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ON DIRAC STRUCTURES ADMITTING A VARIATIONAL APPROACH

OSCAR COSSERAT, ALEXEI KOTOV, CAMILLE LAURENT-GENGOUX,
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ABSTRACT. We discuss the notion of horizontal cohomology for Dirac structures and, more generally, Lie algebroids. We then use this notion to describe the condition allowing a variational formulation of Dirac dynamics.

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INTRODUCTION / MOTIVATION

This paper is a part of series of works by the authors (in various combinations) concerning rather broad subjects of “geometrizing mechanics” and “geometric integrators”. They include attempts to spell-out the underlying geometric structures for a large class of mechanical systems, like it was done decades ago with symplectic structures for Hamiltonian (conservative) systems. They also address the question of application of these geometric construction to design reliable simulation tools for respective classes of mechanical systems.

The precise question we ask ourselves in this paper is mostly motivated by the results of [1], where the appearance of (almost) *Dirac structures* for mechanical systems with constraints is discussed. It can be vaguely formulated as: “given a Dirac structure, what else do we need to know to define meaningful dynamics on it”. The question is in the spirit of the paper [2], where the notion of *Dirac systems* is described in the context of constraints as well as for control theory; it also somehow complements the series of works [3, 4] on a uniform description, using algebroids, of constraint systems in both Hamiltonian and Lagrangian formalisms. All those works, like many others are inspired by the approach to mechanics using double vector bundles introduced in [5, 6]. In this paper, more precisely, we study the cohomological conditions for a system arising from a Dirac structure to admit a variational (Lagrangian) formulation. For convenience, by some language abuse, we will call them *obstructions*, but what we actually mean is *sufficient conditions*. That is for “good cases”, when this obstruction is absent, we explain how a Lagrangian is constructed. This includes some classes of Poisson structures, for which a variational formulation of Hamiltonian mechanics becomes possible. This is also an important step to the construction of Dirac-structure-preserving numerical methods, since having constructed the Lagrangian, under some assumptions, one can profit from the well-established machinery of variational numerical methods.

The paper is organized as follows: We start by recalling some notions of Lie algebroids and give the definition of their cohomology. Then, we describe the main geometric tool – cohomology of Dirac structures, providing some examples. In the second part we explain the relation of this Dirac cohomology and the obstructions to construct a variational formulation for the dynamics on the Dirac structures. We illustrate the construction on some examples and non-examples. To conclude, we explain some ideas about variational integrators and possible application of those in our setting – this is a separate rich topic that we intend to elaborate in another more “mechanically oriented” paper.

1. THE HORIZONTAL COHOMOLOGY OF LIE ALGEBROIDS AND DIRAC STRUCTURES

1.1. Lie algebroids. The notion of Lie algebroids is a simultaneous generalization of tangent bundles and Lie algebras. In this subsection we briefly review the relevant notions. We refer to [7] for a detailed account.

Definition 1.1. Let M be a smooth manifold. A Lie algebroid $(A, \rho, [\cdot, \cdot])$ is given by a finite-dimensional vector bundle A , a vector bundle morphism $\rho: A \rightarrow TM$, called anchor and a (\mathbb{R} -bilinear) Lie bracket on the

sections of A

$$[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$$

satisfying for all $f \in C^\infty(M)$, $s, s' \in \Gamma(A)$:

$$[s, fs'] = f[s, s'] + \rho(s)(f) \cdot s'.$$

It can be shown that the above condition implies that $\rho_* : \Gamma(A) \rightarrow \Gamma(TM) = \mathfrak{X}(M)$ is a Lie algebra homomorphism. Lie algebroids appear in many different settings:

- The tangent bundle TM with its usual bracket and $\rho = id$ is a Lie algebroid.
- Let $F \subset TM$ be an involutive subbundle, i.e. a foliation. Then F is a Lie algebroid with the restricted bracket and the inclusion $F \rightarrow TM$ as anchor.
- Let \mathfrak{g} be a Lie algebra and $v : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ an infinitesimal action (i.e. a Lie algebra homomorphism). Then $\mathfrak{g} \times M$ is a Lie algebroid with bracket induced by the Lie bracket on \mathfrak{g} and anchor $\rho(\xi, p) = v(\xi)(p)$. In particular Lie algebras can be seen as Lie algebroids over a point.
- Let $\pi \in \Gamma(\Lambda^2 TM)$ be a Poisson bivector. Then the cotangent bundle carries a Lie algebroid structure induced by π . This is actually a particular instance of the Lie algebroid associated to a Dirac structure, which we will treat in the next subsection.

Lie algebroids can be alternatively defined as fiberwise linear Poisson structures on vector bundles or as differential graded manifolds of degree 1 (cf. e.g. [8]). In particular, there is a degree 1 differential (the Lichnerowicz differential, [9]) $d_A : \Gamma(\Lambda^\bullet A^*) \rightarrow \Gamma(\Lambda^{\bullet+1} A^*)$, where \bullet denotes an appropriate integer index. This differential is defined by

$$\begin{aligned} (d_A \eta)(\xi_1, \dots, \xi_{n+1}) &= \sum_i (-1)^{i+1} \rho(\xi_i)(\eta(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{n+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{n+1}) \end{aligned}$$

The differential satisfies $d_A^2 = 0$ and induces a cohomology, which is called Lie algebroid cohomology and denoted by $H^\bullet(A)$. The anchor ρ induces a morphism from the usual de Rham cohomology to it: $H_{dR}^\bullet(M) \rightarrow H^\bullet(A)$.

