



HAL
open science

RIGIDITY OF GENERALIZED VEECH 1969/SATAEV 1975 EXTENSIONS OF ROTATIONS

Sébastien Ferenczi, Pascal Hubert

► **To cite this version:**

Sébastien Ferenczi, Pascal Hubert. RIGIDITY OF GENERALIZED VEECH 1969/SATAEV 1975 EXTENSIONS OF ROTATIONS. Journal d'analyse mathématique, 2022. hal-03446721

HAL Id: hal-03446721

<https://hal.science/hal-03446721>

Submitted on 24 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RIGIDITY OF GENERALIZED VEECH 1969/SATAEV 1975 EXTENSIONS OF ROTATIONS

SÉBASTIEN FERENCZI AND PASCAL HUBERT

ABSTRACT. We look at d -point extensions of a rotation of angle α with r marked points, generalizing the examples of Veech 1969 and Sataev 1975, together with the square-tiled interval exchange transformations of [5]. We study the property of rigidity, in function of the Ostrowski expansions of the marked points by α : we prove that T is rigid when α has unbounded partial quotients, and that T is not rigid when the natural coding of the underlying rotation with marked points is linearly recurrent. But there remains an interesting grey zone between these two cases, in which we have only partial results on the rigidity question; they allow us to build the first examples of non linearly recurrent and non rigid interval exchange transformations.

In a founding paper of 1969 [15] W.A. Veech defines an extension of a rotation of angle α to two copies of the torus with a marked point β , the change of copy occurring on the interval $[0, \beta[$ (resp. $[\beta, 1[$ on a variant): for α with unbounded partial quotients and some values of β , they provide examples of minimal non uniquely ergodic interval exchange transformations. These systems were defined again independently, in a generalized way, by E.A. Sataev in 1975, in a beautiful but not very well known paper [13]: by taking r marked points and $r + 1$ copies of the torus, he gets minimal interval exchange transformations with a prescribed number of ergodic invariant measures; also, improvements on Veech's results were introduced by M. Stewart [14] and K.D. Merrill [11]. In the present paper, we study more general systems, by marking r points and taking d copies of the torus, for any $r \geq 1$, $d \geq 2$, and extending the rotation by the symmetric group S_d .

Though in general our marked points are not in $\mathbb{Z}(\alpha)$, we allow one of them to be $1 - \alpha$, so that our systems generalize also the square-tiled interval exchange transformations studied in [5]. In that paper, we focussed on the measure-theoretic property of rigidity, meaning that for some sequence q_n the q_n -th powers of the transformation converge to the identity (Definition 5 below). Experimentally, in the class of interval exchange transformations, the absence of rigidity is difficult to achieve (indeed, by Veech [16] it is true only for a set of measure zero of parameters) and all known examples satisfy also the word-combinatorial property of *linear recurrence* (Definition 3 below) for their natural coding. Indeed, for the systems studied in [5], we proved rigidity is equivalent to absence of linear recurrence, and thus to α having unbounded partial quotients. For the more general systems considered in the present paper, adaptations of the techniques of [5] do allow us to prove that T is rigid (for every invariant measure) and not linearly recurrent when α has unbounded partial quotients, and that T is uniquely ergodic, linearly recurrent, and not rigid when the natural coding of the underlying rotation with marked points is linearly recurrent (under an extra condition on the permutations, we prove also that T is not of rank one); this linear recurrence requires α to have bounded partial quotients and the marked points to satisfy some conditions on their Ostrowski expansions by α .

Date: January 6, 2021.

2010 Mathematics Subject Classification. Primary 37E05; Secondary 37B10.

But, in sharp contrast with [5], there is no similar equivalence in the present class of systems, and this leaves an interesting grey zone, when α has bounded partial quotients but the Ostrowski expansions of the marked points do not satisfy the conditions required for linear recurrence; this means that in the *Rokhlin towers* defined by the rotation (see Section 3.1 below) the marked points β_i come too close either to one another or to the points $0, \alpha, 1 - \alpha$. In these cases we prove some partial results, namely a sufficient condition (Theorem 15) for non-rigidity (for every ergodic invariant measure) and a sufficient condition (Theorem 17) for rigidity (for every invariant measure): the latter puts all the grey zone on the rigid side for Veech 1969, while, with two (or more) marked points, the former allows us to build the first known examples of non linearly recurrent and non rigid interval exchange transformations, answering Question 8 of [5]:

Theorem 1. *There exists a two-point extension of a rotation with two marked points which is a non linearly recurrent and non rigid interval exchange.*

These conditions are enough to give a full characterization of rigidity for the simplest generalizations of Veech 1969, when we take two copies of the torus and a small number of marked points (for higher numbers of marked points, the question is not untractable but the results become extremely tedious to state). In general, the grey zone seems quite complicated, with many different cases using different techniques, and we seem to be far from a complete characterization of rigidity in our class.

1. DEFINITIONS

1.1. Word combinatorics. We begin with basic definitions. We look at finite *words* on a finite alphabet $\mathcal{A} = \{1, \dots, k\}$. A word $w_1 \dots w_s$ has *length* $|w| = s$ (not to be confused with the length of a corresponding interval). The *concatenation* of two words w and w' is denoted by ww' .

Definition 1. A word $w = w_1 \dots w_s$ occurs at place i in a word $v = v_1 \dots v_{s'}$ or an infinite sequence $v = v_1 v_2 \dots$ if $w_1 = v_i, \dots, w_t = v_{i+s-1}$. We say that w is a factor of v .

A language L over \mathcal{A} is a set of words such if w is in L , all its factors are in L . A language L is minimal if for each w in L there exists n such that w occurs in each word of L with n letters.

The language $L(u)$ of an infinite sequence u is the set of its finite factors.

A word w is called *right special*, resp. *left special* if there are at least two different letters x such that wx , resp. xw , is in L . If w is both right special and left special, then w is called *bispecial*.

1.2. Symbolic dynamics and codings.

Definition 2. The symbolic dynamical system associated to a language L is the one-sided shift $S(x_0 x_1 x_2 \dots) = x_1 x_2 \dots$ on the subset X_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $s' < s$, $x_{s'} \dots x_s$ is in L .

For a word $w = w_1 \dots w_s$ in L , the cylinder $[w]$ is the set $\{x \in X_L; x_0 = w_1, \dots, x_{s-1} = w_s\}$.

For a system (X, T) and a finite partition $Z = \{Z_1, \dots, Z_\rho\}$ of X , the trajectory of a point x in X is the infinite sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_n = i$ if $T^n x$ falls into Z_i , $1 \leq i \leq \rho$.

Then $L(Z, T)$ is the language made of all the finite factors of all the trajectories, and $X_{L(Z, T)}$ is the coding of X by Z .

Note that the symbolic dynamical system (X_L, S) is minimal (in the usual sense, every orbit is dense) if and only if the language L is minimal as in Definition 1.

Definition 3. A language L or the symbolic system (X_L, S) is linearly recurrent if there exists K such that in L , every word of length n occurs in every word of length Kn .

1.3. Boshernitzan's criteria for symbolic systems.

Definition 4. For an invariant measure μ on (X_L, S) , let $e_n(S, \mu)$ be the smallest positive measure of the cylinders of length n .

The following sufficient condition for unique ergodicity is known as Boshernitzan's criterion; it is defined, named, and its sufficiency is proved for codings of interval exchange transformations in [17], then this is extended to every symbolic dynamical system in [3].

Proposition 2. If (X_L, S) is minimal, the system is uniquely ergodic if there exists an invariant measure such that $\limsup_{n \rightarrow +\infty} ne_n(S, \mu) > 0$.

The following result on linear recurrence is also due to M. Boshernitzan, but was written by T. Monteil in [6], Exercise 7.14.

Proposition 3. If (X_L, S) is minimal, the system is linearly recurrent if and only if there exists an invariant measure on (X_L, S) such that $\liminf_{n \rightarrow +\infty} ne_n(S, \mu) > 0$.

1.4. **Measure-theoretic properties.** Let (X, T, μ) be a probability-preserving dynamical system.

Definition 5. (X, T, μ) is rigid if there exists a sequence $q_n \rightarrow \infty$ such that for any measurable set A $\mu(T^{q_n} A \Delta A) \rightarrow 0$.

Definition 6. In (X, T) , a Rokhlin tower is a collection of disjoint measurable sets called levels $F, TF, \dots, T^{h-1}F$. F is the basis of the tower.

If X is equipped with a partition P such that each level $T^r F$ is contained in one atom $P_{w(r)}$, the name of the tower is the word $w(0) \dots w(h-1)$.

A symbolic system is generated by families of Rokhlin towers $F_{i,n}, \dots, T^{h_{i,n}-1}F_{i,n}$, $1 \leq i \leq K$, $n \geq 1$, if each level in each tower is contained in a single atom of the partition into cylinders $\{x_0 = i\}$, and for any word W in $L(T)$ there exist i and n such that W occurs in the name (for this partition) of the tower of basis $F_{i,n}$.

If a symbolic system is generated by families of Rokhlin towers, then, for any invariant measure, any measurable set can be approximated in measure by finite unions of levels of towers.

Definition 7. (X, T, μ) is of rank one if there exists a sequence of Rokhlin towers such that the whole σ -algebra is generated by the partitions $\{F_n, TF_n, \dots, T^{h_n-1}F_n, X \setminus \cup_{j=0}^{h_n-1} T^j F_n\}$.

1.5. **Rotations.** The dynamical behavior of a rotation R of angle α on the 1-torus is linked with the Euclid continued fraction expansion of α . We assume the reader is familiar with the notation $\alpha = [0, a_1, a_2, \dots]$; we define in the classical way the convergents $\frac{p_n}{q_n}$ by $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1, p_{n+1} = a_{n+1}p_n + p_{n-1}, q_{n+1} = a_{n+1}q_n + q_{n-1}$. Let $\alpha_n = |q_n\alpha - p_n|$. We recall

Definition 8. α has bounded partial quotients if the a_i are bounded.

Throughout the paper, except when we need more precision, we use C as a generic notation for constants.

2. VEECH AND SATAEV EXAMPLES

Definition 9. The Veech 1969 system is defined, if $Rx = x + \alpha$ modulo 1, by $T(x, s) = (Rx, \sigma(x)s)$, $s = 1, 2$, where

- $\sigma(x) = \sigma_0$ if x is in the interval $[0, \beta]$, σ_0 being the exchange E ,

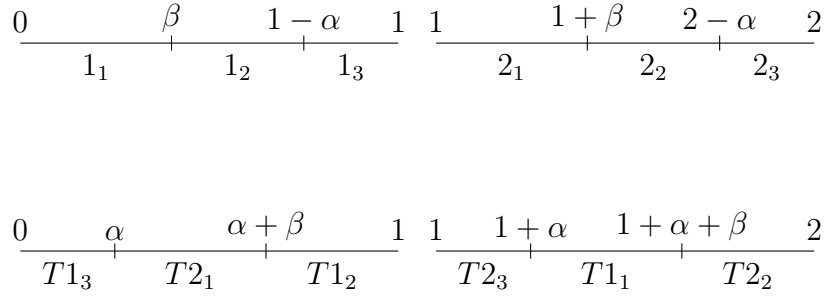


FIGURE 1. Veech 1969

- $\sigma(x) = \sigma_1$ if x is in the interval $[\beta, 1]$, σ_1 being the identity I .

This is defined (in a slightly different terminology) in the famous paper [15], where Veech considers also the variant where $\sigma_0 = I$, $\sigma_1 = E$. We can identify $[0, 1[\times\{s\}$ with $[s - 1, s[$; then T is also an *interval exchange transformation* as in Figure 1 (note that six intervals appear in the picture, but two of them move together thus T is indeed a 5-interval exchange transformation).

We can generalize Veech 1969 naturally by marking several points β_i , and taking more than two copies of the intervals: thus we take $r + 1$ different permutations on $\{1, \dots, d\}$, changing permutation each time we cross a point β_i : these transformations are defined by Sataev [13] in 1975 for $d = r + 1$, clearly without knowledge of Veech's work. Then these systems appear in [14] for $r = 1$ and all d , and in [11] for all r and d , but only in the particular case of extensions by the commutative group $\mathbb{Z}/d\mathbb{Z}$, where all the permutations are circular; some of these systems are also considered in [7].

In general the β_i will be chosen to be rationally independent from α , but if there is only one β and it is equal to $1 - \alpha$, we get the *square-tiled interval exchange transformations* of [5] (though the geometrical model is not the same); thus, to generalize both Veech 1969, Sataev 1975, and the square-tiled interval exchange transformations, we keep the possibility of choosing one of the β_i to be $1 - \alpha$.

Throughout this paper we take α irrational, $0 < \beta_1 < \dots < \beta_r < 1$ irrational, with possibly $\beta_t = 1 - \alpha$; more precisely, *if the index t exists, then $\beta_t = 1 - \alpha$; otherwise $\beta_i \neq 1 - \alpha$ for all i* . We choose $\sigma_0, \dots, \sigma_r$, permutations of $\{1, \dots, d\}$. We always suppose $\sigma_j \neq \sigma_{j+1}$, $0 \leq j \leq r - 1$, as otherwise we could delete some β_i . We take all the β_i , $i \neq t$, and all the $\beta_i - \beta_j$ not in $\mathbb{Z}(\alpha)$. We shall need sometimes another inequality, which generalizes the non-commutation condition used in [5], and which we call the *product inequality*: namely, when $\beta_t = 1 - \alpha$ we ask that $\sigma_r \sigma_{t-1} \neq \sigma_0 \sigma_t$, when $\beta_j \neq 1 - \alpha$ for all j we ask that $\sigma_r \neq \sigma_0$.

Definition 10. *The generalized Veech - Sataev system is defined, if $Rx = x + \alpha$ modulo 1, by $T(x, s) = (Rx, \sigma(x)s)$, $1 \leq s \leq d$, where*

- $\sigma(x) = \sigma_j$ if $\beta_j \leq x < \beta_{j+1}$, $1 \leq j \leq r - 1$,
- $\sigma(x) = \sigma_0$ if $0 \leq x < \beta_1$,
- $\sigma(x) = \sigma_r$ if $\beta_r \leq x < 1$.

T can be seen also as an interval exchange transformation on at most $d(r + 1)$ intervals, or with the following geometric model, generalizing the *Masur-Smillie geometrical model* for Veech 1969

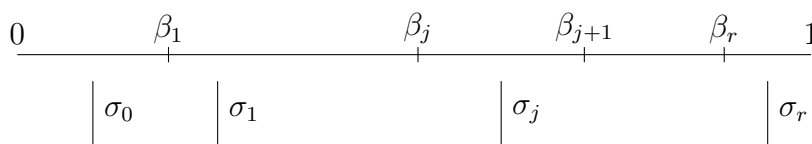


FIGURE 2. Generalized Veech - Sataev

[10]: we build a surface by gluing d tori, the interval $[\beta_i, \beta_{i+1}[$ in the right edge of the s -th torus being glued with the same interval in the left edge of the σ_i -th torus, and mutatis mutandis for the intervals $[0, \beta_1[$ and $[\beta_r, 1[$. Then we take the directional flow of slope α , going from one torus to the other when crossing the gluing lines, and T is its first return map on the union of the d left vertical sides.

This model allows us to give a minimality condition for generalized Veech - Sataev systems.

Proposition 4. *If α and all the β_i are irrational, all the β_i , $i \neq t$, and all the $\beta_i - \beta_j$ are not in $\mathbb{Z}(\alpha)$, an NSC for minimality is that no strict subset of $\{1 \dots d\}$ is invariant by all the σ_i .*

Proof

If a strict subset A of $\{1 \dots d\}$ is invariant by all the σ_i , then $\cup_{i \in A} [0, 1[\times \{i\}$ is invariant by T , and T is not minimal.

