	291	129.41	86.78
50	47	60	80	99	116	133	150	171	196	236	274	142.41	71.63
80	64	79	101	119	135	149	164	181	201	232	260	153.78	60.35
100	73	89	111	128	143	156	170	184	202	229	254	158.75	54.95
200	106	122	140	153	164	173	183	194	206	225	240	173.58	40.48
300	124	137	153	164	172	180	188	197	207	222	234	180.06	33.24
400	136	148	161	170	178	184	191	198	207	219	229	183.95	28.29
500	144	155	167	174	181	187	193	199	207	218	227	186.56	25.17
$(\lambda, K) = (0.2, 2.7677)$													
20	27	36	52	68	85	105	128	157	202	281	371	140.11	127.75
30	38	50	69	86	104	123	145	171	208	272	337	147.35	103.32
50	56	71	92	109	126	143	162	184	214	261	307	157.78	80.72
80	76	92	113	129	144	158	174	192	215	252	286	166.86	65.28
100	87	102	122	138	151	165	179	195	216	248	278	171.42	59.25
200	118	131	147	159	169	179	189	201	215	236	255	182.01	41.86
300	134	145	159	168	177	185	193	203	214	231	246	186.85	34.04
400	144	154	165	174	181	188	196	203	213	228	240	189.64	29.33
500	151	159	170	177	184	190	196	204	212	225	236	191.17	25.94
$(\lambda, K) = (0.5, 2.8966)$													
20	38	49	68	87	107	130	159	197	253	364	493	181.90	181.06
30	52	66	87	105	125	146	172	203	250	335	428	181.46	134.72
50	74	88	109	126	144	163	184	209	245	306	368	184.15	97.45
80	94	108	127	144	159	175	192	213	240	285	329	188.26	75.10
100	103	116	135	149	164	178	194	213	237	276	313	189.27	66.78
200	129	140	155	167	177	188	199	212	229	254	277	193.49	45.93
300	141	151	164	174	183	192	201	211	224	244	262	195.15	37.15
400	149	157	169	178	186	193	202	211	222	238	253	196.15	32.07
500	154	162	172	180	188	195	202	210	220	234	247	196.83	28.69
$(\lambda, K) = (1.0, 2.9221)$													
$m$	5th	10th	20th	30th	40th	50th	60th	70th	80th	90th	95th	$AMRL$	$SDMRL$
20	50	64	86	107	131	158	192	237	304	434	592	220.80	220.45
30	66	81	103	124	146	171	199	235	289	385	493	211.55	155.52
50	86	101	122	141	160	180	204	233	272	338	407	205.29	108.09
80	103	117	137	154	170	187	206	228	258	306	353	202.49	80.95
100	111	125	144	159	174	189	206	226	251	294	335	201.75	70.86
200	133	145	160	172	183	194	206	220	237	264	289	200.39	48.42
300	144	154	167	177	187	196	206	217	231	251	270	199.97	39.00
400	150	159	171	180	189	197	205	214	226	244	259	199.73	33.50
500	155	164	174	183	190	197	205	213	224	239	253	199.77	29.96

Table 2: In-Control *AMRL* and *SDMRL* for different combinations of  $(\lambda, K)$  and  $m$  when  $n = 5$  and  $MRL_0 = 100$

