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Riemannian foliation with dense leaves admitting a Lie extension on a compact connected manifold

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Abstract¹

In this paper we show that if \mathcal{F} is a Riemannian foliation with dense leaves admitting a Lie extension on a compact connected manifold M

Then \mathcal{F} is Lie foliation having dense leaves on a compact connected manifold M .

More specifically every Riemannian foliation \mathcal{F} with dense leaves on a compact connected manifold M admitting a one-codimensional extension is a Lie foliation having dense leaves

1 Introduction

The idea of this article came to us from the papers “*Extension d’un feuilletage de Lie minimal d’une variété compacte*” in Afrika Matematika [5] and “*Riemannian foliation with dense leaves on a compact manifold*” in International Journal of Mathematics and Computer Science [4].

Is shown in ([3], [5]) as any extension of a G -foliation \mathcal{F} having dense leaves on a compact connected manifold M corresponds a Lie subalgebra \mathcal{H} of $\mathcal{G} = \text{Lie}(G)$.

We denote by $\mathcal{F}_{\mathcal{H}}$ the extension corresponding to the subalgebra \mathcal{H} .

The object of this paper is the study of dense leaf Riemannian foliations \mathcal{F} on a connected compact manifold M having a Lie extension $\tilde{\mathcal{F}}$.

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In this article we seek to determine the impact on \mathcal{F} of the existence $\tilde{\mathcal{F}}$ of \mathcal{F} .

To be able to do it , we first establish the following :

(i) We give reminders on extensions of Lie \mathcal{G} -foliation having denses leaves on a compact connected manifold

(ii) We recalled that if \mathcal{F} and \mathcal{F}' are two Riemannian foliations on manifold such as $\mathcal{F} \subset \mathcal{F}'$ then $\ell(M, \mathcal{F}') \subset \ell(M, \mathcal{F})$.

This study shows us that all Riemannian foliation having denses leaves on a compact connected manifold M and admitting a Lie extension is a Lie foliation having denses leaves .

This article is divided into Three parts:

- the first part is devoted to the introduction,
- the second part is devoted to reminders on foliations and on extensions of foliation,
- The third part is devoted to the establishment of the main result

2 Definitions and Reminders

In this paragraph, we reformulate certain definitions and theorems found in [3], [5], [6], [8], and [7] .

Definition 2.1 *Let M be a manifold, a q -codimentional foliation (M, \mathcal{F}) is the data:*

- (i) *of a recovery $(U_i)_{i \in I}$ of M*
- (ii) *of a q -dimentional manifold T*
- (iii) *of submersion $f_i : U_i \rightarrow T$ such as for $U_i \cap U_j \neq \emptyset$ there is a diffeomorphism*

$$\gamma_{ij} : f_i(U_i \cap U_j) \subset T \rightarrow f_j(U_i \cap U_j) \subset T$$

checking

$$f_i = \gamma_{ij} \circ f_j$$

The manifold T is said to be transverse to the foliation (M, \mathcal{F}) and $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ is called a foliated cocycle defining foliation (M, \mathcal{F}) .

Each related fiber of $f_i : U_i \rightarrow T$ is a plate of \mathcal{F} in U_i .

The set of fibers of the submersion $f_i : U_i \rightarrow T$ is the foliation \mathcal{F} on U_i it is noted (U_i, \mathcal{F})

The quotient

$$\mathcal{V}(\mathcal{F}) = \frac{TM}{T\mathcal{F}}$$

is called the transverse bundle of \mathcal{F} .

We note too that

$$\mathcal{V}_x(\mathcal{F}) = \frac{T_x M}{T_x \mathcal{F}}$$

The foliation (U_i, \mathcal{F}) is said simple because it is defined by a submersion (in this case it's f_i) and each related fiber of f_i is called a plate of \mathcal{F} in U_i .

Definition 2.2 An extension of a q -codimensional foliation (M, \mathcal{F}) is a q' -codimensional foliation (M, \mathcal{F}') such as $0 < q' < q$ and the leaves of (M, \mathcal{F}') are union of those of (M, \mathcal{F}) (we note $\mathcal{F} \subset \mathcal{F}'$).

We show that if (M, \mathcal{F}') is a simple extension of a simple foliation (M, \mathcal{F}) and if (M, \mathcal{F}) and (M, \mathcal{F}') are respectively defined by submersions $\pi : M \rightarrow T$ and $\pi' : M \rightarrow T'$, then there exists a submersion $\theta : T \rightarrow T'$ such as $\pi' = \theta \circ \pi$.

We say that the submersion θ is a bond between the foliation (M, \mathcal{F}) and its extension (M, \mathcal{F}') .

It is shown in [3] if the foliation (M, \mathcal{F}) and its extension (M, \mathcal{F}') are respectively defined by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ and $(U_i, f'_i, T', \gamma'_{ij})_{i \in I}$, then we have

$$f'_i = \theta_i \circ f_i$$

and

$$\gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}.$$

That to say θ_s is a bond between the foliation (U_s, \mathcal{F}) and its extension (U_s, \mathcal{F}') .