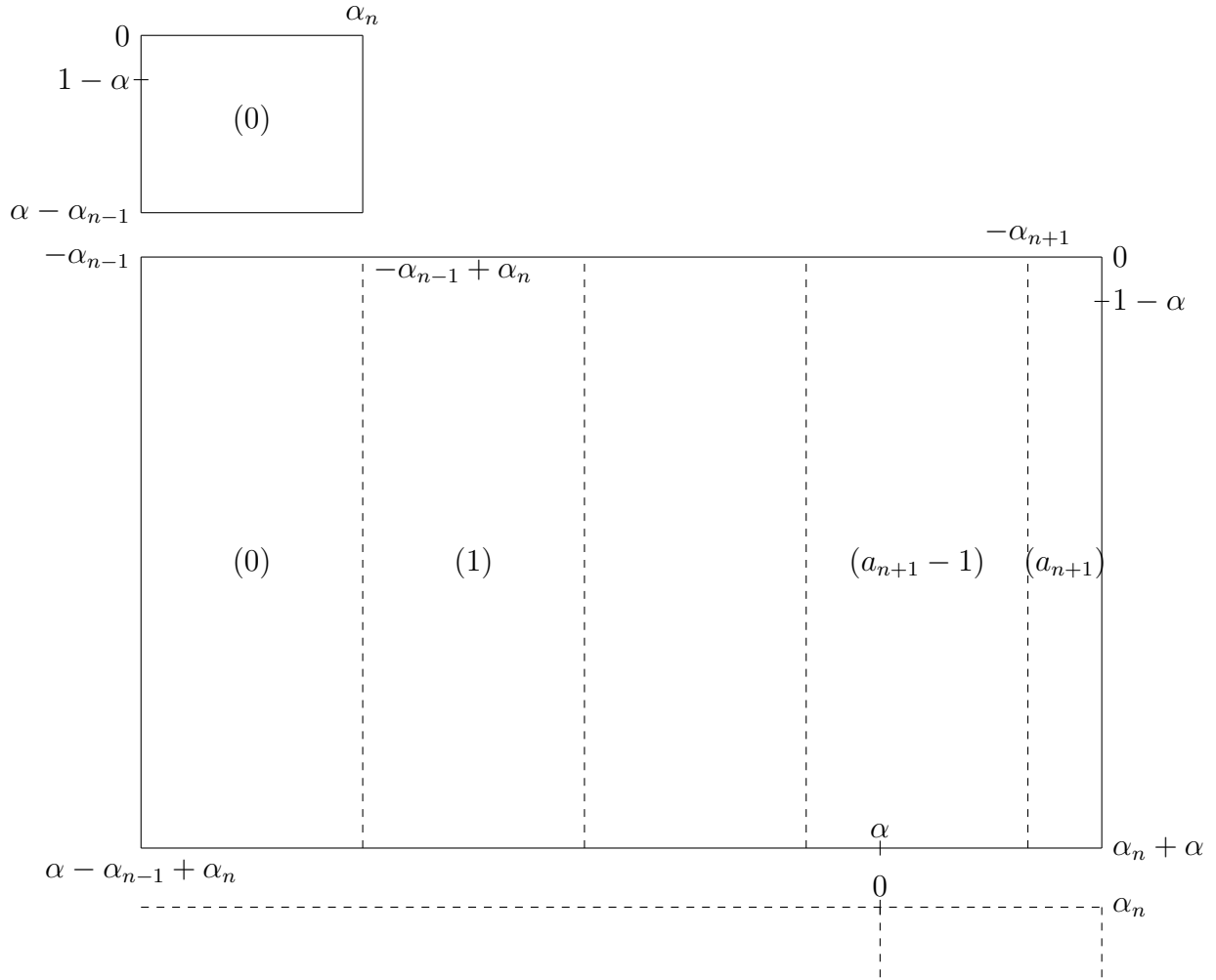
In the other direction, the condition on the permutations ensures that the surface defined above is connected, and the flow is minimal as the conditions on the β_i ensure there is no connection, except possibly (if $\beta_t = 1 - \alpha$) d connections between $1 - \alpha$ and 0, each one staying inside one torus; these connections do not separate the surface into several parts. \square

3. THE ROTATION WITH MARKED POINTS AND THE OSTROWSKI EXPANSION

3.1. Rokhlin towers. The rotation R can be coded either by the partition Z of the interval into $[0, 1 - \alpha[$ and $[1 - \alpha, 1[$, or by the partition Z' of the interval by the points β_1, \dots, β_r . This gives two languages L and L' , and two symbolic systems. The first one is the *natural coding* of R : it is assimilated to R itself and denoted by (X, R) . The second one is called the *rotation with marked points* and denoted by (X', S) .

It is well known, and written for example in [7], that for the rotation R its natural coding is generated by two families of Rokhlin towers, made of intervals. We shall now describe precisely the towers at stage n , or n -towers.

At each stage $n \geq 1$, there are one *large tower* made of q_n intervals (or *levels*) of length α_{n-1} and one *small tower* made of q_{n-1} intervals (or levels) of length α_n . These are described in Figure 3 if n is odd, we make all our comments in that case; the case when $n \geq 2$ is even can be deduced, mutatis mutandis, from Figure 4. Namely, the large tower is represented by the lower rectangle, and the small tower by the upper rectangle. The rotation R sends the basis $[-\alpha_{n-1} + \alpha_n + \alpha, \alpha_n + \alpha[$ to an interval which we put just above it, and call a level of the large tower; this interval is sent by R just above, and so on until, by R^{q_n-1} applied to the basis of the large tower, we reach the top of the large tower, $[-\alpha_{n-1}, 0[$. Then the left part of this top, $[-\alpha_{n-1}, -\alpha_{n-1} + \alpha_n[$ is sent by R onto the basis $[-\alpha_{n-1} + \alpha, -\alpha_{n-1} + \alpha_n + \alpha[$ of the small tower, and we go up in the small tower until, by $R^{q_{n-1}-1}$ applied to the basis of the small tower, we reach the top of the small tower, $[0, \alpha_n[$.

FIGURE 3. Rokhlin n -towers for the rotation, n odd

Where we go next from $[0, \alpha_n[$ or $[-\alpha_{n-1} + \alpha_n, \alpha_n[$ is shown at the bottom of the picture, one application of R goes to the point just above, in the basis of the large tower. For any x , $R^{q_n}x$ is the point situated at distance α_n to the left of x (extending the intervals if one of these points is not in the picture). Note that the three points $1 - \alpha$, 0 and α can be considered as very close together in the n -towers.

Each level of each tower is included in one atom of the partition Z . At the beginning, if $\alpha > \frac{1}{2}$, the large 1-tower has one level, the interval $[1 - \alpha, 1[$ and the small 1-tower has one level, the interval $[0, 1 - \alpha[$, Figure 3 is still valid. If $\alpha < \frac{1}{2}$, the 1-towers, which are still given by Figure 3, are more complicated, but we can define 0-towers: the large 0-tower has one level, the interval $[0, 1 - \alpha[$ and the small 0-tower has one level, the interval $[1 - \alpha, 1[$.

The large tower is partitioned from left to right into $a_{n+1} + 1$ columns, of width α_n except for the last one which is of width α_{n+1} , and except for $\alpha < \frac{1}{2}$, $n = 0$, where there are only a_1 columns and thus Figure 4 does not apply. We denote the columns as in Figure 3 or 4, and include the whole small tower in column (0). The description of R defines immediately the next towers: to get the large $n + 1$ -tower we stack the columns of the n -tower above each other, with the column

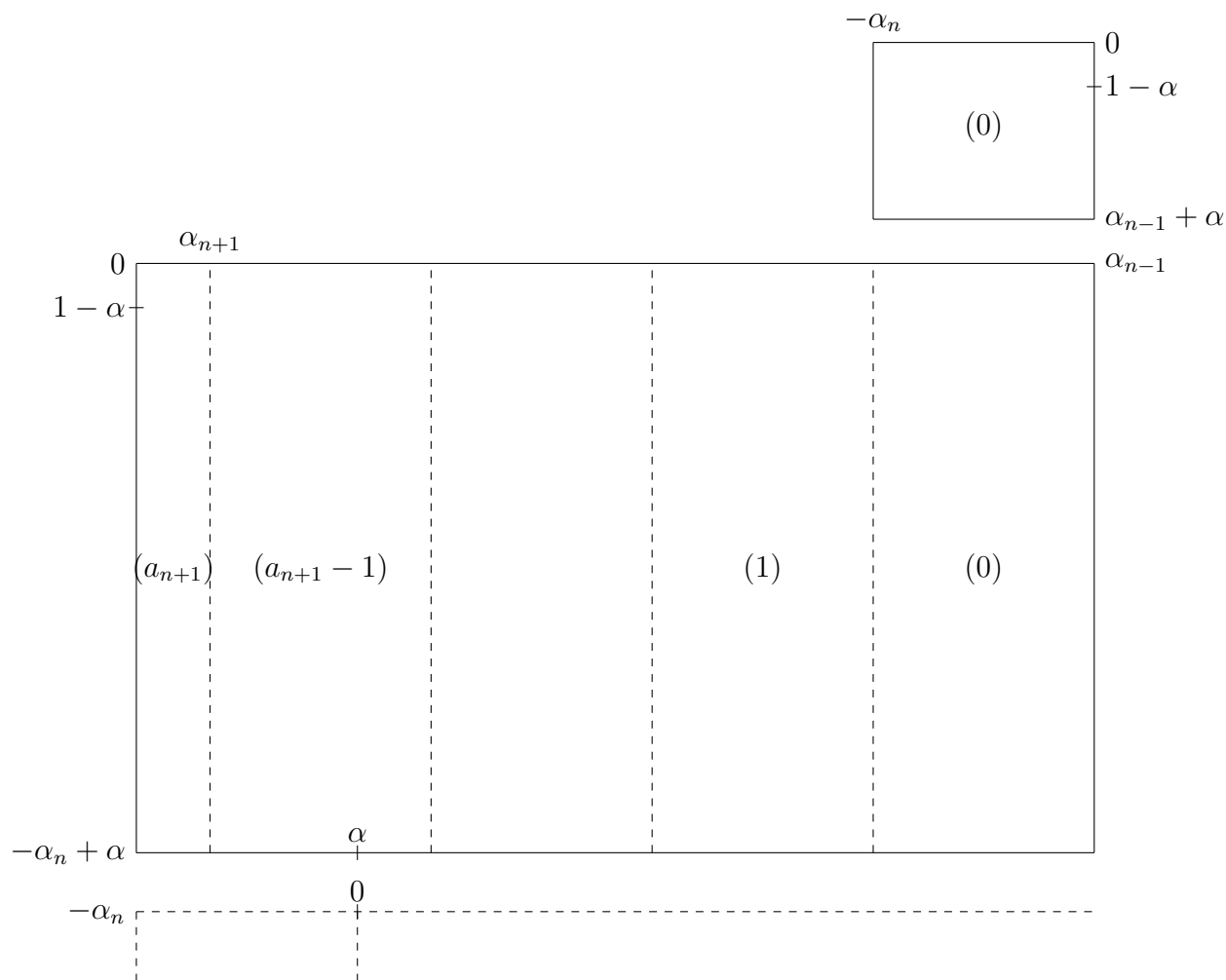


FIGURE 4. Rokhlin n -towers for the rotation, n even

$(a_{n+1} - 1)$ at the bottom, then $(a_{n+1} - 2), \dots, (0)$, and the small n -tower at the top, while the n -column (a_{n+1}) becomes the small $n + 1$ -tower.

Note that all levels are semi-open intervals, closed on the left, open on the right, and thus each column includes its left vertical side and not its right one.

The following lemma will be fundamental in our computations: we shall use it when α has bounded partial quotients, as it quantifies the linear recurrence of the natural coding of the rotation, but we can state and prove it in the general case.

Lemma 5. *Suppose x or y , or both, are in the basis of the large n -tower, and $x = y + z$, $\alpha_{n+1} \leq z \leq \alpha_n$. Then the smallest $k > 0$ such that α lies between $R^k x$ and $R^k y$, with $\alpha \neq R^k x$, is at least $q_n + q_{n-1}$ and at most $q_{n+2} + q_{n+1} + q_n$.*

Proof

y is at a distance z to the left of x ; then for all m $R^m y$ is at the same distance of $R^m x$ on the circle. We make all computations with n odd, the even case is similar. To simplify notations, we write them for x and y which are not on the sides of any n -columns, thus excluding a countable set.

However, we notice they are still valid on this countable set, because of our conventions that the columns are closed on the left, open on the right and we allow $\alpha = R^k y$ when saying α appears between the two orbits.

- (i) Suppose first x is in the basis of the large n -tower and not y . Then x is at a distance at most α_n from the left of the large tower, thus in column (0); y is to the left of the large tower and at less than α_n from it, thus between $-\alpha_{n-1} + \alpha$ and $-\alpha_{n-1} + \alpha_n + \alpha$, thus in the basis of the small tower (see Figure 9 below). y is at a distance d_1 from the left of this basis, x is at a distance $0 < d_2 < z$ from the left of the large tower, with $d_1 + z = d_2 + \alpha_n$. We make $q_n + q_{n-1}$ iterations of R . The orbit of x goes up through the large and small towers, and at the $q_n + q_{n-1}$ -th iteration hits the basis of the large tower, at a point situated d_2 to the right of α . The orbit of y goes up through the small tower, at the q_{n-1} -th iteration hits the basis of the large tower at a point situated d_1 to the right of α , then at the $q_n + q_{n-1}$ -th iteration hits this basis again, α_n left of the previous hit, thus left of α as $d_1 - \alpha_n = d_2 - z < 0$; and before the $q_n + q_{n-1}$ -th iteration α does not appear between the two orbits.
- (ii) Suppose x and y are in the basis of the large tower and in two different columns. Then these columns must be adjacent, and, after at most $a_{n+1}q_n$ iterations of R , during which α does not appear between the two orbits, we are in the situation of case (i).
- (iii) Suppose x and y are in the basis of the large tower and in the same column. This column cannot be column (a_{n+1}) , and x is at distance d_3 from the right of its column. After at most $a_{n+1}q_n + q_{n-1} = q_{n+1}$ iterations of R , during which α does not appear between the two orbits, the orbit of x hits the basis of the large tower, at a point situated d_3 from its right end, and y also, at distance $d_3 + z$ from the right. At this moment, if $d_3 < \alpha_{n+1}$, then the orbit of x is in column (a_{n+1}) and the orbit of y is in column $(a_{n+1} - 1)$ and we are in the situation of case (ii). If $d_3 > \alpha_{n+1}$, the orbits of x and y are in column $(a_{n+1} - 1)$, and we are again in case (iii), but with d_3 replaced by $d_3 - \alpha_{n+1}$. As $d_3 < \alpha_n$ and $\alpha_n = a_{n+2}\alpha_{n+1} + \alpha_{n+2}$, after at most a_{n+2} such laps, during which α does not appear between the two orbits, we are in the situation of case (ii).
- (iv) Suppose finally y is in the basis of the large n -tower and not x . Then x is to the right of the large tower and at less than α_n from it, and y is to the right of α ; after q_n iterations of R , during which α does not appear between the two orbits, we are in the situation of case (ii) or (iii).

