



On Dirac Structures Admitting A Variational Approach

Oscar Cosserat, Alexei Kotov, Camille Laurent-Gengoux, Leonid Ryvkin,
Vladimir Salnikov

► To cite this version:

Oscar Cosserat, Alexei Kotov, Camille Laurent-Gengoux, Leonid Ryvkin, Vladimir Salnikov. On Dirac Structures Admitting A Variational Approach. 2021. hal-03445074v1

HAL Id: hal-03445074

<https://hal.science/hal-03445074v1>

Preprint submitted on 23 Nov 2021 (v1), last revised 19 Dec 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON DIRAC STRUCTURES ADMITTING A VARIATIONAL APPROACH

OSCAR COSSERAT, ALEXEI KOTOV, CAMILLE LAURENT-GENGOUX,
LEONID RYVKIN, AND VLADIMIR SALNIKOV

ABSTRACT. We discuss the notion of basic cohomology for Dirac structures and, more generally, Lie algebroids. We then use this notion to characterize the obstruction to a variational formulation of Dirac dynamics.

CONTENTS

Introduction / motivation	1
1. The basic cohomology of Lie algebroids and Dirac structures	2
1.1. Lie algebroids	2
1.2. The basic cohomology of Lie algebroid	3
1.3. Dirac structures	4
1.4. The natural basic two-cocycle of a Dirac structure	5
1.5. Examples	6
2. A variational approach to exact Dirac structures	7
2.1. Dirac paths	7
2.2. Generalized implicit Lagrangian systems.	8
2.3. Implicit Lagrangian systems with magnetic terms	9
2.4. Applications to numerics	11
Perspectives.	13
Appendix A. Tulczyjew isomorphism(s)	14
Appendix B. Legendre transformation	14
References	15

INTRODUCTION / MOTIVATION

This paper is a part of series of works by the authors (in various combinations) concerning rather broad subjects of “geometrizing mechanics” and “geometric integrators”. They include attempts to spell-out the underlying geometric structures for a large class of mechanical systems, like it was done decades ago with symplectic structures for Hamiltonian (conservative) systems. They also address the question of application of these geometric construction to design reliable simulation tools for respective classes of mechanical systems.

The precise question we ask ourselves in this paper is mostly motivated by the results of [1], where the appearance of (almost) Dirac structures for mechanical systems with constraints is discussed. It can be vaguely formulated as: “given a Dirac structure, what else do we need to know to define meaningful dynamics on it”. More precisely, we study the obstructions for a system arising from a Dirac structure to admit a variational (Lagrangian) formulation.

And for good cases, when this obstruction is absent, we explain how a Lagrangian is constructed. This includes some classes of Poisson structures, for which a variational formulation of Hamiltonian mechanics becomes possible. This is also an important step to the construction of Dirac-structure-preserving numerical methods, since having constructed the Lagrangian, under some assumptions, one can profit from the well-established machinery of variational numerical methods.

The paper is organized as follows: We start by recalling some notions of Lie algebroids and give the definition of their cohomology. Then, we describe the main geometric tool – cohomology of Dirac structures, providing some examples. In the second part we explain the relation of this Dirac cohomology and the obstructions to construct a variational formulation for the dynamics on the Dirac structures. We illustrate the construction on some examples and non-examples. To conclude, we explain some ideas about variational integrators and possible application of those in our setting – this is a separate rich topic that we intend to elaborate in another more “mechanically oriented” paper.

1. THE BASIC COHOMOLOGY OF LIE ALGEBROIDS AND DIRAC STRUCTURES

1.1. Lie algebroids. The notion of Lie algebroids is a simultaneous generalization of tangent bundles and Lie algebras. In this subsection we briefly review the relevant notions. We refer to [2] for a detailed account.

Definition 1.1. Let M be a smooth manifold. A Lie algebroid $(A, \rho, [\cdot, \cdot])$ is given by a finite-dimensional vector bundle A , a vector bundle morphism $\rho : A \rightarrow TM$, called anchor and a (\mathbb{R} -bilinear) Lie bracket on the sections of A

$$[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$$

satisfying for all $f \in C^\infty(M)$, $s, s' \in \Gamma(A)$:

$$[s, fs'] = f[s, s'] + \rho(s)(f) \cdot s'.$$

It can be shown that the above condition implies that $\rho_* : \Gamma(A) \rightarrow \Gamma(TM) = \mathfrak{X}(M)$ is a Lie algebra homomorphism. Lie algebroids appear in many different settings:

- The tangent bundle TM with its usual bracket and $\rho = id$ is a Lie algebroid.

- Let $F \subset TM$ be an involutive subbundle, i.e. a foliation. Then F is a Lie algebroid with the restricted bracket and the inclusion $F \rightarrow TM$ as anchor.
- Let \mathfrak{g} be a Lie algebra and $v : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ an infinitesimal action (i.e. a Lie algebra homomorphism). Then $\mathfrak{g} \times M$ is a Lie algebroid with bracket induced by the Lie bracket on \mathfrak{g} and anchor $\rho(\xi, p) = v(\xi)(p)$. In particular Lie algebras can be seen as Lie algebroids over a point.
- Let $\pi \in \Gamma(\Lambda^2 T^*M)$ be a Poisson bivector. Then the cotangent bundle carries a Lie algebroid structure induced by π . This is actually a particular instance of the Lie algebroid associated to a Dirac structure, which we will treat in the next subsection.