Definition 2.3 Let $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ be a foliated cocycle defining a foliation (M, \mathcal{F}) on a manifold M .

If there is a Riemannian metric g_T on T invariant by the γ_{ij} , we say that \mathcal{F} is a Riemannian foliation. The Riemannian metric g_T is under these conditions called transverse metric of Riemannian foliation \mathcal{F} .

The transverse metric g_T and an identification of the tangent bundle TM to the direct sum of the tangent bundles $T\mathcal{F}$ and normal $\mathcal{V}(\mathcal{F})$ at \mathcal{F} allow to construct a Riemannian metric g on M having the following properties:

- 1) the sub-bundles $T\mathcal{F}$ and $\mathcal{V}(\mathcal{F})$ are orthogonal for g ;
- 2) for each application $f_i : U_i \rightarrow T$ and for each point x of U_i tangent application $T_x f_i$ induces an isometry of the fiber $V_x(\mathcal{F})$ on tangent space $T_{f_i(x)}(T)$.

In this case we identify $\mathcal{V}_x(\mathcal{F})$ and $(T_x \mathcal{F})^\perp$ where $(T_x \mathcal{F})^\perp$ is the orthocomplental of $T_x \mathcal{F}$ relative to the metric g that is to say

$$(T_x \mathcal{F})^\perp = \frac{T_x M}{T_x \mathcal{F}} = \mathcal{V}_x(\mathcal{F}).$$

A metric satisfying conditions 1) and 2) is said to be quasi-fibered with respect to \mathcal{F} .

The existence of a quasi-fibered metric characterizes Riemannian foliations [11].

Proposition 2.4 Let \mathcal{F} be Riemannian foliation defined by the cocycle $(U_i, f_i, T, \gamma_{ij})_{i \in I}$, let \mathcal{F}' be an extension of \mathcal{F} defined by the cocycle $(U_i, f'_i, T', \gamma'_{ij})_{i \in I}$ and let $(\theta_i)_{i \in I}$ be a family of submersions such that

$$f'_i = \theta_i \circ f_i \quad \text{and} \quad \gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij},$$

then \mathcal{F}' is a Riemannian extension of \mathcal{F} if, only if for every $i \in I$, θ_i is a Riemannian submersion.

Definition 2.5 Let $\chi(M)$ be the Lie algebra of vector fields over a manifold M , let X be an element of $\chi(M)$ and let $\chi(\mathcal{F})$ be the Lie algebra of vector fields tangent to the foliation \mathcal{F} .

X is foliated if and only if

$$\forall Y \in \chi(\mathcal{F}), [X, Y] \in \chi(\mathcal{F}).$$

The set of foliated fields is noted $\mathcal{L}(M, \mathcal{F})$, it is a Lie subalgebra of $\chi(M)$.

$\chi(\mathcal{F})$ is an ideal of $\mathcal{L}(M, \mathcal{F})$. We have the following exact sequence:

$$0 \rightarrow \chi(\mathcal{F}) \rightarrow \mathcal{L}(M, \mathcal{F}) \rightarrow \frac{\mathcal{L}(M, \mathcal{F})}{\chi(\mathcal{F})} \rightarrow 0.$$

The quotient

$$\ell(M, \mathcal{F}) = \frac{\mathcal{L}(M, \mathcal{F})}{\chi(\mathcal{F})}$$

is called the Lie algebra of the transverse foliated vector fields of \mathcal{F} . Note that it is a subset of the set of all the sections of the transverse bundle $\mathcal{V}(\mathcal{F})$.

We show in [9] that the following properties are equivalent :

Proposition 2.6 [9]

- i) X is foliated,
- ii) if $(\varphi_t^X)_{|t| < \varepsilon}$ is the local group with a parameter associated with X in the vicinity of an arbitrary point of M , for all t , local diffeomorphism φ_t^X sends a leaf of \mathcal{F} onto a leaf of \mathcal{F} that is to say \mathcal{F} is invariant by φ_t^X ,
- iii) in any open simple distinguished, the local coordinates $(x_1, \dots, x_p, y_1, \dots, y_q)$ is such as the last q components of X are expressed as a function of the only variables y_1, \dots, y_q .

Proposition 2.7 [9] In any simple distinguished open set U , a vector field X is foliated if and only if it is **projectable** into a vector field on the local quotient manifold \overline{U}

Definition 2.8 Let $f \in \mathcal{A}^0(M, \mathcal{F})$ be a differentiable function on M .

We say that f is basic for foliation \mathcal{F} if and only if

$$\forall X \in \chi(\mathcal{F}), Xf \text{ is identically zero.}$$

The set of basic functions for foliation \mathcal{F} is noted $\mathcal{A}_b^0(M, \mathcal{F})$.

