# Supplementary material <br> Damping of a pendulum: <br> An experimental test of the unsteady Stokes friction force on a cylinder 

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#### Abstract

We first explain how we deduce from Loewenberg's publications the parameters describing the force on cylinders with large aspect ratios. We give the parameters necessary to calculate the motion of the two pendulums used in our experiments. We calculate the connection between the damping time constant and the friction force constant $K^{\prime}$, taking into account that the pendulum motion is a rotation around a point and not a translation. We collect the measured values of the damping time constant as a function of the air pressure in the vacuum vessel. The damping time constant has been measured at a very low pressure for both pendulums and we explain why these measurements give an accurate value of the non-hydrodynamic damping time constant. Finally, we reanalyze a series of published measurements of the friction force on cylinders in unsteady motion and we extract from these measurements the (apparent) validity domain of Stokes's result in the plane of Stokes and Reynolds numbers.


## Values of Loewenberg parameters for finite cylinders with a large aspect ratio

In order to calculate the friction force on a cylinder oscillating perpendicularly to its symmetry axis, we need the steady drag coefficient $R_{\perp}^{0}(\phi)$, from which is deduced the unsteady drag coefficient $B_{\perp}^{0}=\left(R_{\perp}^{0}(\phi)\right)^{2} /(6 \pi)$, and the Basset coefficient $B_{\perp}^{\infty}(\phi)$ as functions of the aspect ratio $\phi=L /(2 R)$, where $L$ is the cylinder length and $R$ its radius. We use the results published in two papers by Loewenberg for long cylinders: in a first paper [1], numerical results are plotted in figure 1 and asymptotic expressions, valid in the limit $\phi \rightarrow \infty$, are given in table I while numerical values for $\phi=1,2,5,10$ are given in table I of the second paper [2]. In our experiments, the aspect ratio of the two cylinders are $\phi=13.3$ and 52 . so that we need expressions for $R_{\perp}^{0}(\phi)$ and $B_{\perp}^{\infty}(\phi)$ in the range $\phi=10-60$.

Panel (f) of figure 1 of reference [1] presents a plot of $R_{\perp}^{0}(\phi) / \phi$. The axis legend is $\phi R_{\perp}^{0}(\phi)$ but this is an error as verified by the value for $\phi=10$ given in reference [2]. We have measured the values of $R_{\perp}^{0}(\phi) / \phi$ for $\phi=10-60$ on an expanded image of panel (f). In this range, we have found that $R_{\perp}^{0}(\phi)=(1.039 \pm 0.010) \times R_{\perp}^{0, \text { asy }}(\phi)$ where $R_{\perp}^{0, a s y}$ is its asymptotic expression given by

$$
\begin{equation*}
\frac{R_{\perp}^{0, a s y}(\phi)}{4 \pi \phi}=\frac{2+0.614 / \ln (2 \phi)}{\frac{1}{2}+\ln (2 \phi)}+\frac{0.238}{(\ln (2 \phi))^{3}} \tag{1}
\end{equation*}
$$

The ratio $R_{\perp}^{0}(\phi) / R_{\perp}^{0, \text { asy }}(\phi)$ decreases slowly when $\phi$ increases but, as the variation, of the order of $\pm 1 \%$, is
comparable to the accuracy of Loewenberg's calculations and to the error done in our measurements on his figure 1, we have not taken it into account.

Panel (h) of this figure 1 presents a plot of $B_{\perp}^{\infty}(\phi) / \phi^{2}$ which decreases rapidly when $\phi$ is large: in the range $\phi=10-60$, it is impossible to extract from this plot the value of this quantity with an accuracy better than $\pm 10 \%$. We have used as a starting point the asymptotic expression $B_{\perp}^{\infty}(\phi) /(4 \pi \phi)=2$ when $\phi \rightarrow \infty$, given in table I of Loewenberg's first paper [1]. We have then used the numerical values for $\phi=2,5$ and 10 given in Loewenberg's second paper [2] to fit the ratio of the published values divided by this asymptotic expression. A reasonably good fit is given by

$$
\begin{equation*}
\frac{B_{\perp}^{\infty}(\phi)}{4 \pi \phi}=2+\frac{0.371}{\phi^{3 / 2}} \tag{2}
\end{equation*}
$$

We have collected in table I the values of $r_{\perp}^{0}=$ $R_{\perp}^{0} /(4 \pi \phi), \quad b_{\perp}^{0}=(2 \phi / 3)\left(r_{\perp}^{0}(\phi)\right)^{2} \quad$ and $\quad b_{\perp}^{\infty}=$ $B_{\perp}^{\infty}(\phi) /(4 \pi \phi)$ for the two pendulums.