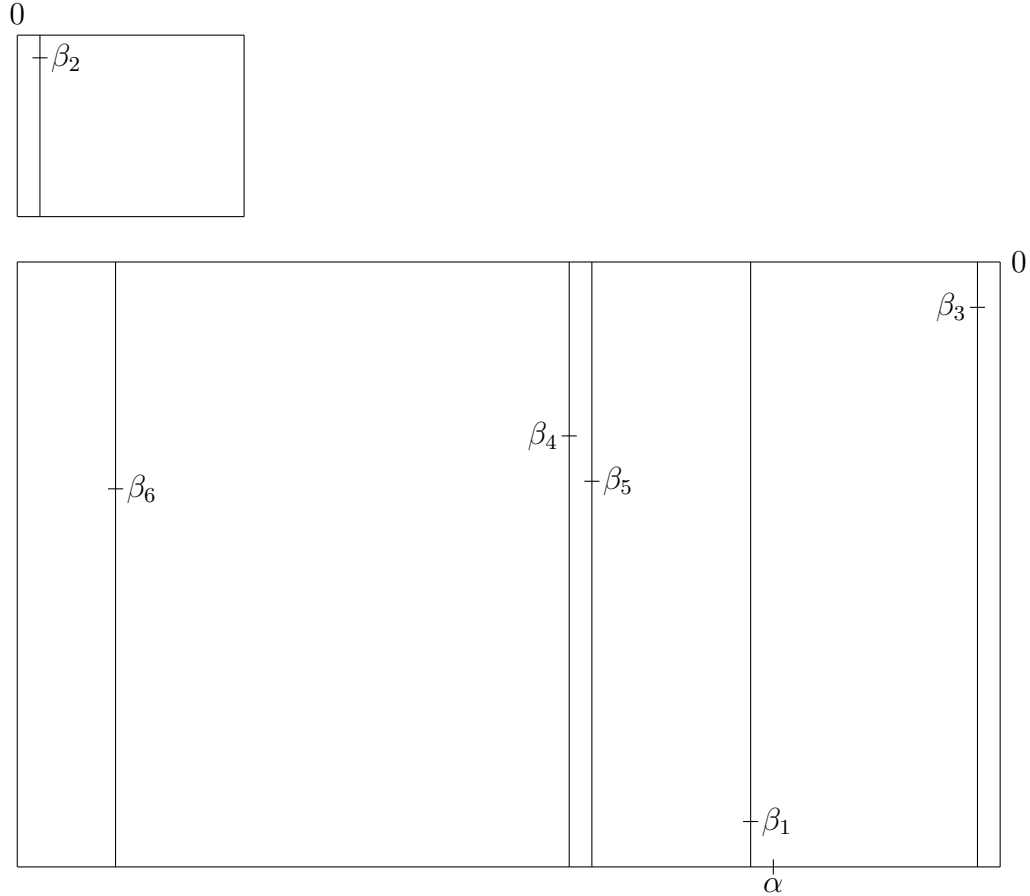
Thus, by taking case (i) for the minimum, and summing our estimates for the maximum, we get the required result. \square

Corollary 6. *Let β and β' be any two points on the circle. Suppose $x = y + z$, $\alpha_{n+1} \leq z \leq \alpha_n$, and β lies between x and y , with $\beta \neq x$. Then the smallest $k > 0$ such that β' lies between $R^k x$ and $R^k y$, with $\beta' \neq R^k x$, is at most $q_{n+2} + q_{n+1} + q_n$.*

Proof

By Lemma 5 this is true for $\beta' = \alpha$ and any β , just because β is in one of the n -towers and is the image of some point in the basis of the large one. As R commutes with every translation, this is true also for any β and β' . \square

3.2. Ostrowski expansion. We now put the points β_i , $i \neq t$, in the picture. By partitioning the two towers for R as in Figure 5 (for odd n), we get $r + 2$ towers generating the rotation with

FIGURE 5. Rokhlin n -towers for the rotation with marked points

marked points S , for which each level of each tower is included in one atom of the partition Z' (if $\beta_t = 1 - \alpha$, only $r + 1$ towers are needed as $1 - \alpha$ is on the side of one tower).

For each $1 \leq i \leq r, i \neq t$, and $n \geq 1$ we define $b_{n+1}(\beta_i)$ as an integer between 0 and a_{n+1} .

Definition 11. $b_{n+1}(\beta_i)$ is $b \neq 0$ if β_i is in column (b) of the large n -tower (for R), and 0 if β_i is either in column (0) of the large n -tower or in the small n -tower.

For odd n (resp. even $n \geq 2$) let $x_n(\beta_i)$ be the (positive) distance of β_i to the left (resp. right) side of the large and small n -towers in Figure 5 (resp. 6).

Proposition 7. For each $i \neq t$, the $b_n(\beta_i)$ are given by a form of alternating Ostrowski expansion of β_i by α , where the Markovian condition is $b_n(\beta_i) = a_n$ implies $b_{n+1}(\beta_i) = 0$. For $i \neq t$, β_i is in $\mathbb{Z}(\alpha)$ if and only if either $b_n(\beta_i) = a_n - 1$ for all n large enough, or $b_{2n}(\beta_i) = a_{2n}$ for all n large enough, or $b_{2n+1}(\beta_i) = a_{2n+1}$ for all n large enough. For $i \neq t, j \neq t$, $\beta_i - \beta_j$ is in $\mathbb{Z}(\alpha)$ if and only if $b_n(\beta_i) = b_n(\beta_j)$ for all n large enough.

Proof

We fix an $i \neq t$. Then

$$b_{n+1}(\beta_i) = \left[\frac{x_n(\beta_i)}{\alpha_n} \right].$$

Now, $x_{n+1}(\beta_i)$ is the distance of β_i to the right (resp. left) side of its n -column if n is odd (resp. even). Thus we get $x_n(\beta_i) = b_{n+1}(\beta_i)\alpha_n + \alpha_n - x_{n+1}(\beta_i)$ if β_i is not in column (a_{n+1}) , $x_n(\beta_i) = b_{n+1}(\beta_i)\alpha_n + \alpha_{n-1} - x_{n+1}(\beta_i)$ if β_i is in column a_{n+1} . Note that if β_i is in column (a_{n+1}) in the large n -tower, then it is in the small $n+1$ -tower. Thus $b_{n+1}(\beta_i) = a_{n+1}$ implies $b_{n+2}(\beta_i) = 0$, and this is the only Markovian condition they have to satisfy.

Thus when $x_n(\beta_i) = b_{n+1}(\beta_i)\alpha_n + \alpha_{n-1} - x_{n+1}(\beta_i)$, then $x_{n+1}(\beta_i) = \alpha_{n+1} - x_{n+2}(\beta_i)$ and $x_n(\beta_i) = a_{n+1}\alpha_n - x_{n+2}(\beta_i)$. Together with the formula when $b_{n+1}(\beta_i) < a_{n+1}$, this gives an expansion $x_1(\beta_i) = \sum_{n \geq 1} (-1)^{n+1} \bar{b}_{n+1} \alpha_n$ with $\bar{b}_n = b_n(\beta_i) + 1$ if $b_n(\beta_i) < a_n$, $\bar{b}_n = b_n$ if $b_n(\beta_i) = a_n$. Thus the \bar{b}_n satisfy the Markovian condition $\bar{b}_{n-1} = a_{n-1}$ if $\bar{b}_n = 0$.

Thus we identify the \bar{b}_n with the alternating Ostrowski expansion of $x_1(\beta_i)$ by α defined in [1]. If $\alpha > \frac{1}{2}$, $x_1(\beta_i)$ is either β_i or $\beta_i + \alpha - 1$; if $\alpha < \frac{1}{2}$, using the 0-towers, we can define $0 \leq b_1(\beta_i) \leq a_1 - 1$ and $x_0(\beta_i)$ in the usual way, so that $x_1(\beta_i) = -x_0(\beta_i) + ((b_1(\beta_i) + 1)\alpha) \wedge (1 - \alpha)$, and $x_0(\beta_i)$ is either $1 - \beta_i$ or $1 - \alpha - \beta_i$. In both cases, we get an expansion of β_i by α , which is $\beta_i = \sum_{n \geq 0} (-1)^{n+1} \bar{b}_{n+1} \alpha_n$ with a suitable \bar{b}_1 , thus our $b_n(\beta_i)$ do provide a form of alternating Ostrowski expansion of β_i by α .

The last conditions come from the fact that if $\beta_i = R^k \alpha$ for $k > 0$ then β_i is in the same column as α , namely column $a_n - 1$, in the $n - 1$ -towers for all n large enough, while if $\beta_i = R^k \alpha$ for $k < 0$ then β_i is in the same column as 0, and this alternates between 0 (in the small tower) and a_n , and in both cases the converse is true by construction of the towers, as the vertical distance from β_i to α (resp. 0) in the $n - 1$ -towers is ultimately constant while the horizontal distance tends to zero with n . Similarly, $\beta_i = R^k \beta_j$ if and only if in the $n - 1$ -towers β_i is in the same column as β_j for all n large enough. \square

As a consequence, we can build β_i with any prescribed sequence $0 \leq b_n(\beta_i) \leq a_n$ satisfying the Markovian condition.

3.3. Linear recurrence. To prove the next theorem, we need some new notations.

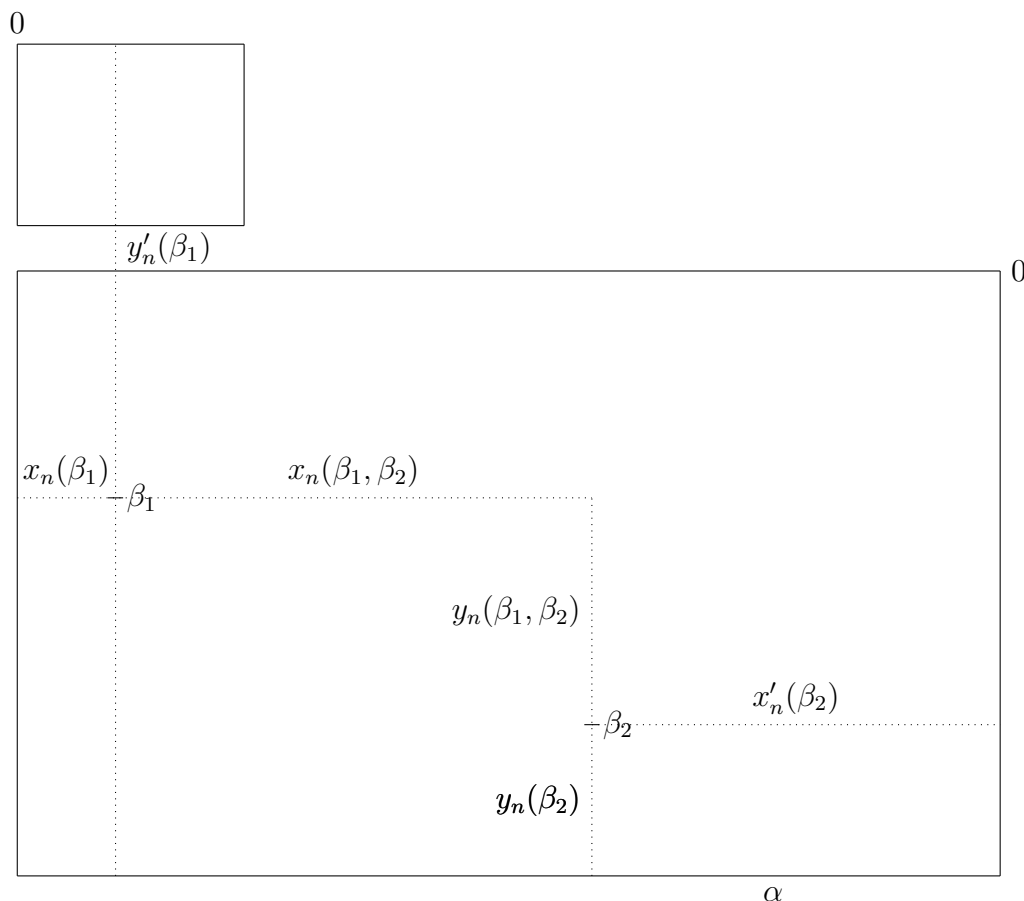
Definition 12. For a given n , each β_i , $i \neq t$, appears in a single position in the n -towers as in Figure 6; it is determined by $x_n(\beta_i)$, from Definition 11. We shall use also

- $y_n(\beta_i) = y$ if $\beta_i = R^y \beta'_i$ where β'_i is in the basis of the large n -tower,
- $x'_n(\beta_i) = \alpha_{n-1} - x_n(\beta_i)$,
- $x_n(\beta_i, \beta_j) = x_n(\beta_j, \beta_i) = |x_n(\beta_i) - x_n(\beta_j)|$,
- $x_n(\beta_i, \alpha) = x_n(\alpha, \beta_i) = |x'_n(\beta_i) - \alpha|$,
- $y'_n(\beta_i) = q_n - y_n(\beta_i)$ if β_i is in the large n -tower, $y'_n(\beta_i) = q_n + q_{n-1} - y_n(\beta_i)$ if β_i is in the small n -tower,
- when β_i is in the small tower, $y''_n(\beta_i) = y_n(\beta_i) - q_n$,
- $y_n(\beta_i, \beta_j) = y_n(\beta_j, \beta_i) = |y_n(\beta_i) - y_n(\beta_j)|$.

It is worth mentioning that β_i and β_j are close to each other in the n -towers vertically either if $y_n(\beta_i, \beta_j)$ is small or if $y_n(\beta_i) + y'_n(\beta_j)$ is small, and β_i and β_j are close to each other in the n -towers horizontally either if $x_n(\beta_i, \beta_j)$ is small or if $x_n(\beta_i) + x'_n(\beta_j)$ is small. Though this will not be mentioned explicitly, each time we claim β_i and β_j are far from each other in one of these senses, this means that we have checked both conditions.

Theorem 8. The symbolic system (X', S) is linearly recurrent if and only all the following conditions are satisfied

- α has bounded partial quotients,

FIGURE 6. Positioning the β_i in the n -towers

- for each $i \neq t$, the number of consecutive n such that $b_n(\beta_i) = a_n - 1$ is bounded,
- for each $i \neq t$, the number of consecutive n such that $b_{2n}(\beta_i) = a_{2n}$ and the number of consecutive n such that $b_{2n+1}(\beta_i) = a_{2n+1}$ are bounded,
- for each $i \neq t$, $j \neq t$ with $j \neq i$, the number of consecutive n such that $b_n(\beta_j) = b_n(\beta_i)$ is bounded.

Proof

We suppose first our conditions are not satisfied.

If α has unbounded partial quotients, there exists n such that $q_n \alpha_n$ is arbitrarily small. In the large n -tower, with α in the basis and $1 - \alpha$ just below 0, we see a cylinder, for the natural coding, of length $q_n - 1$ and Lebesgue measure α_n ; this is a union of cylinders for the coding with marked points, of the same length and of smaller measure. As the Lebesgue measure is the only invariant measure by R , this contradicts linear recurrence by Proposition 3.

If $b_{n+1}(\beta_i) = a_{n+1} - 1$, then by construction of the towers $x_{n+1}(\beta_i, \alpha) = x_n(\beta_i, \alpha)$ and $y_{n+1}(\beta_i) = y_n(\beta_i)$. If this holds for all $M \leq n \leq M+N$, then $x_M(\beta_i, \alpha) = x_{M+N}(\beta_i, \alpha) \leq \alpha_{M+N}$ and $y_{M+N}(\beta_i) = y_M(\beta_i) \leq q_M$. Thus, for example, in the large $M + N$ -tower we see a cylinder (for the coding with marked points) of measure $x_{M+N}(\beta_i, \alpha)$ and length $y_{M+N}(\beta_i)$. The product of these quantities is at most $q_M \alpha_{M+N} \leq \theta^{-N} q_{M+N} \alpha_{M+N} \leq \theta^{-N} C$, where θ is the golden ratio. Thus, if N is allowed to be arbitrarily large, this contradicts linear recurrence by Proposition 3.

