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On infinitely many foliations by caustics in strictly convex non-closed billiards

Alexey Glutsyuk^{*†‡}

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Abstract

Reflection in strictly convex bounded planar billiard acts on the space of oriented lines and preserves a standard area form. A *caustic* is a curve C whose tangent lines are reflected by the billiard to lines tangent to C . The famous Birkhoff conjecture states that the only strictly convex billiards with a foliation by closed caustics near the boundary are ellipses. By Lazutkin's theorem, there always exists a Cantor family of closed caustics approaching the boundary. In the present paper we deal with billiards whose boundary contains a strictly convex embedded (non-closed) curve γ . We prove that there exists a domain adjacent to γ from the convex side that is C^∞ -smoothly foliated by non-closed caustics. This generalizes a previous result by R.Melrose, which yields existence of a local foliation by caustics in a neighborhood of a boundary point. We show that there exists a continuum of above foliations by caustics whose germs at each point in γ are pairwise different. We prove a more general version of this statement in the case, when both γ and the caustics are immersed curves. It also applies to a billiard bounded by a closed strictly convex curve γ and yields infinitely many "immersed" foliations by immersed caustics. For the proof of the above results, we state and prove their analogue for a special class of area-preserving maps generalizing billiard reflections: the so-called C^∞ -lifted strongly billiard-like maps.

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1 Introduction and main results

Billiard reflection from a strictly convex smooth planar curve $\gamma \subset \mathbb{R}^2$ (parametrized by either a circle, or an interval) acts on those oriented lines that are either tangent to γ , or intersect γ transversally at two points. Namely, if a line is tangent to γ , then it is a fixed point of the reflection map. If a line L intersects γ transversally at two points, take the last intersection point B (in the sense of orientation of the line L) and reflect L from $T_B\gamma$ according to the usual reflection law: the angle of incidence is equal to the angle of reflection. By definition, the reflected line oriented at B inside the convex domain adjacent to γ is *the image* of the line L under the billiard reflection map. The reflection map thus defined is also called the *billiard ball map*.

The space of oriented lines in Euclidean plane $\mathbb{R}_{x,y}^2$ carries the standard symplectic form

$$\omega = d\phi \wedge dp, \tag{1.1}$$

where $\phi = \phi(L)$ is the azimuth (the angle of the oriented line L with the x -axis) and $p = p(L)$ is the signed distance of the line L to the origin.

Namely, $p(L) = 0$, if L passes through 0; $p = \text{dist}(L, 0)$, if L does not pass through 0 and orients its tangent circle centered at 0 clockwise; otherwise, $p = -\text{dist}(L, 0)$. It is well-known that

- the symplectic form ω is invariant under affine orientation-preserving isometries;
- *the billiard reflections from all planar curves preserve the symplectic form ω .*

Definition 1.1 A curve C is a *caustic* for the billiard on the curve γ if each line tangent to C is reflected from γ to a line tangent to C . Or equivalently, if the curve of (appropriately oriented) tangent lines to C is an invariant curve for the billiard ball map.

The main result of the paper presented in Subsection 1.1 states that if γ is a C^∞ -smooth strictly convex injectively immersed planar curve parametrized by an interval, then some its neighborhood on its convex side is regularly foliated by caustics, and there are continuum of foliations by caustics having pairwise distinct germs at every point in γ . We prove analogous statement for a non-injectively immersed curve γ , on "immersed foliations" by immersed caustics. In Subsection 1.3 we state its generalization to a special class of area-preserving maps: the so-called C^∞ -lifted strongly billiard-like maps. The plan of the paper is presented in Subsection 1.4. The corresponding background material on symplectic properties of billiard ball map is recalled in Subsection 1.2.

1.1 Main result: an open convex arc has infinitely many foliations by caustics

Theorem 1.2 *Let $\gamma \subset \mathbb{R}^2$ be a strictly convex injectively immersed C^∞ -smooth curve parametrized by an interval. There exists a simply connected domain U adjacent to γ from the convex side that is C^∞ -smoothly foliated by caustics of the curve γ and moreover, admits infinitely many (continuum of) C^∞ -smooth foliations by caustics. At each given point of the curve γ the germs of these foliations are pairwise distinct.*

Remark 1.3 It follows from R.Melrose's result [6, p.184, proposition (7.14)] that each point of the curve γ has an arc neighborhood $\alpha \subset \gamma$ for which there exists a domain U adjacent to α from the convex side and C^∞ -smoothly foliated by caustics. The new result given by Theorem 1.2 is the statement that the latter holds for the whole curve γ and there exist infinitely many distinct foliations by caustics.

Theorem 1.4 *Let $\gamma \subset \mathbb{R}^2$ be a strictly convex C^∞ -smooth curve that is the image of an interval $(0, 1)$ with coordinate x under an **immersion** $\psi : (0, 1) \rightarrow \gamma$. Then there exists a domain $U \subset (0, 1) \times \mathbb{R}_+ \subset \mathbb{R}_{x,y}^2$ adjacent to the interval $J := (0, 1) \times \{0\}$ and a C^∞ -smooth immersion $\Psi : U \cup J \rightarrow \mathbb{R}^2$ extending ψ as a map $J \rightarrow \gamma$, sending U to the convex side from γ and sending horizontal line sections $\{y = \text{const}\} \cap U$ to caustics of the curve γ .*

Definition 1.5 Let $U_1, U_2 \subset \mathbb{R}_x \times (\mathbb{R}_+)_y$ be two domains adjacent to $J = (0, 1) \times \{0\}$. We say that two C^∞ -smooth immersions $\Psi_j : U_j \cup J \rightarrow \mathbb{R}^2$, $j = 1, 2$, with $\Psi_1|_J \equiv \Psi_2|_J$ are *y-equivalent*, if there exist domains $V_j \subset U_j$, $j = 1, 2$, adjacent to J and a C^∞ -smooth diffeomorphism $H : V_1 \cup J \rightarrow V_2 \cup J$, $H|_J = \text{Id}|_J$, **preserving the fibration** $\{y = \text{const}\}$ (mapping fibers to (may be different) fibers) such that $\Psi_1 = \Psi_2 \circ H$. We say that Ψ_1 and Ψ_2 are *y-equivalent at a point* $q \in J$, if there exists a germ of diffeomorphism $H : (\mathbb{R} \times \mathbb{R}_{\geq 0}, q) \rightarrow (\mathbb{R} \times \mathbb{R}_{\geq 0}, q)$ at q satisfying the above statements.

Addendum to Theorem 1.4. *Let γ , ψ and J be the same, as in Theorem 1.4. There exist infinitely many (continuum of) immersions $\Psi : U \cup J \rightarrow \mathbb{R}^2$ (with the same U) satisfying the statements of Theorem 1.4 that are pairwise not y-equivalent at each point in J .*

Remark 1.6 Theorem 1.4 and its addendum also hold in the special case, when γ is a convex curve bijectively parametrized by circle and $\psi : (0, 1) \simeq \mathbb{R} \rightarrow \gamma$ is the universal covering map. Namely, let us extend ψ to the universal covering $\pi : \mathbb{R} \times [0, \varepsilon)$ over a small annulus adjacent to γ from the convex side, set $J := \mathbb{R} \times \{0\}$; $\pi|_J = \psi$. Theorem 1.4 implies that *there exists a domain $V \subset \mathbb{R} \times [0, \varepsilon)$ adjacent to J and a C^∞ -smooth foliation of $V \cup J$ with J being a leaf such that its leaves different from J are projected by π to caustics of the billiard in γ* . The addendum to Theorem 1.4 implies existence of continuum of foliations satisfying the above statement with pairwise distinct germs at each point in J .