TABLE I. Values of $r_{\perp}^{0}, b_{\perp}^{0}$ and $b_{\perp}^{\infty}$ deduced from Loewenberg's data for cylindrical rods.

| Cylinder radius | $R \approx 19 \mathrm{~mm}$ | $R \approx 5 \mathrm{~mm}$ |
| :---: | :---: | :---: |
| Aspect ratio $\phi$ | 13.3 | 52. |
| $r_{\perp}^{0}$ | 0.609 | 0.433 |
| $b_{\perp}^{0}$ | 3.27 | 6.51 |
| $b_{\perp}^{\infty}$ | 2.008 | 2.001 |

## Description of the pendulums and calculation of their dynamical parameters

The pendulum bodies are made of an aluminium alloy 2017 A , with a mass per unit volume $\rho_{p} \approx 2790 \mathrm{~kg} / \mathrm{m}^{3}$. For pendulum 1, the body is a hollow cylinder with an internal radius $R_{\text {int }} \approx 16 \mathrm{~mm}$ and plugs at both ends while, for pendulum 2, the body is a massive cylinder with no holes.

Because of the presence of plugs at both ends, pendulum 1 cannot be simply considered as a hollow cylinder. The top plug is a 16 mm -radius 11 mm -long cylinder exactly inserted in the main cylinder with a 5 mm -radius 10 mm -long cylinder protruding at the top and serving to clamp the spring. The top plug is treated as two objects in the following calculation, object 1 for the 5 mm -radius 10 mm -long cylinder and object 2 for the 16 mm -radius 11 mm -long cylinder. The main cylinder (length 503.6 $\mathrm{mm}, R_{\text {int }} \approx 16 \mathrm{~mm}$ and $R=19.0 \pm 0.05 \mathrm{~mm}$ ) is object 3 . The bottom plug is a 16 mm -radius 12 mm -long cylinder with a $\approx 4 \mathrm{~mm}$-diameter hole is object 4 and the Nd FeB magnet (a $2 \times 3 \times 10 \mathrm{~mm}^{3}$ parallelepiped) which is inserted in the $\approx 4 \mathrm{~mm}$-diameter hole is object 5 . Table II collects the distances $z_{G i}-z_{D}$ of the center of mass of each object and its mass.

TABLE II. For each object making the body of pendulum 1, we give the $z_{G i}-z_{D}$ in mm and its mass $m_{i}$ in $g$.

| Object | $z_{G i}-z_{D}$ | $m_{i}$ |
| :---: | :---: | :---: |
| 1 | 5.0 | 2.20 |
| 2 | 15.5 | 22.3 |
| 3 | 261.9 | 464.4 |
| 4 | 507.6 | 26.9 |
| 5 | 507.6 | 0.45 |

Pendulum 2 is almost exactly cylindrical but we have taken into account the small differences with respect to a cylinder in order to verify that these differences have a negligible effect. We treat as object 1 the 10 mm long top part of the cylinder split in two parts along a diameter connected by a 3 mm diameter screw. The rest of the cylinder is treated as object 2 . The NdFeB magnet (a $2 \times 3 \times 10 \mathrm{~mm}^{3}$ parallelepiped) inserted in $\mathrm{a} \approx 4 \mathrm{~mm}$ diameter hole at the bottom of the cylinder is object 3 . Table III collects the distances $z_{G i}-z_{D}$ of the center of mass of each object and its mass.

TABLE III. For each object part of the body of pendulum 2, we give the distance $z_{G i}-z_{D}$ in mm and its mass $m_{i}$ in g .

| Object $i$ | $z_{G i}-z_{D}$ | $m_{i}$ |
| :---: | :---: | :---: |
| 1 | 5.0 | 2.40 |
| 2 | 266.0 | 112.0 |
| 3 | 517.0 | 0.45 |



FIG. 1. Schematic drawing of the pendulum (not to scale). The spring and the cylindrical body are represented in two positions. The rest position, $\theta=0$, is used to calculate the moment of inertia for rotation around $C$ and the position with $\theta \neq 0$ exhibits the shape of the spring and the center of rotation $C$ of the pendulum body. $O$ is the upper end of the spring where it is clamped in the support and $D$ its lower end, where it is clamped in the pendulum body, while $G$ is the center of mass of the pendulum body. The $x$-axis, the $z$-axis and the oscillation angle $\theta$ are represented. The angle $\theta$ has been grossly exaggerated.

Using these values, we deduce the distance $D G$ of the center of mass of the pendulum body to point $D$, the moment of inertia $I_{G}$ for rotation around $G$. We need the values of these parameters to apply the theory of a spring suspended pendulum [4] and we also need the value of the product $E I_{s}$ of the suspension spring ( $E$ is the Young's modulus of the spring material and $I_{s}$ is the second moment of the area of its cross section). The two pendulums are suspended by the same spring, a $100 \mathrm{~mm}-$ long brass foil with a thickness $c=0.100 \pm 0.01 \mathrm{~mm}$ and a width $d=10.0 \mathrm{~mm}$. This foil is clamped at both ends, at its top $O$ in a supporting piece bolted in the top flange of the vacuum vessel and at its bottom $D$ in the pendulum body. The foil free length is 80 mm . We have measured the product $E I_{s}$ for the foil by measuring its flexion when a force is applied at its end and we have found $E I_{s}=$ $(4.0 \pm 0.5) \times 10^{-5} \mathrm{~N} . \mathrm{m}^{2}$. Then using the theory of a spring suspended pendulum [4], we calculate the distance $L_{1}=C D$ from the rotation center $C$ to $D$ (see figure 1) and finally the moment of inertia $I_{C}$ for rotation around $C$. We have verified that the uncertainty on $E I_{s}$ has a
negligible effect on the value of $L_{1}=C D$ (modification smaller than 0.1 mm ). We also calculate the pendulum oscillation frequency $\omega_{t h} /(2 \pi)$ which is compared to its measured value $\omega_{\text {exp }} /(2 \pi)$. All the parameters describing the two pendulums are listed in table IV.

