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ARNOUX-RAUZY INTERVAL EXCHANGES

PIERRE ARNOUX, JULIEN CASSAIGNE, SÉBASTIEN FERENCZI, AND PASCAL HUBERT

ABSTRACT. The Arnoux-Rauzy systems are defined in [5], both as symbolic systems on three letters and exchange transformations of six intervals on the circle. In connection with a conjecture of S.P. Novikov, we investigate the dynamical properties of these interval exchange transformations, and precise their relation with the symbolic systems, which was known only to be a semi-conjugacy. In order to do this, we define a new system which is an exchange transformation of nine intervals on the line (it was described in [3] for a particular case). Our main result is that the semi-conjugacy determines a measure-theoretic isomorphism (between the three systems) under a diophantine (sufficient) condition, which is satisfied by almost all Arnoux-Rauzy systems for a suitable measure. However, under another condition, the interval exchange transformations are not uniquely ergodic and the isomorphism does not hold for all invariant measures. Finally, we give conditions for these interval exchange transformations to be weakly mixing.

Arnoux-Rauzy dynamical systems were introduced in [5] in order to generalize the fruitful triple interaction between Sturmian sequences and rotation of the 1-torus through the Euclid continued fraction approximation. *Arnoux-Rauzy systems on three symbols*, which we shall call *AR3 systems* in the present paper, are symbolic dynamical systems defined through word-combinatorial conditions, see Definition 7 below. They have been studied from the combinatorial point of view by many authors, see for example [13] [15] [16] [28], and many others. The points in AR3 systems are symbolic sequences (or infinite words) we call *AR3 sequences*. These sequences constitute also the restriction to three-letter alphabets of the class of *Episturmian infinite words*, defined in [18] (an infinite word is standard Episturmian if the set of its prefixes is closed under taking right palindromic closures; it is Episturmian if it has the same set of factors as some standard Episturmian word) and extensively studied, see the surveys [10] and [26].

The first and foremost question on Arnoux-Rauzy systems was to get a geometric representation of the trajectories, the preferred one being as *natural codings* of a rotation of the 2-torus. We say a sequence $\xi = \xi_1 \xi_2 \xi_3 \dots$ on the alphabet $\{a_1, a_2, \dots, a_{d+1}\}$ is a natural coding of a rotation R of \mathbb{T}^d if there exists a fundamental domain Ξ for R , together with a partition $\Xi = \Xi_1 \cup \Xi_2 \cup \dots \cup \Xi_{d+1}$, such that on each Ξ_i the map R is a translation by a vector α_i and, for any point $x \in \Xi$, $R^k(x) \in \Xi_i$ whenever $\xi_k = a_i$. This is a particular case of the *coding* of a system by a partition, which will be defined in Definition 6 below. The *Rauzy gasket* is a set of triples of positive real numbers (a_0, b_0, c_0) which will be defined in Definition 10 below. Every AR3 system defines a rotation R of the 2-torus, and a point in the Rauzy gasket with $a_0 + b_0 + c_0 = 1$, which corresponds, after ordering, to the frequencies of the three symbols in the AR3 sequences, and thus to the measures of the three sets ξ_i which could be used for the coding.

In a famous particular case, the *Tribonacci* sequence, see Definition 8 below, was shown in [34] to be a natural coding of the associated rotation R , and thus the corresponding system is measure-theoretically isomorphic to that rotation. This was generalized to a larger class of Arnoux-Rauzy systems in [4], and recently to almost all Arnoux-Rauzy systems [11], in the sense of the notation

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before Proposition 18 below. On the other hand, [14] provides counter-examples which are not measure-theoretically isomorphic to any rotation, see Section 5 below.

For a general Arnoux-Rauzy system, one has to be content with what looks like a second-best geometric representation built in [5], a coding of a six-interval exchange transformation on the circle. These interval exchange transformations, which we shall call *AR6 systems* in the present paper, are defined for each point of the Rauzy gasket in Definition 11 below. The Tribonacci case corresponds to the point $(\alpha, \alpha^2, 1 - \alpha - \alpha^2)$ in the Rauzy gasket, where α is the unique real number such that $\alpha^3 + \alpha^2 + \alpha = 1$, and for this point Definition 11 gives the *Arnoux-Yoccoz interval exchange transformation* built in [2], which is linked with a pseudo-Anosov map (we refer the reader to [7] for its definition and properties). Note that it is still an open question to find other geometric models, in particular for those Arnoux-Rauzy systems which are not natural codings of rotations of the 2-torus, see for example [35].

However, these six-interval exchange transformations have been recently understood to represent by themselves a very interesting family of systems, as the dimension over the rationals of the set of lengths of the intervals is quite smaller than the number of intervals (namely, three versus six). This kind of interval exchange transformations was pointed out (in a much more general context, in which Arnoux-Rauzy interval exchange transformations can be considered as a kind of baby example) by S.P. Novikov, see [33] [21] [31], also [19] [20] [17]. Novikov's problem is stated in terms of foliations on surfaces and not in terms of interval exchange transformations. Even if Novikov's conjecture and the construction by P. Arnoux and J.-C. Yoccoz were done in the early 80's, the explicit connection between these objects (foliations versus interval exchange transformations) is not well understood. A dictionary between some foliations on surfaces embedded in the 3-dimensional torus and Arnoux-Rauzy interval exchange transformations will be described in a forthcoming paper by I. A. Dynnikov, P. Hubert and A. Skripchenko. This will make a bridge between combinatorics and geometry. In all approaches, the Rauzy gasket plays the role of parameters space. This prompted several authors to make deep studies of this gasket in [6] [8] [9] [27] [25], solving in this special case a conjecture of Novikov, and to look at everything that can be found about this particular family; it was also considered in Lemma 5.9 (attributed to Yoccoz) of [30], a paper related to geometric group theory and the study of automorphisms of free groups. But indeed, a priori not much is known, as these six-interval exchange transformations are only semi-conjugate to the original Arnoux-Rauzy systems. Namely, an AR6 interval exchange transformation admits a coding by a partition into three sets which is an AR3 symbolic system, but this partition is not necessarily a generating partition. Also, as far as we know, the coding by the natural partition into six intervals of the circle cannot be built by *substitutions* (see Definition 3 below), contrarily to its AR3 coding. Hence no property of an AR6 interval exchange transformation can be directly carried out from the underlying AR3 symbolic system. Moreover, all AR3 systems are known to be *minimal* [5] and *uniquely ergodic*, as defined in Definition 5 below, (by M. Boshernitzan's result [12] using the fact that the language complexity is $2n + 1$); but, in stark contrast, deep geometric methods have allowed Dynnikov and Skripchenko [22] to prove, though in a completely different language, the existence of minimal non-uniquely ergodic AR type interval exchange transformations.

The relation between AR6 interval exchange transformations and underlying AR3 symbolic systems was partially tackled in [3], though only in the particular case of Tribonacci, and with a certain lack of details: that paper defines yet another Arnoux-Rauzy interval exchange transformation, this

time on nine intervals. These interval exchange transformations, which we shall call *AR9 systems* in the present paper, are defined (not only for the example of [3]) in Definition 12 below. Note that the interval exchange transformations used in [22] have a similar look; though they are also built, in Lemma 3 (3.5 in the translated issue) of [22], from points in the Rauzy gasket, these do not belong to the class of AR9 systems of our Definition 12 below, as the permutation is completely different, and also the arrival intervals are grouped into connected unions of three, three and three instead of four, two and three; their successive iterates through the Rauzy-Veech induction, described in Figure 10 of [22], do not belong to our AR9 class either, as they are defined by different permutations though for some of them the grouping of arrival intervals is indeed by four, two and three. The link between these two classes of systems is not yet clear.

Starting from an AR9 system, an AR3 appears again as a coding by a partition into three sets, and here the coding of the AR9 by the natural partition into nine intervals can be explicitly generated by a substitution. This is the key for studying ergodic properties of AR9 interval exchange transformations, and extending them to the AR6 interval exchange transformations which appear as factors of AR9. The one stated in [3] is the measure-theoretic isomorphism between the three corresponding systems (AR3, AR6, AR9) in the Tribonacci case, though no proof is offered.

In the present paper, we generalize the construction of AR9 systems to every point in the Rauzy gasket, and their construction by substitutions, using an *induction* process defined in Section 3.2 below; we use them to derive dynamical properties of AR6 and AR9 systems. Our main result is

Theorem 1. *Almost every (in the sense of [11], see the notation before Proposition 18 below) AR9 or AR6 interval exchange transformation is uniquely ergodic and measure-theoretically isomorphic to its AR3 coding.*

This theorem could be deduced (using Lemma 14 below and some extra work) from the ergodicity of the induction process. We choose to derive it from a stronger result, namely an explicit sufficient diophantine condition (Proposition 16 and Theorem 17 below) for measure-theoretic isomorphism between the corresponding AR9, AR6 and AR3 systems; it also implies unique ergodicity for the AR6 and AR9. This condition is satisfied by almost all Arnoux-Rauzy systems (Proposition 18 below), and many explicit examples including Tribonacci, all systems which are periodic points under the induction, and, more generally, all the so-called Arnoux-Rauzy systems with *bounded weak partial quotients* (Proposition 19 below). Thus

Corollary 2. *Almost all AR9 or AR6 interval exchange transformations, including the Tribonacci ones and all those with bounded weak partial quotients, are measure-theoretically isomorphic to rotations of the 2-torus.*

Thus at last we have proved the isomorphism result for the Tribonacci case. This provides the backbone of an answer to Question 9 (asked by G. Forni) in [24] and this was another motivation for the present paper. We recall that a measure-theoretic dynamical system is *rigid* if a sequence of powers converges to the identity in \mathcal{L}^2 .

Corollary 3. *The Arnoux-Yoccoz interval exchange transformation, or else the Tribonacci AR9, provide nontrivial examples of rigid self-induced interval exchange transformations.*

Then we give a class of examples of non-uniquely ergodic AR9 (or AR6) interval exchange transformations, with a somewhat more explicit proof than in [22], and give both examples and counter-examples to the isomorphism problem: these AR9 are measure-theoretically isomorphic

to their AR3 coding if we equip them with an ergodic invariant measure, but of course this cannot hold if we take one of the many non-ergodic measures. Then we show that weak mixing is also present in the class of AR9 (or AR6) systems.

The present paper focuses on symbolic dynamics, in a deliberately pedestrian way, but of course there is geometry underlying and motivating the constructions. In the case of AR9 systems, this is new and generalizes the geometry in [3]; indeed, as the 9-interval exchange transformations studied in [22], these systems are first return maps on some transversal of a foliation: this will appear in the mentioned forthcoming paper by Dynnikov, Hubert and Skripchenko. That paper will provide a detailed description of this phenomenon.

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1. BASIC DEFINITIONS

We look at finite *words* on a finite alphabet $\mathcal{A} = \{1, \dots, k\}$. A word $w_1 \dots w_t$ has *length* $|w| = t$. The *concatenation* of two words w and w' is denoted by ww' .

Definition 1. A word $w = w_1 \dots w_t$ occurs at place i in a word $v = v_1 \dots v_s$ or an infinite sequence $v = v_1 v_2 \dots$ if $w_1 = v_i, \dots, w_t = v_{i+t-1}$. We say that w is a *subword* of v .

Definition 2. A language L over \mathcal{A} is a set of words. In the present paper, all languages are assumed to be *factorial* (if w is in L , all its subwords are in L), and *extendable* (if w is in L , aw is in L for at least one letter a of \mathcal{A} , and wb is in L for at least one letter b of \mathcal{A}).

A language L is *minimal* if for each w in L there exists n such that w occurs in each word of L of length n .

The language $L(u)$ of an infinite sequence u is the set of its finite subwords.

Definition 3. A substitution ψ is a map from an alphabet \mathcal{A} into the set \mathcal{A}^* of finite words on \mathcal{A} ; it extends naturally to a morphism of \mathcal{A}^* for the operation of concatenation.

Definition 4. The symbolic dynamical system associated to a language L is the one-sided shift $S(x_0 x_1 x_2 \dots) = x_1 x_2 \dots$ on the subset Y_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $t < s$, $x_t \dots x_s$ is in L .

Definition 5. A dynamical system (X', U) is *minimal* if every orbit is dense. It is *uniquely ergodic* if it has a unique invariant probability measure.

We remark that a symbolic dynamical system (X_L, S) is minimal if and only if the language L is minimal.

Definition 6. For a dynamical system (X', U) and a finite partition $\{P_1, \dots, P_l\}$ of X' , the *trajectory* of a point x in X' is the infinite sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_n = i$ if $U^n x$ falls into P_i , $1 \leq i \leq l$.

Then if L is the language made of all the finite subwords of all the trajectories, (Y_L, S) is called the coding of (X^l, U) by the partition $\{P_1, \dots, P_l\}$.

2. CLASSICAL ARNOUX-RAUZY SYSTEMS

2.1. AR3 symbolic systems. These systems are the “genuine” Arnoux-Rauzy systems; we take here as a definition their constructive characterization, derived in [5] from the original definition, and modified in the present paper by a renaming of letters and words. Here we choose to name a, b, c , the letters of the alphabet, in such a way that the words of length 2 are aa, ab, ac, ba, ca ; then the following definition is equivalent to the original one.

