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1 Design of topological elastic waveguides

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Topological physics is emerging as an active area of research, addressing fundamental questions on how geometry, symmetry and topology affect physical properties, paving the way towards novel technological applications. Originally investigated in quantum systems, these concepts have been thereafter translated across diverse domains including, electromagnetic, plasmonic, elastic and acoustic waves. Specifically, in elasticity, due to the strong tendency to hybridize of wave modes with different polarization, topological protection is viewed as a revolutionizing approach to design waveguides supporting unique features such as (i) being immune to defects and (ii) suppressing back-scattering during the wave propagation phenomenon. These novel features arise as a consequence of their dispersion surface topology.

This tutorial aims to introduce the theoretical, numerical and experimental frameworks to investigate topological elastic waveguides, discussing the key ideas, first, in the context of discrete systems, and then, in continuous elastic solids. After a comprehensive description of the currently used state of the art scientific techniques, various classes of topological wave phenomena leading to localized waves in elastic architected plates and beams are presented. Implications of the presence of both longitudinal and shear waves in elastic solids are discussed, and the associated challenges, opportunities and strategies to exploit their interplay highlighted. The symmetry conditions required to induce them are discussed using a number of representative examples. Finally, future research directions of this fledgling field are outlined.

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8 I. INTRODUCTION

9 A waveguide is a structure that guides waves with minimal loss of energy by restrict-
10 ing the transmission of energy to one direction. Waveguides in diverse areas of physics
11 exist depending on the kinds of energy they transport, e.g., elastic, acoustic, electromag-
12 netic, plasmonic and electronic. The governing equations associated to the type of wave
13 propagation vary accordingly. For example, wave propagation in elastic media is governed
14 by the Cauchy equations of elastodynamics, in fluids by the Helmholtz equation, while
15 Maxwell and Schrödinger equations govern electromagnetic waves and electron transport,
16 respectively. Note that these equations are distinct in nature, with the Cauchy, Helmholtz
17 and Maxwell equations being systems of hyperbolic partial differential equations while the
18 Schrodinger equation is dispersive¹. The common link between them that is relevant to the
19 present topic is the existence of wave-like solutions in periodic domains that obey the Bloch
20 theorem²⁻⁴.

21 There are key differences between wave types supported in various physical media. Elas-
22 tic solids support both longitudinal (pressure) and transverse (shear) waves. In contrast,
23 fluids support only longitudinal waves⁵ while electromagnetic media only support transverse
24 waves⁴. Elastic waveguides have been extensively investigated both to understand the
25 fundamental properties of materials and for technological applications including sensing,
26 actuation, signal processing and energy conversion or harvesting, to name a few^{6,7}. For
27 instance, sensing applications may include pressure, temperature and strain measurement.
28 Such waveguides are generally comprised of a straight channel, embedded with piezoelec-
29 tric transducers. If the length of the channel, and thus the spacing between the embedded
30 transducers changes, then the frequency of the traversing wave also changes. This shift
31 in frequency is correlated with the quantity to be measured. Wave steering for actuation,
32 energy harvesting, vibration control and structural health monitoring are other widespread
33 applications of elastic waveguides^{7,8}.

34 In this context, over the last two decades, architected materials or metamaterials⁹ have
35 lead to a new class of waveguides exhibiting unique wave control opportunities. For instance,
36 Fig. 1 presents an example of an elastic metamaterial-based waveguide showing energy redi-
37 rection features. The system consists of a plate with periodic circular inclusions arranged in
38 a square lattice. Due to the periodicity, the waveguide does not allow propagation of waves

39 in certain band of frequencies, often referred to as *frequency bandgaps*⁹. In this band, the
40 wave is confined to the strip (straight or L-shaped as illustrated). This example illustrates
41 how designing the geometry of the structure can lead to waveguiding along specific paths.

42 Although the metamaterial paradigm represented an excellent candidate to boost wave-
43 guide applications, in fact they have not found widespread technological diffusion. The pri-
44 mary reason is that their unique behavior is sensitive to the presence of defects and imper-
45 fections. In particular, significant losses arise due to scattering and localization at defect
46 locations or at corners, as illustrated in Fig. 1b. To overcome these problems, recently, a new
47 class of architected structures called *topological waveguides* have been introduced, rapidly
48 evolving as a leading field of research. Such structures exploit the symmetry and topology of
49 their dispersion surfaces to support modes that are immune to defects, to imperfections and
50 that do not suffer from scattering losses. This immunity arises as a consequence of topologi-
51 cal properties, and hence these waves are called *topologically protected*. Originally discovered
52 in $2D$ electron gases in the context of the quantum Hall effect¹⁰, they have been extended to
53 other classical areas of physics in the past decade. Examples include electromagnetic¹¹, plas-
54 monic¹², acoustic¹³, electromechanical¹⁴ and elastic^{15–18} media. Even though these diverse
55 media have very distinct governing equations, the concept of topological modes translate
56 across disciplines because the protection arises from specific symmetry properties of the
57 eigenvalue problem¹⁹, or more specifically, a family of eigenvalue problems as a parameter
58 is varied (see sections IV and V, for the details).

59 Topological modes in elastic media can be broadly classified into two categories. The
60 first one involves active components, like rotating gyroscopes^{22,23}, and it has been primarily
61 demonstrated for discrete elastic media. The second category solely uses passive components,
62 and has been demonstrated in a wide variety of discrete systems such as combination of
63 pendulums and levers²⁴, rotating disks²⁵, mass-spring networks^{26–28}, as well as for continuous
64 elastic media like architected plates^{16,29–31}.

65 This tutorial aims to introduce the fundamental concepts and working principles of such
66 topological waveguides in elastic media, as well as practical steps to design them. It is
67 organized as follows. First, an overview of various waves supported by homogeneous and
68 architected elastic media is presented. This is followed by the description of the most
69 commonly used theoretical, numerical and experimental tools to investigate periodic elastic
70 structures. In section IV, illustrative examples of topological modes are provided using

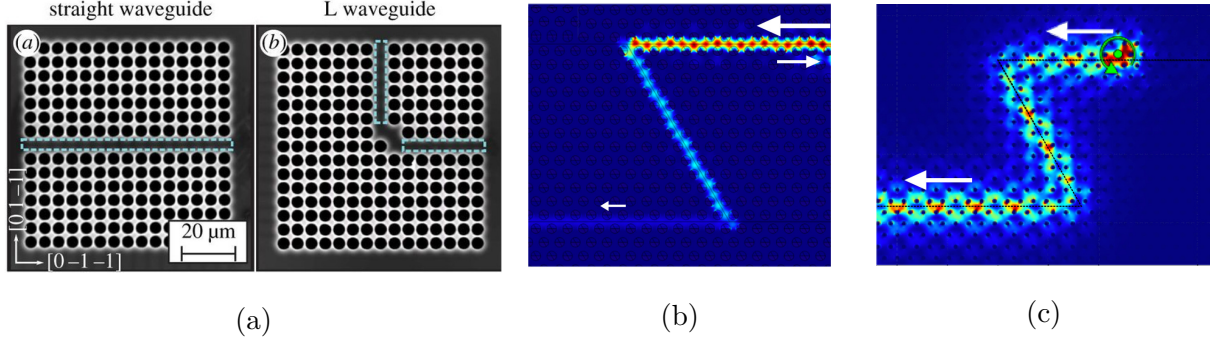


FIG. 1: (a) Example of waveguides allowing energy to propagate towards (left panel) a straight or (right panel) a L-shaped path²⁰. Comparison of the wave field reconstruction for (b) a trivial and (c) a topological photonic waveguide. Wave reflects back at corners in the trivial case while no energy is reflected in the topological one²¹. (Reproduced with permission from Phil. Trans. A 373, 20140364 (2015) and Phys. Rev. Lett. 114, 127401 (2015). Copyright 2015 Royal Society and 2015 American Physical Society.)

71 discrete mass-spring chains and beams. The key ideas leading to the different types of
 72 modes and their combination are described in detail. Section V presents the realization
 73 of various topological modes in continuous elastic media and discusses the key steps and
 74 strategies for their design. Finally, promising future directions are outlined in Sec. VI.

75 II. WAVES IN ELASTIC MEDIA

76 Elastic waves can be defined as disturbances propagating in an elastic solid due to a local
 77 deviation from static mechanical equilibrium conditions³². Such disturbances manifest as
 78 time varying displacement and stress fields in the solid. If the deformations are small enough
 79 (i.e., small displacement assumption holds), the medium can be assumed to be linear elastic
 80 and the wave characteristics do not depend on the disturbance amplitude. Depending on
 81 the domain geometry and boundary conditions (finite or infinite media, presence of free
 82 surfaces) of the elastic medium, various types of elastic waves can be identified.

83 A. Bulk, surface and guided waves in homogeneous media

84 Waves propagating indefinitely in an infinite homogeneous body without being inter-
85 rupted by boundaries or interfaces are called bulk waves. The general elastodynamic wave
86 equation governing the evolution of the displacement field $\mathbf{u}(\mathbf{x}, t)$ has the form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial}{\partial \mathbf{x}} \left(\mathbb{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \quad (1)$$

87 where \mathbb{C} is the fourth-order elasticity tensor and ρ is the density^{33,34}. In a linear, isotropic
88 and elastic medium, this governing equation reduces to the form

$$\rho \ddot{\mathbf{u}} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \quad (2)$$

89 where λ, μ are the material properties called *Lamé constants*. Solutions to Eqn. (2) support
90 one *longitudinal* wave and two *shear* waves with specific wave velocities that are functions of
91 the material's properties and independent of the frequency or the direction of propagation.
92 These waves have distinct types of motion. Longitudinal waves (also called P-waves) are
93 characterized by particle motion alternating compression and stretching of the medium (see
94 left panel of Fig. 2a). Shear waves are characterized by transverse particle movements in
95 alternating direction (see central and right panels of Fig. 2a). In P-waves, energy propagates
96 parallel to the displacement direction of a point, while in the second one, energy propagates
97 perpendicular to a point's displacement direction. In shear waves, displacement occurs in the
98 plane normal to the propagation direction and the wave can be decomposed into horizontal
99 shear wave (SH) and vertical shear wave (SV), as shown in the central and right panels of
100 Fig. 2a.

101 Wave propagation in anisotropic media is governed by the general elastodynamics equa-
102 tion Eqn. (1) and their analysis is considerably more complicated. Examples of anisotropic
103 elastic media include piezoelectric media, natural materials like wood, engineered materials
104 like composites. They can be classified as triclinic, monoclinic, orthotropic, cubic and trans-
105 versely isotropic depending on how isotropy broken along various symmetry axes. Similar
106 to isotropic media, anisotropic media also support three types of waves in an infinite solid.
107 However, there is no clear distinction like longitudinal and shear waves since the particle
108 motion can be at an arbitrary angle to the wave front propagation direction. In addition,
109 this angle and the wave velocity are also direction dependent. The wave front shape is repre-
110 sented by slowness diagrams or surfaces, and they illustrate the variation in wave speed with

111 direction. Anisotropic media with boundaries can also support Rayleigh and guided waves
112 and their analysis often requires numerical procedures. The interested reader is directed to
113 references discussing elastic waves in anisotropic media³⁵. In this tutorial, we will restrict
114 the attention to solids that are isotropic in the bulk at the length scales considered, i.e.,
115 macroscopic.

116 A three-dimensional ($3D$) medium bounded at a surface supports the propagation of
117 waves localized at the free half-space, namely Rayleigh (R) waves and Love (L) waves (refer
118 to Fig. 2b). Rayleigh waves (left panel of Fig. 2b) are characterized by a counterclockwise
119 elliptical motion of the medium particles polarized in the xz plane, if the energy flows along
120 the x direction. The amplitude of these wave decreases exponentially as e^{-bz} with distance
121 z from the free surface. The exponent b is inversely proportional to the wavelength of the
122 wave³³. Love waves are horizontally polarized surface waves (see right panel of Fig. 2b).
123 They exist only when the top layer has a lower shear wave velocity than the semi-infinite
124 media below it. These waves derive from the interference of many shear waves guided
125 by the top elastic layer, i.e. the one with lower velocity characteristics and with the top
126 boundary free. Particle oscillations in Love waves involve alternating transverse movements.
127 The direction of medium particle oscillations is horizontal (for instance in the xz plane) and
128 perpendicular to the direction of propagation (x). As in the case of Rayleigh waves, their
129 wave amplitude decreases with depth.

130 Finally, in some materials, the relation expressed by Eqn. (1) does not fully describe their
131 response when subjected to an elastic strain. Indeed, certain materials become electrically
132 polarized when they are strained. This effect, called the direct piezoelectric effect, manifests
133 experimentally by the appearance of bound electrical charges at the surfaces of a strained
134 medium. It is a linear phenomenon, and the polarization changes sign when the sign of
135 the strain is reversed. Though piezoelectricity is a complex phenomena intimately related
136 to the microscopic structure of the solid, the macroscopic behavior can be quantitatively
137 captured in terms of a rather simple constitutive model. We direct the reader interested in
138 wave propagation in piezoelectric media to additional references³⁶⁻³⁸.

139 Let us now discuss wave propagation in structures where one dimension is much smaller
140 than the other two. Examples include plates and shells, where the thickness is smaller
141 compared to the in-plane dimensions. When a point in a plate (Fig. 2c, for example) is
142 transversely excited, waves propagating from this point encounter the upper and lower free

143 surfaces of the structure, leading to reflection and mode conversion (longitudinal waves to
144 shear waves, and vice versa). The wave field is a superposition of multiple reflections and the
145 resulting generated waves. This wave field can essentially be approximated as propagating
146 in the in-plane directions with a specific displacement field in the direction of the smaller
147 dimension (examples in Fig. 2d). Such waves are called Lamb waves or guided waves.

148 An elastic plate supports an infinite number of guided modes, each with a different
149 displacement field. Among them, the symmetrical and anti-symmetric zero-order modes
150 (shown in Fig. 2d, having symmetric and anti-symmetric transverse displacement field about
151 the plate center-plane) deserve special attention because (i) they are the only modes that
152 exist over the entire frequency spectrum from zero to indefinitely high frequencies, and (ii)
153 in most practical situations they carry more energy than the higher-order modes. Also, in
154 the low frequency range (i.e. when the wavelength is greater than the plate thickness) these
155 modes are called the “extensional mode” and the “flexural mode” respectively, terms that
156 describe the nature of the particle motion.

157 Lamb waves are dispersive, i.e., their wave velocities depends on the frequency in addition
158 to the material and geometric properties of the plate. These wave velocities require numerical
159 procedures and cannot be determined in analytical closed form^{33,34}. Guided waves travel
160 long distances with little attenuation, making them well suited for non-destructive evaluation
161 and structural health monitoring applications. The word *waveguide* arises from structures
162 hosting these types of waves. The interested reader is directed to references that discuss
163 approaches to extract Lamb wave solutions in isotropic, anisotropic, as well as piezoelectric
164 waveguides^{8,33,36}.

165 B. Waves in architected structures

166 In previous section we have seen that the properties of bulk, surface and guided waves
167 are determined by the relationship between the particle motion, wave propagation direction
168 and the constitutive properties of the medium. In homogeneous, isotropic (infinite) media,
169 the energy flow coincides with the wave front of the wave.

170 However, in many practical cases, the assumptions of isotropy and infinite extent are
171 not always satisfied, and often waves propagate in media with geometrical or mechanical
172 discontinuities. When the elastic waves propagate in structures with complex geometries,

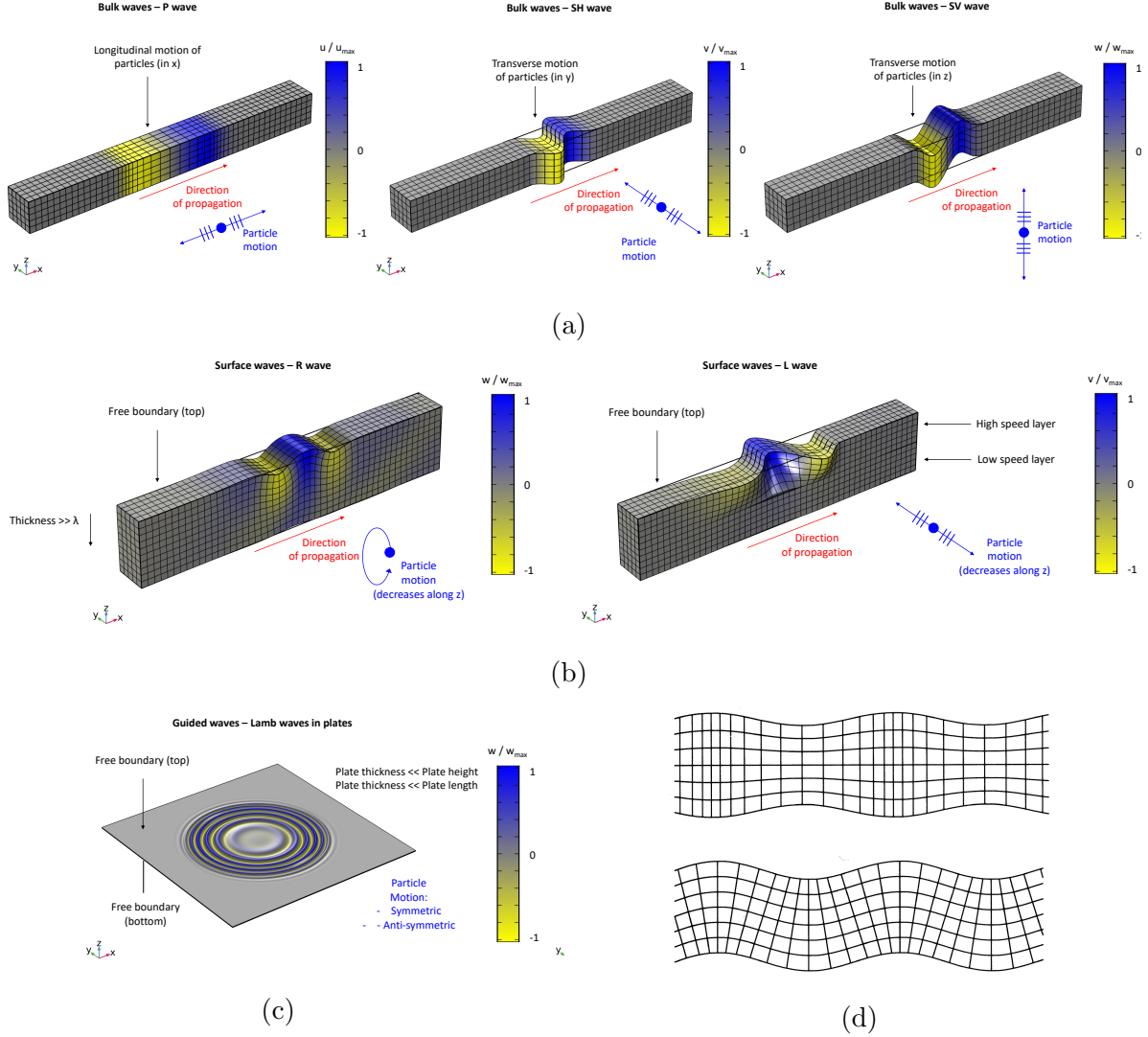


FIG. 2: (a) Bulk Waves propagating in an infinite media and characterized by (left panel) particle motion alternating compression and stretching of the medium, and (center, right panels) transverse particle motion in orthogonal planes. (b) A 3D medium with a free surface allows for the propagation of waves localized at the free half-space. Left panel: Rayleigh waves and right panel: Love waves. (c) Lamb waves in a plate. (d) **First two Lamb modes, top: symmetric and bottom: anti-symmetric about the plate center-plane.**

173 multiple reflections take place due to impedance mismatch at the interfaces and free surfaces,
 174 strongly complicating the energy flow/redistribution. **The most general case is when waves**
 175 **propagate in structures with no spatial inversion symmetry and broken time-reversal sym-**
 176 **metry, i.e., where the material parameters and geometrical design of the system randomly**
 177 **vary both in space and time. A mechanical system has broken time reversal symmetry if**

178 its governing equations change under the transformation $t \rightarrow -t$. Examples include struc-
179 tures with attached spinning gyroscopes^{22,23} and systems with damping. Similarly, spatial
180 inversion symmetry means the structure is invariant under the transformation $\mathbf{x} \rightarrow -\mathbf{x}$.

