HAL open science

4D-Var identification of DMD reduced-order models

Gilles Tissot, Laurent Cordier, Bernd Noack

To cite this version:

Gilles Tissot, Laurent Cordier, Bernd Noack. 4D-Var identification of DMD reduced-order models. CFM 2015 - 22ème Congrès Français de Mécanique, Aug 2015, Lyon, France. hal-03446223

HAL Id: hal-03446223

https://hal.science/hal-03446223v1

Submitted on 24 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

4D-Var identification of DMD Reduced-Order Models

G. TISSOT, L. CORDIER and **B. R. NOACK**

Institut PPRIME, CNRS – Université de Poitiers – ENSMA

F86036 Poitiers cedex, France

Laurent.Cordier@univ-poitiers.fr

Résumé :

Une stratégie de réduction de modèle est cruciale pour contrôler par une approche basée sur des modèles une grande classe d'écoulement. En turbulence, les modèles réduits sont principalement développés par projection de Galerkin des équations de premier principe sur les modes de la décomposition orthogonale aux valeurs propres (POD). La POD est largement utilisée car elle extrait d'une séquence de données une base orthonormale qui capture de manière optimale l'énergie de l'écoulement. Malheureusement, le niveau d'énergie n'est pas nécessairement le critère correct en termes de modélisation dynamique, et le développement d'un système dynamique sur les modes POD conduit parfois à des modèles non pertinents. Ici, la Décomposition en Mode Dynamique (DMD) telle que proposée récemment par Schmid (2010) est utilisée pour déterminer les modes DMD. Un modèle réduit basé sur les modes DMD est alors déterminé par projection de Galerkin des équations de Navier-Stokes sur un ensemble choisi de modes DMD optimisés. Finalement, une approche d'assimilation variationnelle dite 4D-Var est employée pour identifier les coefficients du modèle réduit DMD. Essentiellement, la 4D-Var combine des observations imparfaites, une solution d'ébauche et les équations dynamiques gouvernant le système observé pour déterminer une estimation optimale de l'état vrai du système. L'approche est illustrée sur des données expérimentales obtenues pour un écoulement de sillage de cylindre à Re=13000.

Abstract :

A reduced-order modelling (ROM) strategy is crucial to achieve model-based control in a wide class of flow configurations. In turbulence, ROMs are mostly derived by Galerkin projection of first principles equations onto the proper orthogonal decomposition (POD) modes. POD is widely used since it extracts from a sequence of data an orthonormal basis which captures optimally the flow

energy. Unfortunately, energy level is not necessarily the correct criterion in terms of dynamical modelling and deriving a dynamical system based on POD modes leads sometimes to irrelevant models. Here, the Dynamic Mode Decomposition (DMD) as recently proposed by Schmid (2010) is used to determine the DMD modes. A DMD ROM is then derived by Galerkin projection of the Navier-Stokes equations onto a selected set of optimized-DMD modes. Finally, a four-dimensional variational assimilation approach (4D-Var) is employed to identify the coefficients of the DMD ROM. Essentially, 4D-Var combines imperfect observations, a background solution and the underlying dynamical principles governing the system under observation to determine an optimal estimation of the true state of the system. The methodology is illustrated for a cylinder wake flow studied experimentally at Re=13000.

Mots clefs : Data assimilation ; 4D-Var ; DMD Reduced-Order Models.

1 Introduction

For a turbulent flow, the number of active degrees of freedom is so important that a preliminary step of model reduction is often necessary for having a chance to understand the flow physics or to derive a control strategy. Reduced-order models (ROMs) are well adapted for developing an efficient control strategy. However, finding the appropriate basis for representing the flow in a low-dimensional space is strongly related to a given objective. Indeed, it is somewhat different for a flow to understand the instability mechanisms, to educe the coherent structures mainly responsible for the energy or to represent the non-linear dynamics.