A Lie algebroid always induces a singular foliation on M : The subspace $\rho(A) \subset TM$ is always involutive and – by construction – locally finitely generated, hence the integrability theorem (cf. [10], reviewed in [11]) applies and M has a decomposition into immersed connected submanifolds $M = \sqcup_\alpha N_\alpha$ such that $TN_\alpha = \rho(A)|_{N_\alpha}$ for all N_α . Moreover, the bracket on A restricts to well-defined brackets on $A|_{N_\alpha}$, turning $A|_{N_\alpha} \rightarrow N_\alpha$ into Lie algebroids.

The submanifolds N_α are called leaves (of the foliation induced by the Lie algebroid) and a Lie algebroid is called transitive, if it has only one leaf, i.e. $\rho(A) = TM$ and M is connected.

1.2. The horizontal cohomology of Lie algebroid.

Definition 1.2. Let $A \xrightarrow{\rho} TM$ be a Lie algebroid over the smooth manifold M . We define:

- The subspace of ρ -horizontal forms at $m \in M$ as:

$$(\Lambda^\bullet A_m^*)^{hor} := \{\alpha \in \Lambda^\bullet A_m^* \mid \iota_v \alpha = 0 \ \forall v \in \ker(\rho_m : A_m \rightarrow T_m M)\}$$

- The subspaces of ρ -horizontal forms:

$$\Gamma(\Lambda^\bullet A^*)^{hor} = \{\alpha \in \Gamma(\Lambda^\bullet A^*) \mid \alpha_m \text{ and } (d_A \alpha)_m \text{ are horizontal for all } m\}$$

- the horizontal cohomology of A as the quotient

$$\mathcal{H}_{hor}^\bullet(A) = \frac{\ker(d_A : \Gamma(\Lambda^\bullet A^*)^{hor} \rightarrow \Gamma(\Lambda^{\bullet+1} A^*)^{hor})}{\text{Image}(d_A : \Gamma(\Lambda^{\bullet-1} A^*)^{hor} \rightarrow \Gamma(\Lambda^\bullet A^*)^{hor})}$$

Remark 1.3. Of course, there are natural maps $H_{dR}^\bullet(M) \rightarrow \mathcal{H}_{hor}^\bullet(A)$ and $\mathcal{H}_{hor}^\bullet(A) \rightarrow H^\bullet(A)$. In general, these maps are neither injective nor surjective, as we will see in the sequel.

Example 1.4. When A is a transitive Lie algebroid (i.e. $\rho(A) = TM$), then $\mathcal{H}_{hor}^\bullet(A)$ is isomorphic to the usual de Rham cohomology $H_{dR}^\bullet(M)$. More generally, if $\rho(A)$ is a regular foliation (i.e. if ρ has constant rank), then $\mathcal{H}_{hor}^\bullet(A)$ recovers the longitudinal cohomology of the foliation induced by $\rho(A)$.

The above example actually extends to the following:

Lemma 1.5. Let A be a Lie algebroid and $N \subset M$ a leaf of A and $\eta \in \Gamma((\Lambda^k A^*)^{hor})$ a ρ -horizontal form.

- (1) $\eta|_N$ is a ρ -horizontal k -form on the restricted Lie algebroid $A|_N \rightarrow N$, i.e. it induces a unique k -form $\eta_N \in \Omega^k(N)$.
- (2) η is completely determined by the collection $\{\eta_N \mid N \text{ leaf of } A\}$.
- (3) When η is horizontal, we have $(d_A \eta)_N = d\eta_N$.
- (4) Let $[\eta] = 0 \in \mathcal{H}_{hor}^k(A)$, then $[\eta_N] = 0 \in H_{dR}^k(N)$ for all leaves N of the algebroid A .

Remark 1.6. For $A = F$ the Lie algebroid of a regular foliation, $\mathcal{H}_{hor}^\bullet(A)$ is the longitudinal cohomology along the leaves of A , which must not be confused with the cohomology of the leaf space, or equivariant cohomology, that is the cohomology of forms which are basic with respect to the leaf space.

Remark 1.7. In view of the above example and remark, let us stress that the usual intuition related to the ‘‘horizontal’’ as parallel to the base should be applied very carefully since it is sometimes misleading.

This newly defined cohomology also should not be confused with basic or equivariant cohomology of algebroids (cf. [12, 13]).

1.3. Dirac structures. In this subsection, we briefly review the notion of Dirac structures. For details we refer to [14].

Let M be a manifold. The standard Courant algebroid (exact Courant algebroid with vanishing Ševera class) on M is given by $(\mathbb{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$, where $\mathbb{T}M$ is, as a vector bundle $TM \oplus T^*M$, $\langle \cdot, \cdot \rangle : \mathbb{T}M \otimes \mathbb{T}M \rightarrow \mathbb{R}$ is the standard symmetric pairing $\langle (v, \alpha), (w, \beta) \rangle = \alpha(w) + \beta(v)$ and $[\cdot, \cdot] : \Gamma(\mathbb{T}M) \otimes \Gamma(\mathbb{T}M) \rightarrow \Gamma(\mathbb{T}M)$ is the Courant bracket:

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha - \frac{1}{2}d(\beta(X) - \alpha(Y))).$$

This bracket is skew-symmetric but does not satisfy the Jacobi identity. There is an alternative definition of bracket (the Dorfman bracket), which satisfies the Jacobi identity, but is not skew symmetric. We are now prepared to give the central definition of a Dirac structure:

Definition 1.8. A $\dim(M)$ -dimensional subbundle $D \subset \mathbb{T}M$ is called Dirac structure, if it is isotropic (i.e. $\langle D, D \rangle = 0$) and involutive (i.e. $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$).