	$(\lambda, K) = (0.1, 2.3030)$		$(\lambda, K) = (0.2, 2.5025)$		$(\lambda, K) = (0.5, 2.6619)$		$(\lambda, K) = (1, 2.6980)$	
$m$	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>
20	64.28	40.97	71.83	51.01	88.27	70.40	104.82	85.00
30	69.43	35.65	76.07	42.15	89.53	53.91	102.10	62.22
50	75.87	29.79	81.36	33.57	91.75	40.20	100.55	44.59
70	79.78	26.16	84.69	28.81	93.19	33.53	100.08	36.55
100	83.65	22.54	87.74	24.34	94.64	27.74	99.79	29.94
150	87.43	18.65	90.73	19.93	95.97	22.46	99.65	24.01
200	89.73	16.19	92.46	17.21	96.69	19.35	99.60	20.64
250	91.24	14.36	93.63	15.29	97.25	17.27	99.57	18.38
300	92.38	13.03	94.51	13.92	97.53	15.70	99.56	16.72
400	93.81	11.09	95.55	11.92	98.03	13.55	99.55	14.41
500	94.82	<b>9.72</b>	96.24	10.57	98.30	12.09	99.50	12.87
600	95.53	8.79	96.80	<b>9.56</b>	98.51	11.01	99.55	11.75
700	96.03	8.02	97.12	8.79	98.65	10.18	99.51	10.85
800	96.44	7.42	97.40	8.18	98.69	<b>9.54</b>	99.53	10.11
900	96.76	6.95	97.66	7.68	98.87	8.94	99.49	<b>9.57</b>
1000	97.00	6.51	97.81	7.28	98.92	8.50	99.57	9.04
1100	97.22	6.17	97.97	6.90	98.98	8.12	99.57	8.62
1300	97.54	5.61	98.20	6.34	99.05	7.43	99.46	7.86
1500	97.80	5.19	98.35	5.86	99.09	6.94	99.51	7.38
2000	98.18	4.37	98.63	5.04	99.20	5.98	99.55	6.39
2500	98.44	3.90	98.84	4.51	99.26	5.38	99.52	5.70
3000	98.65	3.51	98.88	4.04	99.28	4.90	99.54	5.18
5000	98.99	2.67	99.15	3.16	99.38	3.78	99.49	3.93

Table 3: In-Control *AMRL* and *SDMRL* for different combinations of  $(\lambda, K)$  and  $m$  when  $n = 5$  and  $MRL_0 = 200$

	$(\lambda, K) = (0.1, 2.5986)$		$(\lambda, K) = (0.2, 2.7677)$		$(\lambda, K) = (0.5, 2.8966)$		$(\lambda, K) = (1, 2.9221)$	
$m$	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>
20	119.08	100.74	140.20	129.91	182.05	182.97	222.06	221.26
30	128.90	86.08	147.40	103.75	181.58	134.44	211.69	155.58
50	142.05	71.82	157.77	81.29	184.32	97.40	205.31	107.86
70	150.55	63.45	164.57	69.65	186.74	80.42	203.08	87.28
100	159.13	55.23	171.28	58.98	189.32	66.25	201.73	70.81
150	167.93	46.42	178.02	48.52	191.88	53.43	200.78	56.46
200	173.45	40.56	182.09	42.02	193.48	45.92	200.42	48.34
250	177.24	36.33	184.87	37.47	194.45	40.90	200.19	42.96
300	180.09	33.10	186.85	34.09	195.20	37.21	200.04	39.05
400	184.00	28.36	189.55	29.26	196.21	32.08	199.90	33.63
500	186.56	25.06	191.23	25.99	196.83	28.62	199.82	29.96
600	188.39	22.60	192.47	23.54	197.25	26.07	199.74	27.30
700	189.78	<b>20.73</b>	193.42	21.68	197.53	24.07	199.70	25.24
800	190.79	19.16	194.07	<b>20.18</b>	197.76	22.50	199.72	23.57
900	191.70	17.90	194.70	18.93	197.95	21.22	199.67	22.22
1000	192.37	16.83	195.07	17.93	198.10	<b>20.10</b>	199.63	21.06
1100	193.01	15.93	195.53	17.07	198.21	19.19	199.64	<b>20.08</b>
1300	193.88	14.47	196.04	15.58	198.42	17.60	199.63	18.41
1500	194.58	13.32	196.49	14.44	198.58	16.35	199.61	17.14
2000	195.77	11.29	197.22	12.42	198.83	14.16	199.56	14.86
2500	196.49	9.94	197.68	11.04	198.98	12.66	199.61	13.24
3000	196.96	8.98	197.97	10.05	199.05	11.55	199.51	12.05
5000	198.00	6.80	198.57	7.72	199.28	8.94	199.50	9.40