181 If a certain regularity is assumed for the spatio-temporal modulation (for instance intro-
182 ducing periodicity), the use of the space-time Floquet theory^{39–42} can be envisaged. So far,
183 non-reciprocity through space-time modulation has been investigated in optical and elec-
184 tromagnetic systems, in the case of mechanical waves, as well as in airborne acoustics^{43–45}.
185 Breaking time-reversal symmetry requires active components that introduce/remove energy
186 into/from the structure.

187 Another emerging area of research concerns the propagation of elastic waves in solely
188 passive media characterized by a high degree of complexity in their geometrical design. Often
189 periodic, i.e., characterized by a unit cell periodically repeated in space, these media are
190 usually referred to as *architected metamaterials*. The periodic architecture of these structures
191 can be limited to one, two or three dimensions, and be at the same scale (ordinary architected
192 structures)⁹ or at multiple scale levels (hierarchical architected structures)^{46,47}. Their static
193 and dynamic properties derive from a tailored geometry and material distribution (creating
194 specific impedance jumps/variatioins), in addition to their material constituents. Compared
195 to waves in conventional materials described in the previous section, architected materials
196 have opened novel ways of manipulating and controlling the propagation of elastic waves.
197 Examples include omnidirectional stop bands or full bandgaps (BGs), negative refraction,
198 wave focusing and perfect transmission of waves at sharp angles^{48–52}.

199 Figure 3 reports an example of an architected elastic waveguide, i.e., namely a plate
200 with circular holes in a hexagonal lattice arrangement. The smallest repetitive block, if
201 one exists, is called *unit cell*, and in the considered case is highlighted in light green in
202 Fig. 3a, and reported in its in-plane and trigonometric view in Figs. 3b,c, respectively. The
203 interplay of periodicity, geometry and material composition within each unit cell can result
204 in effective mechanical properties very distinct from those of its constituent materials. The
205 unconventional dynamic properties may derive from (i) Bragg scattering, i.e., a destructive
206 interference arising from an impedance mismatch proportionally periodic in space to $na/2$,
207 with $n \in \mathbb{N}$ and a the lattice parameter³, or (ii) local resonances⁵³.

208 The information about the wave propagation within these structures is often derived
209 through the examination of the so-called dispersion diagram or dispersion surfaces. In what

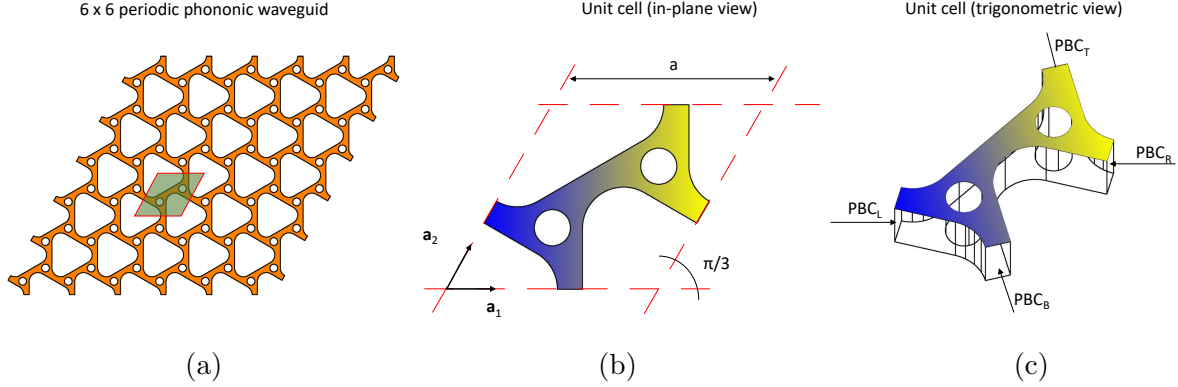


FIG. 3: (a) In-plane view of an architected waveguide consisting of a plate with circular and triangular holes in a periodic arrangement. The unit cell, i.e. the smallest repetitive building block, is highlighted in light green. (b) In-plane and (c) isometric view of the unit cell. \mathbf{a}_1 and \mathbf{a}_2 are the lattice vectors. Dispersion analysis of this periodic structure is conducted by the imposing periodic boundary conditions (PBC) at the domain edges (see section III B for further details).

210 follows, the principal approaches to extract these information are presented.

211 III. METHODS FOR DISPERSION ANALYSIS OF PERIODIC ELASTIC 212 STRUCTURES

213 Dispersion analysis gives the frequency-wave vector $\omega(\boldsymbol{\kappa})$ relation, i.e., the relation be-
 214 tween spatial and temporal periodicity of a traveling wave. An arbitrary wave of sufficient
 215 regularity can be expressed as a superposition of harmonic waves of distinct frequencies. Dis-
 216 persion relation specifies how each individual frequency component propagates in a solid. In
 217 particular, it provides information of the wave vector as a function of the frequency and its
 218 gradient gives the group velocity of the wave. Dispersion relations completely characterize
 219 the dynamic behavior of a linear elastic solid because in conjunction with the superposition
 220 principle, they can be used to predict the propagation of any arbitrary wave. In this section,
 221 we discuss various methods to determine dispersion relations for an architected linear elastic
 222 solid.

223 **A. Analytical techniques**

224 The plane wave expansion method gives a good approximation to the mode shape and
 225 natural frequency of an architected elastic media that is periodic. Let us consider a plane
 226 wave traveling through a periodic elastic structure with frequency ω and wave vector $\boldsymbol{\kappa}$. The
 227 wave vector may be viewed as the spatial analogue of frequency. Just as frequency measures
 228 how rapidly a point in the medium is varying with time, wave vector measures how rapidly
 229 the displacement field is varying in space. Hence, it is a vector having each component κ_p
 230 inversely proportional to the corresponding wavelength along the direction with unit vector
 231 \mathbf{e}_p .

232 The displacement field due to this traveling wave is expressed as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})e^{-i\omega t}. \quad (3)$$

In a periodic domain with lattice vectors \mathbf{a}_p , $\mathbf{U}(\mathbf{x})$ satisfies the Bloch periodicity condition
 $\mathbf{U}(\mathbf{x} + \mathbf{a}_p) = e^{i\boldsymbol{\kappa} \cdot \mathbf{a}_p} \mathbf{U}(\mathbf{x})$ for each periodic direction p and $\mathbf{U}(\mathbf{x})$ is called a Bloch mode. Let
 us derive an expression for $\mathbf{U}(\mathbf{x})$ that will help us to determine the dispersion surfaces. We
 start by considering the function $\mathbf{g}(\mathbf{x}) = \mathbf{U}(\mathbf{x})e^{-i\boldsymbol{\kappa} \cdot \mathbf{x}}$. The Bloch periodicity condition then
 implies that $\mathbf{g}(\mathbf{x})$ is periodic in the unit cell, i.e., $\mathbf{g}(\mathbf{x} + \mathbf{a}_p) = \mathbf{g}(\mathbf{x})$. Indeed,

$$\mathbf{g}(\mathbf{x} + \mathbf{a}_p) = \mathbf{U}(\mathbf{x} + \mathbf{a}_p)e^{-i\boldsymbol{\kappa} \cdot (\mathbf{x} + \mathbf{a}_p)} = e^{i\boldsymbol{\kappa} \cdot \mathbf{a}_p} \mathbf{U}(\mathbf{x})e^{-i\boldsymbol{\kappa} \cdot \mathbf{a}_p} e^{-i\boldsymbol{\kappa} \cdot \mathbf{x}} = \mathbf{U}(\mathbf{x})e^{-i\boldsymbol{\kappa} \cdot \mathbf{x}} = \mathbf{g}(\mathbf{x}).$$

233 Since $\mathbf{g}(\mathbf{x})$ is a periodic function with periodicity of the unit cell, it can be expressed as a
 234 Fourier series in the form

$$\mathbf{g}(\mathbf{x}) = \sum_{m,n,r=-\infty}^{\infty} \mathbf{a}_{m,n,r} e^{i\mathbf{G}_{m,n,r} \cdot \mathbf{x}}, \quad \mathbf{G}_{m,n,r} = m\mathbf{a}_1 + n\mathbf{a}_2 + r\mathbf{a}_3. \quad (4)$$

235 Here m, n and r take integer values and $\mathbf{a}_{m,n,r}$ is a vector with complex coefficients for each
 236 basis function with index (m, n, r) . For periodic beams, we only have \mathbf{a}_1 while for periodic
 237 plates, we only have \mathbf{a}_1 and \mathbf{a}_2 . Using this equation, the displacement field can be written
 238 as

$$\mathbf{u}(\mathbf{x}, t) = \sum_{m,n,r} \mathbf{a}_{m,n,r} e^{i\mathbf{G}_{m,n,r} \cdot \mathbf{x}} e^{-i\omega t}. \quad (5)$$

239 The real part of the right hand side in the above equation gives the displacement field.

240 Equation (5) is the starting point of the plane wave expansion method. The displacement
 241 field can be expressed as a superposition of periodic waves with periodicity of the unit cell,

242 along with a factor $e^{i\boldsymbol{\kappa}\cdot\mathbf{x}}$ that takes into account the wavelength of the wave. The functions
 243 $e^{i\mathbf{G}_{m,n,r}\cdot\mathbf{x}}$ constitute an orthonormal basis over a unit cell and determine the part of the Bloch
 244 mode that fluctuates within the unit cell. To determine the dispersion relation $\omega(\boldsymbol{\kappa})$, we
 245 choose a finite set of basis function by allowing m, n, r to take integer values from $-N$ to N
 246 in Eqn. (5). The coefficients $\mathbf{a}_{m,n,r}$ are determined by substituting the expression for $\mathbf{u}(\mathbf{x}, t)$
 247 into the governing equations, taking the scalar product with each basis function $\mathbf{e}_p e^{i\mathbf{G}_{m,n,r}\cdot\mathbf{x}}$
 248 and integrating over a unit cell. Recall that \mathbf{e}_p is the unit vector along direction with index
 249 p . This procedure leads to a system of linear homogeneous equations defining an eigenvalue
 250 problem and its solution gives the dispersion surfaces. In particular, we impose each value
 251 of the wave vector and determine the corresponding natural frequencies.

252 Let us now illustrate the PWE method with an example. Figure 4a displays a metamate-
 253 rial bar with two alternating materials whose Young's modulus E are different. The darker
 254 material has E 4 times higher than the lighter material. The PWE method can be used to
 255 calculate the dispersion surfaces of this structure. Figure 4b displays the real part of two
 256 basis functions $e^{i(G+\boldsymbol{\kappa})\cdot\mathbf{x}}$ for the PWE method with $\boldsymbol{\kappa} = 2\pi/3L$ and $G = 0, 2\pi/L$. Applying
 257 this method yields the frequencies and corresponding mode shapes of the propagating waves.
 258 The red curve in Fig. 4a shows the displacement field of the first mode shape at a wavenum-
 259 ber $\boldsymbol{\kappa} = 2\pi/3$ over 6 unit cells. For this wavenumber $\boldsymbol{\kappa}$, Bloch periodicity condition implies
 260 that the displacement field is periodic over 3 unit cells, which is consistent with the curves
 261 in Fig. 4a. The reader may refer to additional detailed illustrations on the PWE method
 262 for analysis of propagating^{54–61} and evanescent^{62,63} waves in architected elastic structures.

263 The analytical plane wave expansion method works well for simple geometries where the
 264 mode shape can be represented with a few basis functions N in Eqn. (5). As the unit cells
 265 become more complex, incorporating complicated shaped holes or inclusions, the error in
 266 mode shape represented with a small N is high, while increasing N leads to ill conditioning of
 267 the resulting matrices. To overcome these issues, numerical methods based on finite element
 268 analysis are used to efficiently determine the dispersion surfaces for complex geometries. We
 269 discuss this method in the following subsection.

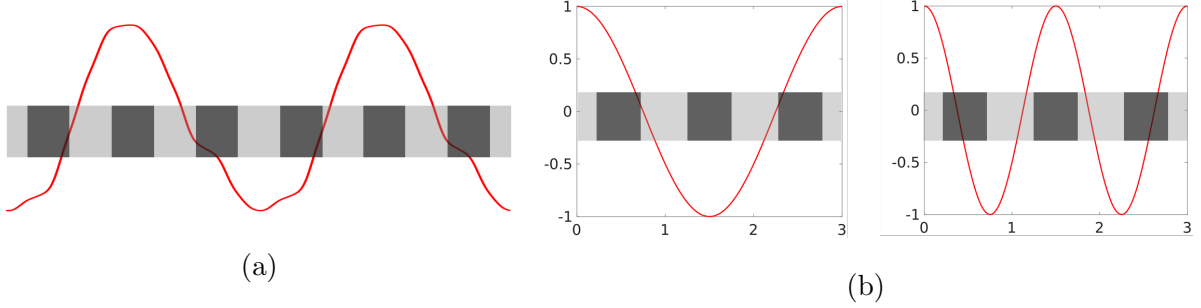


FIG. 4: (a) Mode shape (red curve) of a metamaterial bar (schematic behind the curve) with two materials of different Young's modulus at a wavenumber $\kappa = 2\pi/3L$. Displacement field is shown over 6 unit cells. (b) Real part of typical basis functions

$e^{i(\kappa+G)x}$ in the PWE method with $G = 0$ and $G = 2\pi/L$. Basis functions are shown over 3 unit cells for clarity.

270 B. Numerical techniques

271 The governing equation (2) for a 3D linear elastic solid is considered over a single unit
 272 cell with appropriate boundary conditions to determine the dispersion surfaces. To illustrate
 273 these conditions, let us consider the unit cell in Fig. 3c. The displacement fields at the right
 274 and top boundary are expressed in terms of the corresponding fields at the left and bottom
 275 boundary using Bloch periodicity as

$$\mathbf{u}(\mathbf{x}_R, t) = e^{i\kappa \cdot \mathbf{a}_1} \mathbf{u}(\mathbf{x}_L, t), \quad \mathbf{u}(\mathbf{x}_T, t) = e^{i\kappa \cdot \mathbf{a}_2} \mathbf{u}(\mathbf{x}_B, t). \quad (6)$$

276 Similarly, by force equilibrium at each point on the boundary, the traction \mathbf{t} on the various
 277 surfaces are related as

$$\mathbf{t}(\mathbf{x}_R, t) = -e^{i\kappa \cdot \mathbf{a}_1} \mathbf{t}(\mathbf{x}_L, t), \quad \mathbf{t}(\mathbf{x}_T, t) = -e^{i\kappa \cdot \mathbf{a}_2} \mathbf{t}(\mathbf{x}_B, t). \quad (7)$$

278 The two equations (6) and (7) constitute independent boundary conditions on each bound-
 279 ary. Substituting $\mathbf{u}(\mathbf{x}, t) = e^{i\omega t} \mathbf{U}(\mathbf{x})$ into the governing equations and using these two
 280 conditions leads to a well posed eigenvalue problem. Its solution provides the mode shapes
 281 $\mathbf{U}(\mathbf{x})$ and natural frequencies ω for each wave vector κ . The eigenvalue problem is in the
 282 form of a partial differential equation and it can be solved using numerical techniques.

283 **The finite element method (FEM) is the most widely used numerical method for disper-**
 284 **sion analysis of unit cells with complex geometries. This method involves two key steps to**

285 convert the governing partial differential equation to a system of algebraic equations⁶⁴. The
286 first step is to consider a weak form of the governing equations that results in a variational
287 problem over a function space. The second step is to reduce this variational problem to a
288 system of algebraic equations by *discretization*, i.e., choosing a finite dimensional subspace
289 that approximates the infinite dimensional function space. Discretization involves two com-
290 ponents: (a) meshing, i.e., dividing the domain into smaller sub-domains called elements,
291 and (b) element type, i.e., choosing a set of basis functions up to a specified order for an
292 element. The solution fields in each element lies in the vector space spanned by these basis
293 functions. The mesh should be sufficiently fine with well shaped elements so that the solution
294 can be represented with good resolution using the chosen element type. FEM formulation
295 for dispersion analysis has been done in recent years for beams, plates and solids^{65–67}. To-
296 day, several commercial software packages are available to conduct dispersion analysis using
297 FEM.