A generalization of Theorems 1.2, 1.4 for the so-called C^∞ -lifted strongly billiard-like maps will be stated in Subsection 1.3.

1.2 Background material: symplectic properties of billiard ball map

Let γ be a C^∞ -smooth strictly convex oriented curve in \mathbb{R}^2 parametrized injectively either by an interval, or by circle. Let s be its natural length

parameter respecting its orientation. We identify a point in γ with the corresponding value of the natural parameter s .

Let $\Gamma := T_{=1}\mathbb{R}^2|_\gamma \subset T\mathbb{R}^2_\gamma$ denote the restriction to γ of the unit tangent bundle of the ambient plane \mathbb{R}^2 :

$$\Gamma = \{(q, u) \mid q \in \gamma, u \in T_q\mathbb{R}^2, \|u\| = 1\}.$$

It is a two-dimensional surface parametrized diffeomorphically by $(s, \phi) \in \gamma \times S^1$; here $\phi = \phi(u)$ is the angle of a given unit tangent vector $u \in T_s\mathbb{R}^2$ with the orienting unit tangent vector $\dot{\gamma}(s)$ to γ . The curve

$$\tilde{\gamma} := \{\phi = 0\} = \{(s, \dot{\gamma}(s)) \mid s \in \gamma\}$$

is the graph of the above vector field $\dot{\gamma}$. For every $(q, u) \in \Gamma$ set

$$L(q, u) := \text{the oriented line through } q \text{ directed by the vector } u.$$

We treat the two following cases separately.

Case 1): the curve γ either is parametrized by an interval and goes to infinity in both directions, or is parametrized by circle. That is, it bounds a strictly convex infinite (respectively, bounded) planar domain. Let $\Gamma^0 \subset \Gamma$ denote the neighborhood of the curve $\tilde{\gamma}$ that consists of those $(q, u) \in \Gamma$ that satisfy the following conditions:

- a) the line $L(q, u)$ either intersects γ at two points q and q' , or is the orienting tangent line to γ at q : $u = \dot{\gamma}(s)$; in the latter case we set $q' := q$;
- b) the angle between the oriented line $L(q, u)$ and any of the orienting tangent vectors to γ at q or q' is acute.

Let u' denote the directing unit vector of the line $L(q, u)$ at q' . Consider the two following involutions acting on Γ^0 and Γ respectively:

$$\beta : \Gamma^0 \rightarrow \Gamma^0, \beta(q, u) = (q', u'); \quad \beta^2 = Id;$$

$$I : \Gamma \rightarrow \Gamma \text{ is the reflection from } T_q\gamma : I(q, u) = (q, u^*),$$

where u^* is the vector symmetric to u with respect to the tangent line $T_q\gamma$. Let $\Gamma^0_+ \subset \Gamma^0$ denote the open subset of those pairs (q, u) in which the vector u is directed to the convex side from the curve γ .

Remark 1.7 The domain Γ^0 is β -invariant. It is a topological disk (cylinder), if γ is parametrized by an interval (circle). The domain Γ^0_+ is a topological disk (cylinder) adjacent to $\tilde{\gamma}$.

Let Π_γ denote the open subset of the space of oriented lines in \mathbb{R}^2 consisting of the lines $L(q, u)$ with $(q, u) \in \Gamma_+^0$. The mapping $\Lambda : (q, u) \mapsto L(q, u)$ is a diffeomorphism

$$\Lambda : \Gamma_+^0 \rightarrow \Pi_\gamma$$

It extends to the set $\Gamma_+^0 \cup \tilde{\gamma}$ as a homeomorphism sending each point $(s, \dot{\gamma}(s)) \in \tilde{\gamma}$ to the tangent line $T_s\gamma$ directed by $\dot{\gamma}(s)$.

Remark 1.8 Let \mathcal{T} denote the billiard ball map given by reflection from the curve γ acting on oriented lines. It is well-known that the billiard ball map \mathcal{T} is conjugated by Λ to the product of two involutions

$$\tilde{\delta}_+ := I \circ \beta = \Lambda^{-1} \circ \mathcal{T} \circ \Lambda : \Gamma_+^0 \rightarrow \Gamma.$$

If the curve γ is C^∞ -smooth, then both involutions I and β are C^∞ -smooth on Γ and Γ^0 respectively. Their product is well-defined and smooth on a neighborhood of the curve $\tilde{\gamma}$ and fixes the points of the curve $\tilde{\gamma}$. Both involutions preserve the canonical symplectic form $\sin \phi ds \wedge d\phi$ on $\Gamma \setminus \tilde{\gamma}$, which is known to be the Λ -pullback of the standard symplectic form on the space of oriented lines. See [1, 2, 5, 6, 7, 8]; see also [3, subsection 7.1].

Let us recall another representation of the billiard ball map \mathcal{T} in a chart where it preserves the standard symplectic form. To do this, consider the orthogonal projection $\pi_\perp : (T\mathbb{R}^2)|_\gamma \rightarrow T\gamma$ sending each vector $u \in T_q\mathbb{R}^2$ with $q \in \gamma$ to its orthogonal projection to the tangent line $T_q\gamma$. It projects the unit tangent bundle Γ to the unit ball bundle

$$T_{\leq 1}\gamma := \{(q, w) \mid q \in \gamma, w \in T_q\gamma, \|w\| \leq 1\}.$$

A tangent vector $w = w \frac{\partial}{\partial s} \in T_q\gamma$ will be identified with its coordinate $w = \pm \|w\|$ in the basic vector $\frac{\partial}{\partial s}$. Thus, $\pi_\perp(s, \phi) = (s, \cos \phi)$. Consider the following function and differential form on $T\gamma$:

$$y := 1 - w; \quad \omega := ds \wedge dy. \tag{1.2}$$

The form ω coincides with the standard symplectic form on the tangent bundle $T\gamma$ of the curve γ (considered as a Riemannian manifold equipped with the metric $|ds|^2$ coming from the standard Euclidean metric on \mathbb{R}^2).

The curve $\tilde{\gamma} = \{(s, \dot{\gamma}(s)) \mid s \in \gamma\} = \{w = 1\} = \{y = 0\} \subset T\gamma$ is a component of the boundary $\partial T_{\leq 1}\gamma$. The projection π_\perp sends Γ_+^0 diffeomorphically to a domain in $T_{\leq 1}\gamma$ adjacent to $\tilde{\gamma}$. It extends homeomorphically to $\Gamma_+^0 \cup \tilde{\gamma}$ as the identity map $Id : \tilde{\gamma} \rightarrow \tilde{\gamma}$. Let $\mu_+ : \pi_\perp(\Gamma_+^0 \cup \tilde{\gamma}) \rightarrow \Gamma_+^0 \cup \tilde{\gamma}$ be the inverse to the restriction of the projection π_\perp to $\Gamma_+^0 \cup \tilde{\gamma}$. Set

$$\delta_+ := \pi_\perp \circ \tilde{\delta}_+ \circ \mu_+ = \pi_\perp \circ \Lambda^{-1} \circ \mathcal{T} \circ \Lambda \circ \mu_+. \tag{1.3}$$

Theorem 1.9 ([8, subsection 1.5], [6, 7, 1, 2]; see also [3, theorem 7.3]).
The mapping $\delta_+ : \pi_\perp(\Gamma_+^0) \rightarrow T_{\leq 1}\gamma$ given by (1.3), is symplectic: it preserves the form $\omega = ds \wedge dy$.