Lie algebroids can be alternatively defined as fiberwise linear Poisson structures on vector bundles or as differential graded manifolds of degree 1 (cf. e.g. [3]). In particular, there is a degree 1 differential (the Lichnerowicz differential, [4]) $d_A : \Gamma(\Lambda^\bullet A^*) \rightarrow \Gamma(\Lambda^\bullet A^*)$, defined by

$$(d_A \eta)(\xi_1, \dots, \xi_{n+1}) = \sum_i (-1)^{i+1} \rho(\xi_i)(\eta(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{n+1})) \\ + \sum_{i < j} (-1)^{i+j} \eta([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{n+1})$$

The differential satisfies $d_A^2 = 0$ and induces a cohomology, which is called Lie algebroid cohomology and denoted by $H^\bullet(A)$. The anchor ρ induces a morphism $H_{dR}^\bullet(M) \rightarrow H^\bullet(A)$.

A Lie algebroid always induces a singular foliation on M : The subspace $\rho(A) \subset TM$ is always involutive and - by construction - locally finitely generated, hence the integrability theorem (cf. [5]) applies and M has a decomposition into immersed connected submanifolds $M = \sqcup_\alpha L_\alpha$ such that $TL_\alpha = \rho(A)|_{L_\alpha}$ for all L_α . Moreover, the bracket on A restricts to well-defined brackets on $A|_{L_\alpha}$, turning $A|_{L_\alpha} \rightarrow L_\alpha$ into Lie algebroids.

The submanifolds L_α are called leaves (of the foliation induced by the Lie algebroid) and a Lie algebroid is called transitive, if it has only one leaf, i.e. $\rho(A) = TM$ and M is connected.

1.2. The basic cohomology of Lie algebroid.

Definition 1.2. Let $A \xrightarrow{\rho} TM$ be a Lie algebroid over the smooth manifold M . We define:

- The subspace of ρ -horizontal forms at $m \in M$ as:
 $(\Lambda^\bullet A_m^*)^{hor} := \{\alpha \in \Lambda^\bullet A_m^* \mid \iota_v \alpha = 0 \ \forall v \in \ker(\rho_m : A_m \rightarrow T_m M)\}$
- The subspaces of ρ -basic forms:

$$\Gamma(\Lambda^\bullet A^*)^{bas} = \{\alpha \in \Gamma(\Lambda^\bullet A^*) \mid \alpha_m \text{ and } (d_A \alpha)_m \text{ are horizontal for all } m\}$$

- the basic cohomology of A as the quotient

$$\mathcal{H}_{bas}^\bullet(A) = \frac{\ker(d_A : \Gamma(\Lambda^\bullet A^*)^{bas} \rightarrow \Gamma(\Lambda^\bullet A^*)^{bas})}{\text{Image}(d_A : \Gamma(\Lambda^\bullet A^*)^{bas} \rightarrow \Gamma(\Lambda^\bullet A^*)^{bas})}$$

Remark 1.3. Of course, there are natural maps $H_{dr}^\bullet(M) \rightarrow \mathcal{H}_{bas}^\bullet(A)$ and $\mathcal{H}_{bas}^\bullet(A) \rightarrow H^\bullet(A)$. In general, these maps are neither injective nor surjective, as we will see in the sequel.

Example 1.4. When A is a transitive Lie algebroid (i.e. $\rho(A) = TM$), then $H_{bas}^\bullet(A)$ is isomorphic to the usual de Rham cohomology $H_{dR}^\bullet(M)$. More generally, if $\rho(A)$ is a regular foliation (i.e. if ρ has constant rank), then $H_{bas}^\bullet(A)$ recovers the longitudinal cohomology of the foliation induced by $\rho(A)$.

The above example actually extends to the following:

Lemma 1.5. Let A be an algebroid and $L \subset M$ a leaf of A and $\eta \in \Gamma((\Lambda^k A^*)^{hor})$ a ρ -horizontal form.

- (1) $\eta|_L$ is a horizontal k -form on the restricted Lie algebroid $A|_L \rightarrow L$, i.e. it induces a unique k -form $\eta_L \in \Omega^k(L)$.
- (2) η is completely determined by the collection $\{\eta_L \mid L \text{ leaf of } A\}$.
- (3) When η is basic, we have $(d_A \eta)_L = d\eta_L$.
- (4) Let $[\eta] = 0 \in \mathcal{H}_{bas}^k(A)$, then $[\eta_L] = 0 \in H_{dR}^k(L)$ for all leaves L of the algebroid A .

Remark 1.6. For $A = F$ the Lie algebroid of a regular foliation, $\mathcal{H}_{bas}^\bullet(A)$ is the longitudinal cohomology along the leaves of A , which must not be confused with the cohomology of the leaf space, or equivariant cohomology, that is the cohomology of forms which are basic with respect to the leaf space.

Remark 1.7. In view of the above example and remark, let us stress that the usual intuition related to the “basic” as coming from the base should be applied very carefully since it is sometimes misleading.

1.3. Dirac structures. In this subsection, we briefly review the notion of Dirac structures. For details we refer to [6].

Let M be a manifold. The standard Courant algebroid (exact Courant algebroid with vanishing Ševera class) on M is given by $(\mathbb{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$, where $\mathbb{T}M$ is, as a vector bundle $TM \oplus T^*M$, $\langle \cdot, \cdot \rangle : \mathbb{T}M \otimes \mathbb{T}M \rightarrow \mathbb{R}$ is the standard symmetric pairing $\langle (v, \alpha), (w, \beta) \rangle = \alpha(w) + \beta(v)$ and $[\cdot, \cdot] : \Gamma(\mathbb{T}M) \otimes \Gamma(\mathbb{T}M) \rightarrow \Gamma(\mathbb{T}M)$ is the Courant bracket:

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha)).$$

This bracket is skew-symmetric but does not satisfy the Jacobi identity. There is an alternative definition of bracket (the Dorfman bracket),

which satisfies the Jacobi identity, but is not skew symmetric. We are now prepared to give the central definition of a Dirac structure:

Definition 1.8. A $\dim(M)$ -dimensional subbundle $D \subset \mathbb{T}M$ is called Dirac structure, if it is isotropic (i.e. $\langle D, D \rangle = 0$) and involutive (i.e. $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$).

