Joint Coefficient and Solution Estimation for the 1D Wave Equation: An Observer-Based Solution to Inverse Problems

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***Abstract*—The estimation of an unknown source coefficient and of the solution to a 1-dimensional (1D) wave equation via exponentially convergent state observers is the problem under consideration in this work. The coefficient is assumed to depend on the space variable only and to be polynomial. The main observation information for this inverse problem is the value of the solution to the wave equation in a subinterval of the domain, including also some of its higher-order spatial derivatives. In order to estimate the source coefficient, we turn it into a new state as in finite-dimensional parameter identification approaches. However in this infinite-dimensional setting, this requires the introduction of a novel indirect approach involving an infinite-dimensional state transformation. Sufficient conditions allow the design of a composite observer consisting of an internal observer, which estimates in higher regularity spatial norms both the source term and the solution on a subinterval, and a boundary observer, in order to eventually provide the estimation of the solution everywhere. The observer convergence is proven by means of Lyapunov analysis. An extension of this approach to the case of the identification of a potential in the wave equation is finally considered, which is a nonlinear inverse problem since here, in the equation, the unknown coefficient multiplies the solution.**

Keywords: coefficient identification/estimation in the wave equation, inverse problems, observers for PDEs

I. Introduction

The *inverse problem* of determining unknown coefficients in partial differential equations (PDEs) has been extensively studied during the last decades, see [22]. A lot of the recent results rely on the use of Carleman estimates for the PDE operator under consideration, dating back to the seminal work of [11]. In our work, we are interested in the study of inverse problems for the wave equation, for which Carleman estimates approaches have already given a lot of answers, from uniqueness and stability in the determination of source or potential term (see [35] or [21]) to reconstruction algorithms (e.g. [6]). Other techniques also exist for the same kind of problem, see e.g. [9] for a spectral estimation of eigenvalues approach. For this equation, applications may concern the recovery of properties of the medium when dealing with acoustic waves, ocean and seismic prospection, medical imaging, or geophysics, see [15], [32], see also [2] for the 1D wave equation (inverse problems for bridges, antennas, space-structures, steel-grid reinforcements etc).

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This article has the ambition to propose a different kind of answer to the inverse problem of the determination of a time independent source term in the wave equation. The main goal here will be to design an online estimation of the source coefficient by means of an appropriate observer.

Observation and observer design results for the wave equation have already been given for instance in [27], [31]. Estimating the initial state of infinite-dimensional systems from measurements on a finite interval is solved by use of a sequence of forward and backward observers in [30], see also [17]. One can also mention [13] or [10] for the so-called back-and-forth algorithms. In [18] and references therein, the problem of reconstruction of initial data using observers is considered. In all these works, the reconstruction algorithms are either performed offline or lead to asymptotic, but not exponential, estimation.

The present paper deals with the estimation through time of a source coefficient of the 1D wave equation via Luenberger observers. A novel approach for this rather classical question is proposed here, method that is initially based on a standard trick for constant parameter estimation in finite dimensions: the state of the system is extended with the unknown parameter, in order to get a cascade system in a *canonical* form, for which an observer can be easily designed, see for instance [8]. In the present case, where the role of the parameter is played by a space-varying source coefficient, a similar approach turns out not to be so direct due to the properties of the differential operator of the extended coupled system. An indirect approach is thus proposed, that eventually leads to the exponential estimation in time of the coefficient. In [20], an in-domain constant parameter is estimated for a hyperbolic system via adaptive observer, however, similarly as in finite dimension (see [34]), the estimation is not necessarily exponential. Contrary to this approach, in the present work, the estimation of the potential is exponentially convergent, while the solution to the wave equation is simultaneously estimated.

The so-called indirect approach we will describe in this paper includes the introduction of an infinite-dimensional transformation that maps the wave equation into a cascade system of three coupled PDEs, whose differential operator is decomposed into a part with the same elements on the diagonal and a part acting on the observation of the solution (considered as measurement). Aiming at the final observer design, we assume that the coefficient we seek is a polynomial of the space variable and of even degree, and that observations include the solution and its higher order spatial derivatives in

a subinterval of the domain coming from appropriately strong regularity assumptions. An exponentially convergent composite observer is then designed. It consists of two components : an internal observer on the observed subinterval, providing the estimation of the potential arbitrarily fast, and of the solution in this subinterval, and a boundary observer for the rest of the domain, bringing the rest of the estimation of the solution. The stability proof in higher regularity spatial norms is based on the introduction of appropriate Lyapunov functionals. The current approach is inspired by a recently introduced methodology in [24] to deal with the observer design of under-observed systems of coupled PDEs.

In summary, this work introduces an extension of the classical parameter identification/estimation methodology from finite dimensions to infinite dimensions, more precisely to the problem of coefficient estimation for the wave equation.

The paper is organized as follows. In Section II, we introduce the estimation problem and we present our approach that leads to solvability of the main observer problem. In Section III, we present the joint estimation of a source term and the solution to the wave equation via the proposed observer. We extend this approach to a related nonlinear inverse problem in Section IV, concerning the recovery of a potential in a wave equation under comparable technical assumptions. Section V includes an illustrative simulation, and we provide some conclusions and perspectives in Section VI.

Notation: For a given w in $\mathbb{R}^n$, $|w|$ denotes its usual Euclidean norm. For a matrix A in $\mathbb{R}^{n\times n}$, $A^\top$ denotes its transpose, $|A| := \sup\{|Aw|, |w| = 1\}$ is its induced norm and $\mathrm{Sym}(A) = \frac{A+A^\top}{2}$ stands for its symmetric part. If A is symmetric, $\underline{\mathrm{eig}}(A)$ is its minimum eigenvalue. I_n and 0_n are the identity and zero matrix of dimension n, and $\mathrm{antidiag}_k(A_i)_{i=1,\dots,k}$ denotes the block matrix consisting of k matrices A_i placed on antidiagonal block-row i. By $\{k,\dots,l\}$ we denote the set of $l-k+1$ consecutive integers from k to l. For given $u : [0,+\infty)\times[l,L] \to \mathbb{R}^n$ and time $t \geq 0$ we use the notation $u(t)(x) := u(t,x)$, for all x in $[l,L]$ to refer to the profile at certain time and $\partial_t^i u$ (resp. $\partial_x^i u$) is its partial derivative with respect to t (resp. x) of order i. For a q - times continuously differentiable mapping u we denote $\|u\|_{C^q[l,L]} := \sum_{i=0}^q \max\{|\partial_x^i u(x)|, x \in [l,L]\}$ for the q-norm. Any element $u : [l,L] \to \mathbb{R}^n$ of the Hilbert space $L^2(l,L;\mathbb{R}^n)$ satisfies $\|u\|^2_{L^2(l,L)} = \int_l^L |u(x)|^2 dx < +\infty$. From there, the Sobolev space $H^q(l,L;\mathbb{R}^n)$ gathers functions u such that all their weak derivatives up to order q are also in $L^2(l,L;\mathbb{R}^n)$, and is equipped with the norm $\|u\|_{H^q(l,L;\mathbb{R}^n)} = \sum_{i=0}^q \left(\int_l^L |\partial_x^i u(x)|^2 dx\right)^{1/2}$. Finally, $\mathcal{L}(\mathscr{X})$ denotes the space of bounded linear operators from $\mathscr{X}$ to $\mathscr{X}$ and $\mathbb{R}^N[x]$ denotes the space of polynomials of the variable x and of degree N.

II. Problem Statement and Assumptions

Let us consider the following wave equation:

$$\begin{cases} \partial_t^2 z(t,x) = \partial_x^2 z(t,x) + F(t,x), & t \in \mathbb{R}_+, x \in (0,L) \\ z(t,0) = h_1(t),\ z(t,L) = h_2(t), & t \in \mathbb{R}_+, \\ z(0,x) = z^0(x),\ \partial_t z(0,x) = z^1(x), & x \in (0,L), \end{cases} \tag{1}$$

where $F(t,x)$ is a source term that takes the shape

$$F(t,x) = f(x)R(t,x), \tag{2}$$

where f is an unknown coefficient and R is known. The boundary data h_1 and h_2 are also given, but $\left(z^0, z^1\right)$ are unknown initial data (contrary to several other approaches of inverse problems where they are usually known). The problem we wish to solve here is summarized as follows.

Problem 1: Determine sufficient conditions and define appropriate observations on the solution of the wave equation that will allow the design of an observer system leading to estimation of the unknown coefficient $f(x)$ and the solution $z(t,x)$ of the wave equation simultaneously, converging exponentially in time.

To the best of our knowledge, this problem has not been given solutions yet. The problem of finding a source coefficient is a *linear inverse problem* as the operator mapping the solution to the unknown coefficient is linear (see [35]). Previous approaches to solve initial state or source reconstruction have lead to offline estimation (back-and-forth observers), in contrast with the aim of this work. In [20], an in-domain constant uncertainty is estimated for a class of hyperbolic systems via an adaptive observer with a measurement of the state in the entire domain based on a standard approach for parameter identification in finite-dimensional systems (see also [36]), that we avoid in the present work. Such approaches or observer-based approaches as in [13] might lead to asymptotic convergence of the observer scheme to the unknown parameters, but not necessarily exponentially, as we plan here. Additionally, the problem of estimation of a spatially varying parameter, as the source term $f(x)$ here, is even more complicated.

