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GEVREY REGULARITY AND SUMMABILITY OF THE FORMAL POWER SERIES SOLUTIONS OF THE INHOMOGENEOUS GENERALIZED BOUSSINESQ EQUATIONS

PASCAL REMY

ABSTRACT. In this article, we investigate Gevrey and summability properties of the formal power series solutions of the inhomogeneous generalized Boussinesq equations. We first prove that the inhomogeneity and the formal solutions are together s -Gevrey for any $s \geq 1$, and that the formal solutions are generically 1-Gevrey while the inhomogeneity is s -Gevrey with $s < 1$. In the latter case, we give in particular an explicit example in which the formal solution is s' -Gevrey for no $s' < 1$, that is exactly 1-Gevrey. Then, we give a necessary and sufficient condition under which the formal solutions are 1-summable in a given direction $\arg(t) = \theta$. In addition, we present some technical results on the generalized binomial and multinomial coefficients, which are needed for the proofs of our various results.

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1. INTRODUCTION

1.1. Setting the problem. The nonlinear evolution equations are often used to represent the motion of the isolated waves, localized in a small part of space in

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many fields such as optical fibers, neural physics, solid state physics, hydrodynamics, diffusion process, plasma physics and nonlinear optics (nonlinear heat equation, nonlinear Klein-Gordon equation, nonlinear Euler-Lagrange equation, Burgers equation, Korteweg-de Vries equation, Boussinesq equation, etc.).

When studying such equations, one of the major challenges is the determination of exact solutions, if any exists, and the precise analysis of their properties (dynamic, asymptotic behavior, etc.) in order to have a better understanding of the mechanism of the underlying physical phenomena and dynamic processes.

Thus, for several decades, many analytical methods have been developed in this perspective. For example, in the case of real variables, that is when the variables (t, x) belong to a subset of \mathbb{R}^2 , we can quote, among the many existing techniques, the tanh-sech method, the F-expansion method, the exp-function method, the variational iteration method, etc. More recently, in the case of complex variables, that is when the variables (t, x) belong to a subset of \mathbb{C}^2 , the summation theory has also been used successfully [19, 25, 26]. This theory, initially developed within the framework of the meromorphic ordinary differential equation with an irregular singular point (see for instance [10, 16]), allows the construction of explicit solutions from formal solutions.

In the following, we will be more particularly interested in this theory which we shall apply to the generalized Boussinesq equation.

1.2. The generalized Boussinesq equation. The Boussinesq equation

$$(1.1) \quad \begin{cases} \partial_t^2 u - a \partial_x^4 u - b \partial_x^2 u - c \partial_x^2 (u^n) = 0 \\ a, b, c \text{ nonzero constants} \\ n \geq 2 \text{ an integer} \end{cases}$$

was introduced for the first time in 1872 by J. Boussinesq [4] (the original equation corresponds to the values $a = -1$, $b = c = 1$ and $n = 2$). It allows to model many physical problems such as, for example, the propagation of long waves in shallow water, the propagation of one-dimensional nonlinear lattice-waves, the propagation of vibrations in a nonlinear string, or the propagation of ionic sound waves in a plasma.

In the case of real variables, Eq. (1.1) has already been the subject of many investigations and many results have already been established (see for instance [1, 6, 27] and the references therein). On the other hand, to our knowledge, it does not seem that there are known results when (at least) one of the coefficients a , b , or c is variable, or when the variables t and x are complex.

In the present paper, we are interested in the following inhomogeneous generalized Boussinesq equation

$$(1.2) \quad \begin{cases} \partial_t^2 u - a(t, x) \partial_x^4 u - P(t, x, u) \partial_x^2 u - Q(t, x, u) (\partial_x u)^2 = \tilde{f}(t, x) \\ \partial_t^j u(t, x)|_{t=0} = \varphi_j(x), \quad j = 0, 1 \end{cases}$$

in two variables $(t, x) \in \mathbb{C}^2$, where

- $a(t, x)$ is analytic on a polydisc $D_{\rho_0} \times D_{\rho_1}$ centered at the origin $(0, 0) \in \mathbb{C}^2$ (D_ρ denotes the disc with center $0 \in \mathbb{C}$ and radius $\rho > 0$) and satisfies the condition $a(0, 0) \neq 0$;

- $P(t, x, X)$ and $Q(t, x, X)$ are two polynomials in X with analytic coefficients on $D_{\rho_0} \times D_{\rho_1}$ and with degree less than or equal to a positive integer $d \geq 1$:

$$P(t, x, X) = \sum_{m=0}^d b_{P,m}(t, x) X^m \quad \text{and} \quad Q(t, x, X) = \sum_{m=0}^d b_{Q,m}(t, x) X^m;$$

- the inhomogeneity $\tilde{f}(t, x)$ is a formal power series in t with analytic coefficients in D_{ρ_1} (we denote this by $\tilde{f}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$) which may be smooth, or not¹;
- the initial conditions $\varphi_0(x)$ and $\varphi_1(x)$ are analytic on D_{ρ_1} .

Observe that Eq. (1.2) coincides with Eq. (1.1) for $a(t, x) \equiv a \in \mathbb{C}^*$, $P(t, x, X) = b + cnX^{n-1}$, $Q(t, x, X) = cn(n-1)X^{n-2}$ and $\tilde{f}(t, x) \equiv 0$. Observe also that, in the special case where P and Q are both zero, Eq. (1.2) is reduced to the inhomogeneous linear Euler-Lagrange equation

$$(1.3) \quad \begin{cases} \partial_t^2 u - a(t, x) \partial_x^4 u = \tilde{f}(t, x) \\ \partial_t^j u(t, x)|_{t=0} = \varphi_j(x), \quad j = 0, 1 \end{cases}$$

which allows to model a dynamic inhomogeneous beam with a transverse load.

Considering t as the variable and x as a parameter, we have the following.

Proposition 1.1. *Equation (1.2) admits a unique formal solution $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$.*

Proof. Let us write the coefficients $a(t, x)$, $b_{P,m}(t, x)$ and $b_{Q,m}(t, x)$ for all $m = 0, \dots, d$, and the inhomogeneity $\tilde{f}(t, x)$ in the form

$$\begin{aligned} a(t, x) &= \sum_{j \geq 0} a_{j,*}(x) \frac{t^j}{j!} & \tilde{f}(t, x) &= \sum_{j \geq 0} f_{j,*}(x) \frac{t^j}{j!} \\ b_{P,m}(t, x) &= \sum_{j \geq 0} b_{P,m;j,*}(x) \frac{t^j}{j!} & b_{Q,m}(t, x) &= \sum_{j \geq 0} b_{Q,m;j,*}(x) \frac{t^j}{j!} \end{aligned}$$

with $a_{j,*}(x), b_{P,m;j,*}(x), b_{Q,m;j,*}(x), f_{j,*}(x) \in \mathcal{O}(D_{\rho_1})$ for all $j \geq 0$ and all $m = 0, \dots, d$. Looking for $\tilde{u}(t, x)$ on the same type:

$$\tilde{u}(t, x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} \quad \text{with } u_{j,*}(x) \in \mathcal{O}(D_{\rho_1}) \text{ for all } j \geq 0,$$

one easily checks that its coefficients $u_{j,*}(x)$ are uniquely determined for all $j \geq 0$ by the recurrence relations

$$(1.4) \quad \begin{aligned} u_{j+2,*}(x) &= f_{j,*}(x) + \sum_{j_0+j_1=j} \binom{j}{j_0, j_1} a_{j_0,*}(x) \partial_x^4 u_{j_1,*}(x) \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} \binom{j}{j_0, \dots, j_{m+1}} T_{P,m,j_0, \dots, j_{m+1}}(x) \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} \binom{j}{j_0, \dots, j_{m+2}} T_{Q,m,j_0, \dots, j_{m+2}}(x), \end{aligned}$$

¹We denote \tilde{f} with a tilde to emphasize the possible divergence of the series \tilde{f} .

where $T_{P,m,j_0,\dots,j_{m+1}}(x)$ and $T_{Q,m,j_0,\dots,j_{m+2}}(x)$ are the analytic functions on D_{ρ_1} respectively defined by

$$T_{P,m,j_0,\dots,j_{m+1}}(x) = \begin{cases} b_{P,0;j_0,*}(x)\partial_x^2 u_{j_1,*}(x) & \text{if } m = 0 \\ b_{P,m;j_0,*}(x)u_{j_1,*}(x)\dots u_{j_m,*}(x)\partial_x^2 u_{j_{m+1},*}(x) & \text{if } m \geq 1 \end{cases}$$

and

$$T_{Q,m,j_0,\dots,j_{m+2}}(x) = \begin{cases} b_{Q,0;j_0,*}(x)\partial_x u_{j_1,*}(x)\partial_x u_{j_2,*}(x) & \text{if } m = 0 \\ b_{Q,m;j_0,*}(x)u_{j_1,*}(x)\dots u_{j_m,*}(x)\partial_x u_{j_{m+1},*}(x)\partial_x u_{j_{m+2},*}(x) & \text{if } m \geq 1 \end{cases},$$

together with the two initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0, 1$. As usual, the notation $\binom{a}{a_0, \dots, a_p}$, for any nonnegative real numbers a, a_0, \dots, a_p such that $a_0 + \dots + a_p = a$, stands for the multinomial coefficients (see Section 4). \square

1.3. Known results and aim of the article. In the two previous articles [18, 26] (see also [20], and [17, 19, 24] for more general equations), the author studied, in the framework of the Euler-Lagrange equation (1.3), the Gevrey regularity and the 1-summability of the formal solution $\tilde{u}(t, x)$. More precisely, he proved the two following.

Proposition 1.2 (Gevrey regularity [18]). *Let $\tilde{u}(t, x)$ be the formal solution in $\mathcal{O}(D_{\rho_1})[[t]]$ of Eq. (1.3). Then,*

- (1) $\tilde{u}(t, x)$ and $\tilde{f}(t, x)$ are together s -Gevrey for any $s \geq 1$.
- (2) $\tilde{u}(t, x)$ is generically 1-Gevrey while $\tilde{f}(t, x)$ is s -Gevrey with $s < 1$.

Proposition 1.3 (Summability [26]). *Let $\tilde{u}(t, x)$ be the formal solution in $\mathcal{O}(D_{\rho_1})[[t]]$ of Eq. (1.3). Let $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction issuing from 0. Then,*

- (1) $\tilde{u}(t, x)$ is 1-summable in the direction θ if and only if the inhomogeneity $\tilde{f}(t, x)$ and the formal series $\partial_x^n \tilde{u}(t, x)|_{x=0} \in \mathbb{C}[[t]]$ for $n = 0, 1, 2, 3$ are 1-summable in the direction θ .
- (2) Moreover, the 1-sum $u(t, x)$, if any exists, satisfies Eq. (1.3) in which $\tilde{f}(t, x)$ is replaced by its 1-sum $f(t, x)$ in the direction θ .

In this article, we propose to prove that these two results remain true in the case of the generalized Boussinesq equation (1.2). To do this, we shall use an approach similar to those already developed by the author in [17–19, 23–26] for some nonlinear partial differential equations (see also [3, 20–22] for an approach in the linear case). Let us point out here that, as we shall see below, the terms $u^m(\partial_x u)^2$ of Eq. (1.2), which have never been considered in the previous articles, shall make the calculations much more complicated by revealing new computational and combinatorial situations, which require some technical results on the generalized binomial and multinomial coefficients, that is on the binomial and multinomial coefficients with nonnegative real terms.

The organization of the article is as follows. Section 2 is devoted to the study of the Gevrey regularity of the formal solution $\tilde{u}(t, x)$ of Eq. (1.2). After briefly recalling the definition and some basic properties of the s -Gevrey formal power series in $\mathcal{O}(D_{\rho_1})[[t]]$ which are needed, we prove that $\tilde{u}(t, x)$ and the inhomogeneity $\tilde{f}(t, x)$ of Eq. (1.2) are together s -Gevrey for any $s \geq 1$, and that $\tilde{u}(t, x)$ is generically

1-Gevrey while $\tilde{f}(t, x)$ is s -Gevrey with $s < 1$ (Theorem 2.3). In the latter case, we give in particular an explicit example in which $\tilde{u}(t, x)$ is s' -Gevrey for no $s' < 1$, that is $\tilde{u}(t, x)$ is exactly 1-Gevrey. In Section 3, we investigate the 1-summability of $\tilde{u}(t, x)$ and we give a necessary and sufficient condition under which the formal solution $\tilde{u}(t, x)$ is 1-summable in a given direction $\arg(t) = \theta$. We prove in particular that the statement of Proposition 1.3 above remains true (Theorem 3.5). In the last Section 4, we present all the technical results on the generalized binomial and multinomial coefficients which are needed for the proofs of our two main results. This section can also be read independently of the rest of the article, so as not to burden the main proofs.