Let us look at some examples:

- Let $\omega \in \Omega_{cl}^2(M)$ be a closed 2-form. Then its graph $\Gamma_\omega = \{(v, \iota_v \omega) \mid v \in TM\}$ is a Dirac structure. Any Dirac structure with bijective anchor $D \rightarrow TM$ (i.e. the restriction of the projection $\mathbb{T}M \rightarrow TM$ to D is bijective) can be described by the graph of a closed 2-form.
- Let $\pi \in \Gamma(\Lambda^2 TM)$ be a Poisson structure. Its graph $\Gamma_\omega = \{(\iota_\alpha \pi, \alpha) \mid \alpha \in T^*M\}$ is a Dirac structure. Any Dirac structure with bijective projection $D \rightarrow T^*M$ can be described as the a Poisson bivector.
- Let $F \subset TM$ be an involutive (regular) distribution and $F^\circ \subset T^*M$ its annihilator. Then $D = F \oplus F^\circ$ is a Dirac structure.

Remark 1.9. The closedness of the 2-form ω is essential for Γ_ω to be involutive. However, for a non-closed 2-form, we can consider a twisted Courant algebroid $(\mathbb{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_{d\omega})$ (with a twisting of the Courant bracket using $d\omega$) with respect to which Γ_ω is involutive, i.e. a (twisted) Dirac structure. In this article, we will only work with Dirac structures in the standard Courant algebroid.

Restricted to a Dirac structure D , the Courant bracket becomes a Lie bracket and turns D into a Lie algebroid. We call the (basic) Lie algebroid cohomology of D its (basic) Dirac cohomology. The Dirac structure also induces a canonical basic 2-cocycle that we will now describe.

1.4. The natural basic two-cocycle of a Dirac structure. Let $D \subset \mathbb{T}Q$ be a Dirac structure. We define $\omega_D \in \Gamma(\Lambda^2 D^*)$ by

$$\omega_D((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v).$$

As D is isotropic, we have $\omega_D((v, \alpha), (w, \beta)) = 2\alpha(w) = -2\beta(v)$, i.e. ω_D is horizontal. A computation based on the involutivity of D ([7]), shows that ω_D is closed in Dirac cohomology, i.e. $d_D \omega_D = 0$, and hence ω_D is basic. It thus yields a natural class in $\mathcal{H}_{bas}^2(D)$. Hence, in view of Lemma 1.5 we have:

Lemma 1.10. Let $D \subset \mathbb{T}M$ be a Dirac structure.

- (1) There is a naturally induced basic cocycle $\omega_D \in \Gamma(\Lambda^2 D^*)^{bas}$ associated to any Dirac structure D .
- (2) If $[\omega_D] = 0 \in \mathcal{H}_{bas}^2(D)$, then for any leaf L of D , $[(\omega_D)_L] = 0 \in H_{dR}^2(L)$.

Remark 1.11. The second statement above heavily relies on the fact that we work with the basic cohomology: Even when a primitive of ω_D in $\Gamma(\Lambda^\bullet D^*)$ exists, it has no reason to be horizontal (i.e. to restrict to leaves) in general.

Remark 1.12. The construction of the above 2-form works out, even when D is an *almost-Dirac structure*, i.e. an isotropic $\dim(M)$ -dimensional subbundle of TM (which might not be involutive). However, in this case D can fail to be a Lie algebroid and hence there is no associated Lie algebroid cohomology to lie in.

1.5. Examples. Let us interpret this class in the most important cases.

Example 1.13. $D = \Gamma_\omega \subset TM$ is the graph of a (pre-)symplectic structure ω . Then the Lie algebroid structure on D is isomorphic to the Lie algebroid TM . Hence, the basic cohomology is canonically isomorphic to the de Rham cohomology ($\mathcal{H}_{bas}^\bullet(D) \cong H^\bullet(D) = H_{dR}^\bullet(M)$). The form ω_D corresponds to ω under this isomorphism.

Example 1.14. Let $D = F \times F^\circ \subset TM$, where $F \subset TM$ is a regular foliation. The Lie algebroid structure on D is the induced bracket on F (the usual Lie bracket of vector fields) and the zero bracket on F° . Then $H^\bullet(D) \cong H^\bullet(F) \times \Lambda^\bullet(F^\circ)^*$ and $\mathcal{H}_{bas}^\bullet(D) \cong H^\bullet(F)$. The form ω_D is zero.

Remark 1.15. If $F \subset TM$ is not involutive, then $D = F \times F^\circ$ is still an almost-Dirac structure (cf. Remark 1.12). Even though there is no cohomology, the associated 2-form ω_D is still zero.

Example 1.16. Let $D \subset TM$ be the graph of a Poisson structure π . Then the Lie algebroid D is isomorphic to T^*M and $H^\bullet(D) \cong H_\pi^\bullet(M)$ is known as the Poisson cohomology. The class of ω_D in $H^\bullet(D)$ corresponds to the class of π in $H_\pi^2(M)$. The class of ω in the finer cohomology $\mathcal{H}_{bas}^\bullet(D)$ is zero if and only if $\pi \in \mathfrak{X}^2(M)$ admits a primitive $E \in \mathfrak{X}(M)$, which is tangent to the Poisson structure, i.e. is a section of $\rho(D) \subset TM$.

