



HAL
open science

MINIMALITY AND UNIQUE ERGODICITY OF VEECH 1969 TYPE INTERVAL EXCHANGE TRANSFORMATIONS

Sébastien Ferenczi, Pascal Hubert

► **To cite this version:**

Sébastien Ferenczi, Pascal Hubert. MINIMALITY AND UNIQUE ERGODICITY OF VEECH 1969 TYPE INTERVAL EXCHANGE TRANSFORMATIONS. 2021. hal-03446722

HAL Id: hal-03446722

<https://hal.science/hal-03446722>

Preprint submitted on 24 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MINIMALITY AND UNIQUE ERGODICITY OF VEECH 1969 TYPE INTERVAL EXCHANGE TRANSFORMATIONS

SÉBASTIEN FERENCZI AND PASCAL HUBERT

ABSTRACT. We give conditions for minimality of $\mathbb{Z}/N\mathbb{Z}$ extensions of a rotation of angle α with one marked point, solving the problem for any prime N : for $N = 2$, these correspond to the Veech 1969 examples, for which a necessary and sufficient condition was not known yet. We provide also a word combinatorial criterion of minimality valid for general interval exchange transformations, which applies to $\mathbb{Z}/N\mathbb{Z}$ extensions of any interval exchange transformation with any number of marked points. Then we give a condition for unique ergodicity of these extensions when the initial interval exchange transformation is linearly recurrent and there are one or two marked points.

In a famous paper of 1969 [16], much ahead of its time, W.A. Veech defines an extension of a rotation of angle α by the group $\mathbb{Z}/2\mathbb{Z}$ by marking a point β and going from one copy of the torus to the other one on the interval $[0, \beta[$ (resp. $[\beta, 1[$ on a variant): for α with unbounded partial quotients and some values of β , this gives the first ever examples of minimal non uniquely ergodic interval exchange transformations. These systems were defined again independently, by E.A. Sataev in 1975, in a beautiful but not very well known paper [13], in a more general context by taking q marked points and an extension of the torus by the symmetric group on $q + 1$ elements; this gives minimal interval exchange transformations with a prescribed number of ergodic invariant measures. The cases of extensions of rotations by $\mathbb{Z}/N\mathbb{Z}$ was further studied by Veech [17], then by M. Stewart [15] for one marked point, and by K.D. Merrill [12] for any number of marked points. A geometrical model of Veech 1969 was given later by H. Masur and J. Smillie [10], where the transformation appears as a first return map of a directional flow on a surface made with two tori glued along one edge.

In addition to the spectacular result mentioned above, Veech, followed by Stewart and Merrill, proved several deep results on minimality and unique ergodicity of these transformations. Their proofs are generally arithmetic and based on a very clever study of trajectories of rotations based on the Ostrowski expansion. In the present paper, we aim to improve some of these results by replacing arithmetic arguments by geometrical or word combinatorial ones. We shall consider two levels of generalization, the *Veech N -examples*, which are extensions of rotations with the same step function f taking the two values 0 and 1, but by any group $\mathbb{Z}/N\mathbb{Z}$ and not only $\mathbb{Z}/2\mathbb{Z}$, and the *Veech 1969 type extensions*, where we extend any interval exchange transformation by $\mathbb{Z}/N\mathbb{Z}$ with any step function f .

We study first the minimality of the original Veech 1969: it is proved in [16] that if β is not in $\mathbb{Z}(\alpha)$, the skew product transformation T_f is minimal, but for the remaining cases, Veech could prove only (p. 6 of [16]) that if α and β are irrational, at least one of the two transformations defined by α and β (taking into account the variant described above) is minimal. As far as we know, this result has not been improved in the last fifty years; we can now give a necessary and sufficient condition for minimality, for Veech N -examples for any prime N , which implies Veech's partial

Date: September 1, 2021.

2010 Mathematics Subject Classification. Primary 37E05; Secondary 37A05, 37B10, 37E30, 37E35, 37F34.

results for $N = 2$, see Theorem 1 below. We give also a word combinatorial criterion of minimality for general interval exchange transformations (Theorem 5 below), namely the connectedness of the *Rauzy graph* of words of length $M + 1$, where M is the maximal length of a primitive connection: this becomes interesting for those not satisfying M. Keane's i.d.o.c. condition, such as the Veech 1969 type extensions; for these we can thus give a general criterion of minimality (Theorem 10 below), by a reasoning which can be considered as a word combinatorial version of K. Schmidt's theory of *essential values for cocycles* [14]. Our criterion is then made completely explicit on examples of extensions of rotations with step functions taking two or three values.

As for unique ergodicity, we focus on the case where α has bounded partial quotients. Then, in contrast with the more famous results, we could expect that minimality implies unique ergodicity. For Veech 1969, this was proved to hold in [16] when β is not in $\mathbb{Z}(\alpha)$. Then Merrill [12] proved it for extensions by $\mathbb{Z}/N\mathbb{Z}$ and f with two or three values (under a mild extra condition), but unexpectedly found counter-examples when f has four values, see Section 3.1 below. We can show, again by very different methods, that if we extend a general *linearly recurrent* interval exchange transformation (which generalizes rotations with bounded partial quotients), in the same way with a step function f taking three values (under an additional condition on these) or two values, minimality implies unique ergodicity (Theorems 17 and 21 below). These results are proven by geometric methods and use different versions of Masur's criterion.

1. DEFINITIONS

1.1. Interval exchanges. Our intervals are always left closed, right open, as $[a, b[$.

Definition 1. An r -interval exchange transformation $T(\lambda, \pi)$ with vector $(\lambda_1, \lambda_2, \dots, \lambda_r)$, and permutation π is defined on $[0, \lambda_1 + \dots + \lambda_r = 1[$ by

$$Tx = x + \sum_{\pi^{-1}(j) < \pi^{-1}(i)} \lambda_j - \sum_{j < i} \lambda_j.$$

when x is in the interval

$$\left[\sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \right].$$

For $1 \leq i \leq r - 1$, we call the i -th discontinuity of T the point $\gamma_i = \sum_{j \leq i} \lambda_j$, and the i -th discontinuity of T^{-1} the point $\beta_i = \sum_{\pi(j) \leq \pi(i)} \lambda_j$, while γ_i is the i -th discontinuity of T , namely $\gamma_i = \sum_{j \leq i} \lambda_j$. Then I_i is the interval $[\gamma_{i-1}, \gamma_i[$ if $2 \leq i \leq r - 1$, while $I_1 = [0, \gamma_1[$ and $I_r = [\gamma_{r-1}, 1[$.

Warning: roughly half the texts on interval exchange transformations re-order the subintervals by π^{-1} ; the present definition corresponds to the following ordering of the TI_i : from left to right, $TI_{\pi(1)}, \dots, TI_{\pi(r)}$. In particular, Veech in [18] orders them by π^{-1} , which accounts for its different definition of the following permutation, denoted by σ in that paper. Note also that the point we call discontinuities may not be actual ones, as for some permutations and letters TI_{i+1} may be adjacent to TI_i from the right.

Definition 2. Let $T = T(\lambda, \pi)$, the permutation ξ (σ from [18]) is defined by

- $\xi(j) = \pi(1) - 1$ for $j = 0$,
- $\xi(j) = r$ for $j = \pi(r)$,
- $\xi(j) = \pi(\pi^{-1}(j) + 1) - 1$ for other j .

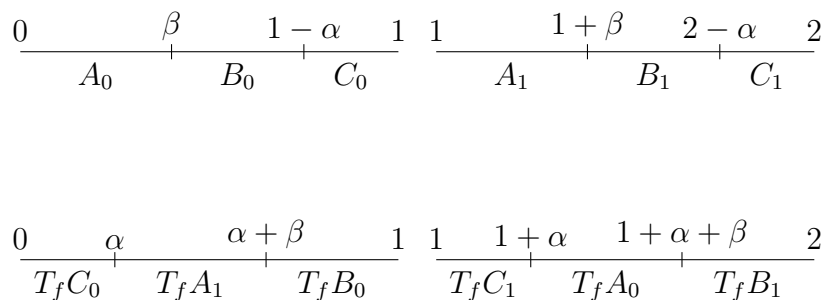


FIGURE 1. Veech 1969

1.2. **Veech 1969.** The famous Veech 1969 example mentioned in the introduction is defined in [16] (in a slightly different terminology) as a two-point extension of the rotation of angle α on the torus. Namely

Definition 3. *The Veech 1969 system is defined, if $Rx = x + \alpha$ modulo 1, by $T_f(x, s) = (Rx, s + f(x))$, $s \in \mathbb{Z}/2\mathbb{Z}$, where*

- $f(x) = 1$ if x is in the interval $[0, \beta]$,
- $f(x) = 0$ if x is in the interval $[\beta, 1]$.

The variant T_{1-f} of Veech 1969 is defined in the same way by replacing f with $1 - f$.

We can identify $[0, 1[\times\{s\}$ with $[s, s + 1[$; then T is also an interval exchange transformation as in Figure 1. Instead of numbering the intervals from 1 to 6, we call them A_0 to C_0 , A_1 to C_1 , as will be used in Theorem 11 below. Note that what we write as a 6-interval exchange transformation could also be written as a 5-interval exchange transformation, by merging the intervals C_0 and A_1 .

We define now the most generalized Veech examples we shall consider in the present paper.

Definition 4. *Let $N \geq 2$ be an integer. Let T be an interval exchange transformation on the interval $I = [0, 1[$, f a piecewise constant function from I to $\mathbb{Z}/N\mathbb{Z}$ defined by*

$$f(x) = \left\{ \begin{array}{ll} a_1 & \text{if } x \in [0, \zeta_1[, \\ a_i & \text{if } x \in [\zeta_{i-1}, \zeta_i[, \quad 1 \leq i \leq q, \\ a_{q+1} & \text{if } x \in [\zeta_q, 1] \end{array} \right\}.$$

where $0 < \zeta_1 < \dots < \zeta_q < 1$ and $a_1, \dots, a_{q+1} \in \mathbb{Z}$. The Veech 1969 type extension of T by f is

$$T_f : \begin{array}{ccc} I \times \mathbb{Z}/N\mathbb{Z} & \rightarrow & I \times \mathbb{Z}/N\mathbb{Z} \\ (x, s) & \mapsto & (Tx, s + f(x)) \end{array}$$

For general dynamical systems, we recall

Definition 5. *A dynamical system is minimal if every orbit is dense, uniquely ergodic if there exists a unique invariant probability measure.*

1.3. **Suspensions and flows.** From an interval exchange transformation, one can construct a family of translation surfaces such that the vertical flow is a suspension flow over the interval exchange transformation (see [18] and [19] for a precise definition). In the following text, we will use a special case of this construction called *Masur's polygon* (see [8]). Minimality (resp. unique ergodicity) of the flow is equivalent to minimality (resp. unique ergodicity) of the associated interval

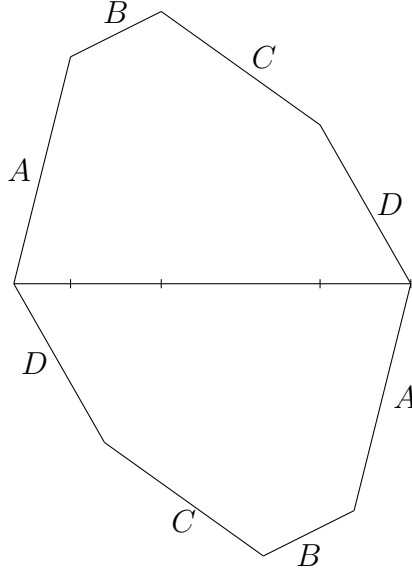


FIGURE 2. Masur's polygon built over an interval exchange transformation with permutation $(4, 3, 2, 1)$.

exchange transformation. In section 2.4, we define a suspension flow over the interval exchange transformation T_f .