If $b_{n+1}(\beta_i) = a_{n+1}$, then β_i is in the small $n + 1$ -tower, with $x'_{n+1}(\beta_i) = x_n(\beta_i)$ and $y'_{n+1}(\beta_i) = y'_n(\beta_i)$, and then $x'_{n+1}(\beta_i) = x_{n+2}(\beta_i)$ and $y'_{n+1}(\beta_i) = y'_{n+2}(\beta_i)$. If this holds for all $M \leq n \leq M + 2N$, $y'_{M+2N}(\beta_i) = y'_M(\beta_i) \leq q_M$. In the large $M + 2N$ -tower we see a cylinder of measure $x_{M+2N}(\beta_i) \leq \alpha_{M+2N}$ and length $y'_{M+2N}(\beta_i)$. If N is allowed to be arbitrarily large, we conclude as in the previous case.

If $b_{n+1}(\beta_i) = b_{n+1}(\beta_j)$, then $x_{n+1}(\beta_i, \beta_j) = x_n(\beta_i, \beta_j)$ and $y_{n+1}(\beta_i, \beta_j) = y_n(\beta_i, \beta_j)$. If this holds for all $M \leq n \leq M + N$, then $y_{M+N}(\beta_i, \beta_j) = y_M(\beta_i, \beta_j) \leq q_M$. In the $M + N$ -towers we see a cylinder of measure $x_{M+N}(\beta_i, \beta_j) \leq \alpha_{M+N}$ and length $y_{M+N}(\beta_i, \beta_j)$. If N is allowed to be arbitrarily large, we conclude as in the previous cases.

We suppose now all our conditions are satisfied. In particular, α has bounded partial quotients.

If $b_{n+1}(\beta_i) \neq a_{n+1} - 1$, then $x_n(\beta_i, \alpha) \geq \alpha_{n+2}$. Otherwise, $x_n(\beta_i, \alpha) = x_m(\beta_i, \alpha)$ for the first $m > n$ for which $b_{m+1}(\beta_i) \neq a_{m+1} - 1$, and we know $m \leq n + K$. Thus we get that for all n ,

$$x_n(\beta_i, \alpha) \geq \alpha_{n+K+2} \geq C\alpha_n.$$

If $b_n(\beta_i) \neq a_n - 1$, then by construction of the towers $y_n(\beta_i) \geq q_{n-1}$. Otherwise, $y_n(\beta_i) = y_m(\beta_i)$ for the last $m < n$ for which $b_m(\beta_i) \neq a_m - 1$, and we know $m \leq n - K$. Thus we get that for all n ,

$$y_n(\beta_i) \geq q_{n-K-1} \geq Cq_n.$$

If $b_{n-1}(\beta_i) \neq a_{n-1} - 1$, by construction of the towers the result $y_{n-1}(\beta_i) \geq Cq_{n-1}$ implies, when $y''_n(\beta_i)$ is defined, that

$$y''_n(\beta_i) \geq Cq_n.$$

If $b_{n+1}(\beta_i) \neq a_{n+1}$, then $x'_n(\beta_i) \geq \alpha_{n+1}$. Otherwise, $x'_n(\beta_i) = x'_m(\beta_i)$ for the first $m > n$ such that $m - n$ is even and $b_{m+1}(\beta_i) \neq a_{m+1}$, and we know $m \leq n + K$. Thus we get that for all n ,

$$x'_n(\beta_i) \geq \alpha_{n+K+1} \geq C\alpha_n.$$

If $b_{n+2}(\beta_i) \neq a_{n+2}$, then $x'_{n+1}(\beta_i) \geq \alpha_{n+2}$ and by construction of the towers $x_n \geq \alpha_{n+2}$ (β_i being far from one side of the $n + 1$ -towers, is far from the opposite side of the n -towers). Otherwise, $x'_{n+1}(\beta_i) = x'_{m+1}(\beta_i)$ for the first $m > n$ such that $m - n$ is even and $b_{m+2}(\beta_i) \neq a_{m+2}$, and we know $m \leq n + K$. Thus we get that for all n ,

$$x_n(\beta_i) \geq \alpha_{n+K+2} \geq C\alpha_n.$$

If $b_{n-1}(\beta_i) \neq a_{n-1}$, then β_i is not in the small $n - 1$ -tower, thus far from the top in the n -towers: we have $y'_n \geq q_{n-1}$. Otherwise, $y_n(\beta_i) = y_m(\beta_i)$ for the last $m < n$ such that $n - m$ is even and $b_{m-1}(\beta_i) \neq a_{m-1}$, and we know $m \leq n - K$. Thus we get that for all n ,

$$y'_n(\beta_i) \geq q_{n-K-1} \geq Cq_n.$$

If $b_{n+1}(\beta_i) \neq b_{n+1}(\beta_j)$, then β_i and β_j are not in the same column in the n -towers. Because of the previous results on x_n and x'_n , each of them is at a distance greater than $C\alpha_n$ from the sides of their column, thus $x_n(\beta_i, \beta_j) \geq C\alpha_n$. Otherwise, $x_n(\beta_i, \beta_j) = x_m(\beta_i, \beta_j)$ for the first $m > n$ for which $b_{m+1}(\beta_i) \neq b_{m+1}(\beta_j)$, and we know $m \leq n + K$. Thus for all n ,

$$x_n(\beta_i) \geq C\alpha_{n+K} \geq C\alpha_n.$$

If $b_n(\beta_i) \neq b_n(\beta_j)$, then by construction of the towers $y_n(\beta_i, \beta_j) \geq q_{n-1}$. Otherwise, $y_n(\beta_i, \beta_j) = y_m(\beta_i, \beta_j)$ for the last $m < n$ for which $b_m(\beta_i) \neq b_m(\beta_j)$, and we know $m \leq n - K$. Thus we get that for all n ,

$$y_n(\beta_i, \beta_j) \geq q_{n-K-1} \geq Cq_n.$$

A cylinder H of length h (for the coding with marked points) is an interval $[y, x[$ for which each iterate by R^{-m} , $1 \leq m \leq h - 1$, is in a single atom of Z' . For a given measure $\mu(H) = x - y$, the minimal value of h is reached when either x and $R_{-h+1}y$, or y and $R_{-h+1}x$, are endpoints of atoms of Z' (otherwise the interval $[y, x[$ could be extended to the left or to the right). We take n such that $\mu(H)$ is smaller than α_n ; then as in the proof of Lemma 5 we see H in the n -towers or less than α_n from the right or left of Figure 7. Then the above computations imply that h is at least Cq_n and $\mu(H)$ at least $C\alpha_n$. Hence we get the linear recurrence from Proposition 3. \square

The following lemma will be used later.

Lemma 9. *If (X', S) is linearly recurrent, when W is a bispecial word in $L(S)$, of length greater than an initial constant C_0 , then if WU is in $L(S)$ with fixed $|U| \leq C|W|$, then U can only be one of two words U_1 and U_2 , where the first letters of U_1 and U_2 are different, possibly the second letters of U_1 and U_2 are different, and then the l -th letters of U_1 and U_2 are the same for $l \leq |U_1| \wedge |U_2|$.*

Proof

This is proved by looking in the towers for S , using the fact that, by the proof of Theorem 8, all $y_n(\beta_i)$, $y'_n(\beta_i)$, $y''_n(\beta_i)$ and $y_n(\beta_i, \beta_j)$ are at least Cq_n . Then W corresponds to a set of trajectories which coincides on $|W|$ consecutive symbols, but some (in particular, the leftmost and rightmost ones) are different on the letter before and the letter after. If all these trajectories are at a distance between α_{n+1} and α_n for some $n \geq 2$, then W can be seen in the n -towers.

As W is right special, it must end just before we see either a β_i , $i \neq t$, or $1 - \alpha$ between the leftmost and rightmost trajectories in W (as in Lemma 5 the rightmost one is allowed to hit the considered β_i or $1 - \alpha$ but not the leftmost). In the first case, these two trajectories disagree on the level containing β_i ; in the second case, the two trajectories disagree left and right of $1 - \alpha$, and on the next letter as they are left and right of 0; in both cases, then they agree again until we see again $1 - \alpha$ or some β_j between the leftmost and rightmost trajectories in W , thus for a length at least Cq_n . As W is left special, it begins just after we see either a β_i , $i \neq t$, or 0 between the leftmost and rightmost trajectories in W , thus by Corollary 6 its length is at most $q_n + q_{n+1} + q_{n+2} \leq C'q_n$, and thus the claimed property is proved. \square

4. RIGIDITY FOR GENERALIZED VEECH -SATAEV

4.1. The natural coding of T . We look now at the *natural coding* of T , namely its coding by the partition into the $d(r + 1)$ intervals used in Definition 10 (though they are not necessarily the intervals of continuity of T , see Figure 1 above), and we call it (Y, T) . We denote by s_i the i -th interval in the s -th copy of $[0, 1[$. A trajectory x of T under this natural coding projects on a trajectory $\phi(x)$ of the rotation with marked points (X', S) , by applying the map $\phi(s_i) = i$ letter to letter. Because all the σ_i are bijective, and their compositions also, as in Lemma 5 of [5] for any word w in $L(T)$, there are exactly d words v such that $\phi(w) = \phi(v)$, and for each of these words either $v = w$ or on the letters $v_i \neq w_i$ for all i .

As (X', S) is generated by the $r + 2$ towers in Figure 5, (Y, T) is generated by $d(r + 2)$ Rokhlin towers. More precisely, by construction of the towers, for all n , the trajectories of the natural coding of R are covered by disjoint occurrences of M_n and P_n , the names of the large and small n -towers. The trajectories of the coding with marked points S are covered by the names of the towers in Figure 5: these are denoted by $P_{n,i}$, $1 \leq i \leq r_1 < r + 2$, and $M_{n,j}$, $r_1 + 1 \leq j \leq r + 2$,

r_1 depending on n (we number them from right to left if n is odd, from left to right otherwise). The trajectories of T are covered by $d(r+2)$ words $P_{n,i,j}$ and $M_{n,i,j}$, $1 \leq j \leq d$ which are all the words which project on $P_{n,i}$ and $M_{n,i}$ by ϕ .

Proposition 10. *If the minimality condition of Proposition 4 is satisfied, (Y, T) is linearly recurrent if and only if (X', S) is linearly recurrent. In this case, (Y, T) is uniquely ergodic.*

Proof

Let $[w]$ be a cylinder for (Y, T) : for the Lebesgue measure μ on both sets we have $\mu[w] = \frac{1}{d}\mu[\phi w]$, and, for any invariant measure ν on (Y, T) , on (X', S) ν projects on μ , the unique invariant measure, thus $\mu[\phi w] = \sum_{\phi v = \phi w} \nu[v] \geq \nu[w]$. Hence, under the assumption of minimality, the result on linear recurrence in both directions comes from Proposition 3, while unique ergodicity comes from Proposition 2. \square

4.2. The non-exotic cases. We use now all the preliminary work to derive results generalizing those in [5]. We do consider these generalizations as non-trivial but do not claim them to be unexpected.

Proposition 11. *If α has unbounded partial quotients, (Y, T) is rigid for any invariant measure.*

Proof

In trajectories of R , by construction of the towers we have $P_{n+1} = P_n^{a_{n+1}} M_n$, $M_{n+1} = P_n$ for all n . Thus $P_{n+2} = (P_n^{a_{n+1}} M_n)^{a_{n+2}} P_n$, $M_{n+2} = P_{n+1} = P_n^{a_{n+1}} M_n$. As M_n is shorter than P_n disjoint occurrences of the word $P_n^{a_{n+1}}$ fill a proportion at least $1 - \frac{2}{a_{n+1}+1}$ of the length of both M_{n+2} and P_{n+2} .

In trajectories of S , the construction of the towers and the above remark imply that a proportion at least $1 - \frac{2}{a_{n+1}+1}$ of the length of all $M_{n+2,j}$ and $P_{n+2,j}$ is covered by concatenations of the type $P_{n,j_1} \dots P_{n,j_{a_{n+1}}}$ of length $q_n a_{n+1}$. Moreover, all these concatenation contain, at the same place, cycles of the form $P_{n,i_j}^{c_{n,j}}$, where the $c_{n,j}$, $1 \leq j \leq r_2 \leq r+1$ (r_2 depending on n) are the successive numbers of n -columns containing no β_l , between two column containing at least one β_l or between the sides of the towers and a column containing at least one β_l (here column (0) is replaced by its intersection with the large tower). Thus $\sum_{j=1}^{r_2} c_{n,j} \geq a_{n+1} - r$.

In trajectories of T , we look at the words which project by ϕ on cycles $P_{n,i}^c$. n and i being fixed, each $P_{n,i,j}$ can be followed by exactly one $P_{n,i,j}$, and thus the $P_{n,i,j}$, $1 \leq j \leq d$, are grouped into at most d disjoint strings, each one containing at most d words $P_{n,i,j}$. After the last $P_{n,i,j}$ of each string, the only $P_{n,i,j'}$ we can see is the first one of the same string. Then, if we move by $T^{d!q_n}$ inside one of the words which project on the cycle $P_{n,i_j}^{c_{n,j}}$, we go to the same level in the same tower of name $P_{n,i,j}$, except if we are in the last $d!$ words projecting on this cycle. In each concatenation $P_{n,j_1} \dots P_{n,j_{a_{n+1}}}$ mentioned above, these ‘‘good’’ words represent $\sum_{j=0}^{r_2} (c_{n,j} - d!) \vee 0 \geq a_{n+1} - r - (r+1)d!$ of the words in $L(T)$ projecting on that concatenation, and thus a proportion at least $0 \vee (1 - \frac{2rd!}{a_{n+1}})$ of the length of all $M_{n+2,i,j}$ and $P_{n+2,i,j}$.

All the levels of the same n -tower have the same measure by a given invariant μ , thus if E is a union of levels of the towers of name $P_{n,i,j}$, we have $\mu(E \Delta T^{d!q_n} E) \leq \frac{2rd!}{a_{n+1}}$. Now for every set E and n large enough, E can be δ_n -approximated (for the invariant measure μ) by unions of levels of the towers of name $P_{n,i,j}$ or $M_{n,i,j}$; but the towers of name $M_{n,i,j}$ have total measure at most $\frac{2}{a_{n+1}}$ since they represent a smaller fraction of the length of all $M_{n+2,i,j}$ and $P_{n+2,i,j}$. Thus

$\mu(E\Delta T^{d q_n} E) \leq \frac{2rd+4}{a_{n+1}} + \delta_n$. Hence if the a_n are unbounded T is rigid. \square

The notion of *average \bar{d} -separation* is defined in [5], where comments and explanations on this and related notions can be found.

Definition 13. For two words of equal length $w = w_1 \dots w_N$ and $w' = w'_1 \dots w'_N$, their *Hamming or \bar{d} -distance* is $\bar{d}(w, w') = \frac{1}{N} \#\{i; w_i \neq w'_i\}$.

A language L on an alphabet \mathcal{A} is *average \bar{d} -separated* for an integer $e \geq 1$ if there exists a language L' on an alphabet \mathcal{A}' , a K to one (for some $K \geq e$) map ϕ from \mathcal{A} to \mathcal{A}' , extended by concatenation to a map ϕ from L to L' , such that for any word w in L , there are exactly K words v such that $\phi(w) = \phi(v)$, and for each of these words either $v = w$ or $\bar{d}(w, v) = 1$, and a constant C , such that if v_i and v'_i , $1 \leq i \leq e$, are words in L , of equal length N , satisfying

- $\sum_{i=1}^e \bar{d}(v_i, v'_i) < C$,
- $\phi(v_i)$ is the same word u for all i ,
- $\phi(v'_i)$ is the same word u' for all i ,
- $v_i \neq v_j$ for $i \neq j$.