We show in [9] that the following properties are equivalent:

- i) f is basic,
- ii) f is constant on each leaf of \mathcal{F} ,
- iii) in any open simple distinguished, local coordinates $(x_1, \dots, x_p, y_1, \dots, y_q)$ is such that f is expressed as a function of only variables y_1, \dots, y_q .

It will be noted in particular that if \mathcal{F} admits a dense leaf then every basic function is constant on M .

Before finishing this part of recall, we give the theorem of the biunivocal correspondence between Lie subalgebras of $\mathcal{G} = \text{Lie}(G)$ and the extensions of a Lie G -foliation having dense leaves on a compact connected manifold ([3], [5]).

Theorem 2.9 ([3], [5]) (one-to-one correspondence between Lie-subalgebra of $\mathcal{G} = \text{Lie}(G)$ and extension of a Lie \mathcal{G} -foliation with dense leaves on a compact connected manifold)

Let (M, \mathcal{F}) be a Lie \mathcal{G} -foliation with dense leaves on a compact connected manifold M and let $\mathcal{G} \simeq \ell(M, \mathcal{F})$ be the Lie algebra of \mathcal{F} -transverse foliated fields that to say the Lie structural algebra \mathcal{G} of \mathcal{F} .

Then:

1- There is a correspondence 1-to-1 between Lie-subalgebra of $\ell(M, \mathcal{F}) \simeq \mathcal{G}$ and the extensions of \mathcal{F} .

2- An extension of \mathcal{F} is a $\frac{\mathcal{G}}{\mathcal{H}}$ transverse riemannian foliation with trivial normal fiber bundle, defined by 1-forme valued in $\frac{\mathcal{G}}{\mathcal{H}}$.

3- An extension of \mathcal{F} is transverselly homogeneous (resp. of Lie) if and only if the Lie subgroup of G corresponding to it is a closed subgroup (resp. a normal subgroup) of G .

Note that $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$ is an isomorphism of Lie algebras. Were is the 1-form of Fedida.

We point out before continuing that the various points of the following remark are already established results and which are important and will be very useful in establishing the main result of this article.

Remark 2.10 ([3], [5]) (the consequences of the one-to-one correspondence between Lie-subalgebra of $\mathcal{G} = \text{Lie}(G)$ and extension of a Lie \mathcal{G} -foliation with dense leaves on a compact connected manifold)

Let $\mathcal{F}_{\mathcal{H}}$ be an extension of a Lie G -foliation with dense leaves on a compact manifold corresponding to the Lie-subalgebra \mathcal{H} of $\mathcal{G} = \text{Lie}(G)$,

i) Let U_{x_o} be an open set of M containing x_o , let \overline{U}_e be an open set of G containing the neutral element e of G , let $f : U_{x_o} \rightarrow \overline{U}_e$ be a submersion defining a Lie foliation \mathcal{F} on a distingue U_{x_o} checking $f(x_o) = e$ and let ω be a one form of Fedida.

We have for every $X_x \in T_x U_{x_o}$,

$$\omega_x(X_x) = (L_{f(x)*})^{-1} \circ f_{*x}(X_x) \quad \text{i.e the darbour differential of } f \text{ is } \omega$$

where $L_{f(x)}$ is the left translation of $f(x)$.

ii) let \tilde{u} be a \mathcal{F} -transverse foliated vector fields obtained from $u \in \mathcal{G}$. we have ([3], [5])

$$\tilde{u}(x) = \left((\omega_x)_{/(T_x \mathcal{F})^\perp} \right)^{-1} (u) \quad \forall x \in M$$

where $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$ is the 1-form of Fedida.

iii) Let $\tilde{\mathcal{H}}$ be the Lie subalgebra of transverse \mathcal{F} -foliated vector fields associated with \mathcal{H} .

We have ([3], [5])

$$(T\mathcal{F})^\perp \cap T\mathcal{F}_{\mathcal{H}} = \tilde{\mathcal{H}}$$

and

$$T\mathcal{F}_{\mathcal{H}} = T\mathcal{F} \oplus \tilde{\mathcal{H}}$$

and for every $x \in M$ ([3], [5])

$$T_x \mathcal{F}_{\mathcal{H}} = T_x \mathcal{F} \oplus ev_x \left(\tilde{\mathcal{H}} \right)$$

where for all vector field X $ev_x(X) = X_x$.

vi) Let $\mathcal{X}(\mathcal{F}_{\mathcal{H}})$ (respectively $\mathcal{X}(\mathcal{F})$) the Lie algebra of vector fields tangent to $\mathcal{F}_{\mathcal{H}}$ (respectively to \mathcal{F}), let $\mathcal{A}^1(M)$ be the space of differentiable functions on M and let $\tilde{\mathcal{H}}$ be the Lie algebra of \mathcal{F} -transverse foliated vector fields obtained from \mathcal{H} .