Table 4: In-Control *AMRL* and *SDMRL* for different combinations of  $(\lambda, K)$  and  $m$  when  $n = 5$  and  $MRL_0 = 500$

	$(\lambda, K) = (0.1, 2.9443)$		$(\lambda, K) = (0.2, 3.0819)$		$(\lambda, K) = (0.5, 3.1809)$		$(\lambda, K) = (1, 3.1972)$	
$m$	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>	<i>AMRL</i>	<i>SDMRL</i>
20	280.70	336.41	352.53	454.53	487.13	663.59	611.64	813.12
30	300.29	271.77	362.48	337.64	470.18	446.60	561.94	520.67
50	330.83	220.70	384.02	254.11	467.36	307.25	530.91	341.30
70	352.39	194.64	400.21	215.49	470.35	249.25	520.00	270.44
100	375.20	170.07	417.31	181.84	474.98	202.95	512.79	216.10
150	399.74	144.42	435.32	149.72	480.47	162.44	507.82	170.55
200	415.70	127.48	446.65	129.96	483.98	139.29	505.57	145.27
250	427.08	115.11	454.54	116.18	486.45	123.81	504.25	128.66
300	435.59	105.43	460.36	105.86	488.32	112.56	503.39	116.66
400	447.62	91.22	468.33	91.19	490.78	96.87	502.42	100.24
500	455.77	81.07	473.55	81.09	492.26	86.33	501.88	89.22
600	461.67	73.46	477.24	73.62	493.34	78.58	501.48	81.20
700	466.09	67.44	480.11	67.82	494.16	72.63	501.21	75.02
800	469.68	62.63	482.22	63.16	494.81	67.86	500.99	70.08
900	472.48	58.57	484.05	59.32	495.32	63.92	500.84	65.99
1000	474.85	55.13	485.39	56.10	495.71	60.53	500.73	62.51
1100	476.84	52.27	486.50	53.31	496.03	57.67	500.62	59.56
1200	478.49	<b>49.70</b>	487.63	50.96	496.33	55.20	500.50	56.97
1300	479.98	47.47	488.42	<b>48.80</b>	496.57	53.00	500.44	54.70
1400	481.23	45.50	489.17	46.91	496.79	51.04	500.43	52.67
1500	482.33	43.72	489.82	45.25	496.92	<b>49.29</b>	500.34	50.85
1600	483.24	42.10	490.34	43.73	497.15	47.72	500.33	<b>49.26</b>
2000	486.24	37.01	492.11	38.82	497.63	42.60	500.20	44.00
2500	488.77	32.61	493.50	34.56	498.00	38.10	500.05	39.30
3000	490.47	29.42	494.50	31.44	498.24	34.77	500.00	35.87
5000	493.99	22.08	496.47	24.15	498.75	26.88	499.87	27.74



Table 5: The relative difference  $\Delta = \frac{|AML - MRL_0|}{MRL_0} \times 100\%$  for different combinations of  $(\lambda, K)$ ,  $m$  and  $MRL_0 \in \{100, 200, 500\}$  when  $n = 5$