2. GEVREY REGULARITY OF $\tilde{u}(t, x)$

As said at the beginning of Section 1.2, we consider the time t as the variable and the space x as a parameter. Thereby, to define the notion of *Gevrey formal power series in $\mathcal{O}(D_{\rho_1})[[t]]$* , one extends the classical notion of *Gevrey formal power series in $\mathbb{C}[[t]]$* to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x . Doing that, any formal power series of $\mathcal{O}(D_{\rho_1})[[t]]$ can be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a convenient disc D_ρ and continuous up to its boundary, equipped with the usual supremum norm. For a general study of the formal power series with coefficients in a Banach space, we refer for instance to [2].

2.1. Main result. Before stating our main result on the Gevrey regularity of the formal solution $\tilde{u}(t, x)$ of Eq. (1.2), let us first recall for the convenience of the reader some definitions and basic properties about the Gevrey formal power series in $\mathcal{O}(D_{\rho_1})[[t]]$, which are needed in the sequel.

Definition 2.1 (s -Gevrey formal series). Let $s \geq 0$ be. A formal power series

$$\tilde{u}(t, x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} \in \mathcal{O}(D_{\rho_1})[[t]]$$

is said to be *Gevrey of order s* (in short, s -Gevrey) if there exist three positive constants $0 < \rho < \rho_1$, $C > 0$ and $K > 0$ such that the inequalities

$$\sup_{|x| \leq \rho} |u_{j,*}(x)| \leq CK^j \Gamma(1 + (s+1)j)$$

hold for all $j \geq 0$. We denote by $\mathcal{O}(D_{\rho_1})[[t]]_s$ the set of all the formal series in $\mathcal{O}(D_{\rho_1})[[t]]$ which are s -Gevrey.

In other words, Definition 2.1 means that $\tilde{u}(t, x)$ is s -Gevrey in t , uniformly in x on a neighborhood of $x = 0 \in \mathbb{C}$.

Observe that the set $\mathbb{C}\{t, x\}$ of germs of analytic functions at the origin of \mathbb{C}^2 coincides with the union $\bigcup_{\rho_1 > 0} \mathcal{O}(D_{\rho_1})[[t]]_0$; in particular, any element of $\mathcal{O}(D_{\rho_1})[[t]]_0$ is convergent and $\mathbb{C}\{t, x\} \cap \mathcal{O}(D_{\rho_1})[[t]] = \mathcal{O}(D_{\rho_1})[[t]]_0$.

Observe also that the sets $\mathcal{O}(D_{\rho_1})[[t]]_s$ are filtered as follows:

$$\mathcal{O}(D_{\rho_1})[[t]]_0 \subset \mathcal{O}(D_{\rho_1})[[t]]_s \subset \mathcal{O}(D_{\rho_1})[[t]]_{s'} \subset \mathcal{O}(D_{\rho_1})[[t]]$$

for all s and s' satisfying $0 < s < s' < +\infty$.

Proposition 2.2 ([2, 20]). *Let $s \geq 0$ be. Then, the set $(\mathcal{O}(D_{\rho_1})[[t]])_s, \partial_t, \partial_x$ is a \mathbb{C} -differential algebra.*

Let us now state the result in view in this section.

Theorem 2.3. *Let $\tilde{u}(t, x)$ be the formal solution in $\mathcal{O}(D_{\rho_1})[[t]]$ of the generalized Boussinesq equation (1.2). Then,*

- (1) $\tilde{u}(t, x)$ and $\tilde{f}(t, x)$ are together s -Gevrey for any $s \geq 1$.
- (2) $\tilde{u}(t, x)$ is generically 1-Gevrey while $\tilde{f}(t, x)$ is s -Gevrey with $s < 1$.

Observe that Theorem 2.3 coincides with Proposition 1.2 in the case of the Euler-Lagrange equation (1.3), that is in the case where the polynomials P and Q are both zero.

Observe also that Theorem 2.3 provides the Gevrey regularity of the classical Boussinesq equation (1.1).

Corollary 2.4. *The formal solution $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ of the Boussinesq equation (1.1) is 1-Gevrey.*

Corollary 2.4 will be improved later for some special values of its coefficients (see Proposition 2.13). The proof of Theorem 2.3 is detailed in the next two sections. The first point is the most technical and the most complicated. Its proof is based on the Nagumo norms, a technique of majorant series and a fixed point procedure (see Section 2.2). As for the second point, it stems both from the first one and from Proposition 2.13 that gives an explicit example for which $\tilde{u}(t, x)$ is s' -Gevrey for no $s' < 1$ while $\tilde{f}(t, x)$ is s -Gevrey with $s < 1$ (see Section 2.3).

2.2. Proof of Theorem 2.3: first point. According to Proposition 2.2, it is clear that

$$\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]_s \Rightarrow \tilde{f}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]_s.$$

Reciprocally, let us fix $s \geq 1$ and let us suppose that the inhomogeneity $\tilde{f}(t, x)$ of Eq. (1.2) is s -Gevrey. By assumption, its coefficients $f_{j,*}(x) \in \mathcal{O}(D_{\rho_1})$ satisfy the following condition (see Definition 2.1): there exist three positive constants $0 < \rho < \rho_1$, $C > 0$ and $K > 0$ such that the inequalities

$$(2.1) \quad |f_{j,*}(x)| \leq CK^j \Gamma(1 + (s+1)j)$$

hold for all $j \geq 0$ and all $|x| \leq \rho$.

We must prove that the coefficients $u_{j,*}(x) \in \mathcal{O}(D_{\rho_1})$ of the formal solution $\tilde{u}(t, x)$ satisfy similar inequalities. The approach we present below is analogous to the ones already developed in [3, 20–22] in the framework of linear partial and integro-differential equations, and in [17, 18, 23, 24] in the case of certain nonlinear equations. It is based on the Nagumo norms [5, 13, 28] and on a technique of majorant series. However, as we shall see, our calculations appear to be much more technical and complicated. Furthermore, the nonlinear terms $u^m(\partial_x u)^2$ generate several new technical combinatorial situations which have never been investigated in the previous articles.

Before starting the calculations, let us first recall for the convenience of the reader the definition of the Nagumo norms and some of their properties which are needed in the sequel.

2.2.1. Nagumo norms.

Definition 2.5. Let $f \in \mathcal{O}(D_{\rho_1})$, $p \geq 0$ and $0 < r < \rho_1$ be. Then, the Nagumo norm $\|f\|_{p,r}$ with indices (p, r) of f is defined by

$$\|f\|_{p,r} := \sup_{|x| \leq r} |f(x) d_r(x)^p|,$$

where $d_r(x)$ denotes the Euclidian distance $d_r(x) := r - |x|$.

Following Proposition 2.6, whose a proof can be found for instance in [22], gives us some properties of the Nagumo norms.

Proposition 2.6. Let $f, g \in \mathcal{O}(D_{\rho_1})$, $p, p' \geq 0$ and $0 < r < \rho_1$ be. Then,

- (1) $\|\cdot\|_{p,r}$ is a norm on $\mathcal{O}(D_{\rho_1})$.
- (2) $|f(x)| \leq \|f\|_{p,r} d_r(x)^{-p}$ for all $|x| < r$.
- (3) $\|f\|_{0,r} = \sup_{|x| \leq r} |f(x)|$ is the usual sup-norm on D_r .
- (4) $\|fg\|_{p+p',r} \leq \|f\|_{p,r} \|g\|_{p',r}$.
- (5) $\|\partial_x f\|_{p+1,r} \leq e(p+1) \|f\|_{p,r}$.

Remark 2.7. Inequalities 4–5 of Proposition 2.6 are the most important properties. Observe besides that the same index r occurs on their both sides, allowing thus to get estimates for the product fg in terms of f and g , and for the derivative $\partial_x f$ in terms of f without having to shrink the disc D_r .

Let us now turn to the proof of the first point of Theorem 2.3.

2.2.2. *First step: some inequalities.* From the recurrence relations (3.1), we first derive for all $j \geq 0$ the identities

$$\begin{aligned} \frac{u_{j+2,*}(x)}{\Gamma(1+(s+1)(j+2))} &= \frac{f_{j,*}(x)}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{j_0+j_1=j} \binom{j}{j_0, j_1} \frac{a_{j_0,*}(x) \partial_x^4 u_{j_1,*}(x)}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} \binom{j}{j_0, \dots, j_{m+1}} \frac{T_{P,m,j_0,\dots,j_{m+1}}(x)}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} \binom{j}{j_0, \dots, j_{m+2}} \frac{T_{Q,m,j_0,\dots,j_{m+2}}(x)}{\Gamma(1+(s+1)(j+2))}, \end{aligned}$$

where

$$T_{P,m,j_0,\dots,j_{m+1}}(x) = \begin{cases} b_{P,m;j_0,*}(x) \partial_x^2 u_{j_1,*}(x) & \text{if } m = 0 \\ b_{P,m;j_0,*}(x) u_{j_1,*}(x) \dots u_{j_m,*}(x) \partial_x^2 u_{j_{m+1},*}(x) & \text{if } m \geq 1 \end{cases}$$

and

$$T_{Q,m,j_0,\dots,j_{m+2}}(x) = \begin{cases} b_{Q,m;j_0,*}(x) \partial_x u_{j_1,*}(x) \partial_x u_{j_2,*}(x) & \text{if } m = 0 \\ b_{Q,m;j_0,*}(x) u_{j_1,*}(x) \dots u_{j_m,*}(x) \partial_x u_{j_{m+1},*}(x) \partial_x u_{j_{m+2},*}(x) & \text{if } m \geq 1 \end{cases},$$

together with the two initial conditions $u_{0,*}(x) = \varphi_0(x)$ and $u_{1,*}(x) = \varphi_1(x)$.

Applying then the Nagumo norms of indices $((s+1)(j+2), \rho)$, we deduce successively from Property 1 and Properties 4-5 of Proposition 2.6 the inequalities

$$\begin{aligned} \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} &\leq \frac{\|f_{j,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{j_0+j_1=j} \binom{j}{j_0, j_1} \frac{\|a_{j_0,*} \partial_x^4 u_{j_1,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} \binom{j}{j_0, \dots, j_{m+1}} \frac{\|T_{P,m,j_0,\dots,j_{m+1}}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} \binom{j}{j_0, \dots, j_{m+2}} \frac{\|T_{Q,m,j_0,\dots,j_{m+2}}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))}, \end{aligned}$$

and, next,

$$\begin{aligned} (2.2) \quad \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} &\leq \frac{\|f_{j,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \\ &+ \sum_{j_0+j_1=j} A_{s,j_0,j_1} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} B_{P,m,s,j_0,\dots,j_{m+1}} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)} \cdots \frac{\|u_{j_{m+1},*}\|_{(s+1)j_{m+1},\rho}}{\Gamma(1+(s+1)j_{m+1})} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} C_{Q,m,s,j_0,\dots,j_{m+2}} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)} \cdots \frac{\|u_{j_{m+2},*}\|_{(s+1)j_{m+2},\rho}}{\Gamma(1+(s+1)j_{m+2})} \end{aligned}$$

for all $j \geq 0$, where the terms A_{s,j_0,j_1} , $B_{P,m,s,j_0,\dots,j_{m+1}}$ and $C_{Q,m,s,j_0,\dots,j_{m+2}}$ are nonnegative and defined by

$$A_{s,j_0,j_1} = \frac{e^4 \|a_{j_0,*}\|_{(s+1)(j_0+2)-4,\rho}}{\Gamma(1+(s+1)j_0)} \frac{\Gamma(1+(s+1)j_1+4)}{\Gamma(1+(s+1)(j_1+2))} \times \frac{\binom{j}{j_0, j_1}}{\binom{(s+1)(j+2)}{(s+1)j_0, (s+1)(j_1+2)}}$$

$$B_{P,m,s,j_0,\dots,j_{m+1}} = \frac{e^2 \|b_{P,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)} \frac{\Gamma(1+(s+1)j_{m+1}+2)}{\Gamma(1+(s+1)(j_{m+1}+2))} \times \frac{\binom{j}{j_0, \dots, j_m, j_{m+1}}}{\binom{(s+1)(j+2)}{(s+1)j_0, \dots, (s+1)j_m, (s+1)(j_{m+1}+2)}}$$

$$C_{Q,m,s,j_0,\dots,j_{m+2}} = \frac{e^2 \|b_{Q,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)} \times \frac{\Gamma(1+(s+1)j_{m+1}+1)\Gamma(1+(s+1)j_{m+2}+1)}{\Gamma(1+(s+1)(j_{m+1}+1)\Gamma(1+(s+1)(j_{m+2}+1)))} \times \frac{\binom{j}{j_0,\dots,j_m,j_{m+1},j_{m+2}}}{\binom{(s+1)(j+2)}{(s+1)j_0,\dots,(s+1)j_m,(s+1)(j_{m+1}+1),(s+1)(j_{m+2}+1)}}$$

Observe that all the norms written in the inequalities (2.2), and especially the norms $\|a_{j_0,*}\|_{(s+1)(j_0+2)-4,\rho}$, $\|b_{P,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}$ and $\|b_{Q,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}$ are well-defined. Indeed, the assumption $s \geq 1$ implies $(s+1)(j_0+2) - 4 \geq 2j_0 \geq 0$ and $(s+1)(j_0+2) - 2 \geq 2j_0 + 2 > 0$.