298 C. Experimental techniques

299 From the experimental point of view, and focusing our attention on the ultrasonic fre-
300 quency range, many methods to excite (indicated by “E” for the sake of brevity in this
301 section) and measure (“M”) elastic waves are widely used. A non-exhaustive list include: (i)
302 conventional and piezoceramic ultrasonic transducers (E/M); (ii) comb-type ultrasonic trans-
303 ducers (E/M); (iii) electromagnetic acoustic transducer (EMAT), for non-contact acoustic
304 wave generation and reception in conducting materials; (iv) magnetostrictive transducers
305 (E/M), exploiting the property of magnetic materials that causes them to change their
306 shape or dimensions during the process of magnetization; (v) impact-controlled approaches
307 (E), such as instrumented impact hammers; (vi) electrodynamic shakers (E); (vii) optical
308 fiber sensors (M); (viii) photoelasticity (M), describing changes in the optical properties of a
309 material under mechanical deformation; (ix) photothermal (M); (ix) scanning laser Doppler
310 vibrometer - SLDV (M), that determines the out-of-plane or normal velocity at each point
311 on the surface of an elastic structure through the Doppler effect. These techniques, which
312 can be selectively chosen according to the specific type of wave to excite/detect (Rayleigh,
313 Lamb, Shear Horizontal, etc.) and the experimental conditions of measurement, can be
314 separated in two categories: those requiring a contact with the sample and those that are

315 contactless.

316 Among the above listed techniques for wave detection, the SLDV is being used increas-
317 ingly often in recent years for several reasons, especially in the case of architected materials.
318 First of all, SLDV, being a contactless measurement technique, eliminates the detrimental
319 effect of adding additional masses in the measurement locations, which can result in a lo-
320 cal change of the structure rigidity. Another advantage of SLDV is its ability to perform
321 measurements automatically in a large number of precisely defined points, almost regardless
322 of the complexity of the geometrical pattern to scan. Also, SLDV allows for 3D measure-
323 ments, allowing object vibration components to be recorded both in the plane perpendicular
324 to the investigated surface and in the one parallel to it. Finally, SLDV allows automatic
325 measurements on a very dense mesh of measurement points, enabling thus the measure-
326 ment of a propagating elastic wave with precision, not only in time but also in space. The
327 measurement frequency range is another positive point of this technique, since it allows for
328 measurements from close to 0 Hz to several MHz, as well as a wide range of vibration veloc-
329 ity amplitudes⁷. All these advantages make SLDV one of the most effective measurement
330 techniques for dynamic characterization of architected materials and elastic waveguides. **In**
331 **contrast, it is more difficult to single out a best suited excitation technique and its choice is**
332 **often a result of several considerations including, (i) frequency range and (ii) type of waves**
333 **to excite, (iii) geometric and (iv) elastic properties of the specimen to investigate. We direct**
334 **the reader to reference books^{7,8} on these aspects.**

335 IV. OVERVIEW OF TOPOLOGICAL MODES

336 There are several classes or types of topological modes depending on the type of symmetry
337 that is broken in a lattice^{19,68}. As discussed in the introduction, they can broadly be classified
338 into two categories: those that break time reversal symmetry and those that preserve it. **This**
339 **classification is based on the simple observation that an elastic media is symmetric or not**
340 **under the transformation $t \rightarrow -t$. The physical meaning is that breaking time-reversal**
341 **symmetry requires active or dissipative components. A more fundamental classification of**
342 **symmetries involves using time-reversal, chiral and particle-hole symmetry operators is used**
343 **in quantum mechanics, where each of these has a physical meaning. In an elastic media,**
344 **applying the corresponding mathematical operators leads to constraint equations that may**

345 not be related to fundamental principles like balance and conservation laws.

346 We introduce the key concepts underlying topological modes through a number of exam-
347 ples of varying complexity. They are all based on discrete lattices or elastic beams with the
348 goal of elucidating the main ideas behind topological protection in a simplified setting^{69,70}.
349 The first example concerns discrete mass-spring chains followed by its extension to contin-
350 uous elastic media: architected beams. Afterwards, an example presenting lattices with a
351 varying parameter is considered (Sec. IV C), showing energy transport from one corner to
352 another. Sec. IV D presents an example of how topological modes can be induced in $2D$ dis-
353 crete lattices, and their relation to the concepts in $1D$ lattices are elucidated in Sec. IV A.
354 We will focus on two distinct classes of modes, namely helical and valley modes in $2D$
355 lattices.

356 A. Illustrative example: topological modes due to spectral flow

357 Let us illustrate the key idea behind topological modes with a simple discrete mass-spring
358 chain example. It is inspired by the SSH model, the first model that was demonstrated to
359 support localized electronic modes due to topological properties^{10,71}. The corresponding
360 ideas are valid and can be extended to continuous elastic media. Consider the infinite mass-
361 spring chain shown in Fig. 5a, having identical masses m and two springs of different stiffness
362 k_1 and k_2 . Note that this chain is a periodic structure. The unit cell of a periodic structure
363 is the smallest unit that generates the lattice by repetitive translation. Identifying a unit
364 cell requires careful consideration to satisfy this lattice generation requirement. The unit
365 cell of this chain has 2 springs (k_1, k_2) and 2 identical masses, labeled a and b .

Let us investigate the dynamic response of the infinite chain. The governing equations for a unit cell indexed n are

$$\begin{aligned} m\ddot{u}_{n,a} + k_1(u_{n,a} - u_{n,b}) + k_2(u_{n,a} - u_{n-1,b}) &= 0, \\ m\ddot{u}_{n,b} + k_1(u_{n,b} - u_{n,a}) + k_2(u_{n,b} - u_{n+1,b}) &= 0. \end{aligned}$$

366 We solve this system of equations in the Fourier domain by imposing a solution of the
367 form $u_{n,\alpha} = e^{i\mu n - i\omega t} U_\alpha$ with $\alpha = \{a, b\}$. Here μ and ω are the wavenumber and frequency,
368 respectively and U_α are the displacement of the two masses in the Fourier domain. The

369 above system of equations then become

$$\begin{pmatrix} k_1 + k_2 & -k_1 - k_2 e^{-i\mu} \\ -k_1 - k_2 e^{i\mu} & k_1 + k_2 \end{pmatrix} \begin{pmatrix} U_a \\ U_b \end{pmatrix} = \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} U_a \\ U_b \end{pmatrix} \quad (8)$$

370 Equation (8) defines an eigenvalue problem and its solution gives the dispersion surfaces
 371 of the structure, illustrated in Fig. 5c. For each wavenumber μ , there are two natural
 372 frequencies ω that are solutions of Eqn. (8). These surfaces characterize traveling wave
 373 solutions in the infinite lattice. Note that there are no real valued wavenumbers μ when Ω
 374 is in the range $(1, \sqrt{2})$ and when $\Omega > \sqrt{2}$, which indicates an absence of traveling waves at
 375 these frequencies.

376 Next, let us consider the dynamic behavior of a finite chain. In general, its natural
 377 frequencies depends on the chain's boundary conditions. To exemplify the ideas, let us
 378 consider a chain in the form of a ring (Fig. 5b) **with M unit cells**. Each mode shape can
 379 be identified with a corresponding traveling wave in the infinite lattice, that satisfies the
 380 condition

$$u_{n,\alpha} = u_{n+M,\alpha}. \quad (9)$$

381 This condition arises in the finite lattice due to the topology of the ring. Let us determine
 382 the wavenumber of the corresponding wave in the infinite lattice. Combining this condition
 383 (Eqn. (9)) with the Bloch periodicity condition leads to the following set of wavenumbers
 384 for a ring with M unit cells

$$u_{n+M,\alpha} = e^{i\mu M} u_{n,\alpha} = u_{n,\alpha} \implies e^{i\mu M} = 1. \quad (10)$$

385 Note that the solution to $e^{i\mu N} = 1$ is $\mu = 2\pi s/N$ for any integer s . Due to the discrete nature
 386 of the lattice, wavenumbers modulo 2π essentially represent the same wave. For example,
 387 the displacement field corresponding to waves with wavenumbers μ and $\mu + 2\pi$ are identical.
 388 A unique set of waves can be identified with wavenumbers lying in the first Brillouin zone.
 389 They are

$$\mu = \frac{2\pi s}{N}, \quad s = 0, \pm 1, \pm 2, \dots \pm \lfloor N/2 \rfloor. \quad (11)$$

390 Here $\lfloor N/2 \rfloor$ is the integer part of $N/2$. Hence each mode on the ring can be identified with
 391 a wavenumber from the set in Eqn. (11).

392 **The red circles in** Fig. 5c display these wavenumber values for a ring with $N = 5$ unit cells.
 393 This ring has $2N = 10$ masses and thus 10 natural frequencies. These natural frequencies

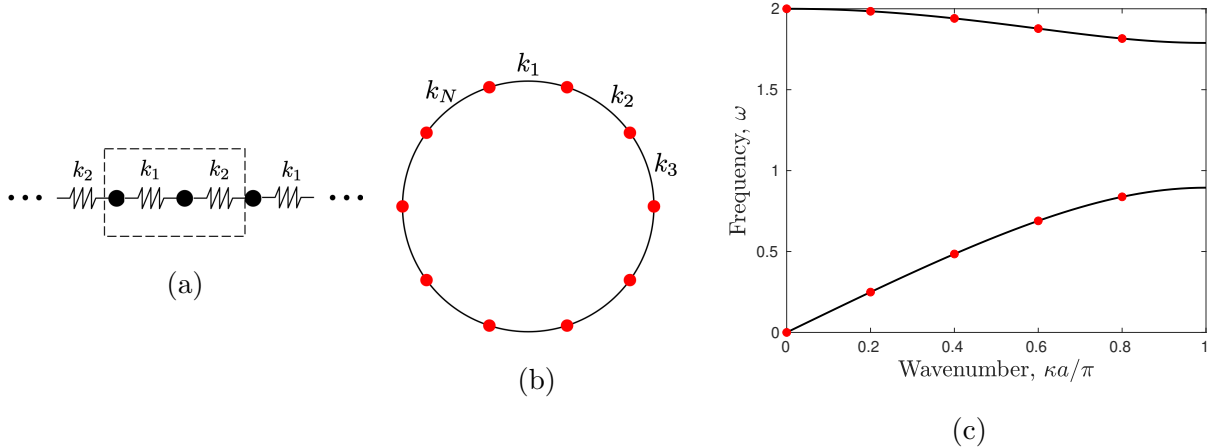


FIG. 5: (a) An infinite lattice with two distinct and alternating spring stiffness values. Dashed rectangle is a unit cell. (b) Hypothetical finite lattice in the form of a ring. (c) Dispersion curves of the lattice along with finite ring frequencies superimposed as red circles.

394 all lie on the dispersion curves and their corresponding wavenumbers take values in the set
 395 derived in Eqn. (10). This set ensures that there are $N = 5$ modes of the ring lattice in
 396 each dispersion branch. To summarize, the key observation is that the topology of the ring
 397 lattice ensures that equal number of its modes lie on each dispersion branch.

398 Let us now consider a ring of $N = 40$ masses with two kinds of unit cells, shown in
 399 Figs. 6a and 6b. We set the r -th spring stiffness in the ring to

$$k_r = 1 + \alpha \cos\left(\frac{2\pi r}{p}\right), \quad r \in \{1, 2, \dots, N\}. \quad (12)$$

400 Here $\alpha < 1$ and p is equal to the number of masses (or springs) in each unit cell. r is the
 401 spring index and it runs from 1 to N as there are N springs in the ring. Setting $p = 4$ and 5
 402 gives the rings whose unit cell schematics are shown in Fig. 6a and 6b. Their corresponding
 403 dispersion curves, along with the natural frequencies of the corresponding ring of N masses
 404 is also shown below (Figs. 6c, 6d). Let us make a couple of observations that are analogous
 405 to the chain with two distinct springs (k_1 and k_2) discussed above. The lattice with $p = 4$
 406 has 4 dispersion curves or branches and the N natural frequencies are distributed equally
 407 in each dispersion branch, with the wavenumbers given by Eqn. (10). Similarly, the lattice
 408 with $p = 5$ has 5 masses per unit cell, 5 dispersion curves and $N/5$ natural frequencies in
 409 each branch.

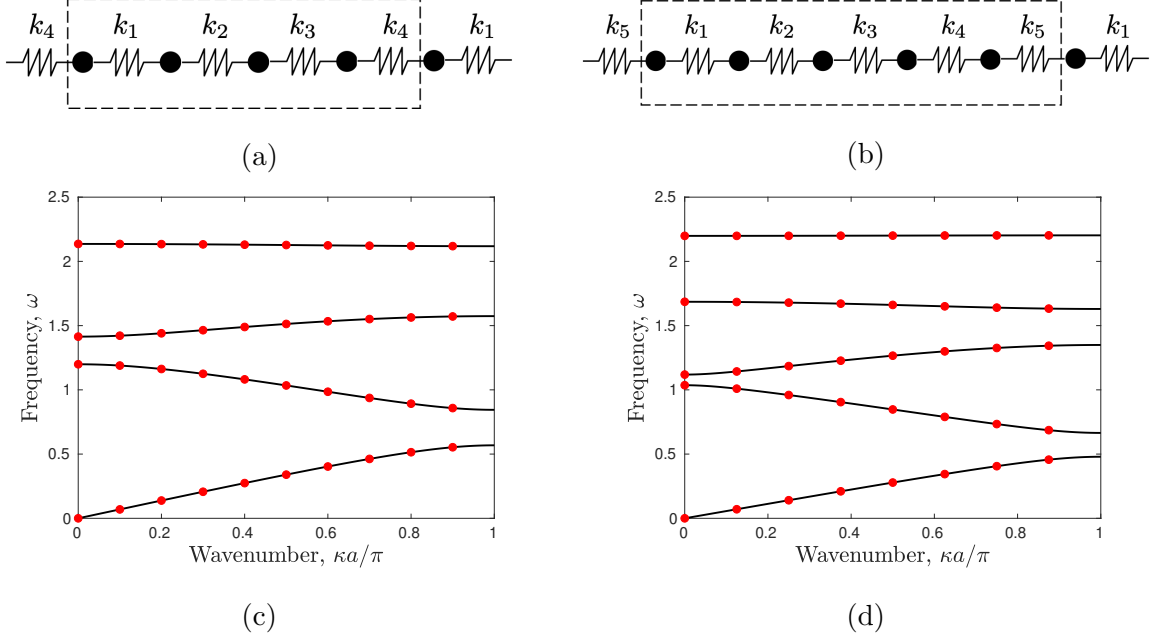


FIG. 6: Unit cells of lattices with (a) 4 and (b) 5 distinct spring stiffness values. Dashed rectangle shows unit cell. (c), (d): their dispersion curves having 4 and 5 branches. Red circles show the frequencies of a finite ring lattice of 40 masses are distributed equally among the dispersion curves: 10 and 8 in each branch, respectively, for the two lattices.

410 Now, let us examine what happens as we transition from the $p = 4$ to the $p = 5$ lattice.
411 We do this by setting $p = 4 + \beta$ in the denominator in Eqn. (12), with β taking values in the
412 set $[0, 1]$. The natural frequencies are given by solving the eigenvalue problem arising from
413 the governing equations for the masses in the ring. Figure 7 displays the natural frequencies
414 of the ring with N masses as a function of β . As β is varied continuously in the set $[0, 1]$, the
415 N natural frequencies also vary continuously. At $\beta = 0$, the $p = 4$ lattice has 4 dispersion
416 bands separated by bandgaps and there are $N/4 = 10$ frequencies in each band. On the
417 other hand, at $\beta = 1$, the $p = 5$ lattice has 5 dispersion bands with $N/5 = 8$ frequencies in
418 each band.

419 Let us analyze how the natural frequencies of the ring vary with β . Let us consider the
420 top dispersion band of the $p = 4$ and $p = 5$ lattices. These two lattices correspond to $\beta = 0$
421 and 1, respectively. At these two β values, all modes lie on the dispersion bands and the
422 bands are separated by bandgaps for both the lattices. The number of frequencies changes
423 from 10 to 8 as $10 - 8 = 2$ modes go to the second band (from the top) in the $p = 5$ lattice.
424 There are 8 modes in the second band for $p = 5$ lattice, 6 modes go from the second band

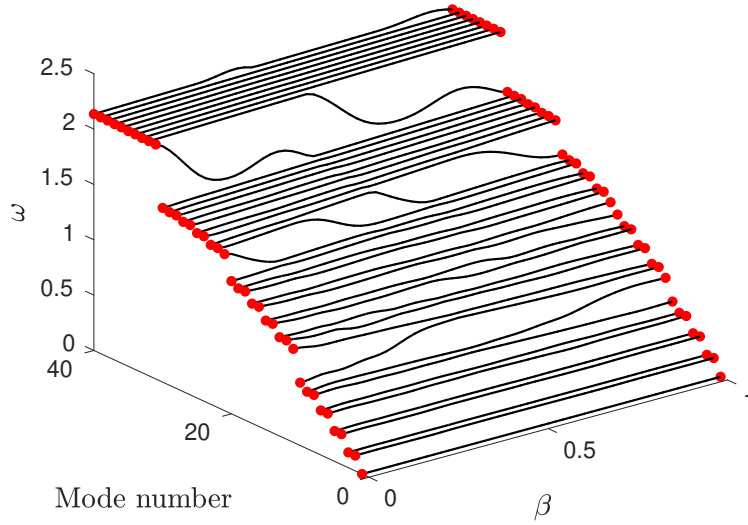


FIG. 7: Change in natural frequency of each mode with parameter β showing a spectral flow across the bandgaps. A number of modes traverse the gap to satisfy the requirement of equal number of modes in each dispersion branch at $\beta = 0$ and 1.

425 along with 2 modes from the top band at $\beta = 0$. Using similar counting arguments, we can
 426 determine how the 10 modes in each of the 4 branches of the $p = 4$ lattice get distributed
 427 into the 5 branches of the $p = 5$ lattice.

428 The key observation is that a number of modes flow across the bands as β varies from 0
 429 to 1 to satisfy the requirement of $N/5 = 8$ and $N/4 = 10$ modes in each dispersion branch in
 430 the $p = 5$ and $p = 4$ lattice, respectively. This flow of natural frequencies, or spectral flow,
 431 arises solely to satisfy the topological constraint of the number of modes in each dispersion
 432 branch. For $\beta = 0$, there are 10 modes in each of the 4 bulk bands and for $\beta = 1$, there are
 433 8 modes in each of the 5 bulk bands. The key topological argument is the following: as we
 434 vary β smoothly in $[0, 1]$, the only way to have a change in the number of modes in the top
 435 bulk band change from 10 to 8 is if $10 - 8 = 2$ modes migrate down to another bulk band.
 436 Thus, two modes traverse the bandgap as β is varied in $[0, 1]$ and these modes are labeled
 437 as topologically protected modes. This migration of modes is independent of the functional
 438 form of $p(\beta)$ in $[4, 5]$, with the only requirement being continuity. It is also independent
 439 of the specific functional form of k_m in Eqn. (12). Indeed, any functional form of k_m that
 440 gives distinct values 4 (or 5) distinct values of spring stiffness and leads to 4 (or 5) distinct
 441 dispersion bands should also exhibit such spectral flow.

