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**Some geometry of irregular connections on curves (slides  
of talk at Conference on Gauge Theory and  
Representation Theory, IAS Princeton, 27/11/2007)**

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Some geometry of  
irregular connections  
on curves

P. Boalch

Guiding principle for seeking hyperKähler manifolds - (Hitchin, Seminar Bourbaki '91):-

“ Look for hyperKähler structures on manifolds with complex symplectic structures ”

Representative collection of examples:-

- ① Resolutions of rational double point singularities
  - ② Coadjoint orbits of complex Lie groups
  - ③ Spaces of representations of a surface group in a complex Lie group
- ⋮

- Here a 'surface group' is the fundamental group  $\pi_1(\Sigma)$  of a compact (Riemann) surface  $\Sigma$

- If  $G$  a complex (reductive) group

$$M = \text{Hom}^{\text{irr}}(\pi_1(\Sigma), G) / G$$

has a natural complex symplectic structure  
(Atiyah-Bott, Goldman, ...)

— independent of complex structure of  $\Sigma$

"symplectic nature of the fundamental group"



# Tannakian categories

- Rough idea: axiomatise properties of categories of representations of groups
- Prove: any Tannakian category  $\cong$  category of representations of some group

Basically:  $\pi_1(\Sigma)$  is the Tannaka group of the (Tannakian) category of vector bundles with holom. connections /  $\Sigma$

Define 'wild fundamental group'  $w\pi_1(\Sigma)$  to be Tannaka group of category of vector bundles with mera connections /  $\Sigma$

- constructed (in some sense) by Martinet-Ramis '91

## Picture

$$\text{Hom}(w\pi, (\Sigma), G)/G$$

much too big

- Has natural Poisson structure & finite dimensional symplectic leaves
- Describe (& study) generic such leaves directly

## Main results

- ① Atiyah-Bott type definition of symplectic str.
  - indept. of cx str. of  $\Sigma$  (& "irreg. types")
  - "symplectic nature of wild fundamental group"
- ② (w. O. Biquard) HyperKähler metrics & correspondence w. mono. Higgs bundles
  - (one direction done earlier by Sabbah)
  - (& simple poles by Simpson, Nakajima, ...)

## Alternative viewpoint

"Hyperkahler metrics on total spaces of  
algebraically completely integrable systems"

- The Hitchin integrable system on moduli spaces of  
Higgs bundles was extended to meromorphic Higgs  
bundles by Bottacin, Markman ~ '94

( $\mathbb{P}^1$  case by Beauville, Adams-Harnad-Hurtubise, ...)



## Plan for rest of talk

- ① Simplest examples
- ② Quick general description of the spaces
- ③ More examples
  - surprising link to Poisson Lie groups
  - appearance of  $G$ -braid groups
- ④ Fourier-Laplace transform (simplest case)
- ⑤ Generalised Deligne-Simpson problem



Examples

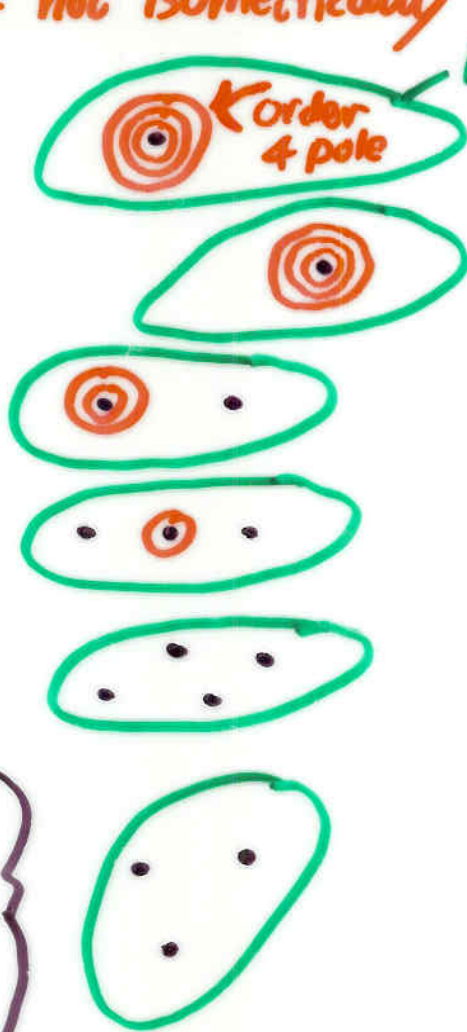
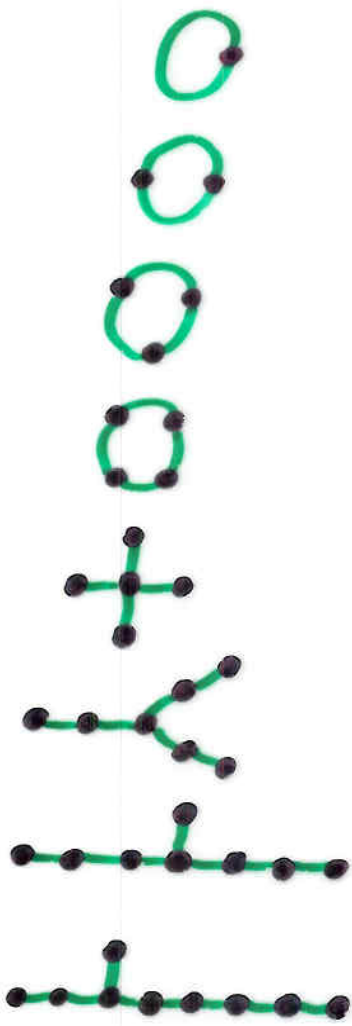
"Quaternionic Curves"