Lemma 2.8. (1) For all $j \geq 0$ and all $j_0, j_1 \geq 0$ such that $j_0 + j_1 = j$,

$$(2.3) \quad A_{s,j_0,j_1} \leq \frac{e^4 \|a_{j_0,*}\|_{(s+1)(j_0+2)-4,\rho}}{\Gamma(1+(s+1)j_0)}$$

(2) For all $m, j \geq 0$ and $j_0, \dots, j_{m+1} \geq 0$ such that $j_0 + \dots + j_{m+1} = j$,

$$(2.4) \quad B_{P,m,s,j_0,\dots,j_{m+1}} \leq \frac{e^2 \|b_{P,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)}$$

(3) For all $m, j \geq 0$ and $j_0, \dots, j_{m+2} \geq 0$ such that $j_0 + \dots + j_{m+2} = j$,

$$(2.5) \quad C_{Q,m,s,j_0,\dots,j_{m+2}} \leq \frac{e^2 \|b_{Q,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)}$$

Proof. (1) Let us first observe that the assumption $s \geq 1$ implies

$$1 + (s+1)(j_0+2) \geq 1 + (s+1)j_0 + 4 \geq 2.$$

Hence, the inequality

$$\frac{\Gamma(1+(s+1)j_1+4)}{\Gamma(1+(s+1)(j_1+2))} \leq 1$$

by increasing of the Gamma function on $[2, +\infty[$.

On the other hand, applying successively the Vandermonde Inequality (see Proposition 4.2) and Proposition 4.4, we get

$$\begin{aligned} \binom{(s+1)(j+2)}{(s+1)j_0, (s+1)(j_1+2)} &\geq \binom{j}{j_0, j_1} \binom{sj+2(s+1)}{sj_0, sj_1+2(s+1)} \\ &\geq \binom{j}{j_0, j_1}; \end{aligned}$$

hence, the inequality

$$\frac{\binom{j}{j_0, j_1}}{\binom{(s+1)(j+2)}{(s+1)j_0, (s+1)(j_1+2)}} \leq 1.$$

Inequality (2.3) follows.

(2)-(3) The proof of inequalities (2.4) and (2.5) is similar and is left to the reader. \square

Applying Lemma 2.8 to the previous relations (2.2), we finally get the inequalities

$$(2.6) \quad \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \leq g_{s,j} + \sum_{j_0+j_1=j} \alpha_{s,j_0} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)}$$

$$+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} \beta_{P,m,s,j_0} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)} \cdots \frac{\|u_{j_{m+1},*}\|_{(s+1)j_{m+1},\rho}}{\Gamma(1+(s+1)j_{m+1})}$$

$$+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} \gamma_{Q,m,s,j_0} \frac{\|u_{j_1,*}\|_{(s+1)j_1,\rho}}{\Gamma(1+(s+1)j_1)} \cdots \frac{\|u_{j_{m+2},*}\|_{(s+1)j_{m+2},\rho}}{\Gamma(1+(s+1)j_{m+2})}$$

for all $j \geq 0$, where the terms $g_{s,j}$, α_{s,j_0} , β_{P,m,s,j_0} and γ_{Q,m,s,j_0} are nonnegative and defined by

$$g_{s,j} = \frac{\|f_{j,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \quad \beta_{P,m,s,j_0} = \frac{e^2 \|b_{P,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)}$$

$$\alpha_{s,j_0} = \frac{e^4 \|a_{j_0,*}\|_{(s+1)(j_0+2)-4,\rho}}{\Gamma(1+(s+1)j_0)} \quad \gamma_{Q,m,s,j_0} = \frac{e^2 \|b_{Q,m;j_0,*}\|_{(s+1)(j_0+2)-2,\rho}}{\Gamma(1+(s+1)j_0)}$$

The following result, which will be useful in the next section, provides some bounds on these various terms.

Lemma 2.9. *There exist two positive constants $C_1, K_1 > 0$ such that the inequalities*

$$0 \leq g_{s,j} \leq C_1 K_1^j \quad 0 \leq \beta_{P,m,s,j} \leq C_1 K_1^j$$

$$0 \leq \alpha_{s,j} \leq C_1 K_1^j \quad 0 \leq \gamma_{Q,m,s,j} \leq C_1 K_1^j$$

hold for all $j \geq 0$ and all $m = 0, \dots, d$.

Proof. Given the hypothesis on the coefficients $f_{j,*}(x)$ of the inhomogeneity $\tilde{f}(t,x)$ (see inequality (2.1) at the beginning of Section 2.2), and the analyticity of the functions $a(t,x)$, $b_{P,m}(t,x)$ and $b_{Q,m}(t,x)$ at the origin of \mathbb{C}^2 , we first have the relations

$$|f_{j,*}(x)| \leq CK^j \Gamma(1+(s+1)j) \quad |b_{P,m;j,*}(x)| \leq C'' K''^j j!$$

$$|a_{j,*}(x)| \leq C' K'^j j! \quad |b_{Q,m;j,*}(x)| \leq C''' K'''^j j!$$

for all $j \geq 0$, all $m \in \{0, \dots, d\}$ and all $|x| \leq \rho$, the height constants $C, K, C', K', C'', K'', C''', K''' > 0$ being independent of j, m and x . Hence, applying the definition of the Nagumo norms and the classical property $\Gamma(a) \leq \Gamma(b)$ for all $a \geq 1$ and all $b \geq \max(2, a)$, the inequalities

$$0 \leq g_{s,j} \leq \frac{CK^j \Gamma(1+(s+1)j) \rho^{(s+1)(j+2)}}{\Gamma(1+(s+1)(j+2))} = C \rho^{2(s+1)} (K \rho^{s+1})^j$$

$$0 \leq \alpha_{s,j} \leq \frac{e^4 C' K'^j j! \rho^{(s+1)(j+2)-4}}{\Gamma(1+(s+1)j)} = e^4 C' \rho^{2(s-1)} (K' \rho^{s+1})^j$$

$$0 \leq \beta_{P,m,s,j} \leq \frac{e^2 C'' K''^j j! \rho^{(s+1)(j+2)-2}}{\Gamma(1+(s+1)j)} = e^2 C'' \rho^{2s} (K'' \rho^{s+1})^j$$

$$0 \leq \beta_{Q,m,s,j} \leq \frac{e^2 C''' K'''^j j! \rho^{(s+1)(j+2)-2}}{\Gamma(1+(s+1)j)} = e^2 C''' \rho^{2s} (K''' \rho^{s+1})^j$$

for all $j \geq 0$ and all $m \in \{0, \dots, d\}$. The constants C_1 and K_1 follow, which achieves the proof. \square

We shall now bound the Nagumo norms $\|u_{j,*}\|_{(s+1)j,\rho}$ for any $j \geq 0$. To do that, we shall proceed similarly as in [3, 17, 18, 20–24] by using a technique of majorant series.

2.2.3. *Second step: a majorant series.* Let us consider the formal power series

$$v(X) = \sum_{j \geq 0} v_j X^j \in \mathbb{R}^+[[X]],$$

where the coefficients v_j are recursively determined from the two initial conditions

$$v_0 = 1 + \|\varphi_0\|_{0,\rho}, \quad v_1 = \frac{\|\varphi_1\|_{s+1,\rho}}{\Gamma(2+s)}$$

by the relations

$$(2.7) \quad v_{j+2} = g_{s,j} + \sum_{j_0+j_1=j} \alpha_{s,j_0} v_{j_1} + \sum_{j_0+\dots+j_{d+2}=j} \delta_{s,j_0} v_{j_1} \dots v_{j_{d+2}}$$

for all $j \geq 0$, where the terms δ_{s,j_0} are nonnegative and defined by

$$\delta_{s,j_0} = \sum_{m=0}^d (\beta_{P,m,s,j_0} + \gamma_{Q,m,s,j_0}).$$

Lemma 2.10. *The following inequalities hold for all $j \geq 0$:*

$$(2.8) \quad 0 \leq \frac{\|u_{j,*}\|_{(s+1)j,\rho}}{\Gamma(1+(s+1)j)} \leq v_j.$$

Proof. According to the initial conditions on the u_j 's and on the v_j 's, the inequalities (2.8) hold for $j = 0$ and $j = 1$. Let us now suppose that these inequalities are true for all $k = 0, \dots, j+1$ for a certain $j \geq 0$. Then, it results from inequalities (2.6) that

$$\begin{aligned} 0 \leq \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} &\leq g_{s,j} + \sum_{j_0+j_1=j} \alpha_{s,j_0} v_{j_1} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+1}=j} \beta_{P,m,s,j_0} v_{j_1} \dots v_{j_{m+1}} \\ &+ \sum_{m=0}^d \sum_{j_0+\dots+j_{m+2}=j} \gamma_{Q,m,s,j_0} v_{j_1} \dots v_{j_{m+2}}. \end{aligned}$$

Let us now observe that

- for all $m \in \{0, \dots, d\}$, any $(m+2)$ -tuple $(j_0, \dots, j_{m+1}) \in \mathbb{N}^{m+2}$ such that $j_0 + \dots + j_{m+1} = j$ can be seen as the $(d+3)$ -tuple $(j_0, \dots, j_{m+1}, j_{m+2}, \dots, j_{d+2}) \in \mathbb{N}^{d+3}$ where $j_{m+2} = \dots = j_{d+2} = 0$;
- for all $m \in \{0, \dots, d-1\}$, any $(m+3)$ -tuple $(j_0, \dots, j_{m+2}) \in \mathbb{N}^{m+3}$ such that $j_0 + \dots + j_{m+2} = j$ can be seen as the $(d+3)$ -tuple $(j_0, \dots, j_{m+2}, j_{m+3}, \dots, j_{d+2}) \in \mathbb{N}^{d+3}$ where $j_{m+3} = \dots = j_{d+2} = 0$.