Reduced-order models based on proper orthogonal decomposition (POD) are the most commonly used as the POD modes are optimal in terms of energy content. Despite this property, POD-based ROM is well known to be inaccurate, essentially due to truncation errors, if the model is not improved (Cordier et al., 2013). Since the energy content is important but is not sufficient in general to catch the dynamical behaviour, we are focusing in this paper on a procedure called optimized DMD recently introduced by Chen et al. (2012). It is a variant of the dynamic mode decomposition (Schmid, 2010) which extracts dynamically relevant flow features from time-resolved experimental or numerical data. Our objective is to derive a ROM which will inherit the dynamical properties of the projection basis.

In Sec. 2, the optimized DMD algorithm is introduced and an improvement of the original algorithm based on the use of a gradient method is shortly described. Section 3 then presents two ways of predicting the temporal behaviour of the

flow outside the time horizon of the snapshots, and ends with the introduction of a data assimilation formalism for combining the two approaches. Finally, in Sec. 4, we present results obtained on PIV data for a cylinder wake in turbulent regime.

2 Optimized DMD

In the classical DMD algorithm (Schmid, 2010), the extraction of a reduced basis by modes' selection is not trivial. Indeed, the non orthogonality of the DMD modes may raise the projection error while increasing the order of the DMD basis. To address these issues, we propose to use the *optimized DMD* as recently introduced by Chen et al. (2012). Let us consider N snapshots $\mathbf{v}_k$ $(k = 1, \ldots, N)$ sampled at a constant time step Δt, the optimized DMD consists in seeking $N_\mathrm{o} < N$ complex scalars $\{\hat{\lambda}_j\}_{j=1}^{N_\mathrm{o}}$ and vectors $\{\hat{\mathbf{\Phi}}_j\}_{j=1}^{N_\mathrm{o}}$ such that

$$\mathbf{v}_k = \sum_{j=1}^{N_\mathrm{o}} \hat{\mathbf{\Phi}}_j \hat{\lambda}_j^{k-1} + \mathbf{r}_k \quad k = 1, \ldots, N \tag{1}$$

and $\Gamma = \sum_{k=1}^{N} \|\mathbf{r}_k\|_2^2$ is minimized. In optimized DMD, the number of modes that is searched is also a parameter of the method leading by construction to a reduced-order model of desirable size. In the original algorithm presented in Chen et al. (2012), the modes were determined with a global optimization technique combining simulated annealing and the Nelder-Mead simplex method. Here, we improved the original algorithm and determined analytically the gradient of Γ with respect to the variation of the eigenvalues $\hat{\lambda}_j$ (Tissot, 2014). Two advantages of this technique are that all the optimization is done in a space of size N_o and that we can use a descent method for increasing the speed of convergence.

3 DMD-based Reduced-Order Model

In the classical DMD algorithm or in optimized DMD, the assumption of linear dynamics leads jointly to the extraction of a basis for the flow and to the introduction of a time propagator. We present in this section how to use these informations for deriving a reduced-order model based on DMD.

3.1 DMD time propagator

By definition, DMD identifies the linear operator which represents at best a sequence of snapshots. A direct consequence of the linear assumption is that each DMD mode contains only a single frequency while POD, which captures

the most energetic structures, gives modes that contain several frequencies. The effect of that on the time stepping operator is clearly visible in the reconstruction equation (1). Indeed, the contribution of each DMD mode is weighted by the corresponding eigenvalue raised to the index of the time step. Hence, if the linear assumption corresponds really to the physical phenomenon (linear or weakly non-linear systems, data lying on a limit cycle) then it has a sense to propagate the state using this operator. The validity of the DMD time propagator is then strongly dependent on the data used for determining the DMD modes, and especially on the linearity assumption.

3.2 Galerkin projection

In (1), the temporal coefficients of the states are only depending on the DMD eigenvalues and on the time index k. This equation can be considered as a pure kinematic description of the flow since the dynamics is artificially introduced through the linear assumption at the heart of DMD. For increasing the probability to derive a model which can represent the long-term flow dynamics, the information that the snapshots are governed by some underlying dynamical principles (Navier-Stokes equations in our case) should be incorporated in the modelling step. This is particularly true when the linearity assumption is questionable.