$\left\{ \begin{array}{l} \dim_{\mathbb{H}} = 1 \\ \text{real 4 mfd's} \end{array} \right.$

- simplest examples have open pieces

$\cong$  Kronheimer's ALE spaces  
Cx. sympl. not isometrically

$A_0$   
 $A_1$   
 $A_2$   
 $A_3$   
 $D_4$   
 $E_6$   
 $E_7$   
 $E_8$



$GL_2$  - not generic  
 $GL_2$   
 $GL_2$   
 $GL_2$   
 $GL_2$   
 $GL_3$   
 $GL_4$   
 $GL_6$  } not generic

cf. Cherkis-Kapustin, ..., Painlevé...

# Quick explicit description of the spaces

$\Sigma$  genus  $g$ ,  $m$  punctures  $\left\{ \begin{array}{l} \text{simple pole} \\ \text{case} \end{array} \right\}$

Choose  $m$  conjugacy classes  $e_i \subset G$

$$M = \text{Hom}_e(\pi_1(\Sigma), G) / G$$

$$\cong \left\{ \underline{A}, \underline{B}, \underline{C} \mid \prod [A_i, B_i] \prod C_j = 1, C_j \in e_j \right\} / G$$

view as multiplicative moment map condition

$$=: \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_g \otimes e_1 \otimes \dots \otimes e_m // G$$

where  $\mathbb{D} = G \times G$  with map  $\mu: \mathbb{D} \rightarrow G; (A, B) \mapsto ABA^{-1}B^{-1}$   
- can construct  $\omega$  sympl. structure algebraically this way  
(quasi-Hamiltonian geometry - Alekseev-Malkin-Meinrenken)

Generic higher order poles :-

(most polar coeff. regular semisimple  $\in \mathfrak{h}_{\text{reg}}$ )

Replace conjugacy classes  $e \in G$  by

$$e = ((U_+ \times U_-)^{k-1} \times G) / H$$

where

$k =$  pole order

$H \subset G$  max. complex torus

$U_{\pm} \subset B_{\pm} \subset G$  full unipotent subgroups  
( $B_+ \cap B_- = H$ )

Choose  $t \in H$ , then 'moment map' is

$$\mu: e \rightarrow G; \quad [ \{s_i^{\pm}\}, c ] \mapsto c s_1^- s_1^+ s_2^- s_2^+ \dots t c^{-1}$$

( $e$  is a quasi-Hamiltonian  $G$ -space)