In particular for $M = \mathbb{R}^2$, the Poisson structure $\pi = x^2 \partial_x \wedge \partial_y$ admits such a primitive $E = x \partial_x$.

For Poisson structures, on each leaf L of D , ω_D restricts to a symplectic form ([7]). By the above discussion, this leads to the following obstruction to the existence of basic primitives of ω_D :

Lemma 1.17. If the basic cohomology class of a Poisson structure vanishes, then the only compact leaves of its foliation have dimension 0.

Let \mathfrak{g} be a Lie algebra. Its dual $M = \mathfrak{g}^*$ carries a natural Poisson structure, whose leaves are the coadjoint orbits of $G \curvearrowright \mathfrak{g}^*$ (cf. e.g. [8]).

In this case $[\pi] \in H_\pi^2(M) = H^2(D)$ is always zero: There exists a (linear) vector field E , such that $[\pi, E] = \pi$. However, E can rarely be chosen to be tangent to the coadjoint orbits. For instance, when \mathfrak{g} is compact and semi-simple, this can never occur.

2. A VARIATIONAL APPROACH TO EXACT DIRAC STRUCTURES

2.1. Dirac paths.

Theorem 2.1. Let $D \subset \mathbb{T}M$ be a Dirac structure over M , $H \in C^\infty(M)$ be a Hamiltonian function and γ a path on M .

Assume that the basic 2-class $[\omega_D]$ vanishes, and let $\theta \in \Gamma(D^*)^{hor}$ be such that $d_D\theta = \omega_D$, then the following statements are equivalent:

- (i) The path γ is a Hamiltonian curve, i.e. $(\dot{\gamma}(t), dH_{\gamma(t)}) \in D$ for all t .
- (ii) All Dirac paths $\zeta : I \rightarrow D$ over γ (i.e. $\rho(\zeta) = \dot{\gamma}$) are critical points among the Dirac paths with the same end points of the following functional:

$$(1) \quad \zeta \mapsto \int_I (\theta_{\gamma(t)}(\zeta(t)) + H(\gamma(t))) dt$$

- (iii) There exists a Dirac path $\zeta : I \rightarrow D$ over γ and the latter is the critical point among Dirac paths with the same end points of the functional (1).

Proof. As $\theta(\zeta)$ does not depend on the choice of a Dirac path ζ over γ , the equivalence of (ii) and (iii) is obvious.

For the equivalence between (i) and (iii), let us first note that we can restrict the functional (1) to a fixed leaf L containing γ . On L , ω_D corresponds to a pre-symplectic form ω_L and θ yields a potential of ω_L , i.e. a 1-form $\theta_L \in \Omega^1(L)$ such that $d\theta_L = \omega_L$. Thus, it suffices to prove the assertion of the theorem for an exact pre-symplectic form $\omega_L = \theta_L$. That is to say, we want to prove, that $\omega_L^b(\dot{\gamma}) = dH_\gamma$ if and only if $\gamma : I \rightarrow L$ is a critical point of

$$\int_I (\theta_L|_{\gamma(t)}(\dot{\gamma}(t)) + H(\gamma(t))) dt$$

among curves with fixed endpoints. For symplectic manifolds this is a classical result [9], we provide a proof (that follows the same lines) for the sake of completeness.

Take a small variation

$$v : \begin{array}{ccc} I \times (-\epsilon, \epsilon) & \rightarrow & M \\ (t, s) & \mapsto & v(t, s) \end{array}$$

of γ with fixed endpoints and set $Y(t) = \frac{\partial}{\partial s}|_{s=0} v \in T_{\gamma(t)}L$. We call $\gamma^*L_Y\theta_L$ the smooth \mathbb{R} -valued function on I defined by

$$\gamma^*L_Y\theta_L(t) = \frac{\partial}{\partial t}\theta_L|_{\gamma(t)}(Y(t)) + d\theta_L(Y(t), \dot{\gamma}(t)).$$

(If Y happens to be the restriction to γ of a vector field Y on L , $\gamma^*L_Y\theta_L(t)dt \in \Omega^1(I)$ is the pull-back through $\gamma: I \rightarrow L$ of $L_Y\theta_L$, hence the notation.)

$$\begin{aligned} & \frac{\partial}{\partial s}\Big|_0 \int_{t \in I} \left(\theta_L|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \\ &= \int_{t \in I} \frac{\partial}{\partial s}\Big|_0 \left(\theta_L|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \\ &= \int_{t \in I} (\gamma^*L_Y\theta_L(t) + dH|_{\gamma(t)}(Y(t))) dt \\ &= \int_{t \in I} (d\theta_L(Y(t), \dot{\gamma}(t)) + dH|_{\gamma(t)}(Y(t))) dt \end{aligned}$$

In the two last equalities, we used integration by parts and the fact that $Y(0) = 0$ and $Y(1) = 0$ since the endpoints of v are fixed, so that $\int_I \frac{\partial}{\partial t}\theta_L|_{\gamma(t)}(Y(t))dt = 0$. Since $Y(t)$ is arbitrary except at end points, this proves the result. \square

2.2. Generalized implicit Lagrangian systems. Let $L: TQ \rightarrow \mathbb{R}$ a (possibly degenerate) Lagrangian.

Definition 2.2. We call Tulczyjew's differential the map $u \mapsto \mathcal{D}_u L := \beta(d_u L)$, where $\beta: T^*TQ \rightarrow T^*T^*Q$ is the Tulczyjew isomorphism (cf. Appendix A). Its image is a submanifold of T^*T^*Q .