In model reduction, projection methods are very often used for deriving reduced-order models. The projection of the Navier-Stokes equations onto the optimized DMD modes $\hat{\mathbf{\Phi}}_j$, leads to a quadratic dynamical system for the time coefficients ν_j $(j = 1, \cdots, N_{\text{Gal}})$ given by:

$$\sum_{j=1}^{N_{\text{Gal}}} G_{ij} \frac{\mathrm{d}\nu_j}{\mathrm{d}t}(t) = C_i + \sum_{j=1}^{N_{\text{Gal}}} L_{ij}\nu_j(t) + \sum_{j=1}^{N_{\text{Gal}}} \sum_{k=1}^{N_{\text{Gal}}} Q_{ijk}\nu_j(t)\nu_k(t) \tag{2}$$

where $N_{\text{Gal}} \leq N_{\text{o}}$ is the number of modes kept in the expansion. Since the optimized DMD modes are not orthonormal, G is a full hermitian matrix, called Gram matrix, and has to be inverted once for integrating in time (2). Finally, the coefficients C_i associated to the pressure term are neglected assuming that the integral of the pressure around the boundaries of the domain is very small.

3.3 Data assimilation

In the previous subsections, two different approaches were presented for determining a reduced-order model based on DMD. The first method (see Sec. 3.1) comes directly from the DMD algorithm and as such is purely kinematic while the second method (see Sec. 3.2) does not incorporate all the dynamical informations coming from DMD. The objective of this section is to combine the two sources of informations for deriving a more representative dynamical system.

Data assimilation (Cordier et al., 2013) is the right framework for combining heterogeneous observations with the underlying dynamical principles governing the system under observation to estimate at best physical quantities. Here, we apply the four-dimensional variational approach of data assimilation (4D-Var). More precisely, we seek for the initial condition perturbation η and the time coefficients $\mathbf{c} = \{C_i, L_{ij}, Q_{ijk}\}$ of (2) such that the solutions of the dynamical model tend to the time coefficients obtained directly by optimized DMD. For improving the numerical convergence of the optimization problem, background errors which penalize the variations between the background states $(0, \mathbf{c}^b)$ and the estimated values are introduced. The corresponding cost functional reads:

$$J(\eta, \mathbf{c}) = \frac{1}{2} \sum_{k=1}^{N_t} \sum_{j=1}^{N_{\mathrm{Gal}}} \left(\nu_j(t_k; \eta, \mathbf{c}) - \hat{\lambda}_j^{k-1} \right)^2 + \frac{\sigma_\eta}{2} \|\eta\|^2 + \frac{\sigma_c}{2} \|\mathbf{c} - \mathbf{c}^b\|^2 \qquad (3)$$

where N_t is the number of time steps for the time horizon of the reduced-order model, and where σ_η and σ_c are penalization terms which give more or less weight in the background solutions. The background states are found by application of the Galerkin projection onto the optimized DMD modes.

4 Results

In this paper, 2D-2C PIV data are considered for a cylinder wake at $Re = 13000$. The database contains $N_s = 1000$ snapshots sampled at the frequency $f_s = 1\,\mathrm{kHz}$. First, the classical DMD algorithm was applied for $N = N_s$. In the case of experimental data, modes' selection becomes extremely hard, and considering too many DMD modes $\mathbf{\Phi}_j$ in the reconstruction of snapshots may lead to high level of errors due to the non-orthogonality of the basis. Optimized DMD has then been performed on the first 256 snapshots ($T = 32$). The optimization problem linked to the optimized DMD is solved through a gradient descent algorithm. The initial conditions are DMD modes selected with the energetic criterion $E_j = \|\mathbf{\Phi}_j\|^2 \frac{e^{2\sigma_j T} - 1}{2\sigma_j T}$ (Tissot, 2014), where $\sigma_j = \ln(|\lambda_j|)/\Delta t$ is the growth rate of the DMD modes. We reconstructed the original snapshots from 7 modes obtained by classical DMD (modes's selection based on E_j) and optimized DMD (see Fig. 1 for the fifth snapshot). As expected, the L^2-norm error of reconstruction is lower for the optimized DMD than for the classical algorithm. For the optimized DMD, we obtain very good filtered approximations of the original snapshots. Finally, a 4D-Var approach has been performed in the optimized DMD subspace for $\sigma_\eta = \sigma_c = 1$. This assimilation was made in a time window of size $T = 32$ and then, the optimal solution was forecast over a time length equal to $3T$. In Fig. 2, we compare the temporal coefficients ν_j obtained by optimized DMD and 4D-Var to the projection of the snapshots on the optimized DMD modes. For the long-term horizon, the 4D-Var solution