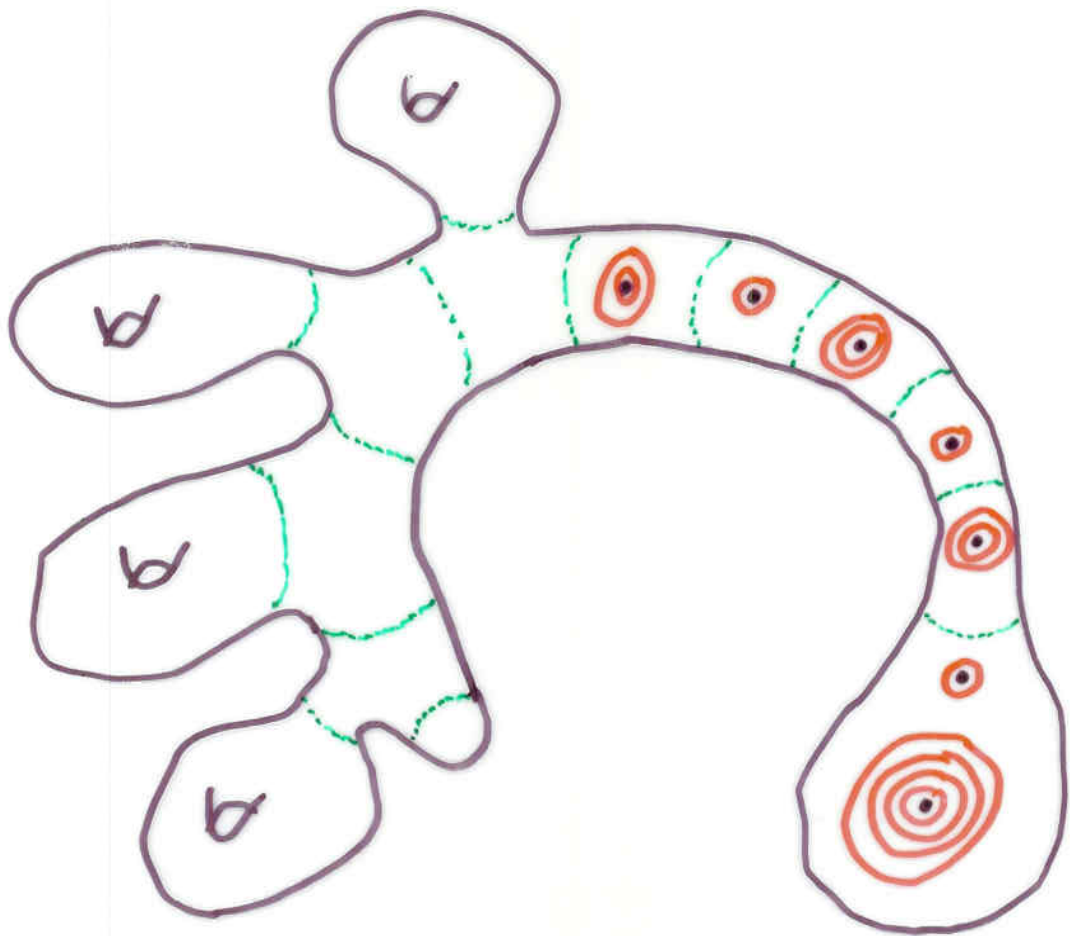


Have such 'generalised conjugacy class'  $e$   
at each pole, then

$$M \cong \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_{\mathfrak{g}} \otimes \mathbb{C}_1 \otimes \dots \otimes \mathbb{C}_m // G$$

- get explicit (algebraic) symplectic structure
- same as that from extended Atiyah-Bott viewpoint





# Local irregular Riemann-Hilbert correspondence (generic, order 2 pole)

- $\Delta \subset \mathbb{C}$  unit disk
- $\mathcal{G} = \{ \text{holomorphic maps } g: \Delta \rightarrow G \}$
- $\mathcal{G}_1 = \{ g \in \mathcal{G} \mid g(0) = 1 \}$
- Fix  $A_0 \in \mathfrak{h}_{\text{reg}}$

Theorem (Balser-Jurkat-Lutz '79, PB '02)

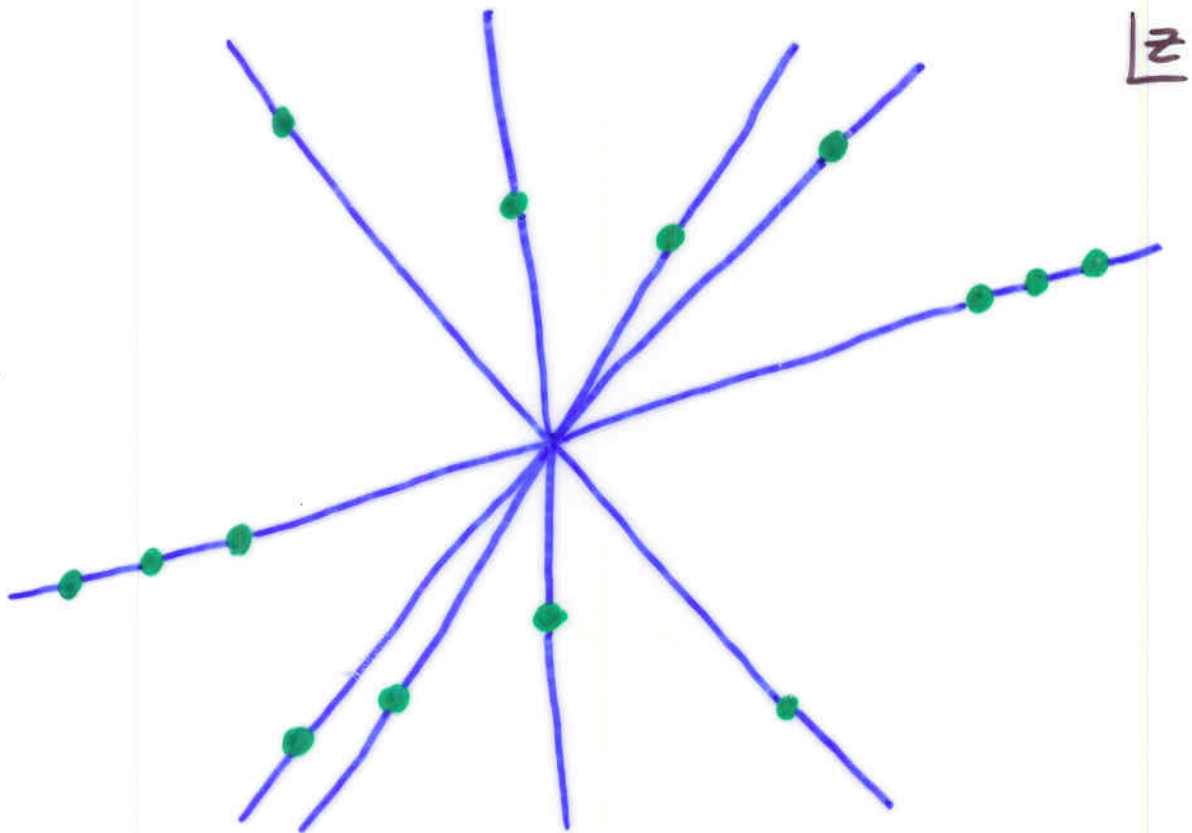
$$\left\{ \left( \frac{A_0}{z^2} + \frac{B}{z} + \mathbb{H} \right) dz \mid \begin{array}{l} B \in \mathfrak{g} \\ \mathbb{H}: \Delta \rightarrow \mathfrak{g} \text{ holom.} \end{array} \right\} / \mathcal{G}_1$$