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the moduli spaces of translation surfaces by its linear action on the polygons. The Teichmüller geodesic flow acts by $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. We recall *Masur's criterion* ([8] and [10]): let (X, ω) be a translation surface, if the projection in the moduli space of Riemann surfaces of the g_t orbit of (X, ω) is not recurrent (leaves every compact subset of the moduli space) then the vertical flow on X is uniquely ergodic. This means that if the vertical flow on X is not ergodic, the length on the shortest closed curve on $g_t(X, \omega)$ tends to zero as t tends to infinity. McMullen's version of Masur's criterion claims that the nonergodicity of the vertical flow on X implies that the length on the shortest closed separating curve on $g_t(X, \omega)$ tends to zero as t tends to infinity (see [11]). We will use this version in the sequel of the paper.

1.4. Word combinatorics. The following definitions will be used mainly in Sections 2.2 and 2.3 below, except the linear recurrence which is important for Section 3 below. We look at finite words on a finite alphabet $\mathcal{A} = \{1, \dots, r\}$.

Definition 6. A word $w_1 \dots w_s$ has length $|w| = s$ (not to be confused with the length of a corresponding interval). Prefixes and suffixes are defined in the usual way. The concatenation of two words w and w' is denoted by ww' .

A word $w = w_1 \dots w_s$ occurs at place i in a word $v = v_1 \dots v_t$ or an infinite sequence $v = v_1 v_2 \dots$ if $w_1 = v_i, \dots, w_s = v_{i+s-1}$. If w occurs at place i in v for some i , we say that w is a factor of v .

A language L over \mathcal{A} is a set of words such if w is in L , all its factors are in L , aw is in L for at least one letter a of \mathcal{A} , and wb is in L for at least one letter b of \mathcal{A} .

A return word of a word w in a language L is a nonempty word v in L such that w is a suffix of wv .

A language L is recurrent if for each w in L there exists at least one return word of w in L .

A language L is minimal if for each w in L there exists n such that w occurs in each word of L with n letters.

The language $L(u)$ of an infinite sequence u is the set of its finite factors.

For a word w in L , we denote by $A(w)$ the set of all letters x such that xw is in L , and by $D(w)$ the set of all letters x such that wx is in L .

A word w in L is called right special, resp. left special if $\#D(w) > 1$, resp. $\#A(w) > 1$. If $w \in L$ is both right special and left special, then w is called bispecial.

Definition 7. The Rauzy graph G_n of a language L is made with vertices w which are all words of length n of L , with an edge from w to w' whenever $w = av$, $w' = vb$ for letters a and b , and the word avb is in L ; the label of this edge is the word avb .

A return path of a language L is a sequence of words of equal length (or equivalently vertices in a Rauzy graph G_n) v, v_1, \dots, v_z such that $v_i \neq v$, $1 \leq i \leq z$, and the word $vs_1\dots s_zs$ is in L , where s_1, \dots, s_z, s are respectively the last letters of v_1, \dots, v_z, v . The word $s_1\dots s_zs$ is called the label of the return path.

Definition 8. The symbolic dynamical system associated to a language L is the one-sided shift $S(x_0x_1x_2\dots) = x_1x_2\dots$ on the subset X_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $t < s$, $x_t\dots x_s$ is in L .

For a word $w = w_0\dots w_{s-1}$ in L , the cylinder $[w]$ is the set $\{x \in X_L; x_0 = w_0, \dots, x_{s-1} = w_{s-1}\}$.

For a system (X, T) and a finite partition $Z = \{Z_1, \dots, Z_r\}$ of X , the trajectory of a point x in X is the infinite sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_n = i$ if $T^n x$ falls into Z_i , $1 \leq i \leq r$.

Then $L(Z, T)$ is the language made of all the finite factors of all the trajectories, and $X_{L(Z, T)}$ is the coding of X by Z .

For an interval exchange T , if we take for Z the partition made by the intervals I_i , $1 \leq i \leq r$, of Definition 1, we denote $L(Z, T)$ by $L(T)$ and call $X_{L(T)}$ the natural coding of T .

Note that the symbolic dynamical system (X_L, S) is minimal (Definition 5) if and only if the language L is minimal.

Definition 9. A language L is linearly recurrent if there exists a constant K such that each word of length n of L occurs in every word of length at least Kn of L .

We say that an interval exchange transformation T is linearly recurrent whenever $L(T)$ is linearly recurrent.

Remark 1. If T is a linearly recurrent interval exchange transformation, and X its Masur's polygon, then the g_t -orbit of X is bounded in the corresponding moduli space of translation surfaces.

2. MINIMALITY

2.1. A minimality criterion for Veech N -examples. In this section we shall prove the following

Theorem 1. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, N a prime number. If $Rx = x + \alpha$ modulo 1, $T_f(x, s) = (Rx, s + f(x))$, $s \in \mathbb{Z}/N\mathbb{Z}$, with

- $f(x) = 1$ if x is in the interval $[0, \beta]$,
- $f(x) = 0$ if x is in the interval $[\beta, 1[$.

Then

- T_f is not minimal if and only if

$$\beta = m\alpha + n$$

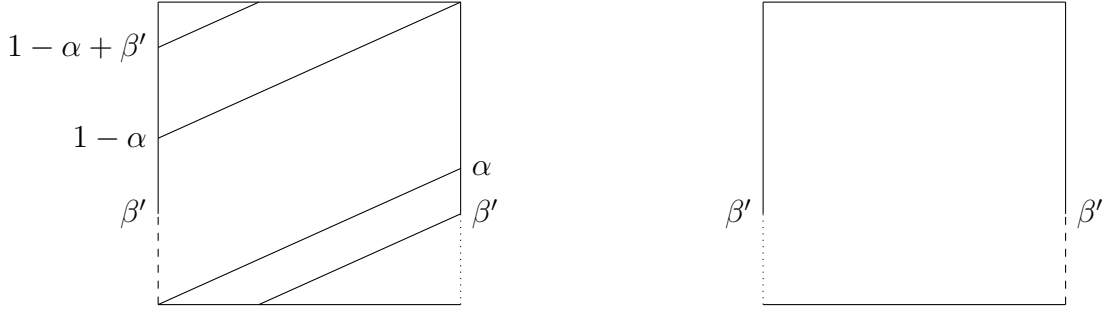


FIGURE 3. Geometrical model for Veech 1969

for some $m \in N\mathbb{Z}$, $n \in N\mathbb{Z}$,

- T_{1-f} is not minimal if and only if

$$\beta = m\alpha + n + 1$$

for some $m \in N\mathbb{Z}$, $n \in N\mathbb{Z}$.

We call T_f the *Veech N -example*. We introduce the *Masur-Smillie geometrical model* defined in [10] for $N = 2$. Consider N copies of the standard two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with two marked points, one is O the origin and the other one P has coordinates $(0, \beta')$. On each copy we make a vertical slit from O to P and we identify the right side of the slit in the copy number i with the left side of the slit in the copy number $i + 1$ (modulo N). This construction yields a translation surface X_N that is a $\mathbb{Z}/N\mathbb{Z}$ covering of \mathbb{T}^2 ramified over the points O and P .

Lemma 2. *We take the directional flow of slope α on X_N . Its first return map T' on the union of the N left vertical sides is conjugate to the Veech N -example T_f if $\beta' = \beta$, to its variant T_{1-f} if $\beta' = 1 - \beta$.*

Proof

T' is an extension of the rotation by α on $[0, 1[$, where the intervals $[1 - \alpha, 1 - \alpha + \beta'[$ are those sent to the next copy of $[0, 1[$. We do not change T' (just changing the fundamental domain for the rotation) if we cut the left parts $[0, 1 - \alpha[$ in each copy, and paste them on the left of 0, then translate the intervals by $\alpha - 1$ to have again $[0, 1[$; then the change of copy occurs for the interval $[0, \beta'[$, thus what we get is T_f for $\beta = \beta'$. For the variant, we cut the right parts $[1 - \alpha + \beta', 1[$ in each copy, and paste them on the left of 0, then translate the intervals by $\alpha - \beta'$ to have again $[0, 1[$, then the change of copy occurs for the interval $[0, 1 - \beta'[$. \square

By Lemma 2, Theorem 1 is equivalent to the following Theorem 3, which we shall now prove.

Theorem 3. *Let ϕ_t^θ be the linear flow in direction θ on X_N and $\alpha = \tan \theta$. Assume that β is irrational and N is prime. If α is rational, the linear flow ϕ_t^θ is periodic. If α is irrational, ϕ_t^θ is non minimal on X_N if and only if there are m and n in $N\mathbb{Z}$ such that*

$$\alpha = \frac{n + \beta}{m}$$

or

$$\alpha = \frac{n - \beta}{m}$$

Remark 2. We assume that β is irrational as otherwise X_N is a square tiled surface and every irrational direction is minimal and uniquely ergodic.

Proof

We first fix some notations. Let us call O_N and P_N the preimages of O respectively P in X_N and $\pi_N : X_N \rightarrow \mathbb{T}_2$ be the covering map. The branching locus of π_N is $\Sigma = \{O, P\}$ and the ramification points are $\Sigma_N = \{O_N, P_N\}$. Denote by ϕ the shift from a copy to the next one, it is an automorphism of the covering of order N ($\phi^N = id$ and $\phi_N \circ \pi_N = \pi_N$).