outperforms the optimized DMD as illustrated by the time evolution of the relative error (Fig. 3).

5 Conclusion

The classical and optimized DMD algorithms were applied with the objective to derive reduced-order models. In the classical DMD algorithm, modes' selection may become arduous since most of the time we do not know if we should privilege mode's energy, frequency behaviour or growth rate. An alternative is to use directly the optimized DMD algorithm as proposed recently by Chen et al. (2012) since the number of modes is also a parameter of the method. In the DMD framework, the time propagation is included in the approach. Hence, we have guaranty that the temporal evolution of the system can be well reproduced over the time window of the snapshots where the linear approximation should hold. To improve this result, we first derive a reduced-order model obtained by Galerkin projection of the Navier-Stokes equations onto the optimized DMD modes. We then introduce, a 4D-Var approach to combine the informations coming from the optimized DMD, and those coming from the Galerkin projection. We showed that 4D-Var solution clearly outperforms the optimized DMD for a long-term prediction.

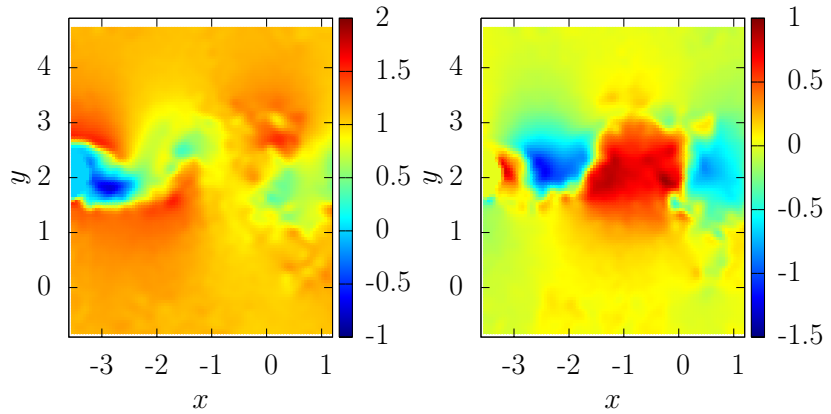
References

K. K. Chen, J. H. Tu, and C. W. Rowley. Variants of Dynamic Mode Decomposition: Boundary Condition, Koopman, and Fourier Analyses. *Journal of Nonlinear Science*, 22(6):887–915, 2012.

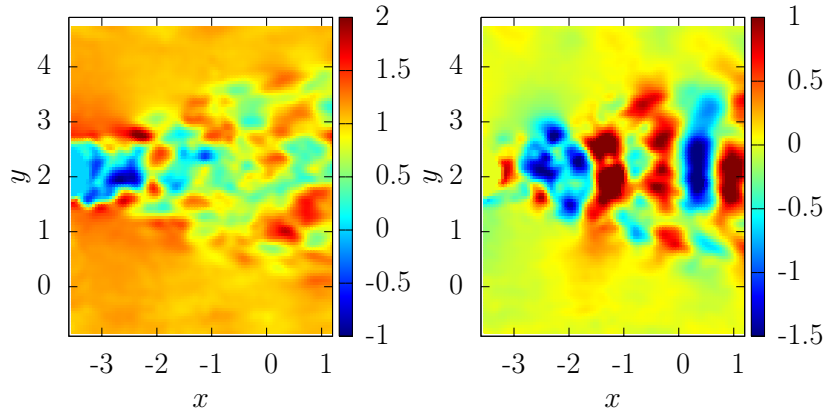
L. Cordier, B. R. Noack, G. Daviller, G. Tissot, G. Lehnasch, J. Delville, M. Balajewicz, and R. Niven. Identification strategies for model-based control. *Experiments in Fluids*, 54(8):1–21, 2013. ISSN 0723-4864.