Proposition 1.10 [3, proposition 7.5]. Let $\kappa(s)$ denote the (geodesic) curvature of the curve γ . The involutions I, β and the mappings $\tilde{\delta}_+, \delta_+$ admit the following (asymptotic) formulas:

$$I(s, \phi) = (s, -\phi), \quad \beta(s, \phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), -\phi + O(\phi^2)), \quad (1.4)$$

$$\tilde{\delta}_+(s, \phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), \phi + O(\phi^2)), \quad (1.5)$$

$$\delta_+(s, y) = (s + 2\sqrt{2}\kappa^{-1}(s)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})). \quad (1.6)$$

The asymptotics are uniform on compact subsets of points $s \in \gamma$, as $\phi \rightarrow 0$ (respectively, as $y \rightarrow 0$).

Case 2). Let γ be parametrized by an interval, but now it does not necessarily go to infinity or bound a region in the plane. Moreover, we allow γ to be an immersed curve that may self-intersect. In this case some lines $L(q, u)$ may intersect γ in more than two points. Now the definition of the subset $\Gamma^0 \subset \Gamma$ should be modified to be the subset of those $(q, u) \in \Gamma$ for which there exists a $q' \in \gamma \cap L(q, u)$ satisfying the conditions a) and b) from Case 1) and such that the arc $(q, q') \subset \gamma$ is disjoint from the line $L(q, u)$, injectively immersed (i.e., without self-intersections) and the orienting tangent vector of the latter arc at each its point has acute angle with $L(q, u)$. (Here q and q' may be not the only points of intersection $\gamma \cap L(q, u)$.)

Remark 1.11 For any given $(q, u) \in \Gamma^0$ the point q' satisfying the conditions from the above paragraph exists, whenever u is close enough to $\dot{\gamma}(q)$ (dependently on q). Whenever it exists, it is unique. All the statements and discussion in the previous Case 1) remain valid in our Case 2). Now the mapping Λ is a local diffeomorphism but not necessarily a global diffeomorphism: an oriented line intersecting γ at more than two points (if any) may correspond to at least two different tuples $(q, u) \in \Gamma_+^0$.

1.3 Generalization to C^∞ -lifted strongly billiard-like maps

In this subsection and in what follows we study the next class of area-preserving mappings introduced in [3] generalizing the billiard maps (1.6).

Definition 1.12 (see [3, definition 7.6]). Let (a, b) be a (may be (semi) infinite) interval in \mathbb{R} with coordinate s . Let $V \subset (a, b) \times \mathbb{R}_+$ be a domain adjacent to the interval $J := (a, b) \times \{0\}$. A mapping $F : V \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0} \subset \mathbb{R}_{s,y}^2$ is called *billiard-like*, if it satisfies the following conditions:

- (i) $F : V \cup J \rightarrow F(V \cup J)$ is a homeomorphism fixing the points in J ;
- (ii) $F|_V$ is a diffeomorphism preserving the standard area form $ds \wedge dy$;
- (iii) F has the asymptotics of the type

$$F(s, y) = (s + w(s)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})), \text{ as } y \rightarrow 0; \quad w(s) > 0, \quad (1.7)$$

uniformly on compact subsets in the s -interval (a, b) ;

- (iv) the variable change

$$(s, y) \mapsto (s, z), \quad y = z^2$$

conjugates F to a map $\tilde{F}(s, z)$ that is smooth at $(a, b) \times \{0\}$.

If, in addition to conditions (i)–(iv), the latter mapping \tilde{F} is a product of two symplectic involutions I and β fixing the points of the line $z = 0$,

$$\tilde{F} = I \circ \beta, \quad I(s, z) = (s, -z),$$

$$\beta(s, z) = (s + w(s)z + O(z^2), -z + O(z^2)), \quad \beta^2 = Id, \quad (1.8)$$

then F will be called a (*strongly*) *billiard-like map*.

If F is strongly billiard-like, and the corresponding involution β (or equivalently, the conjugate map \tilde{F}) is C^∞ -smooth, and C^∞ -smooth at the points of the boundary interval J , then F is called *C^∞ -lifted*. The above definitions make sense for F being a germ of map at the interval J .

Example 1.13 The mapping δ_+ from (1.6) (and its germ at the curve $\{y = 0\}$) is strongly billiard-like in the coordinates (s, y) with $w(s) = 2\sqrt{2}\kappa^{-1}(s)$, see (1.4), (1.5) and (1.6). If the curve γ is C^∞ -smooth, then β and hence, $\tilde{\delta}_+ = I \circ \beta$ are C^∞ -smooth, and hence, δ_+ is C^∞ -lifted.

Proposition 1.14 *The class of (germs at J of) C^∞ -lifted strongly billiard-like maps is invariant under conjugacy by (germs at J of) C^∞ -smooth symplectomorphisms $G : V \cup J \rightarrow G(V \cup J) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ sending J onto an interval in $\mathbb{R} \times \{0\}$. Here $V \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to J .*

Proof Let F be a C^∞ -lifted strongly billiard-like map, $\tilde{F} = I \circ \beta$ be its lifting. Let $V \subset \mathbb{R} \times \mathbb{R}_+$ be a domain adjacent to J , and $G : V \cup J \rightarrow G(V \cup J) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$, be a C^∞ -smooth symplectomorphism as above. Let us denote

$G(s, y) = (\widehat{s}(s, y), \widehat{y}(s, y))$. One has $\widehat{y}(s, 0) \equiv 0$, $\frac{\partial \widehat{s}}{\partial s}(s, 0) > 0$, $\frac{\partial \widehat{y}}{\partial y}(s, 0) > 0$, by definition and orientation-preserving property (symplectomorphicity). Thus, $\widehat{y}(s, y) = yg(s, y)$, where $g(s, y)$ is a positive C^∞ -smooth function on a neighborhood of the interval J in $(a, b) \times \mathbb{R}_{\geq 0}$. The lifting \widetilde{G} of the map G to the variables (s, z) , $y = z^2$, acts as follows:

$$\widetilde{G} : (s, z) \mapsto (\widehat{s}(s, z^2), \widehat{z}(s, z)); \quad \widehat{z} = \sqrt{\widehat{y}(s, z^2)} = z\sqrt{g(s, z^2)}. \quad (1.9)$$

The latter square root is well-defined and C^∞ -smooth. This implies that the map \widetilde{G} is a C^∞ -smooth diffeomorphism of domains with arcs of boundaries corresponding to $V \cup J$ and $G(V \cup J)$. Hence, the lifting $\widetilde{G} \circ \widetilde{F} \circ \widetilde{G}^{-1}$ of the conjugate $F_G := G \circ F \circ G^{-1}$ is a C^∞ -smooth diffeomorphism that is the product of \widetilde{G} -conjugates of the involutions I and β . One has $\widetilde{G} \circ I \circ \widetilde{G}^{-1} = I$, by (1.9); F_G is a symplectomorphism, since so are F and G ;

$$G(s, y) = (\widehat{s}, \widehat{y}) = (\widehat{s}(s, 0) + O(y), g(s, 0)y + O(y^2)), \quad (1.10)$$

by diffeomorphicity. Substituting (1.10) and (1.7) to the expression $F_G = G \circ F \circ G^{-1}$ and denoting $(s, 0) := G^{-1}(\widehat{s}, 0)$, we get

$$F_G(\widehat{s}, \widehat{y}) = (\widehat{s} + \frac{\partial \widehat{s}}{\partial s}(s, 0)w(s)g^{-\frac{1}{2}}(s, 0)(\widehat{y})^{\frac{1}{2}} + O(\widehat{y}), \widehat{y} + O(\widehat{y}^{\frac{3}{2}})).$$