$m$	$MRL_0 = 100$					$MRL_0 = 200$					$MRL_0 = 500$				
	(0.1, 2.3030)	(0.2, 2.5025)	(0.5, 2.6619)	(1, 1.6980)	(0.1, 2.5986)	(0.2, 2.7677)	(0.5, 2.8966)	(1, 2.9221)	(0.1, 2.9443)	(0.2, 3.0819)	(0.5, 3.1809)	(1, 3.1972)	(0.1, 3.1972)	(0.2, 3.1972)	(0.5, 3.1972)
20	35.72%	28.17%	11.73%	4.82%	40.46%	29.90%	8.96%	11.07%	43.86%	29.49%	2.57%	22.33%	43.86%	29.49%	2.57%
30	30.57%	23.93%	10.47%	2.10%	35.55%	26.30%	9.21%	5.85%	39.94%	27.50%	5.96%	12.39%	39.94%	27.50%	5.96%
50	24.13%	18.64%	8.25%	0.55%	28.98%	21.12%	7.84%	2.65%	33.83%	23.20%	6.53%	6.18%	33.83%	23.20%	6.53%
70	20.22%	15.31%	6.81%	0.08%	24.73%	17.71%	6.63%	1.54%	29.52%	19.96%	5.93%	4.00%	29.52%	19.96%	5.93%
100	16.35%	12.26%	5.36%	0.21%	20.44%	14.36%	5.34%	0.86%	24.96%	16.54%	5.00%	2.56%	24.96%	16.54%	5.00%
150	12.57%	9.27%	4.03%	0.35%	16.04%	10.99%	4.06%	0.39%	20.05%	12.94%	3.91%	1.56%	20.05%	12.94%	3.91%
200	10.27%	7.54%	3.31%	0.40%	13.27%	8.95%	3.26%	0.21%	16.86%	10.67%	3.20%	1.11%	16.86%	10.67%	3.20%
250	8.76%	6.37%	2.75%	0.43%	11.38%	7.57%	2.78%	0.09%	14.58%	9.09%	2.71%	0.85%	14.58%	9.09%	2.71%
300	7.62%	5.49%	2.47%	0.44%	9.96%	6.58%	2.40%	0.02%	12.88%	7.93%	2.34%	0.68%	12.88%	7.93%	2.34%
400	6.19%	4.45%	1.97%	0.45%	8.00%	5.23%	1.89%	0.05%	10.48%	6.33%	1.84%	0.48%	10.48%	6.33%	1.84%
500	5.18%	3.76%	1.70%	0.50%	6.72%	4.38%	1.58%	0.09%	8.85%	5.29%	1.55%	0.38%	8.85%	5.29%	1.55%
600	4.47%	3.20%	1.49%	0.45%	5.80%	3.76%	1.38%	0.13%	7.67%	4.55%	1.33%	0.30%	7.67%	4.55%	1.33%
700	3.97%	2.88%	1.35%	0.49%	5.11%	3.29%	1.24%	0.15%	6.78%	3.98%	1.17%	0.24%	6.78%	3.98%	1.17%
800	3.56%	2.60%	1.31%	0.47%	4.61%	2.97%	1.12%	0.14%	6.06%	3.56%	1.04%	0.20%	6.06%	3.56%	1.04%
900	3.24%	2.34%	1.13%	0.51%	4.15%	2.65%	1.02%	0.17%	5.50%	3.19%	0.94%	0.17%	5.50%	3.19%	0.94%
1000	3.00%	2.19%	1.08%	0.43%	3.82%	2.46%	0.95%	0.19%	5.03%	2.92%	0.86%	0.15%	5.03%	2.92%	0.86%
1100	2.78%	2.03%	1.02%	0.43%	3.49%	2.23%	0.89%	0.18%	4.63%	2.70%	0.79%	0.12%	4.63%	2.70%	0.79%
1300	2.46%	1.80%	0.95%	0.54%	3.06%	1.98%	0.79%	0.19%	4.00%	2.32%	0.69%	0.09%	4.00%	2.32%	0.69%
1500	2.20%	1.65%	0.91%	0.49%	2.71%	1.75%	0.71%	0.19%	3.53%	2.04%	0.62%	0.07%	3.53%	2.04%	0.62%
2000	1.82%	1.37%	0.80%	0.45%	2.11%	1.39%	0.59%	0.22%	2.75%	1.58%	0.47%	0.04%	2.75%	1.58%	0.47%
2500	1.56%	1.16%	0.74%	0.48%	1.75%	1.16%	0.51%	0.19%	2.25%	1.30%	0.40%	0.01%	2.25%	1.30%	0.40%
3000	1.35%	1.12%	0.72%	0.46%	1.52%	1.02%	0.48%	0.24%	1.91%	1.10%	0.35%	0.00%	1.91%	1.10%	0.35%
5000	1.01%	0.85%	0.62%	0.51%	1.00%	0.71%	0.36%	0.25%	1.20%	0.71%	0.25%	0.03%	1.20%	0.71%	0.25%