P. J. Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of Fluid Mechanics*, 656:5–28, 7 2010. ISSN 1469-7645.

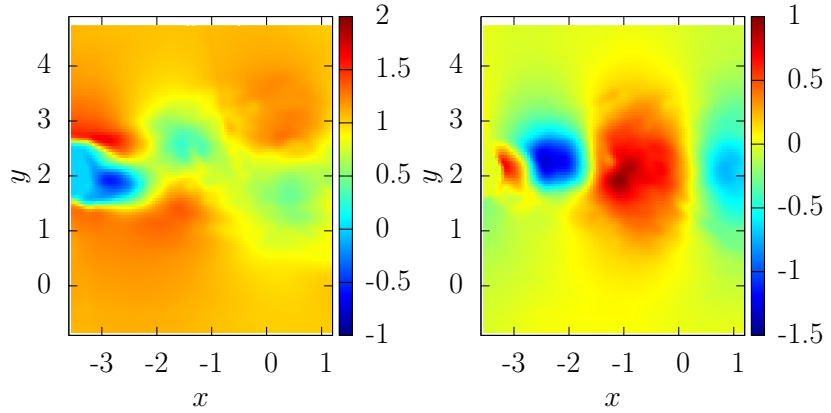
G. Tissot. *Réduction de modèle et contrôle d'écoulements*. PhD thesis, Université de Poitiers, 2014.



(a) Original field.



(b) Reconstruction using "classical" DMD modes.



(c) Reconstruction using optimized DMD modes.

Figure 1: Reconstruction of the snapshot $\mathbf{v}_5$ with 7 modes obtained by classical DMD (modes' selection based on the energetic criterion E_j) and optimized DMD. Streamwise velocity are represented on the left column and transversal velocity on the right column. The corresponding L^2-norm errors are 45.6% (classical DMD) and 15.6% (optimized DMD).