442 In summary, we saw how modifying the parameter β smoothly from 0 to 1 leads to a
 443 spectral flow between the bandgaps. This hypothetical example of a lattice embedded on a
 444 ring shows the simplest realization of modes arising to satisfy the topological constraint of
 445 N/p modes in each dispersion branch. The ring shaped lattice in this hypothetical example
 446 can be extended to real structures where there is an interface or boundary between distinct
 447 lattice types. The mode shapes corresponding to the modes traversing the bandgap will be
 448 localized at such interfaces and boundaries. Indeed, such modes cannot be bulk modes since
 449 their natural frequency is in the bandgap. Since they have a topological origin, such modes
 450 cannot be removed in the presence of a wide class of defects or imperfections.

451 In the subsequent examples in this section, we will see how a similar spectral flow between
 452 bulk bands arises at a boundary of a lattice or at an interface between two lattices. The
 453 bulk dispersion surfaces of these lattices are characterized by topological invariants and such
 454 interface/boundary modes arise when they have different/non-zero topological invariants.

455 B. Localized modes in beams

456 Our next example shows how a spectral flow similar to that discussed in Sec. IV A arises
 457 in an elastic beam with ground springs. Figure 8 displays the schematic of the considered
 458 system, where the ground springs are located at a distance specified by a parameter θ . There
 459 are circles of radius r with centers located distance a apart, with $r < a$. The n -th ground
 460 spring location is obtained by projecting a point at an angle $n\theta$ from the circle to the beam.
 461 Specifically, this location is $x_n = na + r \sin(2\pi n\theta)$ and the periodicity of the resulting lattice
 462 pattern thus depends on θ . If θ is a rational number, for example p/q , where p and q are
 463 co-prime integers, then a unit cell has q springs. On the other hand, if θ is an irrational
 464 number, then there is no repetitive unit cell and the resulting structure is quasiperiodic.

The plane wave expansion method discussed in Sec. III A may be used to determine the natural frequencies of the beam structures considered here. The basis functions in Eqn. (4) running over a single index m are a valid choice, as they are periodic over a unit cell⁶⁹. Figure 9a displays the natural frequencies of a finite beam (red curves) overlaid on the corresponding frequencies for an infinite beam. It shows how the frequencies change with the parameter θ . Note that each vertical slice (constant θ line) is a different lattice. A homogeneous beam (without ground springs) has an infinite number of natural frequencies

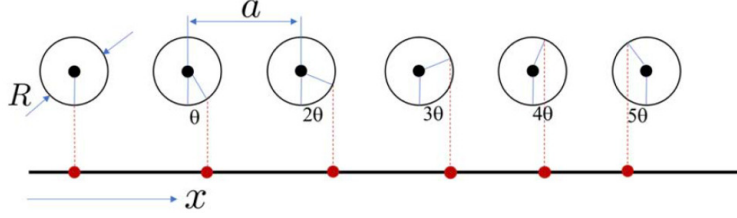


FIG. 8: Schematic of procedure to determine the location of each ground spring by projecting from a circle. Varying the parameter θ gives a family of periodic (quasiperiodic) structures for rational (irrational) θ^{69} .

without an upper bound and its dispersion relation is

$$\omega = \sqrt{\frac{EI}{\rho A}} \kappa^2.$$

465 Here κ is the wavenumber and E, I, ρ and A denote the Young's modulus, area moment of
 466 inertia, density and cross-section area of the beam. Let us first consider the $\theta = 0$ beam
 467 structure. Having a periodic arrangement of ground springs introduces a length scale, the
 468 distance a between adjacent ground springs. This length scale of the resulting unit cell
 469 introduces Bragg scattering bandgaps in the dispersion surface. For a finite beam with N
 470 ground springs and simply supported boundary conditions, there are N modes in this first
 471 band, i.e., below the first bandgap. This observation follows from similar arguments as the
 472 discrete lattice embedded on a ring that was considered in Sec. IV A.

473 Let us restrict attention to this first dispersion band at $\theta = 0$. As the parameter θ
 474 is varied in the set $[0, 1]$, this band splits into a number of smaller bands with bandgaps
 475 between them. The red curves corresponding to the finite beam fall within the bulk bands
 476 as θ takes values in the set $\{p/N : p = 1, 2, \dots, N\}$. For these values of θ , one can double the
 477 length L of the beam and set $u(L + x) = -u(L - x)$ and check by direct substitution that
 478 this displacement field will be an eigenmode of the doubled beam. This transformation now
 479 allows us to identify the mode shapes with that of a corresponding beam shaped in the form
 480 of a ring, i.e., without a boundary. Using similar arguments for the beam as in Sec. IV A,
 481 we arrive at this set $\{p/N : p = 1, 2, \dots, N\}$ of θ values. In addition, we observe spectral
 482 flow between the bulk bands as θ changes between these discrete values.

483 We thus see features similar to that observed in the hypothetical ring lattice in Sec. IV A
 484 in the elastic beam. Furthermore, the number of modes that traverse from one band to the

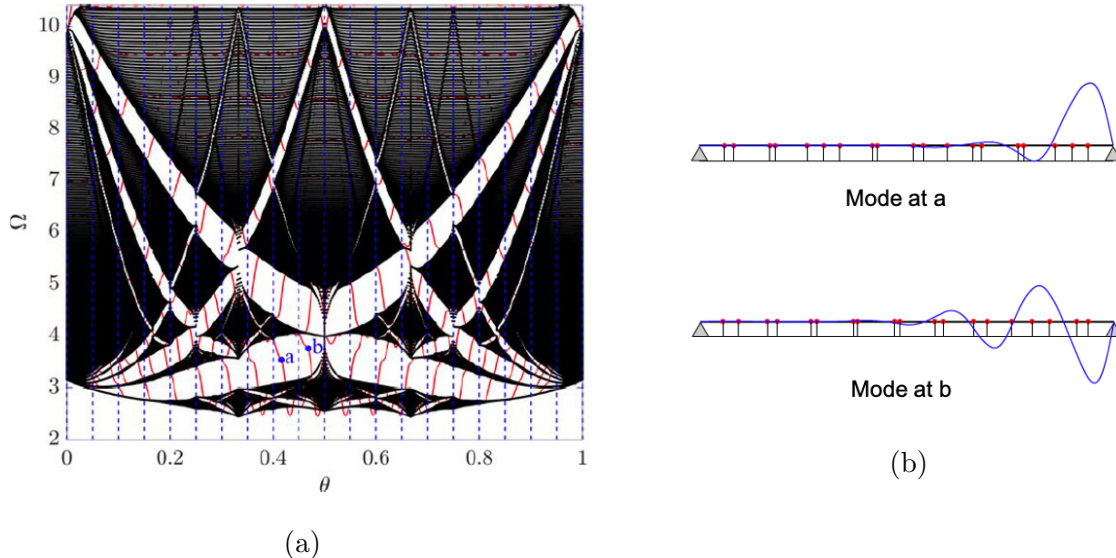


FIG. 9: (a) Frequencies of a finite beam with 20 ground springs (red curves) superimposed on infinite beam frequencies (black curves) for a family of lattices. Each θ is a distinct structure. Spectral flow between bulk bands (contiguous black curves) arises. (b) Mode shapes of two localized modes at points a and b in (a)⁶⁹.

485 other can also be determined using similar counting arguments. For example, let us consider
 486 two beam lattices, one with $\theta = 1/5$ and the other with $\theta = 1/4$. The first band at $\theta = 0$
 487 splits into 5 and 4 dispersion bands, respectively. The first $N = 20$ natural frequencies of the
 488 finite lattice are distributed equally in these dispersion bands. As θ varies between these two
 489 values, we observe a spectral flow between the various bands similar to the discrete lattice
 490 case. In particular, one mode flows from the bottom band of the $\theta = 1/5$ beam lattice to
 491 the second band of the $\theta = 1/4$ lattice to satisfy the constraint of specific number of modes
 492 in each band.

493 Having demonstrated how a similar spectral flow arises in an architected beam, let us
 494 examine the mode shape of this mode. Figure 9b displays two representative mode shapes,
 495 for modes in the bandgap marked as (a) and (b) in Fig. 9a. The modes are localized at
 496 the right boundary of the beam and their displacement amplitude decreases rapidly with
 497 distance from this end. Note that there are $N = 20$ such modes spanning this large bandgap
 498 - having modes marked (a) and (b) - as θ varies from 0 to 1. This number N is equal to
 499 the number of ground springs in the finite beam⁶⁹. For practical applications, it is more
 500 desirable to have a system or structure where the number of localized modes in a bandgap

501 is independent of the system size. The next example achieves this and its topological modes
 502 are independent of the number of unit cells in the lattice structure.

503 C. Topological elastic pumping of localized modes

504 In the last section, we saw how to achieve localized modes at the boundary of an elastic
 505 beam. Let us now study an example of a structure that can support transport of elastic
 506 energy by exploiting such topological modes. This example is inspired by the Thouless
 507 charge pump in quantum mechanics^{72,73}. We start by considering a mass-spring chain whose
 508 stiffness k_n of the n -th spring is

$$k_n = 1 + \alpha \cos\left(\frac{2\pi n}{3} + \phi\right). \quad (13)$$

509 Here $\alpha < 1$ and ϕ takes values in the set $[0, 2\pi]$. There are 3 masses in each unit cell and
 510 hence the lattice has 3 dispersion surfaces.

511 Figure 10a displays the dispersion surfaces for a family of lattices as ϕ varies from 0 to
 512 2π and $\alpha = 0.6$. The bandgaps do not close for small α . Now, let us consider a finite chain
 513 of 61 masses. This chain has $N = 20$ unit cells and an extra mass is added so that we get a
 514 *commensurate lattice*, one whose modes can be identified with modes in the corresponding
 515 infinite chain for $\phi = 0$. Figure 10b displays how the natural frequencies change with ϕ
 516 for this finite chain. Note that there is a mode that spans each bandgap as it flows from
 517 one bulk band to another and then back again. The solid and dashed lines in this bandgap
 518 indicate a mode localized at the left and right boundary, respectively. Figure 10c displays
 519 the mode shape of the mode spanning the first bandgap, showing how it transitions from
 520 one edge to another in the finite chain.

521 The origin of this spectral flow is also topological as the number of modes n_i in each bulk
 522 band changes as ϕ changes by π . For this considered lattice, all the modes lie on dispersion
 523 surfaces at $\phi = 0$ and $\phi = \pi$. The total number of modes is $3N + 1$ and they are distributed
 524 differently in each of the bulk bands for the two lattices $\phi = 0$ and $\phi = \pi$. The number
 525 of modes n_i can be determined directly by explicit calculations. The lattices at $\phi = 0$ and
 526 $\phi = \pi$ are illustrated in Fig. 11a. Note that the unit cells of these two lattices are translated
 527 copies of each other but the key point is that the location of the lowest spring stiffness is
 528 different. This difference leads to differences at the two ends of a finite chain and their

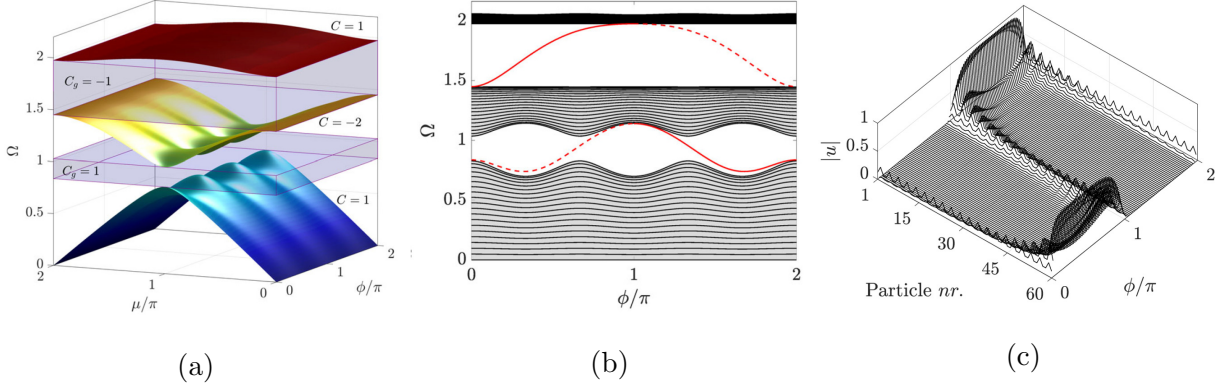


FIG. 10: (a) Dispersion surfaces of a family of lattices as a function of ϕ showing 2 bandgaps between the bulk bands, along with the Chern number. (b) Natural frequencies of a finite chain of $N = 61$ masses showing a spectral flow across the bandgap as ϕ varies. Dashed (solid) curves indicate the mode is localized on the left/right boundary. (c) Displacement magnitude of the localized mode in the second (higher) bandgap showing the transition from right to left boundary⁷⁴. (Reproduced with permission from Phys. Rev. Lett. 123, 034301 (2019). Copyright 2019 American Physical Society.)

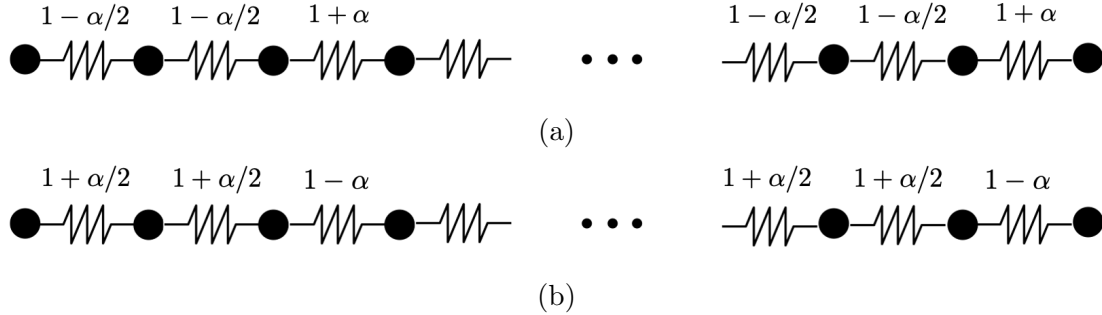


FIG. 11: The finite lattice at (a) $\phi = 0$ and (b) $\phi = \pi$ has distinct stiffness distribution. The number of frequencies lying on the first and third dispersion branches are different for $\phi = 0$ and π . This difference implies a spectral flow in the bandgap as ϕ varies in $[0, \pi]$.

529 resulting natural frequencies. A direct calculation yields $n_1 = N + 1$, $n_2 = n_3 = N$ for the
 530 $\phi = 0$ lattice and $n_1 = n_2 = N$, $n_3 = N + 1$. The only way to have this change in the
 531 discrete number of modes between the two lattices is to have a spectral flow of a mode from
 532 the first band to the second and another mode from the second to the third band as ϕ varies
 533 from 0 to π .

The lattice above shows a specific example where varying a single parameter results in a

spectral flow of a localized mode across the bandgap. It leads to the natural question: what class of lattices exhibit such spectral flow of a localized mode across a bandgap? The answer to this question is related to the Chern number, a topological invariant of a vector field over a manifold⁷⁵. In particular, the eigenvectors that are mode shapes in each band constitute a vector field. If there are n degrees of freedom in a unit cell, then each eigenvector has n components. These n components are complex numbers, with the additional property that if \mathbf{u} is an eigenvector, then $\beta\mathbf{u}$ is also an eigenvector for any nonzero complex number β . Let us consider a family of lattices whose properties vary smoothly with a parameter ϕ such that the dynamic matrix is a periodic function of ϕ , i.e., $\mathbf{D}(\phi) = \mathbf{D}(\phi + 2\pi)$. Let us also define $\Omega = [0, 2\pi] \times [0, 2\pi]$ as the domain in the wavenumber μ and parameter ϕ plane. Dispersion analysis of each lattice in this family gives n eigenvectors for each ϕ and wavenumber μ . There are n dispersion surfaces and hence n vector fields of the corresponding eigenvectors in Ω . The Chern number of a band measures a topological property of the vector field comprised of its eigenvectors. It is given by

$$C = \frac{1}{2\pi i} \int_{\Omega} [\nabla \times (\mathbf{u}^* \cdot \nabla \mathbf{u})]_3 d\Omega.$$

534 Note that the eigenvectors \mathbf{u} of a dispersion band is periodic in this plane with period 2π
535 along each axes. \mathbf{u} is periodic in μ due to the periodicity of the Brillouin zone.

536 The reader may refer to several excellent sources for a detailed derivation and mathe-
537 matical properties of the Chern number⁷⁵⁻⁷⁷. Here we give an intuitive introduction to the
538 physical meaning of this quantity that is relevant to the present topic. The Chern number
539 is equal to the number of singularities in the phase of the vector field in Ω . To see how
540 such a singularity arises and its relation to localized modes at a boundary, let us consider
541 a dispersion surface of a lattice with n degrees of freedom per unit cell. The eigenvector at
542 each (μ, ϕ) can be expressed as a unit vector in \mathbb{C}^n , i.e., an n -component vector over the
543 complex number field.

544 Let us determine how \mathbf{u} and its derivatives change as we move along a path in Ω . Recalling
545 that $\beta\mathbf{u}$ is also an eigenvector for any complex number β , we can still write down a phase
546 independently for each eigenvector, i.e., each member in the set $\{e^{i\theta}\mathbf{u}\}$ with $\|\mathbf{u}\| = 1$ is also
547 an eigenvector. To meaningfully compare \mathbf{u} at two distinct points in the (μ, ϕ) plane, we
548 need to fix a gauge - a generalized coordinate system specifying θ . Otherwise the phase θ
549 at any two distinct points can make the individual components arbitrary. In particular, we

550 need to make a choice for the phase $\theta(\mu, \phi)$ as a function of μ and ϕ . By Stokes theorem,
 551 the Chern number for the vector field $\mathbf{u}(\mu, \phi)$ in the domain Ω becomes

$$C = \int_{\partial\Omega} [\mathbf{n} \times (\mathbf{u}^* \cdot \nabla \mathbf{u})]_3 d\partial\Omega. \quad (14)$$

Here \mathbf{n} is the unit outward normal from Ω at a point on the boundary $\partial\Omega$. If there is a smooth choice possible, then the Chern number becomes zero due to periodicity of $\mathbf{u}(\mu, \phi)$ in Ω . If no such smooth choice of a phase field is possible, then this results in a discontinuity in $\theta(\mu, \phi)$. In particular, it can be shown that⁷⁴

$$\mathbf{u}(\phi = 2\pi, \mu) = e^{is\mu} \mathbf{u}(\phi = 0, \mu).$$

552 Here s is an integer. Substituting the above relation into Eqn. (14) gives $C = s$. The key
 553 message is that C is non-zero only when there is a discontinuity in the phase field θ .