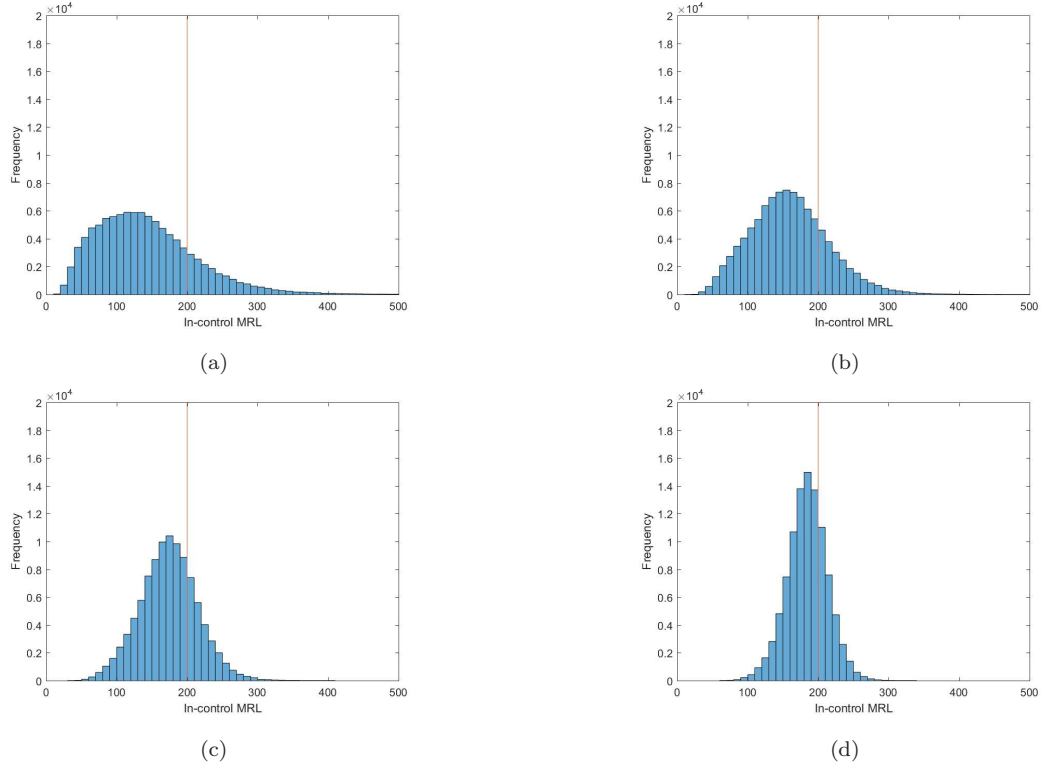


Figure 1: Histograms of the in-control  $MRL$  values for (a)  $m = 50$ , (b)  $m = 100$ , (c)  $m = 200$ , (d)  $m = 400$  when  $(\lambda, K) = (0.1, 2.5986)$ ,  $MRL_0 = 200$  and  $n = 5$ . (The red line corresponds to the  $MRL_0 = 200$  on the horizontal axis)