We have ([3], [5])

$$\mathcal{X}(\mathcal{F}_{\mathcal{H}}) = \mathcal{X}(\mathcal{F}) \oplus \left((\mathcal{A}_b^0(M, \mathcal{F})) \otimes \tilde{\mathcal{H}} \right)$$

where $\mathcal{A}_b^0(M, \mathcal{F})$ is the set of differentiable and \mathcal{F} -basic functions on M

Definition 2.11 [8] A q -codimensional foliation \mathcal{F} on a manifold M is transversely parallelizable if its exist q X_1, X_2, \dots, X_q \mathcal{F} -foliated transverses vectors fields linearly independent at any point of M .

Proposition 2.12 [8] Let M be a connected compact manifold.

A foliation transversely parallelizable having dense leaves on M is a Lie foliation.

Proposition 2.13 [9] Let \mathcal{F} be a q -codimensional Riemannian foliation on a compact connected manifold M , let $\overline{F^{\natural}}$ be the closure of a leaf F^{\natural} of the lifted foliation \mathcal{F}^{\natural} of \mathcal{F} on the orthonormal transverse frame bundle M^{\natural} and let $\phi : \overline{F^{\natural}} \rightarrow M$ be the projection which to a frame at x associates x then:

- i) $\phi(F^{\natural})$ is a leaf of \mathcal{F} and $\overline{\phi(F^{\natural})} = \phi(\overline{F^{\natural}})$
- ii) The map $\phi : \overline{F^{\natural}} \rightarrow \phi(\overline{F^{\natural}})$ is a locally trivial fibration
- iii) In the case where \mathcal{F} is a foliation having dense leaves in M ,

$$\phi : \overline{F^{\natural}} \rightarrow M$$

is a covering

We end these reminders by giving the following proposition which is useful to us in establishing the main result.

Proposition 2.14 Let \mathcal{F}' and \mathcal{F}'' be two Riemannian foliations on a manifold M .

If $\mathcal{F}' \subset \mathcal{F}''$ then $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$.

Proof. Let X be a \mathcal{F}'' -transverse foliated vectors fields , let λ be a \mathcal{F}' -bundle-like metric having λ_T for metric \mathcal{F}' -transverse and let U be an open set of M distinguished both for \mathcal{F}' and \mathcal{F}'' .

It is a question of showing that X is \mathcal{F}' -foliated transverse. For this in this article we will use the fact that [9] *in a distinguished simple open set V , a vector field Y is foliated if and only if it is projectable into a vector field on the \mathcal{F}' -local quotient manifold \overline{V} .*

That said, U being an open set of M distinguished both for \mathcal{F}' and \mathcal{F}'' then [3] there are three submersions $f' : U \rightarrow \overline{U}$, $f'' : U \rightarrow \overline{\overline{U}}$ and $\theta : \overline{U} \rightarrow \overline{\overline{U}}$ such as θ is the bond between (U, \mathcal{F}') and (U, \mathcal{F}'') and that the foliation restrictions (U, \mathcal{F}') and (U, \mathcal{F}'') are defined respectively by f' and f'' .

We therefore have the following diagram which is commutative:

$$\begin{array}{ccc} & & \overline{U} \\ & \nearrow f' & \downarrow \theta \\ U & \xrightarrow{f''} & \overline{\overline{U}} \end{array} .$$

The submersions f' , f'' and θ are Riemannian submersions [3] since they define the Riemannian foliations (U, \mathcal{F}') and (U, \mathcal{F}'') . In addition, the submersion θ is Riemannian because it is the bond between the Riemannian foliations (U, \mathcal{F}') and (U, \mathcal{F}'') [3].

We now consider $P_{(x, \mathcal{F}')}^U$ and $P_{(x, \mathcal{F}'')}^U$ the respective plates of \mathcal{F}' and \mathcal{F}'' in U passing through $x \in U$.

Let $z_1, z_2 \in P_{(x, \mathcal{F}')}^U$ two elements of $P_{(x, \mathcal{F}')}^U$ we have

$$f'(z_1) = f'(z_2)$$

because z_1 , and z_2 belong to the same fiber of f' , Furthermore

$$P_{(x, \mathcal{F}')}^U \subset P_{(x, \mathcal{F}'')}^U$$

from where

$$f''(z_1) = f''(z_2)$$

. As X is \mathcal{F}'' -foliated transverse then

$$f''_{*z_1}(X_{z_1}) = f''_{*z_2}(X_{z_2}) \quad \forall z_1, z_2 \in P_{(x, \mathcal{F}')}^U.$$

So the fact that

$$f'' = \theta \circ f'$$

causes that

$$\theta_{*f'(z_1)}(f'_{*z_1}(X_{z_1})) = \theta_{*f'(z_2)}(f'_{*z_2}(X_{z_2})) \quad \forall z_1, z_2 \in P_{(x, \mathcal{F}')}^U$$

because $f'(z_1) = f'(z_2)$.