TABLE IV. For the two pendulums, this table gives the measured values of the diameter $2 R$, the length $L$ of the pendulum cylindrical body, the aspect ratio $\phi=L /(2 R)$, the distance $D G$ from the clamping point $D$ of the spring at the top of the pendulum body to its center of mass $G$, the mass $m$, the calculated moment of inertia $I_{G}$ for rotation around the center of mass $G$ of the pendulum body, the calculated moment of inertia $I_{C}$ for rotation around the rotation center, the calculated value of the length $L_{1}=C D$, the measured pendulum frequency $\omega_{\text {exp }} /(2 \pi)$ and its calculated value $\omega_{t h} /(2 \pi)$, finally the pendulum period $T$.

| Quantity | pendulum 1 | pendulum 2 |
| :---: | :---: | :---: |
| $2 R(\mathrm{~mm})$ | $38.0 \pm 0.1$ | $9.97 \pm 0.01$ |
| $L(\mathrm{~mm})$ | 514.0 | 522.0 |
| $\phi=L /(2 R)$ | 13.3 | 52. |
| $D G(\mathrm{~mm})$ | 263.1 | 260.4 |
| $M(\mathrm{~g})$ | 516.25 | 114.85 |
| $10^{3} I_{G}\left(\mathrm{~kg} . \mathrm{m}^{2}\right)$ | 13.04 | 2.659 |
| $10^{3} I_{C}\left(\mathrm{~kg} \cdot \mathrm{~m}^{2}\right)$ | 66.95 | 14.37 |
| $L_{1}=C D(\mathrm{~mm})$ | 60.4 | 58.9 |
| $\omega_{\text {exp }} /(2 \pi)(\mathrm{Hz})$ | 0.768 | 0.786 |
| $\omega_{\text {th }} /(2 \pi)(\mathrm{Hz})$ | 0.770 | 0.782 |
| $T(\mathrm{~s})$ | 1.302 | 1.272 |

## Connection between the damping time constant and the friction force constant $K^{\prime}$

In the main text, we have described the pendulum motion by a mass-spring-damper system, because of its simplicity. The exact description takes into account the fact that the pendulum motion is not a translation but a rotation around point $C$ (see figure 1) with an angular amplitude $\theta(t)$ following the differential equation

$$
\begin{equation*}
I_{C} \frac{d^{2} \theta}{d t^{2}}=-k \theta+\Gamma^{h}+\Gamma^{n h} \tag{3}
\end{equation*}
$$

$I_{C}$ is the moment of inertia of the pendulum calculated for a rotation around $C .-k \theta$ is the torque due to the gravity and to the spring. $\Gamma^{h}$ and $\Gamma^{n h}$ are the torques of the hydrodynamic and non-hydrodynamic forces.

If we assume that the pendulum body is a cylinder of internal radius $R_{\text {int }}$ and external radius $R$, of length $L$ with a mass per unit volume $\rho_{p}$, its mass is $M=\pi \rho_{p}\left(R^{2}-R_{\text {int }}^{2}\right) L$ and its moment of inertia $I_{C}$ is given by the following integral written in cylindrical coordinates $r, \varphi, z$

$$
\begin{equation*}
I_{C}=\int_{L_{1}}^{L_{2}} d z \int_{R_{\text {int }}}^{R} r d r \int_{0}^{2 \pi} d \varphi \rho_{p}\left(z^{2}+r^{2} \cos ^{2} \varphi\right) \tag{4}
\end{equation*}
$$

where, in order to simplify the results, we have noted $L_{2}=L_{1}+L$ the distance from $C$ to the bottom of the cylinder

$$
\begin{align*}
I_{C} & =\pi \rho_{p}\left[\left(R^{2}-R_{i n t}^{2}\right) \frac{L_{2}^{3}-L_{1}^{3}}{3}+\frac{R^{4}-R_{i n t}^{4}}{4}(L)\right] \\
& =M\left[\frac{L_{2}^{3}-L_{1}^{3}}{3 L}+\frac{R^{2}+R_{i n t}^{2}}{4}\right] \\
& \approx M \frac{L_{2}^{3}-L_{1}^{3}}{3 L} . \tag{5}
\end{align*}
$$

In the approximate result, the neglected term is a fraction of the order of $(R / L)^{2}$ of the main term and this approximation is excellent for both pendulums.