Remark 1: *Note that the first part of Problem 1 on the coefficient estimation has been given a solution in our introductory work [25]. However, here we provide a solution to the complete problem of joint estimation of the solution and coefficient, which is more complex. At the same time, we have some weaker conditions and we present the results in a more concrete manner compared to this preliminary work, in order for the reader to obtain a complete framework on the solvability of this problem.*

The answer we will be able to give to Problem 1 will require strong assumptions such as the polynomial nature of f and some high regularity of the solution. Dropping these assumptions is still an open problem.

We first assume that the unknown coefficient f belongs to $\mathbb{R}^N[x]$, with $N \in \{0,2,4,\dots\}$, meaning:

$$f(x) = \sum_{i=0}^N f_i x^i, \quad \forall x \in [0,L], \tag{3}$$

with a priori known even degree N and completely unknown coefficients $f_i \in \mathbb{R}$. This assumption on restriction of the space of admissible coefficients f to the finite-dimensional space $\mathbb{R}^N[x]$ is a sufficient condition for the design of an infinite-dimensional observer in the subsequent section.

Consider now a given $\delta \in (0,L]$ that determines the observation interval $[0,\delta]$ in which we assume availability of the observations in the sequel. We will also make the hypothesis that the initial data localized on $[0,\delta]$ and the boundary

conditions of the system have some additional regularity than the usual one assumed for the wave equation. This will induce existence of solutions in $[0,\delta]$ of sufficiently strong regularity. More precisely, we make the following two assumptions.

Assumption 1 (Regularity): *The function R, part of the source term in* (2)*, satisfies*

$$\begin{aligned} R \in & C^{3N+3}\left(\mathbb{R}_+; H^{3N+3}(0,\delta)\right) \cap L^\infty\left(\mathbb{R}_+; H^{3N+3}(0,\delta)\right), \\ R \in & C^1\left(\mathbb{R}_+; H^1(0,L)\right) \cap L^\infty\left(\mathbb{R}_+; H^1(0,L)\right). \end{aligned}$$

Moreover, a bound $c>0$ is a priori known for which

$$\|R\|_{L^\infty(\mathbb{R}_+;H^{N+1}(0,\delta))} \leq c. \tag{4}$$

Simultaneously, R satisfies

$$\inf_{(t,x)\in\mathbb{R}_+\times[0,\delta]} |R(t,x)| > 0. \tag{5}$$

In the sequel, a positive lower bound of $|R|$ coming from (5) *is considered a priori known.*

We furthermore assume that

$$z^0 \in H^{3N+3}(0,\delta), \quad z^1 \in H^{3N+2}(0,\delta)$$

while in the rest of the domain we assume more commonly

$$z^0 \in H^1(0,L), \quad z^1 \in L^2(0,L).$$

We also suppose that h_1 and $z(\cdot,\delta)$ are of class $H^{3N+3}(\mathbb{R}_+)$ and that on the right boundary ($x=L$) we have $h_2 \in H^1(\mathbb{R}_+)$. For initial condition restricted to $[0,\delta]$, namely, for $z^0\,|_{[0,\delta]}$, $z^1\,|_{[0,\delta]}$ we finally assume that they satisfy compatibility conditions of order $3N+3$.

Assumptions on R given by (4)-(5) are stronger versions of classical assumptions in the solvability of linear inverse problems (see e.g. [18]). From Assumption 1, we may invoke well-known existence/uniqueness arguments (see for instance [28], following also a transposition method, since we have inhomogeneous boundary conditions), along with the extra regularity and, thus, we can easily deduce the unique existence of solutions

$$\begin{aligned} z \in & C^{3N+1}\left(\mathbb{R}_+; H^{3N+3}(0,\delta)\right) \\ & \cap\ C^{3N+2}\left(\mathbb{R}_+; H^{3N+2}(0,\delta)\right) \\ & \cap\ C^{3N+3}\left(\mathbb{R}_+; H^{3N+1}(0,\delta)\right), \\ & z \in C^0\left(\mathbb{R}_+, H^1(0,L)\right) \cap C^1\left(\mathbb{R}_+, L^2(0,L)\right). \end{aligned}$$

The following assumption concerns the main observations that we consider in this work to be available in the output of the system.

Assumption 2 (Observations): *As output observations, we consider several measurements of the solution of the wave equation. To start with, the solution on $[0,\delta]$:*

$$z(t,\cdot)\,|_{[0,\delta]}, \quad t\in\mathbb{R}_+. \tag{6a}$$

Given the extra regularity of the system, we also consider as output the spatial derivatives of the state (in the weak sense) of higher order on $\{x=0\}$ and $\{x=\delta\}$:

$$\partial_x^i \partial_t z(t,l), \quad i\in\{0,\dots,2N+1\},\ l=0,\delta. \tag{6b}$$

We further use the internal spatial derivatives (in the weak sense) of order up to $3N+3$ of the solution localized on $[0,\delta]$:

$$\partial_x^i z(t,\cdot)\,|_{[0,\delta]}, \quad i\in\{1,\dots,3N+3\}. \tag{6c}$$

Finally, we assume observation of the following state components on the right boundary $x=L$:

$$z(t,L), \quad \partial_t z(t,L), \quad \partial_x z(t,L), \tag{6d}$$

which are common in observation problems for the wave equation.

Notice that the appearing observations $z(t,0)$ and $z(t,L)$ in Assumption 2 correspond to $h_1(t)$ and $h_2(t)$, respectively. Note also that we avoid observations of noncausal nature, namely, time-derivatives of the solution, except for the first time derivative (6b) on the boundaries, which is natural as it is a part of the state $(z,\partial_t z)$ of the wave. Time-derivatives are not in general available as measurements in observer designs. However, the spatial derivatives of the solution in a part of the domain, which are assumed to be known, can be measured causally and one can consider them as observations in conjunction with the extra regularity of the system. Note also that to solve the inverse problems for the potential of the wave equation as in [5], we need the first spatial derivative of the solution on the boundaries as observation. However, this online estimation problem via observers appears to be more demanding, requiring stronger assumptions. Finally, notice that the observations on the right boundary $x=L$, given by (6d), are only essential for the estimation of the solution to the wave equation and not for the estimation of the source term.

A convenient and standard approach in the parameter identification to design an observer is to write (1) as a cascade system of PDEs, as we are trying to recover the trick followed in finite dimension, where the system's equations are extended by $\partial_t f = 0$, see [8]. In a first step, let us rewrite (1) as a first-order hyperbolic system as follows:

$$\begin{cases} \partial_t u_1(t,x) = -\partial_x u_1(t,x) + u_2(t,x), \\ \partial_t u_2(t,x) = \partial_x u_2(t,x) + |R(t,x)| u_3(t,x), \\ \partial_t u_3(x) = 0, \end{cases} \tag{7}$$

where

$$u_1 = z, \quad u_2 = \partial_t z + \partial_x z, \quad u_3 = f\operatorname{sgn}\left(R(0,0)\right). \tag{8}$$

Remark that $\operatorname{sgn}\left(R(0,0)\right) = \operatorname{sgn}\left(R(t,x)\right)$, for all (t,x) in $\mathbb{R}_+\times[0,\delta]$ as a result of (5) in Assumption 1. It is worth noting at this step that (7) satisfies a cascade form similarly as in the finite dimensions but up to the differential operator. For finite dimensions, such a cascade form (*observability canonical form*) allows Luenberger observer design. Using the abstract realization of (7), the diagonal part of system's operator is written as follows:

$$\begin{pmatrix} -\partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & -\partial_x^{N+1} \end{pmatrix}.$$

Its third element $-\partial_x^{N+1}$ is placed there instead of zero, contrary to what seen in (7), however in accordance with the fact that $\partial_x^{N+1} v_3 = \partial_x^{N+1} f = 0$, due to the polynomial

nature of f. This placement of the nonzero element $-\partial_x^{N+1}$ is made in order to obtain later a differential operator for the observer error equation of the shape $-I_3\partial_x^{N+1}$ in contrast with the three distinct elements appearing above. In fact, the differential operator in the third equation, which should be nonzero, is achieved to appear in the remaining two equations via an appropriate infinite-dimensional transformation, which is explicitly given in a second step as it is explained below.

Internal observability of under-observed linear cascade systems has been studied for instance in [3] or [29]. In these works, it is shown that the presence of distinct elements on the diagonal of the main differential operator creates technical obstructions coming from a problem of algebraic solvability. This observability problem reveals the difficulty that will subsequently appear in observer designs for such systems. Besides this, particular solutions to the problem of observer design of under-observed cascade nonlinear infinite-dimensional systems have been given in [23]. Here, inspired by this work, we employ an analogous strategy noting that (7) is under-observed, since we only measure the first state instead of all its three components. The goal is to write system (7) in a form where its differential operator is decomposed into a part with the same elements on its diagonal plus an operator acting on the first state only, which is the observation z. Following this change of dynamics, the problem of observer design can be solved as in [23] for other coupled systems of PDEs.