$$\cong U_+ \times U_- \times \mathfrak{h} \ni (S_+, S_-, \Lambda) \quad \begin{array}{l} \text{Stokes multipliers} \\ \pi_{\mathfrak{h}}(B) \end{array}$$

- still have action of  $\mathfrak{H} \subset \mathfrak{g}$  "compatible framing at 0"
- to get  $\mathcal{E}$ : include arbitrary framing at point  $1 \in \partial\Delta$   
& quotient by  $\mathfrak{H}$  fixing value of  $\Lambda$  ( $t = e^{2\pi i \Lambda}$ )
- have more intrinsic viewpoint (isom. here depends on choice of sector at 0  $\Rightarrow$  choice of +ve roots)

Roots  $\mathcal{R} \subset \mathfrak{h}^*$   
 $A_0 \in \mathfrak{h}_{\text{reg}}$

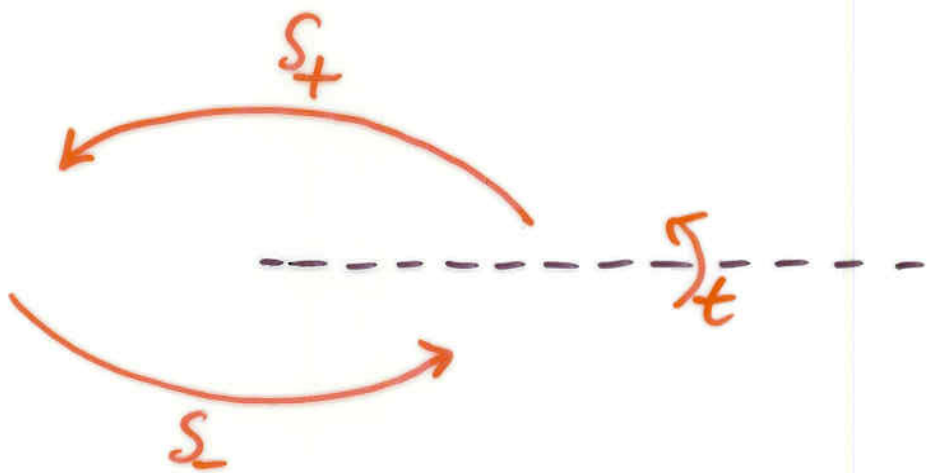
$\langle \mathcal{R}, A_0 \rangle \subset \mathbb{C}^*$



Preferred horizontal section on each sector

$\mathbb{I} : \text{sector} \rightarrow \mathbb{G}$

(asymptotics /  $k$ -summation)





$\rightsquigarrow U_+ \times U_- \times \mathfrak{h}$  inherits complex Poisson str.

Theorem

This Poisson manifold is isomorphic to the standard Poisson Lie group  $G^*$  (dual to  $G$ ).

$$G^* = \left\{ (b_+, b_-, \lambda) \in B_+ \times B_- \times \mathfrak{h} \mid \pi_{\mathfrak{h}}(b_+) = \pi_{\mathfrak{h}}(b_-^{-1}) = e^{\pi i \lambda} \right\}$$

- symplectic leaves got by fixing conjugacy class of

$$b_-^{-1} b_+ = s s_+ t \in G$$

- (well) approximated by setting  $\mathfrak{h} = 0$ ,  $B \in \mathfrak{g} \cong \mathfrak{g}^*$

$$\text{map } \gamma_{A_0} : \mathfrak{g}^* \longrightarrow G^*$$
$$B \longmapsto \left[ \frac{A_0}{z} + \frac{B}{z} \right]$$

is Poisson for any  $A_0 \in \mathfrak{h}_{\text{reg}}$

"dual exponential map"

## More hyperKähler spaces (examples)

- Typically  $\begin{array}{c} \theta \\ \uparrow \\ \mathfrak{g}^* \end{array} \xrightarrow{\vee} \mathcal{L} \subset \mathfrak{g}^*$

Forget framings:  $M \cong \mathcal{L} //_{\mathfrak{h}} H$  (hk)