Let us look at some examples:

- Let $\omega \in \Omega_{cl}^2(M)$ be a closed 2-form. Then its graph $\Gamma_\omega = \{(v, \iota_v\omega) \mid v \in TM\}$ is a Dirac structure. Any Dirac structure with bijective anchor $D \rightarrow TM$ (i.e. the restriction of the projection $\mathbb{T}M \rightarrow TM$ to D is bijective) can be described by the graph of a closed 2-form.
- Let $\pi \in \Gamma(\Lambda^2 TM)$ be a Poisson structure. Its graph $\Gamma_\pi = \{(\iota_\alpha\pi, \alpha) \mid \alpha \in T^*M\}$ is a Dirac structure. Any Dirac structure with bijective projection $D \rightarrow T^*M$ can be described as a Poisson bivector.
- Let $F \subset TM$ be an involutive (regular) distribution and $F^\circ \subset T^*M$ its annihilator. Then $D = F \oplus F^\circ$ is a Dirac structure.

Remark 1.9. The closedness of the 2-form ω is essential for Γ_ω to be involutive. However, for a non-closed 2-form, we can consider a twisted Courant algebroid $(\mathbb{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_{d\omega})$ (with a twisting of the Courant bracket using $d\omega$) with respect to which Γ_ω is involutive, i.e. a (twisted) Dirac structure. In this article, we will only work with Dirac structures in the standard Courant algebroid.

Restricted to a Dirac structure D , the Courant bracket becomes a Lie bracket and turns D into a Lie algebroid. We call the (horizontal) Lie algebroid cohomology of D its (horizontal) Dirac cohomology. The Dirac structure also induces a canonical horizontal 2-cocycle that we will now describe.

1.4. The natural horizontal two-cocycle of a Dirac structure.

Let $D \subset \mathbb{T}M$ be a Dirac structure. We define $\omega_D \in \Gamma(\Lambda^2 D^*)$ by

$$\omega_D((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v).$$

As D is isotropic, we have $\omega_D((v, \alpha), (w, \beta)) = 2\alpha(w) = -2\beta(v)$, i.e. ω_D is horizontal at each point. A computation based on the involutivity of D ([15]), shows that ω_D is closed in Dirac cohomology, i.e. $d_D \omega_D = 0$, and hence ω_D is horizontal.¹ It thus yields a natural class in $\mathcal{H}_{hor}^2(D)$. Hence, in view of Lemma 1.5 we have:

Lemma 1.10. Let $D \subset \mathbb{T}M$ be a Dirac structure.

- (1) There is a naturally induced horizontal cocycle $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ associated to any Dirac structure D .
- (2) If $[\omega_D] = 0 \in \mathcal{H}_{hor}^2(D)$, then for any leaf N of D , $[(\omega_D)_N] = 0 \in H_{dR}^2(N)$.

Remark 1.11. The second statement above heavily relies on the fact that we work with the horizontal cohomology: Even when a primitive of ω_D in $\Gamma(\Lambda^\bullet D^*)$ exists, it has no reason to be horizontal (i.e. to restrict to leaves) in general.

Remark 1.12. The construction of the above 2-form works out, even when D is an *almost-Dirac structure*, i.e. an isotropic $\dim(M)$ -dimensional subbundle of $\mathbb{T}M$ (which might not be involutive). However, in this case D can fail to be a Lie algebroid and hence there is no associated Lie algebroid cohomology to lie in.

1.5. **Examples.** Let us interpret this class in the most important cases.

Example 1.13. $D = \Gamma_\omega \subset \mathbb{T}M$ is the graph of a (pre-)symplectic structure ω . Then the Lie algebroid structure on D is isomorphic to the Lie algebroid TM . Hence, the horizontal cohomology is canonically isomorphic to the de Rham cohomology ($\mathcal{H}_{hor}^\bullet(D) \cong H^\bullet(D) = H_{dR}^\bullet(M)$). The form ω_D corresponds to ω under this isomorphism.

Example 1.14. Let $D = F \times F^\circ \subset \mathbb{T}M$, where $F \subset TM$ is a regular foliation. The Lie algebroid structure on D is the induced bracket on F (the usual Lie bracket of vector fields) and the zero bracket on F° . Then $H^\bullet(D) \cong H^\bullet(F) \times \Lambda^\bullet(F^\circ)^*$ and $\mathcal{H}_{hor}^\bullet(D) \cong H^\bullet(F)$. The form ω_D is zero.

Remark 1.15. If $F \subset TM$ is not involutive, then $D = F \times F^\circ$ is still an almost-Dirac structure (cf. Remark 1.12). Even though there is no cohomology, the associated 2-form ω_D is still zero.