Definition 2.3. We also define a map from TQ to T^*Q by $\mathbb{F}L(v)$ for every $v \in T_q Q$ by

$$\frac{\partial}{\partial t}\Big|_{t=0} L(v + tw) = \langle \mathbb{F}L(v), w \rangle$$

- a) We denote by $\mathfrak{Leg} = \mathbb{F}L(TQ) \subset T^*Q$ the image of $\mathbb{F}L$.
- b) We call partial vector fields on \mathfrak{Leg} sections¹ of $\Gamma(TT^*Q)|_{\mathfrak{Leg}}$.
- c) An integral curve of a partial vector field X on \mathfrak{Leg} is a path $t \mapsto u_t \in TQ$ such that

$$\frac{d}{dt}\mathbb{F}L(u_t) = X_{\mathbb{F}L(u_t)}.$$

- d) An implicit Lagrangian system for an almost Dirac structure $\mathbb{D} \subset TT^*Q$ is a pair (X, L) , with X a partially defined vector field on \mathfrak{Leg} , such that $(X(\mathbb{F}L(u)), \mathcal{D}_u L) \in \mathbb{D}$ for all u in TQ .

¹For E a vector bundle over a manifold X and $Y \subset X$ an arbitrary subset (not necessarily a manifold), we denote by $\Gamma(E)|_Y$ restrictions to Y of smooth sections of E in a neighborhood of Y in X .

Remark 2.4. We do not assume partial vector fields on \mathfrak{Leg} to be tangent to \mathfrak{Leg} in any sense. Of course, if they are not tangent, they may have little integral curves.

Remark 2.5. Particular cases include:

- Usual Hamiltonian dynamics. When $\mathbb{F}L$ is a diffeomorphism and \mathbb{D} is the graph of the canonical symplectic form on T^*Q , implicit Lagrangian systems are pairs (X_H, L) , where X_H is the Hamiltonian vector field of the Legendre transform H of L .
- Constraint dynamics, which actually motivated the construction, we give some details below: Example 2.12.

2.3. Implicit Lagrangian systems with magnetic terms.

Definition 2.6 ([7]). Let $D \subset \mathbb{T}M$ be a subbundle.

- (1) For all $\phi: M' \rightarrow M$, we denote by $\phi^!D$ the set

$$\phi^!D_{m'} := \left\{ (X, \phi^*\beta) \text{ with } X \in T_{m'}M', \beta \in T_{\phi(m')}^*M \text{ s.t. } (\phi_*(X), \beta) \in D_{\phi(m')} \right\}$$

When D is an (almost-)Dirac structure we call $\phi^!D$ the pullback of D .

- (2) Let ω be a 2-form $\omega \in \Omega^2(M)$, we denote by $e^\omega D$ the set

$$e^\omega D = \{ (v, \beta + \iota_v \omega) \mid (v, \beta) \in D \}$$

and call it the gauge transform of D .

Lemma 2.7 (cf. e.g. [7]). Let $D \subset \mathbb{T}M$ be a Dirac structure and M' be a manifold.

- (1) For any smooth map $\phi: M' \rightarrow M$, $\phi^!D$ is a Dirac structure on M' .
- (2) For any closed 2-form $\omega \in \Omega^2(M)$, $e^\omega D$ is a Dirac structure on M .

Given $D \subset \mathbb{T}Q$ be a Dirac structure on Q , this lemma allows to consider (i) its pull back $\pi^!D$ on T^*Q through the canonical base map $\pi: T^*Q \rightarrow Q$, then (ii) consider the gauge transformation $e^\Omega \pi^!D$ of this pull-back with respect to the canonical symplectic 2-form Ω .

Definition 2.8. Let $D \subset \mathbb{T}Q$ be a Dirac structure on Q . We call constrained magnetic Lagrangian system an implicit Lagrangian system for the Dirac structure $\mathbb{D} = e^\Omega \pi^!D \subset \mathbb{T}T^*Q$ as above.

Theorem 2.9. Let $D \subset \mathbb{T}Q$ be a Dirac structure and $L: TQ \rightarrow \mathbb{R}$ a Lagrangian. Assume that the 2-form $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ admits a basic primitive $\theta \in \Gamma(D^*)^{hor}$. Then for $q: I \rightarrow Q$ the following are equivalent:

- a) There exists a Dirac path $\zeta: I \rightarrow D$ such that $\rho(\zeta) = \dot{q}$ which is the critical point among Dirac paths with the same end points of

$$(2) \quad \int_I (L(\rho(\zeta(t))) + \theta(\zeta(t))) dt.$$

b) For all $t \in I$, the following condition holds.

$$(3) \quad \left(\frac{\partial}{\partial t} \mathbb{F}L(\dot{q}(t)), \mathcal{D}_{\dot{q}(t)} L \right) \in \mathbb{D} = e^\Omega \pi^! D.$$

Proof. We claim that it suffices to check the equivalence on a small open set in Q . The second assertion is clearly local in nature and a path is a critical point for the functional (2) if and only if for any $t \in I$ there is a subinterval $t \in I_t \subset I$ on which it is critical.

It suffices therefore to establish the equivalence on an open subset U of Q , on which the Dirac structure takes the following normal form ([10]):

- $U = S \times N$, where $S \subset \mathbb{R}^a$, $N \subset \mathbb{R}^b$ with $q(0)$ lying in $S \times \{0\}$.
- $D|_U = e^\eta((TS \oplus \{0\}) \times \Gamma^\Pi)$, where $\eta \in \Omega_{cl}^2(U)$ and $\Pi \in \mathfrak{X}^2(N)$ is a Poisson bivector field vanishing at 0 and Γ^Π is the corresponding Dirac structure on N .