Definition 7. An AR3 symbolic system is the symbolic system on $\{a, b, c\}$ generated by the three substitutions

- $\sigma_I: a \rightarrow ab, b \rightarrow ac, c \rightarrow a,$
- $\sigma_{II}: a \rightarrow ab, b \rightarrow a, c \rightarrow ac,$
- $\sigma_{III}: a \rightarrow a, b \rightarrow ab, c \rightarrow ac,$

and a directing sequence $r_n, n \in \mathbb{N}^*, r_n \in \{I, II, III\}$, taking the value I infinitely many times.

Namely, it is the symbolic system (Y_3, S) whose language is generated by the words $A_k = \sigma_{r_1} \dots \sigma_{r_k} a, B_k = \sigma_{r_1} \dots \sigma_{r_k} b, C_k = \sigma_{r_1} \dots \sigma_{r_k} c, k \geq 1$. The respective lengths of the words A_k, B_k, C_k will always be denoted by $h_{a,k}, h_{b,k}, h_{c,k}$.

As mentioned in the introduction, (Y_3, S) is minimal and uniquely ergodic: we denote by μ its invariant probability measure.

Note that our modification of the rules changes the usual condition of [5], that each substitution is used infinitely often, to the present condition that σ_I is used infinitely often.

The most famous particular case is the *Tribonacci system*, in which a particular point is the *Tribonacci sequence*:

Definition 8. The Tribonacci system is the AR3 system defined by $r_n = I$ for all n . The Tribonacci sequence is the fixed point of σ_I i.e. the sequence beginning with $(\sigma_i)^k a$ for all k .

2.2. Partial quotients and multiplicative rules. These quantities are defined in [14], but we redefine them here as the notations are different.

Definition 9. We write the directing sequence (r_n) in a unique way as $k_1 - 1 \geq 0$ times the symbol III followed by one symbol I or II , then $k_2 - 1 \geq 0$ times III followed by one I or II etc.... the $k_n \geq 1$ are then called the partial quotients of the system.

The multiplicative times are $m_0 = 0, m_n = k_1 + \dots + k_n, n \geq 1$: they are the times m for which $r_m \neq III$.

Then the words $A_{m_n}, B_{m_n}, C_{m_n}$ can be built by the following *multiplicative rules*, which could also be expressed by substitutions but would need a countable set of them:

- if $r_{m_{n+1}} = I$, we say that the $(n + 1)$ -th multiplicative rule is a rule I_m , and we have
 - $A_{m_{n+1}} = A_{m_n}^{k_{n+1}} B_{m_n},$
 - $B_{m_{n+1}} = A_{m_n}^{k_{n+1}} C_{m_n},$
 - $C_{m_{n+1}} = A_{m_n};$
- if $r_{m_{n+1}} = II$, we say that the $(n + 1)$ -th multiplicative rule is a rule II_m , and
 - $A_{m_{n+1}} = A_{m_n}^{k_{n+1}} B_{m_n},$

$$\begin{aligned} - B_{m_{n+1}} &= A_{m_n}, \\ - C_{m_{n+1}} &= A_{m_n}^{k_{n+1}} C_{m_n}. \end{aligned}$$

For Tribonacci, we have $k_n = 1$ for all n , and all multiplicative rules are I_m .

We recall that in [14], we use different substitutions (called “(additive) concatenation rules” in that paper), and the sequence of multiplicative rules (as defined in that paper) corresponds to the successive number of times we use each substitution: the $(n + 1)$ -th multiplicative rule is of type 1 whenever the m_{n-1} -th and m_{n+1} -th substitutions are different. Then the H_n , G_n and J_n of [14] are exactly the same as respectively A_{m_n} , B_{m_n} and C_{m_n} in the present paper, and types 1 and 2 of [14] correspond to our rules I_m and II_m .

We shall use the inequalities proved in Lemma 7 of [14] at the multiplicative times: namely $h_{b,m_n} \leq 2h_{a,m_n}$ and $h_{c,m_n} \leq 2h_{a,m_n}$. These are not true in general at other (additive) times $p \neq m_n$.

2.3. AR6 interval exchange transformations. These exchange transformations of six intervals on a circle are defined in [2] for Tribonacci, see also [7], and [5] for the general case.

Definition 10. *The Rauzy gasket Γ is the set of triples of positive real numbers (a_0, b_0, c_0) such that, if we define recursively the numbers a_n, b_n, c_n by taking the triple $(a_{n-1} - b_{n-1} - c_{n-1}, b_{n-1}, c_{n-1})$ and reordering it, then for each $n \geq 0$ we have $a_n > b_n > c_n > 0$.*

Note that Γ is not closed and we do not include the boundary in the definition. In view of the induction process in Section 3.2 below, it is easier not to normalize the points in Γ , but all the AR6 and AR9 systems defined below by points (a_0, b_0, c_0) in Γ are not changed, except for an immediate renormalization, if we replace (a_0, b_0, c_0) by any $(\lambda a_0, \lambda b_0, \lambda c_0)$ for $\lambda > 0$.

Definition 11. *An AR6 interval exchange transformation (X_6, T) is defined in the following way from any triple (a_0, b_0, c_0) in Γ : X_6 is a circle of length $2a_0 + 2b_0 + 2c_0$. The circle is partitioned into three intervals of respective lengths $2a_0, 2b_0, 2c_0$, then each one is cut into two halves; the action of T first exchanges by translations respectively the two intervals of length a_0 , the two intervals of length b_0 , the two intervals of length c_0 , then translates everything by $a_0 + b_0 + c_0$, i.e. a half-circle.*

We could also look at the same transformation as an exchange transformation of seven intervals on the interval $[0, 2a_0 + 2b_0 + 2c_0[$, but a better model on the interval will be given in Section 3 below.

Note that the location of the origin on the circle does not change the system up to topological conjugacy and measure-theoretic isomorphism for any invariant measure (in the sense that any invariant measure on one of them can be carried to the other one, and the two measure-theoretic systems are isomorphic). Similarly, the order between the intervals of lengths $2a_0, 2b_0, 2c_0$ on the circle is not mentioned in Definition 11 (the fact that it is not always the same is somewhat understated in [5]). By changing the origin, we can reduce the number of possible orders to two, and the two AR6 interval exchange transformations defined with the same (a_0, b_0, c_0) but different orders of these intervals are conjugate by a symmetry on the circle, thus are also topologically conjugate and measure-theoretically isomorphic for any invariant measure.

For example, when the intervals of lengths $2a_0, 2b_0, 2c_0$ are successive intervals of the circle in that order, T is shown in Figure 1, where on the left circle $a-, a+, b-, \dots$ denote the intervals of length a_0, a_0, b_0, \dots while on the right circle the letters b correspond to the images of these intervals by the transformation. If in Figure 1 we choose to put the origin at the left end of the interval

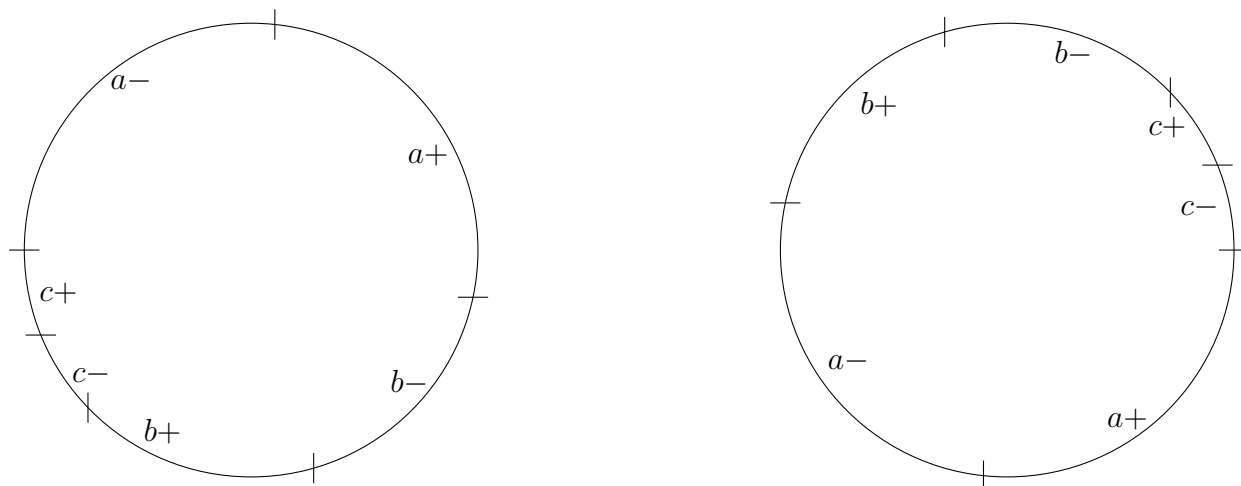


FIGURE 1. AR6 interval exchange transformation

denoted by $a-$, $[0, a_0)$ is sent to $[a_0 + a_0 + b_0 + c_0, 2a_0 + a_0 + b_0 + c_0)$ modulo $2a_0 + 2b_0 + 2c_0$, $[a_0, 2a_0)$ is sent to $[a_0 + b_0 + c_0, a_0 + a_0 + b_0 + c_0)$ modulo $2a_0 + 2b_0 + 2c_0$, etc...

The link between AR3 symbolic systems and AR6 interval exchange transformations, studied in [5], will be described in Section 3.4 below. But, as pointed out in the introduction, we do not know any constructive way to build directly the language of the *natural* coding of the AR6 interval exchange transformation, that is its coding by the partition into its six intervals of continuity on the circle, coded by $a-$, $a+$, $b-$, $b+$, $c-$, $c+$. That is why we need to introduce one more class of Arnoux-Rauzy systems.

2.4. Note on endpoints. One recurring problem when dealing with interval exchange transformations is what to do with interval endpoints? A satisfying answer to this question is given by M. Keane in Section 5 of [29]: by carefully doubling the endpoints and their orbits, he defines a Cantor set on which the transformation becomes a homeomorphism, and shows this is equivalent to taking the natural coding by the partition into defining intervals. In the present paper, to make definitions easier, we do not use Keane's construction, and all intervals are closed on the left, open on the right; but that will introduce technical difficulties, see Remark 1 below.

3. THE NEW SYSTEMS: ARNOUX-RAUZY ON NINE SYMBOLS

3.1. AR9 interval exchange transformations. As mentioned in the introduction, these are defined for the particular case of Tribonacci in [3]. Note that we use the same symbol T for AR9 and AR6 interval exchange transformations in view of Proposition 5 below.

An AR9 interval exchange transformation is defined by a point (a_0, b_0, c_0) in Γ , as an exchange transformation of nine intervals on a union of three disjoint intervals on the line.

Definition 12. For a point (a_0, b_0, c_0) in Γ , an AR9 interval exchange transformation (X_9, T) is defined on the union of three disjoint intervals Ω_0 of length $a_0 + b_0$, Ω'_0 of length $b_0 + c_0$, Ω''_0 of length $a_0 + c_0$.

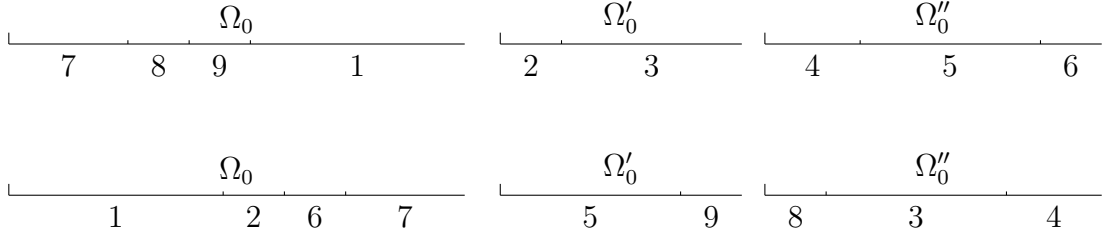


FIGURE 2. AR9 interval exchange transformation

An AR9 interval exchange transformation in the first, second or third order is defined in the following way:

- in the first order, from left to right we see $\Omega_0, \Omega'_0, \Omega''_0$; in the second order, from left to right we see $\Omega'_0, \Omega''_0, \Omega_0$; in the third order, from left to right we see $\Omega''_0, \Omega_0, \Omega'_0$,
- we partition the interval Ω_0 , from left to right, into four intervals of successive lengths $b_0 - c_0, c_0, c_0, a_0 - c_0$, denoted respectively by $I_{7,0}, I_{8,0}, I_{9,0}, I_{1,0}$, and into four intervals of successive lengths $a_0 - c_0, c_0, c_0, b_0 - c_0$, which we define respectively to be $TI_{1,0}, TI_{2,0}, TI_{6,0}, TI_{7,0}$,
- we partition the interval Ω'_0 , from left to right, into two intervals of successive lengths c_0, b_0 , denoted respectively by $I_{2,0}, I_{3,0}$, and into two intervals of successive lengths b_0, c_0 , which we define respectively to be $TI_{5,0}, TI_{9,0}$,
- we partition the interval Ω''_0 from left to right, into three intervals of successive lengths $a_0 - b_0, b_0, c_0$, denoted respectively by $I_{4,0}, I_{5,0}, I_{6,0}$, and into three intervals of successive lengths $c_0, b_0, a_0 - b_0$, which we define respectively to be $TI_{8,0}, TI_{3,0}, TI_{4,0}$.

An AR9 interval exchange transformation in the reversed first, second or third order is defined in the same way, except that in all items above “from left to right” is replaced by “from right to left” (note that all intervals are still closed on the left, open on the right).