Consequently, using the fact that $v_0 \geq 1$, we have

$$0 \leq \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{m+1}} \leq \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{m+1}} v_0^{d-m+1} = \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}}$$

$$0 \leq \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{m+2}} \leq \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{m+2}} v_0^{d-m} = \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}}$$

for all $m \in \{0, \dots, d\}$. Hence, the inequalities

$$\begin{aligned} 0 \leq \sum_{j_0+\dots+j_{m+1}=j} \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{m+1}} &\leq \sum_{\substack{j_0+\dots+j_{d+2}=j \\ j_{m+2}=\dots=j_{d+2}=0}} \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}} \\ &\leq \sum_{j_0+\dots+j_{d+2}=j} \beta_{P,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}} \end{aligned}$$

and

$$\begin{aligned} 0 \leq \sum_{j_0+\dots+j_{m+2}=j} \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{m+2}} &\leq \sum_{\substack{j_0+\dots+j_{d+2}=j \\ j_{m+3}=\dots=j_{d+2}=0}} \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}} \\ &\leq \sum_{j_0+\dots+j_{d+2}=j} \gamma_{Q,m,s,j_0} v_{j_1} \cdots v_{j_{d+2}} \end{aligned}$$

for all $m \in \{0, \dots, d\}$ since all the terms are nonnegative. Using then the definition of the terms δ_{s,j_0} , we get

$$0 \leq \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \leq g_{s,j} + \sum_{j_0+j_1=j} \alpha_{s,j_0} v_{j_1} + \sum_{j_0+\dots+j_{d+2}=j} \delta_{s,j_0} v_{j_1} \cdots v_{j_{d+2}}$$

and thereby

$$0 \leq \frac{\|u_{j+2,*}\|_{(s+1)(j+2),\rho}}{\Gamma(1+(s+1)(j+2))} \leq v_{j+2}$$

by comparing with (2.7), which achieves the proof. \square

Following Proposition 2.11 allows us to bound the v_j 's.

Proposition 2.11. *The formal series $v(X)$ is convergent. In particular, there exist two positive constants $C', K' > 0$ such that $v_j \leq C' K'^j$ for all $j \geq 0$.*

Proof. It is sufficient to prove the convergence of $v(X)$.

First of all, let us start by observing that $v(X)$ is the unique formal power series in X solution of the functional equation

$$(2.9) \quad (1 - X^2 \alpha(X)) v(X) = X^2 \delta(X) (v(X))^{d+2} + g(X),$$

where

$$\alpha(X) = \sum_{j \geq 0} \alpha_{s,j} X^j, \quad \delta(X) = \sum_{j \geq 0} \delta_{s,j} X^j \text{ and}$$

$$g(X) = 1 + \|\varphi_0\|_{0,\rho} + \frac{\|\varphi_1\|_{s+1,\rho}}{\Gamma(2+s)} X + X^2 \sum_{j \geq 0} g_{s,j} X^j.$$

are three convergent power series (see Lemma 2.9) with nonnegative coefficients. In particular, they respectively define three increasing functions on $[0, r_\alpha[$, $[0, r_\delta[$ and $[0, r_g[$, where $r_\alpha > 0$, $r_\delta > 0$ and $r_g > 0$ stand respectively for the radius of convergence of $\alpha(X)$, $\delta(X)$ and $g(X)$. We also denote by $r'_\alpha > 0$ the radius of convergence of the series $1/(1 - X^2 \alpha(X))$.

Notice that the convergence of $v(X)$ is obvious when $\delta(X) \equiv 0$ (case where the polynomials P and Q are both zero), since we have the identity $(1 - X^2\alpha(X))v(X) = g(X)$.

Let us now assume $\delta(X) \not\equiv 0$. To prove the convergence of the formal series $v(X)$, we proceed through a fixed point method as follows. Let us set

$$V(X) = \sum_{i \geq 0} V_i(X)$$

and let us choose the solution of Eq. (2.9) given by the system

$$\begin{cases} (1 - X^2\alpha(X))V_0(X) = g(X) \\ (1 - X^2\alpha(X))V_{i+1}(X) = X^2\delta(X) \sum_{i_1 + \dots + i_{d+2} = i} V_{i_1}(X) \dots V_{i_{d+2}}(X) \quad \text{for } i \geq 0. \end{cases}$$

By induction on $i \geq 0$, we easily check that

$$(2.10) \quad V_i(X) = \frac{C_{i,d+2} X^{2i} (\delta(X))^i (g(X))^{i(d+1)+1}}{(1 - X^2\alpha(X))^{i(d+2)+1}},$$

where the $C_{i,d+2}$'s are the positive constants recursively determined from $C_{0,d+2} := 1$ by the relations

$$C_{i+1,d+2} = \sum_{i_1 + \dots + i_{d+2} = i} C_{i_1,d+2} \dots C_{i_{d+2},d+2}.$$

Thereby, all the V_i 's are analytic functions on the disc with center $0 \in \mathbb{C}$ and radius $\min(r'_\alpha, r_\delta, r_g)$ at least. Moreover, identities (2.10) show us that $V_i(X)$ is of order X^{2i} for all $i \geq 0$. Consequently, the series $V(X)$ makes sense as a formal power series in X and we get $V(X) = v(X)$ by unicity.

We are left to prove the convergence of $V(X)$. To do that, let us choose $0 < r < \min(r'_\alpha, r_\alpha, r_\delta, r_g)$. By definition, the constants $C_{i,d+2}$'s are the generalized Catalan numbers of order $d+2$. We have therefore

$$C_{i,d+2} = \frac{1}{(d+1)i+1} \binom{i(d+2)}{i} \leq 2^{i(d+2)}$$

for all $i \geq 0$ (see [7, 8, 14] for instance). On the other hand, according to the increasing of the functions $\alpha(X)$, $\delta(X)$ and $g(X)$ on $[0, r]$, we derive from identities (2.10) the inequalities

$$|V_i(X)| \leq \frac{g(r)}{1 - r^2\alpha(r)} \left(\frac{2^{d+2}\delta(r)(g(r))^{d+1}}{(1 - r^2\alpha(r))^{d+2}} |X|^2 \right)^i$$

for all $i \geq 0$ and all $|X| \leq r$. Consequently, since $\delta(r) > 0$ and $g(r) > 0$ (we have indeed $g(0) \geq 1$), the series $V(X)$ is normally convergent on any disc with center $0 \in \mathbb{C}$ and radius

$$0 < r' < \min \left(r, \left(\frac{(1 - r^2\alpha(r))^{d+2}}{2^{d+2}\delta(r)(g(r))^{d+1}} \right)^{1/2} \right).$$

This proves the analyticity of $V(X)$ at 0 and ends the proof of Proposition 2.11. \square

According to Lemma 2.10 and Proposition 2.11, we can now bound the Nagumo norms $\|u_{j,*}\|_{(s+1)j,\rho}$.

Corollary 2.12. *Let $C', K' > 0$ be as in Proposition 2.11. Then, the inequalities*

$$\|u_{j,*}\|_{(s+1)j,\rho} \leq C' K'^j \Gamma(1 + (s+1)j)$$

hold for all $j \geq 0$.

We are now able to conclude the proof of the first point of Theorem 2.3.

2.2.4. *Third step: conclusion.* We must prove on the sup-norm of the $u_{j,*}(x)$ estimates similar to the ones on the norms $\|u_{j,*}\|_{(s+1)j,\rho}$ (see Corollary 2.12). To this end, we proceed by shrinking the closed disc $|x| \leq \rho$. Let $0 < \rho' < \rho$. Then, for all $j \geq 0$ and all $|x| \leq \rho'$, we have

$$|u_{j,*}(x)| = \left| u_{j,*}(x) d_\rho(x)^{(s+1)j} \frac{1}{d_\rho(x)^{(s+1)j}} \right| \leq \frac{|u_{j,*}(x) d_\rho(x)^{(s+1)j}|}{(\rho - \rho')^{(s+1)j}} \leq \frac{\|u_{j,*}\|_{(s+1)j,\rho}}{(\rho - \rho')^{(s+1)j}}$$

and, consequently,

$$\sup_{|x| \leq \rho'} |u_{j,*}(x)| \leq C' \left(\frac{K'}{(\rho - \rho')^{s+1}} \right)^j \Gamma(1 + (s+1)j)$$

by applying Corollary 2.12. This ends the proof of the first point of Theorem 2.3.

2.3. Proof of Theorem 2.3: second point. Let us fix $s < 1$. According to the filtration of the s -Gevrey spaces $\mathcal{O}(D_{\rho_1})[[t]]_s$ (see Section 2.1) and the first point of Theorem 2.3, it is clear that we have the following implications:

$$\tilde{f}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]_s \Rightarrow \tilde{f}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]_1 \Rightarrow \tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]_1.$$

To conclude that we can not say better about the Gevrey order of $\tilde{u}(t, x)$, that is $\tilde{u}(t, x)$ is *generically* 1-Gevrey, we need to find an example for which the formal solution $\tilde{u}(t, x)$ of Eq. (1.2) is s' -Gevrey for no $s' < 1$. Proposition 2.13 below provides such an example.

Proposition 2.13. *Let us consider the inhomogeneous Boussinesq equation*

$$(2.11) \quad \begin{cases} \partial_t^2 u - a \partial_x^4 u - b \partial_x^2 u - c \partial_x^2 (u^n) = \tilde{f}(t, x), & a, b, c > 0, \quad n \geq 2 \\ \partial_t^j u(t, x)|_{t=0} = \varphi(x), & j = 0, 1 \end{cases}$$

where $\varphi(x)$ is the analytic function on D_1 defined by

$$\varphi(x) = \frac{1}{1-x}.$$

Suppose that the inhomogeneity $\tilde{f}(t, x)$ is s -Gevrey and satisfies the inequalities $\partial_x^\ell \tilde{f}_{j,*}(0) \geq 0$ for all $\ell, j \geq 0$.

Then, the formal solution $\tilde{u}(t, x)$ of Eq. (2.11) is exactly 1-Gevrey.

Proof. According to the remark above, it is sufficient to prove that $\tilde{u}(t, x)$ is s' -Gevrey for no $s' < 1$.

First of all, let us rewrite Eq. (2.11) in the form

$$\partial_t^2 u - a \partial_x^4 u - (b + cnu^{n-1}) \partial_x^2 u - cn(n-1)u^{n-2}(\partial_x u)^2 = \tilde{f}(t, x).$$

We derive then from the general relations (3.1) that the coefficients $u_{j,*}(x)$ of the formal solution $\tilde{u}(t, x)$ of Eq. (2.11) are recursively determined by the initial

conditions $u_{0,*}(x) = u_{1,*}(x) = \varphi(x)$ and, for all $j \geq 0$ by the relations

$$\begin{aligned} u_{j+2,*}(x) &= f_{j,*}(x) + a\partial_x^4 u_{j,*}(x) + b\partial_x^2 u_{j,*}(x) \\ &+ cn \sum_{j_1+\dots+j_n=j} \binom{j}{j_1, \dots, j_n} u_{j_1,*}(x) \dots u_{j_{n-1},*}(x) \partial_x^2 u_{j_n,*}(x) \\ &+ cn(n-1) \sum_{j_1+\dots+j_n=j} \binom{j}{j_1, \dots, j_n} u_{j_1,*}(x) \dots u_{j_{n-2},*}(x) \partial_x u_{j_{n-1},*}(x) \partial_x u_{j_n,*}(x). \end{aligned}$$

In particular, we easily check that the coefficients $u_{2j,*}(x)$ read for all $j \geq 1$ on the form

$$u_{2j,*}(x) = a^j \partial_x^{4j} \varphi(x) + \text{rem}_j(x),$$

where $\text{rem}_j(x)$ is a linear combination with nonnegative coefficients of terms of the form

$$\prod_{\substack{\ell \in \{0, \dots, 2(j-1)\} \\ d_1, d_2 \geq 0 \\ p_1, p_2, p_3, p_4, p_5, p_6, p_7 \geq 0}} a^{p_1} b^{p_2} c^{p_3} n^{p_4} (n-1)^{p_5} (\partial_x^{d_1} f_{\ell,*}(x))^{p_6} (\partial_x^{d_2} \varphi(x))^{p_7}.$$

Using then our assumptions on the coefficients a , b , c and n , and on the inhomogeneity $\tilde{f}(t, x)$, we finally get the following inequalities:

$$(2.12) \quad u_{2j,*}(0) \geq a^j (4j)!.$$

Let us now suppose that $\tilde{u}(t, x)$ is s' -Gevrey for some $s' < 1$. Then, Definition 2.1 and inequality (2.12) imply

$$1 \leq C \left(\frac{K}{a} \right)^j \frac{\Gamma(1 + 2(s' + 1)j)}{\Gamma(1 + 4j)}$$

for all $j \geq 1$ and some convenient positive constants C and K independent of j . Proposition 2.13 follows since such inequalities are impossible. Indeed, applying the Stirling's Formula, we get

$$(2.13) \quad C \left(\frac{K}{a} \right)^j \frac{\Gamma(1 + 2(s' + 1)j)}{\Gamma(1 + 4j)} \underset{j \rightarrow +\infty}{\sim} C' \left(\frac{K'}{j^{2(1-s')}} \right)^j$$

with

$$C' = C \sqrt{\frac{s'+1}{2}} \quad \text{and} \quad K' = \frac{K e^{2(1-s')} (2(s'+1))^{2(s'+1)}}{256a},$$

and the right hand-side of (2.13) goes to 0 when j tends to infinity. \square

This achieves the proof of the second point of Theorem 2.3.