As $\theta_{*f'(z_1)}$ is a vector isometry then

$$f'_{*z_1}(X_{z_1}) = f'_{*z_2}(X_{z_2}) \quad \forall z_1, z_2 \in P_{(x, \mathcal{F}')}^U$$

It follows from the above that X is projected along the plates of \mathcal{F}' on the local \mathcal{F}' -quotients \bar{U} open set distinguished U for \mathcal{F}' .

Ultimately we can say that

$$\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}').$$

■

3 Main result

Theorem 3.1 *i) Let \mathcal{F} be a Riemannian foliation with dense leaves admitting a Lie extension $\tilde{\mathcal{F}}$ on a compact connected manifold M .*

Then \mathcal{F} is a Lie foliation having dense leaves on the compact connected manifold M .

ii) More specifically every Riemannian foliation \mathcal{F} with dense leaves on a compact connected manifold M admitting a one-codimensional extension is a Lie foliation having dense leaves .

Proof. i) Let $\tilde{\mathcal{F}}$ be a Lie extension of a Riemannian foliation \mathcal{F} having dense leaves on a compact manifold M , let $\overline{F^{\natural}}$ be the closure of a leaf F^{\natural} of the lifted foliation \mathcal{F}^{\natural} of \mathcal{F} on the orthonormal transverse frame bundle M^{\natural} , let $\mathcal{F}_{/\overline{F^{\natural}}}^{\natural}$ be the restriction from \mathcal{F}^{\natural} to $\overline{F^{\natural}}$, let $\phi : \overline{F^{\natural}} \rightarrow M$ be the projection which to a frame at x associates x , let η be a \mathcal{F} -bundle-like métrique, let $\eta^{\natural} = \phi^*(\eta)$ be the metric pull back of the metric η , let $\ell(\overline{F^{\natural}}, \mathcal{F}_{/\overline{F^{\natural}}}^{\natural})$ be the Lie algebra of $\mathcal{F}_{/\overline{F^{\natural}}}^{\natural}$ -foliated transverse vectors fields, let $\ell(M, \mathcal{F})$ be the Lie algébra of \mathcal{F} -transverse foliated vectors fields on M and let \mathcal{G} be the Lie structurale algebra of \mathcal{F} .

The foliation $\phi^*(\tilde{\mathcal{F}}) = \tilde{\mathcal{F}}^{\natural}$ is an extension of Lie \mathcal{G} -foliation Lie $\mathcal{F}_{/\overline{F^{\natural}}}^{\natural}$ [9] having dense leaves on the compact connected manifold $\overline{F^{\natural}}$. Hence according to Theorem 2.9 ([3], [5]) there exists a subalgebra

$\tilde{\mathcal{H}}^{\natural} \subset \ell(\overline{F^{\natural}}, \mathcal{F}_{/\overline{F^{\natural}}}^{\natural}) \simeq \mathcal{G}$ of global $\mathcal{F}_{/\overline{F^{\natural}}}^{\natural}$ -transverses foliated vectors fields such that

$$\begin{aligned} T_{z^{\natural}} \tilde{\mathcal{F}}^{\natural} &= T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \oplus \left(T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \right)^{\perp} \cap T_{z^{\natural}} \tilde{\mathcal{F}}^{\natural} \quad \forall z^{\natural} \in \overline{F^{\natural}} \\ &= T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \oplus ev_{z^{\natural}}(\tilde{\mathcal{H}}^{\natural}) \quad (cf \text{ Remark 2.10 iii}) \end{aligned}$$

where $ev_{z^{\natural}}(X^{\natural}) = X^{\natural}$ for every $X^{\natural} \in \ell(\overline{F^{\natural}}, \mathcal{F}_{/\overline{F^{\natural}}}^{\natural})$.

Because according to ([3], [5]) there exists a Lie subalgebra $\tilde{\mathcal{H}}^{\natural} \subset \ell(\overline{F^{\natural}}, \mathcal{F}_{/\overline{F^{\natural}}}^{\natural})$ such as

$$\forall z^{\natural} \in \overline{F^{\natural}} \quad \left(T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \right)^{\perp} \cap T_{z^{\natural}} \tilde{\mathcal{F}}^{\natural} = ev_{z^{\natural}}(\tilde{\mathcal{H}}^{\natural}) \quad (*).$$

We note before continuing using the covering $\phi : \overline{F^{\natural}} \rightarrow M$ (Cf Proposition 2.13 iii)) that $\forall z^{\natural} \in \overline{F^{\natural}}$

$$T_{z^{\natural}} \phi^* \left(\widetilde{\mathcal{F}} \right) = T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \oplus ev_{z^{\natural}} \left(\widetilde{\mathcal{H}}^{\natural} \right).$$

Thus

$$T\phi^* \left(\widetilde{\mathcal{F}} \right) = T\mathcal{F}_{/\overline{F^{\natural}}}^{\natural} \oplus \left(\widetilde{\mathcal{H}}^{\natural} \right) \quad (**)$$