As a result of the previous reasoning, in order to write the system in an appropriate form for observer design, let us apply a lower triangular state transformation $\mathcal{T} \in \mathcal{L}(\mathscr{X})$ invertible with inverse $\mathcal{T}^{-1} \in \mathcal{L}(\mathscr{X})$, where $\mathscr{X} = H^{3N+3}(0,\delta) \times H^{3N+2}(0,\delta) \times H^{3N+2}(0,\delta)$. The transformation $\mathcal{T} : u\,|_{[0,\delta]} \mapsto \mathcal{T}u$ satisfies

$$\mathcal{T}u = \begin{pmatrix} 1 & 0 & 0 \\ -\partial_x - \partial_x^{N+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{9}$$

This type of transformation is inspired by the previous works [23], [24]. Notice that for the particular case $N = 0$, corresponding to constant f in the space variable, we have $\mathcal{T} = I_3$ implying that we keep the original variables unchanged and system is written as a first-order hyperbolic system.

Now, system (7) writes in the new state variable

$$v = \mathcal{T}u\,|_{[0,\delta]} \tag{10}$$

as a cascade system of PDEs of order $2N+2$ as follows:

$$\begin{cases} \partial_t v_1(t,x) = +\partial_x^{N+1} v_1(t,x) + v_2(t,x), \\ \partial_t v_2(t,x) = -\partial_x^{N+1} v_2(t,x) + |R(t,x)| v_3(t,x) \\ \qquad\qquad + \partial_x^2 v_1(t,x) + \partial_x^{2N+2} v_1(t,x), \\ \partial_t v_3(t,x) = -\partial_x^{N+1} v_3(t,x), \quad \text{in } (0,+\infty)\times(0,\delta), \end{cases}$$

which amounts to

$$\begin{aligned} \partial_t v(t,x) = &-\partial_x^{N+1} v(t,x) \\ &+ A\left[R(t)\right](x) v(t,x) + \mathcal{K}Cv(t)(x), \end{aligned} \tag{11}$$

where

$$A[R] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & |R| \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K} := \begin{pmatrix} 2\partial_x^{N+1} \\ \partial_x^2 + \partial_x^{2N+2} \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

System (11) is written as a cascade system of three coupled PDEs via coupling coefficient $A[R]$. As we noted earlier, differential operator associated to this system consists of a part $-I_3\partial_x^{N+1}$, which is desired for the observer convergence proof, plus an undesired operator $\mathcal{K}$ acting on the first state $v_1 = z$ only. To get rid of it in the observer error dynamics, this second part will be copied in the observer dynamics acting on the measurement $z\,|_{[0,\delta]}$.

The decomposition of the differential operator into these two parts and the subsequent elimination of $\mathcal{K}Cv$ reveals why we demanded knowledge of z and of its spatial derivatives in $[0,\delta]$ from Assumption 2. In fact, even higher-order derivatives than the ones contained in the domain of $\mathcal{K}$ are considered as observations, as the estimation needs to be proven in higher regularity norms. It is worth also noting that the coefficients f_i of the polynomial f are written as follows:

$$f_i = \frac{1}{i!\text{sgn}\left(R(0,0)\right)} \partial_x^i v_3(t,0),$$

which are our target in the estimation problem.

To complete the initial value-boundary conditions problem, we associate to system (11) the following boundary conditions, which stem from boundary conditions of (1) in conjunction with transformations (8)-(9):
$\forall j \in \{0,\dots,N\}$,

$$\partial_x^j v_1(t,0) = \partial_x^j z(t,0), \tag{13a}$$

$$\partial_x^j v_2(t,0) = \partial_t \partial_x^j z(t,0) - \partial_x^{N+1+j} z(t,0), \tag{13b}$$

$$\partial_x^j v_3(0) = \sum_{k=j}^{N} \sigma_{jk} \partial_x^k v_3(\delta), \tag{13c}$$

where

$$\sigma_{jk} := \frac{(-\delta)^{k-j}}{(k-j)!}, \ \forall j \le k \le N, \tag{14}$$

while (13c) is in view of the polynomial properties of $v_3 = f$ coming from its Taylor representation. Notice also that boundary conditions (13a)-(13b) are induced naturally from transformation (9). These boundary conditions involve mappings that have been already considered as measurements by Assumption 2.

III. Observer and main result

In this section, we present our observer of composite form consisting of an internal observer on the subinterval $[0,\delta]$ and a boundary one for its complement $[\delta, L]$. This observer provides a solution to Problem 1 of Section II.

A. Observer design

We first present the internal observer corresponding to the observed subinterval $[0,\delta]$, see Assumption 2. For system (11), written in such an appropriate form as explained previously, we are in a position to propose the following Luenberger internal observer localized in $[0,\delta]$:

$$\begin{aligned}\partial_t\hat{v}(t,x) &= -\partial_x^{N+1}\hat{v}(t,x) + A\left[R(t)\right](x)\hat{v}(t,x)\\ &\quad + \mathcal{K}z(t)(x) + \Theta P^{-1}C^\top\left(z(t,x) - C\hat{v}(t,x)\right),\\ &\qquad \forall (t,x)\in(0,+\infty)\times(0,\delta). \end{aligned} \tag{15}$$

Taking into account the stronger regularity from Assumption 1, we additionally associate to observer state $\hat{v}$ its spatial derivatives $\partial_x^i\hat{v}$ up to order $i=N+1$. Overall, the observer state $\hat{v}^{[N+1]} := \left(\hat{v},\quad \partial_x\hat{v},\quad \cdots,\quad \partial_x^{N+1}\hat{v}\right)^\top$ satisfies the following $(N+2)$ systems resulting from $(N+1)$ differentiations of (15) (incorporating also the case $i=0$, which corresponds to (15)):

$$\begin{aligned}\partial_t\partial_x^i\hat{v} &= -\partial_x^{N+1+i}\hat{v} + \sum_{k=0}^{i}\binom{i}{k}A\left[\partial_x^{i-k}R\right]\partial_x^k\hat{v} + \mathcal{K}\partial_x^i z\\ &\quad + \Theta P^{-1}C^\top\left(\partial_x^i z - C\partial_x^i\hat{v}\right),\quad i\in\{0,\dots,N+1\}. \end{aligned} \tag{16}$$

Boundary conditions for each of the states $\hat{v}_1,\hat{v}_2,\hat{v}_3$ of (16) are assumed to be of the following form:

$$\begin{aligned}
&\partial_x^j\hat{v}_1(t,0) = \partial_x^j z(t,0) + \textstyle\sum_{k=j}^N \sigma_{jk}\partial_x^k\left(\hat{v}_1(t,\delta) - z(t,\delta)\right), && \forall j\in\{i,\dots,N\},\\
&\partial_x^j\hat{v}_1(t,\delta) = \partial_x^j z(t,\delta), && \forall j\in\{N+1,\dots,N+i\};\\
&\partial_x^j\hat{v}_2(t,0) = \partial_t\partial_x^j z(t,0) - \partial_x^{N+1+j}z(t,0)\\
&+\textstyle\sum_{k=j}^N \sigma_{jk}\partial_x^k\left[\hat{v}_2(t,\delta) - \left(\partial_t - \partial_x^{N+1}\right)z(t,\delta)\right], && \forall j\in\{i,\dots,N\},\\
&\partial_x^j\hat{v}_2(t,\delta) = \partial_t\partial_x^j z(t,\delta) - \partial_x^{N+1+j}z(t,\delta), && \forall j\in\{N+1,\dots,N+i\};\\
&\partial_x^j\hat{v}_3(t,0) = \textstyle\sum_{k=j}^N \sigma_{jk}\partial_x^k\hat{v}_3(t,\delta), && \forall j\in\{i,\dots,N\},\\
&\partial_x^j\hat{v}_3(t,\delta) = 0, && \forall j\in\{N+1,\dots,N+i\}.
\end{aligned} \tag{17}$$

We can see observer equations (16)-(17) as a family of observers parametrized by i and corresponding to each spatial derivative $\partial_x^i\hat{v}$. Notice that in both (16) and (17), we injected output correction terms. The ones injected in boundary conditions (17) are needed for the elimination of the boundary terms in the Lyapunov analysis, while the one in the internal dynamics, (15) multiplied by a suitable gain, is used in the observer convergence. This gain consists of matrix Θ given by

$$\Theta := \operatorname{diag}\left\{\theta,\theta^2,\theta^3\right\} \tag{18}$$

with $\theta>0$ a tuning parameter and P a positive definite symmetric matrix satisfying for all $(t,x)\in\mathbb{R}_+\times[0,\delta]$ a Lyapunov matrix inequality of the following form:

$$\operatorname{Sym}\left(PA\left[R(t)\right](x)\right) - C^\top C \le -\eta I_3,$$

for some constant $\eta>0$. Such an inequality is always feasible for A and C given by (12) (structured as an observability canonical form of a finite-dimensional system) and under conditions (4)-(5) in Assumption 1 for function R. The reader can refer to [19] for the feasibility of inequalities of such a type in finite-dimensional observer design and the extension to the present formulation is direct. Matrix P that we choose as a solution of (24) serves as a Lyapunov matrix in the sequel. Note here the necessity of Assumption 2 on the observations as the assumed measurements (6a)-(6c) intervene in the dynamics of observer equation (16), particularly in mapping $\mathcal{K}$ and in the output correction term, while the boundary measurements (6b) intervene in observer boundary conditions (17).