Then $\{1, \dots, N\}$ is the disjoint union of three (possibly empty) integer intervals I_1, J_1, I_2 (in increasing order) such that

- $v_{i, J_1} = v'_{i, J_1}$ for all i ,
- $\sum_{i=1}^e \bar{d}(v_{i, I_1}, v'_{i, I_1}) \geq 1$ if I_1 is nonempty,
- $\sum_{i=1}^e \bar{d}(v_{i, I_2}, v'_{i, I_2}) \geq 1$ if I_2 is nonempty,

where $w_{i, H}$ denotes the word made with the h -th letters of the word w_i for all h in H .

This implies in particular that $\#J_1 \geq N(1 - \sum_{i=1}^e \bar{d}(v_i, v'_i))$.

We call \bar{d} -separation the *average \bar{d} -separation* with $K = e = 1$, $L = L'$, ϕ the identity.

The proof of next proposition will follow step by step the proof of Proposition 44 of [5]. The main difference is that in [5] $L(T)$ projects by ϕ on $L(R)$, while here it projects on the more complicated $L(S)$. Hence Lemma 9 above will replace Lemma 42 of [5].

Proposition 12. *If the product inequality before Definition 10 and the minimality condition of Proposition 4 are satisfied, and the rotation with marked points (X', S) is linearly recurrent, $L(T)$ is average \bar{d} -separated with $e = d$.*

Proof

We take $L' = L(S)$, $K = d$. Let v_i and v'_i be as in Definition 13.

We compare first u and u' ; note that if we see l in some word $\phi(z)$ we see some s_l at the same place on z ; thus $\bar{d}(z, z') \geq \bar{d}(\phi(z), \phi(z'))$ for all z, z' ; in particular, if $\bar{d}(u, u') = 1$, then $\bar{d}(v_i, v'_i) = 1$ for all i and our assertion is proved.

Thus we can assume $\bar{d}(u, u') < 1$. We partition $\{1, \dots, N\}$ into successive integer intervals where u and u' agree or disagree: we get intervals $I_1, J_1, \dots, I_g, J_g, I_{g+1}$, where g is at least 1, the intervals are nonempty except possibly for I_1 or I_{g+1} , or both, and for all j , $u_{J_j} = u'_{J_j}$, and, except if I_j is empty, u_{I_j} and u'_{I_j} are completely different, i.e. their distance \bar{d} is one.

Then for $i \leq g - 1$, the word $u_{J_i} = u'_{J_i}$ is right special in the language $L(S)$, and this word is left special if $i \geq 2$.

(H0) We suppose first that $u_{J_1} = u'_{J_1}$ is also left special and $u_{J_g} = u'_{J_g}$ is also right special.

Then, by Lemma 9, either $\#J_j$ is smaller than a fixed m_1 , or $1 \leq \#I_{j+1} \leq 2$ and

$$\#I_{j+1} + \#J_{j+1} > C\#J_j,$$

Similar considerations for S^{-1} imply that for $j > 1$ either $\#J_j < m_1$, or $1 \leq \#I_j \leq 2$ and $\#J_{j-1} + \#I_j > C\#J_j$.

We look now at the words v_i and v'_i for some i ; by the remark above, v_{i,J_j} and v'_{i,J_j} are completely different if I_j is nonempty. As for v_{i,J_j} and v'_{i,J_j} , they have the same image by ϕ , thus are equal if they begin by the same letter, completely different otherwise.

Moreover, suppose that J_j has length at least m_1 , and $v_{i,J_j} = v'_{i,J_j} = Y(i)$, projecting on a right special word Y in $L(S)$ ending with the letter j ; then $Y(i)$ ends with the letter $s(i)_j$. Bispecial words in $L(S)$ are described in the proof of Lemma 9; if Y ends just before we see a β_i , $i \neq t$, after Y in $L(S)$ we see the letters j_1 or j_2 , these denoting the two adjacent intervals around β_i , then the same j_3 . Thus after $Y(i)$ in $L(T)$ we see the letters $(\sigma s(i))_{j_1}$ or $(\sigma s(i))_{j_2}$ for some σ , then the letters $(\sigma_{j_4} \sigma s(i))_{j_3}$ or $(\sigma_{j_5} \sigma s(i))_{j_3}$, these two permutations denoting the $\sigma(x)$ on the two mentioned intervals. If Y ends just before we see $1 - \alpha$, after Y in $L(S)$ we see the letters j_6 or j_7 , these denoting the two adjacent intervals around $1 - \alpha$, then the letters r or 0 , denoting the two adjacent intervals around 0 , then the same j_8 . Thus after $Y(j)$ in $L(T)$, if $\beta_t = 1 - \alpha$ we see the letters $(\sigma s(i))_{j_6}$ and $(\sigma s(i))_{j_7}$ for some σ , then the letters $(\sigma_{t-1} \sigma s(i))_0$ and $(\sigma_t \sigma s(i))_r$, then the letters $(\sigma_0 \sigma_{t-1} \sigma s(i))_{j_8}$ and $(\sigma_r \sigma_t \sigma s(i))_{j_8}$; if $\beta_j \neq 1 - \alpha$ for all j , we see the letters $(\sigma s(i))_{j_6}$ and $(\sigma s(i))_{j_7}$ for some σ , then the letters $(\sigma' \sigma s(i))_0$ and $(\sigma' \sigma s(i))_r$, for some σ' , then the letters $(\sigma_0 \sigma' \sigma s(i))_{j_8}$ and $(\sigma_r \sigma' \sigma s(i))_{j_8}$. In both cases, this gives us the first letters of the two words $v_{i,J_{j+1}}$ and $v'_{i,J_{j+1}}$.

We estimate $c = \sum_{i=1}^d \bar{d}(v_i, v'_i)$, by looking at the indices in some set $G_j = J_j \cup I_{j+1} \cup J_{j+1}$, for any $1 \leq j \leq g-1$;

- if both $\#J_j$ and $\#J_{j+1}$ are smaller than m_1 the contribution of G_j to the sum c is at least $\frac{1}{2m_1+1}$ as I_{j+1} is nonempty by construction;
- if $\#J_j \geq m_1$, and for at least one i v_{i,J_j} and v'_{i,J_j} are completely different, then the contribution of G_j to c is bigger than $\min(\frac{1}{2}, \frac{C_1}{C_1+1})$ as either $\#J_{j+1} < m_1$ or $\#J_j + \#I_{j+1} > C_1 \#J_{j+1}$;
- if $\#J_j \geq m_1$ and for all i , $v_{i,J_j} = v'_{i,J_j} = Y(i)$; then, because the v_i are all different and project by ϕ on the same word, the $s(i)$ in the last letter of $Y(i)$ takes d different values when i varies; the hypotheses imply that, in the notations of the previous paragraph $\sigma_{j_4} \sigma s(i) \neq \sigma_{j_5} \sigma s(i)$ for at least one i , and $\sigma_0 \sigma_{t-1} \sigma s(i) \neq \sigma_r \sigma_t \sigma s(i)$, resp. $\sigma_0 \sigma' \sigma s(i) \neq \sigma_r \sigma' \sigma s(i)$, for at least one i . This ensures that for this i , $v_{i,J_{j+1}}$ and $v'_{i,J_{j+1}}$ are completely different. As $\#J_{j+1} + \#I_{j+1} > C_1 \#J_j$, the contribution of G_j to c is bigger than C ;
- if $\#J_{j+1} \geq m_1$, we imitate the last two items by looking in the other direction.

Now, if g is even, we can cover $\{1, \dots, Q\}$ by sets G_j and some intermediate i_l , and get that c is at least a constant C . If g is odd and at least 3, by deleting either I_1 and J_1 , or J_g and I_{g+1} , we cover at least half of $\{1, \dots, Q\}$ by sets G_j and some intermediate i_l , and c is at least C .

Thus if $\sum_{i=1}^d \bar{d}(v_i, v'_i)$ is smaller than a constant C , we must have $g = 1$; then if $\sum_{i=1}^d \bar{d}(v_i, v'_i) < 1$, $v_{i, J_1} = v'_{i, J_1}$. Thus if c is smaller than C , we get our conclusion under the extra hypothesis (H0).

For the end of the proof of average \bar{d} -separation, without the hypothesis (H0), we refer the reader to the end of the proof of Proposition 44 in [5], as there is nothing different. \square

As is proved in Theorem 3 of [5], for uniquely ergodic systems average \bar{d} -separation implies non-rigidity, but we shall not use that here, as Theorem 15 below gives a simpler and more general proof. We use now the stronger notion of \bar{d} -separation:

Proposition 13. *If the minimality condition is satisfied, (X', S) is linearly recurrent, and, for all $1 \leq u \leq d$, $\sigma_j(u) \neq \sigma_{j+1}(u)$, $0 \leq j \leq r-1$, $j \neq t$, $\sigma_r \sigma_{t-1}(u) \neq \sigma_0 \sigma_t(u)$ (resp. $\sigma_r(u) \neq \sigma_0(u)$) if $\beta_j \neq 1 - \alpha$ for all j), then $L(T)$ is \bar{d} -separated and (Y, T) is not of rank one.*

Proof

Then in Proposition 12 we can replace $e = d$ by $e = 1$, with the same proof. Then the proof of Theorem 10 of [5] applies without modifications. \square

This last proposition is satisfied in particular for Veech 1969.

4.3. Isolated points. In Theorem 8, the absence of linear recurrence comes from the fact that in the Rokhlin towers some β_i , $i \neq t$, comes “too close”, horizontally and vertically, to α , or 0 (which is close to α), or another β_j . Thus we define an opposite notion, which we call the isolation (in the towers) of these points.

Definition 14. *In the Ostrowski expansion of Proposition 7, for integers $n \geq 1$, $M \geq 1$*

- *for $j \neq t$, β_j is (n, M) -isolated if there exist $n - M \leq m_1 \leq n$, $n - M \leq m_2 \leq n$, m_2 odd, $n - M \leq m_3 \leq n$, m_3 even, $n - M \leq m'_i \leq n$, $1 \leq i \leq r$, $i \neq j$, satisfying $b_{m_1}(\beta_j) \neq a_{m_1} - 1$, $b_{m_2}(\beta_j) \neq a_{m_2}$, $b_{m_3}(\beta_j) \neq a_{m_3}$, $b_{m'_i}(\beta_j) \neq b_{m'_i}(\beta_i)$, for all $1 \leq i \leq r$, $i \neq j$,*
- *α is (n, M) -isolated if for all $1 \leq i \leq r$, there exist $n - M \leq m_i \leq n$, $n - M \leq m'_i \leq n$, m'_i odd, $n - M \leq m''_i \leq n$, m''_i even, satisfying $b_{m_i}(\beta_i) \neq a_{m_i} - 1$, $b_{m'_i}(\beta_i) \neq a_{m'_i}$, $b_{m''_i}(\beta_i) \neq a_{m''_i}$.*

To make statements simpler, we shall write sometimes that *always one of the β_i is isolated* to denote there exists M such that for all m , there exists $1 \leq j \leq r$, $j \neq t$ such that β_j is (m, M) -isolated, and, mutatis mutandis, we shall write that *always one of the β_i or α is isolated*.

Theorem 8 says that T is linearly recurrent whenever always all the β_i , $i \neq t$, and α , are isolated; but weaker assumptions can also be useful. The first one is a sufficient condition for unique ergodicity.

Proposition 14. *If the minimality condition is satisfied, α has bounded partial quotients, and there exists \bar{M} and a sequence $p_k \rightarrow +\infty$ such that for all k , all β_i , $i \neq t$, and α are (p_k, \bar{M}) -isolated, T is uniquely ergodic.*

Proof

By the proof of Theorem 8, this condition implies that $\limsup_{n \rightarrow +\infty} ne_n(S, \mu) > 0$ for the Lebesgue (and unique invariant) measure μ ; thus by the proof of Proposition 10 $\limsup_{n \rightarrow +\infty} ne_n(T, \mu) > 0$

for the Lebesgue measure μ , and we conclude by Proposition 2. \square

Proposition 14 will be used to build uniquely ergodic examples when needed; note that its hypothesis is not equivalent to the unique ergodicity of T : it is not satisfied by Veech 1969 if the sequence $b_n(\beta)$ is made of strings of increasing lengths where either $b_n = a_n - 1$ or $b_n = a_n$, $b_{n+1} = 0$, while T is uniquely ergodic by [15].

Then, as we mentioned above, for our systems non-rigidity will be implied by weaker conditions than the ones ensuring average \bar{d} -separation.

Theorem 15. *Suppose the minimality condition is satisfied, α has bounded partial quotients, and there exists M such that*

- *either for all n , there exists $1 \leq j \leq r$, $j \neq t$, such that β_j is (n, M) -isolated,*
- *or the product inequality is satisfied, and for all n , either α is (n, M) -isolated or there exists $1 \leq j \leq r$, $j \neq t$, such that β_j is (n, M) -isolated,*

then (Y, T) is not rigid for any ergodic invariant measure.

Proof

Let μ be an ergodic invariant measure for T . Assume that (Y, T) is rigid; then there exists a sequence Q_k tending to infinity such that $\mu(D\Delta T^{Q_k}D)$ tends to zero for each of the $d(r+2)$ intervals D defining the natural coding of T . We fix ϵ and k such that for all these intervals

$$\mu(D\Delta T^{Q_k}D) < \epsilon.$$

Let $A_{D,k} = D\Delta T^{Q_k}D$; by the ergodic theorem, for each D and k $\frac{1}{N} \sum_{j=0}^{N-1} 1_{T^j A_{D,k}}(z)$ tends to $\mu(A_{D,k})$, for almost all z ; we can choose a set Λ of full μ -measure on which this convergence holds for all D and k . Thus for all z in Λ and all k , there exists $N_0(k)$ such that for all N larger than $N_0(k)$ and all D ,

$$\frac{1}{N} \sum_{j=0}^{N-1} 1_{T^j A_{D,k}}(z) < \epsilon.$$

By summing these $d(r+2)$ inequalities, we get that

$$\bar{d}(z_0 \dots z_{N-1}, z_{Q_k} \dots z_{Q_k+N-1}) < d(r+2)\epsilon$$

for all $N > N_0(k)$. Moreover, for z in some set Λ' of full μ -measure, we can choose $N_0(k)$ such that for all $N > N_0(k)$ these inequalities are also satisfied if we replace z by any of the d different points z' such that $\phi(z') = \phi(z)$.

We shall now show that this is not possible by estimating $\sum_{i=1}^d \bar{d}(x_0^i \dots x_{N-1}^i, y_0^i \dots y_{N-1}^i)$ for the d points x^i such that $\phi(x^i)$ is a given point x and the d points y^i such that $\phi(y^i)$ is a given point y . We take $n \geq 1$ such that $\alpha_{n+1} \leq x - y \leq \alpha_n$, and N much larger than q_n ; we shall look at the trajectories of x and y in the n -towers.

We partition $\{0, \dots, N-1\}$ into successive integer intervals where x and y agree or disagree: we get intervals $I_1, J_1, \dots, I_h, J_h, I_{h+1}$ as in the proof of Proposition 12; for all l , $x_{J_l} = y_{J_l}$, x_{I_l} and y_{I_l} are either empty or completely different, i.e. their distance \bar{d} is one. Except maybe the first one, each J_l begins after we see α or a β_i , $i \neq t$, between the trajectories of x and y , and ends before we see $1 - \alpha$ or a β_i , $i \neq t$, between the trajectories of x and y .