This implies that the conjugate F_G has type (1.7) and hence, is strongly billiard-like. This proves the proposition. \square

Theorem 1.15 *For every C^∞ -lifted strongly billiard-like map F there exists a domain $U \subset \{y > 0\}$ adjacent to J such that $U \cup J$ admits a C^∞ -smooth F -invariant function \widetilde{h} , $\widetilde{h}|_J \equiv 0$, $\frac{\partial \widetilde{h}}{\partial y} > 0$. (Thus, the foliation $\widetilde{h} = \text{const}$ is a C^∞ -smooth foliation by F -invariant curves, and J is its leaf.) Moreover, there are continuum of C^∞ -smooth F -invariant foliations $\widetilde{h} = \text{const}$ as above on the same union $U \cup J$ whose germs at each point in J are pairwise distinct.*

Addendum to Theorem 1.15: a smooth symplectic normal form for C^∞ -lifted strongly billiard-like maps. *In Theorem 1.15 one can choose a function \widetilde{h} so that there exists a new additional coordinate $\tau = \tau(s, y)$ bijectively parametrizing the interval J such that (τ, \widetilde{h}) are symplectic coordinates on a domain $U \subset \{y > 0\}$ adjacent to J and in these coordinates*

$$F(\tau, \widetilde{h}) = (\tau + \sqrt{\widetilde{h}}, \widetilde{h}). \quad (1.11)$$

Definition 1.16 Let $V \subset (a, b) \times \mathbb{R}_+ \subset \mathbb{R}_{s,y}^2$ be a domain adjacent to an interval $J = (a, b) \times \{0\}$. A C^∞ -smooth function $f(s, y)$ on $V \cup J$ is *y-flat*, if $f(s, 0) \equiv 0$, $f(s, y)$ tends to zero with all its partial derivatives, as $y \rightarrow 0$, and the latter convergence is uniform on compact subsets in the s -interval (a, b) for the function f and for each its individual derivative.

Remark 1.17 In the conditions of the above definition let (x, h) be new C^∞ -smooth coordinates on $V \cup J$ with $h(s, 0) \equiv 0$. Then each y -flat function is h -flat and vice versa. This follows from definition.

The proof of Theorem 1.15 uses Marvizi–Melrose result [5, theorem (3.2)] stating a formal analogue of Theorem 1.15: existence of a F -invariant formal power series $\sum_k h_k(s)y^s$, see Theorem 2.1 below. It implies that in appropriate coordinates (τ, h) the map F takes the form $F(\tau, h) = (\tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h))$. Here $\text{flat}(h)$ is an h -flat function, see the above definition. In the coordinates (τ, ϕ) , $\phi = \sqrt{h}$, the lifted map \tilde{F} takes the form

$$\tilde{F}(\tau, \phi) = (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi)). \quad (1.12)$$

We prove existence of a C^∞ -smooth \tilde{F} -invariant function $\tilde{\phi}$ with $\tilde{\phi} - \phi = \text{flat}(\phi)$ (the next theorem), and then deduce Theorems 1.15, 1.2, 1.4.

Theorem 1.18 *Let $V \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi$ be a domain adjacent to an interval $J = (a, b) \times \{0\}$. Let $\tilde{F} : V \cup J \rightarrow \mathbb{R}_\tau \times (\mathbb{R}_{\geq 0})_\phi$ be a C^∞ -smooth mapping of type (1.12). (Here we do not assume any area-preserving property.) There exists a domain $W \subset V$ adjacent to J and an \tilde{F} -invariant C^∞ -smooth function on $W \cup J$ of the type $\tilde{\phi}(\tau, \phi) = \phi + \text{flat}(\phi)$. (Thus, the foliation $\tilde{\phi} = \text{const}$ is \tilde{F} -invariant, C^∞ -smooth, and J is its leaf.)*

Addendum to Theorem 1.18. *There exist continuum of functions $\tilde{\phi}$ satisfying the statements of Theorem 1.18 such that the corresponding foliations $\tilde{\phi} = \text{const}$ are C^∞ -smooth on the same subset $W \cup J$ and have pairwise distinct germs at each point in J .*

This addendum and non-uniqueness in Theorem 1.15 will be proved using the following proposition.

Proposition 1.19 *Let $J = (a, b) \times \{0\}$, $W \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi$ be a domain adjacent to J . Let $\tilde{F} : W \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a map, as in (1.12). Any two \tilde{F} -invariant foliations (functions, line fields) on W having distinct germs at J have distinct germs at each point in J . The same statement holds for similar objects invariant under a C^∞ -lifted strongly billiard-like map.*

1.4 Plan of the paper

In Subsection 2.1 we recall the above-mentioned Marvizi – Melrose result [5, theorem 3.2] (with proof) yielding C^∞ -smooth coordinates in which $F(\tau, h) = \tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h)$. It implies that the lifted map \tilde{F} , written in the coordinates (τ, ϕ) , $\phi = \sqrt{h}$, takes form (1.12).

Theorem 1.18 will be proved in Subsections 2.2–2.4. To do this, first in Subsection 2.2 we construct a fundamental domain for the map \tilde{F} (a curvilinear sector Δ with vertex at a point in J) and an \tilde{F} -invariant function $\tilde{\phi}$ defined on a bigger sector that is ϕ -flatly close to ϕ on the latter bigger sector. Then in Subsection 2.3 we construct its \tilde{F} -invariant extension along the \tilde{F} -orbits and show that it is well-defined on a domain adjacent to J . In Subsection 2.4 we prove that thus extended function $\tilde{\phi}$ is C^∞ -smooth and ϕ -flatly close to ϕ . This will prove Theorem 1.18.

The existence statement of Theorem 1.15 will be deduced from Theorem 1.18 in Subsection 2.5. Proposition 1.19 and non-uniqueness in Theorem 1.18 (Addendum) and in Theorem 1.15 will be proved in Subsection 2.6. Theorems 1.4 and 1.2 will be proved in Subsection 2.7.

2 Construction of foliation by invariant curves. Proofs of Theorems 1.18, 1.15, 1.2, 1.4 and Proposition 1.19

2.1 Marvizi–Melrose construction of an ”up-to-flat” first integral

Here we recall the following Marvizi–Melrose theorem with proof.

Theorem 2.1 [5, theorem (3.2)]. 1) Let $V \subset (a, b) \times \mathbb{R}_{>0} \subset \mathbb{R}_{x,y}^2$ be a domain adjacent to the interval $J := (a, b) \times \{0\}$. Let $F : V \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a C^∞ -lifted strongly billiard-like map. There exist a domain $U \subset V$ adjacent to J and a real-valued C^∞ -smooth function $h : U \cup J \rightarrow \mathbb{R}_{\geq 0}$, $h|_J \equiv 0$, $\frac{\partial h}{\partial y}|_J > 0$, such that the difference $h \circ F - h$ is C^∞ -smooth and y -flat. Moreover, one can normalize h as above so that the mapping F coincides, up to y -flat terms, with the time 1 map of the flow of the Hamiltonian vector field with the Hamiltonian function $\frac{2}{3}h^{\frac{3}{2}}$. This normalization determines the Taylor series $h(s, y) = \sum_{k=1}^{+\infty} h_k(s)y^k$ uniquely.