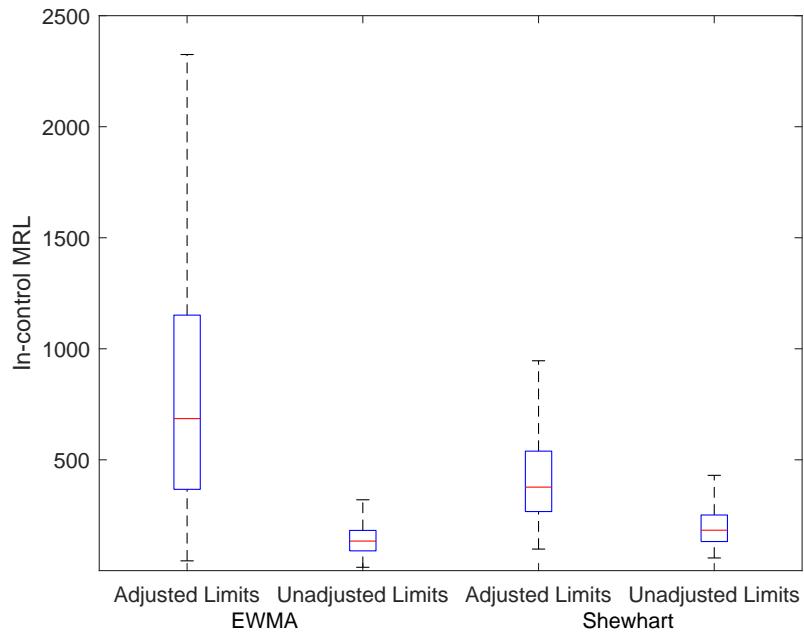


Figure 2: The boxplots of the in-control  $MRL$  values for the EWMA and Shewhart  $\bar{X}$  charts with the adjusted and unadjusted limits.

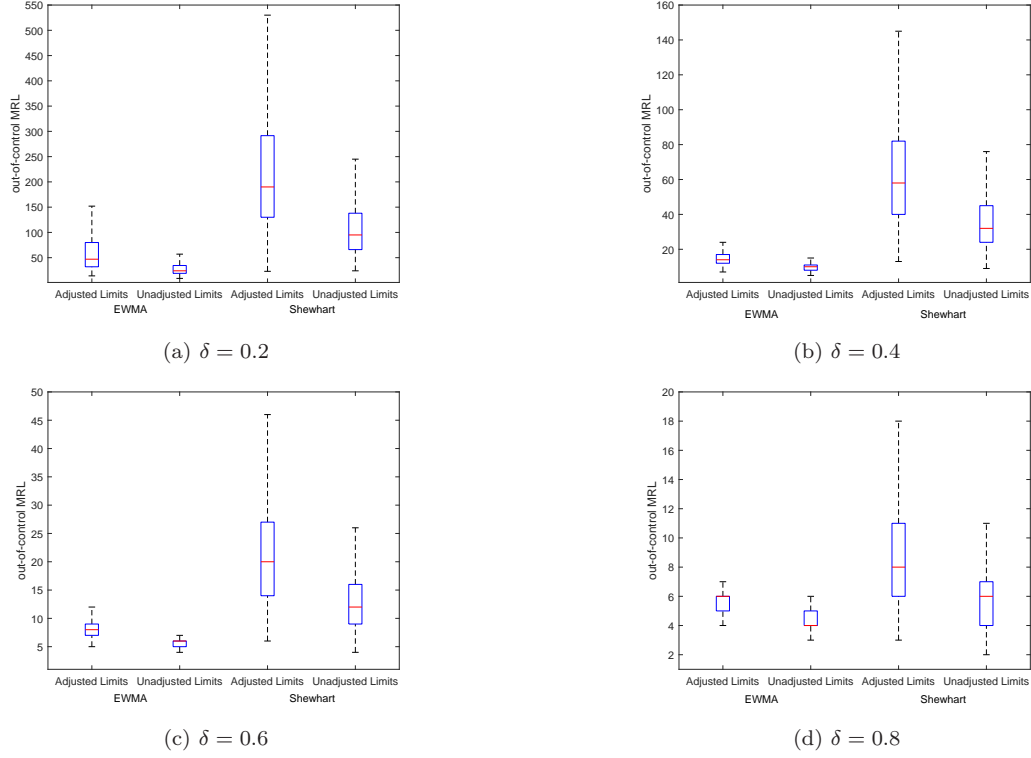


Figure 3: The boxplots of the out-of-control  $MRL$  values for the EWMA and Shewhart  $\bar{X}$  charts with the adjusted and unadjusted limits for  $\delta \in \{0.2, 0.4, 0.6, 0.8\}$