Hence the equality (**) shows that the Lie subalgebra $\widetilde{\mathcal{H}}^{\natural}$ is invariant by right translations R_g with $g \in K$ (where K is the stural group of covering ϕ) because $T\phi^* \left(\widetilde{\mathcal{F}} \right)$ and $T\mathcal{F}_{/\overline{F^{\natural}}}^{\natural}$ are invariant by right translations R_g

From this we can say that there is a vector space of transverse vector fields $\widetilde{\mathcal{H}}$ on M such that

$$\widetilde{\mathcal{H}}^{\natural} = \phi^* \left(\widetilde{\mathcal{H}} \right) \text{ that is to say } \phi_* \left(\widetilde{\mathcal{H}}^{\natural} \right) = \widetilde{\mathcal{H}}$$

where $\phi^* \left(\widetilde{\mathcal{H}} \right)$ is the pulled-back vector space of transverse vector fields $\widetilde{\mathcal{H}}$.

Let us now show that any transverse vector field $\widetilde{h} \in \widetilde{\mathcal{H}}$ is \mathcal{F} -foliated transverse.

Let \widetilde{h} be a vector field of $\widetilde{\mathcal{H}}$, let Y be a vector field of $\mathcal{X}(\mathcal{F})$ and let $x \in M$ be an element of M .

There exist an open local set U containing x and trivializing the covering $\phi : \overline{F^{\natural}} \rightarrow M$ and there exist an open set U^{\natural} checking:

- i) $U^{\natural} \subset \phi^{-1}(U)$
- ii) $\phi_{/U^{\natural}} : U^{\natural} \rightarrow U$ is a diffeomorphism

We note that there exist a vector space of transverse vector fields $\widetilde{\mathcal{H}}_{/U}$ on U such that

$$\phi_* \left(\widetilde{\mathcal{H}}_{/U^{\natural}}^{\natural} \right) = \widetilde{\mathcal{H}}_{/U}.$$

From where there is $\widetilde{h}_{/U^{\natural}}^{\natural} \in \widetilde{\mathcal{H}}_{/U^{\natural}}^{\natural}$ such that

$$\left(\phi_{/U^{\natural}} \right)_* \left(\widetilde{h}_{/U^{\natural}}^{\natural} \right) = \widetilde{h}_{/U}.$$

As there is $Y^{\natural} \in \mathcal{X}(\mathcal{F}^{\natural})$ such that $\left(\phi_{/U^{\natural}} \right)_* \left(Y_{/U^{\natural}}^{\natural} \right) = Y_{/U}$ then

$$\begin{aligned} \left[Y_{/U}, \widetilde{h}_{/U} \right] &= \left[\left(\phi_{/U^{\natural}} \right)_* \left(Y_{/U^{\natural}}^{\natural} \right), \left(\phi_{/U^{\natural}} \right)_* \left(\widetilde{h}_{/U^{\natural}}^{\natural} \right) \right] \\ &= \left(\phi_{/U^{\natural}} \right)_* \left(\left[Y_{/U^{\natural}}^{\natural}, \widetilde{h}_{/U^{\natural}}^{\natural} \right] \right). \end{aligned}$$

But $\left[Y_{/U^{\natural}}^{\natural}, \widetilde{h}_{/U^{\natural}}^{\natural} \right] \in \mathcal{X}(\mathcal{F}_{/U^{\natural}}^{\natural})$ so $\left[Y_{/U}, \widetilde{h}_{/U} \right] \in \mathcal{X}(\mathcal{F}_{/U})$.

Thus $\left[Y, \widetilde{h} \right]_x \in T_x \mathcal{F}$ because $\left[Y, \widetilde{h} \right]_x = \left[Y, \widetilde{h} \right](x) = \left[Y, \widetilde{h} \right]_{/U}(x) = \left[Y_{/U}, \widetilde{h}_{/U} \right](x)$

and $\left[Y_{/U}, \widetilde{h}_{/U} \right] \in \mathcal{X}(\mathcal{F}_{/U})$.

It follows from the above that any vector field $\tilde{h} \in \tilde{\mathcal{H}}$ is \mathcal{F} -foliated transverse vector field.

Let $\tilde{h} \in \tilde{\mathcal{H}}$ be a \mathcal{F} -foliated transverse of $\tilde{\mathcal{H}}$, let's show now that if there is $x_0 \in M$ such that $\tilde{h}(x_0) = 0$ then $\tilde{h} = 0$.

Suppose there is $x_0 \in M$ such that $\tilde{h}(x_0) = 0$.