The relative homology group $H_1(X, \Sigma_N, \mathbb{R})$ has real dimension $2N + 1$. Let Δ and Γ be the vertical and horizontal cycles in \mathbb{T}^2 , a basis of $H_1(X, \Sigma_N, \mathbb{R})$ is given by

$$\{\Delta_1, \dots, \Delta_N, \Gamma_1, \dots, \Gamma_N, V_N\}$$

where $\Pi_N^{-1}[\Delta] = \{\Delta_1, \dots, \Delta_N\}$, $\Pi_N^{-1}[\Gamma] = \{\Gamma_1, \dots, \Gamma_N\}$, V_N is a preimage of the vertical saddle connection V on the torus. We number these cycles so that, for $1 \leq k \leq N$, $\Delta_k = \phi_*^k(\Delta_1)$ and $\Gamma_k = \phi_*^k(\Gamma_1)$.

Let $H_1^+(X_N, \Sigma_N, \mathbb{Z})$ be the sublattice of $H_1(X, \Sigma_N, \mathbb{R})$ invariant by ϕ . $H_1^+(X_N, \Sigma_N, \mathbb{Z})$ is identified to $H_1(\mathbb{T}^2, \Sigma, \mathbb{Z})$ by $[\gamma] \rightarrow \Pi_N^{-1}[\gamma]$. In plain terms,

$$H_1^+(X_N, \Sigma_N, \mathbb{Z}) = \mathbb{Z}(G_1 + \dots + G_N) \oplus \mathbb{Z}(\Delta_1 + \dots + \Delta_N) \oplus \mathbb{Z}V$$

Remark that, if the slope α is rational, ϕ_t^θ is periodic since X_N is a covering of \mathbb{T}^2 . In the rest of the proof, we assume that α is irrational. If the flow in direction θ is not minimal, the surface is decomposed into connected components separated by union of saddle connections of direction θ . Since α is irrational, there is neither a connection from O_N to O_N nor from P_N to P_N . Moreover since β is irrational, in direction θ , if there is a connection from O_N to P_N , there is no connection from P_N to O_N . Thus, projecting on \mathbb{T}^2 , there is only one saddle connection from O to P (resp. from P to O). Let us analyse the value of α when there is a saddle connection from O to P (the other case is analogous).

Lemma 4. Let σ be a saddle connection from O to P in direction θ and $\sigma_1, \dots, \sigma_N$ its preimages by π_N . If a subfamily of $\sigma_1, \dots, \sigma_N$ is a separating multicurve, then $[\sigma_1] = \dots = [\sigma_N]$.

Proof

Assume that

$$(1) \quad [\tau] = [\sigma_{i_1} + \dots + \sigma_{i_t} - \sigma_{i_{t+1}} - \dots - \sigma_{i_{t+t'}}] = 0 \text{ in } H_1(X_N, \Sigma_N, \mathbb{Z}).$$

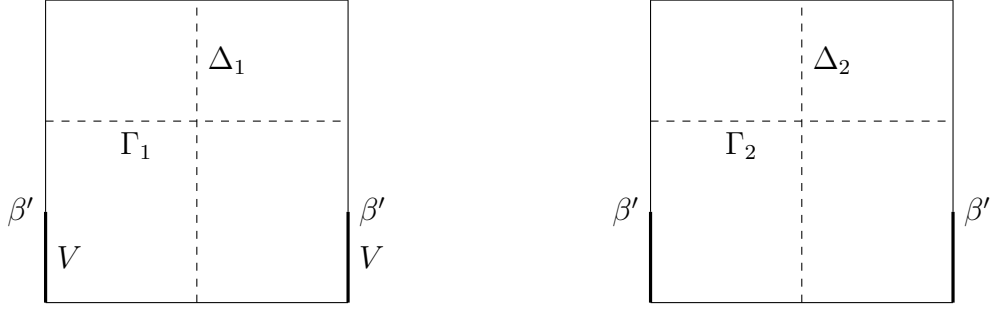
In the above formula, $t = t'$ since $\int_\tau \omega = 0$ where ω is the pull-back of dz on X_N .

Equation (1) is equivalent to

$$(2) \quad \sum_{i=1}^N \varepsilon_i [\sigma_i] = 0$$

with $\varepsilon_i \in \{0, 1, -1\}$ with $\sum_{i=1}^N \varepsilon_i = 0$.

Applying ϕ to this equation, we get the following system of N equations: $\sum_{i=1}^N \varepsilon_i [\sigma_{i+k-1}] = 0$ for $1 \leq k \leq N$.

FIGURE 4. The basis of the homology for $N = 2$

This means that $\sum_{i=1}^N \varepsilon_{i-k+1} [\sigma_i] = 0$. Let \bar{C} be the $N \times N$ matrix with $\bar{C}_{k,i} = \varepsilon_{i-k+1}$. Let \bar{B} be the $N \times (2N + 1)$ matrix with columns $[\sigma_1], \dots, [\sigma_N]$ in the basis of $H_1(X, \Sigma_N, \mathbb{R})$ described above. We get

$$\bar{C}^t \bar{B} = 0$$

Consequently the image of ${}^t \bar{B}$ is contained in the kernel of \bar{C} .

Moreover \bar{C} is a circulating matrix, thus conjugated to the diagonal matrix with diagonal coefficients $(P_C(1), P_C(\zeta), \dots, P_C(\zeta^{N-1}))$, with $\zeta = \exp \frac{2i\pi}{N}$ and $P_C(X) = \varepsilon_1 + \varepsilon_2 X + \dots + \varepsilon_N X^{N-1}$. We have $P_C(1) = 0$ since $\sum_{i=0}^N \varepsilon_i = 0$. But, if $k \neq 0$, $P_C(\zeta^k) \neq 0$ since the degree of P_C is at most $N - 1$ and it is not the cyclotomic polynomial $\sum_{i=0}^{N-1} X^i$. Therefore the dimension of the kernel of \bar{C} is one and the rank of \bar{B} is also one. Since, $[\sigma_1] = \dots = [\sigma_N]$ is solution of (2), it is the only one. This proves Lemma 4. \square

Remark 3. Let E be the real vector space generated by $[\sigma_1], \dots, [\sigma_N]$. The action of ϕ induces a linear representation ρ of $\mathbb{Z}/N\mathbb{Z}$ on E by $\rho(k)[\sigma_i] = \phi^k(\sigma_i) = \sigma_{k+i}$. This representation is a quotient of the regular one. Equation (1) implies that ρ is not the regular representation since $\dim_{\mathbb{R}} E < N$. Lemma 4 proves that, when N is a prime number, ρ is the trivial representation.

End of proof of Theorem 3

By Lemma 4, if the linear flow ϕ_t^θ is not minimal then $[\sigma_1] = \dots = [\sigma_N]$. This yields $i(\sigma, \Gamma) = i(\sigma_1 + \dots + \sigma_n, \Gamma_1) = Ni(\sigma_1, \Gamma_1)$. By the same reasoning, we get that $i(\sigma, \Delta)$ is a multiple of N , thus the slope of σ is of the form $\frac{n+\beta}{m}$ for n and m multiples of N .

Conversely, if $\alpha = \frac{n+\beta}{m}$, $i(\sigma, \Gamma) = m$ and $i(\sigma, \Delta) = n$. Assume that n is not a multiple of N . $i(\sigma_1 + \dots + \sigma_N, \Gamma_1) = n$, therefore, for some $1 \leq i < j \leq N$, $i(\sigma_i, \Gamma_1) \neq i(\sigma_j, \Gamma_1)$ and $[\sigma_i] \neq [\sigma_j]$. By Lemma 4, this implies that no union of the σ_j is a separating curve. \square

Remark 4. For more general $N = p_1^{a_1} \dots p_k^{a_k}$, with p_i prime numbers, the same reasoning (looking at the intermediate covering with p_i copies) proves that the directional flow is not minimal if $\alpha = \frac{np_i + \beta}{mp_i}$ for n and m in \mathbb{Z} . Thus we make the following conjecture, for which the “if” parts are known to hold. Some cases will be solved in Section 2.3 below.

Conjecture 1. If α is irrational,

- the flow ϕ_t^θ is not minimal on X_N if and only if $\alpha = \frac{n+\beta}{m}$,
- T_f is not minimal if and only if $\beta = m\alpha + n$,

- T_{1-f} is not minimal if and only if $\beta = m\alpha + n + 1$, for some $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, with $(N, m, n) \neq 1$.

2.2. A word combinatorial minimality criterion for general interval exchange transformations. In this section, we turn to general interval exchange transformations as in Section 1.1 above. The great majority of those studied in the literature satisfy M. Keane's *i.d.o.c. condition*, namely the $r - 1$ negative orbits $\{T^{-n}\gamma_i, n \geq 0, 1 \leq i \leq r - 1\}$ of the discontinuities of T are infinite disjoint sets; if T is i.d.o.c. it has no periodic point and no *connection* of the form $\gamma_j = T^m\beta_i$. The i.d.o.c. condition was shown in [7] to be a sufficient condition for minimality. But we have just seen in Theorem 1 above examples of interval exchange transformations with nontrivial connections, and they can be minimal or not minimal; to determine when they are minimal, we can look whether some saddle connections disconnect a surface, as in Section 2.1. Here we propose a word combinatorial method, by looking whether connections disconnect the Rauzy graphs of Definition 7. As far as we know, this criterion is new, though it could be deduced from P. Arnoux's result, [1] Chapter 1 Lemma 4, stating that, for an aperiodic interval exchange transformation T , there is a partition of the interval into finitely many invariant sets, on any of which T is minimal. The late M. Boshernitzan had an algorithm for deciding minimality, which is somewhat related to ours but focuses more on finding the connections.

Theorem 5. *Let T be an interval exchange transformation with no periodic point, and let M be the largest integer n for which, for some i and j , there is a primitive connection, namely $T^n\beta_i = \gamma_j$ with no $T^p\beta_i = \gamma_k$ for any k and $0 \leq p < n$, or $M = 0$ if there is no such connection. T is minimal if and only if the Rauzy graph G_{M+1} of the language $L(T)$ defined by its natural coding is connected (as a non-oriented graph).*

The proof will come from a sequence of intermediate results, some of which are valid in more general contexts.

Lemma 6. *Let T be an interval exchange transformation.*

Every left special word w in $L(T)$ is a prefix of at least one infinite sequence \mathcal{O}_i , where $(\mathcal{O}_i)_n = j$ whenever $T^n\beta_i$ is in I_j , for $n \geq 0$, $1 \leq i \leq r - 1$.

A word w of length m is bispecial if and only if

- *at least one β_i is in the interior of $[w]$,*
- *at least one $T^{-m}\gamma_j$ is in the interior of $[w]$.*

For any word w of length m , no $T^{-p}\gamma_k$ is in the interior of $[w]$ for any k and $0 \leq p < m$.