Suppose that for some $j \neq t$ β_j is (n, M) -isolated. We group the I_l and J_l into intervals $K_g = I_{l_{-(g)}} \cup J_{l_{-(g)}} \cup I_{l_{-(g)+1}} \cup J_{l_{-(g)+1}} \dots \cup I_{l_{+(g)}} \cup J_{l_{+(g)}}$ where $J_{l_{-(g)}}$ begins after β_j , $J_{l_{+(g)}}$ ends before β_j , and no other J_l inside K_g has any of these two properties. By Corollary 6, for all g $\#K_g \leq 2(q_n + q_{n+1} + q_{n+2}) \leq C_1 q_n$, while $\#K_g \geq q_n$ because two times where β_j is between the trajectories of x and y are separated by at least q_n . Also, by the proof of Theorem 8, $y_n(\beta_j)$, $y'_n(\beta_j)$, $y''_n(\beta_j)$ and all $y_n(\beta_i, \beta_j)$, $i \neq j$, are at least $C_2 q_n$, thus for each g we have $\#J_{l_{-(g)}} \geq C_2 q_n$, $\#J_{l_{+(g)}} \geq C_2 q_n$. Now, for each i , $x_{J_{l_{+(g)}}}^i$ and $y_{J_{l_{+(g)}}}^i$ are either equal or completely different. If for at least one i they are completely different, then $\sum_{i=1}^d \bar{d}(x_{J_{l_{+(g)}}}^i, y_{J_{l_{+(g)}}}^i) \geq 1$ and $\sum_{i=1}^d \bar{d}(x_{K_g}^i, y_{K_g}^i) \geq C_3$. Otherwise, we deduce the first letters of $x_{J_{l_{-(g+1)}}}^i$ and $y_{J_{l_{-(g+1)}}}^i$ from the common last letter of $x_{J_{l_{+(g)}}}^i$ and $y_{J_{l_{+(g)}}}^i$ as in the proof of Proposition 12 above, and find that they must be different for at least one i , because the permutations σ_{j_1} which is $\sigma(x)$ on the interval left of β_j and σ_{j_2} on the interval right of β_j have different values on at least one point. Then $\sum_{i=1}^d \bar{d}(x_{J_{l_{-(g+1)}}}^i, y_{J_{l_{-(g+1)}}}^i) \geq 1$ and $\sum_{i=1}^d \bar{d}(x_{K_{g+1}}^i, y_{K_{g+1}}^i) \geq C_3$. Thus we have always $\sum_{i=1}^d \bar{d}(x_{K_g \cup K_{g+1}}^i, y_{K_g \cup K_{g+1}}^i) \geq C_4$. We extend $\{0, \dots, N-1\}$ by at most $C_1 q_n$ on the left and on the right to a set K' made with an even number of K_g ; then $\sum_{i=1}^d \bar{d}(x_{K'}^i, y_{K'}^i) \geq C_5$ and $\sum_{i=1}^d \bar{d}(x_0^i \dots x_{N-1}^i, y_0^i \dots y_{N-1}^i) \geq C_5 - \frac{2C_1 q_n}{N}$.

Suppose that α is (n, M) -isolated: then we make a similar reasoning. Now our interval K_g are defined by $J_{l_{-(g)}}$ begins after α , $J_{l_{+(g)}}$ ends before $1 - \alpha$, and no other J_l inside K_g has any of these two properties. To get that the first letters of some $x_{J_{l_{-(g+1)}}}^i$ and $y_{J_{l_{-(g+1)}}}^i$ must be different, we use that $\sigma_0 \sigma_{t-1}$ and $\sigma_r \sigma_t$, resp. σ_0 and σ_r if $\beta_j \neq 1 - \alpha$ for all j , have different values on at least one point. By the proof of Theorem 8, all $y_n(\beta_i)$, $y'_n(\beta_i)$, $y''_n(\beta_i)$, $1 \leq i \leq r$, are at least $C_2 q_n$. And we get again $\sum_{i=1}^d \bar{d}(x_0^i \dots x_{N-1}^i, y_0^i \dots y_{N-1}^i) \geq C_5 - \frac{2C_1 q_n}{N}$.

Under the hypotheses of the theorem, this last relations holds for all n and all x and y with $\alpha_{n+1} \leq x - y \leq \alpha_n$, thus this contradicts rigidity. \square

Theorem 15 applies in particular when (X', S) is linearly recurrent, even when $\sigma_0 \sigma_t = \sigma_r \sigma_{t-1}$, resp. $\sigma_0 = \sigma_r$ if $\beta_j \neq 1 - \alpha$ for all j (as soon as there is at least one $\beta_j \neq 1 - \alpha$, otherwise we are in the cases of [5]). As mentioned in the introduction, this gives the first known examples of non rigid non linearly recurrent interval exchange transformations (note that we could get further examples for any $r \geq 2$ and $d \geq 2$):

Proof of Theorem 1

Suppose the conditions of Theorem 15 are satisfied but not those of Theorem 8. This is possible for example if we build $\beta_1 \neq 1 - \alpha$ and $\beta_2 \neq 1 - \alpha$ with prescribed Ostrowski expansions such that, for a fixed M , for all m β_2 is (m, M) -isolated, while there are unbounded strings of consecutive $b_n(\beta_1) = a_n - 1$. Then (X', S) is not linearly recurrent and (Y, T) is not rigid, and not linearly recurrent by Proposition 10. Unique ergodicity will be satisfied by Proposition 14 if we ensure β_1 is (p_k, M) -isolated for a sequence p_k .

Now, if we take $r = 2$ and $d = 2$, with β_1 and β_2 as above, and we alternate between the two possible permutations, the identity and the exchange, changing when we cross β_1 and β_2 , we get the examples claimed in the theorem. \square

When (X', S) is not linearly recurrent, Lemma 9 is not satisfied, and we do not know whether (Y, T) is average \bar{d} -separated.

4.4. In the grey zone: rigidity. We call grey zone the cases when α has bounded partial quotients, but (X', S) is not linearly recurrent. We could conclude to non-rigidity when the hypotheses of Theorem 15 are satisfied, but there are still many other cases. When Theorem 15 does not apply, then for all n some β_i and β_j and/or $\beta_i, i \neq t$, and α are too close in the n -towers. The simplest case is when all the β_i come close to α simultaneously.

Definition 15. We say that all the β_i cluster on α if there exist two sequences m_k and N_k , tending to infinity, with $m_k + N_k < m_{k+1}$, such that for all $1 \leq i \leq r, i \neq t$, we have

- either $b_n(\beta_i) = a_n - 1$ for all $m_k \leq n \leq m_k + N_k$,
- or $b_n(\beta_i) = a_n$ for all even $m_k \leq n \leq m_k + N_k$,
- or $b_n(\beta_i) = a_n$ for all odd $m_k \leq n \leq m_k + N_k$.

We recall that $T^n(x, s) = (R^n x, \psi_n(x)s)$ where

$$\psi_n(x) = \sigma(R^{n-1}x) \dots \sigma(x).$$

Lemma 16. Suppose that for a given n , for all $i \neq t$, either $x_n(\beta_i, \alpha) < \epsilon\alpha_n$ and $y_n(\beta_i) < \epsilon q_n$, or $x_n(\beta_i) < \epsilon\alpha_n$ and $y'_n(\beta_i) < \epsilon q_n$, or $x'_n(\beta_i) < \epsilon\alpha_n$ and $y_n(\beta_i) < \epsilon q_n$; for $0 \leq h \leq q_n - 1$ we call $\tau_{h,n}$ the permutation $\sigma(x_h)$ when if n is odd x_h is the leftmost (resp. if n is even x_h is the rightmost) point of the level h in the large n -tower (the basis being level 0). Then, on a set Ξ_n of measure at least $1 - 6\epsilon$, whenever x is in level h of the large or small n -tower, then

$$\psi_{q_n}(x) = \theta_{h,n} = \tau_{h-1,n} \dots \tau_{0,n} \tau_{q_n-1,n} \dots \tau_{h,n}.$$

Proof

We do the proof for n odd. We delete the set Ξ_n , of small measure as claimed, made with the x in any of the five following sets:

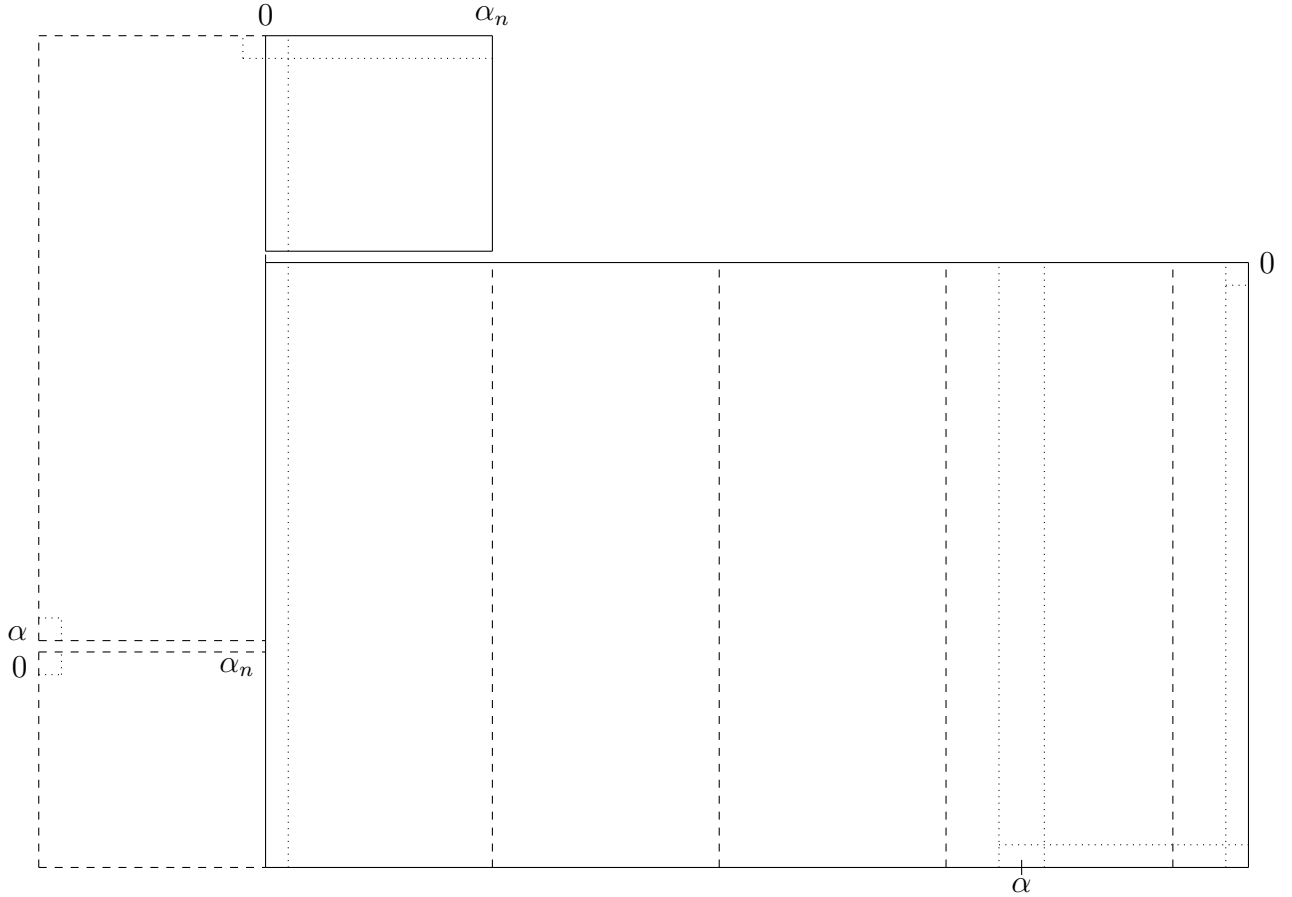
- the images by $R^m, 0 \leq m \leq q_n - 1$, of $[\alpha_n + \alpha - \epsilon\alpha_n, \alpha_n + \alpha[$,
- the images by $R^m, 0 \leq m \leq q_n - 1$, of $[\alpha - \epsilon\alpha_n, \alpha + \epsilon\alpha_n[$,
- the images by $R^m, 0 \leq m \leq \epsilon q_n$, of $[\alpha, \alpha_n + \alpha[$,
- the images by $R^m, 0 \leq m \leq q_n + q_{n-1} - 1$, of $[\alpha + \alpha_n - \alpha_{n-1}, \alpha + \alpha_n - \alpha_{n-1} + \epsilon\alpha_n[$,
- the images by $R^m, q_{n-1} - \epsilon q_n \leq m \leq q_{n-1}$, of $[\alpha - \alpha_{n-1}, \alpha - \alpha_{n-1} + \alpha_n[$.

In Figure 7, we show the n -towers and what we see less than α_n to their left. The set we delete is between dotted lines, or between dotted line and sides, in the n -towers; the β_i are confined to the small rectangles near α and 0 (remember $1 - \alpha$ is just below 0).

If x is in the large n -tower but not in column (0), using the above exclusions, we see that whenever the orbit of x is in level g , there is no $\beta_i, 1 - \alpha$ or 0 between this trajectory and x_g , and thus the contribution of level g to $\psi_{q_n}(x)$ is $\tau_{g,n}$, and the claimed formula holds.

If x is in the q_{n-1} first levels in column (0), the contribution of level g is $\tau_{g,n}$ until we reach the top of the large n -tower; then the orbit of x crosses levels 0, 1, ... of the small n -tower, staying to the right of $\beta_i, 1 - \alpha$ or 0. Hence there is no $\beta_i, 1 - \alpha$ or 0 between this right part of level g of the small n -tower and the left part of level g of the large n -tower, the contribution of this level g to $\psi_{q_n}(x)$ is $\tau_{g,n}$, and our formula holds.

If x is in column (0) (either in the large or in the small n -tower) above the q_{n-1} first levels but below the upper ϵq_n levels (of this column, that is of the small n -tower), we continue the reasoning

FIGURE 7. All β_i cluster on α

of the previous paragraph. The contributions are the expected ones until we reach the top of the small n -tower, whose contribution is $\tau_{q_{n-1}-1,n}$. Then the orbit of x crosses levels $0, 1, \dots$ of the large n -tower, staying to the right of $\beta_i, 1 - \alpha$ or 0 (because we have excluded that x is in the leftmost part of width $\epsilon\alpha_n$ of column (0)), until we reach the $q_n - 1$ -th iterate of x , which is still at least ϵq_n levels below the top. As long as $g \leq q_n - q_{n-1} - 1$, there is no $\beta_i, 1 - \alpha$ or 0 between the right part of level g of the large n -tower and the left part of level $g + q_{n-1}$ of the large n -tower, thus the contributions are the expected ones until the orbit of x reaches level $q_n - q_{n-1} - 1$ of the large n -tower, whose contribution is $\tau_{q_{n-1}}$. Then for $g \geq q_n + q_{n-1}$, there is no $\beta_i, 1 - \alpha$ or 0 between this right part of level g of the large n -tower and the left part of level $g + q_{n-1} - q_n$ of the small n -tower, there is no $\beta_i, 1 - \alpha$ or 0 in this level $g + q_n - q_{n-1}$ of the small n -tower because the orbit of xt has not reached the upper ϵq_n levels, there is no $\beta_i, 1 - \alpha$ or 0 between this level $g + q_n - q_{n-1}$ of the small n -tower and the left part of the same level of the large n -tower. Thus the contributions are still as expected and our result holds. \square

Theorem 17. *If α has bounded partial quotients, all the β_i cluster on α , $\sigma_k \sigma_j = \sigma_j \sigma_k$ for all j, k , then (Y, T) is rigid for any invariant measure.*

Proof

For any k , we choose $n = m_k + \lceil \frac{N_k}{2} \rceil$. For a given ϵ , by the proof of Theorem 8, if k is large enough the hypotheses of Lemma 16 are satisfied, and its results hold with Ξ_n and $\theta_{h,n}$.