Considering the covering $\phi : \overline{F^{\natural}} \rightarrow M$, the pulled-back vector field $\phi^*(\tilde{h})$ is a vector field on $\overline{F^{\natural}}$. It is easy to see that $\phi^*(\tilde{h})$ is invariant by right translations R_g and is a \mathcal{F} -foliated transverse vector field. We take note that

$$\forall x_0^{\natural} \in \phi^{-1}(x_0), \quad \phi^*(\tilde{h})(x_0^{\natural}) = 0.$$

As $\mathcal{F}_{/\overline{F^{\natural}}}$ is a Lie foliation having dense leaves, then there exists a Lie $\mathcal{F}_{/\overline{F^{\natural}}}$ -parallelism $(X_1^{\natural}, X_2^{\natural}, \dots, X_k^{\natural})$ and constants $(c_i)_{1 \leq i \leq k}$ for which we have

$$\phi^*(\tilde{h}) = \sum_{i=1}^k c_i X_i^{\natural}$$

and

$$0 = \phi^*(\tilde{h})(x_0^{\natural}) = \sum_{i=1}^k c_i X_i^{\natural}(x_0^{\natural}).$$

This last equality entails that $c_i = 0$ for all i and it shows that

$$\phi^*(\tilde{h}) = 0.$$

Thus

$$\tilde{h} = \phi_*(\phi^*(\tilde{h})) = \phi_*(0) = 0.$$

That said we can consider $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_r)$ a base of $\tilde{\mathcal{H}}$ checking

$$\tilde{h}_i(x) \neq 0 \quad \forall x \in M$$

for all i and this base will be useful to us in the following.

Now consider the equality (*).

Equality (*)

$$\left(T_{z^{\natural}} \mathcal{F}_{/\overline{F^{\natural}}}^{\natural}\right)^{\perp} \cap T_{z^{\natural}} \tilde{\mathcal{F}}^{\natural} = \text{ev}_{z^{\natural}}(\tilde{\mathcal{H}}^{\natural}) \quad (*)$$

causes that

$$\left(T \mathcal{F}_{/\overline{F^{\natural}}}^{\natural}\right)^{\perp} \cap T \tilde{\mathcal{F}}^{\natural} = \tilde{\mathcal{H}}^{\natural} \quad (**).$$

So we have

$$\begin{aligned}
\tilde{\mathcal{H}} &= \phi_{*z^{\natural}}(\tilde{\mathcal{H}}^{\natural}) \\
&= \phi_{*z^{\natural}}\left(ev_{z^{\natural}}\left(\tilde{\mathcal{H}}^{\natural}\right)\right) \\
&= \phi_{*z^{\natural}}\left(\left(T_{z^{\natural}}\mathcal{F}^{\natural}\right)^{\perp} \cap T_{z^{\natural}}\tilde{\mathcal{F}}^{\natural}\right) \\
&= (T\mathcal{F})^{\perp} \cap T\tilde{\mathcal{F}}
\end{aligned}$$

As $\mathcal{F} \subset \tilde{\mathcal{F}}$ then

$$(T\mathcal{F})^{\perp} = \left((T\mathcal{F})^{\perp} \cap T\tilde{\mathcal{F}}\right) \oplus \left(T\tilde{\mathcal{F}}\right)^{\perp}$$

But

$$\left(T\tilde{\mathcal{F}}\right)^{\perp} = \left(T\tilde{\mathcal{F}}\right)^{\perp} \cap TM$$

so

$$(T\mathcal{F})^{\perp} = \tilde{\mathcal{H}} \oplus \left(\left(T\tilde{\mathcal{F}}\right)^{\perp} \cap TM\right) \quad (***)$$

As $\tilde{\mathcal{F}}$ is a Lie foliation so its exist a Lie algebra $\tilde{\mathcal{K}}$ of $\tilde{\mathcal{F}}$ -transverse foliated vector fields such as

$$\left(T\tilde{\mathcal{F}}\right)^{\perp} \cap TM = \tilde{\mathcal{K}}$$

Note that $\tilde{\mathcal{K}} \subset \ell(M, \mathcal{F})$ because

$$\tilde{\mathcal{K}} \subset \ell(M, \tilde{\mathcal{F}}) \subset \ell(M, \mathcal{F}) \quad (cf \text{ Proposition 2.14})$$

That said we have according to (***)

$$\begin{aligned}
(T\mathcal{F})^{\perp} &= \tilde{\mathcal{H}} \oplus \left(\left(T\tilde{\mathcal{F}}\right)^{\perp} \cap TM\right) \\
&= \tilde{\mathcal{H}} \oplus \tilde{\mathcal{K}}
\end{aligned}$$

Consider $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_r)$ a base of $\tilde{\mathcal{H}}$ checking

$$\tilde{h}_i(x) \neq 0 \quad \forall x \in M$$

for all i and $(\tilde{k}_{r+1}, \tilde{k}_{r+2}, \dots, \tilde{k}_{r+s})$ a Lie parallelism of $\tilde{\mathcal{F}}$. It is easy to see that $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_r, \tilde{k}_{r+1}, \tilde{k}_{r+2}, \dots, \tilde{k}_{r+s})$ is a parallelism of \mathcal{F} -foliated transverse vector fields, that to say,

$$(T\mathcal{F})^{\perp} = \langle \tilde{h}_1 \rangle \oplus \langle \tilde{h}_2 \rangle \oplus \dots \oplus \langle \tilde{h}_r \rangle \oplus \langle \tilde{k}_{r+1} \rangle \oplus \langle \tilde{k}_{r+2} \rangle \oplus \dots \oplus \langle \tilde{k}_{r+s} \rangle$$