It is clear from the definition that two AR9 interval exchange transformations defined with the same (a_0, b_0, c_0) but different actual locations on the line of the intervals $\Omega_0, \Omega'_0, \Omega''_0$ (equivalently, different locations of the origin and gaps between the intervals), or different orders, are conjugate by a map which is continuous except on a finite number of points. Thus they are measure-theoretically isomorphic for any invariant measure, in the sense of Section 2.3 above; all will be topologically isomorphic if we suppose no two of the intervals $\Omega_0, \Omega'_0, \Omega''_0$ are adjacent.

We could also define AR9 interval exchange transformations on the circle, gluing $\Omega_0, \Omega'_0, \Omega''_0$ as in Proposition 5 below, but we prefer to define them on the line as (contrarily to the AR6 case) there is no need to add an interval. If we choose the $\Omega_0, \Omega'_0, \Omega''_0$ to be adjacent, and this is allowed by our definition, we get examples of “usual” nine-interval exchange transformations as in [29], defined on one interval. But as we shall see below this adjacency will not be conserved by induction, so we have to use the more general family. We shall check that all our results, in particular Lemma 11 below, which states the adjacency of certain intervals, is true whatever the gaps between $\Omega_0, \Omega'_0, \Omega''_0$.

For example, an AR9 interval exchange transformation in the first order is shown in Figure 2, where i in the upper part corresponds to $I_{i,0}$ and i in the lower part corresponds to $TI_{i,0}$. An example in the reversed second order is shown in Figure 3.

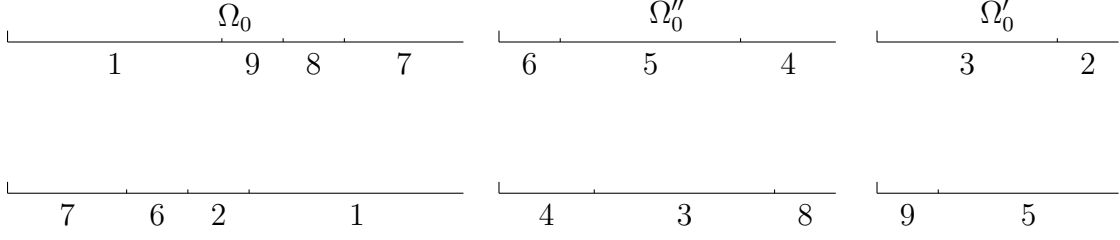


FIGURE 3. AR9 interval exchange transformation in reversed order

3.2. Induction. Now, we take an AR9 system (X_9, T) ; to fix ideas, we suppose it is in the first order. Let T_1 be the induced map of T on $I_{1,0} \cup I_{2,0} \cup I_{3,0} \cup I_{4,0}$. We define $a_1 > b_1 > c_1$ as the triple $(a_0 - b_0 - c_0, b_0, c_0)$ after reordering. Then there are three cases, which we tackle by growing order of difficulty.

3.2.1. Induction step case III: $a_1 = a_0 - b_0 - c_0$. Then $b_1 = b_0, c_1 = c_0$.

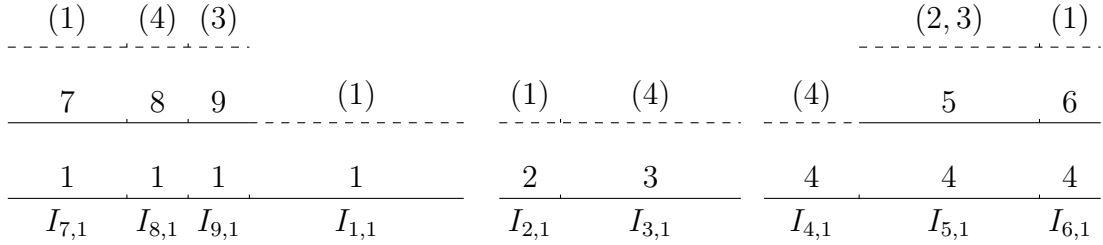


FIGURE 4. Induction Case III

The situation is essentially described in Figure 4. The induction set $I_{1,0} \cup I_{2,0} \cup I_{3,0} \cup I_{4,0}$ is a disjoint union of three intervals which we denote by $\Omega_1, \Omega'_1, \Omega''_1$, and is further cut into nine new intervals $I_{i,1}$, whose respective lengths are, from left to right, $b_1 - c_1, c_1, c_1, a_1 - c_1, c_1, b_1, a_1 - b_1, b_1, c_1$. Then T acts on the picture as a move upwards, until we reach again the induction set, which is marked by dashed lines. Each interval of the picture is labelled by j above if it is in $I_{j,0}$; the labels are between parentheses for the dashed intervals, as they will not be used further (note that $T_1 I_{5,1} = T^2 I_{5,1}$ is the union of a (full) subinterval of $I_{2,0}$ with a (left) subinterval of $I_{3,0}$, hence the ambiguous label). Thus for example $I_{7,1}$ is sent by T onto $I_{7,0}$, then by an iteration of T into $I_{1,0}$, hence $T_1 = T^2$ on $I_{7,1}$. And we check that T_1 is indeed an AR9 interval exchange transformation defined by (a_1, b_1, c_1) on the union of $\Omega_1, \Omega'_1, \Omega''_1$; the order is still the first one.

3.2.2. Induction step case I: $c_1 = a_0 - b_0 - c_0$. Then $a_1 = b_0, b_1 = c_0$.

The length of each $I_{i,1}$ in Figure 5 is the same as in case III. T_1 is an AR9 interval exchange transformation defined by (a_1, b_1, c_1) , in the third order.

3.2.3. Induction step case II: $b_1 = a_0 - b_0 - c_0$. Then $a_1 = b_0, c_1 = c_0$.

The length of each $I_{i,1}$ in Figure 6 is the same as in case III. T_1 is an AR9 interval exchange transformation defined by (a_1, b_1, c_1) , in the reversed second order.

The same computations work if we start from an AR9 in the second order. We get the same pictures, in the second order in Case III, the first order in Case I, the reversed first order in Case

(1)	(4)	(3)	(3)		(2, 3)	(3)	(1)
7	8	9	9	(1)	(4)	5	5 6
1	1	1	2	2	3	3	4 4
$I_{4,1}$	$I_{5,1}$	$I_{6,1}$	$I_{7,1}$	$I_{8,1}$	$I_{9,1}$	$I_{1,1}$	$I_{2,1}$ $I_{3,1}$

FIGURE 5. Induction Case I

(1)	(4)	(3)				(2, 3)	(3)	(1)
7	8	9	(1)	(1)	(4)	5	5	6
1	1	1	1	2	3	3	4	4
$I_{1,1}$	$I_{9,1}$	$I_{8,1}$	$I_{7,1}$	$I_{6,1}$	$I_{5,1}$	$I_{4,1}$	$I_{3,1}$	$I_{2,1}$

FIGURE 6. Induction Case II

II. When we start from the third order, we get the same pictures, in the third order in Case III, the reversed third order in Case II, and the second order in Case I. If we start from a reversed order, just reverse the orientation of the pictures.

We can now iterate the induction: starting with $T_0 = T$, we define T_k as the induced map of T_{k-1} on the set $\cup_{i=1}^4 I_{i,k-1}$, which we denote by $J_{a,k-1}$. It defines intervals $\Omega_k, \Omega'_k, \Omega''_k, I_{i,k}$, with $\cup_{i=1}^9 I_{i,k} = J_{a,k-1}$. The points (a_k, b_k, c_k) in the Rauzy gasket have been defined in [5], where the same induction process is described for AR6 interval exchange transformations. This process gives an algorithm of simultaneous approximation of (a_0, b_0, c_0) , which is called the *Arnoux-Rauzy algorithm*. Our induction on AR9 interval exchange transformations gives also an algorithm of simultaneous approximation of $(a_0 + b_0, b_0 + c_0, a_0 + c_0)$ by the lengths of the intervals $\Omega_k, \Omega'_k, \Omega''_k$: this turns out to be the *fully subtractive algorithm* where the smallest of the numbers is subtracted from the other two [32].

3.3. AR9 symbolic systems.

Definition 13. An AR9 symbolic system (Y_9, S) is the natural coding of an AR9 interval exchange transformation (X_9, T) , that is its coding by the partition into $I_{i,0}$, $1 \leq i \leq 9$; we denote by ψ the map associating to each point $x \in X_9$ its trajectory in Y_9 .

Remark 1. Because of the way we deal with the endpoints, see Section 2.4 above, ψ is injective but not surjective; we have $Y_9 = \psi(X_9) \cup D_9$, where D_9 is a countable set made with the improper trajectories of the right endpoints of the intervals $I_{i,0}$ and their negative orbits. These are the limits, in the product topology of $\{1, \dots, 9\}^{\mathbb{N}}$, in which Y_9 is closed, of trajectories of points approaching these endpoints from the left, and similarly for their pre-images.

Proposition 4. *For each (a_0, b_0, c_0) in Γ , the AR9 symbolic system associated to any AR9 interval exchange transformation defined by (a_0, b_0, c_0) is the symbolic system on $\{1, \dots, 9\}$ generated (in the sense of Definition 7 above) by the three substitutions*

- $\sigma'_I: 1 \rightarrow 35, 2 \rightarrow 45, 3 \rightarrow 46, 4 \rightarrow 17, 5 \rightarrow 18, 6 \rightarrow 19, 7 \rightarrow 29, 8 \rightarrow 2, 9 \rightarrow 3,$
- $\sigma'_{II}: 1 \rightarrow 17, 2 \rightarrow 46, 3 \rightarrow 45, 4 \rightarrow 35, 5 \rightarrow 3, 6 \rightarrow 2, 7 \rightarrow 1, 8 \rightarrow 19, 9 \rightarrow 18,$
- $\sigma'_{III}: 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 4, 5 \rightarrow 45, 6 \rightarrow 46, 7 \rightarrow 17, 8 \rightarrow 18, 9 \rightarrow 19.$

and a directing sequence $r_n, n \in \mathbb{N}^*, r_n \in \{I, II, III\}$, defined by $r_n = I$ if $a_n = a_{n-1} - b_{n-1} - c_{n-1}$, $r_n = II$ if $b_n = a_{n-1} - b_{n-1} - c_{n-1}$, $r_n = III$ if $c_n = a_{n-1} - b_{n-1} - c_{n-1}$; r_n takes the value I infinitely many times.

Any system defined in this way is an AR9 symbolic system.

Proof

We iterate the induction of Section 3.2, and call $i_k, 1 \leq i \leq 9$, the trajectory under T of any point x in $I_{i,k}$ between the time 0 and the first return time of x in $J_{a,k-1}$, coded by the partition into $I_{i,k}, 1 \leq i \leq 9$. Because of the induction steps, $i_k = \sigma'_{r_1} \dots \sigma'_{r_k} i$; if we iterate the induction infinitely many times, the words 1_k to $9_k, k \geq 0$, generate the language of T . As $a_n > b_n > c_n > 0, r_n = I$ infinitely often.

It is actually proved in [5] that the construction of r_n gives a surjective map from Γ to the set of sequences $r_n, n \in \mathbb{N}^*, r_n \in \{I, II, III\}$ where r_n takes the value I infinitely many times, and that two triples have the same image if and only if they are proportional. This proves our last assertion. \square

Thus the AR9 symbolic system does not depend on the location or the order of the intervals $\Omega_0, \Omega'_0, \Omega''_0$. The common length of the words $1_k, 2_k, 3_k, 4_k$, is $h_{a,k}$ defined in Section 2.1, $h_{b,k}$ is the common length of the words $5_k, 6_k, 7_k, h_{c,k}$ the common length of the words $8_k, 9_k$.

In the Tribonacci case, the directing sequence $r_n = I$ for all n corresponds to the point $(\alpha, \alpha^2, 1 - \alpha - \alpha^2)$ mentioned in the introduction (or to any of its multiples), where α is the unique real number such that $\alpha^3 + \alpha^2 + \alpha = 1$.

The multiplicative rules of Section 2.2 above extend immediately to AR9 systems, in the following way

- if the $(n + 1)$ -th multiplicative rule is a rule I_m ,
 - $1_{m_{n+1}} = 3_{m_n} 4_{m_n}^{k_{n+1}-1} 5_{m_n},$
 - $2_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 5_{m_n},$
 - $3_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 6_{m_n},$
 - $4_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 7_{m_n},$
 - $5_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 8_{m_n},$
 - $6_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 9_{m_n},$
 - $7_{m_{n+1}} = 2_{m_n} 1_{m_n}^{k_{n+1}-1} 9_{m_n},$
 - $8_{m_{n+1}} = 2_{m_n},$
 - $9_{m_{n+1}} = 3_{m_n};$
- if the $(n + 1)$ -th multiplicative rule is a rule II_m ,
 - $1_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 7_{m_n},$
 - $2_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 6_{m_n},$

$$\begin{aligned}
- 3_{m_{n+1}} &= 4_{m_n}^{k_{n+1}} 5_{m_n}, \\
- 4_{m_{n+1}} &= 3_{m_n} 4_{m_n}^{k_{n+1}-1} 5_{m_n}, \\
- 5_{m_{n+1}} &= 3_{m_n}, \\
- 6_{m_{n+1}} &= 2_{m_n}, \\
- 7_{m_{n+1}} &= 1_{m_n}, \\
- 8_{m_{n+1}} &= 1_{m_n}^{k_{n+1}} 9_{m_n}, \\
- 9_{m_{n+1}} &= 1_{m_n}^{k_{n+1}} 8_{m_n}.
\end{aligned}$$

3.4. Relations between Arnoux-Rauzy systems. Starting from a point (a_0, b_0, c_0) in Γ , we have defined two geometric systems, (X_9, T) and (X_6, T) .