Remark 2: *The determination of matrix P solving* (24) *depends only on the knowledge of the bounds of $A[R]$ coming from* (4)-(5) *in Assumption 1 without this requiring a priori knowledge of $R(t,x)$ however. In order to weaken this assumption even more and avoid a priori knowledge of these bounds, we might alternatively follow a different approach as in [7] concerning finite dimensions, where a time-varying Lyapunov matrix $P(t)$ is considered instead.*

We now note that the observer system (16)-(17) is localized in the space subinterval $[0,\delta]$ and not in the whole space $[0,L]$. This observer provides an estimate of the source coefficient f and the solution z to the wave equation in the part $[0,\delta]$ only, as it is proven in the sequel. However, since f is polynomial, it is written as

$$f(x) = \sum_{i=0}^{N}\frac{\partial_x^i v_3(0)}{i!\operatorname{sgn}\left(R(0,0)\right)}x^i,\quad x\in[0,L] \tag{19}$$

and, also, its estimation in $[0,\delta]$ is extended everywhere in $[0,L]$ by analytic continuation and is given by

$$\hat{f}(t,x) = \sum_{i=0}^{N}\frac{\partial_x^i\hat{v}_3(t,0)}{i!\operatorname{sgn}\left(R(0,0)\right)}x^i,\quad \text{in } \mathbb{R}_+\times[0,L]. \tag{20}$$

This way, we get the estimation of the source term f on the whole spatial domain.

As a final step, we propose a boundary observer of system (7) on the partition $[\delta,L]$. This boundary observer is constructed by copying standard techniques from Lyapunov analysis for hyperbolic systems (e.g. [4]). In such a way, we obtain the estimation of the yet unknown solution z to the wave equation on the whole spatial domain. The proposed boundary observer with state $\hat{\hat{u}} = \begin{pmatrix}\hat{\hat{u}}_1 & \hat{\hat{u}}_2\end{pmatrix}^\top$ satisfies the following equations in $\mathbb{R}_+\times(\delta,L)$:

$$\begin{cases}\partial_t\hat{\hat{u}}_1(t,x) = -\partial_x\hat{\hat{u}}_1(t,x) + \hat{\hat{u}}_2(t,x),\\ \partial_t\hat{\hat{u}}_2(t,x) = \partial_x\hat{\hat{u}}_2(t,x) + R(t,x)\hat{f}(t,x),\end{cases} \tag{21}$$

with the following boundary conditions:

$$\begin{aligned}\begin{pmatrix}\hat{\hat{u}}_1(t,\delta)\\ \hat{\hat{u}}_2(t,L)\end{pmatrix} &= \begin{pmatrix}\hat{v}_1(t,\delta)\\ \partial_t z(t,L) + \partial_x z(t,L)\end{pmatrix}\\ &\quad + k\begin{pmatrix}0\\ \hat{\hat{u}}_1(t,L) - z(t,L)\end{pmatrix}\end{aligned} \tag{22}$$

where $k\in\mathbb{R}$ is an appropriately chosen constant gain.

This second observer (21)-(22) of state $\hat{\hat{u}}\in H^1(\delta,L)\times L^2(\delta,L)$ can be seen as a boundary observer of the wave equation. It exploits the estimates $\hat{f}(t,x)$ and $\hat{v}_1(t,\delta)$ coming from the internal observer equations (16)-(17) and also the observations $z(t,L),\partial_t z(t,L)$, and $\partial_x(t,L)$ coming from Assumption 2. In other words, state components of the internal

observer are used as inputs for the boundary observer, while system's output (boundary observations) are utilized in the output correction term inside boundary conditions (22). Gain k can be tuned to increase the exponential decay of the observer or even to draw near a finite time observer for case $k = 0$.

This work elaborates the convergence of the composite observer, consisting of parts (16)-(17) and (21)-(22), which estimates the source component f and the solution z in the whole space domain. The overall observer state ξ is written as

$$\begin{aligned} \xi &:= (\hat{z}, \hat{f})^\top, \text{ with } \hat{z} := \begin{cases} \hat{v}_1, & x \in [0, \delta] \\ \hat{\hat{u}}_1, & x \in [\delta, L] \end{cases} \\ \text{and } \hat{f} &= \sum_{i=0}^{N} \frac{\partial_x^i \hat{v}_3(0)}{i! \text{sgn}\left(R(0,0)\right)} x^i, \quad x \in [0, L], \end{aligned} \tag{23}$$

where $\hat{v}$ satisfies (16)-(17), and $\hat{\hat{u}}$ satisfies (21)-(22).

Note that by virtue of the first boundary condition in (22) (on $x = \delta$), observer state $\hat{z}$ is continuous with respect to the space variable.

We are now in a position to state our main result, which gives a solution to Problem 1 in Section II.

Theorem 1: *Consider wave equation* (1) *with unknown source term F given by* (2) *consisting of a known function R and an unknown polynomial $f(x) = \sum_{i=0}^{N} f_i x^i$ of known even degree N. Suppose that Assumption 1 holds and that observations in Assumption 2 are available at system's output. Let $P \succ 0$ satisfying*

$$Sym\left(PA\left[R(t)\right](x)\right) - C^\top C \leq -\eta I_3, \tag{24}$$

for some $\eta > 0$. Denote also $v^0 = \left(z^0, z^1 - \partial_x^{N+1} z^0, f\right)^\top$ with (z^0, z^1) the unknown initial condition of the wave equation. Then, the following results (i) and (ii) hold about the unique solutions ξ to composite observer (23) *assuming also that initial condition $\hat{v}^0(\cdot) := \hat{v}(0, \cdot)$ of internal observer* (16)-(17) *belongs to $H^{2N+2}\left(0, \delta; \mathbb{R}^3\right)$ and satisfies compatibility conditions of order $2N + 2$ and also that initial condition $(\hat{\hat{u}}_1(0, \cdot), \hat{\hat{u}}_2(0, \cdot))$ of boundary observer* (21)-(22) *belongs to $H^1(\delta, L) \times L^2(\delta, L)$.*

(i) For gain $\theta \geq 1$, internal observer (16)-(17) *provides an estimate for the source component via $\hat{v}_3$, in the sense that there exist $\kappa, \ell > 0$ such that*

$$\begin{aligned} &\left| f_i - \frac{1}{i! sgn\left(R(0,0)\right)} \partial_x^i \hat{v}_3(t, 0) \right| \\ &\qquad \leq \ell \theta^2 e^{-\theta \kappa t} \| v^0 - \hat{v}^0 \|_{H^{N+1}(0,\delta;\mathbb{R}^3)}, \end{aligned} \tag{25}$$

for $i = 0, \ldots, N$.

(ii) For appropriate choice of boundary observer gain k small enough and choice of internal observer gain θ large enough, composite observer (23) *provides an exponential estimation of the solution z, namely, there exist $\alpha, \beta > 0$ such that*

$$\begin{aligned} \| z(t, \cdot) - \hat{z}(t, \cdot) \|_{L^2(0,L)} \leq \alpha e^{-\beta t} \Big(&\theta^2 \| v^0 - \hat{v}^0 \|_{H^{N+1}(0,\delta;\mathbb{R}^3)} \\ &+ \| u(0, \cdot) - \hat{\hat{u}}(0, \cdot) \|_{L^2(\delta, L; \mathbb{R}^2)} \Big). \end{aligned} \tag{26}$$

B. Proof of the Observer Convergence

We prove here both parts (i) and (ii) of Theorem 1.

(i) We prove the convergence of the internal observer (16)-(17). First, we easily deduce global existence and uniqueness of solutions to (16)-(17) under Assumption 1 by invoking classical arguments from the theory of PDEs (Lumers-Philipps theorem for instance). More particularly, for initial condition $\hat{v}^0 \in H^{2N+2}\left(0, \delta; \mathbb{R}^3\right)$ satisfying compatibility conditions of order $2N + 2$ there exist unique classical solutions $\hat{v}$ to (16)-(17) belonging to $C^0\left(\mathbb{R}_+; H^{2N+2}\left(0, \delta; \mathbb{R}^3\right)\right) \cap C^1\left(\mathbb{R}_+; H^{N+1}\left(0, \delta; \mathbb{R}^3\right)\right)$.