Proof

$w = w_0 \dots w_{m-1}$ is left special if and only if aw and $a'w$ exist for $a \neq a'$ which is equivalent to say $[w]$ intersects both $T[aw]$ and $T[a'w]$, or equivalently the interior of $[w]$ contains one or several β_i . Thus in particular w is a prefix of one or several \mathcal{O}_i .

Similarly, w is right special if and only if the interior of $[w]$ contains one or several $T^{-m}\gamma_j$. But it cannot contain any $T^{-p}\gamma_k$ for $0 \leq p \leq m - 1$ as $T^p[w]$ is included in the cylinder $[w_p]$. \square

Lemma 7. *Let T be an interval exchange transformation. Let w be a bispecial word in $L(T)$. Then we can write $A(w) = \{a_1, \dots, a_p\}$ so that, for any k , $D(a_k w) \cap D(a_{k+1} w)$ has at most one element.*

If $D(a_k w) \cap D(a_{k+1} w)$ is empty, there is a primitive connection of length equal to the length of w .

Proof

As in [6] and Figure 5, such a $[w]$ is partitioned by the intervals $T[a_i w]$, $a_i \in A(w)$, which we

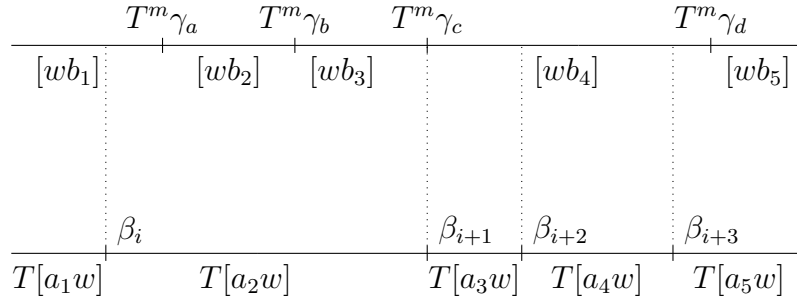


FIGURE 5. A bispecial interval

order from left to right, and by the intervals $[wb_j]$, $b_j \in D(w)$, which we order also from left to right. If the right end of $T[a_k w]$ falls in the interior of a $[wb_l]$, then $D(a_k w) \cap D(a_{k+1} w)$ has one element, b_l . If the right end of $T[a_k w]$ falls at an end of a $[wb_l]$, then $D(a_k w) \cap D(a_{k+1} w)$ is empty, and we have $\gamma_j = T^m \beta_i$ for some i and j , and m the length of w . In this case there is a connection of length m , and it is primitive as by Lemma 6 we cannot have any $\gamma_k = T^p \beta_i$ for $0 \leq p < m$. \square

Lemma 8. *Let L be a language. If for all bispecial words of L of length n we can write $A(w) = \{a_1, \dots, a_p\}$ and, for any k , $D(a_k w) \cap D(a_{k+1} w)$ has at least one element, and if G_n is connected, then G_{n+1} is connected.*

Proof As G_n is connected, G_{n+1} will be connected if two of its vertices are in the same connected component whenever they are labels of two consecutive edges of G_n . Suppose this fails: then for some w , a and b , aw and wb are not in the same connected component of G_{n+1} ; then, in particular awb is not in L . But as every word is extendable to the left and right, there exist a' and b' such that $a'wb$ and awb' are in L , thus $a \neq a'$, $b \neq b'$, and w is bispecial. Then, the a_i being as in the hypothesis, if $a_1 w$ is in some connected component U of G_{n+1} , so is wx for every x in $D(a_1 w)$, hence also wx is in U for one element of $D(a_2 w)$, and so on, hence all the $a_i w$ and all wy for y in $D(w)$ are in U , and this is a contradiction. \square

Remark 5. *Identifying all words awb for a bispecial w and letters a, b , is called resolving the bispecial w . In the standard classification of bispecial words [3], the bispecial words in Lemma 7 and Lemma 8 are the length of particular cases of neutral bispecials when each $D(a_k w) \cap D(a_{k+1} w)$ has exactly one element, and of weak bispecials when each $D(a_k w) \cap D(a_{k+1} w)$ has at most one element and at least one of these intersections is empty.*

What these lemmas prove is that, for languages of interval exchange transformations, only a weak bispecial can disconnect the Rauzy graphs, and this corresponds to a primitive connection. The examples in Theorem 11 below show that this condition is not sufficient, such a connection and bispecial may (Figure 7) or may not (Figure 8) disconnect the Rauzy graphs.

Proposition 9. *An interval exchange transformation with no periodic point is minimal if and only if all the Rauzy graphs G_n of the language $L(T)$ are connected.*

Proof

We notice first that the minimality of T and the minimality of its natural coding are equivalent, as cylinders, resp. intervals, form a basis of the topologies.

If any Rauzy graph is not connected, this contradicts the minimality of $L(T)$, and hence of T .

In the other direction, we note first that $L(T)$ is recurrent (Definition 6), as otherwise some interval $[w]$ would be disjoint from all $T^n[w]$, $n \neq 0$, and could not have positive length. Suppose now T has no periodic point and all Rauzy graphs are connected.

Suppose there is a word w in L for which no right extension is right special. Then, after w , we can see only one infinite sequence of letters. As $L(T)$ is recurrent, w occurs a second time in it, so this gives an ultimately periodic trajectory, which is excluded as T has no periodic point. Similarly, each word of $L(T)$ has a left extension which is left special, thus by Lemma 6 the words of $L(T)$ are the factors of the $r - 1$ infinite sequences \mathcal{O}_i . Let w be a word of $L(T)$: w must be a suffix of infinitely many words of $L(T)$, because every word of $L(T)$ can be extended infinitely many times to the left; thus w occurs infinitely often in at least one \mathcal{O}_i ; hence all the prefixes of \mathcal{O}_i occur infinitely often in at least one \mathcal{O}_j , and if $j \neq i$, then we can drop \mathcal{O}_i and use only the other \mathcal{O}_j to generate $L(T)$. Thus $L(T)$ is the union of the languages $L(\bar{W}_i)$, $1 \leq i \leq q \leq r - 1$, made with the factors of one-sided infinite sequences \bar{W}_i such that $L(\bar{W}_i)$ is recurrent: each factor of \bar{W}_i occurs at infinitely many places in \bar{W}_i .

We fix i . Let w be a factor of \bar{W}_i ; we look at the possible return words of w in $L(\bar{W}_i)$; starting from w , we shall return to w (because of the recurrence of $L(\bar{W}_i)$), but there are many possible paths; bifurcations correspond to right special words of the form wv ; by the reasoning of Lemma 6 applied to T^{-1} , these correspond to suffixes of the negative orbits of the γ_i , and, if there are more than $r - 1$ of them, some wv_1 and wv_2 are different suffixes of the same orbit; thus there is an occurrence of w in the shortest suffix which is not the initial occurrence, thus we have already returned to w before the bifurcation; hence we cannot see more than $r - 1$ bifurcations, with at most $r - 1$ choices for each one, before returning to w . Thus there are at most r^2 ways of going from one occurrence of w in \bar{W}_i to the next one. Hence any factor w occurs in \bar{W}_i at infinitely many places with bounded gaps, and $L(\bar{W}_i)$ is minimal.

Suppose $q \geq 2$. Let $L' = L(\bar{W}_1) \cup \dots \cup L(\bar{W}_{q-1})$. Suppose that for some n there is no word of length n in $L' \cap L(\bar{W}_q)$; then we cannot make a (non-oriented) path in G_n between a word of L' and a word of $L(\bar{W}_q)$, as this would include an edge, either from w to w' or from w' to w , for some w in $L(\bar{W}_q)$ and w' in L' , and the label of this edge cannot be in $L(T)$. Thus G_n is not connected, contradiction.

Thus $L' \cap L(\bar{W}_q)$ has words of all lengths, and as $L(\bar{W}_q)$ is minimal we get $L(\bar{W}_q) \subset L'$ and we do not need \bar{W}_q . We can do the same for the other \bar{W}_i , $i \geq 2$ and we get $L(T) = L(\bar{W}_1)$ which is minimal. \square

Proof of Theorem 5

It follows from Proposition 9, as Lemma 7 and Lemma 8 imply that if G_{M+1} is connected so are all the G_n . \square

Note that a part of this proof has been used already twice by the first author, namely to prove Theorem 2.9 of [5] and Theorem 2.28 of [4] (in the i.d.o.c. case, under assumptions on the language, without knowing a priori that it comes from an interval exchange transformation) but in both papers many details are missing, thus we take this opportunity to provide a complete proof; the unexplained condition on the words of length 2 in [4] is just the connectedness of the Rauzy

graph of length 1.

2.3. Minimality criteria for Veech 1969 type extensions.

Theorem 10. *Let T be a minimal aperiodic r -interval exchange transformation with discontinuities (as in Definition 1.1) γ_i , $1 \leq i \leq r - 1$, and T_f its Veech 1969 type extension as in Definition 4. Let*

- γ'_j , $1 \leq j \leq \bar{q}$, be all the different points γ_i and ζ_j , ordered from left to right.
- M be the maximal length of a primitive connection $\gamma'_j = T^m(T\gamma'_i)$, and 0 if there is no such connection.
- R be \bar{q} minus the number of different primitive connections $\gamma'_j = T^m(T\gamma'_i)$.
- \bar{L} be the language of the coding of T by the points γ'_i , where the interval $[\gamma'_i, \gamma'_{i+1}[$ is coded by the symbol $A^{(i+1)}$, $1 \leq i \leq \bar{q} + 1$, $[0, \gamma'_1[$ by $A^{(1)}$, $[\gamma'_{\bar{q}}, 1[$ by $A^{(\bar{q}+1)}$.
- \tilde{f} be the map associating to the symbol $A^{(i)}$ the value of the function f on the interval coded by $A^{(i)}$.

We choose a vertex w of the Rauzy graph G_{M+1} of the language \bar{L} . Then it has exactly $R + 1$ different return paths, whose labels are words $U_1 \dots U_{R+1}$. Let $\xi_{i,j}$ be the number of occurrences of the symbol A^i in the word U_j , $1 \leq i \leq \bar{q} + 1$, $1 \leq j \leq R + 1$. Let

$$d_j = \sum_{i=1}^{\bar{q}+1} \xi_{i,j} \tilde{f}(A^{(i)}), \quad 1 \leq j \leq R + 1.$$

Then T_f is minimal if and only if $(N, d_1, \dots, d_{R+1}) = 1$ (if $d_l = 0$, we consider it has a common factor with every integer).

Note that d_j can also be written as

$$d_j = \sum_{i=1}^{q+1} \xi'_{i,j} a_j,$$

where the $\xi'_{i,j}$ depend only on the initial interval exchange transformation T and the marked points ζ_i .