Proposition 5. *An AR9 interval exchange transformation defined by (a_0, b_0, c_0) is conjugated to an AR6 interval exchange transformation defined by (a_0, b_0, c_0) by a map which is continuous except on a finite number of points, and thus gives a measure-theoretic isomorphism for each invariant measure, and any AR6 interval exchange transformation is conjugate to an AR9 in this way.*

Proof

By gluing together the three intervals $\Omega_0, \Omega'_0, \Omega''_0$ we define a map ϕ'_6 sending X_9 to a circle of length $2a_0 + 2b_0 + 2c_0$: for example, in the first order, we identify the right end of Ω_0 with the left end of Ω'_0 , the right end of Ω'_0 with the left end of Ω''_0 , the right end of Ω''_0 with the left end of Ω_0 . This conjugates (X_9, T) to a system (X_6, T) which is exactly the AR6 interval exchange transformation defined in Section 2.3 above: its intervals of continuity are the $\phi'_6(J_{j,0})$, $j \in \{a-, a+, b-, b+, c-, c+\}$ where $J_{a-,0} = I_{1,0} \cup I_{2,0}$, $J_{a+,0} = I_{3,0} \cup I_{4,0}$, $J_{b-,0} = I_{5,0}$, $J_{b+,0} = I_{6,0} \cup I_{7,0}$, $J_{c-,0} = I_{8,0}$, $J_{c+,0} = I_{9,0}$. It is immediate that every AR6 interval exchange transformation can be built in this way. \square

As in Proposition 4, any point in Γ defines a directing sequence (r_n) . Each directing sequence defines two symbolic systems, (Y_9, S) and (Y_3, S) .

Proposition 6. *The coding of an AR9 symbolic system defined by (a_0, b_0, c_0) , by the partition into three sets $J_{a,0} = I_{1,0} \cup I_{2,0} \cup I_{3,0} \cup I_{4,0}$, $J_{b,0} = I_{5,0} \cup I_{6,0} \cup I_{7,0}$, $J_{c,0} = I_{8,0} \cup I_{9,0}$, is the AR3 symbolic system defined by the directing sequence in Proposition 4, and all AR3 symbolic systems can be built in this way.*

Proof

We define the letter-to-letter map ϕ by $\phi(1) = \phi(2) = \phi(3) = \phi(4) = a$, $\phi(5) = \phi(6) = \phi(7) = b$, $\phi(8) = \phi(9) = c$. If we build the words A_k, B_k, C_k in Definition 7 with a directing sequence (r_n) and the words 1_k to 9_k in the proof of Proposition 4, we get inductively that for all k , $\phi(1_k) = \phi(2_k) = \phi(3_k) = \phi(4_k) = A_k$, $\phi(5_k) = \phi(6_k) = \phi(7_k) = B_k$, $\phi(8_k) = \phi(9_k) = C_k$. By the induction steps of Section 3.2, A_k , resp. B_k , resp. C_k is the trajectory, under the coding by the partition into three sets, of any point x in $I_{i,k}$, $1 \leq i \leq 4$, resp. $5 \leq i \leq 7$, resp. $8 \leq i \leq 9$, between the time 0 and the first return time of x in $J_{a,k-1}$, and the words A_k, B_k, C_k , $k \geq 0$, generate the language of T .

The last assertion comes again from the fact, known from [5], that each directing sequence defines (up to homothety) a point in Γ . \square

Corollary 7. *An AR9 symbolic system has an AR3 symbolic system defined by the same directing sequence as a factor, and all AR3 symbolic systems can be built in this way.*

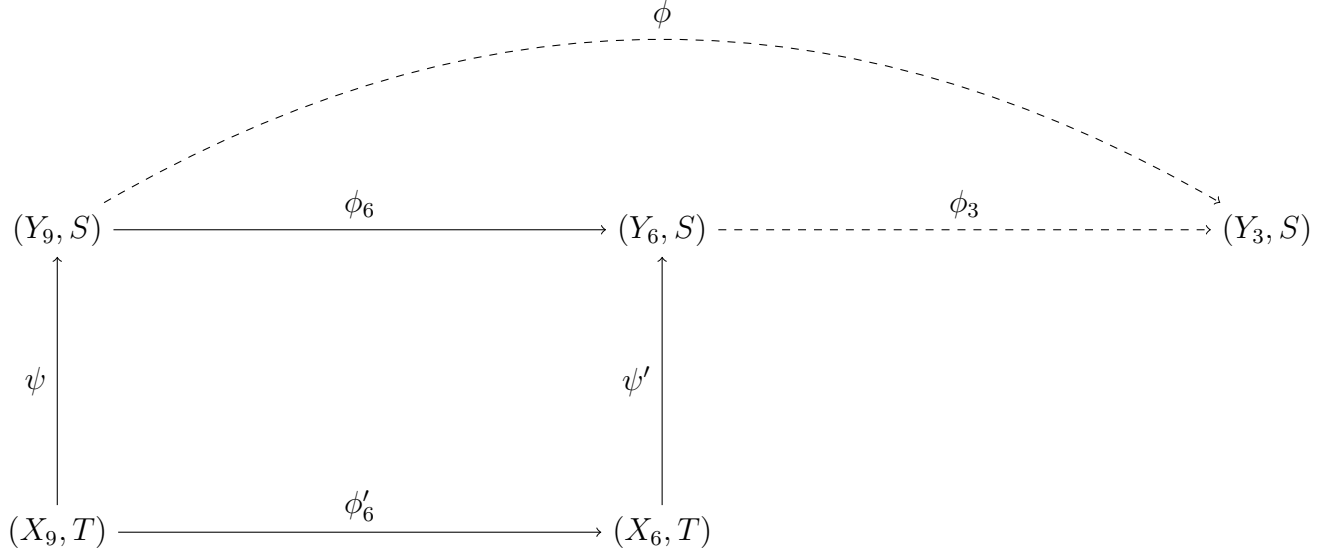


FIGURE 7. The five AR systems

Proof

These are two codings of the same AR9 interval exchange transformation, and the partition into nine intervals is finer than the partition into three sets. \square

The map associating to a point in (X_9, S) its coding in (Y_3, S) is just $\phi\psi$, where ψ is defined in Definition 13 and ϕ in the proof of Proposition 6. As in Remark 1, we have $\phi\psi(X_9) = Y_3 \setminus D'_3$ for the countable set D'_3 made with improper trajectories; note that $D'_3 \subset D_3$ where $D_3 = \phi(D_9)$. $\phi\psi$ conjugates the map T on X_9 with the shift S on X_3 . To use the vocabulary of [5], $\phi\psi$ is called a *semi-conjugacy*; as is pointed out in the introduction above, this does not give a one-to-one correspondence between points. Similarly, ϕ conjugates the shifts on Y_9 and Y_3 , with $\phi(Y_9) = Y_3$; it is also a semi-conjugacy, and not injective, see Proposition 13 below.

We can also define the AR6 symbolic system (Y_6, S) on $\{a-, a+, b-, b+, c-, c+\}$, as the natural coding ψ_6 , of (X_6, T) , by its six intervals of continuity; we have $Y_6 = \psi_6(X_6) \cup D_6$ for a countable set D_6 . We can write $\phi = \phi_3 \circ \phi_6$, with $\phi_6(1) = \phi_6(2) = a-$, $\phi_6(3) = \phi_6(4) = a+$, $\phi_6(5) = b-$, $\phi_6(6) = \phi_6(7) = b+$, $\phi_6(8) = c-$, $\phi_6(9) = c+$, and $\phi_3(j-) = \phi_3(j+) = j$ for $j = a, b, c$.

In the same way as Proposition 6, we could reprove the main result of [5]: the coding of an AR6 interval exchange transformation defined by (a_0, b_0, c_0) , by the partition into three sets $\phi'_6(J_{a-,0} \cup J_{a+,0})$, $\phi'_6(J_{b-,0} \cup J_{b+,0})$, $\phi'_6(J_{c-,0} \cup J_{c+,0})$, is the AR3 symbolic system defined by the directing sequence of Proposition 4, and all AR3 symbolic systems can be built in this way. Thus (Y_6, S) appears as an intermediate coding between the AR3 and AR9 symbolic systems; because of Proposition 5, ϕ_6 , applied letter to letter, is invertible except on a countable set (included in $\phi_6(D_9)$), and conjugates (Y_9, S) and (Y_6, S) , which are thus measure-theoretically isomorphic for each invariant measure.

In Figure 7, the four systems linked by full edges are, for all our purposes, the same system; the nature of the dashed edges will be investigated in the remainder of this paper.

As was already mentioned, we do not know any way to build the trajectories in Y_6 as in Definition 7 or Proposition 4; but they can be deduced from the trajectories in Y_9 by applying ϕ_6 letter to letter, and that was the main objective of the theory of AR9 systems. However, in general it will be easier to work directly on AR9 systems and then derive the properties of AR6 systems.

At this stage, it may be useful to recall the various notations we use, for which we had to make choices because of the number of systems we have defined and some long pre-existing notations. a, b, c are always the three symbols on which AR3 systems are built. But a_k, b_k, c_k , for any k , are real numbers, describing lengths of intervals. A_k, B_k, C_k are the words used to build AR3 systems, of lengths (i.e. number of letters) $h_{a,k}, h_{b,k}, h_{c,k}$. 1 to 9 are the symbols on which AR9 symbolic systems are built, 1_k to 9_k are the words used to build them, their lengths are among $h_{a,k}, h_{b,k}, h_{c,k}$. Interval lengths for AR9 systems, when needed, are defined in terms of a_k, b_k, c_k . Roman numerals are used to number substitutions and rules to build words.

4. DYNAMICAL PROPERTIES

4.1. Minimality. By using the condition that $r_n = I$ for infinitely many n , the minimality of AR3 symbolic systems and AR6 interval exchange transformations is shown in [5]. The minimality of AR6 symbolic systems follows, as the minimality of an interval exchange transformation is equivalent to the minimality of its natural coding, small intervals corresponding to small cylinders.

Proposition 8. *Any AR9 system is minimal.*

Proof

We show it for the symbolic systems, the minimality of the interval exchange transformations follows from the remark just above. We want to show that in the language of (Y_9, S) any word w occurs in any long enough word. It is enough to show that for all n and $1 \leq i \leq 9$ there exists N such that i_n occurs in every j_N , $1 \leq j \leq 9$.

For example we take $i = 1$. Through σ'_{III} i_n occurs in i_{n+1} for all i , as we are after sufficient conditions we can ignore these rules. We start from 1_n ; it occurs in 1_{n+1} through any number of σ'_{II} , so we wait until the first σ'_I (we know it exists), in which 1_{p_1} occurs in $4_{p_1+1}, 5_{p_1+1}, 6_{p_1+1}$.

We follow these three words until just before the next σ'_I : if there has been no σ'_{II} , we have to track $4_{p_2}, 5_{p_2}, 6_{p_2}$; if there has been one σ'_{II} , the words into which at least one of $4_{p_1+1}, 5_{p_1+1}, 6_{p_1+1}$ occur are $2_{p_2}, 3_{p_2}, 4_{p_2}$; if there have been two σ'_{II} or more, these words are $2_{p_2}, 3_{p_2}, 4_{p_2}, 5_{p_2}, 6_{p_2}$. So in the worst case we have to track either $2_{p_2}, 3_{p_2}, 4_{p_2}$ or $4_{p_2}, 5_{p_2}, 6_{p_2}$. After the σ'_I , these occur either in $1_{p_2+1}, 2_{p_2+1}, 3_{p_2+1}$ or in a larger set of words.

Again we follow these three words until just before the next σ'_I : if there have been no σ'_{II} , we have to track $1_{p_3}, 2_{p_3}, 3_{p_3}$; if there has been one σ'_{II} , the words to track are $1_{p_3}, 4_{p_3}, 5_{p_3}, 6_{p_3}, 7_{p_3}, 8_{p_3}, 9_{p_3}$; if there have been two σ'_{II} , these words are $1_{p_3}, 2_{p_3}, 3_{p_3}, 4_{p_3}, 7_{p_3}, 8_{p_3}, 9_{p_3}$; if there have been at least three σ'_{II} , we have already won (i_n occurs in all the j_{p_3}).

$1_{p_3}, 4_{p_3}, 5_{p_3}, 6_{p_3}, 7_{p_3}, 8_{p_3}, 9_{p_3}$ after σ'_I give $1_{p_4}, 2_{p_4}, 3_{p_4}, 4_{p_4}, 5_{p_4}, 6_{p_4}, 7_{p_4}$ which are conserved by any number of σ'_{II} , and give every word after the next σ'_I .

$1_{p_3}, 2_{p_3}, 3_{p_3}, 4_{p_3}, 7_{p_3}, 8_{p_3}, 9_{p_3}$ give everything after σ'_I .

$1_{p_3}, 2_{p_3}, 3_{p_3}$ after σ'_I give $1_{p_4}, 4_{p_4}, 5_{p_4}, 6_{p_4}, 7_{p_4}, 8_{p_4}, 9_{p_4}$ (with which we win after another σ'_I , as just above), after one σ'_{II} $1_{p_4}, 2_{p_4}, 3_{p_4}, 4_{p_4}, 7_{p_4}, 8_{p_4}, 9_{p_4}$ which will give everything after σ'_I , after two σ'_{II} everything.