By definition, for all h and h' , $\theta_{h',n}$ is of the form $\theta' \theta_{h,n} \theta'^{-1}$ where θ' is some composition of the $\sigma(x)$. As all these commute, $\theta_{h,n}$ is a constant θ_n for all h , and $\psi_{q_n}(x) = \theta_n$ for all x in Ξ_n .

Moreover, if x is in level h in the n -towers, $R^{q_n}x$ is in level $h - q_{n-1}$ if x is in column (0) between levels q_{n-1} and $q_n + q_{n-1} - 1$, in level $h + q_n - q_{n-1}$ if x is in the small tower, in level h if x is in any other level. Thus, again as the $\sigma(x)$ commute, $\psi_{q_n}(R^{q_n}x) = \theta_n$ for all x in $R^{-q_n}\Xi_n$, and similarly $\psi_{q_n}(R^{l q_n}x) = \theta_n$ for all x in $R^{-l q_n}\Xi_n$, hence $\psi_{l q_n}(x) = \theta_n^l$ for all x in $\bigcap_{l'=0}^{l-1} R^{-l' q_n}\Xi_n$. Let $1 \leq \zeta_n \leq d!$ be the order of the permutation θ_n : then $\psi_{\zeta_n q_n}(x)$ is the identity for x in a set of measure at least $1 - 6d!\epsilon$.

As also $|R^{\zeta_n q_n}x - x| < \frac{C d!}{q_n}$, we get that the sequence $\zeta_n q_n$ is a rigidity sequence for (Y, T) . \square

Theorem 17 is valid in particular when the permutations σ_i correspond to the addition of some elements of $\mathbb{Z}/d\mathbb{Z}$, as in [15], [14] or [11]. The same technique, with more work, applies when the σ_i do not commute, but only in some very particular cases.

Proposition 18. *Suppose $d = 3$, $r = 1$, with one marked point $\beta \neq 1 - \alpha$, and the two values of $\sigma(x)$ are a transposition and a circular permutation. For every α with bounded partial quotients, we can find β such that T is rigid for any invariant measure.*

Proof

For this, we use again the quantity $\psi_n(x) = \sigma(R^{n-1}x) \dots \sigma(x)$. For any of our systems with $r = 1$, we can define by recursion three quantities:

- if β is in the large n -tower, and n is odd, $\psi_{1,n}$, resp. $\psi_{2,n}$, is the value of ψ_{q_n} on the basis of the large n -tower left, resp. right, of the vertical of β , $\psi_{3,n}$ is the value of $\psi_{q_{n-1}}$ on the basis of the small n -tower;
- if β is in the large n -tower, and n is even, $\psi_{1,n}$, resp. $\psi_{2,n}$, is the value of ψ_{q_n} on the basis of the large n -tower right, resp. left, of the vertical of β , $\psi_{3,n}$ is the value of $\psi_{q_{n-1}}$ on the basis of the small n -tower;
- if β is in the small n -tower, and n is odd, $\psi_{2,n}$, resp. $\psi_{3,n}$, is the value of $\psi_{q_{n-1}}$ on the basis of the small n -tower left, resp. right, of the vertical of β , $\psi_{1,n}$ is the value of ψ_{q_n} on the basis of the large n -tower;
- if β is in the small n -tower, and n is even, $\psi_{2,n}$, resp. $\psi_{3,n}$, is the value of $\psi_{q_{n-1}}$ on the basis of the small n -tower right, resp. left, of the vertical of β , $\psi_{1,n}$ is the value of ψ_{q_n} on the basis of the large n -tower;

The construction of the towers implies that

- if β is in the large n -tower and $b_{n+1}(\beta) \neq a_{n+1}$,

$$\begin{aligned} \psi_{1,n+1} &= \psi_{3,n} \psi_{1,n}^{b_{n+1}} \psi_{2,n}^{a_{n+1}-b_{n+1}}, \\ \psi_{2,n+1} &= \psi_{3,n} \psi_{1,n}^{b_{n+1}+1} \psi_{2,n}^{a_{n+1}-b_{n+1}-1}, \\ \psi_{3,n+1} &= \psi_{2,n}; \end{aligned}$$
- if β is in the large n -tower and $b_{n+1}(\beta) = a_{n+1}$,

$$\begin{aligned} \psi_{1,n+1} &= \psi_{3,n} \psi_{1,n}^{a_{n+1}} = \psi_{3,n} \psi_{1,n}^{b_{n+1}} \psi_{2,n}^{a_{n+1}-b_{n+1}}, \\ \psi_{2,n+1} &= \psi_{2,n}, \\ \psi_{3,n+1} &= \psi_{1,n}; \end{aligned}$$

- if β is in the small n -tower,

$$\psi_{1,n+1} = \psi_{3,n}\psi_{1,n}^{a_{n+1}},$$

$$\psi_{2,n+1} = \psi_{2,n}\psi_{1,n}^{a_{n+1}},$$

$$\psi_{3,n+1} = \psi_{1,n}.$$

Given α , we shall build a β clustering on α , such that for infinitely many n with β close to α in the n -tower both $\psi_{1,n}$ and $\psi_{1,n-1}$ are circular permutations, or equivalently have signature 1.

We build β by its $b_n(\beta)$. We put $N_0 = 0$ and choose an $M_0 > N_0 + 2$; for $N_0 + 1 \leq n \leq M_0 - 1$, we choose any $0 \leq b_n(\beta) \leq a_n - 1$, so that β stays in the large n -tower, which implies in particular, because of the hypothesis on $\sigma(x)$ and the definition of T , that $\psi_1(n)$ and $\psi_2(n)$ have opposite signatures. If ψ_{1,M_0-1} has signature $+1$, we put $M'_0 = M_0 - 1$. Otherwise, we choose $0 \leq b_{M_0}(\beta) \leq a_{M_0} - 1$; then if ψ_{1,M_0} has signature $+1$, we put $M'_0 = M_0$. If both ψ_{1,M_0} and ψ_{1,M_0-1} have signature -1 , $\psi_{3,M_0} = \psi_{2,M_0-1}$ and ψ_{2,M_0} have signature $+1$, and the signature of ψ_{1,M_0+1} is $(-1)^{b_{M_0+1}}$; if we choose b_{M_0+1} even this will be $+1$. We choose an even $b_{M_0+1} < a_{M_0+1}$ (this is always possible as we may take $b_{M_0+1} = 0$), and put $M'_0 = M_0 + 1$.

Thus in all cases ψ_{1,M'_0} has signature $+1$ and β is in the large M'_0 -tower. If ψ_{1,M'_0-1} has also signature $+1$, we define $N'_0 = M'_0$. Otherwise, ψ_{1,M'_0-1} has signature -1 , $\psi_{3,M'_0} = \psi_{2,M'_0-1}$ has signature $+1$, ψ_{2,M'_0} has signature -1 , and the signature of ψ_{1,M_0+1} is $(-1)^{a_{M_0+1}-b_{M_0+1}}$, and we choose $b_{M_0+1}(\beta)$ so that ψ_{1,M_0+1} has signature $+1$; if $a_{M_0+1} > 1$, we can do it such that β is in the large $M_0 + 1$ -tower and put $N'_0 = M_0 + 1$. If $a_{M_0+1} = 1$, we choose $b_{M_0+1} = 1$ and β is in the small $M_0 + 1$ -tower. Using the recursion formulas above, we get that ψ_{2,M'_0+1} has signature -1 , ψ_{3,M'_0+1} has signature $+1$, ψ_{1,M'_0+2} has signature $+1$, ψ_{3,M'_0+2} has signature $+1$, and β is in the large $M'_0 + 2$ -tower. We put $N'_0 = M_0 + 2$.

Then we choose $N_1 > N'_0$ and for $N'_0 \leq n \leq N_1$ we choose $b_n(\beta) = a_n - 1$. The recursion formulas imply that, for all those n , $\psi_{1,n}$ has signature $+1$, $\psi_{2,n}$ has signature -1 , $\psi_{3,n}$ has signature -1 . Then we choose $b_{N_1+1} \neq a_{N_1+1} - 1$ (which may imply that β is in the small $N_1 + 1$ -tower), and start the same process again with N_0 replaced by N_1 . Thus we define sequences $N_k \leq M_k \leq M'_k \leq N'_k < N_{k+1}$, and we choose N_{k+1} so that $N_{k+1} - N'_k$ tends to infinity.

We can now adapt the proof of Theorem 17. For any k , we choose $n = \lfloor \frac{N_{k+1} - N'_k}{2} \rfloor$. For a given ϵ , by the proof of Theorem 8, if k is large enough the hypotheses of Lemma 16 are satisfied, and its results hold with Ξ_n and $\theta_{h,n}$. Moreover, by the proof of Lemma 15, $\theta_{0,n} = \psi_{1,n}$. All this is still true if we replace n by $n - 1$.

By definition, for all h , $\theta_{h',n}$ is of the form $\theta'_{0,n}\theta'^{-1}$ where θ' is some composition of the $\sigma(x)$. Thus the signature of $\theta_{h,n}$ is $+1$ for all h , and so is the signature of $\theta_{h,n-1}$. This implies that all the $\theta_{h,n}$ and $\theta_{h,n-1}$ are circular permutations, and thus commute.

We conclude as in Theorem 17 that $\psi_{3q_n}(x)$ is some $\theta_{h,n}^3$, and thus the identity, for all x in a set of measure at least $1 - C\epsilon$, and that the sequence $3q_n$ is a rigidity sequence for (Y, T) . \square

4.5. The cases $d = 2$. In these cases, which constitute the most immediate generalizations of Veech 1969, the two possible permutations are the identity I and the exchange E . Not only they commute, but, if we have two sequences of such permutations $\sigma_{i,l} \neq \sigma_{i,r}$ for all $1 \leq i \leq K$, then $\sigma_{K,l} \dots \sigma_{1,l}$ and $\sigma_{K,r} \dots \sigma_{1,r}$ are equal if K is even, different if K is odd. Thus, as we shall see in the two following propositions, *a cluster of an even number of marked points (different from $1 - \alpha$) behaves as if there was no marked point at all, and a cluster of an odd number of such marked*

points behaves as an isolated marked point. In theory, with both these properties together with Theorems 15 and 17, we could solve completely the question of rigidity for $d = 2$ and any number of marked points. However, as the reader may be convinced by studying Proposition 20 below, a full result would be unduly complicated to state, let alone to prove, so we shall limit ourselves to a complete study of the cases when $1 \leq r \leq 3$, and of some examples for $r = 4$. These examples in Proposition 20 provide *non-rigid examples which do not satisfy the hypotheses of Theorem 15.*

Proposition 19. *If α has bounded partial quotients and T satisfies the minimality condition, for $d = 2$ and at most three marked points different from $1 - \alpha$, either Theorem 15 applies or (Y, T) is rigid for any invariant measure.*

More precisely:

- *if $r = 1$ and $\beta \neq 1 - \alpha$, (the Veech 1969 case), or $r = 2$ and $\beta_t = 1 - \alpha$, (Y, T) is non-rigid if (X', S) is linearly recurrent, rigid for any invariant measure otherwise;*
- *if $r = 2$ and $\beta_j \neq 1 - \alpha$ for all j , or $r = 3$ and $\beta_t = 1 - \alpha$, (Y, T) is non-rigid for any ergodic invariant measure if always one of the β_i is isolated, rigid for any invariant measure otherwise;*
- *if $r = 3$ and $\beta_j \neq 1 - \alpha$ for all j , or $r = 4$ and $\beta_t = 1 - \alpha$, (Y, T) is non-rigid for any ergodic invariant measure if always α or one of the β_i is isolated, rigid for any invariant measure otherwise.*

Proof

For Veech 1969, Theorem 15 does not apply if and only if (X', S) is not linearly recurrent, and then we can use Theorem 17 to get rigidity. This is true also when $r = 2$ and $\beta_t = 1 - \alpha$, as we change permutation, from I to E or from E to I , when we cross β_i , thus $\sigma_0 = \sigma_r$, $\sigma_t \neq \sigma_{t-1}$, and the product inequality is satisfied.

When $r = 2$ and $\beta_j \neq 1 - \alpha$ for all j , $\sigma_0 = \sigma_r$; when $r = 3$ and $\beta_t = 1 - \alpha$, $\sigma_t \neq \sigma_{t-1}$, $\sigma_0 \neq \sigma_r$, hence in both these cases the product inequality is not satisfied. Thus Theorem 15 applies only when always one of the β_i is isolated, and Theorem 17 applies when all β_i cluster on α . There remains the case where α is always isolated but β_1 and β_2 can be very close. In that case, we choose an n such that $x_n(\beta_1, \beta_2) < \epsilon\alpha_n$ and $y_n(\beta_1, \beta_2) < \epsilon q_n$. Suppose for example that β_2 is higher than β_1 in the n -towers; let $\sigma_{i,l}$, resp. $\sigma_{i,r}$, be the permutation $\sigma(x)$ on the left (resp. right) of β_i on the same level of the n -towers, $i = 1, 2$, let $\sigma_{j_1}, \dots, \sigma_{j_h}$ be the values of $\sigma(x)$ on the successive levels between β_1 and β_2 . Then $\sigma_{i,l} \neq \sigma_{i,r}$ for $i = 1, 2$, and thus $\sigma_{2,l}\sigma_{j_h}, \dots, \sigma_{j_1}\sigma_{1,l} = \sigma_{2,r}\sigma_{j_h}, \dots, \sigma_{j_1}\sigma_{1,r}$ by the remark at the beginning of Section 4.5 and commutation. Hence we can make the same reasoning as in Lemma 16: supposing for example that in the n -towers β_2 is higher than β_1 and to its right, $\beta_i = R^{h_i}\beta'_i$, $i = 1, 2$, with β'_i in the basis of the large n -tower; we delete a small set made with the images by R^m , $0 \leq m \leq q_n - 1$, of $[\beta'_1, \beta'_2]$, the images by R^m , $h_1 \leq m \leq h_2$, of the basis of the large n -tower, and the upper two levels of the small n -tower. Then for the non-deleted x we get the same formula as in Lemma 16, and, as in Theorem 17 we conclude that $2q_n$ is a rigidity sequence for T .

When $r = 3$ and $\beta_j \neq 1 - \alpha$ for all j , $\sigma_0 \neq \sigma_r$; when $r = 4$ and $\beta_t = 1 - \alpha$, we have $\sigma_t \neq \sigma_{t-1}$ and $\sigma_0 = \sigma_r$, hence in both these cases the product inequality is always satisfied. Therefore the only case when we cannot apply Theorem 15 or Theorem 17 is when α and the β_i are never isolated, but the β_i do not cluster on α ; thus infinitely often α is close to one of the β_i , for example β_3 , while β_1 and β_2 are very close. For such an n , the reasoning of the last case applies again, and, by deleting all what we have deleted in this case and all we have deleted in Lemma 16, for the

non-deleted x we get the same formula as in Lemma 16, and, as in Theorem 17 we conclude that $2q_n$ is a rigidity sequence for T . \square

In the examples of the next proposition, one of the β_i (to make things simpler, we take always the same one, β_1) will be close to α infinitely often, while the other three will be always far from α and β_1 but infinitely often close together. This allows non-rigidity though none of our β_i or α is always isolated.