The torques $\Gamma^{h}$ and $\Gamma^{n h}$ are given by

$$
\begin{align*}
\Gamma_{h} & =\int_{L_{1}}^{L_{2}} \frac{d f^{h}}{d z} z d z  \tag{6}\\
\Gamma^{n h} & =\int_{L_{1}}^{L_{2}} \frac{d f^{n h}}{d z} z d z \tag{7}
\end{align*}
$$

where $d f^{h} / d z$ is equal to

$$
\begin{equation*}
\frac{d f^{h}}{d z}=\frac{f^{h}}{L}=-2 \pi \eta\left[K^{\prime} \frac{d x}{d t}+\frac{K}{\omega} \frac{d^{2} x}{d t^{2}}\right] \tag{8}
\end{equation*}
$$

The non-hydrodynamic force $d f^{n h} / d z$ is taken equal to

$$
\begin{equation*}
\frac{d f^{n h}}{d z}=\frac{f^{n h}}{L} \equiv-\eta^{n h} \frac{d x}{d t} \tag{9}
\end{equation*}
$$

which defines the parameter $\eta^{n h}$. The local velocity and acceleration are expressed as a function of the first and second derivatives of $\theta$ by $d x / d t=z d \theta / d t$ and $d^{2} x / d t^{2}=$ $z d^{2} \theta / d t^{2}$

$$
\begin{align*}
\Gamma_{h} & =-2 \pi \eta\left[K^{\prime} \frac{d \theta}{d t}+\frac{K}{\omega} \frac{d^{2} \theta}{d t^{2}}\right] \int_{L_{1}}^{L_{2}} z^{2} d z  \tag{10}\\
\Gamma^{n h} & =-\eta^{n h} \frac{d \theta}{d t} \int_{L_{1}}^{L_{2}} z^{2} d z \tag{11}
\end{align*}
$$

As $\int_{L_{1}}^{L_{2}} z^{2} d z=\left(L_{2}^{3}-L_{1}^{3}\right) / 3$, eq. (3) becomes

$$
\begin{align*}
& \left(I_{C}+I_{C}^{h}\right) \frac{d^{2} \theta}{d t^{2}}+\left(2 \pi \eta K^{\prime}+\eta_{n h}\right) \frac{L_{2}^{3}-L_{1}^{3}}{3} \frac{d \theta}{d t}+k \theta=0 \\
& I_{C}=M \frac{L_{2}^{3}-L_{1}^{3}}{3 L} \\
& I_{C}^{h}=\pi \rho R^{2} L\left[2 m_{\perp}^{A}+b_{\perp}^{\infty} \frac{\delta}{R}\right] \frac{L_{2}^{3}-L_{1}^{3}}{3} \tag{12}
\end{align*}
$$

Because the air mass per unit volume $\rho$ is in factor, the moment of inertia $I_{C}^{h}$ decreases when the air pressure decreases ( $\delta$ increases but only like $1 / \sqrt{\rho}$ ). For the largest pressure used in our experiments $P \approx 1000 \mathrm{mbar}, \delta<R$ and $I_{C}^{h}$ is dominated by the term proportional to $m_{\perp}^{A}$. As $m_{\perp}^{A} \approx 0.5$, the ratio $I_{C}^{h} / I_{C}$ is approximately equal to

$$
\begin{equation*}
\frac{I_{C}^{h}}{I_{C}} \approx \frac{\rho}{\rho_{p}} \times \frac{R^{2}}{R^{2}-R_{i n t}^{2}}=\frac{M^{h}}{M} \tag{13}
\end{equation*}
$$

where $M^{h}=\pi \rho R^{2} L$ is the added mass. At the pressure $P \approx 1000 \mathrm{mbar}$, this ratio is equal to $M^{h} / M \approx 1.2 \times 10^{-3}$ for the $R \approx 19 \mathrm{~mm}$-pendulum (for which $R_{\text {int }} \approx 16 \mathrm{~mm}$ ) and $M^{h} / M \approx 0.5 \times 10^{-3}$ for the $R \approx 5 \mathrm{~mm}$-pendulum (which is a solid cylinder i.e. $R_{\text {int }}=0 \mathrm{~mm}$ ). From now on, we will neglect $I_{C}^{h}$. The solution of equation (12) is

$$
\begin{align*}
\theta(t) & =\theta_{\max }(t) \sin (\omega t) \\
\text { with } \theta_{\max }(t) & =\theta_{\max }(0) \exp (-t / \tau) \tag{14}
\end{align*}
$$

with $\omega \approx \sqrt{k / I_{C}}$. The damping time constant $\tau$ is given by

$$
\begin{align*}
\frac{1}{\tau} & =\frac{1}{\tau^{h}}+\frac{1}{\tau^{n h}}  \tag{15}\\
\frac{1}{\tau^{h}} & =\frac{\pi \eta K^{\prime}}{I_{C}} \frac{L_{2}^{3}-L_{1}^{3}}{3}  \tag{16}\\
\frac{1}{\tau^{n h}} & =\frac{\eta^{n h}}{2 I_{C}} \frac{L_{2}^{3}-L_{1}^{3}}{3} \tag{17}
\end{align*}
$$

We explain below how we measure the non-hydrodynamic damping time constant $\tau^{n h}$. If we replace $I_{C}$ by its value (5, the relation between $\tau^{h}$ and $\eta K^{\prime}$ can be simplified

$$
\begin{equation*}
\frac{1}{\tau^{h}}=\frac{\eta K^{\prime}}{\rho_{p}\left(R^{2}-R_{i n t}^{2}\right)} \tag{18}
\end{equation*}
$$