Example 1.16. Let $D \subset \mathbb{T}M$ be the graph of a Poisson structure π . Then the Lie algebroid D is isomorphic to T^*M and $H^\bullet(D) \cong H_\pi^\bullet(M)$

¹Note that from now on we write horizontal as a shorthand for ρ -horizontal, since the anchor map is no longer explicitly used.

is known as the Poisson cohomology (see for example [16]). The class of ω_D in $H^\bullet(D)$ corresponds to the class of π in $H_\pi^2(M)$. The class of ω in the finer cohomology $\mathcal{H}_{bas}^\bullet(D)$ is zero if and only if $\pi \in \mathfrak{X}^2(M)$ admits a primitive $E \in \mathfrak{X}(M)$ (a vector field E satisfying $L_E\pi = \pi$), which is tangent to the Poisson structure, i.e. is a section of $\rho(D) \subset TM$.

For instance for $M = \mathbb{R}^2$, the Poisson structure $\pi = x^2\partial_x \wedge \partial_y$ admits such a primitive $E = x\partial_x$.

Also, on each leaf N of D , ω_D restricts to a symplectic form ([15]). And hence there are no exact symplectic structures on compact manifolds, this leads to the following obstruction to the existence of horizontal primitives of ω_D : for a Poisson structure with vanishing cohomology class, the only compact leaves of its symplectic foliation are points.

Example 1.17. Here is a classical type of Poisson structures: let \mathfrak{g} be a Lie algebra. Its dual $M = \mathfrak{g}^*$ carries a natural Poisson structure, whose leaves are the coadjoint orbits of $G \curvearrowright \mathfrak{g}^*$ (cf. e.g. [17]).

In this case $[\pi] \in H_\pi^2(M) = H^2(D)$ is always zero: There exists a (linear) vector field E , such that $[\pi, E] = \pi$. However, E can rarely be chosen to be tangent to the coadjoint orbits. For instance, when \mathfrak{g} is compact and semi-simple, this can never occur.

2. A VARIATIONAL APPROACH TO EXACT DIRAC STRUCTURES

2.1. Dirac paths.

Theorem 2.1. Let $D \subset TM$ be a Dirac structure over M , $H \in C^\infty(M)$ be a Hamiltonian function and γ a path on M .

Assume that the horizontal 2-class $[\omega_D]$ vanishes, and let $\theta \in \Gamma(D^*)^{hor}$ be such that $d_D\theta = \omega_D$, then the following statements are equivalent:

- (i) The path γ is a Hamiltonian curve, i.e. $(\dot{\gamma}(t), dH_{\gamma(t)}) \in D$ for all t .
- (ii) All Dirac paths $\zeta : I \rightarrow D$ over γ (i.e. $\rho(\zeta) = \dot{\gamma}$) are critical points among the Dirac paths with the same end points of the following functional:

$$(1) \quad \zeta \mapsto \int_I (\theta_{\gamma(t)}(\zeta(t)) + H(\gamma(t))) dt$$

- (iii) There exists a Dirac path $\zeta : I \rightarrow D$ over γ and the latter is the critical point among Dirac paths with the same end points of the functional (1).

Proof. As $\theta(\zeta)$ does not depend on the choice of a Dirac path ζ over γ , the equivalence of (ii) and (iii) is obvious.

For the equivalence between (i) and (iii), let us first note that we can restrict the functional (1) to a fixed leaf N containing γ . On N , ω_D corresponds to a pre-symplectic form ω_N and θ yields a potential of ω_N , i.e. a 1-form $\theta_N \in \Omega^1(N)$ such that $d\theta_N = \omega_N$. Thus, it suffices

to prove the assertion of the theorem for an exact pre-symplectic form $\omega_N = d\theta_N$. That is to say, we want to prove, that $\omega_N^b(\dot{\gamma}) = dH_\gamma$ if and only if $\gamma : I \rightarrow N$ is a critical point of

$$\int_I (\theta_N|_{\gamma(t)}(\dot{\gamma}(t)) + H(\gamma(t))) dt$$

among curves with fixed endpoints. For symplectic manifolds this is a classical result [18], we provide a proof (that follows the same lines) for the sake of completeness.

Take a small variation

$$v : \begin{array}{ccc} I \times (-\epsilon, \epsilon) & \rightarrow & N \\ (t, s) & \mapsto & v(t, s) \end{array}$$

of γ with fixed endpoints and set $Y(t) = \frac{\partial}{\partial s}|_{s=0} v \in T_{\gamma(t)}N$. We call $\gamma^* L_Y \theta_N$ the smooth \mathbb{R} -valued function on I defined by

$$\gamma^* L_Y \theta_N(t) = \frac{\partial}{\partial t} \theta_N|_{\gamma(t)}(Y(t)) + d\theta_N(Y(t), \dot{\gamma}(t)).$$

(If Y happens to be the restriction to γ of a vector field Y on N , $\gamma^* L_Y \theta_N(t) dt \in \Omega^1(I)$ is the pull-back through $\gamma : I \rightarrow N$ of $L_Y \theta_N$, hence the notation.)

$$\begin{aligned} & \frac{\partial}{\partial s} \Big|_0 \int_{t \in I} \left(\theta_N|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \\ &= \int_{t \in I} \frac{\partial}{\partial s} \Big|_0 \left(\theta_N|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \\ &= \int_{t \in I} (\gamma^* L_Y \theta_N(t) + dH|_{\gamma(t)}(Y(t))) dt \\ &= \int_{t \in I} (d\theta_N(Y(t), \dot{\gamma}(t)) + dH|_{\gamma(t)}(Y(t))) dt \end{aligned}$$

In the two last equalities, we used integration by parts and the fact that $Y(0) = 0$ and $Y(1) = 0$ since the endpoints of v are fixed, so that $\int_I \frac{\partial}{\partial t} \theta_N|_{\gamma(t)}(Y(t)) dt = 0$. Since $Y(t)$ is arbitrary except at end points, this proves the result. \square

Remark 2.2. In the above proof, we actually do not require the equality $d\theta = \omega$ and the horizontality of θ globally. We only need both properties along L .