Proposition 20. *Suppose $d = 2$, we have four marked points $\beta_1, \beta_2, \beta_3, \beta_4$ different from $1 - \alpha$, the minimality condition is satisfied, α has bounded partial quotients, there exist M_0 and two sequences m_k and N_k , tending to infinity, with $m_k + N_k < m_{k+1}$, such that*

- for all k and all $m_k \leq n \leq m_k + N_k$, $b_n(\beta_1) = a_n - 1$, $b_n(\beta_2) = b_n(\beta_3) = b_n(\beta_4)$,
- for all k and all $m_k \leq n \leq m_k + N_k + M_0$, there exists $n - M_0 \leq m'_1 \leq n$ such that $b_{m'_1}(\beta_2) \neq a_{m'_1} - 1$,
- for all k and all $m_k \leq n \leq m_k + N_k + M_0$, there exists an even $n - M_0 \leq m'_2 \leq n$ such that $b_{m'_2}(\beta_2) \neq a_{m'_2}$,
- for all k and all $m_k \leq n \leq m_k + N_k + M_0$, there exists an odd $n - M_0 \leq m'_3 \leq n$ such that $b_{m'_3}(\beta_2) \neq a_{m'_3}$,
- for all k and all $m_k + N_k + M_0 \leq n \leq m_{k+1}$, β_2 is (n, M_0) -isolated.

Then T is not rigid for any ergodic invariant measure.

Proof

By the proof of Theorem 8, there exists a fixed constant C_0 , depending only on the size of the partial quotients of α , such that,

- for any β , if there exist $n - 2M_0 \leq m'_1 \leq n$, $n - 2M_0 \leq m'_2 \leq n$, $n - 2M_0 \leq m'_3 \leq n$ such that m'_2 is even, m'_3 is odd, $b_{m'_1}(\beta) \neq a_{m'_1} - 1$, $b_{m'_2}(\beta) \neq a_{m'_2}$, $b_{m'_3}(\beta) \neq a_{m'_3}$, then both $y_n(\beta)$ and $y'_n(\beta)$ are at least C_0q_n ;
- for any $\beta \neq \beta'$, if there exists $n - 2M_0 \leq m'_4 \leq n$ such that $b_{m'_4}(\beta) \neq b_{m'_4}(\beta')$, then $y_n(\beta, \beta')$ is at least C_0q_n .

Our hypotheses ensure that for our system, the first result holds for every n with $\beta = \beta_2$, and also (because of the values of $b_n(\beta_1)$, $b_n(\beta_2)$, $b_n(\beta_3)$, $b_n(\beta_4)$ for $m_k \leq n \leq m_k + N_k$) that both results hold for $\beta = \beta_2, \beta = \beta_3, \beta = \beta_4, \beta' = \beta_1$ for $m_k + M_0 \leq n \leq m_k + N_k + M_0$ (that is why we have chosen $2M_0$ to define C_0).

Using the other part of the proof of Theorem 8, we choose $M_1 > M_0$, depending only on the size of the partial quotients of α , such that,

- for any $\beta \neq \beta'$, if $b_{m'}(\beta) = b_{m'}(\beta')$ for all $n \leq m' \leq n + M_1$, $x_n(\beta, \beta') \leq \frac{\alpha_{n+1}}{4}$ (remember that $\alpha_{n+1} \geq C\alpha_n$),
- for any $\beta \neq \beta'$, if $b_{m'}(\beta) = b_{m'}(\beta')$ for all $n - M_1 \leq m' \leq n$, $y_n(\beta, \beta') \leq \frac{C_0q_n}{2}$,
- for any β , if $b_{m'}(\beta) = a_{m'} - 1$ for all $n - M_1 \leq m' \leq n$, $y_n \leq \frac{C_0q_n}{2}$.

Now we make the beginning of the proof of Theorem 15 above: to contradict rigidity, we have to estimate $\sum_{i=1}^d \bar{d}(x_0^i \dots x_{N-1}^i, y_0^i \dots y_{N-1}^i)$ for the d points x^i such that $\phi(x^i)$ is a given point x and the d points y^i such that $\phi(y^i)$ is a given point y . We take $n \geq 1$ such that $\alpha_{n+1} \leq \rho = x - y \leq \alpha_n$, and N much larger than q_n ; we shall look at the trajectories of x and y in the n -towers.

Suppose $m_k + M_1 \leq n \leq m_k + N_k - M_1$. For this n , we place $\beta_2, \beta_3, \beta_4$ in the n -towers. We call β the one which is lowest, β'' the highest, β' the middle one. As in the proof of Theorem 15,

we cut $\{0, \dots, N-1\}$ into intervals I_l and J_l and group them into intervals $K_g = I_{l_-(g)} \cup J_{l_-(g)} \cup I_{l_-(g)+1} \cup J_{l_-(g)+1} \dots \cup I_{l_+(g)} \cup J_{l_+(g)}$ where $J_{l_-(g)}$ begins after β , $J_{l_+(g)}$ ends before β , and no other J_l inside K_g has any of these two properties. We have again that for all $g \# K_g \leq C_1 q_n$. and $\# K_g \geq q_n$.

The beginning of $J_{l_-(g)}$ and the end of $J_{l_+(g)}$ correspond to a j such that β is between $T^j x$ and $T^j y$, which is equivalent to $T^j y \in [\beta - \rho, \beta[$; by the ergodic theorem, for N large, there are about $\rho N \geq \alpha_{n+1} N$ such indices j . We call ‘‘bad’’ those j for which $T^j y$ is in $[\beta - \frac{\alpha_{n+1}}{4}, \beta[$ or $T^j y$ is in $[\beta - \rho, \beta - \rho + \frac{\alpha_{n+1}}{4}[$, which correspond at most to about $N \frac{\alpha_{n+1}}{2}$ indices. By deleting all K_g for which $J_{l_+(g)}$ ends before a bad j , we keep at least half of the intervals K_g . Again, we look at the transition between K_g and K_{g+1} for the non-deleted K_g . The beginning of $J_{l_+(g)}$ is α or a β_i ; the possible one making $J_{l_+(g)}$ shortest is either α or β_1 , which is at least $C_0 q_n$ far (vertically) from β ; thus $\# J_{l_+(g)}$ is at least $C_0 q_n$. For each i , $x_{J_{l_+(g)}}^i$ and $y_{J_{l_+(g)}}^i$ are either equal or completely different. If for at least one i they are completely different, this gives a contribution of 1 to the global \bar{d} -sum on the length of $J_{l_+(g)}$.

Now, by our hypothesis, both β' and β'' are $\frac{\alpha_{n+1}}{4}$ close (horizontally) to β and $\frac{C_0 q_n}{2}$ close (vertically) to β . Thus the fact that our K_g has not been deleted guarantees that after seeing β between $T^j y$ and $T^j x$, we shall see β' between $T^{j'} y$ and $T^{j'} x$, β'' between $T^{j''} y$ and $T^{j''} x$, with $j < j' < j'' < j + \frac{C_0 q_n}{2}$; and we do not see either $1 - \alpha$ or β_1 before as we are far enough from the top of the towers. Thus $J_{l_-(g+1)+2}$ begins with β'' , and ends before a point which, in the case that makes it shortest, is either β_1 or $1 - \alpha$ and is at least $\frac{C_0 q_n}{2}$ far (vertically) from β'' .

If $x_{J_{l_+(g)}}^i$ and $y_{J_{l_+(g)}}^i$ are equal for all i , we shall deduce from their common last letter the first letters of $x_{J_{l_-(g+1)+2}}^i$ and $y_{J_{l_-(g+1)+2}}^i$ as in the proof of Proposition 12 above. For that we use again the remark at the beginning of Section 4.5: let $\sigma_{1,l}, \sigma_{2,l}, \sigma_{3,l}$, resp. $\sigma_{1,r}, \sigma_{2,r}, \sigma_{3,r}$ be the permutations $\sigma(x)$ on the left (resp. right) of β, β', β'' on the same level of the n -towers. The two permutations involved in computing the letter we want are, by commutation, $\sigma \sigma_{3,l} \sigma_{2,l} \sigma_{1,l}$ and $\sigma \sigma_{3,r} \sigma_{2,r} \sigma_{1,r}$ for a fixed σ , and these are different. This gives a contribution of 1 to the global \bar{d} -sum on the length of $J_{l_-(g+1)} + 2$.

Thus, for each non-deleted K_g , there is a contribution of 1 to the global \bar{d} -sum on a length at least $\frac{C_0 q_n}{2} \geq \frac{C_0}{2C_1} \# K_g$. The non-deleted K_g make a proportion at least $\frac{1}{2C_1}$ of $\{0, \dots, N-1\}$, thus the global \bar{d} -sum cannot be close to 0.

Suppose now $m_k + N_k + M_0 \leq n \leq m_{k+1} + M_1$. Then β_2 is $(n, M_0 + M_1)$ isolated and, after fixing x and y we conclude as in the proof of Theorem 15 that the global \bar{d} -sum cannot be close to 0.

Suppose now $m_k + N_k - M_1 \leq n \leq m_k + N_k + M_0$. For these n , our hypotheses ensure that there exist $n \leq m'_4 \leq n + M_1 + M_0$ such that $b_{m'_4}(\beta_2) \neq b_{m'_4}(\beta_3)$, $n \leq m'_5 \leq n + M_1 + M_0$ such that $b_{m'_5}(\beta_2) \neq b_{m'_5}(\beta_4)$. By the proof of Theorem 8, this implies that both $x_n(\beta_2, \beta_3)$ and $x_n(\beta_2, \beta_4)$ are at least $C_2 \alpha_n$.

We fix an n and place $\beta_2, \beta_3, \beta_4$ in the n -towers. Again, we fix $x = y + \rho$, define the I_l and J_l .

- If β_2 is the leftmost of the points $\beta_2, \beta_3, \beta_4$. By the ergodic theorem, for N large, there are about $\rho N \leq \alpha_n N$ indices j such that $\beta_2 \leq T^j y \leq \beta_2 + \rho$, and at least about $C_2 \alpha_n N$ indices j such that $\beta_2 \leq T^j y \leq \beta_2 + C_2 \alpha_n$;

- if β_2 is the rightmost of the points $\beta_2, \beta_3, \beta_4$. By the ergodic theorem, for N large, there are about $\rho N \leq \alpha_n N$ indices j such that $\beta_2 - \rho \leq T^j x \leq \beta_2$, and at least about $C_2 \alpha_n N$ indices j such that $\beta_2 - C_2 \alpha_n \leq T^j x \leq \beta_2$;
- If β_2 is the middle one of the points $\beta_2, \beta_3, \beta_4$, suppose for example β_3 is the leftmost one. By the ergodic theorem, for N large, there are about $\rho N \leq \alpha_n N$ indices j such that $\beta_3 \leq T^j y \leq \beta_3 + \rho$, and at least about $C_2 \alpha_n N$ indices j such that $\beta_3 \leq T^j y \leq \beta_3 + C_2 \alpha_n$.

We group the I_l and J_l in intervals K_g , using $\beta = \beta_2$ in the first two cases, $\beta = \beta_3$ in the last case. Take the first case for example: for a proportion at least C_2 of the K_g , $J_{l+(g)}$ ends at a j such that $T^j x$ is to the right of β_2 , and between β_2 and the verticals of β_3 and β_4 . Hence for these j we cannot see β_3 or β_4 between the trajectories of x and y before j and after the basis of the towers, or after j and before the top of the towers; thus for these K_g the permutations giving the first letter of $\#J_{l-(g+1)}$ are the same as when β_2 is isolated. The vertical distances from β_2 to α , $1 - \alpha$ and β_1 being bounded from below as in the previous case, both $\#J_{l+(g)}$ and $\#J_{l-(g+1)}$ are at least $C_0 q_n$; thus for this proportion C_2 of the K_g there is a contribution of 1 to the global \bar{d} -sum on a length at least $C_0 q_n \geq \frac{C_0}{C_1} \#K_g$. The other cases are similar, and we conclude that the global \bar{d} -sum cannot be close to 0. \square

Note that in the particular case of $d = 2$, two different permutations are different on all points, so we could make the above reasonings on each $\bar{d}(x_0^i \dots x_{N-1}^i, y_0^i \dots y_{N-1}^i)$, but that would not simplify significantly the computations.

We can make examples satisfying the hypotheses of Proposition 20 for every value of α . For example, if all a_n are equal to 1, for $m_k \leq n \leq m_k + N_k$, $b_n(\beta_1)$ will always be 0 while $b_n(\beta_2) = b_n(\beta_3) = b_n(\beta_4)$ can be successively 1, 0, 0, 1, 0, 0, 1, 0, 0, ...

REFERENCES

- [1] P. ARNOUX, S. FERENCZI, P. HUBERT: Trajectories of rotations, *Acta Arith.* 87 (1999), no. 3, 209–217.
- [2] M. BOSHERNITZAN: Rank two interval exchange transformations, *Ergodic Theory Dynam. Systems* 8 (1988), no. 3, 379–394.
- [3] M. BOSHERNITZAN: A condition for unique ergodicity of minimal symbolic flows, *Ergodic Theory Dynam. Systems* 12 (1992), no. 3, 425–428.
- [4] S. FERENCZI: Systems of finite rank, *Colloq. Math.* 73 (1997), 35–65.
- [5] S. FERENCZI, P. HUBERT: Rigidity of interval exchanges, *J. Mod. Dyn.* 14 (2019), 153–177.
- [6] S. FERENCZI, T. MONTEIL: Infinite words with uniform frequencies, and invariant measures, *Combinatorics, automata and number theory*, 373–409, *Encyclopedia Math. Appl.*, 135 (2010), Cambridge Univ. Press, Cambridge.
- [7] M. GUENAI, F. PARREAU: Valeurs propres de transformations liées aux rotations irrationnelles et aux fonctions en escalier (eigenvalues of transformations arising from irrational rotations and step functions, (French), preprint, arXiv: 0605250.
- [8] H. MASUR: Interval exchange transformations and measured foliations, *Annals of Mathematics*, 115 (1982), 169–200.
- [9] H. MASUR: Hausdorff dimension of divergent Teichmüller geodesics, *Trans. Amer. Math. Soc.* 324 (1991), no. 1, 235–254.
- [10] H. MASUR, S. TABACHNIKOV: Rational billiards and flat structures, *Handbook of dynamical systems*, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.
- [11] K. D. MERRILL: Cohomology of step functions under irrational rotations, *Isr. J. of Math.* 52 (1985), 320–340.
- [12] D. ROBERTSON: Mild mixing of certain interval exchange transformations, *Ergodic Theory Dynam. Systems* 39 (2019), no. 1, 248–256.
- [13] E. A. SATAEV: The number of invariant measures for flows on orientable surfaces, (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975), no. 4, 860–878, translated in *Mathematics of the USSR-Izvestiya*, 9 (1975), 813–830.

- [14] M. STEWART: Irregularities of uniform distribution, *Acta Math. Acad. Sc. Hung.* 37 (1981), 1–39.
- [15] W. A. VEECH: Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2, *Trans. Amer. Math. Soc.* 140, (1969), 1–33.
- [16] W. A. VEECH: A criterion for a process to be prime, *Monatsh. Math.* 94 (1982), no. 4, 335–341.
- [17] W. A. VEECH: Boshernitzan’s criterion for unique ergodicity of an interval exchange transformation, *Ergodic Theory Dynam. Systems* 7 (1987), no. 1, 149–153.

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE MARSEILLE, I2M - UMR 7373, 13453 MARSEILLE, FRANCE.

E-mail address: ssferenczi@gmail.com

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE MARSEILLE, I2M - UMR 7373, 13453 MARSEILLE, FRANCE.

E-mail address: hubert.pascal@gmail.com