Then, using eq. (15), we relate $\eta K^{\prime}$ to the measured value $\tau_{\text {exp }}$ of the damping time constant $\tau$

$$
\begin{align*}
\eta K^{\prime} & =\rho_{p}\left(R^{2}-R_{i n t}^{2}\right)\left(\frac{1}{\tau^{e x p}}-\frac{1}{\tau^{n h}}\right) \\
& =\frac{M}{\pi L}\left(\frac{1}{\tau^{e x p}}-\frac{1}{\tau^{n h}}\right) \tag{19}
\end{align*}
$$

which is the result given in our paper (equation 26). In the case of pendulum 1 , which is a hollow cylinder with two plugs, the value of $I_{C}$ given by eq. (5) is not very accurate. We rather use eq. (16) with the moment of inertia $I_{C}$ calculated by summing the contributions of the various parts of the pendulum body. The relation between $\eta K^{\prime}$ and the measured damping time constant $\tau_{\exp }$ then becomes

$$
\begin{equation*}
\eta K^{\prime}=\frac{3 I_{C}}{\pi\left(L_{2}^{3}-L_{1}^{3}\right)} \times\left(\frac{1}{\tau^{e x p}}-\frac{1}{\tau^{n h}}\right) \tag{20}
\end{equation*}
$$

which is the result given in our paper (equation 28).

## Measurement of the non-hydrodynamic damping time constant

In order to measure the non-hydrodynamic damping time constant, we measure the damping time constant at a very low pressure. Here, we prove that with a pressure $P<10^{-5}$ mbar, the damping effect due to the gas is negligible with respect to the non-hydrodynamic damping effects.

Cagnoli et al. [3] give an equation relating the residual value of the damping time constant $\tau_{h}$ to the gas pressure $P$, in the so-called molecular regime, when the mean free path of the gas molecule is larger than the size of the vacuum tank. Although this equation has been calculated for a pendulum shape different from the one of our pendulums, we may use it to get a estimate of $\tau_{h}$ in the molecular reqime. Expressed with our notations, this equation is

$$
\begin{equation*}
\tau_{h}=\frac{8 M}{2 R L P} \sqrt{\frac{\pi k_{B} T}{8 m_{m o l}}} \tag{21}
\end{equation*}
$$

where the square root is the mean velocity of the gas molecule, $k_{B}$ being the Boltzmann constant and $m_{m o l}$ the molecular mass. This velocity is $\approx 180 \mathrm{~m} / \mathrm{s}$ for air at the set-up temperature $T \approx 295 \mathrm{~K}$. With $R, L$ and $m$ given in table IV and a pressure $P=10^{-5} \mathrm{mbar}=10^{-3}$ Pa , we find $\tau_{h} \approx 4 \times 10^{7} \mathrm{~s}$ for pendulum 1 and $\tau_{h} \approx 3 \times 10^{7}$ s for pendulum 2 .

The damping time constant measured with a pressure smaller than $10^{-5}$ mbar is $\tau_{\text {exp }}=(1.99 \pm 0.1) \times 10^{5} \mathrm{~s}$ for the pendulum 1 and $\tau_{\text {exp }}=(2.01 \pm 0.08) \times 10^{5} \mathrm{~s}$ for pendulum 2. As our estimated value $\tau_{h}$ is more than $100-$ times larger than $\tau_{\text {exp }}$ in these very low pressure experiments, these experiments give an accurate measurement of $\tau_{n h} \approx(2 \pm 0.1) \times 10^{5} \mathrm{~s}$ for both pendulums. We use this value to extract $\tau_{h}$ from the values of $\tau_{\text {exp }}$ listed in tables V and VI. As $\tau_{n h}$ is larger than $\sim 40 \tau_{\text {exp }}$ in all our experiments, the difference between $\tau_{\text {exp }}$ and $\tau_{h}$ is very small, at most $2.5 \%$ and the $5 \%$ uncertainty on $\tau_{n h}$ has a negligible effect on the uncertainty on $\tau_{h}$.

## Measured values of the damping time constants $\tau^{e x p}$ and of $\tau^{h}$.

The damping time constant $\tau^{e x p}$ of the two pendulums have been measured as a function of the pressure $P$ in mbar. Their values are collected in tables V and VI which also give the the hydrodynamic damping time constant $\tau^{h}$ given by

$$
\begin{equation*}
\frac{1}{\tau^{h}}=\frac{1}{\tau^{e x p}}-\frac{1}{\tau^{n h}} \tag{22}
\end{equation*}
$$

deduced from eq. (26) of our main text and using $\tau^{n h}=$ $2 \times 10^{5} \mathrm{~s}$.