2) The analogue of the above statement holds if J is replaced by the coordinate circle $S_s^1 = S_s^1 \times \{0\}$ lying in the cylinder $C := S_s^1 \times [0, \varepsilon)$ equipped with the standard area form and F is a strongly billiard-like map $C \rightarrow S^1 \times$

$\mathbb{R}_{\geq 0}$. In this case the coefficients $h_k(s)$ of the above normalized series are well-defined and C^∞ -smooth on the circle S_s^1 .

3) Let h be the function normalized as in Statement 1). Let τ denote the time function for the Hamiltonian vector field with the Hamiltonian function h . In the coordinates (τ, h) (which are symplectic) the map F takes the form

$$F : (\tau, h) \mapsto (\tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h)). \quad (2.1)$$

Proof The strongly billiard-like mapping F has the form (1.7):

$$F(s, y) = (s + w(s)\sqrt{y} + O(y), y + q(s)y^{\frac{3}{2}} + O(y^2)). \quad (2.2)$$

Its lifting $\tilde{F}(s, z)$, $z = \sqrt{y}$, has the form

$$\tilde{F}(s, z) = (s + w(s)z + O(z^2), z + \frac{q(s)}{2}z^2 + O(z^3)). \quad (2.3)$$

The mapping $\tilde{F}(s, z)$ admits an asymptotic Taylor series in z , and $F(s, y)$ admits an asymptotic Puiseux series in y involving powers $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. The coefficients of both series are C^∞ -smooth functions in s . Therefore, the mapping F acts by the formula $h \mapsto h \circ F$ not only on functions, but also on formal Puiseux series. It transform each power series $h = \sum_{k=1}^{+\infty} h_k(s)y^k$ with coefficients being C^∞ -smooth functions on (a, b) to a Puiseux series of the above type. Our goal is to find an F -invariant power series (or equivalently, an \tilde{F} -invariant even power series $\sum_{k=1}^{+\infty} h_k(s)z^{2k}$) and then to choose its C^∞ -smooth representative. To do this, we use the following formula for the function $q(s)$ in (2.2), see [4, formula (1.2)], [3, formula (7.18)], which follows from area-preserving property:

$$q(s) = -\frac{2}{3}w'(s). \quad (2.4)$$

Step 1: constructing an even series $\sum_{k=1}^{+\infty} g_k z^{2k}$ whose \tilde{F} -image is also an even series. We construct its coefficients $g_k(s)$ by induction as follows.

Induction base: $k = 1$. Let us find a function $g_1(s)$ such that the \tilde{F} -image of the function $g_1(s)z^2$ contains no z^3 -term. This is equivalent to the statement saying that the function $g_1(s + w(s)z)(z + \frac{q(s)}{2}z^2)^2$ contains no z^3 -term, which is in its turn equivalent to the differential equation

$$g_1'(s)w(s) + q(s)g_1(s) = 0, \quad q(s) = -\frac{2}{3}w'(s),$$

which has a unique solution $g_1(s) = w^{\frac{2}{3}}(s)$ up to constant factor. (Note that $w^{\frac{2}{3}}(s)y$ is a well-known function: the second Lazutkin coordinate [4, 5].)

Induction step in the case, when $J = (a, b) \times \{0\}$ is an interval. Let we have already found an even Taylor polynomial $G_{n-1}(s, z) := \sum_{k=1}^{n-1} g_k(s)z^{2k}$, $n \geq 2$, such that the asymptotic Taylor series in z of the function $G_{n-1} \circ \tilde{F}$ contains no odd powers of z of degrees no greater than $2n - 1$. Let us construct $g_n(s)$, set $G_n(s, z) := \sum_{k=1}^n g_k(s)z^{2k}$, so that

$$G_n \circ \tilde{F} - G_n \text{ contains no } z^{2n+1} \text{ - term.} \quad (2.5)$$

Note that $G_n \circ \tilde{F} - G_n$ obviously cannot contain odd powers of degrees less than $2n$. Let $b(s)z^{2n+1}$ denote the degree $2n + 1$ term in the Taylor series of the function $G_{n-1} \circ F$. Condition (2.5) is equivalent to the differential equation

$$g'_n(s)w(s) - \frac{2n}{3}w'(s)g_n(s) = -b(s), \quad (2.6)$$

which always has a solution $g_n(s)$ well-defined on the interval (a, b) .

Step 2. Constructing an \tilde{F} -invariant series. The mapping \tilde{F} is the product $I \circ \beta$ of two involutions: $I(s, z) = (s, -z)$ and β . Let $g := \sum_{k=1}^{+\infty} g_k(s)z^{2k}$ be the series constructed on Step 1. One has

$$g \circ \tilde{F} = (g \circ I) \circ \beta = g \circ \beta, \quad (2.7)$$

since the series g is even. The series (2.7) is even (Step 1). Hence, the series

$$t := g + g \circ \beta$$

is even and β -invariant by construction. Therefore, it is \tilde{F} -invariant. Its first coefficient is equal to $2g_1(s) = 2w^{\frac{2}{3}}(s) > 0$, by construction. We denote the \tilde{F} -invariant series thus constructed by $t := \sum_{k=1}^{+\infty} t_k(s)z^{2k}$.

Step 3: symplectic coordinates and normalization. Let $t(s, y)$ be a function representing the series $\sum_{k=1}^{+\infty} t_k(s)y^k$, which is obtained from the latter series (given by Step 2) by the variable change $y = z^2$. It is defined on a domain U adjacent to J and C^∞ -smooth on $U \cup J$; $t|_J \equiv 0$, $\frac{\partial t}{\partial y}|_J > 0$. Let H_t denote the corresponding Hamiltonian vector field. Fix an arbitrary C^∞ -smooth function θ such that $d\theta(H_t) \equiv 1$, $\theta|_{s=0} = 0$: a time function for the vector field H_t . Then (θ, t) are symplectic coordinates for the form $\omega = dx \wedge dy$: $\omega = d\theta \wedge dt$. Shrinking U (keeping it adjacent to J) we can and will consider that they are global coordinates on $U \cup J$. The difference $t \circ F - t$ is t -flat, by construction, and hence, so is $dF(H_t) - H_t$. Therefore, in the coordinates (θ, t) the symplectic map F takes the form

$$F : (\theta, t) \mapsto (\theta + \xi(t), t) + \text{flat}(t). \quad (2.8)$$

In the new coordinates (θ, t) the map F is C^∞ -lifted strongly billiard-like, as in the old coordinates (s, y) , by Proposition 1.14.

Claim 1. *The function $\xi(t)$ in (2.8) has the form $\xi(t) = \sqrt{t}\psi(t)$, where $\psi(t)$ is a C^∞ -smooth function on a segment $[0, \varepsilon]$, $\varepsilon > 0$, $\psi \geq 0$, $\psi(0) > 0$.*

Proof Let \tilde{F} denote the lifting of the map F to the coordinates (θ, ζ) , $\zeta = \sqrt{t}$. One has $\tilde{F} = I \circ \beta$, where $I(\theta, \zeta) = (\theta, -\zeta)$ and β is a symplectic involution, $\beta(\theta, 0) \equiv (\theta, 0)$. The involution β takes the form

$$\beta(\theta, \zeta) = (\theta + q(\zeta), -\zeta) + \text{flat}(\zeta), \quad q(\zeta) = \xi(\zeta^2) \text{ for } \zeta > 0. \quad (2.9)$$

The function $q(\zeta)$ should be C^∞ -smooth, as is β , and $q'(0) > 0$ (strong billiard-likedness). The condition saying that β is an involution implies that $q(\zeta) + q(-\zeta) = \text{flat}(\zeta)$. This in its turn implies that $q(\zeta) = \zeta\psi(\zeta^2) + \text{flat}(\zeta)$, where ψ is a C^∞ -smooth function; $\psi(0) = q'(0) > 0$. This together with (2.9) implies the statement of the claim. \square