Similar (shorter, as we can use what we already proved about 1_n and successive others) chasing arguments take care of the other i_n . \square

4.2. Rokhlin towers.

Definition 14. In a system (X', U) , a Rokhlin tower is a collection of disjoint measurable sets $F, UF, \dots, U^{h-1}F$ ($U^j F$ is called level j of the tower, F is called the base, h the height of the tower). A slice of τ is a union of whole levels $U^{p_1} F \dots U^{p_l} F$, and a column of τ is a union of all sublevels $G, \dots, U^{h-1}G$ for a subset G of F . We shall usually write “the tower τ ” as a shortened form of “the tower for which the union of the levels is the set τ ”.

Proposition 9. In an AR9 interval exchange transformation (X_9, T) , there are nine sequences of towers $\tau_{i,k}$, respectively of base $I_{i,k}$, and height equal to the length of the word i_k , $1 \leq i \leq 9$, $k \geq 0$. The union of all the levels for fixed k is X_9 , and every point x in X_9 is determined by the sequence $\iota(x, k), \eta(x, k)$ such that x is in $T^{\eta(x,k)} I_{\iota(x,k),k}$, $k \geq 0$. This remains true if we restrict k to a subsequence, for example the m_n . All levels of these towers are intervals.

Proof

From the induction steps in Section 3.2, we deduce that the $\tau_{i,k}$ are indeed Rokhlin towers, whose union of levels for fixed k is indeed X_9 . All these levels are intervals and their lengths are smaller than a_k , which tends to zero when k goes to infinity, hence the result. \square

Figures 4, 5, 6 going from stage 0 to stage 1 show how the towers at order 1 are made from the towers at order 0 by *cutting and stacking*. This cutting and stacking is done in the same way from stage k to stage $k + 1$; it is dictated by the induction as above, and can be read on the rules giving the words 1_{k+1} to 9_{k+1} as concatenations of the words 1_k to 9_k , which are deduced from the substitutions σ'_I to σ'_{III} : for example, when $r_{k+1} = I$, σ'_I is applied, and we deduce from $1 \rightarrow 35$ that $1_{k+1} = 3_k 5_k$, and the tower $\tau_{1,k+1}$ is made by a column of $\tau_{5,k}$ stacked above a column of $\tau_{3,k}$.

Corollary 10. In (Y_9, S) , the $\tau'_{i,k} = \psi(\tau_{i,k})$, $i = 1, \dots, 9$, form nine sequences of Rokhlin towers. If D_9 is the countable set defined in Remark 1, every point y in $Y_9 \setminus D_9$ is determined by the sequences $\iota(y, k), \eta(y, k)$ such that y is in $S^{\eta(y,k)} \psi(I_{\iota(y,k),k})$, $k \geq 0$.

In (X_9, T) , there exist three sequences of Rokhlin towers $\tau_{a,k}, \tau_{b,k}, \tau_{c,k}$, respectively of bases $J_{a,k}, J_{b,k}, J_{c,k}$, and heights equal to $h_{a,k}, h_{b,k}, h_{c,k}$, $k \geq 0$, where $J_{a,k} = I_{1,k} \cup I_{2,k} \cup I_{3,k} \cup I_{4,k}$, $J_{b,k} = I_{5,k} \cup I_{6,k} \cup I_{7,k}$, $J_{c,k} = I_{8,k} \cup I_{9,k}$. The union of all their levels for fixed k is X_9 .

In the AR3 system (Y_3, S) , the $\tau'_{j,k} = \phi\psi(\tau_{j,k})$, $j = a, b, c$, form three sequences of Rokhlin towers; if $D_3 = \phi(D_9)$, every point x in $Y_3 \setminus D_3$ is determined by the sequences $\iota'(y, k), \eta(y, k)$ such that y is in $S^{\eta(y,k)} \phi\psi(J_{\iota'(y,k),k})$, $k \geq 0$.

Proof

The first assertion comes from Proposition 9 translated by ψ to the symbolic system. The second assertion comes from the definition of the $J_{j,k}$ and the values of the heights. The third assertion comes from the first one and the fact that for all k ϕ sends $\psi(I_{i,k})$ to $\psi(J_{a,k})$ if $i = 1, 2, 3, 4$, $\psi(J_{b,k})$ if $i = 5, 6, 7$, $\psi(J_{c,k})$ if $i = 8, 9$. and similarly for the other levels. \square

Remark 2. We can also build directly (slightly) enlarged versions of the various towers τ' in the symbolic systems. This is done in [14] for the $\tau'_{j,k}$, $j = a, b, c$, by induction on cylinders which are

the closure of $\phi\psi(J_{a,k})$ in the topology of the symbolic systems, and can be done in the same way for the $\tau'_{i,k}$, $i = 1, \dots, 9$, by induction on unions of cylinders which are the closure of $\psi(J_{a,k})$. These enlarged towers are closed and include also improper trajectories; but we do not need that for our results, for which countable sets can be neglected, and in any case points of D_3 must be taken into account, see Remark 3 below.

The towers $\tau'_{i,k}$, $i = 1, \dots, 9$, can be built by cutting and stacking with the same rules as the $\tau_{i,k}$. The $\tau_{j,k}$ or $\tau'_{j,k}$, $j = a, b, c$, can be built by cutting and stacking, using the concatenation rules generating the words A_k to C_k , deduced from the substitutions σ_I to σ_{III} ; we shall also use the multiplicative rules to build more quickly these towers at multiplicative times, as is shown in Figures 9 and 10 below.

Lemma 11. *For every k , the sets $T^j I_{2,k}$ and $T^j I_{3,k}$, $0 \leq j \leq h_{a,k} - 1$, resp. $T^j I_{5,k}$ and $T^j i_{6,k}$, $0 \leq j \leq h_{b,k} - 1$, resp. $T^j I_{8,k}$ and $T^j i_{9,k}$, $0 \leq j \leq h_{c,k} - 1$, are adjacent intervals.*

Proof

We make the induction hypothesis that our result is true at order k and that $T^j I_{2,k}$, $T^j I_{5,k}$, r $T^j I_{8,k}$ are the leftmost of the respective two adjacent intervals when T_k is not in a reversed order, the rightmost if T_k is in a reversed order.

This is true for $k = 0$, whatever the order. The induction step from k to $k + 1$ describes also the way the towers at order $k + 1$ are built from the towers at order k .

Take for example Case I when T_k is not in a reversed order. The new tower 8 is made by taking a right subinterval of the base $I_{2,k}$ of the old tower 2, and keeping the corresponding part of all the levels of the old tower 2; The new tower 9 is made by taking a left subinterval of the base $I_{3,k}$ of the old tower 3, and keeping the corresponding part of all the levels of the old tower 3. Thus all corresponding levels of the new towers 8 and 9 are adjacent as those of the old towers 2 and 3 were, and the levels of the new tower 8 are to the left of those of the new tower 9.

The new tower 2 is made by taking a left subinterval of the base $I_{4,k}$ of the old tower 4, and keeping the corresponding part of all the levels of the old tower 4, until the top; above that we stack a right subinterval of $I_{5,k}$, and the corresponding part of all the levels of the old tower 5. The new tower 3 is made by taking a right subinterval of $I_{4,k}$, and keeping the corresponding part of all the levels of the old tower 4, until the top; above that we stack $I_{6,k}$, and all the levels of the old tower 6. Thus all corresponding levels of the new towers 2 and 3 are adjacent as those of the old towers 5 and 6 were, while the levels of the old tower 4 are intervals, and the levels of the new tower 2 are to the left of those of the new tower 3.

The new tower 6 is made by taking a right subinterval of $I_{1,k}$, and keeping the corresponding part of all the levels of the old tower 1, until the top; above that we stack a left subinterval of $I_{9,k}$, and the corresponding part of all the levels of the old tower 9. The new tower 5 is made by taking a subinterval of $I_{1,k}$ just left of the previous one, and keeping the corresponding part of all the levels of the old tower 1, until the top; above that we stack the subinterval $I_{8,k}$, and all the levels of the old tower 8. Thus all corresponding levels of the new towers 5 and 6 are adjacent as those of the old towers 8 and 9 were, while the levels of the old tower 1 are intervals, and the levels of the new tower 5 are to the left of those of the new tower 6.

The other cases are deduced from this one by changing suitably the recursion rules (according to Figures 4,5,6) and the orientation. For example, we build the new tower 8 in two further cases.

In Case I when T_k is in a reversed order, the new tower 8 is made by taking a left subinterval of the base $I_{2,k}$ of the old tower 2, and keeping the corresponding part of all the levels of the old tower 2. At the end, the levels of the new tower 8 will be to the right of those of the new tower 9. In case III when T_k is not in a reversed order, the new tower 8 is made by taking a subinterval of $I_{1,k}$ just left of the one used for the new tower 9, and keeping the corresponding part of all the levels of the old tower 1, until the top; above that we stack the subinterval $I_{8,k}$, and all the levels of the old tower 8. The levels of the new tower 8 will be to the left of those of the new tower 9. \square

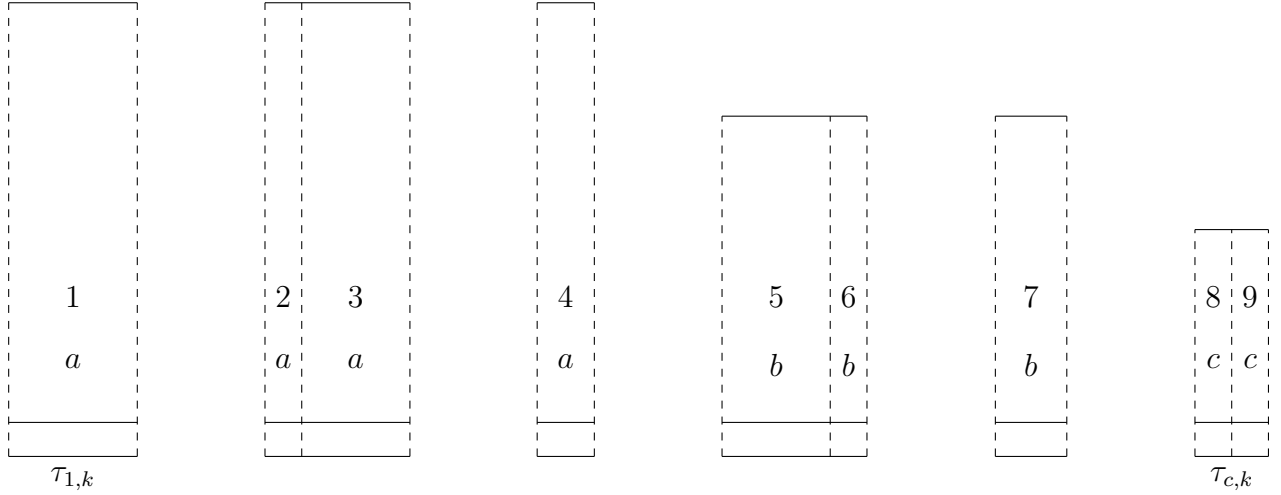


FIGURE 8. Rokhlin towers in X_9

An immediate consequence is best seen on Figure 8:

Corollary 12. *Each level of the towers $\tau_{c,k}$ is an interval, each level of the towers $\tau_{b,k}$ is a union of at most two intervals, each level of the towers $\tau_{a,k}$ is a union of at most three intervals.*

Note that the $J_{j,k}$ and their images are not intervals for $j = a, b$, except maybe for the first values of k , with some choices of $\Omega_0, \Omega'_0, \Omega''_0$, but even in that case, for example if they are in the first order, $J_{b,0}$ is not an interval. Similarly, except maybe for the first values of k , the levels of the towers $\tau_{b,k}$ are not intervals, the levels of the towers $\tau_{a,k}$ are not unions of less than three intervals.

4.3. Isomorphism.

Definition 15. *For $i = 1, 2, 3$, let $E_i \subset Y_3$ be the set of points which have i pre-images under ϕ .*

Proposition 13. *$Y_3 \setminus D_3 \subset E_1 \cup E_2 \cup E_3$. E_3 is countable. If $\mu(E_1) < 1$, then for any invariant probability μ' the system (Y_9, S, μ') is a two-point extension of (Y_3, S, μ)*

Proof

If we know a point y in Y_3 , we know the sequences $\iota'(y, k)$ in $\{a, b, c\}$ and $0 \leq \eta(y, k) \leq h_{\iota'(y,k),k} - 1$ such that y is in $S^{\eta(y,k)} \phi \psi J_{\iota'(y,k),k}$ for all k . Except if y is in D_3 , there exist points x in X_9 such that $\phi \psi(x) = y$, and the pre-images of y by ϕ are the points $\psi(x)$. Because of the way $\phi \psi$ acts on the towers, all these x must be in $T^{\eta(y,k)} I_{\iota(x,k),k}$ where $\iota'(y, k) = \phi(\iota(x, k))$. y being fixed, for a given k , all possible x are in at most three of the intervals of Figure 8 above. If $\iota'(y, k) = a$, all possible x are either in $T^{\eta(y,k)} I_{1,k}$, or in $T^{\eta(y,k)} I_{2,k} \cup T^{\eta(y,k)} I_{3,k}$, or in $T^{\eta(y,k)} I_{4,k}$, and similarly

there are only two possible intervals if $l'(y, k) = b$, and one if $l'(y, k) = c$. If there exist more than three such points x , two of them must be infinitely often in the same interval, thus must be the same as the intersection of infinitely many of these intervals defines at most one point. Thus we get our first assertion.