Values of the measured values $K^{\prime e x p}$ and comparison to the theoretical values

In this section we give the measured values of $K^{\text {exp }}$ deduced from the hydrodynamic damping time constant $\tau^{h}$ and we compare these measurements to Stokes' $K^{\prime}$ value and to the finite cylinder $K^{\prime}$ value given by eq. (12) of our paper. The uncertainty on the measured value

TABLE V. For the experiments with the $R \approx 19 \mathrm{~mm}$ pendulum, this table collects the values of the pressure $P$, of the damping time constant $\tau^{e x p}$ and of the hydrodynamic damping time constant $\tau^{h}$.

| $P$ (mbar) | $\tau^{e x p}(\mathrm{~s})$ | $\tau^{h}(\mathrm{~s})$ |
| :---: | :---: | :---: |
| 1036.8 | $1129 \pm 23$ | $1136 \pm 23$ |
| 843.5 | $1234 \pm 25$ | $1242 \pm 25$ |
| 664.9 | $1332 \pm 27$ | $1341 \pm 27$ |
| 506.1 | $1649 \pm 33$ | $1663 \pm 34$ |
| 371.9 | $1822 \pm 36$ | $1839 \pm 37$ |
| 258.1 | $1996 \pm 40$ | $2017 \pm 41$ |
| 166.2 | $2426 \pm 49$ | $2457 \pm 50$ |
| 111.6 | $2908 \pm 58$ | $2953 \pm 59$ |
| 52.3 | $3977 \pm 80$ | $4062 \pm 82$ |
| 25.7 | $5124 \pm 103$ | $5266 \pm 105$ |

TABLE VI. For the experiments with the $R \approx 5 \mathrm{~mm}$ pendulum, this table collects the values of the pressure $P$, of the damping time constant $\tau^{e x p}$ and of the hydrodynamic damping time constant $\tau^{h}$.

| $P(\mathrm{mbar})$ | $\tau^{e x p}(\mathrm{~s})$ | $\tau^{h}(\mathrm{~s})$ |
| :---: | :---: | :---: |
| 1033.4 | $775 \pm 16$ | $778 \pm 16$ |
| 641.1 | $942 \pm 19$ | $947 \pm 19$ |
| 338.0 | $1199 \pm 24$ | $1206 \pm 24$ |
| 131.0 | $1661 \pm 33$ | $1675 \pm 35$ |
| 20.2 | $3007 \pm 60$ | $3053 \pm 61$ |
| 9.75 | $3485 \pm 70$ | $3546 \pm 71$ |

$K^{\prime e x p}$ is estimated near $3 \%, 2 \%$ due to the uncertainty on the measurement of $\tau^{h}$ and $1 \%$ due to uncertainty on the value of the viscosity $\eta$ (see the main paper).

TABLE VII. For the $R \approx 19 \mathrm{~mm}$ pendulum, this table collects the values of the pressure $P$, of viscous penetration depth $\delta$, of the measured value $K^{\prime e x p}$ and compares it to Stokes' $K^{\prime}$ value and to the finite cylinder $K^{\prime}$ value.

| $P(\mathrm{mbar})$ | $\delta(\mathrm{mm})$ | $K^{\prime \text { exp }}$ | Stokes $K^{\prime}$ | finite cylinder $K^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1036.8 | 2.512 | 17.25 | 16.11 | 16.74 |
| 843.5 | 2.785 | 15.76 | 14.62 | 15.25 |
| 664.9 | 3.137 | 14.58 | 13.09 | 13.70 |
| 506.1 | 3.597 | 11.75 | 11.54 | 12.15 |
| 371.9 | 4.194 | 10.63 | 10.03 | 10.62 |
| 258.1 | 5.035 | 9.68 | 8.52 | 9.08 |
| 166.2 | 6.27 | 7.95 | 7.02 | 7.56 |
| 111.6 | 7.66 | 6.61 | 5.92 | 6.44 |
| 52.3 | 11.19 | 4.80 | 4.33 | 4.80 |
| 25.7 | 15.96 | 3.71 | 3.29 | 3.71 |

TABLE VIII. For the $R \approx 5 \mathrm{~mm}$ pendulum, this table collects the values of the pressure $P$, of viscous penetration depth $\delta$, of the measured value $K^{\prime e x p}$ and compares it to the measured value $K^{\prime e x p}$ and compares it to Stokes' $K^{\prime}$ value and to the finite cylinder $K^{\prime}$ value.

| $P(\mathrm{mbar})$ | $\delta(\mathrm{mm})$ | $K^{\prime \text { exp }}$ | Stokes $K^{\prime}$ | finite cylinder $K^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1033.4 | 2.49 | 4.88 | 4.95 | 5.39 |
| 641.1 | 3.16 | 4.01 | 4.09 | 4.52 |
| 338.0 | 4.35 | 3.15 | 3.20 | 3.63 |
| 131.0 | 6.99 | 2.27 | 2.29 | 2.71 |
| 20.2 | 17.8 | 1.24 | 1.33 | 1.66 |
| 9.75 | 25.6 | 1.07 | 1.12 | 1.39 |

## Discussion of previous experiments

The experiments discussed here have been chosen because they give information on the validity domain of Stokes' results for cylinders as a function of the Reynolds and Stokes numbers.