Proof of Theorem 10

By taking $0 \leq s \leq N - 1$ in Definition 4 and identifying each $I \times \{s\}$ with the interval $[s, s + 1[$, we can view T_f as an $N(\bar{q} + 1)$ -interval exchange transformation. T_f has no periodic point as this would project on a periodic point for T . We get its natural coding by coding with the symbol $A_s^{(i)}$ the product of the interval coded by $A^{(i)}$ by the set $\{s\}$, $s \in \mathbb{Z}/N\mathbb{Z}$, getting the language L . Thus, by Theorem 5, the minimality of T_f is equivalent to the connectedness of the Rauzy graph $G_{M+1}(L)$, as the maximal length of a primitive connection is still M .

As T is minimal, $G_{M+1}(\bar{L})$ is connected. The words in $G_{M+1}(L)$ project on those of $G_{M+1}(\bar{L})$ by replacing each letter $A_s^{(i)}$ by $A^{(i)}$, $1 \leq i \leq \bar{q} + 1$, $s \in \mathbb{Z}/N\mathbb{Z}$. The word $A^{(i)}A^{(j)}$ exists in \bar{L} if and only if $A_s^{(i)}A_{s+\tilde{f}(A^{(i)})}^{(j)}$ exists in L for all s (see Figure 1 above). Thus if w in \bar{L} has $A^{(i)}$ as its last letter, it is the projection of exactly N words W_s in L , with last letter respectively $A_s^{(i)}$ for each $s \in \mathbb{Z}/N\mathbb{Z}$, and each connected component of G_{M+1} contains at least one of the W_s .

The words of length $M + 1$ of \bar{L} are produced by a $\bar{q} + 1$ -interval exchange transformation with connections, and behave like those produced by an $R + 1$ -interval exchange transformation without connections, thus, by the reasoning of [7] p.28, each corresponding interval J can be cut up into intervals J_1, \dots, J_{R+1} such that the $T^j J_j$, $1 \leq j \leq R + 1$, $0 \leq j \leq s_j$, are pairwise disjoint and cover I . Thus each of these words has $R + 1$ different return paths.

Fix such a w in \bar{L} . Starting from a W_s projecting on w , by following return paths, we get that all the words W_{s+d_j} are in the same connected component of $G_{M+1}(L)$ as W_s , $1 \leq j \leq R + 1$; by iterating the process, so are all the words $W_{s+\sum z_j d_j}$ for integers z_j . Thus if N and all the d_j are globally coprime, all the W_s , $s \in \mathbb{Z}/N\mathbb{Z}$ are in the same connected component, and $G_{M+1}(L)$ is connected. Otherwise, N and the d_j have a common factor h ; but every path from W_s to $W_{s'}$ in $G_{M+1}(L)$ projects on a path from w to w in $G_{M+1}(\bar{L})$, which is a concatenation of return paths of w ; thus W_s and $W_{s'}$ are in the same connected component if and only if $s - s'$ is a multiple of h , and $G_{M+1}(L)$ is not connected (see Figures 6, 7, 8 for examples). \square

Theorem 10 reduces the study of the minimality of T_f to the computation of $G_{M+1}(\bar{L})$, which can be done without difficulty for any given example. However, general results are not so easy to get, we shall now give some for extensions of rotations by functions with two values.

Theorem 11. *Let $0 < \alpha < 1$ be irrational, T the rotation of angle α viewed as a 2-interval exchange with discontinuity $1 - \alpha$. Let T_f be as in Definition 4, with $q = 1$. Then*

- (i) *if $0 < \zeta_1 = m\alpha < 1$, T_f is minimal if and only if $(N, ma_1, a_2) = 1$,*
- (ii) *if $0 < \zeta_1 = 1 - m\alpha < 1$, T_f is minimal if and only if $(N, a_1, ma_2) = 1$,*
- (iii) *if $0 < \zeta_1 = m\alpha - m + 1 < 1$, T_f is minimal if and only if $(N, a_1, ma_2) = 1$,*
- (iv) *if $0 < \zeta_1 = m - m\alpha < 1$, T_f is minimal if and only if $(N, ma_1, a_2) = 1$,*
- (v) *if $0 < \zeta_1 = 2 - m\alpha < 1$, $m \geq 2$, and $\alpha < \frac{1}{m-1}$, T_f is minimal if and only if $(N, a_2 + (m - 2)a_1, ma_1) = 1$,*
- (vi) *if $0 < \zeta_1 = m\alpha - 1 < 1$, $m \geq 2$, and $\alpha < \frac{1}{m-1}$, T_f is minimal if and only if $(N, a_1 + (m - 2)a_2, ma_2) = 1$.*

Proof

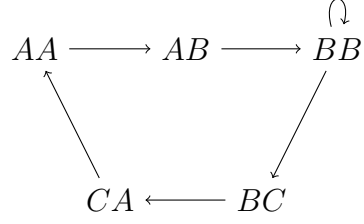
Item (i):

we begin with the first item, the only one for which we shall actually compute the Rauzy graphs. Thus $\zeta_1 = m\alpha$, then $\bar{q} = 2$, $R = 1$. The γ'_i are ζ_1 and $1 - \alpha$; for commodity, we use symbols A, B, C instead of $A^{(1)}, A^{(2)}, A^{(3)}$, and corresponding A_s, B_s, C_s . The longest primitive connection is between α and $m\alpha$, and thus $M = m - 1$. All Rauzy graphs are computed for \bar{L} . The evolution of general Rauzy graphs is described in [2]: when G_l is known, the vertices of G_{l+1} are the labels of the edges of G_l , and the only extra information we need to build the edges of G_{k+1} is the resolution of the bispecials, see Remark 5.

We suppose first $m \geq 2$.

First subcase: $\alpha < \frac{1}{m+2}$

Then the image of ζ_1 by T is $(m + 1)\alpha$, and $0 < \alpha < m\alpha < (m + 1)\alpha < 1 - \alpha < 1$. The words of length 2 are CA, AA, AB, BB, BC . The words A, AA, \dots, A^l correspond to intervals of length $m\alpha, (m - 1)\alpha, \dots, (m - l + 1)\alpha$, thus A^l exists in \bar{L} if and only if $l \leq m$; for $l \leq m - 2$, A^l is bispecial, preceded by C (on the left of the corresponding interval) and A , separated by α , and followed by A (left) and B , separated by $m\alpha$; as there is no connection of length l , such an A^l is a neutral bispecial, and its resolution gives necessarily the words A^{l+2}, CA^{l+1} and $A^{l+1}B$; because



We prove now (iii) from (i).

FIGURE 6. $G_2(\bar{L})$, $\beta = 2\alpha$, $\alpha < \frac{1}{5}$



FIGURE 7. $G_2(L)$, $\beta = 2\alpha$, $\alpha < \frac{1}{5}$, $N = 2$, $a_1 = 1$, $a_2 = 0$

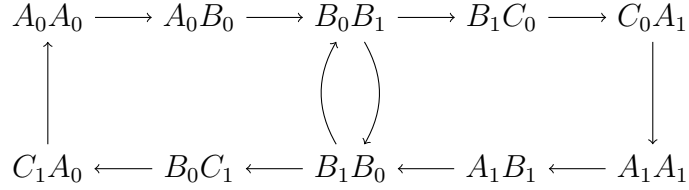


FIGURE 8. $G_2(L)$, $\beta = 2\alpha$, $\alpha < \frac{1}{5}$, $N = 2$, $a_1 = 0$, $a_2 = 1$

of the connection $T^{m-1}\alpha = m\alpha$, A^{m-1} is a weak bispecial, and only the words CA^m and A^mB exist. The words B, BB, \dots, B^l correspond to intervals of length $1 - (m+1)\alpha$ other special word, $1 - (m+2)\alpha, \dots, 1 - (m+l)\alpha$, thus there is a $p \geq 2$ such that B^l exists in \bar{L} if and only if $l \leq p$; for $l \leq p-2$, B^l is a neutral bispecial, resolved by the words B^{l+2}, AB^{l+1} and $B^{l+1}C$; B^{p-1} is also a neutral bispecial, but resolved by the words $AB^{p-1}C, AB^p$ and B^pC ; thus in G_p B^p is not special, but AB^{p-1} is right special and $B^{p-1}C$ is left special.

Suppose first that $p \geq m+1$ (this is true whenever $\alpha < \frac{1}{2m+1}$, which is the case in Figures 6, 7, 8). Then G_1 has two bispecial words, A and B , no right or left special word except these, and G_1 can be described as four paths (defined and labelled in the same way as the return paths of Definition 7), one from A to A labelled A , one from A to B labelled B , one from B to A labelled CA , one from B to B labelled B . The resolution of the bispecials implies that for all $l \leq m-1$ G_l has two bispecial words, A^l and B^l , no right or left special word except these, and G_l can be described as four paths, one from A^l to A^l labelled A , one from A^l to B^l labelled B^l , one from B^l to A^l labelled CA^l , one from B^l to B^l labelled B . Then in G_m there is one bispecial word, B^m , and no right or left special word except this; the return paths of B^m are the single edge around B^m , labelled B , and a path going through A^m , labelled CA^mB^m . As $\tilde{f}(A) = a_1$,

$\tilde{f}(B) = \tilde{f}(C) = a_2$, by Theorem 10 (as T has no periodic point) T_f is minimal if and only if $(N, a_2, ma_1 + (m+1)a_2) = 1$, which is equivalent to the claimed condition.

Suppose now $p \leq m$. G_l is as described above for $l \leq p-1$. G_p has one bispecial word, A^p , one right special word AB^{p-1} , one left special word $B^{p-1}C$, no right or left special word except these, and G_p can be described as four paths, one from A^p to A^p labelled A , one from A^p to AB^{p-1} labelled B^{p-1} , and two from AB^{p-1} to A^p labelled CA^p and BCA^p . Then for $p \leq l \leq m-1$ the situation remains the same, as no new bispecial is created since the left and right special words are separated by the bispecial A^l ; the right special becomes $A^{l-p+1}B^{p-1}$, the left special word $B^{p-1}CA^{l-p}$, the labels of the paths A , B^{p-1} , CA^l and BCA^l . Then in G_m the structure is the same except there is no edge around A^m , there is no bispecial word and the right special has two return paths labelled $B^{p-1}CA^m$ and B^pCA^m . This is true also for $p = m$. Thus T_f is minimal if and only if $(N, ma_1 + pa_2, ma_1 + (p+1)a_2) = 1$, which is equivalent to the claimed condition.