We have to find a function $h(s, y)$, $h(s, 0) \equiv 0$, such that the Hamiltonian vector field with the Hamiltonian function $\frac{2}{3}h^{\frac{3}{2}}$ coincides with $\xi(t)\frac{\partial}{\partial\theta}$: this function will satisfy the normalization statement of Theorem 2.1, part 1), by construction. We are looking for it as a function depending only on t : $h(s, y) = \chi(t)$. The above Hamiltonian vector field is then equal to $\sqrt{\chi(t)}\chi'(t)\frac{\partial}{\partial\theta}$. Thus, we have to solve the equation

$$\chi^{\frac{1}{2}}(t)\chi'(t) = \xi(t) = \sqrt{t}\psi(t), \quad \chi(0) = 0.$$

Its solution $\chi(t)$ is given by the formula

$$\chi(t) = \left(\frac{3}{2} \int_0^t \sqrt{p}\psi(p)dp \right)^{\frac{2}{3}}.$$

This is a C^∞ -smooth function, by construction and smoothness of the function $\psi(t)$. One has $\frac{\partial h}{\partial y}|_J > 0$, since $\chi'(0) = \psi(0) > 0$ and $\frac{\partial t}{\partial y}(s, 0) = 2g_1(s) = 2w^{\frac{2}{3}}(s) > 0$, by construction. Uniqueness of the Taylor series in y of the function $h(s, y)$ satisfying the above Hamiltonian vector field statement follows directly, as in [5, p.383]. Statement 1) of Theorem 2.1 is proved. Statement 3) follows immediately from Statement 1), since in the coordinates (τ, h) , see Statement 3), the Hamiltonian field with the Hamiltonian function $\frac{2}{3}h^{\frac{3}{2}}$ is equal to $(\sqrt{h}, 0)$. Statement 2) (case, when J is a circle and F is defined on a cylinder bounded by J) says that the Taylor coefficients of the series in y of the function $h(s, y)$ are well-defined functions on the circle J . This follows from its the above uniqueness statement. Theorem 2.1 is proved. \square

2.2 Step 1. Construction of an invariant function on a neighborhood of fundamental domain

Here we give the first step of the proof of Theorem 1.18. We consider a fundamental sector Δ for the map \tilde{F} that is bounded by the segment $K = [0, \frac{\eta}{2}]$ of the ϕ -axis, by its \tilde{F} -image and by the straightline segment connecting their ends. We construct an \tilde{F} -invariant function $\tilde{\phi}$ that is ϕ -flatly close to ϕ on a sectorial neighborhood $S_{\chi, \eta}$ of $\overline{\Delta} \setminus \{(0, 0)\}$. See Fig. 1.

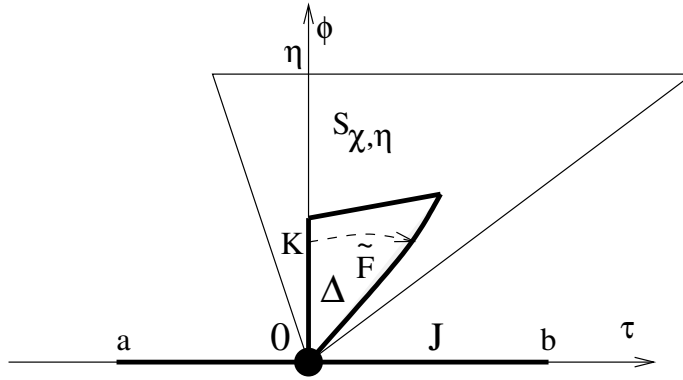


Figure 1: The fundamental domain Δ and its sectorial neighborhood $S_{\chi, \eta}$.

Without loss of generality we consider that the τ -interval contains the origin: $a < 0 < b$. Fix a number χ , $0 < \chi < \frac{1}{2}$. Consider the sectors

$$S_\chi = \{-\chi\phi < \tau < (1 + \chi)\phi\} \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi, \quad (2.10)$$

$$S_{\chi, \eta} := S_\chi \cap \{0 < \phi < \eta\}$$

The domain $S_{\chi, \eta}$ will be the above-mentioned neighborhood of fundamental sector, where we construct an \tilde{F} -invariant function.

Proposition 2.2 *For every $\chi \in (0, \frac{1}{2})$ and $\eta > 0$ small enough dependently on \tilde{F} and χ the following statements hold.*

- (i) *The maps $\tilde{F}^{\pm 1}$, $\tilde{F}^{\pm 2}$ are well-defined on $S_{\chi, 2\eta}$.*
- (ii) *The domains $S_{\chi, 2\eta}$ and $\tilde{F}^2(S_{\chi, 2\eta})$ are disjoint; the latter lies on the right from the former.*
- (iii) *The segment $K := \{0\} \times [0, \frac{\eta}{2}] \in \mathbb{R}_{\tau, \phi}^2$ and its image $\tilde{F}(K)$ intersect just by the origin; $\tilde{F}(K)$ lies on the right from K . The domain $\Delta \subset S_{\chi, 2\eta}$*

bounded by K , $\tilde{F}(K)$ and the straightline segment connecting the endpoints of the arcs K and $\tilde{F}(K)$ distinct from $(0,0)$ is a fundamental domain for the map \tilde{F} . See Fig. 1.

Proof One has

$$d\tilde{F}(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.11)$$

The latter differential sends each line $\{\tau = \zeta\phi\}$ to the line $\{\tau = (\zeta + 1)\phi\}$. This implies that for every $\eta > 0$ small enough statements (i)–(iii) hold. \square

Proposition 2.3 *For every $\chi \in (0, \frac{1}{2})$ and $\eta > 0$ small enough dependently on \tilde{F} and χ there exists a C^∞ -smooth and \tilde{F} -invariant function $\tilde{\phi}(\tau, \phi)$ on $S_{\chi, \eta}$ such that the difference $\tilde{\phi}(\tau, \phi) - \phi$ is ϕ -flat on $S_{\chi, \eta}$: that is, the latter difference tends to zero with all its partial derivatives, as $(\tau, \phi) \in S_{\chi, \eta}$ tends to zero. There exist a continuum of functions $\tilde{\phi}$ satisfying the above statements and without critical points on the same sector $S_{\chi, \eta}$ for which the germs at $(0,0)$ of the foliations $\tilde{\phi} = \text{const}$ are pairwise distinct.*

Proof Let $\nu : S_\chi \rightarrow \mathbb{R}$ denote the function

$$\nu := \frac{\tau}{\phi},$$

whose level curves are lines through the origin. The interval of values of the function ν on S_χ is $M := (-\chi, 1 + \chi)$. Fix a

$$\sigma > 0, \quad 2\sigma < \frac{1}{2} - \chi. \quad (2.12)$$

Consider the covering of the interval M by the intervals

$$\left(-\chi, \frac{1}{2} + \sigma\right), \quad \left(\frac{1}{2} - \sigma, 1 + \chi\right)$$

and a corresponding partition of unity ρ_1, ρ_2 :

$$\rho_1 \equiv 1 \text{ on } \left(-\chi, \frac{1}{2} - \sigma\right); \quad \rho_2 \equiv 1 \text{ on } \left(\frac{1}{2} + \sigma, 1 + \chi\right); \quad (2.13)$$

$$\rho_1, \rho_2 \geq 0, \quad \rho_1 + \rho_2 \equiv 1 \text{ on } M = (-\chi, 1 + \chi).$$