The above enables us to treat a much wider class of Dirac structures than in section 1.5, especially singular ones.

Example 2.3. Consider \mathbb{R}^4 with the Poisson structure $\pi = (x^2 + y^2)\partial_x \wedge \partial_y + \partial_z \wedge \partial_w$. Along the singular leaf $L = \{x = y = 0\}$ the vector field $w\partial_w$ is horizontal and a primitive for π , hence the above remark applies and we obtain a variational characterization of paths.

2.2. Generalized implicit Lagrangian systems. Let $L: TQ \rightarrow \mathbb{R}$ a (possibly degenerate) Lagrangian.

Remark 2.4. Note that in this section and in the next one the base manifold will be systematically Q instead of M used before – this is to stress the fact that in the Lagrangian picture it is necessary to lift the construction to double vector bundles, and thus the relevant Dirac structures will be over Q or T^*Q depending on the context. The general facts about Dirac structures will still be formulated with the base manifold denoted by M .

Definition 2.5. We call Tulczyjew’s differential the map $u \mapsto \mathcal{D}_u L := \beta(d_u L)$, where $\beta: T^*TQ \rightarrow T^*T^*Q$ is the Tulczyjew isomorphism (cf. Appendix A). Its image is a submanifold of T^*T^*Q .

Definition 2.6. We also define a map from TQ to T^*Q by $\mathbb{F}L(v)$ for every $v \in T_q Q$ by

$$\left. \frac{\partial}{\partial t} \right|_{t=0} L(v + tw) = \langle \mathbb{F}L(v), w \rangle$$

- a) We denote by $\mathfrak{Leg} = \mathbb{F}L(TQ) \subset T^*Q$ the image of $\mathbb{F}L$.
- b) We call partial vector fields on \mathfrak{Leg} sections² of $\Gamma(T(T^*Q))|_{\mathfrak{Leg}}$.
- c) An integral curve of a partial vector field X on \mathfrak{Leg} is a path $t \mapsto u_t \in TQ$ such that

$$\frac{d}{dt} \mathbb{F}L(u_t) = X_{\mathbb{F}L(u_t)}.$$

- d) An implicit Lagrangian system for an almost Dirac structure $\mathbb{D} \subset \mathbb{T}T^*Q$ is a pair (X, L) , with X a partially defined vector field on \mathfrak{Leg} , such that $(X(\mathbb{F}L(u)), \mathcal{D}_u L) \in \mathbb{D}$ for all u in TQ .

Remark 2.7. We do not assume partial vector fields on \mathfrak{Leg} to be tangent to \mathfrak{Leg} in any sense. Of course, if they are not tangent, they may have little integral curves.

Remark 2.8. Particular cases include:

- Usual Hamiltonian dynamics. When $\mathbb{F}L$ is a diffeomorphism and \mathbb{D} is the graph of the canonical symplectic form on T^*Q , implicit Lagrangian systems are pairs (X_H, L) , where X_H is the Hamiltonian vector field of H , the Hamiltonian function associated to L via the Legendre transform.
- Constraint dynamics, which actually motivated the construction, we give some details below: Example 2.15.

²For E a vector bundle over a manifold X and $Y \subset X$ an arbitrary subset (not necessarily a manifold), we denote by $\Gamma(E)|_Y$ restrictions to Y of smooth sections of E in a neighborhood of Y in X .

2.3. Implicit Lagrangian systems with magnetic terms.

Definition 2.9 ([15]). Let $D \subset \mathbb{T}M$ be a subbundle.

(1) For all $\phi: M' \rightarrow M$, we denote by $\phi^!D$ the set

$$\phi^!D_{m'} := \left\{ (X, \phi^*\beta) \text{ with } X \in T_{m'}M', \beta \in T_{\phi(m')}^*M \text{ s.t. } (\phi_*(X), \beta) \in D_{\phi(m')} \right\}$$

When D is an (almost-)Dirac structure we call $\phi^!D$ the pullback of D .

(2) Let ω be a 2-form $\omega \in \Omega^2(M)$, we denote by $e^\omega D$ the set

$$e^\omega D = \{(v, \beta + \iota_v \omega) \mid (v, \beta) \in D\}$$

and call it the gauge transform of D .

Lemma 2.10 (cf. e.g. [15]). Let $D \subset \mathbb{T}M$ be a Dirac structure and M' be a manifold.

(1) For any smooth map $\phi: M' \rightarrow M$, $\phi^!D$ is a Dirac structure on M' .

(2) For any closed 2-form $\omega \in \Omega^2(M)$, $e^\omega D$ is a Dirac structure on M .

Given $D \subset \mathbb{T}Q$ a Dirac structure on Q , this lemma allows to consider (i) its pull back $\pi^!D$ on T^*Q through the canonical base map $\pi: T^*Q \rightarrow Q$, then (ii) consider the gauge transformation $e^\Omega \pi^!D$ of this pull-back with respect to the canonical symplectic 2-form Ω .

Definition 2.11. Let $D \subset \mathbb{T}Q$ be a Dirac structure on Q . We call constrained magnetic Lagrangian system an implicit Lagrangian system for the Dirac structure $\mathbb{D} = e^\Omega \pi^!D \subset \mathbb{T}T^*Q$ as above.