As $S \times \{0\}$ is a leaf of D , $q(t) \in S \times \{0\}$ for all t . By our assumptions on D

$$\int_I (L(\rho(\zeta(t))) + \theta(\zeta(t))) dt = \int_I (L(\dot{q}) + \theta^S(\dot{q})) dt,$$

where $\theta^S \in \Omega^1(S)$ is the one-form on S induced by θ and hence satisfies $d\theta = i_S^* \eta$ where $i_S : S \rightarrow U$, $i_S(s) = (s, 0)$.

We will denote by L^S the restriction of L to TS and write $q = (q^S, 0)$. With these conventions, the above functional reads:

$$\int_I (L^S(\dot{q}^S) + \theta^S(\dot{q}^S)) dt.$$

The classical Euler-Lagrange-theorem with magnetic term implies that being a critical point of this functional is equivalent to

$$(4) \quad \left(\frac{\partial}{\partial t} \mathbb{F}L^S(\dot{q}^S(t)), \mathcal{D}_{\dot{q}^S(t)} L^S \right) \in e^\Omega \pi^! e^{\eta^S} (TS \oplus \{0\}).$$

Via the isomorphism $\mathbb{T}T^*U = \mathbb{T}T^*S \times \mathbb{T}T^*N$, the assertion (3) decomposes as two conditions, the first one (on S) being (4). The second condition (on N) is always satisfied, as one can verify by a straightforward computation in local coordinates which relies on the fact that $\pi_N(q(t)) = 0$ for all t .

□

Corollary 2.10. Let Q, L, D be as in Theorem 2.9 and (X, L) an implicit Lagrangian system. Then any integral curve γ of X is the base path of a critical point of (2).

Example 2.11 (Classical symplectic magnetic terms). Let Q be any manifold, $\omega \in \Omega_{cl}^2(Q)$ and $D = \Gamma_\omega \subset \mathbb{T}Q$. Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. In this case $e^\Omega \pi^! D = \Gamma_{\Omega + \pi^* \omega} \subset \mathbb{T}T^*Q$.

As $\mathcal{H}_{basic}^\bullet(D) = H_{dR}^\bullet(M)$, (cf. Example 1.13), the 2-form on D admits a basic potential if and only if ω is de-Rham exact, i.e. $\omega = d\theta$, $\theta \in \Omega^1(Q)$.

Let us assume that the Legendre transform $\mathbb{F}L : TQ \rightarrow T^*Q$ is bijective, and denote the Legendre transform of the Lagrangian by H , i.e.

$$H(p) = \langle p, (\mathbb{F}L)^{-1}p \rangle - L \circ (\mathbb{F}L)^{-1}(p)$$

In this case $\mathcal{D}L$ is simply dH . Theorem 2.9 yields that the critical points of $L(q, \dot{q}) + \theta(\dot{q})$ correspond under the Legendre transform to integral curves of the Hamiltonian flow of H for the symplectic structure $\Omega + \pi^*\omega$. Corollary 2.10 states that (X, dH) is an implicit Lagrangian system with respect to $e^\Omega \pi^! D$ if and only if the vector field X is the Hamiltonian vector field of H with respect to $\Omega + \pi^*\omega$.

Example 2.12 (Holonomic constraints as a regular foliation). Let $F \subset TQ$ be a regular foliation. As discussed in Example 1.14, the Dirac structure $D = F \oplus F^\circ$ always admits a basic potential, as the 2-form in $\Lambda^2 D^*$ is zero (there is no magnetic term). Then $\pi^! D$ is the Dirac structure associated to the pullback foliation $\pi^{-1}(F)$ and

$$e^\Omega \pi^! D = \{(w, \alpha) \in TT^*Q \oplus T^*T^*Q \mid \pi_*(w) \in F, \alpha - \Omega^b w \in \pi^{-1}(F)^\circ\}$$

Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. Then Theorem 2.9 and Corollary 2.10 yield that the integral curves of any implicit Lagrangian system $(X, \mathcal{D}L)$ for $e^\Omega \pi^! D$ are critical points of L among curves that are tangent to F . The condition (3) translates directly to the Euler-Lagrange equations for a system subject to holonomic constraints, which are classically spelled-out using the Lagrange multipliers [11].

Remark 2.13. Holonomic and non-holonomic constraints. Note that the result above concerns the so-called holonomic constraints, i.e. the conditions defining the constraints do not depend essentially on the velocities of the system. Geometrically this means that the foliation F comes from an integrable constraint distribution $\Delta \subset TQ$. Simple mechanical examples and counterexamples can be constructed by “rolling without slipping” problems: They are often formulated as an orthogonality condition on the velocity at the contact point – the condition is integrable for the rolling disk but not for a rolling ball. Under some extra assumptions the non-holonomic constraints ([12]) can still be treated in the variational approach, but in this setting the result is more subtle, since as mentioned above, remarks 1.12 and 1.15, the non-integrable almost Dirac structures are very different from the homological perspective.

2.4. Applications to numerics. One of the motivations for the above construction is its potential application to design appropriate structure preserving numerical methods – so called geometric integrators.

Historically, the first example of those are the symplectic numerical methods, they are known since several decades, and are now state of the art for Hamiltonian systems ([13]). The key idea is that in the continuous setting the Hamiltonian flow not only preserves the level sets of the Hamiltonian function, but also leaves invariant the symplectic form. It is thus natural to mimic this property for the discrete flow, i.e. computing the trajectory numerically one wants to take the symplectic form into account. And since it is actually the same symplectic form that defines the dynamics of the system, one can reverse the argument: a flow preserving the symplectic form will “respect” the level sets of the Hamiltonian defining it.