3. 1-SUMMABILITY OF $\tilde{u}(t, x)$

In this section, we consider the critical value $s = 1$ and we are interested in the 1-summability of the formal solution $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ of the generalized Boussinesq equation (1.2). More precisely, our aim is to make explicit a necessary and sufficient condition under which $\tilde{u}(t, x)$ is 1-summable in a given direction $\arg(t) = \theta$. Before starting our calculations, let us recall the definition and some properties of the 1-summability.

3.1. 1-summable formal series. Still considering t as the variable and x as a parameter, one extends, in the similar way as the s -Gevrey formal series (see Definition 2.1), the classical notion of 1-summability of formal series in $\mathbb{C}[[t]]$ to the notion of 1-summability of formal series in $\mathcal{O}(D_{\rho_1})[[t]]$ in requiring similar conditions, the estimates being however uniform with respect to x .

Among the many equivalent definitions of 1-summability in a given direction $\arg(t) = \theta$ at $t = 0$, we choose in this article a generalization of Ramis' definition which states that a formal series $\tilde{g}(t, x) \in \mathbb{C}[[t]]$ is 1-summable in direction θ if there exists a holomorphic function g which is 1-Gevrey asymptotic to \tilde{g} in an open sector $\Sigma_{\theta, > \pi}$ bisected by θ and with opening larger than π [15, Def. 3.1]. To express the 1-Gevrey asymptotic, there also exist various equivalent ways. We choose here the one which sets conditions on the successive derivatives of g (see [11, p. 171] or [15, Thm. 2.4] for instance).

Definition 3.1 (1-summability). A formal series $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ is said to be 1-summable in the direction $\arg(t) = \theta$ if there exist a sector $\Sigma_{\theta, > \pi}$, a radius $0 < r_1 \leq \rho_1$ and a function $u(t, x)$ called 1-sum of $\tilde{u}(t, x)$ in the direction θ such that

- (1) u is defined and holomorphic on $\Sigma_{\theta, > \pi} \times D_{r_1}$;
- (2) for any $x \in D_{r_1}$, the map $t \mapsto u(t, x)$ has $\tilde{u}(t, x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!}$ as Taylor series at 0 on $\Sigma_{\theta, > \pi}$;
- (3) for any proper² subsector $\Sigma \Subset \Sigma_{\theta, > \pi}$, there exist two positive constants $C, K > 0$ such that, for all $\ell \geq 0$, all $t \in \Sigma$ and all $x \in D_{r_1}$,

$$|\partial_t^\ell u(t, x)| \leq CK^\ell \Gamma(1 + 2\ell) = CK^\ell (2\ell)!.$$

We denote by $\mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}$ the subset of $\mathcal{O}(D_{\rho_1})[[t]]$ made of all the 1-summable formal series in the direction $\arg(t) = \theta$.

Notice that, for any fixed $x \in D_{r_1}$, the 1-summability of $\tilde{u}(t, x)$ coincides with the classical 1-summability. Consequently, Watson's lemma [10, Theorem 5.1.3] implies the unicity of its 1-sum, if any exists.

Notice also that the 1-sum of a 1-summable formal series $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}$ may be analytic with respect to x on a disc D_{r_1} smaller than the common disc D_{ρ_1} of analyticity of the coefficients $u_{j,*}(x)$ of $\tilde{u}(t, x)$.

Denote by $\partial_t^{-1} \tilde{u}$ (resp. $\partial_x^{-1} \tilde{u}$) the anti-derivative of \tilde{u} with respect to t (resp. x) which vanishes at $t = 0$ (resp. $x = 0$). Following Proposition 3.2, which is proved for instance in [3, Prop. 3.2], specifies the algebraic structure of $\mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}$.

Proposition 3.2. *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then, $(\mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}, \partial_t, \partial_x)$ is a \mathbb{C} -differential algebra stable under the anti-derivatives ∂_t^{-1} and ∂_x^{-1} .*

With respect to t , the 1-sum $u(t, x)$ of a 1-summable series $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}$ is analytic on an open sector for which there is no control on the angular opening except that it must be larger than π (hence, it contains a closed sector $\overline{\Sigma}_{\theta, \pi}$ bisected by θ and with opening π) and no control on the radius except that it must be positive. Thereby, the 1-sum $u(t, x)$ is well-defined as a section of the sheaf of analytic functions in (t, x) on a germ of closed sector of opening π (that is, a closed

²A subsector Σ of a sector Σ' is said to be a *proper subsector* and one denotes $\Sigma \Subset \Sigma'$ if its closure in \mathbb{C} is contained in $\Sigma' \cup \{0\}$.

interval $\bar{I}_{\theta,\pi}$ of length π on the circle S^1 of directions issuing from 0; see [12, 1.1] or [9, I.2]) times $\{0\}$ (in the plane \mathbb{C} of the variable x). We denote by $\mathcal{O}_{\bar{I}_{\theta,\pi} \times \{0\}}$ the space of such sections.

Corollary 3.3. *The operator of 1-summation*

$$\begin{aligned} \mathcal{S}_{1;\theta} : \mathcal{O}(D_{\rho_1})\{t\}_{1;\theta} &\longrightarrow \mathcal{O}_{\bar{I}_{\theta,\pi} \times \{0\}} \\ \tilde{u}(t, x) &\longmapsto u(t, x) \end{aligned}$$

is a homomorphism of \mathbb{C} -differential algebras for the derivations ∂_t and ∂_x . Moreover, it commutes with the anti-derivations ∂_t^{-1} and ∂_x^{-1} .

The following result investigates the 1-summability of the analytic functions at the origin $(0, 0) \in \mathbb{C}^2$.

Proposition 3.4. *Let $a(t, x)$ be an analytic function on a polydisc $D_{\rho_0} \times D_{\rho_1}$. Then, $a(t, x) \in \mathcal{O}(D_{\rho_1})\{t\}_{1;\theta}$ for any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.*

Proof. Let us fix a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and four radii $0 < \rho_j'' < \rho_j' < \rho_j$ with $j = 0, 1$. Let us first start by observing that $a(t, x)$ is its own Taylor series at $(0, 0)$ on $D_{\rho_0} \times D_{\rho_1}$. On the other hand, we derive from the Cauchy Integral Formula

$$\partial_t^\ell a(t, x) = \frac{\ell!}{(2i\pi)^2} \int_{\substack{|t'-t|=\rho_0'-\rho_0'' \\ |x'-x|=\rho_1'-\rho_1''}} \frac{a(t', x')}{(t'-t)^{\ell+1}(x'-x)} dt' dx'$$

the inequalities

$$|\partial_t^\ell a(t, x)| \leq \alpha \left(\frac{1}{\rho_0' - \rho_0''} \right)^\ell \ell!$$

for all $\ell \geq 0$ and all $(t, x) \in D_{\rho_0''} \times D_{\rho_1''}$, where α stands for the maximum of $|a(t, x)|$ on the closed polydisc $\bar{D}_{\rho_0'} \times \bar{D}_{\rho_1'}$ (\bar{D}_ρ denotes the closed disc with center 0 $\in \mathbb{C}$ and radius $\rho > 0$). Observing then that $\ell! \leq (2\ell)!$ for all $\ell \geq 0$, we finally get the inequalities

$$|\partial_t^\ell a(t, x)| \leq \alpha \left(\frac{1}{\rho_0' - \rho_0''} \right)^\ell (2\ell)!$$

for all $\ell \geq 0$ and all $(t, x) \in D_{\rho_0''} \times D_{\rho_1''}$. The choice of an arbitrary sector $\Sigma_{\theta, >\pi} \subset D_{\rho_0''}$ ends the proof of Proposition 3.4. \square

Let us now turn to the study of the 1-summability of the formal solution $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ of Eq. (1.2).

3.2. Main result. Before stating our main result, let us start with a preliminary remark. Let us write the coefficients $a(t, x)$, $b_{P,m}(t, x)$ and $b_{Q,m}(t, x)$ in the form

$$\begin{aligned} a(t, x) &= \sum_{n \geq 0} a_{*,n}(t) \frac{x^n}{n!} \\ b_{P,m}(t, x) &= \sum_{n \geq 0} b_{P,m;*,n}(t) \frac{x^n}{n!}, \quad b_{Q,m}(t, x) = \sum_{n \geq 0} b_{Q,m;*,n}(t) \frac{x^n}{n!} \end{aligned}$$

with $a_{*,n}(t), b_{P,m;*,n}(t), b_{Q,m;*,n}(t) \in \mathcal{O}(D_{\rho_0})$ for all $n \geq 0$ and all $m = 0, \dots, d$. Let us also write the formal solution $\tilde{u}(t, x)$ and the inhomogeneity $\tilde{f}(t, x)$ in the same way:

$$\tilde{u}(t, x) = \sum_{n \geq 0} \tilde{u}_{*,n}(t) \frac{x^n}{n!}, \quad \tilde{f}(t, x) = \sum_{n \geq 0} \tilde{f}_{*,n}(t) \frac{x^n}{n!}.$$

Observe that the coefficients $\tilde{u}_{*,n}(t)$ and $\tilde{f}_{*,n}(t)$ are divergent in general (hence, the notation with a tilde). By identifying the terms in x^n in Eq. (1.2), we get the identities

$$(3.1) \quad a_{*,0}(t)\tilde{u}_{*,n+4}(t) = \partial_t^2 \tilde{u}_{*,n}(t) - \tilde{f}_{*,n}(t) - \sum_{\substack{n_0+n_1=n \\ n_1 \neq n}} \binom{n}{n_0, n_1} a_{*,n_0}(t) \tilde{u}_{*,n_1+4}(t) \\ - \sum_{m=0}^d \sum_{n_0+\dots+n_{m+1}=n} \binom{n}{n_0, \dots, n_{m+1}} \tilde{T}_{P,m,n_0,\dots,n_{m+1}}(t) \\ - \sum_{m=0}^d \sum_{n_0+\dots+n_{m+2}=n} \binom{n}{n_0, \dots, n_{m+2}} \tilde{T}_{Q,m,n_0,\dots,n_{m+2}}(t)$$

for all $n \geq 0$, where $\tilde{T}_{P,m,n_0,\dots,n_{m+1}}(t)$ and $\tilde{T}_{Q,m,n_0,\dots,n_{m+2}}(t)$ are the formal power series in t respectively defined by

$$\tilde{T}_{P,m,n_0,\dots,n_{m+1}}(t) = \begin{cases} b_{P,0;*,n_0}(t) \tilde{u}_{*,n_1+2}(t) & \text{if } m = 0 \\ b_{P,m;*,n_0}(t) \tilde{u}_{*,n_1}(t) \dots \tilde{u}_{*,n_m}(t) \tilde{u}_{*,n_{m+1}+2}(t) & \text{if } m \geq 1 \end{cases}$$

and

$$\tilde{T}_{Q,m,n_0,\dots,n_{m+2}}(t) = \begin{cases} b_{Q,0;*,n_0}(t) \tilde{u}_{*,n_1+1}(t) \tilde{u}_{*,n_2+1}(t) & \text{if } m = 0 \\ b_{Q,0;*,n_0}(t) \tilde{u}_{*,n_1}(t) \dots \tilde{u}_{*,n_m}(t) \tilde{u}_{*,n_{m+1}+1}(t) \tilde{u}_{*,n_{m+2}+1}(t) & \text{if } m \geq 1 \end{cases}$$

By assumption, $a_{*,0}(0) = a(0,0) \neq 0$. Then, $1/a_{*,0}(t)$ is well-defined in $\mathbb{C}[[t]]$ and, consequently, each coefficient $\tilde{u}_{*,n}(t)$ is uniquely determined from the inhomogeneity $\tilde{f}(t, x)$ and from the formal series $\tilde{u}_{*,n'}(t)$ with $n' = 0, 1, 2, 3$. In particular, the same applies to the formal solution $\tilde{u}(t, x)$.