(partial compactification of  $\theta //_{\mathfrak{h}} H$ )

- 2 order 2 poles on  $\mathbb{P}^1$  :-

with compatible framings  $\tilde{M} \cong \Gamma$

"double symplectic groupoid" - Lu-Weinstein

& have  $T^*G \hookrightarrow \Gamma$

Forget framings:  $M \cong H // \Gamma // H$  (hk)

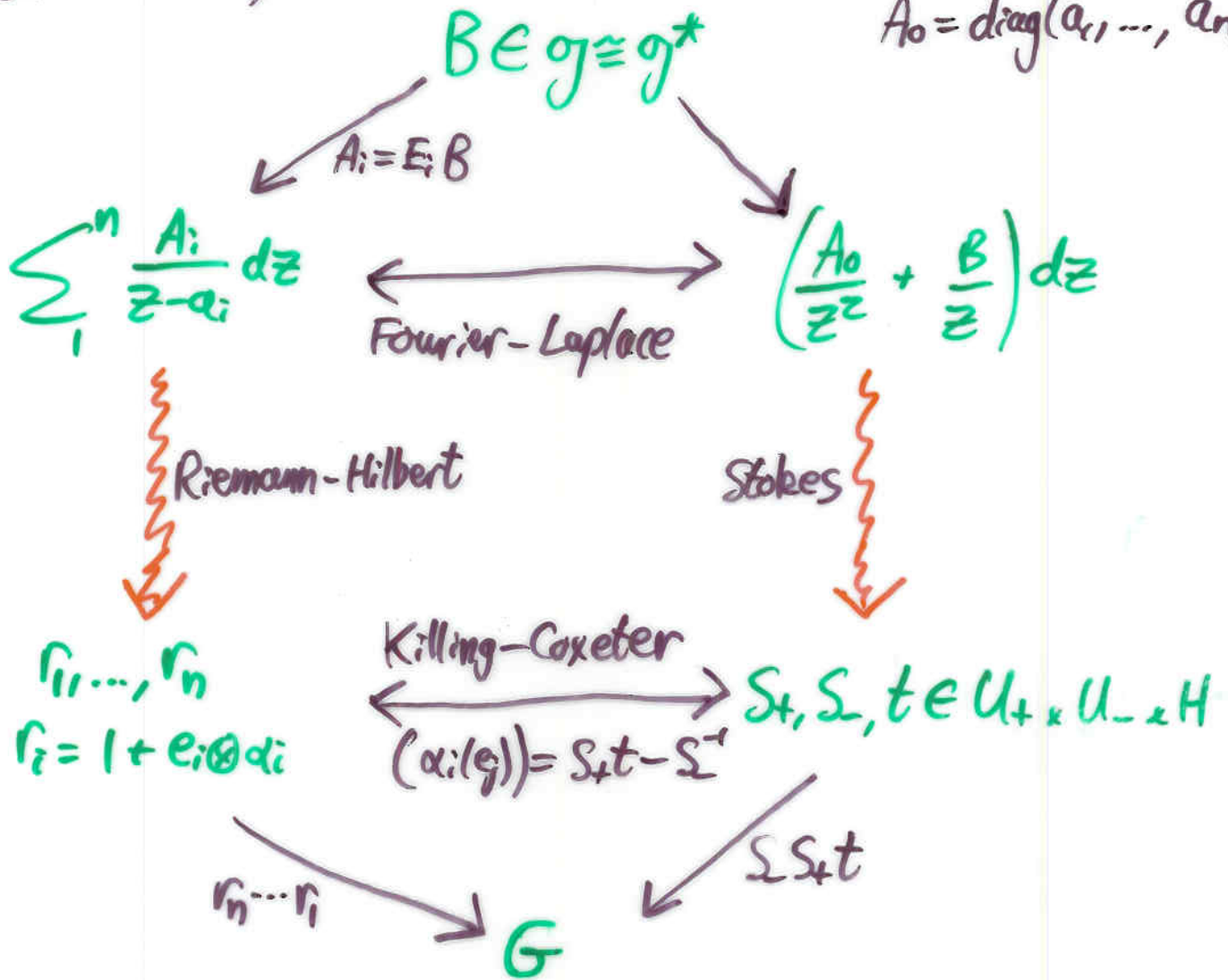
(partial compactification of  $H // T^*G // H$ )

# Fourier-Laplace transform

( $G \subset n$ ,  $IP^1$ , 1 order 2 pole + 1 order 1 pole)

(Balser-Jurkat-Lutz '81)

$$A_0 = \text{diag}(a_1, \dots, a_n)$$



- Get isomorphisms of moduli spaces
- Led to search for independent general construction
- "Fourier-Laplace is an isometry" now proved by S. Szabo '85
- original interest was via Dubrovin's work on semi-simple Frobenius manifolds (~ '95)



# Deligne-Simpson problem

( $G \subset \mathrm{GL}_n$ ,  $\mathbb{P}^1$ , simple poles)

"When are the spaces non-empty?"