Second subcase: $\frac{1}{m+2} < \alpha < \frac{1}{m+1}$

The image of ζ_1 by T is $(m+1)\alpha$, and $0 < \alpha < m\alpha < 1 - \alpha < (m+1)\alpha < 1$. The words of length 2 are AA, AB, AC, BC, CA . A^l corresponds to an interval of length $(m-l+1)\alpha$ and exists whenever $l \leq m$. For $l \leq m-2$, A^l is a neutral bispecial, preceded from left to right by C, A , followed from left to right by A, B, C , and its resolution gives $CA^{l+1}, A^{l+2}, A^{l+1}B, A^{l+1}C$. Then A^{m-1} is a weak bispecial, its resolution gives CA^m, A^mB, A^mC .

For $1 \leq l \leq m-1$, G_l has one bispecial word A^l , and one left special CA^{l-1} , and can be described by the three return paths of A^l , labelled A, BCA^l, CA^l . Then in G_m the structure is the same except there is no edge around A^m , thus there is one right special A^m , one left special CA^{m-1} , and A^m has two return paths labelled BCA^m and CA^m . As $\tilde{f}(A) = a_1, \tilde{f}(B) = \tilde{f}(C) = a_2$, by Theorem 10 T_f is minimal if and only if $(N, ma_1 + a_2, ma_1 + 2a_2) = 1$, which is equivalent to the claimed condition.

Third subcase: $\frac{1}{m+1} < \alpha < \frac{1}{m}$

The image of ζ_1 by T is $(m+1)\alpha - 1$, and $0 < (m+1)\alpha - 1 < \alpha < 1 - \alpha < m\alpha < 1$. The words of length 2 are AA, AB, AC, BA, CA . A^l corresponds to an interval of length $1 - l\alpha$ and exists whenever $l \leq m$. For $l \leq m-2$, A^l is a neutral bispecial, preceded from left to right by B, C, A , followed from left to right by A, B, C , and its resolution gives $BA^{l+1}, CA^{l+1}, A^{l+2}, A^{l+1}B, A^{l+1}C$. Then A^{m-1} is a weak bispecial, its resolution gives $BA^m, CA^{m-1}B, A^mC$ and either CA^m or $BA^{m-1}B$ (depending on further information about α , which is not needed for our purpose).

For $1 \leq l \leq m-1$, G_l has one bispecial word A^l , and can be described by the three return paths of A^l , labelled A, BCA^l, CA^l . In the first case of the resolution of the bispecial A^{m-1} , G_m has one right special CA^{m-1} , one left special A^m , and CA^{m-1} has two return paths labelled ACA^{m-1} and $BA^{m-1}ACA^{m-1}$. In the second case of the resolution of the bispecial A^{m-1} , G_m has one right special BA^{m-1} , one left special $A^{m-1}B$, and BA^{m-1} has two return paths labelled BA^{m-1} and $ACA^{m-1}BA^{m-1}$.

As $\tilde{f}(A) = \tilde{f}(B) = a_1, \tilde{f}(C) = a_2$, by Theorem 10 T_f is minimal if and only if in the first case $(N, ma_1 + a_2, 2ma_1 + a_2) = 1$, and in the second case $(N, ma_1, 2ma_1 + a_2) = 1$, which is equivalent to the claimed condition in both cases.

If $m = 1$ there is a connection of length 0, and, in the same way as above, we check by hand on G_1 that our assertion is satisfied.

Items (ii) to (vi):

by a straightforward generalization of Lemma 2, the minimality of T_f defined by $\alpha, \zeta_1, a_1, a_2$ is equivalent to the minimality of T_f defined by $\alpha, 1 - \zeta_1, a_2, a_1$, thus (ii) is deduced from (i), (iv) from (iii), (vi) from (v).

We prove now (iii) from (i). Starting from T_f defined by $\alpha, \zeta_1, a_1, a_2$, we get the minimality of T_f defined by $1 - \alpha, 1 - \zeta_1, a_2, a_1$ by using the Rauzy graphs again. We write the interval exchange transformation corresponding to the rotation by α with marked point ζ_1 , and its natural coding by A, B, C from left to right; reversing the orientation by applying the map $x \rightarrow 1 - x$, we get the same language for the rotation by $1 - \alpha$ with marked point $1 - \zeta_1$, and its natural coding by A, B, C from right to left (the intervals are now open on the left, closed on the right, which changes some trajectories but not the language); to deduce the new T_f from this coding, we use $\tilde{f} = a_2$ on either A , or A and B , a_1 elsewhere, whenever the initial \tilde{f} was a_1 respectively on A , or A and B , a_2 elsewhere. The lengths of the connections are not changed, hence by Theorem 10 the minimality of the two extensions is equivalent. We apply this result starting from (i), getting the rotation of angle $0 < \bar{\alpha} = 1 - \alpha < 1$ and $\zeta_1 = 1 - m\alpha = m\bar{\alpha} - m + 1$, thus we get (iii).

To get (v) we start from the last subcase of (i) with m replaced by $m - 1$; thus $\frac{1}{m} < \alpha < \frac{1}{m-1}$, the image of $\zeta_1 = (m - 1)\alpha$ by T is $m\alpha - 1$ (as on the interval containing ζ_1 the rotation has the form $x \rightarrow x + \alpha - 1$), and $0 < m\alpha - 1 < \alpha < 1 - \alpha < (m - 1)\alpha < 1$. We found that in G_{m-1} the two return words of the bispecial are labelled either ACA^{m-2} and $BA^{m-2}ACA^{m-2}$ or BA^{m-2} and $ACA^{m-2}BA^{m-2}$. We write the interval exchange transformation corresponding to the rotation by α with marked point ζ_1 , and first conjugate it by $x \rightarrow 1 - x$, getting an interval exchange transformation T' , coding it by A, B, C from right to left, hence getting the same language, as in the above paragraph; the discontinuities are $0 < 1 - (m - 1)\alpha < \alpha < 1$, delimiting from left to right intervals coded by C, B, A , thus they are respectively the cylinders $[C], [B], [A]$, and their images are $0 < 1 - \alpha < 2 - m\alpha < 1$, which are the endpoints, from left to right, of the intervals $T'[A], T'[C], T'[B]$.

We look now at $\bar{T} = T'^{-1}$, which is just the interval exchange transformation corresponding to the rotation by α with marked point $2 - m\alpha$; its natural coding is its coding by the intervals $T'[A], T'[B], T'[C]$, and gives the same language as its coding by the intervals $[A], [B], [C]$. The previous reasoning shows that language is the language \bar{L} of the last subcase of (i) (for $m - 1$) up to retrograding ($w_1 \dots w_s \rightarrow w_s \dots w_1$) of words. We apply Theorem 10 for \bar{T} and the function \bar{f} which is a_1 on $T'[A]$ and $T'[C]$, a_2 on $T'[B]$. For $\zeta_1 = 2 - m\alpha$, the longest primitive connection is from $1 - (m - 1)\alpha$ (it would be $2 - (m - 1)\alpha$ if $\alpha > \frac{1}{m-1}$) to $1 - \alpha$, of length $m - 2$ so we need only the knowledge of the return paths in G_{m-1} ; these have the same labels as for $\zeta_1 = (m - 1)\alpha$, up to two operations on words, retrograding ($w_1 \dots w_s \rightarrow w_s \dots w_1$) and circular conjugacies ($w_1 \dots w_s \rightarrow w_k \dots w_s w_1 \dots w_{k-1}$), to take into account that the labels are computed from the last letters of words in the return paths, and these may be replaced by first letters in some of the operations we made), which do not change the ξ_j of Theorem 10. Thus when the labels for $\zeta_1 = (m - 1)\alpha$ are ACA^{m-2} and $BA^{m-2}ACA^{m-2}$, we get the condition that N, ma_1 and $(2m - 2)a_1 + a_2$ are coprime, and the other case is equivalent, proving (v). \square

Remark 6. As we got (v) from the last subcase of (i) for $m - 1$, the first two subcases of (i) for $m - 1$ give a word combinatorial proof of (ii) for m ; deducing geometrically (ii) from (i) for m is shorter, but cannot be proved with the Rauzy graphs as the maximal length of a primitive connection is not the same; note we do not know how to deduce the Rauzy graphs of (vi) from any other case. Similarly, the case where ζ_1 is not in $\mathbb{Z}(\alpha)$ is easy to solve with our method as $M = 0$ (the result appears already in Veech [16]) but it will be dealt with geometrically, and more generally, in Proposition 13 below.

Remark 7. The Veech N -examples T_f of Section 2.1 correspond to $a_1 = 1, a_2 = 0$, and their variant T_{1-f} to $a_1 = 0, a_2 = 1$. Thus, with the help of Lemma 2 for the flow, we can prove Conjecture 1 for the ζ_1 in (i) to (vi). In cases (v) and (vi) if N is odd, both the Veech N -example and its variant are always minimal, but our result gives new nontrivial examples of non-minimality for some a_1 and a_2 .

Remark 8. For the general case $\zeta_1 = m\alpha + n$, the minimality criterion seems to depend heavily on the congruence of m modulo n . For $m \equiv 1$, we offer the following Conjecture 2, generalizing (vi) and checked by hand computations for small values. For other cases, we only checked that, if $\zeta_1 = 2\alpha - 2$ or $\zeta_1 = 4\alpha - 2$, T_f is minimal if and only if $(N, 2a_1, a_2) = 1$, and that, if $\zeta_1 = 6\alpha - 2$, T_f is minimal if and only if $(N, 2a_1, 3a_2) = 1$. Other values of ζ_1 can be deduced by changing α and ζ_1 to $1 - \alpha$ and $1 - \zeta_1$, or α and $1 - \zeta_1$, as in the proof of Theorem 11.

Conjecture 2. Let $0 < \alpha < 1$ be irrational, T the rotation of angle α viewed as a 2-interval exchange with discontinuity $1 - \alpha$. Let T_f be as in Definition 4, with $q = 1$. If $0 < \zeta_1 = m\alpha - n < 1$, $m = \bar{m}n + 1$, $\bar{m} \geq 1$, T_f is minimal if and only if $(N, a_1 + (\bar{m} - 1)a_2, ma_2) = 1$.

We end this section by giving a few examples for extensions of rotations by functions with three values.

Proposition 12. Let $0 < \alpha < 1$ be irrational, T the rotation of angle α viewed as a 2-interval exchange with discontinuity $1 - \alpha$. Let T_f be as in Definition 4, with $q = 2$. Then

- if $0 < \zeta_1 = 2\alpha < \zeta_2 < 1$, $\zeta_2 \notin \mathbb{Z}(\alpha)$, T_f is minimal if and only if $(N, 2a_1, a_2, a_3) = 1$,
- if $0 < \zeta_1 < \zeta_2 = 2\alpha < 1$, $\zeta_1 \notin \mathbb{Z}(\alpha)$, T_f is minimal if and only if $(N, a_1 + a_2, 2a_2, a_3) = 1$,
- if $0 < \zeta_1 = 2\alpha < \zeta_2 = 3\alpha < 1$, T_f is minimal if and only if $(N, 2a_1 + a_2, a_3) = 1$.