By the same reasoning, if $y \in Y_3 \setminus D_3$ is in $\tau'_{c,k} \cup \tau'_{b,k}$ for infinitely many k , then y is in $E_1 \cup E_2$. Thus if y is in $E_3 \setminus D_3$, y is in $\tau'_{a,k}$ for all $k \geq k_0$. By the rules of construction by cutting and stacking, this implies that for all $k \geq k_0$ $\eta(y, k)$ takes the same value η_0 , thus any pre-image of y by $\phi\psi$ is in $\bigcap_{k \geq k_0} T^{\eta_0} J_{a,k}$. For $\eta_0 = 0$, it is shown in [5] that this intersection consists indeed of three distinct points, whose images by ψ are not in D_6 and which have the same images by $\phi\psi$, thus $E_3 \setminus D_3$ consists of the union of the positive orbits of these three points, which proves our second assertion.

Thus $\mu(E_1 \cup E_2) = 1$, and if $\mu(E_1) < 1$ the number of pre-images by ϕ is two on a set of positive measure, thus almost everywhere by ergodicity, and this is our third assertion. \square

This implies in particular that AR9 (and AR6) systems have at most two ergodic invariant probability measures. This can also be proved by Dynnikov and Skripchenko by a symmetry argument on the underlying AR6 interval exchange transformation: there is an involution which takes any invariant measure to an invariant one, and there is only one invariant measure that is symmetric with respect to this involution.

Lemma 14. *Let y be in $Y_3 \setminus D_3$. If y is in $\tau'_{c,k}$ for infinitely many k , then y is in E_1 .*

Proof

Under the hypothesis, as in the proof of Proposition 13, for infinitely many k all the pre-images of y by $\phi\psi$ are in an interval, of length c_k . Thus the intersection of infinitely many of these intervals defines at most one point. \square

Remark 3. *If we enlarge the towers to cover all Y_3 as in [14] and Remark 2 above, the generalization of Lemma 14 does not hold for $y \in D_3$. Indeed, the point x_0 separating $I_{8,0}$ from $I_{9,0}$ defines one trajectory in $\psi(X_9)$ and one improper trajectory (as in Remark 1), and both these trajectories have the same image y_0 by ϕ , though we can check that, for example in the Tribonacci case, y_0 is in the enlarged $\tau'_{c,k}$ for infinitely many k . However, it is true that every point in Y_3 has at most three pre-images by ϕ , as the only candidates to have more are the points which are in the enlarged $\tau'_{a,k}$ for all $k \geq k_0$, and their pre-images do not give rise to improper trajectories.*

At this stage, one can ask whether the condition to be in $\tau'_{c,k}$ for infinitely many k is necessary for y to be in E_1 . Hopefully, a necessary and sufficient condition will be given in a further paper, but the following lemma gives already a negative answer for many systems including Tribonacci.

Lemma 15. *Suppose that,*

- (i) *either for an infinite sequence s_j , the $(s_j + 2)$ -th multiplicative rule is I_m with $k_{s_j+2} = 1$,*
- (ii) *or for an infinite sequence s_j the $(s_j + 2)$ -th multiplicative rules is I_m and the $(s_j + 1)$ -th multiplicative rule is II_m with $k_{s_j+1} = 1$.*

Let y be in $Y_3 \setminus D_3$. If we are in case (i) and for infinitely many j y is in $\tau'_{b,m_{s_j+1}} \cap \tau'_{b,m_{s_j+3}}$, or if we are in case (ii) and for infinitely many j y is in $\tau'_{b,m_{s_j}} \cap \tau'_{b,m_{s_j+3}}$, then y is in E_1 .

Proof

A pre-image x of y by ϕ is in $\tau'_{5,m_{s_j+3}}$, $\tau'_{6,m_{s_j+3}}$, or $\tau'_{7,m_{s_j+3}}$.

Going from m_{s_j+2} to m_{s_j+3} , we have a number (possibly zero) of σ'_{III} followed by a σ'_I or σ'_{II} .

- Suppose this last substitution is σ'_{II} : the construction of the towers by σ'_{II} implies that x is in $\tau'_{1,m_{s_j+3}-1}$, $\tau'_{2,m_{s_j+3}-1}$, or $\tau'_{3,m_{s_j+3}-1}$; then either the absence of σ'_{III} or the construction of the towers by σ'_{III} imply that x is in $\tau'_{1,p}$, $\tau'_{2,p}$ or $\tau'_{3,p}$ at all stages $m_{s_j+2} \leq p \leq m_{s_j+3} - 1$.
- Suppose now this substitution is σ'_I : the construction of the towers by σ'_I implies that x is in $\tau'_{1,m_{s_j+3}-1}$, $\tau'_{2,m_{s_j+3}-1}$, $\tau'_{8,m_{s_j+3}-1}$ or $\tau'_{9,m_{s_j+3}-1}$. In the last two cases, x is in $\tau'_{c,m_{s_j+3}-1}$ and if this happens infinitely often we conclude by Lemma 14 that y is in E_1 . Otherwise, either the absence of σ'_{III} or the construction of the towers by σ'_{III} imply that x is in $\tau'_{1,p}$ or $\tau'_{2,p}$ at all stages $m_{s_j+2} \leq p \leq m_{s_j+3} - 1$.

Thus in both remaining cases x is in $\tau'_{1,m_{s_j+2}}$, $\tau'_{2,m_{s_j+2}}$, or $\tau'_{3,m_{s_j+2}}$.

Going from m_{s_j+1} to m_{s_j+2} , we have a number of σ'_{III} followed by a σ'_I ; the construction of the towers by σ'_I implies that x is in $\tau'_{3,m_{s_j+2}-1}$, $\tau'_{4,m_{s_j+2}-1}$, $\tau'_{5,m_{s_j+2}-1}$, or $\tau'_{6,m_{s_j+2}-1}$. We are in the last two cases whenever x is in $\tau'_{b,m_{s_j+2}-1}$, and then, if we know its level in that tower, that puts x in a single level of $\tau'_{5,m_{s_j+2}-1} \cup \tau'_{6,m_{s_j+2}-1}$, which puts the possible pre-images of y by $\phi\psi$ in a small interval by Lemma 11; if this happens infinitely often we conclude as in Lemma 14 that y is in E_1 . Otherwise, either the absence of σ'_{III} or the construction of the towers by σ'_{III} imply that x is in $\tau'_{3,p}$ or $\tau'_{4,p}$ at all stages $m_{s_j+1} \leq p \leq m_{s_j+2} - 1$: this is excluded by the hypotheses in case (i), thus our result is proved in that case.

Finally, in case (ii), going from m_{s_j} to m_{s_j+1} by a single σ'_{II} and knowing y is in $\tau'_{b,m_{s_j}}$, we get that x must be in $\tau'_{5,m_{s_j}}$, and the knowledge of its level in $\tau'_{b,m_{s_j}}$ puts the possible pre-images of y by $\phi\psi$ in a small interval, thus we conclude as in Lemma 14. \square

Note that Lemma 15 gives only sufficient conditions, the same reasoning can produce many others. It will not be used further, as Lemma 14 is enough to prove

Proposition 16. *Let*

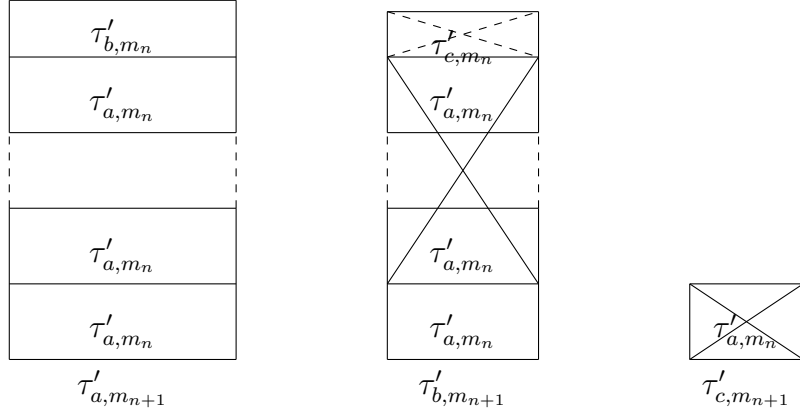
- $\xi_n = \frac{1}{k_{n+2}}$ if the $(n+1)$ -th multiplicative rule is I_m and $k_{n+1} \geq 2$,
- $\xi_n = \frac{1}{3^l k_{n+2} \dots k_{n+l+1}}$ if the $(n+1)$ -th multiplicative rule is I_m with $k_{n+1} = 1$ or II_m , and the next multiplicative rule I_m is the $n+l$ -th, $l \geq 2$.

Suppose $\sum \xi_n = +\infty$. Let Z be the set of y in Y_3 , such that y is not in $\tau'_{c,k}$ for all k large enough. Then $\mu(Z) = 0$ for the unique invariant measure μ .

Proof

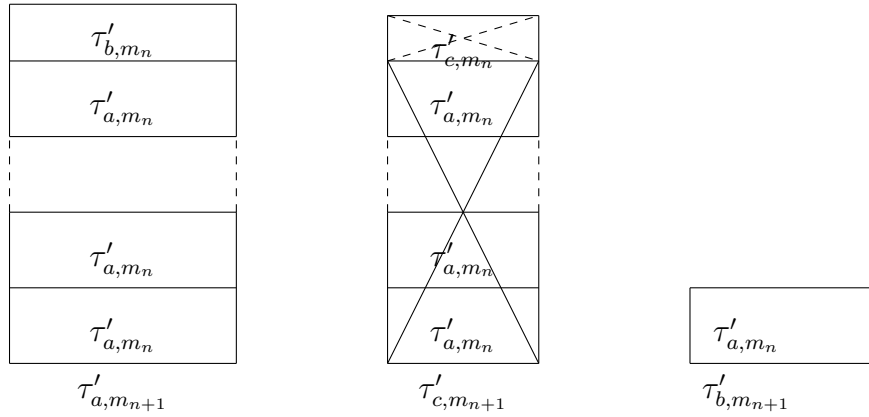
We fix a multiplicative time m_{n_0} , and for $n \geq n_0$ we define Z_n to be the set of y which are not in $\tau'_{c,k}$ for all $m_{n_0} \leq k \leq m_n$, $n \geq n_0$, and V_n such that $Z_n \setminus V_n = Z_{n+1}$. We have $Z_{n_0} = \tau'_{a,m_{n_0}} \cup \tau'_{b,m_{n_0}}$.

At each additive time $m_n \leq k < m_{n+1}$, the new tower $\tau'_{c,k+1}$ is made with $\tau'_{c,k}$ stacked above one column of $\tau'_{a,k}$; $\tau'_{c,m_{n+1}-1}$ is made with τ'_{c,m_n} stacked above $k_{n+1} - 1$ columns of τ'_{a,m_n} . Then, if the $(n+1)$ -th multiplicative rule is II_m , $\tau'_{c,m_{n+1}}$ is made with τ'_{c,m_n} stacked above k_{n+1} columns of τ'_{a,m_n} ; if the $(n+1)$ -th multiplicative rule is I_m , $\tau'_{c,m_{n+1}}$ is made with the last remaining column of τ'_{a,m_n} . Then V_n is made either with $k_{n+1} - 1$ columns of τ'_{a,m_n} stacked above τ'_{c,m_n} plus the last column of τ'_{a,m_n} , or with $k_{n+1} - 1$ columns of τ'_{a,m_n} stacked above τ'_{c,m_n} . In both cases, V_n is a union of slices of $\tau'_{a,m_{n+1}}$ and $\tau'_{c,m_{n+1}}$.

FIGURE 9. Cutting and stacking I_m

Assume, that Z_n is a union of slices of τ'_{a,m_n} and τ'_{b,m_n} (this is true for $n = n_0$); then Z_n is also a union of slices of $\tau'_{a,m_{n+1}}$ and $\tau'_{b,m_{n+1}}$, and thus so is Z_{n+1} . In all cases, $Z_{n+1} \cap \tau'_{a,m_{n+1}}$ is made with all $Z_n \cap \tau'_{b,m_n}$ and the intersection of $Z_n \cap \tau'_{a,m_n}$, with k_{n+1} columns of τ'_{a,m_n} whose levels have measure $a_{m_{n+1}}$. $Z_{n+1} \cap \tau'_{b,m_{n+1}}$ is the intersection of $Z_n \cap \tau'_{a,m_n}$ with one column of τ'_{a,m_n} whose levels have measure $b_{m_{n+1}}$. Thus we have always, for $n \geq n_0 + 1$, $\mu(Z_n \cap \tau'_{a,m_n}) \geq \mu(Z_n \cap \tau'_{b,m_n})$.

Figures 9 and 10 give a schematic view (note that the levels of the towers are not intervals, even when carried to (X_9, T) , see Figure 8 above) of what is used in the proof. The crossed parts form V_n , which has been deleted from Z_n to get Z_{n+1} ; the τ'_{c,m_n} , crossed by dashed lines, have been deleted at an earlier stage.

FIGURE 10. Cutting and stacking II_m

We want now to estimate the measure of V_n .