① Additive version: fix orbits  $\theta_1, \dots, \theta_m \subset \mathfrak{g}$

$\exists ? A_1, \dots, A_m$  s.t.  $A_i \in \theta_i$ ,  $\sum A_i = 0$ , irreducible

$$\mathcal{M}^* = \theta_1 \times \dots \times \theta_m // G \ni \left[ \sum \frac{A_i}{z-a_i} dz \right]$$

② Multiplicative version: fix classes  $e_1, \dots, e_m \in G$

$\exists ? c_1, \dots, c_m$  s.t.  $c_i \in e_i$ ,  $\prod c_i = 1$ , irred.

$$M = e_1 \oplus \dots \oplus e_m // G$$

Lots of progress made by Crawley-Boevey (solved ①, announced ②)  
using link to quiver varieties (star-shaped quivers)

(Riemann-Hilbert maps ① into ②, if non-resonant)



## More general problem

( $GL_n$ ,  $\mathbb{P}^1$ , 1 pole of order  $z$  +  $m$  simple poles)

now leading coef.  $A_0 \in \mathfrak{h}$ , not necessarily  $\mathfrak{h}_{reg}$

Let  $H(\underline{n}) = \text{stab}(A_0) \cong \prod GL_{n_i} \subset GL_n$

$\underline{n} = (n_1, \dots, n_k)$ ,  $\sum n_i = n$  (eigenvalue mults)

$\mathfrak{h}(\underline{n}) = \text{Lie}(H(\underline{n})) = \ker(\text{ad}_{A_0})$

① Fix adjoint orbits  $\mathcal{O}_1, \dots, \mathcal{O}_m \subset \mathfrak{g}$  &  $\check{\mathcal{O}} \subset \mathfrak{h}(\underline{n})$   
( $\check{\mathcal{O}} \cong \check{\mathcal{O}}_1 \times \dots \times \check{\mathcal{O}}_k$ ,  $\check{\mathcal{O}}_i \subset \mathfrak{g}(n_i)$ )

$\exists ? A_1, \dots, A_m$  s.t.  $A_i \in \mathcal{O}_i$ ,  $\pi_{\mathfrak{h}(\underline{n})}(\sum A_i) \in \check{\mathcal{O}}$ , irred.

$$\mathcal{M}^* = \mathcal{O}_1 \times \dots \times \mathcal{O}_m //_{\check{\mathcal{O}}} \mathfrak{h}(\underline{n})$$

Key point: These are also quiver varieties

( $A_0$  scalar ( $k=1$ ), or (by Fourier-Laplace)  $m=1$  get original problem)

(2) Choose conjugacy classes  $e_1, \dots, e_m \subset G$ ,  $\check{e} \subset H(\underline{n})$   
( $\check{e} \cong \check{e}_1 \times \dots \times \check{e}_r$ ,  $\check{e}_i \subset GL_{n_i}$ )

$\exists? c_1, \dots, c_m$  s.t.  $c_i \in e_i$ , " $\vec{\pi}_{H(\underline{n})}(\vec{\pi} c_i) \in \check{e}$ "  
+ irreducible

$\rightarrow$  explicitly: need ①  $\vec{\pi} c_i \in \underline{n}$ -big cell  
 $U_+(\underline{n}) H(\underline{n}) U_-(\underline{n})$

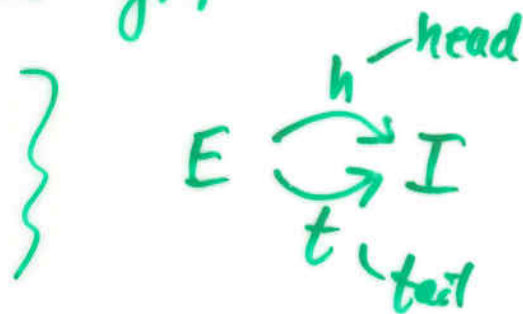
②  $H(\underline{n})$  component is  
in  $\check{e}$

# (affine) Quiver Varieties ( ... Nakajima)

Quiver = directed finite graph

$E =$  edges

$I =$  vertices



Choose  $\kappa$  vector space  $V_i$  ( $\forall i \in I$ )  $d_i = \dim V_i \Rightarrow \underline{d}$

Maps in both directions:  $V = T^* \bigoplus_{e \in E} \text{Hom}(V_{t(e)}, V_{h(e)})$   
 $\mathbb{V}(g, g')$

$$G = G(V) = \prod_{i \in I} GL(V_i) \curvearrowright V$$

Moment map  $\mu$ , with components  $\mu_i: V \rightarrow \text{End}(V_i)$ ;

$$\mu_i(g, g') = \sum_{e|h(e)=i} g_e g'_e - \sum_{e|t(e)=i} g'_e g_e$$

Choose scalar at each node  $\underline{\lambda} = \{ \lambda_i \in \mathbb{C} \mid i \in I \}$

$$QV(\underline{\lambda}, \underline{d}) = \mathbb{V} //_{\underline{\lambda}} G(V)$$

e.g. • ALE spaces, quiver = affine Dynkin diagram (Kronheimer)

• Closures of adjoint orbits  $\Theta \subset \mathfrak{gl}(n)$  (Kraft-Process, ...)