Theorem 2.12. Let $D \subset \mathbb{T}Q$ be a Dirac structure and $L: TQ \rightarrow \mathbb{R}$ a Lagrangian. Assume that the 2-form $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ admits a horizontal primitive $\theta \in \Gamma(D^*)^{hor}$. Then for $q: I \rightarrow Q$ the following are equivalent:

a) There exists a Dirac path $\zeta: I \rightarrow D$ such that $\rho(\zeta) = \dot{q}$ which is the critical point among Dirac paths with the same end points of

$$(2) \quad \int_I (L(\rho(\zeta(t))) + \theta(\zeta(t))) dt.$$

b) For all $t \in I$, the following condition holds.

$$(3) \quad \left(\frac{\partial}{\partial t} \mathbb{F}L(\dot{q}(t)), \mathcal{D}_{\dot{q}(t)} L \right) \in \mathbb{D} = e^\Omega \pi^!D.$$

Proof. We claim that it suffices to check the equivalence on a small open set in Q . The second assertion is clearly local in nature and a path is a critical point for the functional (2) if and only if for any $t \in I$ there is a subinterval $t \in I_t \subset I$ on which it is critical.

It suffices therefore to establish the equivalence on an open subset U of Q , on which the Dirac structure takes the following normal form ([19]):

- $U = S \times N$, where $S \subset \mathbb{R}^a$, $N \subset \mathbb{R}^b$ with $q(0)$ lying in $S \times \{0\}$.
- $D|_U = e^\eta((TS \oplus \{0\}) \times \Gamma^\Pi)$, where $\eta \in \Omega_{cl}^2(U)$ and $\Pi \in \mathfrak{X}^2(N)$ is a Poisson bivector field vanishing at 0 and Γ^Π is the corresponding Dirac structure on N .

As $S \times \{0\}$ is a leaf of D , $q(t) \in S \times \{0\}$ for all t . By our assumptions on D

$$\int_I (L(\rho(\zeta(t))) + \theta(\zeta(t))) dt = \int_I (L(\dot{q}) + \theta^S(\dot{q})) dt,$$

where $\theta^S \in \Omega^1(S)$ is the one-form on S induced by θ and hence satisfies $d\theta = i_S^* \eta$ where $i_S : S \rightarrow U$, $i_S(s) = (s, 0)$.

We will denote by L^S the restriction of L to TS and write $q = (q^S, 0)$. With these conventions, the above functional reads:

$$\int_I (L^S(\dot{q}^S) + \theta^S(\dot{q}^S)) dt.$$

The classical Euler-Lagrange theorem ([20]) with magnetic term implies that being a critical point of this functional is equivalent to

$$(4) \quad \left(\frac{\partial}{\partial t} \mathbb{F}L^S(\dot{q}^S(t)), \mathcal{D}_{\dot{q}^S(t)} L^S \right) \in e^{\Omega} \pi^! e^{\eta^S} (TS \oplus \{0\}).$$

Via the isomorphism $\mathbb{T}T^*U = \mathbb{T}T^*S \times \mathbb{T}T^*N$, the assertion (3) decomposes as two conditions, the first one (on S) being (4). The second condition (on N) is always satisfied, as one can verify by a straightforward computation in local coordinates which relies on the fact that $\pi_N(q(t)) = 0$ for all t .

□

Corollary 2.13. Let Q, L, D be as in Theorem 2.12 and (X, L) an implicit Lagrangian system. Then any integral curve γ of X is the base path of a critical point of (2).

Example 2.14 (Classical symplectic magnetic terms). Let Q be any manifold, $\omega \in \Omega_{cl}^2(Q)$ and $D = \Gamma_\omega \subset \mathbb{T}Q$. Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. In this case $e^{\Omega} \pi^! D = \Gamma_{\Omega + \pi^* \omega} \subset \mathbb{T}T^*Q$.

As $\mathcal{H}_{basic}^\bullet(D) = H_{dR}^\bullet(M)$, (cf. Example 1.13), the 2-form on D admits a basic potential if and only if ω is de-Rham exact, i.e. $\omega = d\theta$, $\theta \in \Omega^1(Q)$.

Let us assume that the Legendre transform $\mathbb{F}L : TQ \rightarrow T^*Q$ is bijective, and denote the Legendre transform of the Lagrangian by H , i.e.

$$H(p) = \langle p, (\mathbb{F}L)^{-1}p \rangle - L \circ (\mathbb{F}L)^{-1}(p)$$

In this case $\mathcal{D}L$ is simply dH . Theorem 2.12 yields that the critical points of $L(q, \dot{q}) + \theta(\dot{q})$ correspond under the Legendre transform to integral curves of the Hamiltonian flow of H for the symplectic structure $\Omega + \pi^*\omega$. Corollary 2.13 states that (X, dH) is an implicit Lagrangian system with respect to $e^\Omega\pi^!D$ if and only if the vector field X is the Hamiltonian vector field of H with respect to $\Omega + \pi^*\omega$.