Set

$$\begin{aligned} \tilde{\phi}(x) &:= \rho_1(\nu(x))\phi(x) + \rho_2(\nu(x))\phi \circ \tilde{F}^{-1}(x) \\ &= \phi(x) + \rho_2(\nu(x))(\phi \circ \tilde{F}^{-1}(x) - \phi(x)). \end{aligned} \quad (2.14)$$

Proposition 2.4 *For every fixed $\chi \in (0, \frac{1}{2})$, $\sigma \in (0, \frac{1}{2}(\frac{1}{2} - \chi))$ and every η small enough (dependently on χ and σ) the function $\tilde{\phi}$ given by (2.14) is well-defined on $S_{\chi, \eta}$ and \tilde{F} -invariant: if $x, \tilde{F}(x) \in S_{\chi, \eta}$, then $\tilde{\phi}(\tilde{F}(x)) = \tilde{\phi}(x)$. It is C^∞ -smooth, and the difference $\tilde{\phi}(x) - \phi(x)$ is ϕ -flat on $S_{\chi, \eta}$.*

Proof Recall that \tilde{F} satisfies asymptotic formula (1.12):

$$\tilde{F}(\tau, \phi) = (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi)).$$

Well-definedness and C^∞ -smoothness of the function $\tilde{\phi}$ on $S_{\chi, \eta}$ for small η are obvious. Its ϕ -flatness on $S_{\chi, \eta}$ follows from formula (2.14), ϕ -flatness of the difference $\phi \circ \tilde{F} - \phi$, see (1.12), and the fact that the function $\nu(\tau, \phi) = \frac{\tau}{\phi}$ has partial derivatives of at most polynomial growth in ϕ , as $(\tau, \phi) \rightarrow 0$ along the sector $S_{\chi, \eta}$. Let us prove \tilde{F} -invariance, whenever η is small enough. For every $\delta > 0$ and every $\eta > 0$ small enough (dependently on δ) the inclusion $\tilde{F}(x) \in S_{\chi, \eta}$ implies that $\tau(x) \leq (\chi + \delta)\phi(x)$, by formula (1.12) recalled above. Therefore the inclusion $x, \tilde{F}(x) \in S_{\chi, \eta}$ implies that x lies in the sector $\{-\chi\phi < \tau < (\chi + \delta)\phi\}$. Choosing $\delta < \sigma$ we get that on the latter sector $\rho_1 \circ \nu \equiv 1$ and $\rho_2 \circ \nu \equiv 0$, since $\chi + \delta < \chi + \sigma < \frac{1}{2} - \sigma$, see (2.12), and by (2.13). Hence, $\tilde{\phi}(x) = \phi(x)$, by (2.14). Similarly applying the above argument "in the inverse time" yields that the inclusion $x, \tilde{F}(x) \in S_{\chi, \eta}$ implies that $\tilde{F}(x)$ lies in the sector $\{(1 - \chi - \delta)\phi < \tau < (1 + \chi)\phi\}$. On the latter sector one has $\rho_1 \circ \nu \equiv 0$, $\rho_2 \circ \nu \equiv 1$, by (2.13) and since

$$1 - \chi - \delta > 1 - \chi - \sigma = 1 - \chi + \sigma - 2\sigma > 1 - \chi + \sigma - \frac{1}{2} + \chi = \frac{1}{2} + \sigma.$$

Therefore, $\tilde{\phi}(\tilde{F}(x)) = \phi \circ \tilde{F}^{-1}(\tilde{F}(x)) = \phi(x)$, by (2.14). Finally we get that $\tilde{\phi}(x) = \tilde{\phi} \circ \tilde{F}(x)$, and hence $\tilde{\phi}$ is \tilde{F} -invariant. The proposition is proved. \square

Let us now prove non-uniqueness of germ at $(0, 0)$ of foliation $\tilde{\phi} = \text{const}$. The point $\frac{1}{2}$ is the midpoint of the interval $M = (-\chi, 1 + \chi)$ defining S_χ . One has $\frac{1}{2} \pm 1 \notin M$, since $0 < \chi < \frac{1}{2}$. Fix a small $\theta > 0$, set $Y := (\frac{1}{2} - \theta, \frac{1}{2} + \theta)$, so that the intervals $Y \pm 1$ lie outside the interval M , on distance greater than θ from its boundary points. Set $SY := \{(\tau, \phi) \mid \frac{\tau}{\phi} \in Y\}$. If η is small enough, then $\tilde{F}^{\pm 1}(x) \notin S_\chi$ for every $x \in S_{\chi, \eta} \cap SY$, by (1.12). Fix a partition of unity (2.13). Fix a positive C^∞ -smooth ϕ -flat function $g(\phi) < 1$, say $g(\phi) = \exp(-\frac{1}{\phi})$. Take an arbitrary C^∞ -smooth bump function $\rho_3(\nu) \geq 0$ supported in Y , $\rho_3 \not\equiv 0$, $\rho_3, \rho_3' \leq 1$. In the definition (2.14) of the function $\tilde{\phi}$ let us add the new term $\eta^2 \rho_3(\nu(x))g(\phi(x))$ to the right-hand side. This changes $\tilde{\phi}$ only inside the sector SY and will change

neither its \tilde{F} -invariance, nor its smoothness and ϕ -flatness, by construction. One can choose a continuum of different functions ρ_3 as above. If η is small enough, then all the corresponding functions $\tilde{\phi}$ are C^∞ -smooth on $S_{\chi,\eta}$ and have no critical points there. For any two different functions ρ_3 the corresponding germs of foliations $\tilde{\phi} = \text{const}$ at the origin are different: the corresponding functions $\tilde{\phi}$ coincide on $S_{\chi,\eta} \setminus SY$, while the germs of their restrictions to SY differ. The cardinality of the set of all the functions in a given finite number of variables is continuum. This implies that there exist a continuum of functions $\tilde{\phi}$ satisfying the statements of Proposition 2.3 for which the corresponding germs of foliations $\tilde{\phi} = \text{const}$ are distinct. Proposition 2.3 is proved. \square

2.3 Step 2. Extension by dynamics

Here we show that an \tilde{F} -invariant function $\tilde{\phi}$ constructed above on a neighborhood of the fundamental domain Δ extends along \tilde{F} -orbits to an \tilde{F} -invariant function on a domain W adjacent to $J = (a, b) \times \{0\} \subset \mathbb{R}_{\tau,\phi}^2$. It suffices to prove that the function $\tilde{\phi}$ extends as above to a rectangle $(a', b') \times (0, \eta')$ adjacent to arbitrary relatively compact subinterval $J' = (a', b') \times \{0\} \Subset J$. The union of the above rectangles corresponding to an exhaustion of J by a sequence of subintervals J' yields a domain W adjacent to all of J , where the extended function is defined. Therefore, we make the following convention.

Convention 2.5 Everywhere below we identify the interval $J = (a, b) \times \{0\}$ with (a, b) and sometimes we denote $J = (a, b) \subset \mathbb{R}$. We will consider that there exists a $\delta > 0$ such that $\tilde{F}^{\pm 1}$ are diffeomorphisms of the rectangle $J \times [0, \delta] \subset \mathbb{R}_{\tau,\phi}^2$ onto its images, and the ϕ -flat terms in its asymptotic formula (1.12) are *uniformly ϕ -flat*: the difference $\tilde{F}(\tau, \phi) - (\tau + \phi, \phi)$ converges to zero uniformly in $\tau \in J$, and any its partial derivative (of any order) also converges to zero uniformly. Indeed, the flat terms in question are uniform on compact subsets in J . Hence, one can achieve their uniformity replacing J by its relatively compact subinterval. Under this assumption the above difference and its differential are both uniformly $o(\phi^m)$ in $\tau \in J$ for each individual $m \in \mathbb{N}$. We also consider that J is a finite interval: a, b are finite.