Proof

We take for example the first case with α small enough, so that $0 < \alpha < 2\alpha < 3\alpha < \zeta_2 < \zeta_2 + \alpha < 1 - \alpha < 1$. We code by A, B, C, D . The longest primitive connection is from α to 2α , so we look at G_2 . The words of length 2 are $AA, AB, BB, BC, CC, CD, DA$. The weak bispecial A is resolved by DAA, AAB ; the neutral bispecials B and C are resolved by $ABB, BBC, BBB, BCC, CCD, CC$ if we take α small enough for BBB and CCC to exist. In G_2 , BB and CC are bispecial. There are one-edge paths from BB to BB , labelled B , and from CC to CC , labelled C , a path from BB to CC labelled CC , and a path from CC to BB labelled $DAABB$. The three return paths of BB are labelled $B, C^p DABB$ and $C^{p-1} DABB$ where p is such that C^p exists but not C^{p-1} , which gives our result as \tilde{f} is a_1 on A, a_2 on B, a_3 on C and D .

For the second case and small values of α and ζ_1 , $0 < \zeta - 1 < \alpha < \zeta_1 + \alpha < 2\alpha < 3\alpha < 1 - \alpha < 1$, we get G_2 with one left special CD , one right special BC , and one bispecial CC with return paths labelled $C, DBBCC$, and $DABCC$, while \tilde{f} is a_1 on A, a_2 on B, a_3 on C and D .

In the third case, the longest primitive connection is from α to 2α (the one from α to 3α is not primitive as there is a connection from 3α to 3α) so we look at G_2 . For $0 < \alpha < 2\alpha < 3\alpha < 4\alpha <$

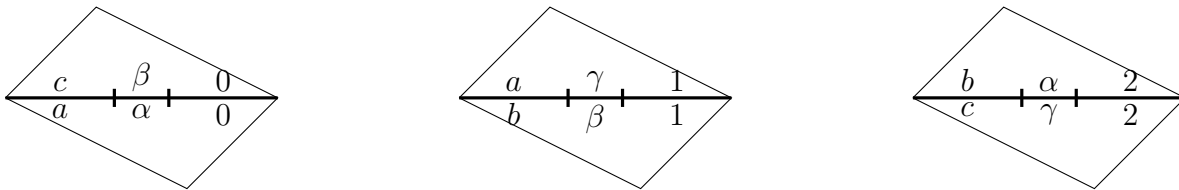


FIGURE 9. In this example, T is a rotation, f takes 3 values: $1, -1, 0$. The suspension flow over T_f is made by 3 tori glued along slits. The identifications of the slits are given by the latin and greek letters and the numbers.

$1 - \alpha < 1$, G_2 has one bispecial CC , and, if CCC exists, the return paths of CC are labelled C and $DAABC$.

Other subcases are similar. □

Remark 9. For $0 < \zeta_1 = 2\alpha < \zeta_2 < 1$, $\zeta_2 \notin \mathbb{Z}(\alpha)$, $a_1 = 1$, $a_2 = 0$, $a_3 = 2$, $N = 4$, we get a non-minimal example which will be used in Section 3.1; we can see its non-minimality also by noticing that $a_2 = a_3$ in $\mathbb{Z}/2\mathbb{Z}$, which would kill the discontinuity ζ_2 if $N = 2$; indeed, for $N = 4$ T_f admits as a factor the Veech 1969 transformation for $\beta = 2\alpha$, and this is not minimal by Theorem 1 above. If we start from $0 < \zeta_1 < \zeta_2 = 2\alpha < 1$ we do the same trick, and get the same factor, by taking $a_1 = 1$, $a_2 = 3$, $a_3 = 0$, $N = 4$, and indeed T_f is not minimal by Proposition 12. This gives a justification for the somewhat surprising lack of symmetry in the criteria in the first and second case of that proposition.

2.4. A general geometrical model and criterion. Let T_f be as in Definition 4. We define now a geometrical model for T_f . We construct the Masur polygon over T . We call X the resulting translation surface and $O, P_1, \dots, P_q, \Omega$ the points in X on the horizontal line with coordinates $0, \zeta_1, \dots, \zeta_q, 1$. We make $q + 1$ slits, V_1 from O to P_1 , V_i from P_{i-1} to P_i , $1 \leq i \leq q$, and V_{q+1} from P_q to Ω . We take N copies of X with these horizontal slits. We glue the top side of a slit of type V_i in the copy k with the bottom of the slit of the same type in the copy $k + a_i$. This construction leads to a surface Y that is a $\mathbb{Z}/N\mathbb{Z}$ -covering of X ramified over O, P_1, \dots, P_q and Ω . Then T_f is the first return map on the union of the horizontal segments (the preimages of the segment $[0, 1]$) of the vertical flow in Y .

Definition 10. Denote by Π the covering $Y \rightarrow X$, we say that T_f is fully ramified if the ramification index of Π at all points $O, P_1, \dots, P_q, \Omega$ is equal to N , which means that each point has exactly one preimage by Π .¹

Proposition 13. Let T_f be as in Definition 4. Suppose the negative orbits under T of all the γ_j and the ζ_i are infinite disjoint sets; then T_f is minimal if and only if $(N, a_1, \dots, a_q) = 1$.

Proof

If (N, a_1, \dots, a_q) have a common factor h , then the set $I \times H$ is invariant by T_f , where H is the (strict) subgroup of $\mathbb{Z}/N\mathbb{Z}$ generated by h .

In the other direction, we use the geometrical model above. The condition on the a_i ensures that they generate the additive group $\mathbb{Z}/N\mathbb{Z}$, thus the surface is connected, and the flow is minimal as

¹The points O and Ω are not always different points on the surface X .

the conditions on the ζ_i and γ_j ensure there is no connection. \square

3. UNIQUE ERGODICITY

3.1. Merrill's results revisited. We find it useful to recall two of Merrill's results which are relevant to our general problematic, while translating them in a more dynamical vocabulary and giving some precisions on their validity.

We deduce from [12], Theorem 3.1 plus the first lines of p. 323, the following

Proposition 14. *Let T_f be as in Definition 4, with T a rotation of irrational angle α on the 1-torus, for α with bounded partial quotients. Let $\zeta_0 = 0$, $\zeta_{q+1} = 1$, $e_i = a_{i+1} - a_i$, $e_q = a_0 - a_q$. Suppose at least one of the $\zeta_i - \zeta_j$ is not in $\mathbb{Z}(\alpha)$, and that, for any $d \neq 0$ in $\mathbb{Z}/N\mathbb{Z}$, no sum of $q' < q$ different de_i is zero. Then T_f is minimal and uniquely ergodic.*

Remark 10. *The hypotheses of Theorem 3.1 of [12] are the same as those of Proposition 14 above, but only for $d = 1$, and thus they are satisfied if $q = 3$ (whatever the a_i provided $a_i \neq a_{i+1}$ for all i) if one of the distances $\zeta_i - \zeta_j$ is not in $\mathbb{Z}(\alpha)$; these imply that f is not a coboundary, but, as is only understated in [12], to get minimality and unique ergodicity of T_f we need them for all values of d . For example, if we take the first example of Remark 9, the hypotheses above are satisfied for $d = 1$ but not for $d = 2$, and indeed T_f is not minimal and hence not uniquely ergodic.*

As mentioned in the introduction, Merrill built a very surprising counter-example in her Theorem 3.2; but again, only coboundary properties are mentioned, and the notion of minimality is absent in [12]. Thus, to derive the following proposition from Merrill's work, we need a sufficient condition of minimality, for example Proposition 13 of the present paper.

Proposition 15. *We can find α with bounded partial quotients, ζ_1 and ζ_2 such that, for $\zeta_3 = \zeta_1 + \zeta_2 - 1$, $a_1 = a_3 = 0$, $a_3 = -a_4 = 1$, any $N \geq 2$, T_f is minimal and not uniquely ergodic.*

3.2. Functions with three values.

Proposition 16. *Let T be an interval exchange transformation, ξ the permutation of Definition 2, $O_\xi(0)$ the ξ -orbit of 0, and T_f as in Definition 4 with $q = 2$. T_f is fully ramified if and only if $(a_1 - a_2, N) = (a_2 - a_3, N) = (|O_\xi(0)|, N) = 1$ and*

- $(a_1, N) = 1$, if 0 and 1 do not belong to the same ξ -orbit,
- $(a_3 - a_1, N) = 1$, if 0 and 1 belong to the same ξ -orbit.

Proof

The condition on the a'_i s is exactly that $|\Pi^{-1}(O)| = |\Pi^{-1}(P)| = |\Pi^{-1}(Q)| = 1$. \square

Theorem 17. *Assume that T is linearly recurrent, fully ramified and that T_f is minimal. Then T_f is uniquely ergodic.*

We shall prove a geometrical version of this result. We use the geometrical model built in Section 2.4 above; P_1 and P_2 are denoted by P and Q for commodity.

Theorem 18. *Let X be a translation surface, assume that its g_t -orbit is bounded in its stratum. Let Y be a $\mathbb{Z}/N\mathbb{Z}$ -cover of X ramified over two regular points P and Q and at least one singularity O with ramification index N at these 3 points. If the vertical flow on Y is minimal then it is uniquely ergodic.*

Proof of Theorem 18

Let g_t be the Teichmüller geodesic flow. Then T is linearly recurrent if and only if the g_t -orbit of X is bounded in the ambient stratum. To prove the ergodicity of the vertical flow on Y , we use McMullen's version of Masur's criterion, see Section 1.3 above. It is enough to prove that there is a sequence of times (t_n) tending to infinity such that $g_{t_n}Y$ has no short separating curve. Assuming this fact, a limit point of $g_{t_n}Y$ in the Deligne-Mumford compactification is connected and McMullen's theorem implies that the vertical flow on Y is uniquely ergodic. The proof is based on two arguments.

The first one is the topological

Lemma 19. *Let σ be a saddle connection on X joining two points where the covering is ramified (O, P, Q, Ω) . The preimages in Y of σ do not disconnect the surface Y .*

This lemma does not depend on the number of branched points.

Proof

Without loss of generality, assume that σ is a saddle connection joining O and P . Let ϕ be the covering automorphism. Connected components of $Y \setminus \Pi^{-1}(\sigma)$ are exchanged by ϕ since $\Pi^{-1}(\sigma)$ is invariant by ϕ . Let \hat{C} be the connected component of \hat{Q} in $Y \setminus \Pi^{-1}(\sigma)$. Since Q is fully ramified, \hat{Q} is invariant by ϕ . Thus the connected component \hat{C} is invariant by ϕ .