We suppose first that the $(n + 1)$ -th multiplicative rule is I_m . If $k_{n+1} \geq 2$, V_n is a slice of $\tau'_{b,m_{n+1}}$ of height $(k_{n+1} - 1)h_{a,m_n}$. If $k_{n+1} = 1$, V_n is $\tau'_{c,m_{n+1}}$.

Suppose $k_{n+1} \geq 2$. Then we need to estimate $\mu(\tau'_{b,m_{n+1}})$; we notice that $\mu(\tau'_{a,m_{n+2}}) \geq \frac{1}{5}$, because this tower is wider than the two others, and at least half as high by the estimate at the end of Section 2.2. $\tau'_{b,m_{n+1}}$ is a slice of $\tau'_{a,m_{n+2}}$ of height $h_{b,m_{n+1}}$, while $h_{a,m_{n+2}} = k_{n+2}h_{a,m_{n+1}} + h_{b,m_{n+1}}$. From $h_{b,m_{n+1}} = k_{n+1}h_{a,m_n} + h_{c,m_n}$, $h_{a,m_{n+1}} = k_{n+1}h_{a,m_n} + h_{b,m_n}$, we get $h_{b,m_{n+1}} \geq \frac{k_{n+1}}{k_{n+1}+2}h_{a,m_{n+1}} \geq \frac{1}{2}h_{a,m_{n+1}}$, and $\mu(\tau'_{b,m_{n+1}}) \geq \frac{1}{10(k_{n+2}+2)}$. Now V_n is a slice of $\tau'_{b,m_{n+1}}$ of relative height at least $\frac{k_{n+1}-1}{k_{n+1}+2} \geq \frac{1}{3}$, and we get $\mu(V_n) \geq \frac{1}{30(k_{n+2}+2)}$.

If $k_{n+1} = 1$, we take first $l = 2$: the $n+2$ -th multiplicative rule is also I_m . Then $\tau'_{c,m_{n+1}}$ is a slice of $\tau'_{b,m_{n+2}}$ of height $h_{c,m_{n+1}}$, while $h_{b,m_{n+2}} = k_{n+2}h_{a,m_{n+1}} + h_{c,m_{n+1}}$. We have $h_{a,m_{n+1}} = h_{a,m_n} + h_{b,m_n}$, $h_{c,m_{n+1}} = h_{a,m_n}$, thus $h_{c,m_{n+1}} \geq \frac{1}{3}h_{a,m_{n+1}}$, thus we get $\mu(\tau'_{c,m_{n+1}}) \geq \frac{1}{3(k_{n+2}+2)}\mu(\tau'_{b,m_{n+2}})$. Then $\mu(\tau'_{b,m_{n+2}})$ is estimated just as $\mu(\tau'_{b,m_{n+1}})$ in the case above, with the only difference that k_{n+2} may be equal to one: we get it is at least $\frac{1}{15(k_{n+3}+2)}$, and thus $\mu(V_n) \geq \frac{1}{45(k_{n+2}+2)(k_{n+3}+2)}$.

For larger values of l we iterate this method, looking at $\tau'_{c,m_{n+1}}$, inside ... inside $\tau'_{c,m_{n+l-1}}$, inside $\tau'_{b,m_{n+l}}$, inside $\tau'_{a,m_{n+l+1}}$. Estimating the measures gives us first factors $\frac{1}{k_{n+2}+2} \dots \frac{1}{k_{n+l+1}+2}$, but depend also on the comparison of successive heights of towers, which brings factors $\frac{1}{3}$.

If the $(n+1)$ -th multiplicative rule is II_m , V_n is a slice of $\tau'_{c,m_{n+1}}$ of height at least $\frac{1}{3}h_{c,m_{n+1}}$ and we estimate its measure in the same way.

In all cases V_n is a column of τ'_{a,m_n} while $Z_n \cap \tau'_{a,m_n}$ is a slice of τ'_{a,m_n} , and for any column Λ and slice Λ' of the same tower we have $\mu(\Lambda \cup \Lambda') = \mu(\Lambda)\mu(\Lambda')$. Thus $\mu(V_n \cap Z_n) \geq \mu(Z_n \cap \tau'_{a,m_n} \cap V_n) = \mu(V_n)\mu(Z_n \cap \tau'_{a,m_n}) \geq \frac{1}{2}\mu(V_n)\mu(Z_n)$, and $\mu(Z_{n+1}) \leq \mu(Z_n)(1 - \frac{1}{2}\mu(V_n)) \leq \mu(Z_n)(1 - K\xi_n)$ for some constant K , and we conclude by a Borel-Cantelli argument, namely $\mu(Z_{n+1}) \leq \mu(Z_{n_0}) \prod_{n>n_0} (1 - K\xi_n)$, thus $\mu(Z) = 0$ because of our hypothesis. \square

We turn now to the isomorphism problem: as E_3 is nonempty, the best we can hope is to replace the semi-conjugacies in Section 3.4 by measure-theoretic isomorphisms.

Theorem 17. *Under the hypothesis of Proposition 16, an AR9 or AR6 symbolic system or interval exchange transformation is uniquely ergodic and measure-theoretically isomorphic to its AR3 coding.*

Proof

Then, by Proposition 16 and Lemma 14 ϕ is invertible almost everywhere. Thus ϕ provides a measure-theoretic isomorphism between (Y_3, S, μ) and (Y_9, S, μ') for any normalized invariant measure μ' . Such an invariant measure μ' can be defined also on (X_9, T) as ψ is invertible almost everywhere, and ψ provides a measure-theoretic isomorphism between (X_9, T, μ') and (Y_9, S, μ') . In particular, any such measure μ' has to be ergodic, hence the unique ergodicity. The results extend then to the intermediate coding (Y_6, S, μ') and to its geometric model (X_6, T, μ') . \square

Now we introduce a notation as in [11]: we consider measures on all infinite sequences of symbols I, II, III and take any shift invariant ergodic probability measure ν which assigns positive measure to each cylinder; by identifying an AR3, AR6, or AR9 system with its defining sequence (r_n) , we can define ν on the set of all AR3, AR6, or AR9 systems.

In particular, one of these measures coincides with the Abramov-Rokhlin projection to the base of the measure of maximal entropy for the suspension flow of the Rauzy gasket built in [8], see also [9].

Proposition 18. *The hypothesis of Proposition 16 is satisfied by ν -almost every AR3, AR6, or AR9 system.*

Proof

This hypothesis is satisfied in particular if for infinitely many n we have $k_{n+1} = 2$ and $k_{n+2} = 1$, which is satisfied in particular if for infinitely many p we have $r_p = I$, $r_{p+1} = r_{p+2} = III$, $r_{p+3} = r_{p+4} = I$. As this cylinder has positive measure and ν is ergodic, this is true for ν -almost every sequence (r_n) . \square

This completes the proof of Theorem 1 above. But the sufficient condition in Proposition 16 gives also the isomorphism (and unique ergodicity) for many explicit examples; while the first set of values of ξ_n is enough to prove Proposition 18 above, with the help of the second set of values we can prove the following.

Proposition 19. *The hypothesis of Proposition 16 is satisfied by all Arnoux-Rauzy systems where the k_n are bounded (in [11] these are said to have bounded weak partial quotients).*

Proof

If the $n + 2$ -th multiplicative rule is I_m , then ξ_n is either $\frac{1}{k_{n+2}}$ or $\frac{1}{3^2 k_{n+2} k_{n+3}}$. As there are infinitely many rules I_m , we get infinitely many n for which $\xi_n \geq \frac{1}{9K_0^2}$ if all the k_n are bounded by K_0 . \square

This completes the proof of Corollary 2 above; then Corollary 3 is proved by using the measure-theoretic isomorphism between the Tribonacci AR3 and a rotation of the 2-torus [34] and the fact that such a rotation is always rigid.

4.4. Non unique ergodicity.

Theorem 20. *If $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$, each corresponding AR9 or AR6 symbolic system or interval exchange transformation is not uniquely ergodic; it has two ergodic invariant probability measures; it is measure-theoretically isomorphic to its AR3 coding if and only if it is equipped with an ergodic measure.*

Proof

Let μ' be any normalized invariant measure on (Y_9, S) . We first show that at multiplicative times all towers have very small measure except τ'_{1,m_n} and τ'_{4,m_n} .

Indeed, from the multiplicative rules of Section 2.2 we get that τ'_{b,m_n} is a slice of $\tau'_{a,m_{n+1}}$ of height h_{b,m_n} , hence $\mu(\tau'_{b,m_n}) \leq \frac{2}{k_{n+1}-1}$, while τ'_{c,m_n} is a slice of either $\tau'_{b,m_{n+1}}$ or $\tau'_{c,m_{n+1}}$, of height h_{c,m_n} , hence $\mu(\tau'_{c,m_n}) \leq \frac{2}{k_{n+1}-1}$; and $\mu'(\tau'_{i,m_n}) \leq \mu'(\psi\tau_{b,m_n}) = \mu(\tau'_{b,m_n})$ for $i = 5, 6, 7$, $\mu'(\tau'_{i,m_n}) \leq \mu'(\psi\tau_{c,m_n}) = \mu(\tau'_{c,m_n})$ for $i = 8, 9$.

Now, from the multiplicative rules at the end of Section 3.3 we get that τ'_{2,m_n} is either $\tau'_{6,m_{n+1}}$ or the union of $\tau'_{8,m_{n+1}}$ with a slice of $\tau'_{7,m_{n+1}}$, thus $\mu'(\tau'_{2,m_n}) \leq \frac{4}{k_{n+2}-1}$. Finally τ'_{3,m_n} is either the union of $\tau'_{9,m_{n+1}}$ with a slice of $\tau'_{1,m_{n+1}}$ of relative height at most $\frac{1}{k_{n+1}-1}$, or the union of $\tau'_{5,m_{n+1}}$ with a slice of $\tau'_{4,m_{n+1}}$ of relative height at most $\frac{1}{k_{n+1}-1}$: in both cases $\mu'(\tau'_{3,m_n}) \leq \frac{3}{k_{n+1}-1}$.

Thus, the condition $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$ implies that for any invariant measure μ' , the system (Y_9, S, μ') is such that μ' -almost every point y in Y_9 is determined by the sequences $l''(y, k)$, $\eta(y, k)$ such that y is in level $\eta(y, k)$ of the tower $\tau'_{l''(y,k),k}$, $l''(y, k) \in \{1, 4\}$. We say that (Y_9, S, μ') is generated by two sequences of towers, and such a measure-theoretic system is said to be a system

of *rank* (at most) two; as a system of rank at most two or by Proposition 13 above, (Y_9, S) has at most two ergodic invariant probability measures.

At multiplicative times, we define recursively $(\tau'_{1,m_n}, \tau'_{4,m_n}) = e^l(\tau'_{1,m_n}, \tau'_{4,m_n})$ if l is the total number of rules I_m (strictly) before the n -th multiplicative rule and e is the exchange. Then for each n , τ'_{1,m_n} makes all but a very small part of $\tau'_{1,m_{n+1}}$, τ'_{4,m_n} makes all but a very small part of $\tau'_{4,m_{n+1}}$, and all the other τ'_{i,m_n} , $i \neq 1, 4$ have very small measure.

We define a new symbolic system $(\bar{X}, \bar{T}, \bar{\mu})$ on the alphabet $\{a, s\}$ by the words $D_0 = a$, $D_{n+1} = s^{h_{a,m_n}} D_n^{k_{n+1}-1} s^{h_{b,m_n}}$. By a standard argument, see [23], we can build towers $\bar{\tau}'_n$ in \bar{X} , $\bar{\tau}'_{n+1}$ being obtained from $\bar{\tau}'_n$ by cutting it into $k_{n+1} - 1$ equal columns, stacking them above each other, stacking below them h_{a,m_n} new levels called *spacer levels*, and stacking above them h_{b,m_n} new levels called *spacers*; almost every point x in \bar{X} is determined by the sequence $\eta'(x, n)$ such that y is in level $\eta'(x, n)$ of the tower $\bar{\tau}'_n$. $(\bar{X}, \bar{T}, \bar{\mu})$ is a *system of rank one*, as it can be generated by a single family of towers.

As it is explained in more details in [1], we can build a map ϕ_1 from \bar{X} to Y_9 by sending the j -th level of the tower $\bar{\tau}'_n$ to the j -th level of the tower τ'_{1,m_n} : it is consistent, defined almost everywhere and one-to-one. By taking the image of $\bar{\mu}$ by ϕ_1 , we build a measure-theoretic isomorphism between the rank one system $(\bar{X}, \bar{T}, \bar{\mu})$ and (Y_9, S) equipped with some invariant probability measure μ_1 ; μ_1 is ergodic as $\bar{\mu}$ is. We do the same for another map ϕ_4 , which sends the j -th level of $\bar{\tau}'_n$ to the j -th level of τ'_{4,m_n} , defining an ergodic μ_4 . Now, $\mu_1(\tau'_{1,m_n})$ and $\mu_4(\tau'_{4,m_n})$ are close to 1, $\mu_1(\tau'_{4,m_n})$ and $\mu_4(\tau'_{1,m_n})$ are close to 0 for n large enough, thus there exists n for which $\mu_1(\tau'_{1,m_n}) \neq \mu_4(\tau'_{1,m_n})$, thus $\mu_1 \neq \mu_4$ on (Y_9, S) .

The results extend immediately to (X_9, T) , and to the AR6 systems, to which we carry μ_1 and μ_4 .