Example 2.15 (Holonomic constraints as a regular foliation). Let $F \subset TQ$ be a regular foliation. As discussed in Example 1.14, the Dirac structure $D = F \oplus F^\circ$ always admits a horizontal primitive, as the 2-form in $\Lambda^2 D^*$ is zero (there is no magnetic term). Then $\pi^!D$ is the Dirac structure associated to the pullback foliation $\pi^{-1}(F)$ and

$$e^\Omega\pi^!D = \{(w, \alpha) \in TT^*Q \oplus T^*T^*Q \mid \pi_*(w) \in F, \alpha - \Omega^b w \in \pi^{-1}(F)^\circ\}$$

Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. Then Theorem 2.12 and Corollary 2.13 yield that the integral curves of any implicit Lagrangian system $(X, \mathcal{D}L)$ for $e^\Omega\pi^!D$ are critical points of L among curves that are tangent to F . The condition (3) translates directly to the Euler-Lagrange equations for a system subject to holonomic constraints, which are classically spelled-out using the Lagrange multipliers [21].

Remark 2.16. Holonomic and non-holonomic constraints. Note that the result above concerns the so-called holonomic constraints, i.e. the conditions defining the constraints do not depend essentially on the velocities of the system. Geometrically this means that the foliation F comes from an integrable constraint distribution $\Delta \subset TQ$. Simple mechanical examples and counterexamples can be constructed by “rolling without slipping” problems: They are often formulated as an orthogonality condition on the velocity at the contact point – the condition is integrable for the rolling disk but not for a rolling ball. Under some extra assumptions the non-holonomic constraints can still be treated in the variational approach ([22]), though with no geometric interpretation. In this setting our result is more subtle, since as mentioned above, remarks 1.12 and 1.15, the non-integrable almost Dirac structures are very different from the cohomological perspective. Formally, we cannot speak of an obstruction class, since the “differential” does not square to zero. However when some primitive can be defined, parts of theorems 2.1 and 2.12 are still valid.

2.4. Applications to numerics. One of the motivations for the above construction is its potential application to design appropriate structure preserving numerical methods – so called geometric integrators.

Historically, the first example of those are the symplectic numerical methods, they are known since several decades, and are now state of the art for Hamiltonian systems ([23]). The key idea is that in the continuous setting the Hamiltonian flow not only preserves the level sets of the Hamiltonian function, but also leaves invariant the symplectic

form. It is thus natural to mimic this property for the discrete flow, i.e. computing the trajectory numerically one wants to take the symplectic form into account. And since it is actually the same symplectic form that defines the dynamics of the system, one can reverse the argument: a flow preserving the symplectic form will “respect” the level sets of the Hamiltonian defining it.

The Lagrangian counterpart of this picture is related to so-called variational integrators ([24]), the idea is rather natural as well. Instead of considering a continuous Lagrangian and searching for its extrema along all the paths with fixed endpoints:

$$(5) \quad \inf \mathcal{L} \equiv \inf \int_0^T L(q(t), \dot{q}(t)) dt$$

one defines the discrete version L_d of the integrant L as follows:

$$L_d(q_{n+1}, q_n, v_n) := \Delta t_n L(q_n, v_n).$$

Here $q_n \equiv q(t_n)$, v_n is some approximation of $\dot{q}(t_n)$ depending on q_n and q_{n+1} ; and Δt_n are the time intervals between q_n and q_{n+1} , not necessarily all equal. One then defines the discrete analogue of variational principle (DVP), i.e. studies the trajectories $(q_0, q_1 \dots, q_{n-1}, q_n)$ extremizing

$$\mathcal{L}_d = \sum_{n=0}^N L_d,$$

subject to $q_0 = q(0)$ and $q_N = q(T)$. For conservative mechanical systems one can recover usual symplectic methods with this variational approach, and it is actually more universal, since the timestep is allowed to vary as well.

A similar strategy can be applied whenever the variational principle can be formulated. For example, in [24] the case of systems with constraints is explored, which motivated some parts of this paper; later on similar ideas were explored for continuous media problems (see e.g. [25]) Hence, the results of Sections 2.2 and 2.3 on the dynamics on Dirac structures fit to the picture perfectly: they basically say that as soon the cohomological obstruction is absent, one can formulate the Dirac dynamics with a variational approach. In Equation (5) one merely replaces the path $q(t)$ in the configuration manifold by a Dirac path $\zeta(t)$. In the continuous setting the Dirac paths preserve the Dirac structure by definition, the variational formulation permits to guarantee this property for the trajectory computed numerically.

There is however an important detail to mention: the folkloric perception of geometric integrators as “preservation of the geometric structure guarantees preservation of physical properties” is slightly simplified. For instance in the symplectic case, it is not the original Hamiltonian that is preserved, but its discrete version, for which one can estimate the difference [26]. The phenomenon is even more subtle in the variational case. In fact, saying that satisfying the discrete variational

principle (DVP) results in preserving some quantities of the system is no longer that straightforward. In the generic case the DVP will only give the relations between different variables of the system, but they will still depend on the choice of discretization or approximation of some of them. It may (and often does) also happen that the choice of the discretization to preserve the structure exactly is technically very difficult or even a priori impossible. This means that the correct statements will concern rather preservation of geometric structures up to some order of discretization step.

A typical example of this situation is provided by the so-called constraint algorithms: for dynamical systems, the methods to take into account the constraints expressed as algebraic conditions on dynamical variables. When it is impossible to explicitly resolve the constraints, i.e. introduce the dynamical variables satisfying them automatically, there are essentially two approaches: introduce the penalization terms with Lagrange multipliers and discretize them appropriately or “project” the solution to the level set of the algebraic conditions at each time step. However, to the best of our knowledge, there are very few proven theorems on how the discrete version of the system satisfies the constraints. We have tried to fill some gaps in empirical observations that one sees in literature. For example ([27]) the Dirac structure based algorithm ([22]) in the absence of constraints is naturally symplectic. And some partial results on how to construct *pseudo-geometric* integrators preserving the conditions *up to some order* are given in [28].