We are now able to state the main result in view in this section.

Theorem 3.5. *Let $\tilde{u}(t, x)$ be the formal solution in $\mathcal{O}(D_{\rho_1})[[t]]$ of the generalized Boussinesq equation (1.2). Let $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction issuing from 0. Then,*

- (1) $\tilde{u}(t, x)$ is 1-summable in the direction θ if and only if the inhomogeneity $\tilde{f}(t, x)$ and the formal series $\tilde{u}_{*,n}(t) = \partial_x^n \tilde{u}(t, x)|_{x=0} \in \mathbb{C}[[t]]$ for $n = 0, 1, 2, 3$ are 1-summable in the direction θ .
- (2) Moreover, the 1-sum $u(t, x)$, if any exists, satisfies Eq. (1.2) in which $\tilde{f}(t, x)$ is replaced by its 1-sum $f(t, x)$ in the direction θ .

Observe that Theorem 3.5 coincides with Proposition 1.3 in the case of the Euler-Lagrange equation (1.3), that is in the case where the polynomials P and Q are both zero.

3.3. Proof of Theorem 3.5. First of all, let us observe that the necessary condition of the first point is straightforward from Proposition 3.2, and that the second point stems obvious from Corollary 3.3. Consequently, we are left to prove the sufficient condition of the first point. To this end, we fix from now on a direction θ and we suppose that the inhomogeneity $\tilde{f}(t, x)$ and the formal power series $\tilde{u}_{*,n}(t) = \partial_x^n \tilde{u}(t, x)|_{x=0} \in \mathbb{C}[[t]]$ for $n = 0, 1, 2, 3$ are all 1-summable in the direction θ . To prove that the formal solution $\tilde{u}(t, x)$ is also 1-summable in this direction,

we shall proceed through a fixed point method similar to the ones already used by W. Balser and M. Loday-Richaud in [3] and by the author in [19–22, 25, 26].

3.3.1. *First step: an associated equation.* Let us set

$$\tilde{v}(t, x) = \tilde{u}_{*,0}(t) + \tilde{u}_{*,1}(t) \frac{x}{1!} + \tilde{u}_{*,2}(t) \frac{x^2}{2!} + \tilde{u}_{*,3}(t) \frac{x^3}{3!}$$

and let us introduce the formal series $\tilde{w}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ defined by the relation

$$\tilde{u}(t, x) = \tilde{v}(t, x) + \partial_x^{-4} \tilde{w}(t, x).$$

With these notations, Eq. (1.2) becomes

$$(3.2) \quad \tilde{w} - \tilde{H}(t, x, \tilde{w}) = \tilde{g}(t, x),$$

where

$$\begin{aligned} \tilde{H}(t, x, \tilde{w}) = & A(t, x) \partial_t^2 \partial_x^{-4} \tilde{w} - \sum_{k=0}^d \tilde{B}_{P,k}(t, x) (\partial_x^{-4} \tilde{w})^k \partial_x^{-2} \tilde{w} \\ & - \sum_{k=0}^d \tilde{B}_{Q,k}(t, x) (\partial_x^{-4} \tilde{w})^k (\partial_x^{-3} \tilde{w})^2 - \sum_{k=0}^d \tilde{C}_{Q,k}(t, x) (\partial_x^{-4} \tilde{w})^k \partial_x^{-3} \tilde{w} \\ & - \sum_{k=1}^d \tilde{D}_k(t, x) (\partial_x^{-4} \tilde{w})^k \end{aligned}$$

with

$$\begin{aligned} A(t, x) &= \frac{1}{a(t, x)}, \quad \tilde{B}_{\bullet,k}(t, x) = \sum_{m=k}^{d_\bullet} \binom{m}{k} \frac{b_{\bullet,m}(t, x)}{a(t, x)} \tilde{v}^{m-k}(t, x), \\ \tilde{C}_{Q,k}(t, x) &= 2\tilde{B}_{Q,k}(t, x) \partial_x \tilde{v}(t, x), \\ \tilde{D}_k(t, x) &= \tilde{B}_{P,k}(t, x) \partial_x^2 \tilde{v}(t, x) + \tilde{B}_{Q,k}(t, x) (\partial_x \tilde{v}(t, x))^2, \end{aligned}$$

and where

$$\tilde{g}(t, x) = A(t, x) (\partial_t^2 \tilde{v} - P(t, x, \tilde{v}) \partial_x^2 \tilde{v} - Q(t, x, \tilde{v}) (\partial_x \tilde{v})^2 - \tilde{f}(t, x)).$$

Observe that $\tilde{w}(t, x)$ is actually the unique formal series solution of Eq. (3.2) (reason as in Proposition 1.1 by exchanging the role of t and x). Observe also that, thanks to the initial assumption $a(0, 0) \neq 0$ (see page 2), the function $A(t, x)$ is well-defined and analytic on a polydisc $D_{\rho'_0} \times D_{\rho'_1}$ with convenient radii $0 < \rho'_j \leq \rho_j$ for $j = 0, 1$; hence, the coefficients $\tilde{B}_{\bullet,k}(t, x)$, $\tilde{C}_{Q,k}(t, x)$ and $\tilde{D}_k(t, x)$, and the inhomogeneity $\tilde{g}(t, x)$ belong to $\mathcal{O}(D_{\rho'_1})[[t]]$. Moreover, taking into account the fact that the formal series $\tilde{f}(t, x)$ and $\tilde{v}(t, x)$ are both 1-summable in the direction θ , it is the same for all these formal series (Propositions 3.2 and 3.4). Consequently, identity (3.2) above tells us that it is sufficient to prove that the formal series $\tilde{w}(t, x)$ is also 1-summable in the direction θ . To do that, we shall proceed in a similar way as [3, 19–22, 25, 26] by using a fixed point method.

3.3.2. *Second step: the fixed point procedure.* Let us set

$$\tilde{W}(t, x) = \sum_{\mu \geq 0} \tilde{w}_\mu(t, x)$$

and let us choose the solution of Eq. (3.2) recursively determined by the relations

$$(3.3) \quad \begin{cases} \tilde{w}_0(t, x) = \tilde{g}(t, x) \\ \tilde{w}_{\mu+1}(t, x) = \tilde{H}'(t, x, \tilde{w}_0, \dots, \tilde{w}_\mu) \quad \text{for all } \mu \geq 0 \end{cases}$$

where

$$\begin{aligned} \tilde{H}'(t, x, \tilde{w}_0, \dots, \tilde{w}_\mu) &= A(t, x) \partial_t^2 \partial_x^{-4} \tilde{w}_\mu \\ &\quad - \sum_{k=0}^d \sum_{\mu_1 + \dots + \mu_{k+1} = \mu} \left(\tilde{B}_{P,k}(t, x) \left(\prod_{i=1}^k \partial_x^{-4} \tilde{w}_{\mu_i} \right) \partial_x^{-2} \tilde{w}_{\mu_{k+1}} \right) \\ &\quad - \sum_{k=0}^d \sum_{\mu_1 + \dots + \mu_{k+2} = \mu} \left(\tilde{B}_{Q,k}(t, x) \left(\prod_{i=1}^k \partial_x^{-4} \tilde{w}_{\mu_i} \right) \partial_x^{-3} \tilde{w}_{\mu_{k+1}} \partial_x^{-3} \tilde{w}_{\mu_{k+2}} \right) \\ &\quad - \sum_{k=0}^d \sum_{\mu_1 + \dots + \mu_{k+1} = \mu} \left(\tilde{C}_{Q,k}(t, x) \left(\prod_{i=1}^k \partial_x^{-4} \tilde{w}_{\mu_i} \right) \partial_x^{-3} \tilde{w}_{\mu_{k+1}} \right) \\ &\quad - \sum_{k=1}^d \sum_{\mu_1 + \dots + \mu_k = \mu} \left(\tilde{D}_k(t, x) \left(\prod_{i=1}^k \partial_x^{-4} \tilde{w}_{\mu_i} \right) \right) \end{aligned}$$

with the classical convention that the products over i are 1 when $k = 0$.

Observe that $\tilde{w}_\mu(t, x) \in \mathcal{O}(D_{\rho'_1})[[t]]$ for all $\mu \geq 0$. Observe also that the $\tilde{w}_\mu(t, x)$'s are of order $O(x^{2\mu})$ in x for all $\mu \geq 0$. Thereby, the series $\tilde{W}(t, x)$ itself makes sense as a formal series in t and x and, consequently, $\tilde{W}(t, x) = \tilde{w}(t, x)$ by unicity.

Let us now respectively denote by $w_0(t, x)$, $B_{\bullet,k}(t, x)$, $C_{Q,k}(t, x)$ and $D_k(t, x)$ the 1-sums of $\tilde{w}_0(t, x)$, $\tilde{B}_{\bullet,k}(t, x)$, $\tilde{C}_{Q,k}(t, x)$ and $\tilde{D}_k(t, x)$ in the direction θ and, for all $\mu > 0$, let $w_\mu(t, x)$ be determined by the relations (3.3) in which the formal series $\tilde{B}_{\bullet,k}(t, x)$, $\tilde{C}_{Q,k}(t, x)$ and $\tilde{D}_k(t, x)$ are respectively replaced by $B_{\bullet,k}(t, x)$, $C_{Q,k}(t, x)$ and $D_k(t, x)$, and all the \tilde{w}_μ by w_μ . By construction, all the functions $w_\mu(t, x)$ are defined and holomorphic on a common domain $\Sigma_{\theta, > \pi} \times D_{\rho'_1}$ with a convenient radius $0 < \rho''_1 \leq \rho'_1$.

To end the proof, it remains to prove that the series $\sum_{\mu \geq 0} w_\mu(t, x)$ is convergent and that its sum $w(t, x)$ is the 1-sum of $\tilde{w}(t, x)$ in the direction θ . To do that, we shall now give estimates on the functions $w_\mu(t, x)$.

3.3.3. *Third step: some estimates on the $w_\mu(t, x)$'s.* According to Definition 3.1 and Proposition 3.4, there exists a radius $0 < r'_1 \leq \min(1, \rho''_1)$ such that, for any proper subsector $\Sigma \Subset \Sigma_{\theta, > \pi}$, there exist two positive constants $C > 0$ and $K \geq 1$ such that

$$(3.4) \quad \begin{cases} |\partial_t^\ell w_0(t, x)| \leq CK^\ell (2\ell)! \\ |\partial_t^\ell A(t, x)| \leq CK^\ell (2\ell)!, & |\partial_t^\ell B_{\bullet,k}(t, x)| \leq CK^\ell (2\ell)! \\ |\partial_t^\ell C_{Q,k}(t, x)| \leq CK^\ell (2\ell)!, & |\partial_t^\ell D_k(t, x)| \leq CK^\ell (2\ell)! \end{cases}$$

for all $\ell \geq 0$, all $k = 0, \dots, d$, and all $(t, x) \in \Sigma \times D_{r'_1}$.

Let us now fix a proper subsector $\Sigma \Subset \Sigma_{\theta, > \pi}$ and let us choose the constants C and K as above. Proposition 3.6 below provides us some first estimates on the derivatives $\partial_t^\ell w_\mu(t, x)$ on $\Sigma \times D_{r_1'}$.

Proposition 3.6. *Let $(P_\mu(x))_{\mu \geq 0}$ be the sequence of polynomials in $\mathbb{R}^+[x]$ recursively determined from $P_0(x) \equiv 1$ by the relations*

$$(3.5) \quad \begin{aligned} P_{\mu+1}(x) &= 3C\partial_x^{-4}P_\mu(x) \\ &+ \sum_{k=0}^d \left(\frac{(3C)^{k+1}}{(4\mu+4)!} \sum_{\mu_1+\dots+\mu_{k+1}=\mu} \left(\prod_{i=1}^{k+1} (4\mu_i)! \right) \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(x) \right) \partial_x^{-2}P_{\mu_{k+1}}(x) \right) \\ &+ \sum_{k=0}^d \left(\frac{(3C)^{k+2}}{(4\mu+4)!} \sum_{\mu_1+\dots+\mu_{k+2}=\mu} \left(\prod_{i=1}^{k+2} (4\mu_i)! \right) \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(x) \right) \left(\prod_{i=k+1}^{k+2} \partial_x^{-3}P_{\mu_i}(x) \right) \right) \\ &+ \sum_{k=0}^d \left(\frac{(3C)^{k+1}}{(4\mu+4)!} \sum_{\mu_1+\dots+\mu_{k+1}=\mu} \left(\prod_{i=1}^{k+1} (4\mu_i)! \right) \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(x) \right) \partial_x^{-3}P_{\mu_{k+1}}(x) \right) \\ &\quad + \sum_{k=1}^d \left(\frac{(3C)^k}{(4\mu+4)!} \sum_{\mu_1+\dots+\mu_k=\mu} \left(\prod_{i=1}^k (4\mu_i)! \partial_x^{-4}P_{\mu_i}(x) \right) \right) \end{aligned}$$

for all $\mu \geq 0$, where the products over i are 1 when $k = 0$ as usual.