Now, the AR3 coding (Y_3, S, μ) is also a system of rank one, generated by the towers τ'_{a,m_n} . The construction of the towers $\bar{\tau}'_n$, is a modification of the construction of the τ'_{a,m_n} , in which we replace a small proportion of the levels of τ'_{a,m_n} by spacer levels (as defined above). This modification does not change the system, thus as in [1] (Y_3, S, μ) is measure-theoretically isomorphic to $(\bar{X}, \bar{T}, \bar{\mu})$, thus to both (Y_9, S, μ_1) and (Y_9, S, μ_4) ; but it cannot be measure-theoretically isomorphic to a non-ergodic (Y_9, S, μ') . And the same reasoning holds for the other AR9 or AR6 systems considered. \square

Note that in this only family of counter-examples we have, the two-point extension of Proposition 13 is rather degenerate, being ergodic only when the measure is concentrated on one copy of the factor.

5. WEAK MIXING

Definition 16. *If (X', U, μ_0) is a finite measure-preserving dynamical system, a real number $0 \leq \theta < 1$ is a measurable eigenvalue (denoted additively) if there exists a non-constant f in $\mathcal{L}^1(X', \mathbb{R}/\mathbb{Z})$ such that $f \circ U = f + \theta$ (in $\mathcal{L}^1(X', \mathbb{R}/\mathbb{Z})$); f is then an eigenfunction for the eigenvalue θ .*

As constants are not eigenfunctions, $\theta = 0$ is not an eigenvalue if U is ergodic.

(X', U, μ_0) is weakly mixing if it has no measurable eigenvalue.

The existence of weak mixing for AR3 systems, proved in [14], came as a surprise. This existence persists for AR9 (and AR6) systems, because under the hypothesis $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$, by Theorem 20 above the AR9 or AR6 system equipped with one of its ergodic measures is isomorphic to its AR3 coding, while by Theorem 2 of [14] this AR3 system is weakly mixing. The sufficient condition given in [14] for weak mixing of AR3 systems is weaker than the condition $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$: we shall show now that under this sufficient condition the AR9 systems are also weakly mixing, for any ergodic invariant measure. But indeed this raises more questions than it gives answers, as we shall see in the discussion below.

Proposition 21. *An ergodic AR9 or AR6 system is weakly mixing if*

- k_{n_i+2} is unbounded,

-

$$\sum_{i=1}^{+\infty} \frac{1}{k_{n_i+1}} < +\infty,$$

-

$$\sum_{i=1}^{+\infty} \frac{1}{k_{n_i}} < +\infty,$$

where the n_i are the $n \geq 1$ for which the n -th multiplicative rule is I_m .

Proof

The only difference between the present proof and the proof in [14] is in the beginning. Namely, to prove Proposition 10 of [14], we use the fact that when we move by $S^{h_{a,m_n}}$ inside a substantial slice of $\tau'_{a,m_{n+1}}$, we arrive at the same level in τ'_{a,m_n} ; here we need the stronger result that for all $i = 1, 2, 3, 4$, when we move by $S^{h_{a,m_n}}$ inside a substantial slice of $\tau'_{i,m_{n+1}}$, we arrive at the same level in some τ'_{j,m_n} . This in turn involves some technical difficulties when k_{n+1} is small, obliging us to use our hypotheses on the k_n at that stage, which was not necessary in [14]. Thus Proposition 10 of [14] is replaced by

Lemma 22. *If θ is a measurable eigenvalue for an AR9 symbolic system (Y_9, S, μ') satisfying the hypotheses of Proposition 21, $k_{n+1} \|h_{a,m_n} \theta\| \rightarrow 0$ when $n \rightarrow +\infty$, where $\| \cdot \|$ denotes the distance to the nearest integer.*

Proof

Let f be an eigenfunction for the eigenvalue θ ; for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ there exists f_n , which satisfies $\int \|f - f_n\| d\mu < \varepsilon$ and is constant on each level of each tower $\tau'_{i,m_{n-2}}$, $\tau'_{i,m_{n-1}}$, and τ'_{i,m_n} , $i = 1, \dots, 9$.

Suppose first $k_{n+1} \geq 3$. Let j be any integer with $0 \leq j \leq \left\lfloor \frac{k_{n+1}-1}{2} \right\rfloor$.

Suppose for example the $(n+1)$ -th multiplicative rule is I_m ; we have the concatenation rule $1_{m_{n+1}} = 3_{m_n} 4_{m_n}^{k_{n+1}-1} 5_{m_n}$. Let τ''_n be the slice of $\tau'_{1,m_{n+1}}$ consisting of levels from h_{a,m_n} to $h_{a,m_n} + \left\lfloor \frac{k_{n+1}-1}{2} \right\rfloor h_{a,m_n} - 1$; it has relative height at least $\frac{1}{5}$.

By construction, for any point x in τ''_n , $S^{jh_{a,m_n}} x$ is in the tower $\tau'_{1,m_{n+1}}$, and in the same level of the tower τ'_{4,m_n} as x . Thus for μ' -almost every $x \in \tau''_n$, $f_n(S^{jh_{a,m_n}} x) = f_n(x)$ while $f(S^{jh_{a,m_n}} x) = \theta j h_{a,m_n} + f(x)$; we have

$$\int_{\tau''_n} \|f_n \circ S^{jh_{a,m_n}} - j\theta h_{a,m_n} - f_n\| d\mu' = \int_{\tau''_n} \|j\theta h_{a,m_n}\| d\mu' = \|j\theta h_{a,m_n}\| \mu(\tau''_n)$$

and

$$\int_{\tau''_n} \|f_n \circ S^{jh_{a,m_n}} - j\theta h_{a,m_n} - f_n\| d\mu' \leq \int_{\tau''_n} \|f_n \circ S^{jh_{a,m_n}} - f \circ S^{jh_{a,m_n}}\| d\mu' + \int_{\tau''_n} \|f_n - f\| d\mu' < 2\varepsilon.$$

Thus we get $\|j\theta h_{a,m_n}\| \mu'(\tau'_{1,m_{n+1}}) < 10\varepsilon$, for $n > N(\varepsilon)$ and any integer $0 \leq j \leq \left\lfloor \frac{k_{n+1}-1}{2} \right\rfloor$.

The same result holds when the $(n+1)$ -th multiplicative rule is II_m , with concatenation rule $1_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 7_{m_n}$: just τ'_{1,m_n} replaces τ'_{4,m_n} . And the same construction, mutatis mutandis, works with $\tau'_{1,m_{n+1}}$ replaced by $\tau'_{i,m_{n+1}}$, $i = 2, 3, 4$. Summing the four inequalities and taking into account that $\sum_{i=1}^4 \mu'(\tau'_{i,m_{n+1}}) = \mu(\tau'_{a,m_{n+1}}) \geq \frac{1}{5}$, we get $\|j\theta h_{a,m_n}\| < 50\varepsilon$ for $0 \leq j \leq \left\lfloor \frac{k_{n+1}-1}{2} \right\rfloor$, hence $\|j\theta h_{a,m_n}\| < 200\varepsilon$ for $0 \leq j \leq k_{n+1}$.

We continue exactly as in [14]. Let $\varepsilon < \frac{1}{1000}$, and suppose $\|k_{n+1}\theta h_{a,m_n}\| \neq k_{n+1}\|\theta h_{a,m_n}\|$: let i be the smallest $0 \leq j \leq k_{n+1}$ such that $\|j\theta h_{a,m_n}\| \neq j\|\theta h_{a,m_n}\|$, then $i \geq 2$ and $\|(i-1)\theta h_{a,m_n}\| = (i-1)\|\theta h_{a,m_n}\|$, thus $i\|\theta h_{a,m_n}\| = (i-1)\|\theta h_{a,m_n}\| + \|\theta h_{a,m_n}\| = \|(i-1)\theta h_{a,m_n}\| + \|\theta h_{a,m_n}\| < 400\varepsilon < \frac{1}{2}$ thus $\|i\theta h_{a,m_n}\| = \|(i\|\theta h_{a,m_n}\|)\| = i\|\theta h_{a,m_n}\|$, contradiction. Thus we get $k_{n+1}\|\theta h_{a,m_n}\| < 200\varepsilon$ for $n > N(\varepsilon)$.

Suppose now $k_{n+1} = 2$; then, except maybe for a finite number of values of n , the hypotheses imply that the n -th multiplicative rule is II_m . Note also that we need only to prove $\|\theta h_{a,m_n}\| < C\varepsilon$. For concatenation rules such as $1_{m_{n+1}} = 1_{m_n}^2 7_{m_n}$, we see that S iterated by the length of 1_{m_n} , namely h_{a,m_n} , sends to itself each level of τ'_{1,m_n} if we start from the first slice τ'_{1,m_n} in $\tau'_{1,m_{n+1}}$, whose height is comparable (by some constant) to the height of $\tau'_{1,m_{n+1}}$, thus we can write the reasoning which leads to $\|\theta h_{a,m_n}\| \mu'(\tau'_{1,m_{n+1}}) < C\varepsilon$.

If there is no square in the concatenation rule, its right member is $3_{m_n} 4_{m_n} 5_{m_n}$, which is equal to $4_{m_{n-1}}^{k_n} 5_{m_{n-1}} 3_{m_{n-1}} 4_{m_{n-1}}^{k_n-1} 5_{m_{n-1}} 3_{m_{n-1}}$; if $k_n \geq 2$ we iterate S by the length of $4_{m_{n-1}}^{k_n} 5_{m_{n-1}}$, which is h_{a,m_n} , starting from the $4_{m_{n-1}}^{k_n-1}$, at the end of $4_{m_{n-1}}^{k_n}$; if $k_n = 1$ we use the length of $3_{m_{n-1}} 5_{m_{n-1}}$, which is h_{a,m_n} , starting from the first $3_{m_{n-1}}$. In both cases, the iteration of S by the chosen quantity will send levels of some $\tau'_{i,m_{n-1}}$ to themselves, thus our choice of f_n allows to write the usual reasoning, and to complete the case $k_{n+1} = 2$.

Suppose $k_{n+1} = k_n = 1$. Then, again for n large enough, the n -th and $n-1$ -th multiplicative rules are II_m . The concatenations we look for are $1_{m_n} 7_{m_n} = 1_{m_{n-1}} 7_{m_{n-1}} 1_{m_{n-1}}$, $4_{m_n} 5_{m_n} = 3_{m_{n-1}} 5_{m_{n-1}} 3_{m_{n-1}}$, $3_{m_n} 5_{m_n} = 4_{m_{n-1}} 5_{m_{n-1}} 3_{m_{n-1}}$, $4_{m_n} 6_{m_n} = 3_{m_{n-1}} 5_{m_{n-1}} 2_{m_{n-1}}$. In the first one, h_{a,m_n} , which is the length of $1_{m_{n-1}} 7_{m_{n-1}}$, can be used to iterate S starting from the first $1_{m_{n-1}}$, and similarly in the second one. The last ones are equal to $3_{m_{n-2}} 4_{m_{n-2}}^{k_{n-1}-1} 5_{m_{n-2}} 3_{m_{n-2}} 4_{m_{n-2}}^{k_{n-1}} 5_{m_{n-2}}$ and $4_{m_{n-2}}^{k_{n-1}-1} 5_{m_{n-2}} 3_{m_{n-2}} 4_{m_{n-2}}^{k_{n-1}} 6_{m_{n-2}}$: in both cases h_{a,m_n} , which is the length of $4_{m_{n-2}}^{k_{n-1}-1} 5_{m_{n-2}} 3_{m_{n-2}}$, can be used to iterate S starting from a sizable slice of the tower.

Suppose $k_{n+1} = 1$ but $k_n \geq 2$. Then $h_{a,m_n} = h_{a,m_{n+1}} - h_{a,m_{n-1}}$. We have $\|\theta h_{a,m_{n-1}}\| < C\varepsilon$ because $k_n \geq 2$, and $\|\theta h_{a,m_{n+1}}\| < C\varepsilon$ either because $k_{n+2} \geq 2$ or because $k_{n+1} = k_{n+2} = 1$, thus we conclude. \square

Then the (nontrivial) Sections 3 and 4 of [14] prove that, under the hypotheses of Proposition 21, the condition $k_{n+1}\|h_{a,m_n}\theta\| \rightarrow 0$ gives no possible θ except $\theta = 0$, which is excluded because

of the ergodicity of the system. The same reasoning applies to the other AR9 or AR6 systems. \square

We do not know whether this sufficient condition gives interesting new examples. It might help to find a weakly mixing AR9 system for which $\mu(E_1) = 1$ in the AR3 coding, but this we were not able to achieve. Indeed, starting from Lemma 14 as in Section 4.3, we are able to build such AR9 systems under the condition $\sum_{i=1}^{+\infty} \frac{1}{k_{n_i+1}} = +\infty$. while $\sum_{i=1}^{+\infty} \frac{1}{k_{n_i}}$ may be finite; we could also get these conditions by starting from Lemma 15 and imitating the proof of Proposition 16; this falls short of being compatible with the conditions of Proposition 21. Indeed, we conjecture that these conditions are not compatible with $\mu(E_1) = 1$, and not even with unique ergodicity; whether these conditions are necessary for weak mixing is not known either. It would be also very interesting to find a uniquely ergodic weakly mixing AR9, or a weakly mixing AR9 which is not isomorphic to its AR3 coding.

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