Let us define a the scaled observer error by

$$\varepsilon := \Theta^{-1}\left(\hat{v} - v\right)$$

and also define

$$E^{[N+1]} := \begin{pmatrix} \varepsilon & \partial_x \varepsilon & \cdots & \partial_x^{N+1} \varepsilon \end{pmatrix}^\top.$$

We can easily see that $E^{[N+1]}$ satisfies the following equations in $\mathbb{R}_+ \times (0, \delta)$:

$$\begin{aligned} \partial_x^i \partial_t \varepsilon = &-\partial_x^{N+1+i} \varepsilon + \theta \sum_{k=0}^{i-1} \binom{i}{k} A\left[\partial_x^{i-k} R\right] \partial_x^k \varepsilon \\ &+ \theta \left(A\left[R\right] - P^{-1} C^\top C\right) \partial_x^i \varepsilon, \end{aligned} \tag{27a}$$

and for all $i \in \{0, \ldots, N+1\}$ and boundary conditions

$$\begin{aligned} \partial_x^j \varepsilon(t, 0) =& \sum_{k=j}^{N} \sigma_{jk} \partial_x^k \varepsilon(t, \delta), \quad \forall j \in \{i, \ldots, N\}, \\ \partial_x^j \varepsilon(t, \delta) =& 0, \quad \forall j \in \{N+1, \ldots, N+i\}, \end{aligned} \tag{27b}$$

which by virtue the dynamics (27a), lead additionally to the following implicit boundary conditions:

$$\partial_x^j \varepsilon(t, 0) = 0, \quad \forall j \in \{N+1, \ldots, N+i\}. \tag{28}$$

To obtain these implicit boundary conditions (28), we first differentiate with respect to t the first equation in (27b) and we obtain

$$\partial_x^j \partial_t \varepsilon(t, 0) = \sum_{k=j}^{N} \sigma_{jk} \partial_x^k \partial_t \varepsilon(t, \delta), \quad \forall j \in \{i, \ldots, N\}.$$

After substituting (27a) in the above equation, we get

$$\begin{aligned} &- \partial_x^{N+1+j} \varepsilon(t, 0) + \theta \sum_{k=0}^{j-1} \binom{j}{k} A\left[\partial_x^{j-k} R(t)\right](0) \partial_x^k \varepsilon(t, 0) \\ &\quad + \theta \left(A\left[R(t)\right](0) - P^{-1} C^\top C\right) \partial_x^j \varepsilon(t, 0) \\ &\quad = \sum_{k=j}^{N} \sigma_{kj} \Big(- \partial_x^{N+1+k} \varepsilon(t, \delta) \\ &\quad + \theta \sum_{i=0}^{k-1} \binom{k}{i} A\left[\partial_x^{k-i} R(t)\right](\delta) \partial_x^i \varepsilon(t, \delta) \\ &\quad + \theta \left(A\left[R(t)\right](\delta) - P^{-1} C^\top C\right) \partial_x^k \varepsilon(t, \delta) \Big). \end{aligned}$$

Now, by virtue of the first equation in (27b), the second and third term of the left-hand side is cancelled out with the second and third term of the right-hand side of the above equation,

respectively. Also, by virtue of the second equation in (27b), the first term $\sum_{k=j}^{N}\sigma_{kj}\partial_x^{N+1+k}\varepsilon(t,\delta)$ in the above equation is equal to zero. As a consequence of cancelling out all these terms, we get $\partial_x^{N+1+j}\varepsilon(t,0)=0$, for all $j=0,\dots,N$, which in turn leads to implicit boundary conditions (28).

Let us now introduce a Lyapunov functional $\mathcal{V}_1 : H^{N+1}(0,\delta)\to\mathbb{R}_+$ by

$$\mathcal{V}_1[\varepsilon]=\sum_{i=0}^{N+1}\rho_i\int_0^\delta \partial_x^i\varepsilon^\top(x)P\partial_x^i\varepsilon(x)dx, \tag{29}$$

with $\rho_0=1$ and $\rho_i>0$, for i in $\{1,\dots,N+1\}$ to be chosen appropriately and matrix $P\succ 0$ satisfying (24). Taking the time derivative of $V_1(t):=\mathcal{V}_1[\varepsilon(t)], t\geq 0$ along the solutions of (27) and substituting the corresponding dynamics, we obtain

$$\begin{aligned}\dot V_1=&-\sum_{i=0}^{N+1}\rho_i\int_0^\delta\big(\partial_x^{N+1+i}\varepsilon^\top(x)P\partial_x^i\varepsilon(x)\\&\qquad+\partial_x^i\varepsilon^\top(x)P\partial_x^{N+1+i}\varepsilon(x)\big)\,dx\\&-\theta\int_0^\delta\left(E^{[N+1]}(x)\right)^\top\mathcal{M}\left[R\right](x)E^{[N+1]}(x)dx,\end{aligned} \tag{30}$$

where $\mathcal{M}\left[R\right]=\left(\mathcal{M}_{ij}\right)_{(N+2)\times(N+2)}\left[R\right]$ with

$$\begin{aligned}\mathcal{M}_{ii}\left[R\right]&=-2\rho_{i-1}\left(\mathrm{Sym}\left(PA\left[R\right]\right)-C^\top C\right),\\&\qquad\qquad\forall i\in\{1,\dots,N+2\}\\\mathcal{M}_{ij}[R]=\mathcal{M}_{ji}^\top[R]&=-\rho_{i-1}\binom{i-1}{j-1}PA\left[\partial_x^{i-j}R\right],\\&\forall i\in\{j+1,\dots,N+2\},j\in\{1,\dots,N+2\}.\end{aligned}$$

Notice that by using the Schur complement, Lyapunov equation (24), and the uniform boundedness of R in $H^{N+1}(0,\delta)$, we can always choose $\rho_i\in(0,1]$, for i in $\{1,\dots,N+1\}$ such that $\mathcal{M}$ becomes positive for all R satisfying (4)-(5) in Assumption 1. More precisely, we can choose constants $\rho_i\in(0,1]$ such that

$$\gamma:=\inf_{R\in L^\infty(\mathbb{R}_+;H^{N+1}(0,\delta))}\frac{w^\top\mathcal{M}[R]w}{|w|^2}>0,\forall w\in\mathbb{R}^{3N+6}\setminus\mathbf{0}.$$

Therefore, applying repeated integrations by parts in (30) and by the fact that N is even, we obtain

$$\dot V_1\leq\sum_{i=0}^{N+1}\rho_i\left[\Pi_i(x)\right]_0^\delta-\theta\frac{\gamma}{|P|}V_1,$$

where

$$\Pi_i(x):=\left(\partial_x^iE^{[N]}(x)\right)^\top\mathcal{P}\partial_x^iE^{[N]}(x),$$

with $\mathcal{P}:=\mathrm{antidiag}_{N+1}\left((-1)^jP\right)_{j=1,\dots,N+1}$.

Next, substituting boundary conditions of ε given by (27b) and by use of (28), we get for i in $\{0,\dots,N+1\}$

$$\begin{aligned}\left[\Pi_i(x)\right]_0^\delta&=\left(\partial_x^iE^{[N]}(\delta)\right)^\top\Omega_i\partial_x^iE^{[N]}(\delta);\\\Omega_i:=&\mathrm{antidiag}_{N+1}\left(\Xi_{ij}\right)_{j=1,\dots,N+1}\\&-\Sigma_i^\top\mathrm{antidiag}_{N+1}\left(\Xi_{ij}\right)_{j=1,\dots,N+1}\Sigma_i,\end{aligned}$$

where

$$\Xi_{ij}:=\begin{cases}\mathbf{0}_3, j=1,\dots,i\text{ and }j=N+2-i,\dots,N+1\\(-1)^jP, j=i+1,\dots,N+1-i\end{cases},$$

Σ_i is a block matrix given by

$$\Sigma_i:=(\sigma_{jk}I_3)_{j,k=i,\dots,N+i},$$

while σ_{jk} are given by (14) for $j\leq k\leq N$ and $\sigma_{jk}\equiv 0$ for $k\leq j-1\leq N$ and for $k=j\in\{N+1,\dots,2N+1\}$. After appropriate calculations, it turns out that $\Omega_i=0$, $\forall i=0,\dots,N+1$, leading eventually to

$$\dot V_1\leq-2\theta\kappa V_1;\quad\kappa:=\frac{\gamma}{2|P|}. \tag{31}$$

By use of the Grönwall Lemma, (8), (10), and (31), we obtain

$$\begin{aligned}\theta^{-1}\|\left(z(t,\cdot)-\hat v_1(t,\cdot)\right)\|_{H^{N+1}(0,\delta)}&\\+\theta^{-2}\|\partial_tz(t,\cdot)-\partial_x^{N+1}z(t,\cdot)-\hat v_2(t,\cdot)\|_{H^{N+1}(0,\delta)}&\\+\theta^{-3}\|f(\cdot)\mathrm{sgn}\left(R(0,0)\right)-\hat v_3(t,\cdot)\|_{H^{N+1}(0,\delta)}&\\\leq\gamma_1\theta^{-1}e^{-\theta\kappa t}\|v^0-\hat v^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)},&\end{aligned} \tag{32}$$

where $\gamma_1:=\frac{1}{\min\{\rho_i,i=0,\dots,N+1\}}\sqrt{3\frac{|P|}{\mathrm{eig}(P)}}$.