With the approach of the current paper we now understand why the naive attempts to increase the order of the constraint-based methods (like e.g. [29]) do not produce the desired results: roughly speaking the obtained integrators fail to be geometric/variational in the proper sense of the word. A way out would be to formulate the DVP for the calculus of variations in a more general case ([20]), and then apply it to the context of Theorems 2.1 and 2.12.

PERSPECTIVES.

In this paper we have defined the basic Dirac cohomology, which permits to describe an explicit and verifiable condition for variational formulation of dynamics on Dirac structures.

As mentioned in the last part of the paper, on top of purely mathematical interest, this construction should be useful to design more reliable tools for numerical integration of the flow of dynamical systems on Dirac structures. Those in turn naturally appear when studying constraint, interacting or dissipative mechanical systems, which are not in the range of classical Hamiltonian formalism. We expect the results of this paper to provide a unified approach to those and in particular an extension of the observations from Section 2.4 to arbitrary Dirac structures.

Let us also mention that, since Poisson manifolds provide an example of Dirac structures, this approach is useful to construct some Poisson integrators. In the context of this paper there is no conceptual difference between the Dirac structures coming from constraint distributions or from symplectic foliations of Poisson manifolds. The constructed discretizations should thus preserve the symplectic leaves. This is somewhat complementary to the strategy of [30], where the main tool is rather Hamiltonian dynamics and symplectic groupoids.

We suppose that these questions are somewhat technical and interest a more applied community than the audience of this journal, hence we intend to devote a separate paper ([31]) to the description of the discrete variational principal for the general case and the related discussion of the implementation issues.

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APPENDIX A. TULCZYJEW ISOMORPHISM(S)

For self-containedness of this paper, we recall here the isomorphisms established by W. Tulczyjew ([5, 6]) between double (co)tangent bundles (at least one “co” should be present). The most non-trivial one is

$$\kappa: TT^*Q \rightarrow T^*TQ,$$

the construction works as follows: start with the double vector bundle TTQ ,

denote $TQ \xrightarrow{p_1} Q$ and $T(TQ) \xrightarrow{p_2} TQ$ with the respective duality pairings $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. There is a canonical flip $\sigma: TTQ \rightarrow TTQ$, then the mapping κ is (implicitly but canonically) defined by imposing

$$\langle \kappa(a), b \rangle_2 = \langle a, \sigma(b) \rangle_1$$

Remark A.1. This flip can be seen as a sort of Schwarz Lemma: for every smooth map $\Sigma(s, t) := \mathbb{R}^2 \rightarrow Q$ (defined in a neighborhood of $(0, 0)$):

- (1) $t \mapsto \frac{\partial \Sigma(0,t)}{\partial s}$ is a path in TQ starting from $\frac{\partial \Sigma}{\partial s}(0,0)$. Its first jet at 0 belongs to $T_{\frac{\partial \Sigma}{\partial s}(0,0)}(TQ)$.
- (2) $s \mapsto \frac{\partial \Sigma(s,0)}{\partial t}$ is a path in TQ starting from $\frac{\partial \Sigma}{\partial t}(0,0)$. Its first jet at 0 belongs to $T_{\frac{\partial \Sigma}{\partial t}(0,0)}(TQ)$.

The canonical flip exchanges both.

All the others are obtained by post- or pre-composing with $\Omega^b: TT^*Q \rightarrow T^*T^*Q$ or its inverse, where Ω is the canonical symplectic form on T^*Q .

We will be mostly interested in the isomorphism

$$\beta \equiv \omega^b \circ \kappa^{-1}: T^*TQ \rightarrow T^*T^*Q$$

which is a particular case of the canonical isomorphism also called Tulczyjew isomorphism $T^*E \simeq T^*E^*$ for any vector bundle E ([32]).

APPENDIX B. LEGENDRE TRANSFORMATION

Throughout this section, E is a vector bundle over Q equipped with a smooth function $L: U \subset E \rightarrow \mathbb{R}$ called *Lagrangian*.

Recall that if a smooth function f on an open convex subset $U \subset V$ of a vector space is strictly convex, then its differential, defined for all $v \in U$ by:

$$\mathbb{F}f: V \rightarrow V^*$$

$$v \mapsto \left(e \mapsto \left. \frac{d}{dt} \right|_{t=0} f(v + te) \right)$$

is a diffeomorphism from U onto its image. If the restriction of L to any fiber is strictly convex, then $\mathbb{F}L: U \subset E \hookrightarrow E^*$ is a diffeomorphism from U to its image U' .

We define the Legendre transform $H \in C^\infty(U')$ of L to be the unique function satisfying $H(\alpha) + L(v) = \langle \alpha, v \rangle$ for all $\alpha \in E^*$ with $\alpha = \mathbb{F}L(v)$.

Proposition B.1. The Legendre transform and the Tulczyjew isomorphism are related by the equality

$$\beta(d_e L) = d_{\mathbb{F}L(v)} H.$$

When L is not strictly convex, the Legendre transform need not exist, however the set $\{\beta(d_e L) \mid e \in U\}$ is a Lagrangian submanifold of T^*E^* . It appears throughout the text as the image of $\mathcal{D}L$.

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