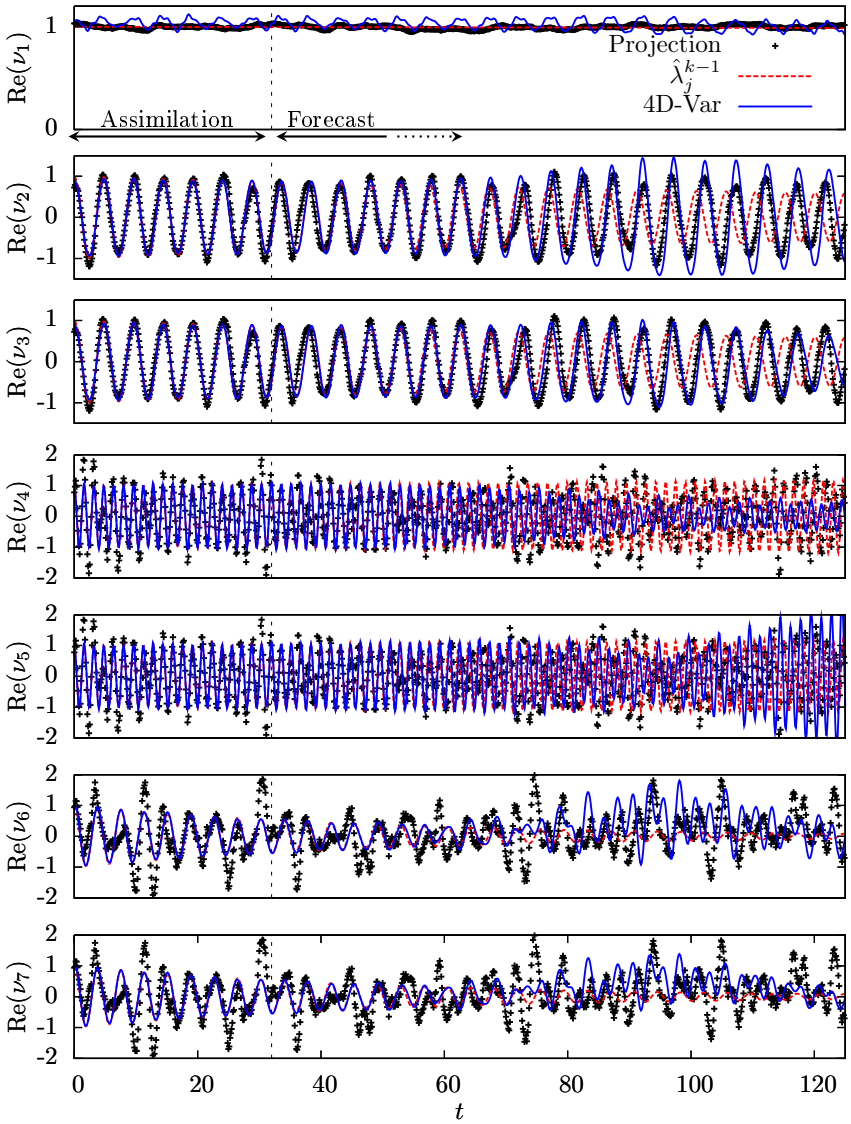


Figure 2: Comparison of temporal coefficients ν_j (real parts) obtained by optimized DMD ($\hat{\lambda}_j^{k-1}$), by 4D-Var and by projection onto the optimized DMD modes. 4D-Var is solved in the 7th order optimized DMD space over the assimilation window $[0, 32]$. The optimal solution is then used to forecast the flow state.

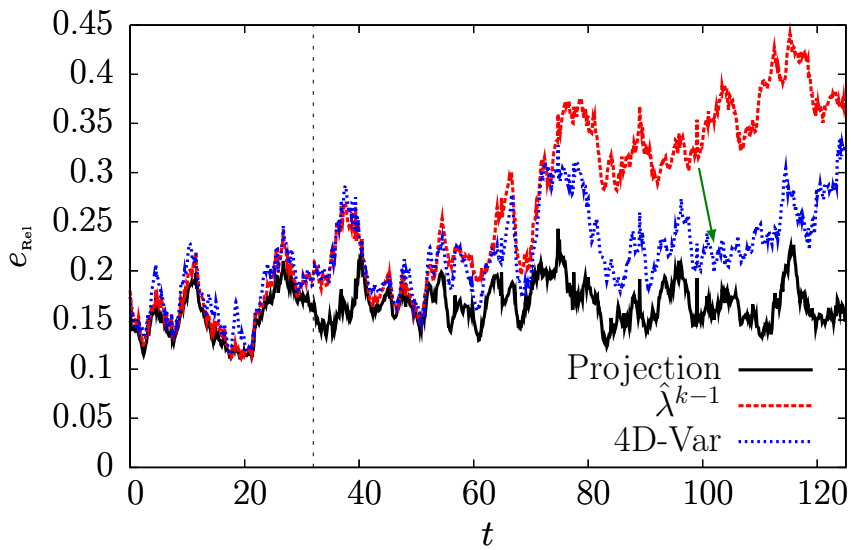


Figure 3: Instantaneous relative errors of the projection onto the 7 optimized DMD modes (black), of the reconstruction of the propagated linear dynamics (red) and of the reconstruction of the dynamics analyzed by 4D-Var (blue). The green arrow represents the decrease of error when the 4D-Var is applied instead of the linear propagation. The vertical dashed line corresponds to the end of the assimilation time horizon and the beginning of the forecast horizon.