Stability inequality (32) indicates that $\hat v_1$ provides an estimate of $z\ |_{[0,\delta]}$ and $\hat v_3$ an estimate of the unknown f in the H^{N+1} spatial norm. We also observe that (32) has lead to a phenomenon of *loss of derivatives*. Indeed, in order to estimate the solution of the wave equation, we demand higher order of regularity $(3N+3)$ of the solution than the order of the convergent derivatives of the observer error $(N+1)$. Such phenomena appear when studying the controllability of underactuated coupled systems (see [3]). Finally, by invoking the continuous embedding $H^{N+1}(0,\delta)\hookrightarrow C^N([0,\delta])$ (see [1] on Sobolev embeddings and Morrey's inequality), we get by (32)

$$\begin{aligned}\|z(t,\cdot)-\hat v_1(t,\cdot)\|_{C^N([0,\delta])}&\\\leq c_0\gamma_1e^{-\theta\kappa t}\|v^0-\hat v^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)},&\end{aligned} \tag{33}$$

$$\begin{aligned}\|f(\cdot)\mathrm{sgn}\left(R(0,0)\right)-\hat v_3(t,\cdot)\|_{C^N([0,\delta])}&\\\leq c_0\gamma_1\theta^2e^{-\theta\kappa t}\|v^0-\hat v^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)},&\end{aligned} \tag{34}$$

with $c_0>0$ an embedding constant.

The latter shows that the coefficients of polynomial $f(\cdot)$, given by $f_i=\frac{f^{(i)}(0)}{i!}$, are approximated by $\frac{1}{i!\mathrm{sgn}(R(0,0))}\partial_x^i\hat v_3(t,0)$ for all i in $\{0,\dots,N\}$ in an exponentially fast manner. By use of (8)-(9), we deduce (25) with $\ell=c_0\gamma_1$.

(ii) Focusing now on the spatial subdomain $[\delta,L]$, we define observer error by

$$\tilde u=\begin{pmatrix}\hat u_1-u_1\\\hat u_2-u_2\end{pmatrix},\quad x\in[\delta,L].$$

For systems of conservation laws, a method for estimation has been developed in [12] and we adopt similar techniques here. The error $\tilde u$ satisfies the following equations in $\mathbb{R}_+\times(\delta,L)$:

$$\begin{aligned}\partial_t\tilde u(t,x)=&\begin{pmatrix}-1&0\\0&1\end{pmatrix}\partial_x\tilde u(t,x)+\begin{pmatrix}0&1\\0&0\end{pmatrix}\tilde u(t,x)\\&+R(t,x)\begin{pmatrix}0\\1\end{pmatrix}\left(\hat f(t,x)-f(x)\right),\end{aligned} \tag{35}$$

with boundary conditions

$$\begin{aligned}\tilde{u}_1(t,\delta) =&\theta\varepsilon_1(t,\delta),\\ \tilde{u}_2(t,L) =&k\tilde{u}_1(t,L).\end{aligned} \tag{36}$$

Introduce a Lyapunov functional $\mathcal{V}_2 : H^1(\delta, L) \to \mathbb{R}_+$ as

$$\mathcal{V}_2[\tilde{u}] = \frac{1}{2}\int_\delta^L \tilde{u}^\top(x)\text{diag}\{e^{-\mu x}, e^{\mu x}\}\tilde{u}(x)dx, \tag{37}$$

with $\mu > 0$ to be chosen appropriately. Taking the time derivative of $V_2(t) := \mathcal{V}_2[\tilde{u}(t)]$, along the dynamics (35) for $t \geq 0$, one obtains

$$\begin{aligned}\dot{V}_2 =&\int_\delta^L \tilde{u}^\top(x)\text{diag}\{-e^{-\mu x}, e^{\mu x}\}\partial_x\tilde{u}(x)dx\\ &+\frac{1}{2}\int_\delta^L e^{-\mu x}\tilde{u}^\top(x)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\tilde{u}(x)dx\\ &+\int_\delta^L e^{\mu x}R(t,x)\tilde{f}(x)\tilde{u}_2(x)dx,\\ =&I_1 + I_2 + I_3,\end{aligned} \tag{38}$$

where $\tilde{f} := \hat{f} - f$ and each of I_1, I_2, and I_3 corresponds to each of the three integral terms in (38). The first integral term can be rewritten after applying integration by parts as

$$I_1 = [G(x)]_\delta^L - \frac{\mu}{2}\int_\delta^L \tilde{u}^\top(x)\text{diag}\{e^{-\mu x}, e^{\mu x}\}\tilde{u}(x)dx, \tag{39}$$

where $G(x) := \frac{1}{2}\tilde{u}^\top(x)\text{diag}\{-e^{-\mu x}, e^{\mu x}\}\tilde{u}(x)$.
The boundary term $[G(x)]_\delta^L$ is written after using boundary conditions (36) as follows:

$$[G(x)]_\delta^L = \tilde{u}_1^2(L)\left(k^2e^{\mu L} - e^{-\mu L}\right) + \theta^2e^{-\mu\delta}\varepsilon_1^2(\delta) - \tilde{u}_2^2(\delta)e^{\mu\delta}.$$

Then, for choice of gain parameter k small enough, namely,

$$|k| \leq e^{-\mu L},$$

we obtain

$$[G(x)]_\delta^L \leq \theta^2e^{-\mu\delta}\varepsilon_1^2(\delta). \tag{40}$$

Considering second integral term, it is easy to see that it can be bounded as follows:

$$I_2 \leq \frac{e^{-\mu\delta}}{2}\int_\delta^L \tilde{u}^\top(x)\begin{pmatrix}e^{-\mu x} & 0\\ 0 & e^{\mu x}\end{pmatrix}\tilde{u}(x)dx = e^{-\mu\delta}V_2. \tag{41}$$

Moreover, the third term of the time-derivative (38) can be bounded as a result of Assumption 1 on function R, the Cauchy-Schwartz and Young's inequalities, and (37) as

$$\begin{aligned}I_3 =&\int_\delta^L e^{\mu x}R(t,x)\tilde{f}(x)\tilde{u}_2(x)dx\\ &\leq \bar{c}e^{\frac{\mu}{2}L}\|\tilde{f}\|_{L^2(\delta,L)}\left(\int_\delta^L e^{\mu x}\tilde{u}_2^2(x)dx\right)^{1/2}\\ &\leq \frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\|\tilde{f}\|_{L^2(\delta,L)}^2 + \frac{\epsilon_0}{2}\int_\delta^L e^{\mu x}\tilde{u}_2^2(x)dx\\ &\leq \frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\|\tilde{f}\|_{L^2(\delta,L)}^2 + \epsilon_0V_2,\end{aligned} \tag{42}$$

for some ϵ_0 chosen in $(0, 1)$ and

$$\bar{c} := \sup_{(t,x)\in\mathbb{R}_+\times[\delta,L]}|R(t,x)|,$$

which makes sense from Assumption 1.

Overall, $\dot{V}_2$ is bounded by use of (38)-(42) as follows:

$$\dot{V}_2 \leq -\left(\mu - e^{-\mu\delta} - \epsilon_0\right)V_2 + \Omega(t),$$

with

$$\Omega(t) := \frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\|\tilde{f}\|_{L^2(\delta,L)}^2 + \theta^2e^{-\mu\delta}\varepsilon_1^2(t,\delta).$$

By choosing parameter μ large enough such that

$$\mu - e^{-\mu\delta} \geq \epsilon_0,$$

we obtain after applying the Grönwall Lemma

$$V_2(t) \leq e^{-\mu_0 t}V_2(0) + \int_0^t e^{-\mu_0(t-s)}\Omega(s)ds, \tag{43}$$

for some $0 < \mu_0 < \mu - e^{-\mu\delta} - \epsilon_0$. We now notice that the difference $\|\tilde{f}\|_{L^2(\delta,L)}$ is bounded as in (32), resulting from (19)-(20), whereas the error of the internal observer $\varepsilon_1(t,\delta)$ is bounded, according to (33), as follows:

$$\begin{aligned}|\varepsilon_1(t,\delta)| \leq&\|\varepsilon_1(t,\cdot)\|_{C^0[0,\delta]}\\ &\leq c_0\gamma_1\theta^{-1}e^{-\theta\kappa t}\|v^0 - \hat{v}^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)}.\end{aligned}$$

By virtue of these bounds, we can bound $\Omega(s)$ for all $s \geq 0$ as follows:

$$\Omega(s) \leq \left(\frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\theta^4 + c_0^2\right)\gamma_1^2e^{-2\theta\kappa s}\|v^0 - \hat{v}^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)}^2. \tag{44}$$

From there, and choosing the gain θ large enough, namely,

$$\theta > \frac{\mu_0}{2\kappa},$$

(43) brings

$$\begin{aligned}V_2(t) \leq&e^{-\mu_0 t}V_2(0) + \left(\frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\theta^4 + c_0^2\right)\gamma_1^2\\ &\times\frac{e^{-2\theta\kappa t} - e^{-\mu_0 t}}{\mu_0 - 2\theta\kappa}\|v^0 - \hat{v}^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)}^2.\end{aligned}$$

Therefore, by use of (37) and trivial inequalities we get

$$\begin{aligned}\|\tilde{u}(t,\cdot)\|_{L^2(\delta,L;\mathbb{R}^2)} \leq& e^{\mu L}e^{-\frac{\mu_0}{2}t}\|u(0,\cdot) - \hat{u}(0,\cdot)\|_{L^2(\delta,L;\mathbb{R}^2)}\\ &+\sqrt{\frac{\frac{1}{2\epsilon_0}\bar{c}e^{\mu L}\theta^4 + c_0^2}{2\theta\kappa - \mu_0}}\gamma_1e^{\frac{\mu}{2}L}e^{-\frac{\mu_0}{2}t}\|v^0 - \hat{v}^0\|_{H^{N+1}(0,\delta;\mathbb{R}^3)}.\end{aligned} \tag{45}$$

The above corresponds to the exponential estimation of the solution $u_1 = z$ on the subinterval $[\delta, L]$.