## The experiments of Martin [5]

Martin [5] has studied the damping of various strings oscillating in water as well in other liquids. We use here only the results of his figure 18 which presents the inverse of the logarithmic decrement noted $\vartheta$ as a function of the square root of the oscillation frequency. The experiments were made with three steel strings oscillating in pure water at $18^{\circ} \mathrm{C}$. With our notations, $\vartheta=2 \pi /(\omega \tau)$ where $\omega$ is the oscillation angular frequency and $\tau$ the damping time constant. Using Stokes' result, the logarithmic decrement is given by

$$
\begin{equation*}
\vartheta=\frac{2 \pi \eta}{\rho_{\text {string }} R^{2} \omega} \times K^{\prime} \tag{23}
\end{equation*}
$$

where $\rho_{\text {string }}$ is the density of the string material. Because the string radius $R \gg \delta$, we keep only the leading term of the expression $K^{\prime}$ given by eq. (3) of our paper

$$
\begin{equation*}
\vartheta \approx \frac{4 \pi \eta}{\rho_{\text {string }} R \delta \omega}=\frac{2 \pi}{\rho_{\text {string }} R} \sqrt{\frac{2 \eta \rho_{\text {fluid }}}{\omega}} \tag{24}
\end{equation*}
$$

where $\delta$ has been replaced by its value in the last form. The two predictions of this equation were verified by Martin, namely that $1 / \vartheta=C \sqrt{\omega}$ and that $C \propto 1 / R$.

Table IX collects the information on these experiments. Martin has measured the logarithmic decrement for various frequencies in the range $250-1600 \mathrm{~Hz}$. Using the values $\eta=1.061 \times 10^{-3}$ Pa.s and $\rho_{\text {fluid }}=998 \mathrm{~kg} / \mathrm{m}^{3}$ for pure water at $18^{\circ} \mathrm{C}$, we have verified his results are in very good agreement with Stokes' theory. The values of $\delta$ are $\delta=3.67 \times 10^{-2} \mathrm{~mm}$ for a frequency equal to 250 Hz and $\delta=1.45 \times 10^{-2} \mathrm{~mm}$ for a frequency equal to 1600 Hz. Following figure 5 of Martin's paper, the maximum
oscillation amplitude (corresponding to the middle of the string assuming that the string oscillation corresponds to its lowest resonance) is 0.054 mm and we use this amplitude to calculate the Reynolds number Re. As Re increases proportionally to the frequency, we evaluate St and Re for the 1600 Hz frequency. Martin's experiments are in the validity range of Stokes' result but possibly not at its upper limit: we conclude that the boundary of the validity domain is for a Reynolds number verifying $\operatorname{Re}_{\max }>7.4 \sqrt{\mathrm{St}}$ for $400<\mathrm{St}<3100$.

TABLE IX. Experiments of Martin: for each string oscillating at a 1600 Hz frequency, we give its radius $R$, an approximate value of the aspect ratio $\phi=L /(2 R)$ (the exact value of the length $L$ is not given in Martin's paper but a table indicates that this length is between 12.5 and 40 cm ; here we use $L=25$ cm ), the ratio $R / \delta$, the Stokes number $\mathrm{St}=4(R / \delta)^{2}$, the Reynolds number Re, the values of the ratio $x_{0} / R=\mathrm{Re} / \mathrm{St}$ and $x_{0} / \delta=\operatorname{Re} / \sqrt{\mathrm{St}}$.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline R(\mathrm{~mm}) & \phi & R / \delta & \mathrm{St} & \operatorname{Re} & x_{0} / R & x_{0} / \delta \\
\hline 0.15 & 830 & 10.3 & 424 & 152 & 0.36 & 7.4 \\
\hline 0.255 & 500 & 17.6 & 1239 & 262 & 0.21 & 7.4 \\
\hline 0.405 & 125 & 27.9 & 3113 & 414 & 0.13 & 7.4 \\
\hline
\end{array}
$$

## The experiment of Williams and Hussey [6, 7]

In his thesis [6], Williams presents the results of 28 experiments corresponding to different values of the ratio $\delta / R$. A selection of 13 experiments has been published [7]. For each of the selected experiments, the publication gives the maximum oscillation amplitude for which the force is a linear function of the amplitude. We discuss here only the 13 selected experiments. The friction force and the added inertia were measured and they were found in good agreement with Stokes' results. Various cylinders were used with radii in the $1.59-6.35 \mathrm{~mm}$ range and the length immersed in the fluid was varied from $\sim 20$ to $\sim 100 \mathrm{~mm}$, in order to correct for the end effects by extrapolation. The aspect ratio $\phi$ was in the range $8-31$.

Table X collects the values of $R / \delta$, of the maximum Reynolds number $\operatorname{Re}_{\text {max }}$, of the Stokes number St, from which we have deduced $x_{0 \max } / R$ and $x_{0 \max } / \delta$. The uncertainty on $\mathrm{Re}_{\max }$ is large, roughly $\pm 25 \%$, and the fluctuations between experiments with similar values of $R / \delta$ are considerable and irregular. The corresponding points in the St-Re plane are near a line of slope 1 , $\mathrm{Re}_{\max } \approx 0.75 \mathrm{St}$ with $0.2<\mathrm{St}<50$.