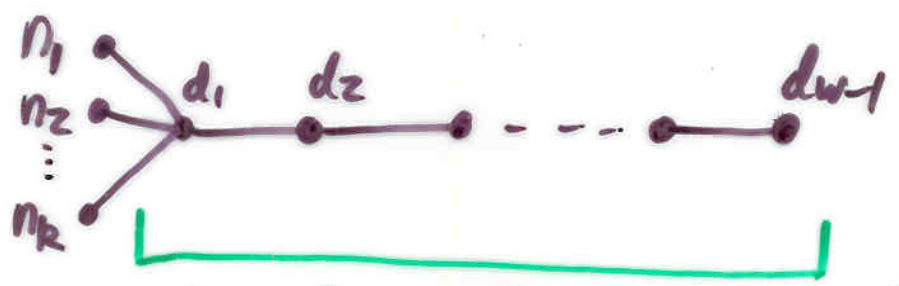


$d_i = \text{rank}(A - x_1) \dots (A - x_i)$  where  
 $A \in \Theta$ ,  $x_1, \dots, x_w$  roots of min. poly of  $A$   
 (param.s  $\simeq$  evals)

Equivalently take graph:



or:



$\sum n_i = n$

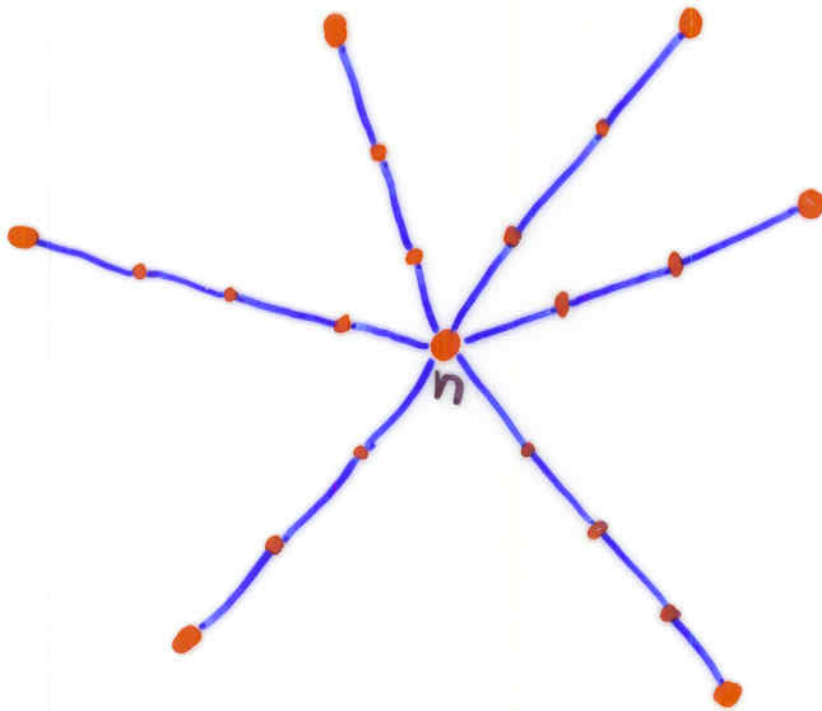
and just symplectic quotient by the groups at these nodes



## Star shaped quivers

$\mathcal{O}_1, \dots, \mathcal{O}_m \Rightarrow m$  legs 

with residual  $G_n$  action.



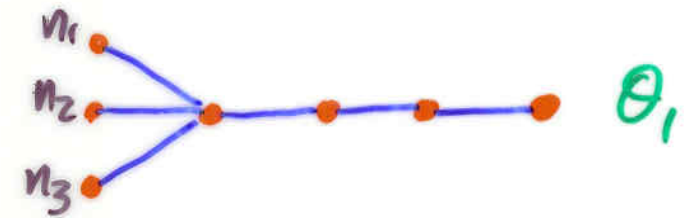
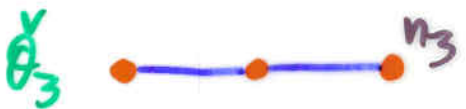
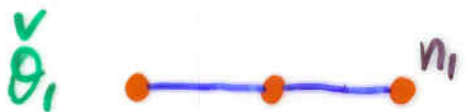
$$QV \cong \mathcal{O}_1 \times \dots \times \mathcal{O}_m // G_n \quad (\text{if orbits closed})$$