554 Now, let us consider a family of finite lattices generated by varying ϕ from 0 to 2π . If the
 555 Chern number of a band is $C = s \neq 0$, then there are s modes traversing the bandgap above
 556 it as ϕ changes in $[0, 2\pi]$. A simple way to understand this spectral flow is to note that a
 557 zero Chern number implies that the modes in the finite lattice can be smoothly deformed
 558 as ϕ varies in $[0, 2\pi]$. Similarly, a nonzero Chern number implies that all modes cannot
 559 be smoothly deformed or mapped as we vary ϕ . Hence the only possibility is that some
 560 modes lie in multiple bands when $\phi = 0$ and π . As ϕ varies in $[0, 2\pi]$, these modes traverse
 561 the bandgap and this traversal is the observed spectral flow between bulk bands. The net
 562 number of such modes equals the number of singularities in the phase field, i.e., the absolute
 563 value of the Chern number. The sign of the Chern number indicates whether the modes
 564 traverse from or to the band as ϕ varies in $[0, \pi]$. Hence any lattice whose dispersion band
 565 has a non-zero Chern number s in the (μ, ϕ) supports a net number $|s|$ of modes traversing
 566 the bandgap to the bulk band above it.

567 Let us see how this family of localized modes can be used to achieve transport of elastic
 568 waves. Figure 12 displays an array of beams coupled with continuous distributed springs
 569 along their length. The key idea is to have a family of lattices with varying ϕ so that the
 570 localized mode moves from one boundary to another. The springs have stiffness analogous
 571 to Eqn. (12) with the ϕ varying along the length of the chain. Hence each segment of the
 572 distributed spring at a distance z along the length is equivalent to a discrete elastic chain
 573 at a value $\phi(z)$. If ϕ is varied from $\pi/3$ to $2\pi/3$, we note that the localized mode shifts

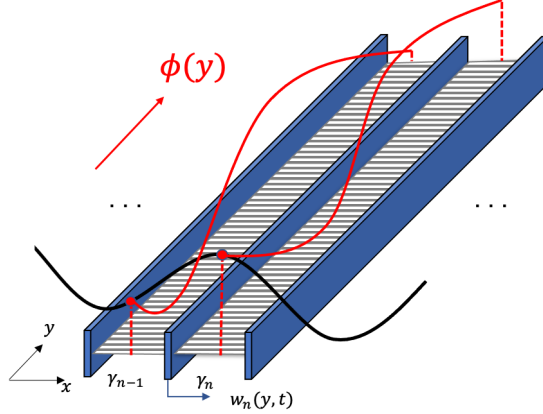


FIG. 12: Schematic of an array of beams coupled with distributed springs whose stiffness is given by Eqn. (13). Along the beam, ϕ varies from $\pi/3$ to $2\pi/3$ so that the localized mode in this chain of distributed springs traverses from left to right boundary.

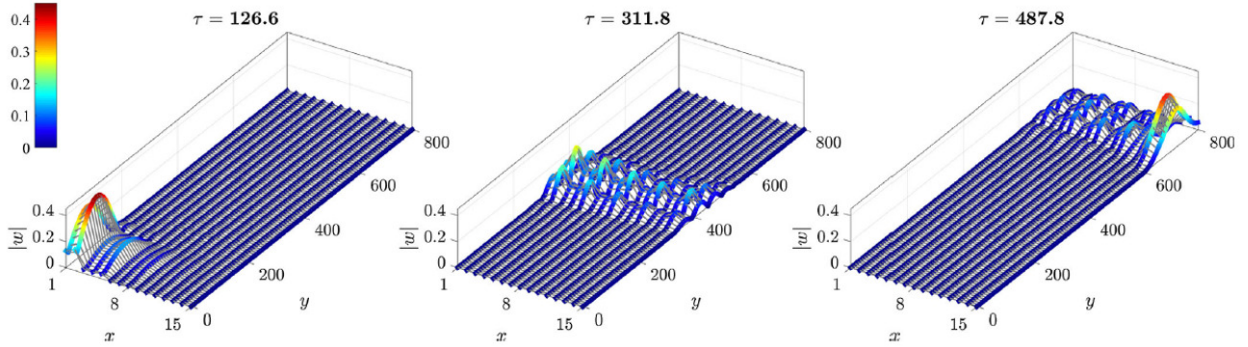


FIG. 13: Snapshots of beam displacements at 3 distinct time instants showing how the wave transitions from a left localized to a bulk wave to finally a right localized wave⁷⁴. (Reproduced with permission from Phys. Rev. Lett. 123, 034301 (2019). Copyright 2019 American Physical Society.)

574 from the right to the left boundary, as illustrated by the distinct time snapshots of the
 575 displacement field in Fig. 13. This transport happens when the structure is subjected to a
 576 dynamic excitation with frequency in the first bandgap. An experimental demonstration of
 577 this wave propagation was achieved in⁷⁸ in a plate whose bending stiffness is modulated by
 578 varying its thickness. This example shows how to achieve energy transport from one corner
 579 to another in an elastic structure. The next example illustrates wave transport between any
 580 two points on the boundary of a 2D mechanical lattice.

581 D. Mechanical spin Hall effect

582 The mechanical analogue of the quantum Hall effect allows for one way transport along
583 the boundaries of a structure. It is inspired by works in quantum mechanics starting with
584 the Haldane model⁷⁹, where electrons move in one direction along the edges of a $2D$ electron
585 gas. In recent years, several works have achieved mechanical analogues of this effect in both
586 periodic and random media. The basic idea is the same as was discussed for localized modes
587 in the family of lattices in Sec. IV C. Instead of a $1D$ family of lattices, we have a $2D$ lattice
588 with the wave vector along the direction tangential to the boundary playing the role of ϕ .
589 Such modes are achieved in lattices whose dispersion bands have a nonzero Chern number.
590 It requires using springs with imaginary stiffness that break time reversal symmetry and is
591 realized using active mechanical components like rotating gyroscopes^{22,23}.

592 To illustrate this concept, let us study the hexagonal lattice in Fig. 14a. It consists of
593 rotating gyroscopes attached to the masses at each lattice site. There are two dispersion
594 bands for this lattice with Chern numbers $+1$ and -1 for the lower and upper bands, respec-
595 tively. These numbers are evaluated by considering the vector field of mode shapes for each
596 band in the two-dimensional wave-vector domain $\boldsymbol{\kappa} = (\kappa_x, \kappa_y)$.

597 To demonstrate the existence of edge modes in such lattices, a common technique is
598 to consider the dispersion of a single strip of finite width shown in Fig. 14b. Periodic
599 boundary conditions are imposed on the long inclined boundaries and the lattice is thus
600 one dimensional with wavenumber equal to the wave-vector component κ_x . Figure 14b
601 displays the dispersion curves for such a finite strip. Most of the modes of the strip lie
602 in the frequency range of the bulk dispersion surfaces obtained with a single unit cell. In
603 addition, two modes traverse the bandgap as κ_x varies from 0 to 2π . Their corresponding
604 mode shapes are localized on each boundary.

605 The reason for this spectral flow is the same as the concept discussed in Sec. IV C. A
606 non-zero Chern number implies the existence of a spectral flow across a bandgap to satisfy
607 the condition that a singularity exists in the phase field. This lattice thus supports edge
608 modes localized at the boundary for all frequencies in the bandgap. In addition, they have
609 a positive (negative) group velocity on the top (bottom) surface. Indeed, recall that the
610 group velocity is the gradient of the frequency with respect to the wave vector⁸⁰. Hence,
611 these modes propagate only one way along each boundary and in effect, traverse clockwise

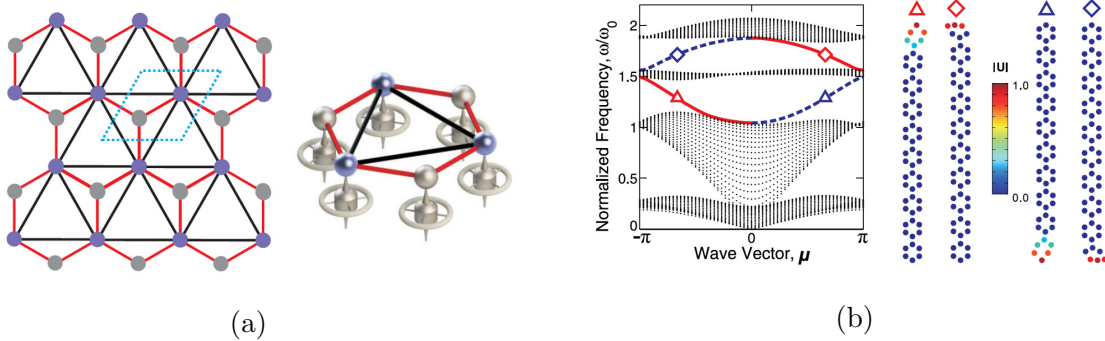


FIG. 14: Elastic lattice supporting chiral edge modes. (a) Hexagonal lattice with rotating gyroscopes at each node. (b) Dispersion curves for a strip as a unit cell showing one way modes in the bandgap. The red (blue) modes are localized at the top (bottom) surface and traverse counter-clockwise around the boundary²³. (Reproduced with permission from Phys. Rev. Lett. 115, 104302 (2015). Copyright 2015 American Physical Society.)

612 around a finite lattice.

613 Note that there are no continuous analogues of this effect in conventional elastic media.
 614 This hexagonal lattice of Haldane was extended to derive another class of topological edge
 615 modes called helical modes and the associated phenomena is called the quantum spin Hall
 616 effect. It essentially consists of two copies of the Haldane model lattice and the total system
 617 preserves time reversal symmetry. The quantum Hamiltonian \mathbf{H} , which is the analogue of
 618 the dynamic stiffness matrix $\mathbf{M}^{-1}\mathbf{K}$ of a mechanical system, has the form

$$\mathbf{H}(\boldsymbol{\kappa}) = \begin{pmatrix} h(\boldsymbol{\kappa}) & 0 \\ 0 & -h^*(\boldsymbol{\kappa}) \end{pmatrix} \quad (15)$$

619 Figure 15 displays examples of discrete mechanical lattices that support such helical edge
 620 modes. The advantage of this class of modes is that they do not require active components
 621 and can be realized in conventional linear elastic media. The key idea is that two copies of
 622 any lattice whose dispersion bands have non-zero Chern number can be used to construct a
 623 lattice that supports helical edge modes. The Chern number of each band in the combined
 624 assembly is zero. Susstrunk and Huber used this concept on a square Hofstadter lattice
 625 with 6 degrees of freedom per unit cell to achieve a mechanical analogue of this quantum
 626 spin Hall effect. They obtained the first experimental observation of helical modes in a
 627 mechanical system that has 2 copies of a square lattice with 6 degrees of freedom per unit

628 cell (Fig. 15a). The experimental setup consists of an array of interconnected pendulums
 629 and levers to mimic springs with positive and negative stiffness. Their key idea to handle
 630 the imaginary terms in $h(\boldsymbol{\kappa})$ was to use the unitary transformation

$$\mathbf{U} = \mathbf{I}_N \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad (16)$$

631 to convert the Hamiltonian to a matrix with only real terms. Here N is the number of unit
 632 cells of a finite lattice. The dynamic stiffness matrix $\mathbf{D} = \mathbf{U}^\dagger \mathbf{H} \mathbf{U}$ has the same eigenvalue
 633 spectrum as \mathbf{H} . In addition, if a wavefunction \mathbf{u} is localized in the quantum case, its
 634 corresponding mechanical mode shape $\mathbf{U}\mathbf{u}$ will also remain localized in the mechanical
 635 lattice. This preservation of localization is due to the block diagonal form of \mathbf{U} . This
 636 form ensures that the mode shape (eigenvector of \mathbf{D}) components in each unit cell can be
 637 expressed as a linear function of the corresponding eigenvector components of \mathbf{H} in the same
 638 unit cell.

Let us discuss in detail the design constructed by one of the authors of this tutorial
 in²⁵. Figure 15b displays a unit cell consisting of rotating disks connected by bars. Each
 disk has one degree of freedom and can rotate in its plane. Our starting point is the
 Hamiltonian used by Kane and Mele⁸¹, that supports helical edge modes at the boundary
 of a hexagonal lattice. Their model has two copies of hexagonal lattice and each unit cell
 thus has 4 degrees of freedom. These degrees of freedom are coupled through real and
 imaginary interaction terms and these couplings are analogous to springs connecting masses
 in a mechanical system. However, in a discrete mechanical system, it is desirable that all
 spring stiffness values are real. To achieve this condition of real coupling terms, we apply
 the transformation \mathbf{U} in Eqn. (16) to get a stiffness matrix with all real terms. In particular,
 after applying Bloch periodicity condition, the stiffness matrix for the hexagonal lattice unit
 cell takes the form

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^* & \mathbf{K}_1 \end{pmatrix},$$

with

$$\begin{aligned} \mathbf{K}_1 &= (3k + 6\lambda)\mathbf{I}_2 - k(1 + e^{i\boldsymbol{\kappa}\cdot\mathbf{a}_1} + e^{i\boldsymbol{\kappa}\cdot\mathbf{a}_2})\sigma_x, \\ \mathbf{K}_2 &= \lambda(e^{i\boldsymbol{\kappa}\cdot\mathbf{a}_1} - e^{i\boldsymbol{\kappa}\cdot\mathbf{a}_2} + e^{-i\boldsymbol{\kappa}\cdot\mathbf{a}_2} - e^{-i\boldsymbol{\kappa}\cdot\mathbf{a}_1} + e^{-i\boldsymbol{\kappa}\cdot(\mathbf{a}_1-\mathbf{a}_2)} - e^{i\boldsymbol{\kappa}\cdot(\mathbf{a}_1-\mathbf{a}_2)})\sigma_z. \end{aligned}$$

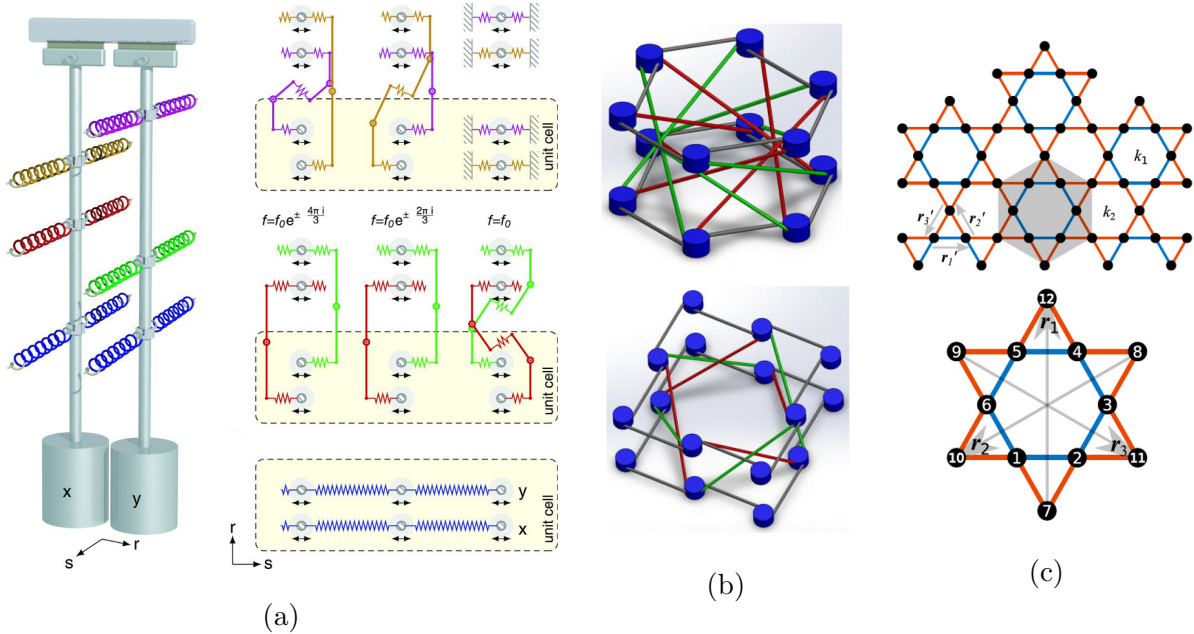


FIG. 15: Examples of discrete mechanical lattices that support helical edge modes. (a) Pendulums with springs²⁴, (b) rotating disks²⁵ and (c) spring mass lattice⁸². (Reproduced with permission from Science 349, 47-50 (2015), J. Appl. Phys. 99, 084305 (2016) and Phys. Rev. B 98, 094302 (2018). Copyright 2015 Amer. Assoc. Adv. Science and 2018 American Physical Society.)

639 Here σ_x and σ_z are Pauli matrices and k , $\pm\lambda$ are the stiffness values of the intra- and inter-
 640 layer coupling springs, respectively. Figure 15b (top) displays a schematic of a hexagon of
 641 the resulting lattice along with the connectivity between the disks.