The Lagrangian counterpart of this picture is related to so-called variational integrators ([14]), the idea is rather natural as well. Instead of considering a continuous Lagrangian and searching for its extrema among all the paths with fixed endpoints:

$$(5) \quad \inf \mathcal{L} \equiv \inf \int_0^T L(q(t), \dot{q}(t)) dt$$

one defines the discrete version of the integrant

$$L_d(q_{n+1}, q_n, v_n) := \Delta t_n L(q_n, v_n).$$

Here $q_n \equiv q(t_n)$, v_n is some approximation of $\dot{q}(t_n)$ depending on q_n and q_{n+1} ; and Δt_n are the time intervals between q_n and q_{n+1} , not necessarily all equal. One then defines the discrete analogue of variational principle (DVP), i.e. studies the trajectories $(q_0, q_1, \dots, q_{n-1}, q_n)$ extremizing

$$\mathcal{L}_d = \sum_{n=0}^N L_d,$$

subject to $q_0 = q(0)$ and $q_N = q(T)$. For conservative mechanical systems one can recover usual symplectic methods with this variational approach, and it is actually more universal, since the timestep is allowed to vary as well.

A similar strategy can be applied whenever the variational principle can be formulated. For example, in [14] the case of systems with constraints is explored, which motivated some parts of this paper. Hence, the results of Sections 2.2 and 2.3 on the dynamics on Dirac structures fit to the picture perfectly: they basically say that as soon the cohomological obstruction is absent, one can formulate the Dirac dynamics with a variational approach. In Equation (5) one merely replaces the path $q(t)$ in the configuration manifold by a Dirac path $\zeta(t)$. In the continuous setting the Dirac paths preserve the Dirac structure by definition, the variational formulation permits to guarantee this property for the trajectory computed numerically.

There is however an important detail to mention: the folkloric perception of geometric integrators as “preservation of the geometric structure guarantees preservation of physical properties” is slightly simplified. For instance in the symplectic case, it is not the original Hamiltonian that is preserved, but its discrete version, for which one can estimate the difference [15]. The phenomenon is even more subtle in the variational case. In fact, saying that satisfying the discrete variational principle (DVP) results in preserving some quantities of the system is no longer that straightforward. In the generic case the DVP will only give the relations between different variables of the system, but they will still depend on the choice of discretization or approximation of some of them. It may (and often does) also happen that the choice of the discretization to preserve the structure exactly is technically very difficult or even a priori impossible. This means that the correct statements will concern rather preservation of geometric structures up to some order of discretization step. However there are very few proven theorems on the matter, but rather empirical observations; some partial results in the context of systems with constraints are given in [16].

PERSPECTIVES.

In this paper we have defined the basic Dirac cohomology, which permits to characterize the obstruction to variational formulation of dynamics on Dirac structures.

As mentioned in the last part of the paper, on top of purely mathematical interest, this construction should be useful to design more reliable tools for numerical integration of the flow of dynamical systems on Dirac structures. Those in turn naturally appear when studying constraint, interacting or dissipative mechanical systems, which are not in the range of classical Hamiltonian formalism.

Let us also mention that, since Poisson manifolds provide an example of Dirac structures, this approach is useful to construct some Poisson integrators. It is somewhat complementary to the strategy of [17], where the main tool is rather Hamiltonian dynamics and symplectic groupoids.

We suppose that these questions interest a more applied community than the audience of this journal, hence we intend to devote a separate paper to the general discussion of discretization issues.

Acknowledgments:

We appreciate inspiring discussions with Dina Razafindralandy and Aziz Hamdouni at early stages of this work. We are thankful to Tilmann Wurzbacher for valuable comments on the manuscript.

We also thank the Erwin Schrödinger International Institute for Mathematics and Physics for hosting the “Geometry for Higher Spin Gravity:

Conformal Structures, PDEs, and Q-manifolds” program, that permitted all the authors to gather in the same room and finish the manuscript.

This work has been supported by the CNRS 80Prime project “GraNum”.

APPENDIX A. TULCZYJEW ISOMORPHISM(S)

For self-containedness of this paper, we recall here the isomorphisms established by W. Tulczyjew between double (co)tangent bundles (at least one “co” should be present). The most non-trivial one is

$$\kappa: TT^*Q \rightarrow T^*TQ,$$

the construction works as follows: start with the double bundle TTQ , denote $TQ \xrightarrow{p_1} Q$ and $T(TQ) \xrightarrow{p_2} TQ$ with the respective duality pairings $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. There is a canonical flip $\sigma: TTQ \rightarrow TTQ$, then the mapping κ is (implicitly but canonically) defined by imposing

$$\langle \kappa(a), b \rangle_2 = \langle a, \sigma(b) \rangle_1$$

Remark A.1. This flip can be seen as a sort of Schwarz Lemma: for every smooth map $\Sigma(s, t) := \mathbb{R}^2 \rightarrow Q$ (defined in a neighborhood of $(0, 0)$):

- (1) $t \mapsto \frac{\partial \Sigma(0, t)}{\partial s}$ is a path in TQ starting from $\frac{\partial \Sigma}{\partial s}(0, 0)$. Its first jet at 0 belongs to $T_{\frac{\partial \Sigma}{\partial s}(0, 0)}(TQ)$.
- (2) $s \mapsto \frac{\partial \Sigma(s, 0)}{\partial t}$ is a path in TQ starting from $\frac{\partial \Sigma}{\partial t}(0, 0)$. Its first jet at 0 belongs to $T_{\frac{\partial \Sigma}{\partial t}(0, 0)}(TQ)$.

The canonical flip exchanges both.

All the others are obtained by post- or pre-composing with $\Omega^b: TT^*Q \rightarrow T^*T^*Q$ or its inverse, where Ω is the canonical symplectic form on T^*Q .