Then, the following inequalities

$$(3.6) \quad |\partial_t^\ell w_\mu(t, x)| \leq CK^{2\mu+\ell}(4\mu+2\ell)!P_\mu(|x|)$$

hold for all $\ell, \mu \geq 0$ and all $(t, x) \in \Sigma \times D_{r_1'}$.

Proof. The proof proceeds by recursion on μ . The case $\mu = 0$ is straightforward from the first inequality of (3.4). Let us now suppose that the inequalities (3.6) hold for all the functions $w_j(t, x)$ with $j = 0, \dots, \mu$ for a certain $\mu \geq 0$.

According to the relations (3.3), we derive from the generalized Leibniz Formula, from the inequalities (3.4) and (3.6), and from the fact that $K \geq 1$ the identities

$$\begin{aligned} |\partial_t^\ell w_{\mu+1}(t, x)| &\leq CK^{2\mu+2+\ell}(4\mu+4+2\ell)! \times \left[S_{\ell, \mu} \partial_x^{-4}P_\mu(|x|) \right. \\ &+ \sum_{k=0}^d \sum_{\mu_1+\dots+\mu_{k+1}=\mu} \left(S'_{k+1, \ell, \mu, \mu_1, \dots, \mu_{k+1}} \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(|x|) \right) \partial_x^{-2}P_{\mu_{k+1}}(|x|) \right) \\ &+ \sum_{k=0}^d \sum_{\mu_1+\dots+\mu_{k+2}=\mu} \left(S'_{k+2, \ell, \mu, \mu_1, \dots, \mu_{k+2}} \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(|x|) \right) \left(\prod_{i=k+1}^{k+2} \partial_x^{-3}P_{\mu_i}(|x|) \right) \right) \\ &+ \sum_{k=0}^d \sum_{\mu_1+\dots+\mu_{k+1}=\mu} \left(S'_{k+1, \ell, \mu, \mu_1, \dots, \mu_{k+1}} \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(|x|) \right) \partial_x^{-3}P_{\mu_{k+1}}(|x|) \right) \\ &\quad \left. + \sum_{k=1}^d \sum_{\mu_1+\dots+\mu_k=\mu} \left(S'_{k, \ell, \mu, \mu_1, \dots, \mu_k} \left(\prod_{i=1}^k \partial_x^{-4}P_{\mu_i}(|x|) \right) \right) \right] \end{aligned}$$

for all $\ell \geq 0$ and all $(t, x) \in \Sigma \times D_{r'_1}$, where

$$S_{\ell, \mu} = C \sum_{\ell_0 + \ell_1 = \ell} \frac{\binom{\ell}{\ell_0, \ell_1}}{\binom{4\mu + 4 + 2\ell}{2\ell_0, 4\mu + 4 + 2\ell_1}}$$

and, for all $p \geq 1$ and all nonnegative integers μ_1, \dots, μ_p such that $\mu_1 + \dots + \mu_p = \mu$:

$$S'_{p, \ell, \mu, \mu_1, \dots, \mu_p} = \frac{C^p}{4!} \sum_{\ell_0 + \ell_1 + \dots + \ell_p = \ell} \frac{\binom{\ell}{\ell_0, \ell_1, \dots, \ell_p}}{\binom{4\mu + 4 + 2\ell}{2\ell_0, 4\mu_1 + 2\ell_1, \dots, 4\mu_p + 2\ell_p, 4}}.$$

Inequality (3.6) for $w_{\mu+1}(t, x)$ follows then from Lemma 3.7 below, which achieves the proof. \square

Lemma 3.7. (1) Let $\ell, \mu \geq 0$. Then,

$$S_{\ell, \mu} \leq 3C.$$

(2) Let $\ell, \mu \geq 0$, $p \geq 1$, and (μ_1, \dots, μ_p) a p -tuple of nonnegative integers such that $\mu_1 + \dots + \mu_p = \mu$. Then,

$$S'_{p, \ell, \mu, \mu_1, \dots, \mu_p} \leq (3C)^p \frac{(4\mu_1)! \dots (4\mu_p)!}{(4\mu + 4)!}.$$

Proof. (1) Using the Vandermonde Inequality (see Proposition 4.2)

$$\binom{4\mu + 4 + 2\ell}{2\ell_0, 4\mu + 4 + 2\ell_1} \geq \binom{\ell}{\ell_0, \ell_1}^2 \binom{4\mu + 4}{0, 4\mu + 4} = \binom{\ell}{\ell_0, \ell_1}^2,$$

we get

$$S_{\ell, \mu} \leq C \sum_{\ell_0 + \ell_1 = \ell} \frac{1}{\binom{\ell}{\ell_0, \ell_1}}$$

and thereby $S_{\ell, \mu} \leq 3C$ by applying Proposition 4.5.

(2) The second point is proved in the same way and is left to the reader. \square

Let us now bound the polynomials P_μ .

Proposition 3.8. Let B be the positive real number defined by

$$B = 48C + (3C)^{d+1} \zeta(2)^d (2 + 3C\zeta(2)) + 9C(1 + C\zeta(2)) \sum_{k=0}^{d-1} (3C\zeta(2))^k,$$

where ζ stands for the Riemann Zeta function. Then, the inequalities

$$0 \leq P_\mu(x) \leq \frac{(Bx^2)^\mu}{(4\mu)!}$$

hold for all $\mu \geq 0$ and all $x \in [0, 1]$.

Proof. The left inequality is obvious since all the coefficients of the $P_\mu(x)$'s are nonnegative. The right inequality is proved by recursion on $\mu \geq 0$ as follows.

The case $\mu = 0$ is clear since $P_0(x) \equiv 1$. Let us now suppose that Proposition 3.8 holds for all the polynomials $P_k(x)$ with $k \in \{0, \dots, \mu\}$ for a certain $\mu \geq 0$, and let us prove it for the polynomial $P_{\mu+1}(x)$.

Applying the recurrence relation (3.5) and the fact that $x \in [0, 1]$, we first get

$$P_{\mu+1}(x) \leq B' \frac{B^\mu x^{2\mu+2}}{(4\mu+4)!}$$

for all $x \in [0, 1]$, where B' is the positive real number defined by

$$B' = 3C\alpha_\mu + 2\beta_{d+1,\mu} + \beta_{d+2,\mu} + \sum_{k=0}^{d-1} (3\beta_{k+1,\mu} + \beta_{k+2,\mu})$$

with

$$\alpha_\mu = \frac{(4\mu+1)(4\mu+2)(4\mu+3)(4\mu+4)}{(2\mu+1)(2\mu+2)(2\mu+3)(2\mu+4)} \quad \text{and}$$

$$\beta_{p,\mu} = \begin{cases} 3C & \text{if } p = 1 \\ (3C)^p \sum_{\mu_1+\dots+\mu_p=\mu} \frac{1}{\prod_{j=1}^{p-1} (\mu_j+1)^2} & \text{if } p \geq 2. \end{cases}$$

The constant α_μ is easily bounded as follows:

$$\alpha_\mu = \frac{2^4 \left(2\mu + \frac{1}{2}\right) (2\mu+1) \left(2\mu + \frac{3}{2}\right) (2\mu+2)}{(2\mu+1)(2\mu+2)(2\mu+3)(2\mu+4)} \leq 2^4 = 16.$$

Let us now prove that $\beta_{p,\mu} \leq (3C)^p \zeta(2)^{p-1}$ for all $p \geq 1$ and all $\mu \geq 0$. This is obvious when $p = 1$ and stems from the following calculations when $p \geq 2$:

$$\begin{aligned} \beta_{p,\mu} &\leq (3C)^p \sum_{\mu_1=0}^{\mu} \dots \sum_{\mu_{p-1}=0}^{\mu} \frac{1}{\prod_{j=1}^{p-1} (\mu_j+1)^2} = (3C)^p \left(\sum_{\mu'=0}^{\mu} \frac{1}{(\mu'+1)^2} \right)^{p-1} \\ &\leq (3C)^p \left(\sum_{\mu'=1}^{+\infty} \frac{1}{\mu'^2} \right)^{p-1} \\ &= (3C)^p \zeta(2)^{p-1}. \end{aligned}$$

Consequently, $B' \leq B$ and the sought inequality on $P_{\mu+1}(x)$ follows, which ends the proof of Proposition 3.8. \square

We are now able to improve the bounds of the functions $\partial_t^\ell w_\mu(t, x)$ given in Proposition 3.6.

Corollary 3.9. *Let us set $K_1 = 4K$ and $c = 16BK^2$. Then, the inequalities*

$$|\partial_t^\ell w_\mu(t, x)| \leq CK_1^\ell (2\ell)! (c|x|^2)^\mu$$

hold for all $\ell, \mu \geq 0$ and all $(t, x) \in \Sigma \times D_{r'_1}$.

Proof. Applying Propositions 3.6 and 3.8, we get

$$|\partial_t^\ell w_\mu(t, x)| \leq CK^{2\mu+\ell}(4\mu+2\ell)! \frac{B^\mu |x|^{2\mu}}{(4\mu)!} = CK^\ell(2\ell)! \binom{4\mu+2\ell}{4\mu, 2\ell} (BK^2|x|^2)^\mu$$

for all $\ell, \mu \geq 0$ and all $(t, x) \in \Sigma \times D_{r'_1}$ (recall that the radius r'_1 was chosen so that $r'_1 \leq 1$). Lemma 3.9 follows then by using the fact that $\binom{4\mu+2\ell}{4\mu, 2\ell} \leq 2^{4\mu+2\ell}$. \square

We are now able to complete the proof of Theorem 3.5.

3.3.4. *Fourth step: conclusion.* Let us now choose for Σ a sector containing a proper subsector Σ' bisected by the direction θ and opening larger than π (recall that such a choice is already possible by definition of a proper subsector, see Footnote 2). Let us also choose a radius $0 < r_1 < \min(r'_1, 1/\sqrt{c})$ and let us set $C_1 := C \sum_{\mu \geq 0} (cr_1^2)^\mu \in \mathbb{R}_+^*$.

Thanks to Corollary 3.9, the series $\sum_{\mu \geq 0} \partial_t^\ell w_\mu(t, x)$ are normally convergent on $\Sigma \times D_{r_1}$ for all $\ell \geq 0$ and satisfy the inequalities

$$\sum_{\mu \geq 0} |\partial_t^\ell w_\mu(t, x)| \leq C_1 K_1^\ell (2\ell)!$$

for all $(t, x) \in \Sigma \times D_{r_1}$. In particular, the sum $w(t, x)$ of the series $\sum_{\mu \geq 0} w_\mu(t, x)$ is well-defined, holomorphic on $\Sigma \times D_{r_1}$ and satisfies the inequalities

$$|\partial_t^\ell w(t, x)| \leq C_1 K_1^\ell (2\ell)!$$

for all $\ell \geq 0$ and all $(t, x) \in \Sigma \times D_{r_1}$. Hence, Conditions 1 and 3 of Definition 3.1 hold, since $\Sigma' \Subset \Sigma$.