Finally, combining (45) (for the part of the solution on $[\delta, L]$, proven here) with (32) (for the part $[0, \delta]$ proven in the part (i)), we deduce (26).

The proof is complete.

Remark 3: *Composite observer* (23) *provides an exponentially fast estimation of the source coefficient f and the solution z. For the coefficient f, the estimation can be arbitrarily fast by appropriate choice of the parameter θ in the dynamics*

of the internal observer (16)-(22) *as seen by stability inequality* (25)*. However, for the solution z to the wave equation, whose estimation relies on boundary observer* (21)-(22) *in (δ, L), its estimation on the whole domain cannot become arbitrarily fast and it is limited by the wave propagation. In* (45)*, we see that the maximal convergence rate (finite-time convergence achieved for observer gain $k = 0$) is independent of the choice of observer gain θ and, thus, cannot be tuned. Finally, we see that the choice of the right endpoint δ of the observation interval $[0, \delta]$ plays a role in the estimation of the coefficient, as seen by* (31)*, where convergence rate coefficient κ depends on γ, which in turn depends on δ, noting that the further $R(t, x)$ is away from zero on $[0, \delta]$, the larger κ is.*

Remark 4: *We designed the first internal observer with a structure resembling to high-gain observers for finite dimensions (see* (16)-(17) *and the form of observer gain matrix* Θ (18)*). High-gain observer designs have been extended to infinite dimensions (see for instance [23], [24]) aiming at tackling the presence of nonlinearities and unknown perturbations in system dynamics. In Section III, this design could be seemingly replaced by a standard Luenberger observer design. However, we use this high-gain design that would be indispensable to deal with possible perturbations appearing in the error dynamics if we intended to prove an Input-to-State (ISS) estimate with respect to a generic (not essentially polynomial) source coefficient. Additionally, the high-gain observer structure that we introduced in this work could be used to deal with an alternative setting where the wave equation was semilinear with Lipschitz nonlinearities of the form $h(z, \partial_t z)$. These topics will be considered in future works.*

Remark 5: *We notice that by* (32)*, additionally to the solution z, its time-derivative $\partial_t z$ is also estimated in $[0, \delta]$, whereas in the subinterval $[\delta, L]$ such an estimation is not obtained, because of the use of a weighted L^2 spatial norm (instead of an H^1 one) as a Lyapunov functional* (37)*.*

IV. Extension to a nonlinear inverse problem

In this section, we discuss the solvability of a closely related inverse problem for the wave equation via exponential observers for the case of an unknown parameter multiplying the state. Solutions to such a nonlinear inverse problem [26] involve the recovery of the potential of the wave equation (see e.g. [35], [6],).

Consider again wave equation (1). In the present formulation, term F is no longer considered a source term but it is assumed to be state-dependent and be written as follows:

$$F(t, x) = -q(x)z(t, x), \tag{46}$$

where q represents the unknown parameter, called potential, to be estimated.

We wish here to solve the first part of Problem 1 with unknown source f substituted by potential q. We provide a solution to this problem concerning the exponential estimation of the potential, but not yet the simultaneous estimation of the solution to the wave equation. Intuition coming from the solvability of inverse problems (see [26]) suggests that this nonlinear inverse problem would require stronger assumptions on the solution to the wave equation than the ones in Section II. The result of this section is a consequence of the analysis presented in [25] concerning the recovery of the potential. To deal with this problem, we replace the conditions on $R(t, x)$ in Section II with similar conditions on $z(t, x)$. Compared to our preliminary work [25] on the coefficient estimation, the conditions here are slightly weaker. The proof is not given in full details as it is a direct adaptation of the proof of Theorem 1. We keep here the same assumptions on the polynomial nature of the unknown parameter q as in Section II and the same observation of the solution on a part of the domain $[0, \delta]$ (avoiding nevertheless the observations on $x = L$ given in (6d), which are only needed for the estimation of the solution). Note that the case of polynomial-type potential is not uncommon in physical realizations, especially in quantum mechanics, see for instance [16], where a use of an even polynomial as a potential is justified in the Schrödinger equation.

Let us state now the hypothesis we need to make.

Assumption 3: *We assume that*

$$z^0 \in H^{3N+3}(0, \delta), \quad z^1 \in H^{3N+2}(0, \delta)$$

while in the rest of the domain we assume more commonly

$$z^0 \in H^1(0, L), \quad z^1 \in L^2(0, L).$$

We also suppose that h_1 and $z(\cdot, \delta) \in H^{3N+3}(\mathbb{R}_+)$ and that $h_2 \in H^1(\mathbb{R}_+)$. For initial condition restricted to $[0, \delta]$, namely, for $z^0 \,|_{[0,\delta]}, z^1 \,|_{[0,\delta]}$ we assume that they satisfy compatibility conditions of order $3N + 3$ with h_1.

Furthermore, we suppose that the solution to the wave equation is uniformly bounded on $[0, \delta]$ as follows:

$$z \in L^\infty\left(\mathbb{R}_+; H^{N+1}(0, \delta)\right) \tag{47}$$

and that it also holds

$$\inf_{(t,x)\in\mathbb{R}_+\times[0,\delta]} |z(t, x)| > 0 \tag{48}$$

implying simultaneously the compatibility with z^0.

Some positive upper and lower bounds of $|z|$ coming from (47)-(48) *are considered a priori known.*

The nature of assumptions in (47)-(48) are revealed in classical works on the solvability of inverse problems for the potential of the wave equation, where some slightly weaker conditions are assumed (see [5], [6]), i.e. that only the infimum of a norm of the initial condition is strictly positive.

Next, consider $P \succ 0$ satisfying for all $(t, x) \in \mathbb{R}_+ \times [0, \delta]$

$$\mathrm{Sym}\left(PA\left[z(t)\right](x)\right) - C^\top C \leq -\eta I_3 \tag{49}$$

for some $\eta > 0$, with $A[z]$ given by (12). Similarly, as in the reasoning of Section III, the above inequality is feasible for z satisfying (47)-(48).

Now, we propose an internal observer to estimate the unknown potential q satisfying the following equation in $\mathbb{R}_+ \times (0, \delta)$, for all $i \in \{0, \dots, N + 1\}$,

$$\begin{aligned}\partial_t \partial_x^i \hat{v} = -\partial_x^{N+1+i}\hat{v} + \sum_{k=0}^{i} \binom{i}{k} A\left[\partial_x^{i-k} z\right] \partial_x^k \hat{v} + \mathcal{K}\partial_x^i z \\ + \Theta P^{-1} C^\top \left(\partial_x^i z - C\partial_x^i \hat{v}\right)\end{aligned} \tag{50}$$

with Θ given by (18), C and $\mathcal{K}$ as in (12). We also associate to this observer the same boundary conditions as in (17).

The result of this section is stated as follows.

Theorem 2: *Consider wave equation* (1) *where F is given by* (46) *with potential $q(x) = \sum_{i=0}^{N} q_i x^i$ an unknown polynomial of known even degree N. Assume that Assumption 3 holds and that observations in Assumption 2 are available in system's output. Let $P \succ 0$ satisfy* (49) *for some $\eta > 0$. Denote also $v^0 = (z^0, z^1 - \partial_x^{N+1} z^0, q)^\top$, where (z^0, z^1) is the unknown initial condition of the wave equation. Then, the following result holds about the unique solution $\hat{v}$ to internal observer* (50)-(17) *with initial condition $\hat{v}^0(\cdot) := \hat{v}(0,\cdot)$ belonging to $H^{2N+2}\left(0,\delta;\mathbb{R}^3\right)$ and satisfying compatibility conditions of order $2N+2$.*

For gain $\theta \geq 1$, $\hat{v}$ provides an estimate for the potential via $\hat{v}_3$, in the sense that there exist $\kappa, \ell > 0$ such that for $i = 0, \ldots, N$,

$$\left| q_i - \frac{1}{i! sgn\left(z^0(0)\right)} \partial_x^i \hat{v}_3(t,0) \right| \leq \ell \theta^2 e^{-\theta\kappa t} \| v^0 - \hat{v}^0 \|_{H^{N+1}(0,\delta;\mathbb{R}^3)}.$$

We present here a sketch of the proof of Theorem 2.