642 Figure 16a displays the dispersion diagram for the lattice with $\lambda = 0$, i.e., no interlayer
 643 coupling springs. The dispersion diagram is projected onto the $\kappa_y = 0$ plane. There is
 644 no bandgap and the two dispersion surfaces touch. Figure 16b displays the corresponding
 645 dispersion diagram of a unit cell with interlayer coupling $\lambda = 0.2$ showing a bandgap now
 646 opens up. To see the presence of topological modes at a boundary, we again examine the
 647 dispersion curves of a lattice that is finite in one direction and infinite in another, similar
 648 to the lattice in Fig. 14b. They are illustrated in Fig. 16c and have two modes spanning
 649 the bandgap in addition to the bulk bands seen in Fig. 16b. The difference between these
 650 two dispersion curves (Figs. 16b, 16c) is the effect of a finite boundary. Indeed, the mode
 651 shapes corresponding to the modes in the bandgap are localized at a boundary. Note how

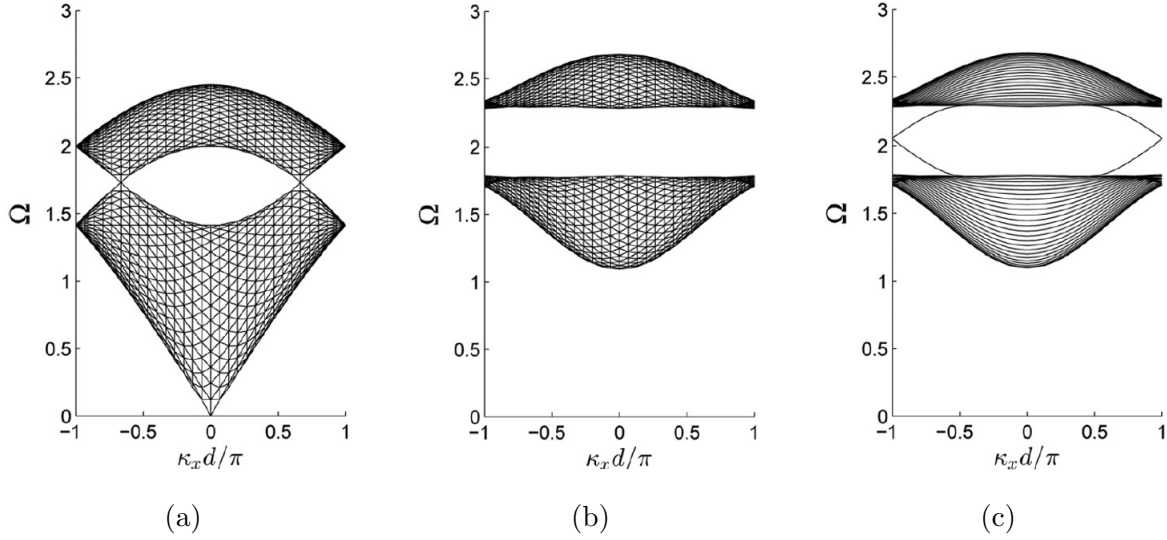


FIG. 16: Dispersion of a unit cell (a) without and (b) with interlayer coupling, showing how Dirac points break to generate a bandgap. (c) Dispersion of a finite strip showing modes spanning the bandgap²⁵. (Reproduced from J. Appl. Phys., 119, 084305 (2016), with the permission of AIP Publishing.)

652 they span the entire bandgap. In contrast to the chiral edge modes in Fig. 14b, there are
 653 two modes with opposite group velocities traversing the bandgap.

654 Let us now see the transient behavior of a finite lattice comprised of these unit cells
 655 subjected to narrow band excitation (like a tone burst) in the bandgap frequency. Figure 17
 656 displays the snapshots of displacement field at various time instants for each of these waves.
 657 They are localized at the boundary and their magnitude decays rapidly away from it. These
 658 discrete models serve to illustrate the key concept behind such waves in mechanical lattices,
 659 but they may not extend straightforwardly to continuous elastic media. In Sec. VB, we will
 660 show an example of helical waves in architected plates whose design is guided by symmetry
 661 considerations. Before that, let us turn attention to studying another class of topological
 662 modes: valley modes, that are supported at the interface between two hexagonal lattice.

663 E. Mechanical valley Hall effect

Valley modes are a class of topological modes that arise due to symmetry properties of Dirac cones in hexagonal lattices^{57,83–86}. 2D periodic lattices with specific symmetry

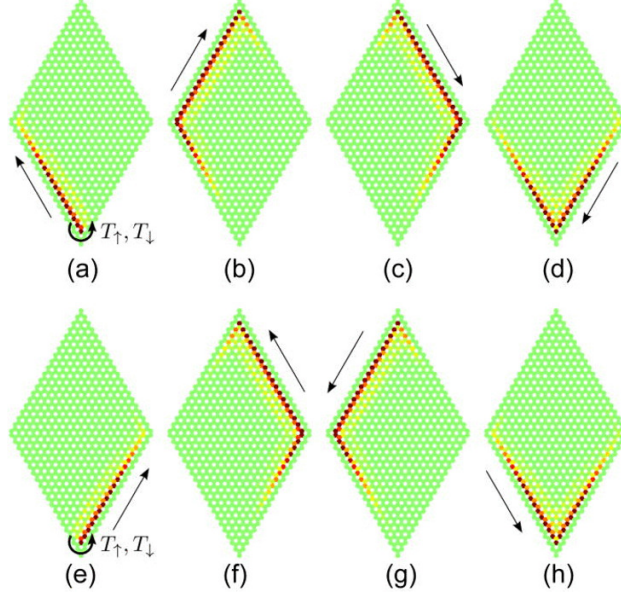


FIG. 17: Snapshots of displacement field at various time instants. Wave propagates clockwise (a-d) or counter-clockwise (e-h) depending on which mode is excited²⁵. (Reproduced from J. Appl. Phys., 119, 084305 (2016), with the permission of AIP Publishing.)

properties exhibit singular features called Dirac cones in their dispersion surfaces. To see an example of how they arise, let us consider the hexagonal discrete mass-spring lattice shown in Fig. 18a. Each mass has one degree of freedom and can move out of plane. Each unit cell has 2 sub-lattice sites, indexed by a, b , and thus 2 degrees of freedom. The governing equations for a unit cell indexed p, q are given by

$$\begin{aligned}
 m_a \ddot{u}_{p,q,a} + k(u_{p,q,a} - u_{p,q,b}) + k(u_{p,q,a} - u_{p-1,q,b}) + k(u_{p,q,a} - u_{p,q-1,b}) &= 0, \\
 m_b \ddot{u}_{p,q,b} + k(u_{p,q,b} - u_{p,q,a}) + k(u_{p,q,b} - u_{p+1,q,a}) + k(u_{p,q,b} - u_{p,q+1,a}) &= 0.
 \end{aligned}$$

664 Figure 18b displays the two dispersion surfaces over the first Brillouin zone. They touch
 665 at the six high symmetry K points. In the vicinity of these points, the dispersion surface
 666 resembles a two cones whose apex touch and these points are called Dirac points. Figure 18c
 667 displays the dispersion surfaces along the path sketched at the base in Fig. 18b. Such Dirac
 668 cones are at the heart of unique physical properties of graphene⁸⁷. The unique property of
 669 Dirac cones is that the mode density at that frequency is very low but the group velocity is
 670 finite, equal to the gradient of dispersion dispersion surface near the cone tip.

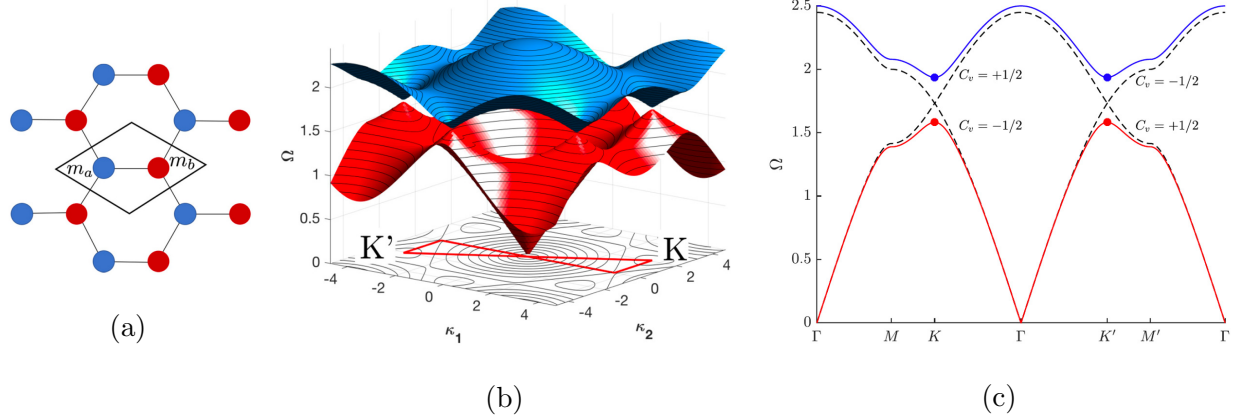


FIG. 18: (a) Hexagonal lattice schematic with unit cell. (b) Dispersion surfaces when $m_a = m_b$ showing Dirac cones at the K points. (c) Dispersion curves over a path through the high symmetry points for the case $m_a \neq m_b$ (solid curves) and $m_a = m_b$ (dashed curves). The valley Chern numbers C_v for each band at the K points are indicated⁸⁸. (Reproduced with permission from Phys. Rev. B 96, 134307 (2017). Copyright 2017 American Physical Society.)

671 Let us see what happens if we break inversion symmetry by making the masses at the two
 672 sub-lattice sites different. **The broken inversion symmetry is about the unit cell center, i.e.,**
 673 **the structure is different under the transformation $\mathbf{x} \rightarrow -\mathbf{x}$.** The degeneracy of the two
 674 modes at the K points break and a bandgap opens up. Again, similar to the case of Sec. IV A
 675 and Sec. IV D, the vector field associated with the bands have topological properties that
 676 lead to the existence of localized modes spanning the bandgap frequencies. In contrast to
 677 the earlier cases, where such modes are localized at the domain boundary, these modes are
 678 localized at the interface between two lattices that are inverted copies of each other.

679 Let us start by considering the lattice shown in Fig. 19. The strip is infinite along the
 680 x -direction but finite along the y -direction. The boundary masses at both ends are fixed.
 681 The masses at the two sub-lattice sites (red and white circles) are different. There is an
 682 interface at the center of the strip and the unit cells on both sides are inverted copies of each
 683 other. This lattice is periodic along the x -direction and the dashed parallelogram shows a
 684 unit cell of this lattice.

685 Let us analyze the dispersion behavior of this finite strip lattice. Let the light and heavy
 686 mass values be $m_1 = 1$ and $m_2 = 2$, respectively. There are two types of lattices depending

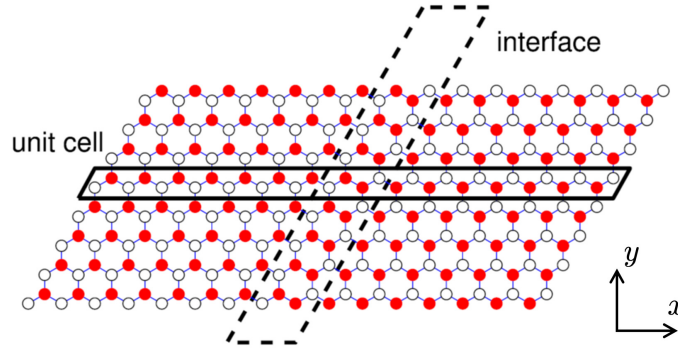


FIG. 19: Schematic of a lattice periodic along horizontal edges and finite along tilted edges. Red and white circles have distinct masses. The interface separates two parts that are inverted copies of each other⁵⁷.

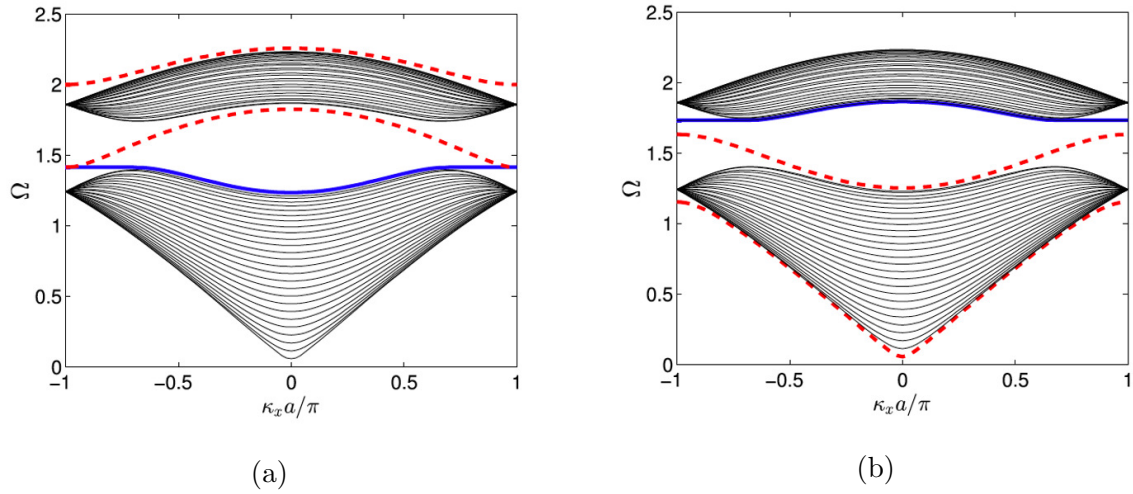


FIG. 20: Dispersion diagrams of a finite strip with an interface having adjacent (a) light and (b) heavy masses. Mode shapes of the dashed red and solid blue curves are localized at the interface and boundary, respectively⁵⁷.

687 on whether there are adjacent light or heavy masses at the interfaces. Figure 20 displays the
 688 dispersion curves for both cases. The dashed lines indicate modes localized at the interface
 689 while the solid blue lines are modes localized at the boundary. The remaining two contiguous
 690 sets of curves correspond to modes spanning the entire unit cell, i.e., bulk modes. The key
 691 observation is that the mode shape and frequency-wavenumber relation of the localized mode
 692 can change depending on the interface type, but there is a mode in the bandgap for both
 693 cases.

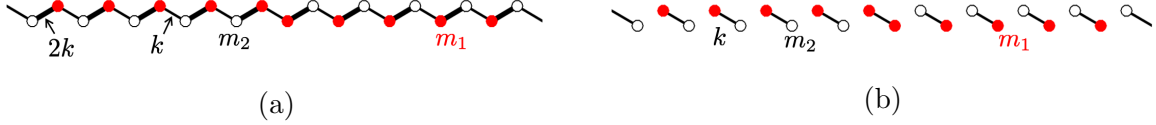


FIG. 21: Schematic of finite strip lattice with interface at the high symmetry points (a) $\kappa_x = 0$ and (b) $\kappa_x = \pi/a$ after applying Bloch periodicity. The two configurations have distinct number of frequencies lying in each bulk band, implying a mode traverses the bandgap as κ_x varies in $[0, \pi/a]$.

694 There are two different approaches to understand the existence of these localized modes
 695 at such interfaces, similar to the two distinct approaches presented in Sec. IV C. The first
 696 approach analyzes the behavior at distinct high symmetry points to infer the existence of a
 697 spectral flow in the bandgap, while the second approach uses topological invariants to infer
 698 localized modes in the bandgap.

699 In the first approach, let us consider the dynamic behavior of the lattice at wavenumbers
 700 $\kappa_x = 0$ and $\kappa_x = \pi/a$. Here a is the unit cell length. Using Bloch periodicity conditions, we
 701 see that all unit cells have identical displacement fields at $\kappa_x = 0$, while adjacent unit cells
 702 have displacement field of opposite sign at $\kappa_x = \pi/a$. These relations lead to the unit cell
 703 transforming into the lattices shown in Fig. 21a and 21b for $\kappa_x = 0$ and π/a , respectively.
 704 Thick lines indicate that the stiffness value is $k = 2$. These transformed lattices are obtained
 705 by using the above displacement field relations at these wavenumbers and the connectivity
 706 of the hexagonal lattice. Similar to the case of topological pumping in Sec. IV C, direct
 707 calculations show that the frequency distribution in each of the bulk bands is different
 708 at these two wavenumbers. The difference in frequency arises for both kinds of interfaces:
 709 adjacent light masses and adjacent heavy masses, and it manifests as a spectral flow between
 710 the bulk bands. Again, there is a topological invariant that captures this difference in mode
 711 distribution and predicts the existence of localized modes. This invariant is the valley Chern
 712 number and it is equal to the integral of the Berry curvature in the vicinity of the K-point.
 713 Detailed derivations of this invariant are presented in⁵⁷. **It should be noted that having**
 714 **distinct masses that break inversion symmetry is one possible way to achieve topological**
 715 **bandgaps. Recent works⁸⁹ have demonstrated band inversion in spring mass models that**
 716 **preserve inversion symmetry.**

717 V. EDGE MODES IN ELASTIC PLATES

718 Having discussed various kinds of topological modes localized at interfaces and bound-
719 aries in architected beams and discrete $2D$ mechanical lattices, let us now see examples of
720 how these concepts can be extended to architected plates. We will consider 3 examples in
721 increasing order of structural complexity. The first one supports valley Hall modes in thin
722 plates using zero order bending or A_0 Lamb waves. The second example achieves helical
723 edge waves using the hybridization of multiple higher order Lamb wave modes in a plate
724 with complex-shaped blind and through holes. The final example extends this plate design
725 to support both helical and valley modes, along with splitting or redirection of a wave at a
726 junction depending on its polarization.

727 A. Elastic valley Hall modes

728 In Sec. IV E, we discussed the discrete mechanical analogue of the quantum valley Hall
729 effect in lattices that have C_3 rotational symmetry but broken inversion symmetry. The key
730 idea is that localized modes arise at the interface of two such lattices that are inverted copies
731 of each other. Let us see how to extend this discrete concept to continuous elastic media.

732 Architected thin plates supporting Lamb waves provide a way to realize a wide class of
733 waves that are continuous analogues of $2D$ discrete mechanical lattices. Examples include
734 hexagonal and Kagome lattices in^{88,90,91}. The common aspect in these designs is that they
735 all satisfy the aforementioned symmetry conditions for the discrete case. Let us discuss in
736 detail the design in Fig. 22a. It consists of a thin plate with hexagon shaped holes drilled
737 into it in a periodic arrangement. Masses are attached at nodal locations in a corresponding
738 discrete hexagonal lattice. The masses attached at the two sub-lattice sites in the unit cell
739 are different and this difference breaks inversion symmetry of the lattice.