Therefore \hat{C} is the preimage of a subset of X denoted by \mathcal{C} . Since the boundaries of \hat{C} are preimages of σ the boundary of \mathcal{C} is σ . A saddle connection cannot separate a surface, thus \mathcal{C} is equal to $X \setminus \sigma$. Consequently $\hat{C} = Y \setminus \Pi^{-1}(\sigma)$, which means that $Y \setminus \Pi^{-1}(\sigma)$ is connected. This proves the lemma. \square

Remark 11. *The same result holds for preimages of saddle connection joining a ramification point P or Q to a singularity of X .*

We also need a description of short separating curves on Y . Since T is linearly recurrent, there is $\delta > 0$ such that the length of every closed curve on $g_t(X)$ is larger than 2δ for every $t \geq 0$. A closed curve on $g_t(Y)$ or a saddle connection on $g_t(X)$ is short if its length is less than $\delta/6$. Two points are *close together* if their distance is no more than $\delta/6$. We now use the fact that there are exactly two regular branched points.

Lemma 20.

- (1) *Let $\tau \geq 0$, there is at most one saddle connection joining P (resp. Q) to a singularity in $g_\tau(X)$ and there is at most one short saddle connection from P to Q .*
- (2) *Assume that $\hat{\gamma}$ is a short curve on $g_\tau(Y)$ and $\gamma = \Pi(\hat{\gamma})$ then*
 - *either P and Q are close together and close to a singularity Z and γ contains at least two components of the short segments joining P, Q and Z*
 - *or P and Q are not close together, but P is close to a singularity Z and Q to $Z' \neq Z$.*

Proof

First of all, by the hypothesis, there is at most one singularity close to P . If there exist two short segments from P to a singularity Z , there is a loop from Z to Z of length less than δ which is impossible.

Now, assume that there is a short separating curve $\hat{\gamma}$ on $g_\tau(Y)$. The curve $\gamma = \Pi\hat{\gamma}$ is a union of saddle connections joining P or Q to singularities of $g_\tau(X)$ or saddle connections from P to Q . By Lemma 19, both P and Q belong to γ . Using the first item of this lemma, we get the dichotomy

described in (2). □

End of proof of Theorem 18

We assume that there is a short $\hat{\gamma}$ separating curve on $g_\tau(Y)$ for some $\tau \geq 0$. We use the description of these curves obtained in Lemma 20.

First case Assume the distance between P and Q on $g_\tau(X)$ is larger than $\delta/3$ then $\gamma = \Pi(\hat{\gamma})$ is a union of saddle connections γ' joining P to a singularity Z in $g_\tau(X)$ and γ'' joining Q to a singularity $Z' \neq Z$. Since the vertical flow on Y is minimal, γ is not vertical. Thus the length of γ on $g_t(X)$ is a continuous function of t tending to infinity with t . Thus there is $t > \tau$ with $1/6 < |\gamma'| < 1/3$ and $|\gamma''| < 1/3$ (or vice versa) in $g_t(X)$. In $g_t(X)$, the point P is at distance at least $\delta/6$ from each singularity in $g_t(X)$ and from Q . This yields that γ'' is the only potentially short saddle connection on $g_t(X)$. Thus there is no short separating curve on $g_t(Y)$.

Second case P and Q are close together and close to a singularity denoted by Z . Then γ is the union of at least two of the three short segments joining P, Q, Z . By the same reasoning as in the previous case, γ is not vertical. Assume for instance that the segment γ' joining Z to P is not vertical and belongs to γ . For some $t > \tau$, in $g_t(X)$, $1/6 < |\gamma'| < 1/3$. This yields that there is at most one short saddle connection on $g_t(X)$: the segment joining P to Q or a segment from Q to a singularity. Consequently, there is no short separating curve on $g_t(X)$.

Therefore for every τ , there is a $t > \tau$ such that $g_t(X)$ has no short separating curve. This proves Theorem 18 applying McMullen's version of Masur's criterion. □

This theorem gives new examples of uniquely ergodic extensions, by using a sufficient condition of minimality, for example Proposition 13 above. However, if we start from a rotation, we do not get more examples than Merrill, and our criterion for unique ergodicity is intrinsically linked to the interval exchange structure, as the following example shows.

Example

Take T to be a rotation of angle α , viewed as an exchange of two intervals, separated by $1 - \alpha$, and two marked points satisfying $0 < \alpha < \zeta_1 < \zeta_1 + \alpha < \zeta_2 < \zeta_2 + \alpha < 1 - \alpha$, $N = 4$, $a_1 = 1, a_2 = 0, a_3 = 3$. T_f is a 16-interval exchange; the images by π of 1 to 16 are respectively 16, 5, 2, 15, 4, 9, 6, 3, 8, 13, 10, 7, 12, 1, 14, 11, and the images by ξ of 0 to 16 are respectively 15, 13, 14, 7, 8, 1, 2, 11, 12, 5, 6, 16, 0, 9, 10, 3, 4. Under ξ , the orbit of 0 has length 9 and is distinct from the orbit of 1; thus T_f is fully ramified, and uniquely ergodic if minimal. But if we take the same rotation but translate the origin by ζ_1 , we get a topologically isomorphic transformation T'_f , with different π' and η' , $\zeta'_1 = \zeta_2 - \zeta_1, \zeta'_2 = 1 - \zeta_1, a'_1 = a_2, a'_2 = a_3, a'_3 = a_1$, thus T'_f is not fully ramified as $a'_3 - a'_2$ is not coprime with N .

Thus the property of being fully ramified is not invariant by topological isomorphism, and does not give a necessary and sufficient condition of unique ergodicity when T is linearly recurrent.

3.3. Functions with two values. In this section we deal with the case when the function f has only two values. This situation is simpler than the one studied in the previous section.

Theorem 21. *Let $N \geq 2$ be an integer and T_f as in Definition 4 with $q = 1$. Assume that T is linearly recurrent and that T_f is minimal then T_f is uniquely ergodic.*

The proof follows the lines of the one given in the previous section and is more elementary. We only give a sketch of it. By the same argument, we only prove a geometrical version of Theorem 21.

Theorem 22. *Let X be a translation surface, assume that its g_t -orbit is bounded in its stratum. Let Y be a $\mathbb{Z}/N\mathbb{Z}$ -cover of X ramified over one regular point P and some singularities. If the vertical flow on Y is minimal then it is uniquely ergodic.*

Proof Let P be the point denoted by P_1 in Section 2.4, P^1, \dots, P^s its preimages. If there is a short curve on the $g_t(Y)$, it is a union of segments joining the P^i to singularities. This means that P is close to a singular point in X . On X marked at P , there is only one short saddle connection joining P to a singularity O . This saddle connection is not contracted to zero on $g_t(X)$ when t tends to infinity since the vertical flow on Y is minimal. Thus there is some t_1 larger than t_0 , such that the segment from O to P is not short in $g_{t_1}(X)$ and, as in the previous proof, P is not close to any other singularity in $g_{t_1}(X)$. Therefore there is no short curve on $g_t(Y)$. By Masur's criterion, the linear flow on Y is uniquely ergodic. \square

REFERENCES

- [1] P. ARNOUX: Un invariant pour les échanges d'intervalles et les flots sur les surfaces, (in French) *Thèse de 3e cycle*, Reims, 1981.
- [2] P. ARNOUX, G. RAUZY: Représentation géométrique de suites de complexité $2n+1$. (French) [Geometric representation of sequences of complexity $2n+1$] *Bull. Soc. Math. France* 119 (1991), no. 2, 199–215.
- [3] J. CASSAIGNE, F. NICOLAS: Factor complexity. *Combinatorics, automata and number theory*, 163–247, *Encyclopedia Math. Appl.*, 135, Cambridge Univ. Press, Cambridge, 2010.
- [4] S. FERENCZI: A generalization of the self-dual induction to every interval exchange transformation. *Ann. Inst. Fourier (Grenoble)* 64 (2014), no. 5, 1947–2002.
- [5] S. FERENCZI, L.Q. ZAMBONI: Structure of K -interval exchange transformations: induction, trajectories, and distance theorems. *J. Anal. Math.* 112 (2010), 289–328.
- [6] S. FERENCZI, L.Q. ZAMBONI: Languages of k -interval exchange transformations. *Bull. Lond. Math. Soc.* 40 (2008), no. 4, 705–714.
- [7] M.S. KEANE: Interval exchange transformations, *Math. Zeitsch.* 141 (1975), p. 25–31.
- [8] H. MASUR: Interval exchange transformations and measured foliations, *Ann. of Math. (2)* 115 (1982), 169–200.
- [9] H. MASUR: Hausdorff dimension of divergent Teichmüller geodesics, *Trans. Amer. Math. Soc.* 324 (1991), no. 1, 235–254.
- [10] H. MASUR, S. TABACHNIKOV: Rational billiards and flat structures, *Handbook of dynamical systems*, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.
- [11] C. MCMULLEN: Diophantine and ergodic foliations on surfaces, *J. Topol.* 6 (2013), no. 2, 349–360.
- [12] K. D. MERRILL: Cohomology of step functions under irrational rotations, *Israel J. Math.* 52 (1985), 320–340.
- [13] E. A. SATAEV: The number of invariant measures for flows on orientable surfaces, (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975), 860–878, translated in *Mathematics of the USSR-Izvestiya*, 9 (1975), 813–830.
- [14] K. SCHMIDT: Cocycles on ergodic transformation groups. *Macmillan Lectures in Mathematics*, Vol. 1. Macmillan Company of India, Ltd., Delhi, 1977. 202 pp.
- [15] M. STEWART: Irregularities of uniform distribution, *Acta Math. Acad. Sc. Hung.* 37 (1981), 1–39.
- [16] W. A. VEECH: Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2, *Trans. Amer. Math. Soc.* 140, (1969), 1–33.
- [17] W. A. VEECH: Finite group extensions of irrational rotations, *Israel J. Math.* 21 (1975), 240–259.
- [18] W. A. VEECH: Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math. (2)* 115 (1982), no. 1, 201–242.
- [19] J.C. YOCCOZ: Continued fraction algorithms for interval exchange maps: an introduction, *Frontiers in number theory, physics, and geometry. I*, 401–435, Springer, Berlin, 2006.

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE MARSEILLE, I2M - UMR 7373, 13453 MARSEILLE, FRANCE.

Email address: ssferenczi@gmail.com

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE MARSEILLE, I2M - UMR 7373, 13453 MARSEILLE, FRANCE.

Email address: hubert.pascal@gmail.com