Proof: The proof follows the steps of the one given in [25]. The objective is to adopt the same methodology as in the part (i) of the proof in Section III-B, except for the fact that z substitutes R in the observer equations. First, notice that under Assumption 3, similarly as in the reasoning in the proof of Theorem 1 (see Section III-B), existence/uniqueness of solutions to the wave and to the observer equations are guaranteed for all times $t \geq 0$. These solutions are regular enough as in the proof of Theorem 1. Second, for the stability analysis, we need to transform the wave equation into a first-order hyperbolic system, similarly as in Section II by using the transformation

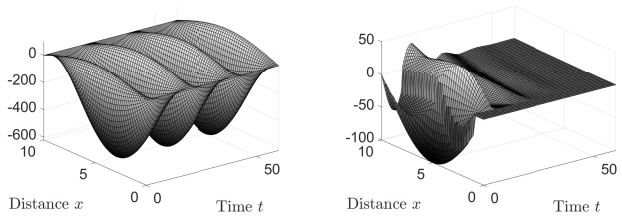
$$u_1 = z, \qquad u_2 = \partial_t z + \partial_x z, \qquad u_3 = -q \text{sgn}\left(z^0(0)\right),$$

noting that as a consequence of (48) in Assumption 3, we have $\text{sgn}\left(z(t,x)\right) = \text{sgn}\left(z^0(0)\right)$, for all (t,x) in $\mathbb{R}_+ \times [0,\delta]$. Then, system writes exactly as (7) and by applying the same transformation as in Section II given by (10), system amounts to

$$\begin{aligned} \partial_t v(t,x) =& -\partial_x^{N+1} v(t,x) \\ &+ A\left[v_1(t)\right](x) v(t,x) + \mathcal{K} C v(t)(x), \end{aligned} \tag{51}$$

with boundary conditions given again by (13). Notice that the above transformed cascade system of PDEs is semilinear in contrast with (11), which was linear. Then, the stability analysis is similar to the one in the proof of part (i) of Theorem 1 in Section III-B noting that we ought to substitute $R(t,x)$ by $z(t,x)$ taking also into account sufficient conditions in Assumption 3. □

Remark 6: In Theorem 2, we presented a result on the estimation of the potential of the wave equation via an internal observer, similarly as in the part (i) of Theorem 1 in Section III on the estimation of the source coefficient. However, we did not give a complete answer to Problem 1, since we did not provide a simultaneous estimation for the solution z to the wave equation as in the second part (ii) of Theorem 1. It seems that a similar global estimation result cannot hold for this case and only a local result might be proven, as this is limited by the nonlinear nature of this problem.



(a) Solution of system (1). (b) Error on the solution.

Fig. 1: Simulations on the whole interval $[0, L]$.

V. Simulation

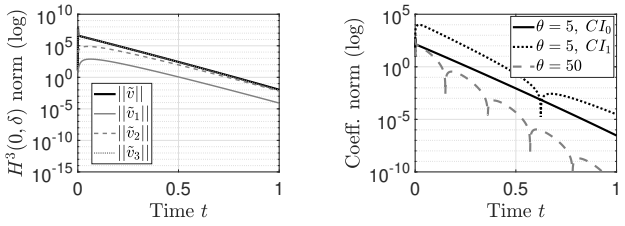
In this section, the composite observer designed in this paper for the joint estimation of the source coefficient and the solution in Section III is set up numerically for an example of polynomial of order $N = 2$. To avoid numerical issues, consistent and unconditionally stable schemes are chosen. For the first observer, which is an Airy's type equation, the Crank Nicholson right-winded numerical schema $\mathcal{D}^+\mathcal{D}^+\mathcal{D}^-$ is selected. For the second observer, which is constituted of two transport equations, state $\hat{u}_1$, $\hat{u}_2$ are respectively discretized with $\mathcal{D}^-$ and $\mathcal{D}^+$. For more details in that direction, one can refer to [14] which deals with Courant-Friedrich-Lewy conditions for several "finite-difference θ-schemes" dedicated to hyperbolic systems of type $(\partial_t + \partial_x^{2p+1})u = 0$, according to the parity of p.

Consider system (1) on the interval $[0, 10]$ with source term $F(t,x) = x(x-10)$, homogeneous boundary conditions $h_1 = h_2 = 0$ and compatible initial condition $\begin{pmatrix} z^0(x) \\ z^1(x) \end{pmatrix} = \begin{pmatrix} 100 sin(\frac{\pi x}{10}) \\ 0 \end{pmatrix}$. On Fig. 1a, the corresponding solution is drawn. The main goal is to reconstruct the three polynomial coefficients $\{1, -10, 0\}$ as well as the solution. For that, measurements as the ones in Assumption 2 are available only on $[0, \delta = 1]$ and on the right boundary at $L = 10$. Thanks to the composite observer (23), the task can be realized and Fig. 1b represents the observer errors : both internal and boundary observers converge.

It is also worth noticing that the convergence is faster on $[0,1]$ than the one on $[1,10]$. This is always possible since the decay rate of the internal observer $\theta\kappa$ can be larger than the decay rate of the second observer β.

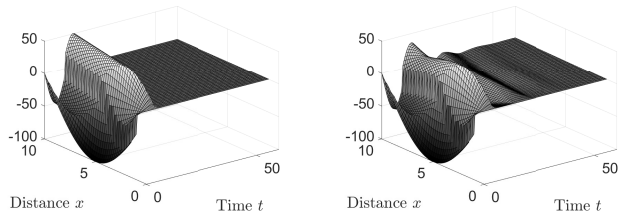
Let us now present more details on the behavior of each observer separately. By selecting our simple example, the main idea is also to highlight the relevance of each gain parameter and to understand their influence on the reconstruction process.

For the internal observer on $[0,1]$, observer error system (27) is considered. Error components gather the estimate of the state of the wave equation $\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \hat{z} - z \\ \partial_t \hat{z} - \partial_x^3 \hat{z} - \partial_t z + \partial_x^3 z \end{pmatrix}|_{[0,1]}$ and the estimate of the source term $\tilde{v}_3 = \hat{v}_3 - v_3$. On Fig. 2a, the H^3 norm of $\tilde{v}$ with respect to the time is depicted and it converges exponentially to



(a) State error for $\theta = 5$. (b) Coefficient error.

Fig. 2: First observer on $[0, \delta]$.



(a) $k = 0$. (b) $k = 0.1$.

Fig. 3: Second observer on $[\delta, L]$.

zero, as demonstrated in (32). An underlying result is the exponential convergence of each polynomial coefficient $\frac{\hat{v}_3^{(i)}(0)}{i!}$ for $i \in \{0, 1, 2\}$ towards the expected ones.

Then, on Fig. 2b the norm of $\sum_{i=0}^{2} \frac{\hat{v}_3^{(i)}(0)}{i!}$ is drawn in log scale and illustrates the statement (i) in Theorem 1. By increasing the control gain θ (see dashed curve), one also confirms that the decay rate increases. Moreover, as an exponential observer, it converges with the same convergence rate independently of initial condition (see dotted curve). Thanks to these coefficient estimations and (19)-(20), the source term can finally be reconstructed on the whole domain.

For the boundary observer on $[1, 10]$, the solution $\hat{\hat{u}}_1 - u_1 \,|_{[1,10]}$ of the difference between (21) and (7) is represented on Fig. 1b with $k = 0.1$. Once again, the state converges exponentially to zero, as mentioned in Theorem 1 (ii). Nevertheless, it is interesting to note that the dynamics of the observer are enforced by the dynamics of the system itself ($\Delta t > 20$), which leads to much slower convergence than the previous internal observer. Lastly, the role played by the observer gain can be interpreted. On Fig. 3, the estimation seems to be faster when control gain $|k|$ decreases, probably at the price of robustness with respect to parameter uncertainties.

Simulations have been also done on the nonlinear inverse problem of the estimation of the potential (see Section IV) and can be found in [25].

VI. Conclusion

The problem of simultaneous estimation of an unknown polynomial coefficient and the solution of the 1D wave equation was solved via a composite exponential observer. The first internal observer is performed in an observed subinterval of the domain and provides an estimate of the source coefficient and of a localized part of the solution. At the same time, in combination with a boundary observer, an estimation of the wave solution is obtained on the rest of the spatial domain. For this purpose, after augmenting the state with the unknown coefficient, the wave equation was written as a cascade system of PDEs of higher order. The measurement included the solution of the wave equation and its spatial derivatives on a subinterval of the domain. Some extensions were finally given on the estimation of a potential (nonlinear inverse problem).

In our future works, we will examine an alternative strategy on observer design for nonlinear inverse problems inspired by a finite-dimensional approach for nonlinear systems appearing in [33], since solutions for such problems were only partially established in the present work. Future research will also include the extension of this approach to 2-dimensional domains and some links of such an observer design approach with the internal observability of cascade systems.

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