740 Figure 22b displays the corresponding dispersion bands for a single unit cell. It is com-
741 puted using finite element analysis and it shows a bandgap opening up due to the breaking
742 of inversion symmetry at the K -point. Note that the bending modes that form a Dirac cone
743 breaks to form a bandgap, but it is not a complete isolated bandgap as there are axial and in
744 plane-shear modes that traverse this frequency range. However, due to their distinct mode
745 shapes, when the bending mode is excited, a negligible fraction of energy is converted to

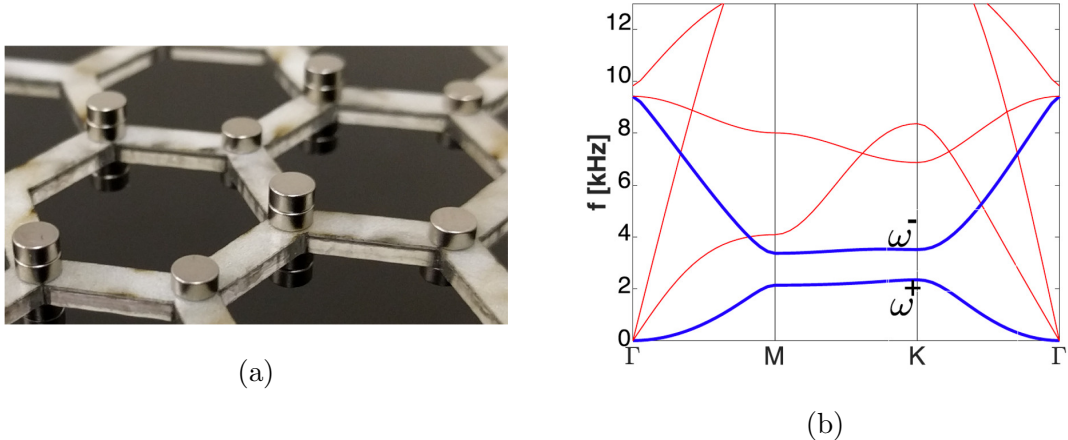


FIG. 22: (a) Designed structure is a thin plate with hexagonal holes. Different masses are attached to the sub-lattice sites to break inversion symmetry. (b) Dispersion curves over the boundary of the IBZ for the unit cell. Blue curves are out of plane bending modes while red curves are in plane longitudinal and shear modes⁸⁸. (Reproduced with permission from Phys. Rev. B 96, 134307 (2017). Copyright 2017 American Physical Society.)

746 these modes.

747 Figure 23 displays the dispersion diagram for a finite strip, calculated using the plane
 748 wave expansion method. The frequency is expressed in non-dimensional form by normalizing
 749 it as $\Omega = \omega \sqrt{EI/\rho A}$ and the wave vector is normalized by the unit cell length a . The modes
 750 are bounded by two curves $\Omega = \mu^2$ and $\Omega = \pi^2 - \mu^2$. This is because the group velocity at
 751 each frequency ω is bounded by the corresponding value for a homogeneous plate. Similar to
 752 the discrete case, there are two kinds of interface, depending on whether there are adjacent
 753 heavy or light masses at the junction. Figures 23c and 23d display the mode shapes at each
 754 kind of interface. These modes are called elastic valley modes and they span the bandgap
 755 frequencies for sufficiently small values of the mass difference. Note how they are localized
 756 at the interface and their amplitude decays rapidly away from it.

757 Let us now see an experimental demonstration of how such waves are able to navigate
 758 sharp bends without scattering losses. The structure is made of acrylic and magnetic cylin-
 759 ders are attached on both sides that attract each other. The unit cell size is 18.4 mm.
 760 The magnetic force between cylinders on adjacent lattice sites is assumed to be negligible.
 761 Figure 24a displays the experimental setup of an N-shaped topological waveguide realized
 762 by creating an interface. The red and cyan circles indicate locations with low and high

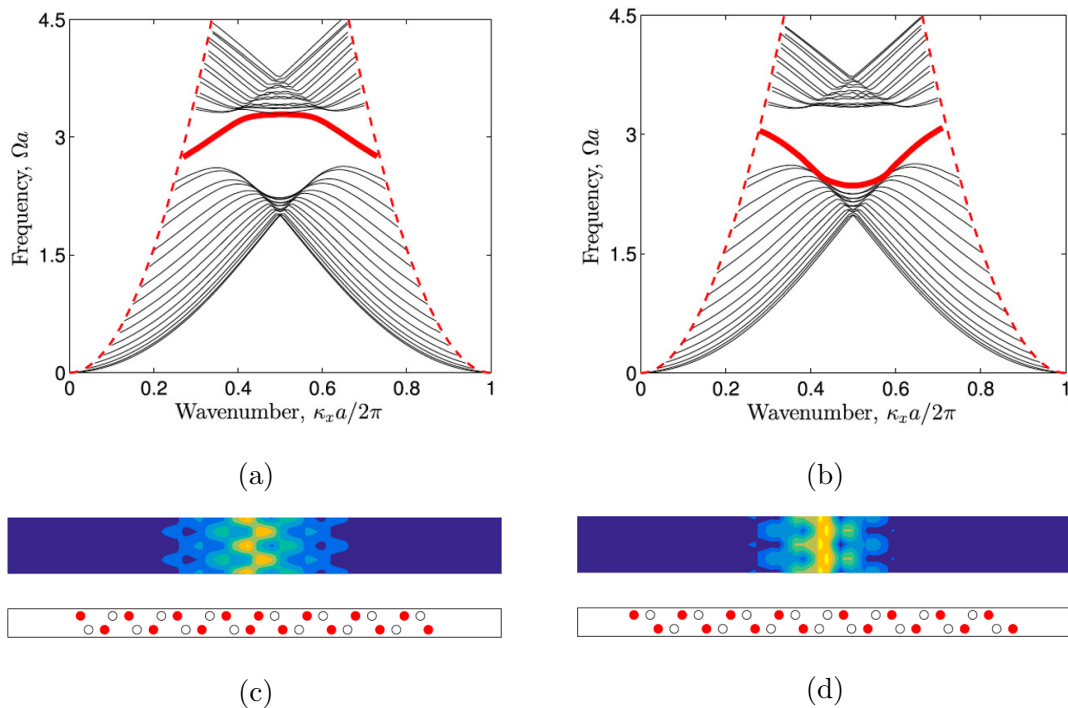


FIG. 23: Dispersion diagram for a finite strip having an interface with adjacent (a) light and (b) heavy masses. In both cases, a mode is localized at the interface with frequency in the bandgap. (c,d) Typical mode shapes of this mode and schematic of the finite strip⁵⁷.

763 masses, respectively. There are two adjacent light masses and the unit cells on each side are
 764 inverted copies of each other. The bottom end of the waveguide is excited with a windowed
 765 tone burst excitation and the velocity field is measured throughout the lattice with sufficient
 766 spacing between adjacent points using an SLDV. Figure 24b displays the amplitude of the
 767 out-of-plane velocity component after the waves have passed through. The wave decays in
 768 amplitude as the plate is made of acrylic, but an examination of the reconstructed tempo-
 769 ral evolution of the dynamics shows the absence of back-scattering as the wave navigates
 770 CORNERS.

771 As discussed earlier, topologically protected modes exist at the interface between lattices
 772 with distinct topological invariants. In this case, the relevant invariant is the valley Chern
 773 number. In contrast, if the lattices on both sides of an interface have identical invariants,
 774 then no modes are guaranteed to exist. Localized defect modes can exist, but they typically
 775 do not span the bandgap and these modes are sensitive to the geometry of the interface.
 776 Such interfaces are topologically trivial and are termed trivial waveguides. The following

777 example illustrates their dynamic behavior. The lattices on either side of the interface are
778 identical in the structure of Fig. 24c. Hence the topological invariants of the two lattices are
779 identical and this interface is a trivial waveguide. The corresponding transient response
780 shows that no wave navigates the corners. Thus comparing its response with that of the
781 topological case shows the efficacy and superiority of the topological waveguide.

782 This example shows how valley modes can be induced in continuous elastic media using
783 architected thin plates. Such valley modes have also been demonstrated in several other
784 systems, including in small scale structures^{92,93}, reconfigurable waveguides with shunted
785 piezoelectric patches⁹¹, in acoustic lattices with sound waves⁹⁴⁻⁹⁶ and in plates with bolts or
786 stubs^{58,97}.

787 B. Elastic spin Hall modes

788 In Sec. IV D, the key ideas behind the mechanical analogue of the quantum spin Hall
789 effect were discussed in the case of discrete lattice-like structures. Recall that this effect
790 leads to topological helical modes at boundaries. In this section, a general procedure or
791 recipe to obtain helical topologically protected edge modes in continuous passive elastic
792 waveguides is thoroughly detailed.

793 In principle, in the case of a system made of solely passive components, time-reversal
794 symmetry is preserved, and a phononic analogue of chiral edge states is precluded. However,
795 as proposed in the pioneering work of Mousavi et al.⁹⁸, helical edge modes can be achieved
796 in passive elastic waveguides as well by exploiting the fact that various Lamb modes are
797 characterized by different polarization. Specifically, they showed that guided symmetric
798 (S0) and anti-symmetric (A0) modes in plates are excellent candidates to achieve an elastic
799 version of the quantum spin Hall effect (i.e., designing a system supporting two effective
800 spins for Lamb waves over a sufficiently broad bandwidth).

801 Geometry and material modifications, in general, affect the dispersion behavior and the
802 associated band structure of plates. For example, breaking translation symmetry by in-
803 troducing periodic holes or inclusions may open bandgaps, which, however, may not be
804 complete and may therefore affect only some of the modes. The opening of bandgaps sup-
805 porting topological modes also requires the structure to maintain a specific symmetry. The
806 general procedure to obtain topologically protected helical edge modes in continuous elastic

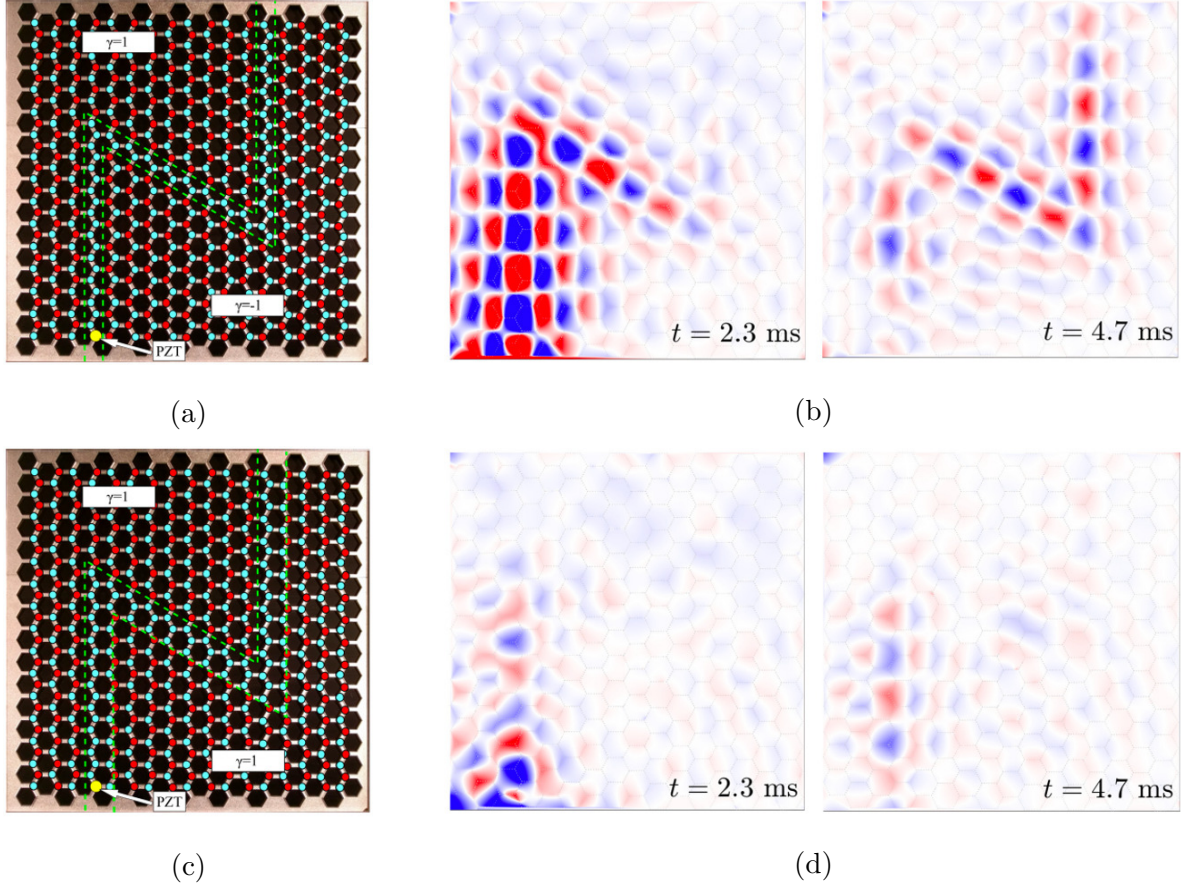


FIG. 24: N-shaped (a) topological and (c) trivial waveguide. Red and cyan circles have different mass amounts. (b,d): Their corresponding dynamic response at distinct time instants showing the wave navigates sharp bends in the topological waveguide⁸⁸. (Reproduced with permission from Phys. Rev. B 96, 134307 (2017). Copyright 2017 American Physical Society.)

807 plates can be decomposed into the following steps:

- 808 • step 1 - Design a unit cell whose dispersion surfaces have an isolated double Dirac
809 cone (Fig. 25);
- 810 • step 2 - Open a bandgap by breaking mid-plane symmetry of the unit cell (Fig. 26);
- 811 • step 3 - Create an interface using the unit cells of step 2. The unit cells on either side
812 are mirror images of each other about the plate mid-plane. (Fig. 27a).

813 We will show below how applying the above steps to a patterned continuous elastic plate
814 leads to helical topological protected edge modes. As mentioned, the first step is to create

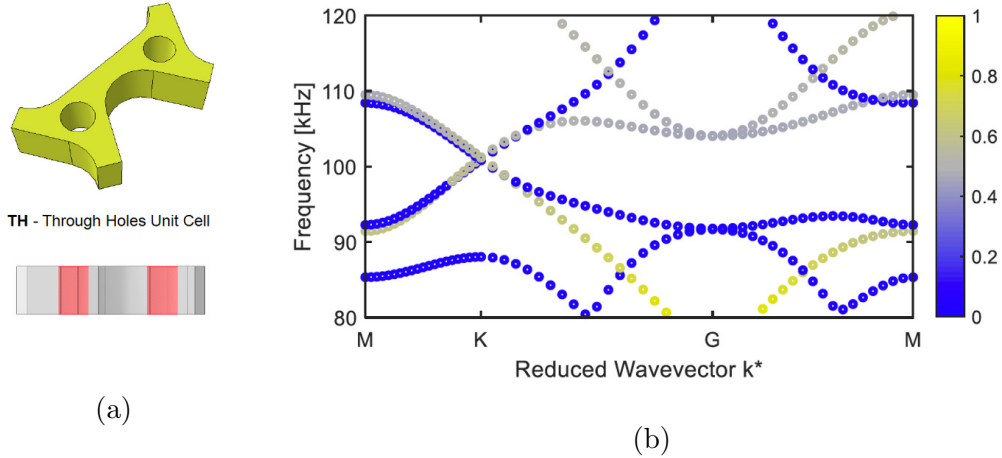


FIG. 25: Step 1:(a) perspective and cross-sectional view of an architected unit cell with through the thickness holes - TH. (b) Its dispersion curves have a double Dirac cone and the corresponding modes have different polarization (in-plane and out-of-plane, represented respectively by the colors blue and grey/yellow in the right panel).

815 an isolated Dirac-like dispersion curves for the two polarized Lamb modes in the absence
 816 of σ_h symmetry (or reflection symmetry about the mid-plane of the plate) breaking. Let
 817 us consider an architected unit cell as the one showed in Fig. 25a. The specific patterning
 818 creates a graphene-like band structure for elastic waves. Specifically, the in-plane hexagonal
 819 symmetry provides Dirac dispersion for waves with an accidental degeneracy at K and K'
 820 points (Dirac points). While the plate geometrically resembles the twisted Kagome lattice
 821 described in⁹⁹, it is a continuous medium with no lumped elements such as point masses
 822 and springs can be identified. The associated dispersion surfaces have a double Dirac cone
 823 of modes with different polarization (in-plane and out-of-plane, represented respectively by
 824 the colors blue and grey/yellow in the right panel). Matching the frequency and the slope
 825 (group velocity) of Dirac cones associated with a symmetric mode and an anti-symmetric
 826 mode in a frequency range with no other modes is analogous to emulating the two spin
 827 states in graphene. Kane and Mele⁸¹ showed that quantum spin Hall effect and helical edge
 828 modes arise when strong spin orbital coupling is introduced.

829 Once isolated double Dirac cones are achieved, a bandgap can be opened by by replacing
 830 the through-the-thickness holes by blind holes, i.e., holes not spanning the whole thickness
 831 of the plate, as illustrated in the perspective and cross-sectional view of Fig. 26a. Blind

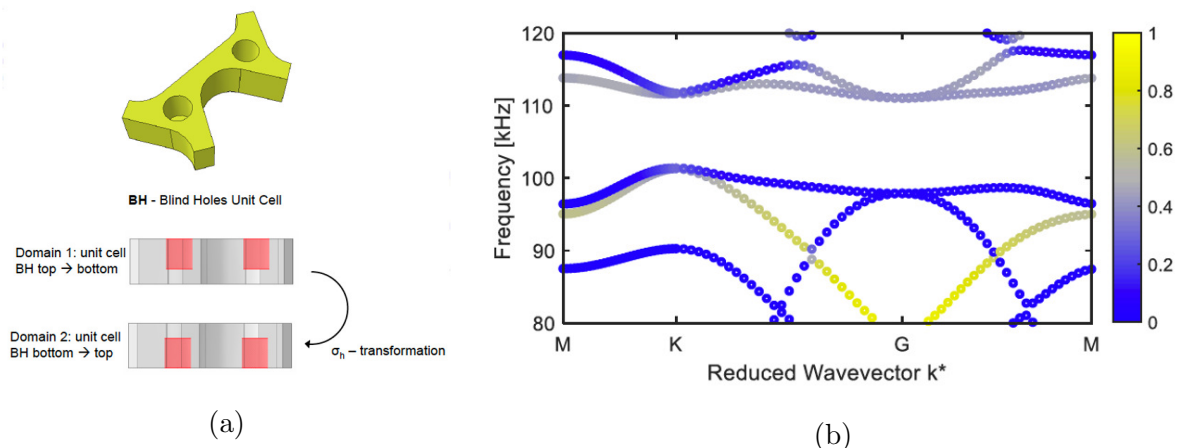


FIG. 26: Step 2: (a) perspective and cross-sectional view of a unit cell with blind holes BH. Unit cells for domain 1 and domain 2 differ by a σ_h transformation, with blind holes being on the top vs bottom surface. (b) Modes are hybridized at the high symmetry point K, breaking σ_h symmetry.