Which implies as \mathcal{F} being with dense leaves then according to definition 2.11 and propostion 2.12 that \mathcal{F} is a Lie $(\tilde{\mathcal{H}} \oplus \tilde{\mathcal{K}})$ -foliation having dense leaves.

ii) Let $\tilde{\mathcal{F}}$ be a one-codimensional extension of a Riemannian foliation \mathcal{F} having dense leaves on a compact connected manifold M .

We have

$$(T\tilde{\mathcal{F}})^\perp = (T\mathcal{F})^\perp \cap TM$$

is a one-dimensional bundle. We denote by \tilde{X} the unified field that orients $(T\tilde{\mathcal{F}})^\perp \cap TM$. We have also $\tilde{X} \in \ell(M, \tilde{\mathcal{F}})$ and

$$(T\tilde{\mathcal{F}})^\perp = \langle \tilde{X} \rangle.$$

It follows from the above according to definition 2.11 and proposition 2.12 that $\tilde{\mathcal{F}}$ is a Lie extension of the Riemannian foliation \mathcal{F} having dense leaves on a compact connected manifold M . Therefore according to 1) \mathcal{F} is a Lie foliation having dense leaves. ■

Remark 3.2 (deduced from the proof of the previous theorem)

Let \mathcal{F} be a Riemannian foliation with dense leaves on a compact connected manifold M , Let $X \in \ell(M, \mathcal{F})$ be a \mathcal{F} -foliated transverse vector field and let $\tilde{\mathcal{F}}$ be an extension of \mathcal{F} .

i) If it exists $x_0 \in M$ such that $X(x_0) = 0$ then $X = 0$.

ii) If it exists $x_0 \in M$ such that $X_{x_0} \in T_{x_0}\tilde{\mathcal{F}}$ then $X_x \in T_x\tilde{\mathcal{F}} \forall x \in M$.

Indeed, the fact we have

$$\begin{aligned} (T\mathcal{F})^\perp &= ((T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}) \oplus ((T\mathcal{F})^\perp \cap (T\tilde{\mathcal{F}})^\perp) \\ &= ((T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}) \oplus (T\tilde{\mathcal{F}})^\perp \end{aligned}$$

and this last equality implies that they exist two \mathcal{F} -transverses foliated vector fields $X_{(T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}} \in (T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}$ and $X_{(T\tilde{\mathcal{F}})^\perp} \in (T\tilde{\mathcal{F}})^\perp$

(indeed, $X_{(T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}} \in \ell(M, \tilde{\mathcal{F}}) \subset \ell(M, \mathcal{F})$ cf proposition 2.14) such that

$$X = X_{(T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}} + X_{(T\tilde{\mathcal{F}})^\perp}.$$

As $X_{x_0} \in T_{x_0}\tilde{\mathcal{F}}$ then $X_{(T\tilde{\mathcal{F}})^\perp}(x_0) = 0$. From where we have according to Remark 3.2 i)

$$X_{(T\tilde{\mathcal{F}})^\perp} = 0.$$

Thus $X = X_{(T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}} \in (T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}}$.

Which means that $X_x \in T_x\tilde{\mathcal{F}} \forall x \in M$.

Remark 3.3 (deduced from the proof of the previous theorem)

Let \mathcal{F} be a Riemannian foliation with dense leaves on a compact connected manifold M and let $\ell(M, \mathcal{F})$ be the Lie algebra of \mathcal{F} -transverse foliated vector fields.

Then every extension $\tilde{\mathcal{F}}$ of \mathcal{F} corresponds to a Lie subalgebra $\tilde{\mathcal{H}} \subset \ell(M, \mathcal{F})$ such that

$$(T\mathcal{F})^\perp \cap T\tilde{\mathcal{F}} = \tilde{\mathcal{H}}$$

and

$$T\tilde{\mathcal{F}} = T\mathcal{F} \oplus \tilde{\mathcal{H}}$$

Remark 3.4 Let \mathcal{F} be a Riemannian foliation with dense leaves on a compact connected manifold M .

By using the preceding remark it is easy to see that there is a one-to-one correspondence between the extensions of \mathcal{F} and the Lie subalgebras of $\ell(M, \mathcal{F})$. Indeed, if $\tilde{\mathcal{H}}$ is a Lie subalgebra of $\ell(M, \mathcal{F})$, the differential system

$$x \rightarrow T_x \mathcal{F} \oplus_{ev_x} (\tilde{\mathcal{H}})$$

is integrable and defines an extension of \mathcal{F} .

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