# Supernova shaped quivers

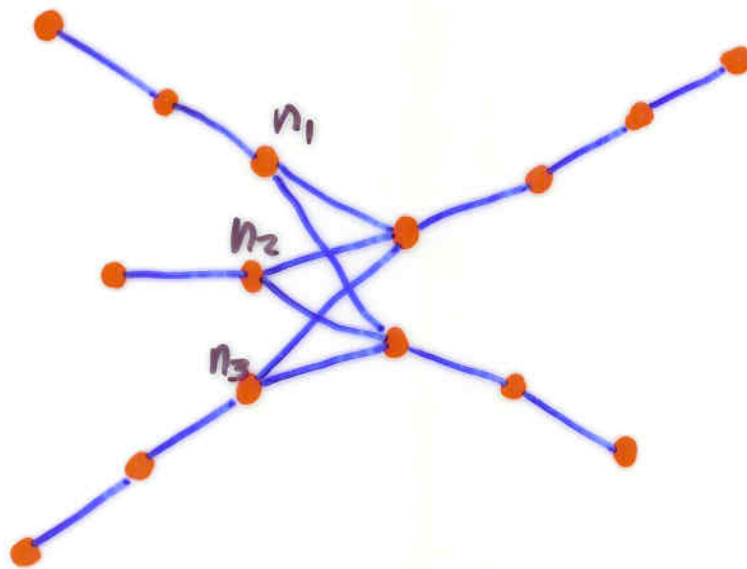
want  $QV = \Theta_1 \times \dots \times \Theta_m \parallel_{\check{\Theta}} H(\underline{n})$

$$\left\{ \begin{array}{l} \sum_i^k n_i = n \\ \check{\Theta} = \prod_i^k \check{\Theta}_i \\ \check{\Theta}_i \subset \mathfrak{gl}(n_i) \end{array} \right.$$

$m=2, k=3:-$

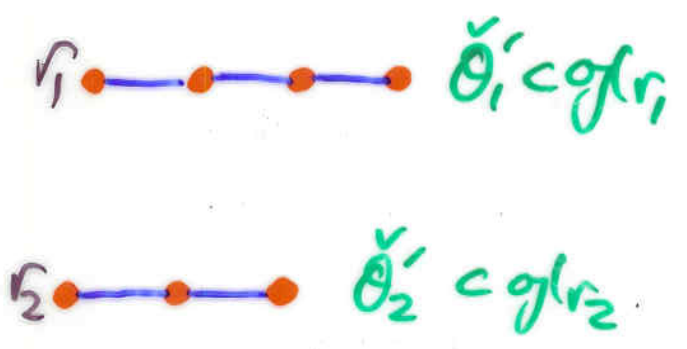
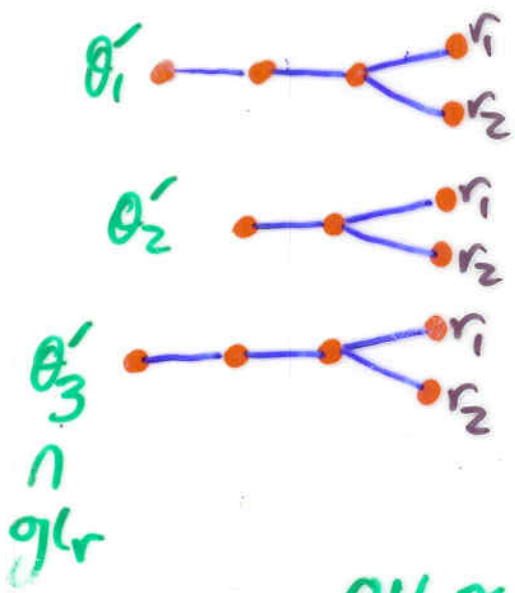
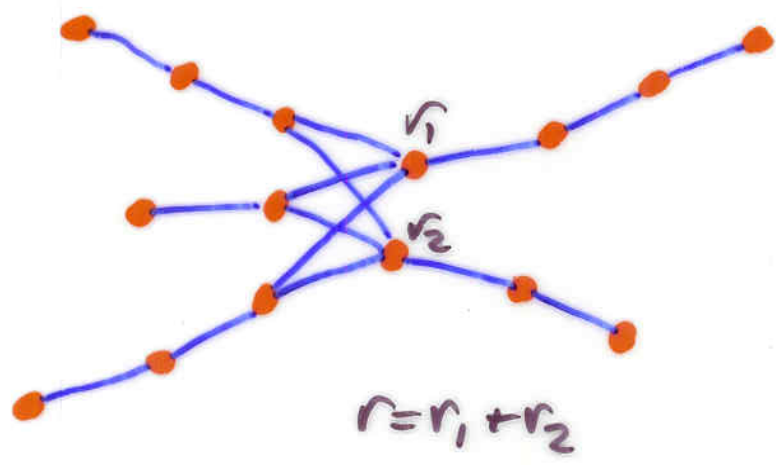


-residual  $H(\underline{n})$  action



(e.g.  $\hat{A}_3$  = )

Dual viewpoint (cf. Harnad)



so  $QU \cong \theta'_1 \times \theta'_2 \times \theta'_3 //_{\theta'_1} H(r)$  as well

Claim This duality is the Fourier-Laplace transform

(Generalisation of Gelfand-MacPherson duality)