To prove the second condition of Definition 3.1, we proceed as follows. The Removable Singularities Theorem implies the existence of $\lim_{\substack{t \rightarrow 0 \\ t \in \Sigma}} \partial_t^\ell w(t, x)$ for all $x \in$

D_{r_1} and, thereby, the existence of the Taylor series of w at 0 on Σ' for all $x \in D_{r_1}$ (see for instance [11, Cor. 1.1.3.3]; see also [10, Prop. 1.1.11]). On the other hand, considering recurrence relations (3.3), it is clear that $w(t, x)$ satisfies Eq. (3.2) where all the formal coefficients and the inhomogeneity are replaced by their 1-sums in the direction θ and, consequently, so does its Taylor series. Then, since Eq. (3.2) has a unique formal series solution $\tilde{w}(t, x)$ (see the remark page 19 just after the definition of Eq. (3.2)), we then conclude that the Taylor expansion of $w(t, x)$ is $\tilde{w}(t, x)$. Hence, Condition 2 of Definition 3.1 holds.

This achieves the proof of the sufficient condition of the first point of Theorem 3.5, which ends its full proof.

4. APPENDIX: SOME TECHNICAL RESULTS ON THE BINOMIAL AND MULTINOMIAL COEFFICIENTS

In combinatorial analysis, the binomial coefficients $\binom{n}{m}$ and the multinomial coefficients $\binom{n}{n_1, \dots, n_q}$ are defined for any nonnegative integers $0 \leq m \leq n$ and

any tuples (n, n_1, \dots, n_q) of nonnegative integers satisfying $q \geq 2$ and $n_1 + \dots + n_q = n$ by the relations

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \quad \text{and} \quad \binom{n}{n_1, \dots, n_q} = \frac{n!}{n_1! \dots n_q!}.$$

They respectively denote the number of ways of choosing m objects from a collection of n distinct objects without regard to order, and the number of ways of putting $n = n_1 + \dots + n_q$ different objects into q different boxes with n_i in the i -th box for all $i = 1, \dots, q$.

Using the fact that $n! = \Gamma(1+n)$ for any integer $n \geq 0$, one can easily extend the definitions of these coefficients to the case where their terms are no longer necessarily integers by setting

$$(4.1) \quad \binom{a}{b} = \frac{\Gamma(1+a)}{\Gamma(1+b)\Gamma(1+a-b)}$$

for any nonnegative real numbers $0 \leq b \leq a$ and

$$(4.2) \quad \binom{a}{a_1, \dots, a_q} = \frac{\Gamma(1+a)}{\Gamma(1+a_1) \dots \Gamma(1+a_q)} = \frac{\Gamma(1+a)}{\prod_{i=1}^q \Gamma(1+a_i)}$$

for any tuples (a, a_1, \dots, a_q) of nonnegative real numbers satisfying $q \geq 2$ and $a_1 + \dots + a_q = a$. Observe that all these coefficients are positive. Observe also that one has the following decomposition

$$(4.3) \quad \binom{a}{a_1, \dots, a_q} = \prod_{i=2}^q \binom{a_1 + \dots + a_i}{a_1 + \dots + a_{i-1}}.$$

The four propositions below extend to the generalized binomial coefficients (4.1) and the generalized multinomial coefficients (4.2) some well-known results in combinatorial analysis.

In the proof of Theorems 2.3 and 3.5, we essentially use the inequalities stated in Propositions 4.2, 4.4 and 4.5. The result of Proposition 4.1 is useful for the proof of Proposition 4.2.

Proposition 4.1 (Pascal Formula). *Let $0 \leq b \leq a$ be two nonnegative real numbers and $1 \leq m \leq n$ two nonnegative integers. Then,*

$$(4.4) \quad \binom{a+n+1}{b+m} = \binom{a+n}{b+m} + \binom{a+n}{b+m-1}.$$

Proof. We compute:

$$\begin{aligned}
\binom{a+n}{b+m} + \binom{a+n}{b+m-1} &= \frac{\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m)} \\
&\quad + \frac{\Gamma(1+a+n)}{\Gamma(1+b+m-1)\Gamma(1+a-b+n-m+1)} \\
&= \frac{(a-b+n-m+1)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&\quad + \frac{(b+m)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \frac{(a+n+1)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \frac{\Gamma(1+a+n+1)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \binom{a+n+1}{b+m};
\end{aligned}$$

hence, the identity (4.4). \square

Proposition 4.2 (Vandermonde Inequality).

- (1) (*Binomial case*) Let $0 \leq b \leq a$ be two nonnegative real numbers and $0 \leq m \leq n$ two nonnegative integers. Then,

$$(4.5) \quad \binom{a+n}{b+m} \geq \binom{a}{b} \binom{n}{m}.$$

- (2) (*Multinomial case*) Let $q \geq 2$ be an integer, (a, a_1, \dots, a_q) a tuple of nonnegative real numbers and (n, n_1, \dots, n_q) a tuple of nonnegative integers such that $a_1 + \dots + a_q = a$ and $n_1 + \dots + n_q = n$. Then,

$$(4.6) \quad \binom{a+n}{a_1+n_1, \dots, a_q+n_q} \geq \binom{a}{a_1, \dots, a_q} \binom{n}{n_1, \dots, n_q}.$$

Proof. \star *First point.* The inequality (4.5) is clear for $n = m = 0$. Let us now fix $0 \leq b \leq a$ and let us prove by induction on $n \geq 1$ the property

$$(\mathcal{P}_n) : \forall m \in \{0, \dots, n\}, \binom{a+n}{b+m} \geq \binom{a}{b} \binom{n}{m}.$$

A direct calculation gives us the property (\mathcal{P}_1) :

$$\begin{aligned}
\binom{a+1}{b} &= \frac{\Gamma(1+a+1)}{\Gamma(1+b)\Gamma(1+a+1-b)} = \frac{a+1}{a+1-b} \binom{a}{b} \geq \binom{a}{b} = \binom{a}{b} \binom{1}{0}, \\
\binom{a+1}{b+1} &= \frac{\Gamma(1+a+1)}{\Gamma(1+b+1)\Gamma(1+a-b)} = \frac{a+1}{b+1} \binom{a}{b} \geq \binom{a}{b} = \binom{a}{b} \binom{1}{1}.
\end{aligned}$$

Assuming now the property (\mathcal{P}_n) for a certain $n \geq 1$, let us prove the property (\mathcal{P}_{n+1}) . As for the property (\mathcal{P}_1) , the sought inequality stems from a direct

calculation when $m = 0$ and $m = n + 1$:

$$\begin{aligned} \binom{a+n+1}{b} &= \frac{\Gamma(1+a+n+1)}{\Gamma(1+b)\Gamma(1+a+n+1-b)} = \left(\prod_{k=1}^{n+1} \frac{a+k}{a+k-b} \right) \binom{a}{b} \\ &\geq \binom{a}{b} = \binom{a}{b} \binom{n+1}{0}, \end{aligned}$$

$$\begin{aligned} \binom{a+n+1}{b+n+1} &= \frac{\Gamma(1+a+n+1)}{\Gamma(1+b+n+1)\Gamma(1+a-b)} = \left(\prod_{k=1}^{n+1} \frac{a+k}{b+k} \right) \binom{a}{b} \\ &\geq \binom{a}{b} = \binom{a}{b} \binom{n+1}{n+1}. \end{aligned}$$

When $m \in \{1, \dots, n\}$, it stems from Proposition 4.1 and the property (\mathcal{P}_n) as follows:

$$\begin{aligned} \binom{a+n+1}{b+m} &= \binom{a+n}{b+m} + \binom{a+n}{b+m-1} \geq \binom{a}{b} \binom{n}{m} + \binom{a}{b} \binom{n}{m-1} \\ &= \binom{a}{b} \left(\binom{n}{m} + \binom{n}{m-1} \right) \\ &= \binom{a}{b} \binom{n+1}{m}. \end{aligned}$$

This ends the induction and proves the first point of Proposition 4.2.

★ *Second point.* Let us apply the relation (4.3) and the inequality (4.5) to each factor of the product. We get

$$\begin{aligned} \binom{a+n}{a_1+n_1, \dots, a_q+n_q} &= \prod_{i=2}^q \binom{a_1+\dots+a_i+n_1+\dots+n_i}{a_1+\dots+a_{i-1}+n_1+\dots+n_{i-1}} \\ &\geq \prod_{i=2}^q \binom{a_1+\dots+a_i}{a_1+\dots+a_{i-1}} \binom{n_1+\dots+n_i}{n_1+\dots+n_{i-1}} \\ &= \left(\prod_{i=2}^q \binom{a_1+\dots+a_i}{a_1+\dots+a_{i-1}} \right) \left(\prod_{i=2}^q \binom{n_1+\dots+n_i}{n_1+\dots+n_{i-1}} \right). \end{aligned}$$

The inequality (4.6) follows then by applying again the relation (4.3), which ends the proof of the second point of Proposition 4.2. \square

Remark 4.3. When all the terms a , b , n and m are nonnegative integers, inequality (4.5) is also a direct consequence of the Chu-Vandermonde Identity

$$(4.7) \quad \binom{a+n}{b+m} = \sum_{k+\ell=b+m} \binom{a}{k} \binom{n}{\ell},$$

since all the coefficients in the sum are nonnegative. However, this proof fails in our general case where a and b are no longer integers, since some terms in the sum of (4.7) may now be negative.

Proposition 4.4 (Variations of the binomial coefficients). *Let $b \geq 0$. Then, the function $\text{Bin}_b : a \in [b, +\infty[\rightarrow \binom{a}{b}$ is increasing on $[b, +\infty[$. In particular, $\text{Bin}_b(a) \geq \text{Bin}_b(b) = 1$ for all $a \geq b$.*

Proof. The derivative of Bin_b is defined for all $a \in [b, +\infty[$ by the identity

$$\text{Bin}'_b(a) = \binom{a}{b} (\Psi(1+a) - \Psi(1+a-b)),$$

where $\Psi = \Gamma'/\Gamma$ is the Psi (or Digamma) function. Since the function $\ln \Gamma$ is convex on $]0, +\infty[$, the latter is increasing on $]0, +\infty[$ and Proposition 4.4 follows from Lagrange Theorem. \square

Proposition 4.5 (Sum of the inverses of binomial and multinomial coefficients).

(1) (*Binomial case*) The following inequality holds for all integers $n \geq 0$:

$$(4.8) \quad \sum_{m=0}^n \frac{1}{\binom{n}{m}} \leq 3.$$

(2) (*Multinomial case*) The following inequality holds for all integers $q \geq 2$ and $n \geq 0$:

$$(4.9) \quad \sum_{n_1 + \dots + n_q = n} \frac{1}{\binom{n}{n_1, \dots, n_q}} \leq 3^{q-1}.$$

Proof. \star *First point.* The inequality (4.8) is obvious when $n \in \{0, 1\}$:

$$\sum_{m=0}^n \frac{1}{\binom{n}{m}} = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \end{cases} \leq 3.$$

Let us now assume $n \geq 2$ and let us write the left hand-side of (4.8) in the form

$$\sum_{m=0}^n \frac{1}{\binom{n}{m}} = \frac{1}{\binom{n}{0}} + \sum_{m=1}^{n-1} \frac{1}{\binom{n}{m}} + \frac{1}{\binom{n}{n}} = 2 + \sum_{m=1}^{n-1} \frac{1}{\binom{n}{m}}.$$

Inequality (4.8) follows then by observing that all the terms $\binom{n}{m}$ are $\geq \binom{n}{1}$ for $m = 1, \dots, n-1$. We get indeed the following relations:

$$\sum_{m=0}^n \frac{1}{\binom{n}{m}} \leq 2 + \frac{n-1}{\binom{n}{1}} = 2 + \frac{n-1}{n} \leq 3.$$

\star *Second point.* Applying the relation (4.3) and setting $n'_k = n_1 + \dots + n_k$ for all $k = 1, \dots, q-1$, we first get the identities

$$\begin{aligned} \sum_{n_1 + \dots + n_q = n} \frac{1}{\binom{n}{n_1, \dots, n_q}} &= \sum_{n_1 + \dots + n_{q-1} \leq n} \frac{1}{\binom{n}{n_1 + \dots + n_{q-1}} \dots \binom{n_1 + n_2}{n_1}} \\ &= \sum_{n'_{q-1}=0}^n \sum_{n'_{q-2}=0}^{n'_{q-1}} \dots \sum_{n'_1=0}^{n'_2} \frac{1}{\binom{n}{n'_{q-1}} \binom{n'_{q-1}}{n'_{q-2}} \dots \binom{n'_2}{n'_1}}. \end{aligned}$$

Inequality (4.9) stems then from the inequality (4.8) which we apply $q-1$ times. This completes the proof. \square

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