832 holes (BH) through part of the plate thickness break the σ_h symmetry while preserving the
 833 original C_{3v} symmetry. This modification induces a coupling between the modes spanning
 834 the Dirac points: the in-plane polarized and out-of-plane polarized modes. This coupling
 835 is analogous to the spin orbital interaction in the quantum spin Hall effect. In contrast to
 836 valley modes where the two sets of modes are associated with two distinct valleys (K, K'),
 837 the spin Hall effect results from the hybridization of two distinct sets of modes spanning
 838 each of the high-symmetry K points.

839 This type of macroscopic geometrical modification produces a unit cell with broken σ_h
 840 symmetry, leading to mode hybridization at the high symmetry point K . Applying a σ_h -
 841 transformation to such a unit cell then gives the geometries for domain 1 and domain 2 (blind
 842 holes are inverted with respect to the mid-plane of the unit cell). The interface between them,
 843 as shown in the schematic in Fig. 27a is an elastic waveguide supporting localized helical
 844 edge modes in the bandgap frequencies. Figure 27b displays the numerical (white lines) and
 845 experimental measurements (energy spots) of the two distinct helical modes propagating
 846 in the waveguide. Experimental reconstruction of the helical edge modes characterized
 847 by a clockwise and counterclockwise variation of phase of the displacement field as the
 848 waves propagate from left to right. Fig. 27c illustrates a snapshot of the displacement

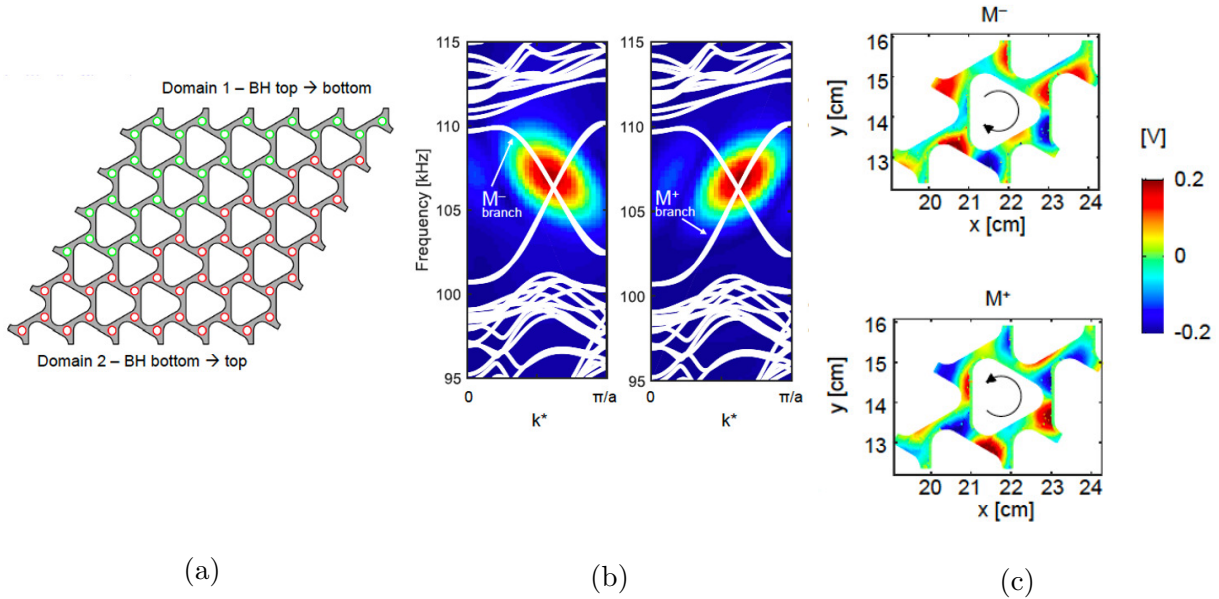


FIG. 27: Step 3: (a) schematic of a continuous elastic waveguide supporting helical modes at the interface between two unit cells related by a σ_h transformation. (b) Numerical (white lines) and experimental measurements (energy spots) of the two distinct helical modes. (c) Experimental reconstruction of the helical edge modes characterized by a clockwise and counterclockwise displacement phase field variation with respect to the direction of wave propagation (left to right)¹⁶.

849 field. Refer to¹⁶ for more details on the geometrical parameters of the unit cell and movie
 850 files of the experimental full field reconstruction. Hence there are two helical modes with
 851 opposite polarization. They have opposite group velocities for each wavenumber κ of waves
 852 propagating along the interface.

853 C. Combined valley and spin Hall modes

854 In this section we show how to design an elastic waveguide supporting multiple classes of
 855 edge modes (helical and valley). Specifically we will show a recipe to construct a continuous
 856 elastic waveguide capable of splitting equal-frequency helical edge waves differing on the
 857 basis of their polarization when they impinge on distinct interfaces at a common junction.
 858 The starting point is the same as for helical modes: a unit cell showing a double Dirac
 859 cone degeneracy, as the one examined in the previous section, and illustrated again in

860 Fig. 28a for clarity. From this unit cell, specific geometric modifications are introduced
861 so as σ_v and σ_h symmetries, i.e., inversion and reflection symmetries, are selectively broken
862 in specific portions of an elastic waveguide, as shown in Fig. 29a. The introduced geometric
863 perturbations produce topological bandgaps that, respectively, support helical and valley
864 modes in a common frequency range. Specifically, replacing the through holes of Fig. 28a
865 with blind holes of 0.9 times the height of the plate, as shown in Fig. 28b, breaks the σ_h
866 symmetry. We denote the configuration with the blind holes on the top (bottom) surface
867 as H^+ (H^-). The interface between H^+ and H^- supports two helical edge modes spanning
868 the gap with positive (Φ^+) and negative (Φ^-) group velocity, respectively (Fig. 29b).

869 Next, we break the σ_v symmetry or reflection symmetry about a vertical plane parallel
870 to a lattice vector direction. This is done by making the holes in each unit cell of different
871 radii, namely r and R (see Fig. 28c). Inverting the position of the larger and smaller radii
872 allows us to identify two distinct unit cells, denoted as Vr and VR (Fig. 28c). Contrary
873 to the previous case, an interface that separates two σ_v -transformed copies of the structure
874 supports a single valley mode, with positive or negative group velocity, depending on the
875 type of interface, i.e., with two adjacent holes of diameter r or R , respectively. Besides, the
876 interfaces between structures supporting helical and valley modes will still support a single
877 hybrid edge mode with either positive (Ψ^+) or negative (Ψ^-) group velocity, as shown in
878 Figs. 29c,d. The edge modes are denoted by the index $+$ ($-$) according to their positive
879 (negative) group velocity when the wavenumber κ is in $[0, \pi/a]$. Note that this wavenumber
880 is for waves propagating along an interface between two kinds of unit cells. In Figs. 29b-d,
881 bulk modes are shaded in grey, while the edge states are denoted by the black, blue and red
882 circles, depending on the types of interface: H^+/H^- , H^+/V^R , and H^-, V^r , respectively.

883 Finally, Figs. 30a,b show the numerical distribution of the von Mises stress field resulting
884 from harmonic excitation at 98 kHz, i.e. within the bulk gap. The excitation is applied at
885 the location shown by the white dot as an out-of-plane displacement distribution and the
886 calculations clearly illustrate the possibility to preferentially excite one of the two modes
887 and to remotely select the interface along which the wave will propagate once impinging
888 the Y-shaped junction. Colors indicate the von Mises stress, ranging from zero (blue) to
889 maximum (red). Refer to⁹⁰ for the geometrical details of the unit cells and further details.
890 Such wave splitting shows the potential for novel elastic wave manipulation capabilities.

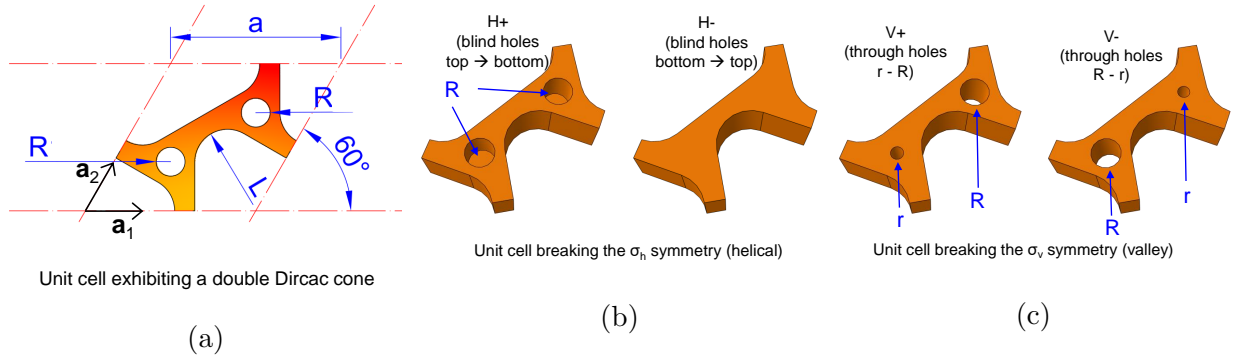


FIG. 28: (a) Top view of the designed unit cell with through-the-thickness holes whose dispersion has a double Dirac cone (see Fig. 25b). (b) Perspective view of two unit cells with broken σ_h symmetry. H+ has blind holes on the top surface and H- has blind holes at the bottom. They both break Dirac cones and open a bandgap (see Fig. 26b). (c) Perspective view of the unit cells with broken σ_v symmetry. V+ and V- had different size through holes. Both unit cells also open a bandgap in the same frequency range.

891 VI. OUTLOOK AND FUTURE DIRECTIONS

892 The past decade has seen an explosive growth in this research field, motivated by both an
 893 exploration of fundamental wave phenomena in elastic media and aimed at specific techno-
 894 logical applications. The key realization in our opinion is that interfaces and defect modes
 895 can be designed in a systematic way by incorporating features derived from topological
 896 considerations. The examples discussed above showed how one can achieve robust and
 897 backscattering free waveguides that have sharp bends and corners. **Different dimensions**
 898 **exhibit distinct topological phenomena: for instance the existence of a non-zero Zak phase**
 899 **in 1D systems and Weyl points in 3D systems.** In this article, we have primarily focused
 900 **on topological phenomena in elastic beams and plates that can be characterized by a Chern**
 901 **number.** Now, let us we outline some promising future research directions.

902 Reconfigurable waveguides^{83,100} are just beginning to be explored and remain to be real-
 903 ized in a variety of elastic waveguides. Similarly, multiphysics interactions, including between
 904 elastic and electromagnetic domains, for example in piezoelectric media^{101,102}, or between
 905 fluids and structures¹⁰³, may open avenues for controlling waves in one media with another.
 906 Extending unit cell designs to three dimensional periodic structures will significantly expand
 907 the design space as well as allow for 3D manipulation and control of elastic waves. They may

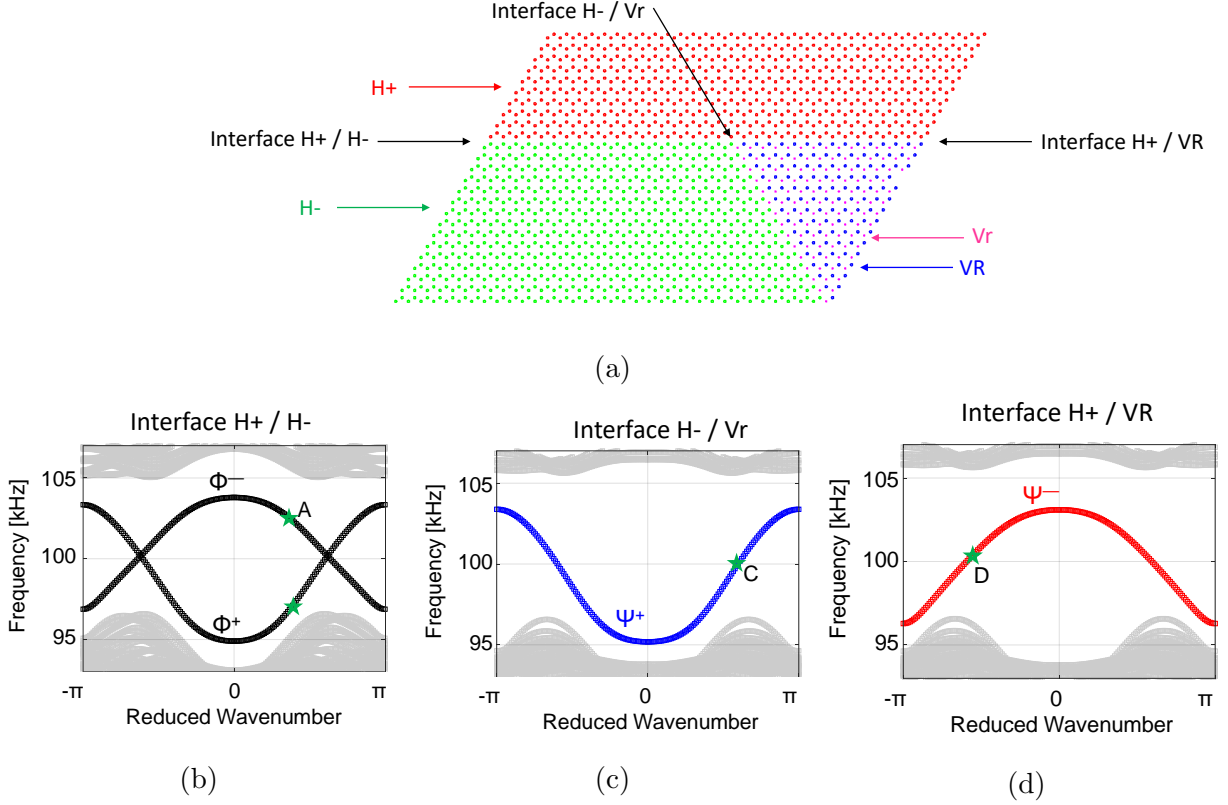


FIG. 29: (a) Digital model to assist the drilling machine for the manufacturing of the waveguide. Holes in red (green) define the H^+ (H^-) unit cell where blind holes are drilled. Hole depth is 0.9 times the height of the plate. Holes in blue (violet) define the V^R (V^r) unit cells where holes are drilled through the whole thickness of the plate, but that have different radii. (b - d) Dispersion diagrams for non-trivial stripes with the following types of interfaces: H^+/H^- , H^+/V^R , and H^-, V^r , respectively. The band structures are computed considering the strips periodic in the horizontal direction and made of 10 unit cells on each side of the domain wall. The bulk modes are reported as gray dots while the interface modes in black, blue and red dotted lines, respectively. The edge modes are denoted by the index $+$ ($-$) according to their positive (negative) group velocity with respect to the propagation direction⁹⁰. (Reproduced with permission from Phys. Rev. B 100, 024304 (2019). Copyright 2019 American Physical Society.)

908 allow for exploiting Weyl points¹⁰⁴ or can realize constructions like fragile topology¹⁰⁵ that
 909 exhibit spectral flow at interfaces. Similarly, quasiperiodic^{106,107} and non-periodic¹⁰⁸ metas-
 910 tructures offer rich opportunities for achieving unique static and dynamic behaviors. Such

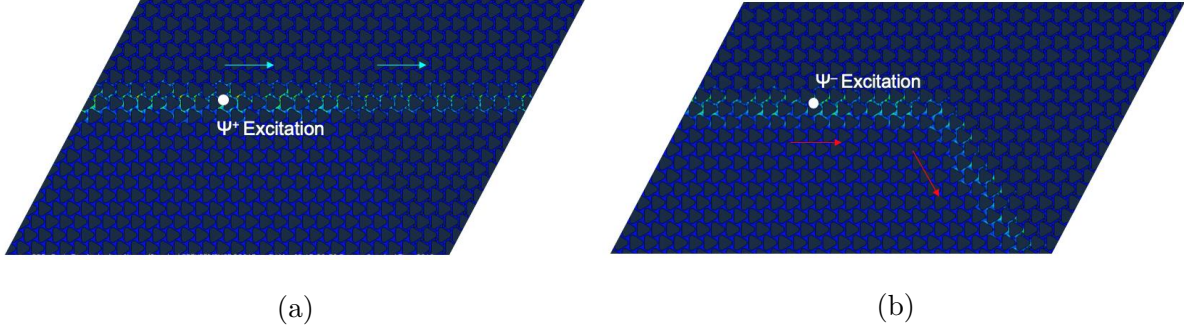


FIG. 30: (a - b) Numerical distribution of the von Mises stress field resulting from harmonic excitation at 98 kHz, i.e. within the bulk gap. The excitation is applied at the location shown by the white dot as an out-of-plane displacement. The calculations illustrate the possibility of preferentially exciting one of the two modes and of thus remotely selecting the interface along which the wave will propagate after it impinges on the Y -shaped junction.

911 concepts have recently started to be investigated in static media and in $1D$ beam systems.
 912 Extending unit cell concepts to non-periodic, complex geometries¹⁰⁹, higher dimensions and
 913 higher order topological modes¹¹⁰ will open avenues for unique wave manipulation proper-
 914 ties. In particular, quasi- and non-periodic structures exhibit for bulk isotropic behavior as
 915 opposed to anisotropic behavior that is associated with periodic media, for example C_3 , C_4
 916 media in $2D$ structures.

917 On the theoretical side, several questions similarly remain unanswered, having to do with
 918 the role of nonlinearities. Notably, nonlinear^{111–113} and non-Hermitian^{114–116} elastic struc-
 919 tures have exciting potential for realizing novel wave phenomena. The topological properties
 920 of such systems remain to be investigated and they may allow for solitons, frequency conver-
 921 sion, localized skin modes, unidirectional reflection in elastic structures like beams, plates
 922 and shells^{117,118}. In this regard, the advent of $3D$ ¹¹⁹ and $4D$ printing^{120,121} has opened novel
 923 avenues for fabricating complex shaped structures that were beyond the reach of conven-
 924 tional manufacturing methods. **Synthetic high dimensional spaces have been created in other**
 925 **physical domains by modulating geometric parameters in time¹²² or by accessing additional**
 926 **degrees of freedom¹²³. Such techniques can lead to the realization of topological wave phe-**
 927 **nomena associated with the corresponding higher dimensions in elastic media too.** We
 928 anticipate that these directions will thus lead to fruitful fundamental and applied research

929 in the coming years and will lead to the discovery of novel and exciting effects in the domain
930 of topological elastic waves.

931

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936 **DATA AVAILABILITY**

937 The data that support the findings of this study are available from the corresponding
938 author upon reasonable request.

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