We will be mostly interested in the isomorphism

$$\beta \equiv \omega^b \circ \kappa^{-1}: T^*TQ \rightarrow T^*T^*Q$$

which is a particular case of the canonical isomorphism also called Tulczyjew isomorphism $T^*E \simeq T^*E^*$ for any vector bundle E .

APPENDIX B. LEGENDRE TRANSFORMATION

Throughout this section, E is a vector bundle over M equipped with a smooth function $L: U \subset E \rightarrow \mathbb{R}$ called *Lagrangian*.

Recall that if a smooth function f on an open convex subset $U \subset V$ of a vector space is strictly convex, then its differential, defined for all $v \in U$ by:

$$\begin{aligned} \mathbb{F}f: V &\rightarrow V^* \\ v &\mapsto \left(e \mapsto \left. \frac{d}{dt} \right|_{t=0} f(v + te) \right) \end{aligned}$$

is a diffeomorphism from U onto its image. If the restriction of L to any fiber is strictly convex, then $\mathbb{F}L : U \subset E \hookrightarrow E^*$ is a diffeomorphism from U to its image U' .

We define the Legendre transform $H \in C^\infty(U')$ of L to be the unique function satisfying $H(\alpha) + L(v) = \langle \alpha, v \rangle$ for all $\alpha \in E^*$ with $\alpha = \mathbb{F}L(v)$.

Proposition B.1. The Legendre transform and the Tulczyjew isomorphism are related by the equality

$$\beta(d_e L) = d_{\mathbb{F}L(v)} H.$$

When L is not strictly convex, the Legendre transform need not exist, however the set $\{\beta(d_e L) \mid e \in U\}$ is a Lagrangian submanifold of T^*E^* . It appears throughout the text as the image of $\mathcal{D}L$.

REFERENCES

- [1] Dina Razafindralandy, Vladimir Salnikov, Aziz Hamdouni, and Ahmad Deeb. Some robust integrators for large time dynamics. *Adv. Model. and Simul. in Eng. Sci.*, 6(5), 2019.
- [2] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124. Cambridge University Press, Cambridge. London Mathematical Society, London, 1987.
- [3] A. Yu. Vaintrob. Lie algebroids and homological vector fields. *Uspekhi Mat. Nauk*, 52(2(314)):161–162, 1997.
- [4] Andre Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. *J. Differ. Geom.*, 12:253–300, 1977.
- [5] R. Hermann. On the accessibility problem in control theory. *Int. Symp. Non-Linear Differ. Equations and Non-Linear Mech.*, 325-332 (1963)., 1963.
- [6] Theodore James Courant. Dirac manifolds. *Trans. Am. Math. Soc.*, 319(2):631–661, 1990.
- [7] Henrique Bursztyn. A brief introduction to Dirac manifolds. In *Geometric and topological methods for quantum field theory*, pages 4–38. Cambridge Univ. Press, Cambridge, 2013.
- [8] Camille Laurent-Gengoux, Anne Pichereau, and Pol Vanhaecke. *Poisson structures*, volume 347. Berlin: Springer, 2012.
- [9] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.
- [10] Christian Blohmann. Removable presymplectic singularities and the local splitting of Dirac structures. *Int. Math. Res. Not. IMRN*, (23):7344–7374, 2017.
- [11] Joseph Louis Lagrange. *Mcanique Analytique*. Mallet-Bachelier, 1855.
- [12] Hiroaki Yoshimura and Jerrold E. Marsden. Dirac structures in Lagrangian mechanics. I. Implicit Lagrangian systems. *J. Geom. Phys.*, 57(1):133–156, 2006.
- [13] Haruo Yoshida. Construction of higher order symplectic integrators. *Phys. Lett. A*, 150(5-7):262–268, 1990.
- [14] J. E. Marsden and M. West. Discrete mechanics and variational integrators. *Acta Numer.*, 10:357–514, 2001.

- [15] Dina Razafindralandy, Aziz Hamdouni, and Marx Chhay. A review of some geometric integrators. *Advanced Modeling and Simulation in Engineering Sciences*, 5(1):16, December 2018.
- [16] Daria Loziienko, Aziz Hamdouni, and Vladimir Salnikov. Construction of pseudo-geometric integrators. *Program Comput Soft*, 2, 2022.
- [17] Oscar Cosserat. Discretizations respecting Poisson structures. *In preparation*, 2021.

E-mail address: `oscar.cosserat@univ-lr.fr`

LASIE – CNRS & UNIVERSITY OF LA ROCHELLE, AV. MICHEL CRÉPEAU,
17042 LA ROCHELLE CEDEX 1, FRANCE

E-mail address: `oleksii.kotov@uhk.cz`

FACULTY OF SCIENCE, UNIVERSITY OF HRADEC KRALOVE, ROKITANSKEHO
62, HRADEC KRALOVE 50003, CZECH REPUBLIC

E-mail address: `camille.laurent-gengoux@univ-lorraine.fr`

INSTITUT ELIE CARTAN DE LORRAINE (IECL), UMR 7502 – 3 RUE AUGUSTIN
FRESNEL, 57000 TECHNOPÔLE METZ, FRANCE

E-mail address: `leonid.ryvkin@mathematik.uni-goettingen.de`

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, INSTITUT FÜR MATHEMATIK, BUN-
SENSTR. 3-5, 37073 GÖTTINGEN

E-mail address: `vladimir.salnikov@univ-lr.fr`

LASIE – CNRS & LA ROCHELLE UNIVERSITY, AV. MICHEL CRÉPEAU, 17042
LA ROCHELLE CEDEX 1, FRANCE