## The experiment of Stuart and Woodgate [8]

Stuart and Woodgate [8] describe an experiment measuring the damping of the oscillation of a pendulum by air friction force. The cylinder radius is 25.4 mm and its

TABLE X. Experiments of Williams and Hussey: for each experiment, we give the ratio $R / \delta$, the Stokes number St , the critical Reynolds number $\mathrm{Re}_{\text {max }}$, the ratio $x_{0 \text { max }} / R=$ $\mathrm{Re}_{\max } / \mathrm{St}$ and $x_{0 \text { max }} / \delta$.

| $R / \delta$ | St | $\mathrm{Re}_{\max }$ | $x_{0 \max } / R$ | $x_{0 \max } / \delta$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.243 | 0.237 | $0.59 \pm 0.10$ | $2.52 \pm 0.43$ | 0.60 |
| 0.252 | 0.255 | $0.24 \pm 0.06$ | $0.99 \pm 0.26$ | 0.24 |
| 0.282 | 0.319 | $0.18 \pm 0.04$ | $0.58 \pm 0.12$ | 0.16 |
| 0.291 | 0.338 | $0.17 \pm 0.04$ | $0.54 \pm 0.11$ | 0.15 |
| 0.368 | 0.541 | $0.43 \pm 0.11$ | $0.79 \pm 0.20$ | 0.29 |
| 0.373 | 0.557 | $0.39 \pm 0.08$ | $0.70 \pm 0.15$ | 0.26 |
| 0.417 | 0.694 | $0.50 \pm 0.10$ | $0.70 \pm 0.14$ | 0.30 |
| 0.518 | 1.07 | $1.2 \pm 0.4$ | $1.09 \pm 0.40$ | 0.58 |
| 0.562 | 1.26 | $1.2 \pm 0.3$ | $0.97 \pm 0.25$ | 0.53 |
| 0.571 | 1.31 | $1.2 \pm 0.2$ | $0.95 \pm 0.11$ | 0.52 |
| 1.117 | 4.99 | $2.6 \pm 0.7$ | $0.54 \pm 0.15$ | 0.58 |
| 1.155 | 5.33 | $1.9 \pm 0.8$ | $0.37 \pm 0.15$ | 0.41 |
| 3.497 | 48.9 | $39 \pm 6$ | $0.73 \pm 0.12$ | 2.79 |

length $L=1219 \mathrm{~mm}$, corresponding to an aspect ratio $\phi=24$. The friction force is found to be independent of the oscillation amplitude up to a 4 mrad amplitude and the measured value, when the air pressure is close to 1000 mbar, is in good agreement with Stokes theoretical result. Using the parameters of this experiment, we have calculated the Stokes number $\mathrm{St}=384$ and the maximum Reynolds number $\operatorname{Re}_{\max }=68.5$ corresponding to the bottom of the pendulum. We deduce from these values $x_{0 \max } / R=0.18$ and $x_{0 \max } / \delta=1.7$.

## The experiment of Berg et al. [9]

Berg et al. [9] have studied, by experiments and by numerical calculations, the force exerted on a cylinder as a function of $x_{0} / \delta$ in the case $R \ll \delta$. The experiments were made with a viscosity meter made of an oscillating grid made of roughly cylindrical wires of radius $R \approx 13.4 \mu \mathrm{~m}$ separated by a distance $L=847 \mu \mathrm{~m}$, corresponding to an aspect ratio $\phi \approx 32$. The calculations were done with infinite cylinders. The excellent sensitivity of this viscosity meter enables the comparison of the force as a function of the amplitude $x_{0}$ and of the oscillation frequency which means as a function of the viscous penetration depth. This experiment gives a very precise measurement of the dependence of the force with the oscillation amplitude $x_{0}$ i.e. with the Reynolds number $\operatorname{Re}=4 x_{0} R / \delta^{2}$.

Figure 3 of this paper proves the measured force for a very small oscillation amplitude $x_{0}$ is in good agreement with Stokes's result, a $6 \%$ difference being attributed to the imperfect knowledge of the wire linear mass density. When $x_{0}$ increases, the force is not a linear function of
the amplitude $x_{0}$ and these authors then define a complex function $C\left(x_{0} / \delta, R / \delta\right)$ which measures the deviation from linearity of the dependence of the force with $x_{0}$. All the results are expressed by the magnitude and the phase of the function $C\left(x_{0} / \delta, R / \delta\right)$.

Figure 4 (its horizontal axis is mislabeled and its label should be $x_{0} / \delta$ ) shows the variations of the function $C\left(x_{0} / \delta, R / \delta\right)$ with $x_{0} / \delta$ : this variation is almost independent of $R / \delta$ which decreases from $\delta=171 \mu \mathrm{~m}$ for a 0.5 Hz oscillation frequency down to $\delta=35 \mu \mathrm{~m}$ for a 12 Hz oscillation frequency. Then figure 13 of this paper shows the variations of $C\left(x_{0} / \delta, 0\right)$, with experimental and numerical results in excellent agreement. From this figure, we deduce that the deviation from linearity is below $1 \%$ if $x_{0} / \delta<0.6$ (and below $10 \%$ if $x_{0} / \delta<2.24$ ). This result is true over the studied range of $R / \delta$ value namely $0.08<R / \delta<0.38$ corresponding to the range of Stokes number $0.024<\mathrm{St}<0.58$. In this range, the above discussion means that Stokes' result is valid within $1 \%$ if $x_{0} / \delta<0.6$ i.e. if $\operatorname{Re}<\operatorname{Re}_{\max } \approx 1.2 \sqrt{\mathrm{St}}$.
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