The next proposition describes asymptotics of two-sided \tilde{F} -orbits.

Proposition 2.6 *For every η small enough and $x := (\tau_0, \phi_0) \in J \times [0, \eta)$*

- a) the iterates $\tilde{F}^j(x) = (\tau_j, \phi_j)$ are well defined for all $j \geq 0$, $j \leq N_+$, where $N_+ = N_+(x)$ is the maximal number j for which $\tau_j < b$;
- b) the inverse iterates $\tilde{F}^{-j}(x) = (\tau_{-j}, \phi_{-j})$ are well-defined for all $j \leq N_-$ where $N_- = N_-(x)$ is the maximal number j for which $\tau_{-j} > a$;
- c) $\phi_j = \phi_0(1 + o(1))$ uniformly in τ_0 and $j \in [-N_-, N_+]$, as $\phi_0 \rightarrow 0$;
- d) the points τ_j form an asymptotic arithmetic progression: $\tau_{j+1} - \tau_j = \phi_0(1 + o(1))$ uniformly in $\tau_0 \in J$ and in $j \in [-N_-, N_+ - 1]$, as $\phi_0 \rightarrow 0$.

Proof Consider two lines and segments through x :

$$L_{\pm}(x) := \{\phi = \phi_0 \pm \phi_0^4(\tau - \tau_0)\}, \quad \lambda_{\pm} := L_{\pm} \cap J \times [0, 2\eta).$$

Claim 2. For every $x = (\tau_0, \phi_0) \in J \times [0, 2\eta)$ with ϕ_0 small enough

- e) the image $\tilde{F}(\lambda_{\pm})$ is disjoint from λ_{\pm} and lies on its right;
- f) the image $\tilde{F}^{-1}(\lambda_{\pm})$ is disjoint from λ_{\pm} and lies on its left.
- g) the right sector $S_+(x)$ bounded by the right subintervals in λ_{\pm} with vertex x is \tilde{F} -invariant;
- h) the left sector $S_-(x)$ bounded by the left subintervals in λ_{\pm} with vertex x is \tilde{F}^{-1} -invariant.

Proof If η is small enough, then $\tilde{F}^{\pm 1}$ are well-defined on $J \times [0, 3\eta)$. If ϕ_0 is small enough, then each λ_{\pm} is projected to all of J , and the ϕ -coordinates of all its points are uniformly asymptotically equivalent to ϕ_0 (finiteness of J). The map \tilde{F} moves a point $z = (\tau, \phi) \in \lambda_{\pm}$ to $y := (\tau + \phi, \phi)$ up to a ϕ -flat term, which is $o(\phi_0^m)$ for every $m \in \mathbb{N}$. On the other hand, the distance of the latter point y to the line L_{\pm} is equal to $\phi \simeq \phi_0$ times the $|\sin|$ of the azimuth of the line L_{\pm} . The latter azimuth is asymptotic to ϕ_0^4 , and hence, is greater than $\frac{1}{2}\phi_0^4$, whenever ϕ_0 is small enough. Thus, $\text{dist}(y, L_{\pm}) \geq \frac{1}{3}\phi_0^5$. Therefore, adding a term $o(\phi_0^m)$, $m \geq 5$, to y will not allow to cross L_{\pm} , and we will get a point lying on the same, right side from the line L_{\pm} , as y . The case of inverse iterates is treated analogously. Statements e) and f) are proved. They immediately imply statements g) and h). \square

Let $\eta \in (0, \frac{1}{8})$ be small enough so that \tilde{F} is defined on the rectangle $\Pi := J \times [0, 3\eta)$ and for every $x \in \Pi$ with $\phi_0 = \phi(x) \in [0, 2\eta]$ the sector $S_+(x)$ contains the points $x_j = \tilde{F}^j(x)$ until they go out of Π (Claim 2 g)). The intersection $S_+(x) \cap \partial\Pi$ is contained in the right lateral side $\{b\} \times [0, 3\eta)$. Therefore, the first j for which x_j goes out of Π is the one for which $\tau(x_j) \geq b$. This proves Statement a) of Proposition 2.6. The proof of Statement b) is analogous. For every $x \in \Pi$ with $\phi_0 = \phi(x)$ small enough the above inclusion $x_j \in S_+(x)$ holds for $j = 1, \dots, N_+$. It implies Statement c) for the above j , by the definition of the sector S_+ . The proof of Statement c)

for $j = -N_-, \dots, -1$ is analogous. Statement d) follows from Statement c), since $\tau \circ \tilde{F}(x) - \tau(x) = \phi(x) + \text{flat}(\phi(x))$, see (1.12). Proposition 2.6 is proved. \square

Corollary 2.7 1) For every $\eta > 0$ small enough each point $x = (\tau_0, \phi_0) \in J \times [0, \frac{2\eta}{3})$ has two-sided orbit lying in $J \times [0, \eta)$ and consisting of points x_j , $j \in [-N_-(x), N_+(x)]$, with $\phi_j \simeq \phi_0$, as $\phi_0 \rightarrow 0$; the latter asymptotics is uniform in the above j and in $\tau_0 \in J$.

2) Let Δ denote the fundamental domain (curvilinear triangle) for the map \tilde{F} from Proposition 2.2, Statement (iii). Let $\hat{\Delta}$ denote the complement of the closure $\overline{\Delta}$ to the union of its vertex $(0, 0)$ and the opposite side. If $\eta > 0$ is small enough, then the domain W saturated by the above two-sided orbits of points in $\hat{\Delta}$ lies in $J \times [0, \frac{2\eta}{3})$ and contains the strip $J \times (0, \frac{\eta}{4})$.

3) The orbit of each point in W contains either a unique point lying in the fundamental domain Δ , or two subsequent points lying in its lateral boundary curves (glued by \tilde{F}).

4) Each \tilde{F} -invariant function $\tilde{\phi}$ on $\hat{\Delta}$ extends to a unique \tilde{F} -invariant function on W as a function constant along the latter orbits.

The corollary follows immediately from Proposition 2.6. Step 2 is done.

2.4 Step 3. Regularity and flatness. End of proof of Theorem 1.18

Here we will prove the following lemma, which will imply Theorem 1.18.

Lemma 2.8 Let in Corollary 2.7 the function $\tilde{\phi}$ on $\hat{\Delta}$ be the restriction to $\hat{\Delta}$ of a C^∞ -smooth \tilde{F} -invariant function defined on a neighborhood of $\hat{\Delta}$. Let the function $\tilde{\phi}(\tau, \phi) - \phi$ be flat on $\hat{\Delta}$: it tends to zero with all its partial derivatives, as $(\tau, \phi) \in \hat{\Delta}$ tends to zero. Consider its extension to the above domain W from Corollary 2.7, Statement 4), and let us denote the extended function by the same symbol $\tilde{\phi}$. The difference $\tilde{\phi}(\tau, \phi) - \phi$ is C^∞ -smooth on $W \cup J$, and it is uniformly ϕ -flat (see Convention 2.5).

Proof For every point $x = (\tau, \phi) \in W$ there exists a $N = N(x) \in \mathbb{Z}$ such that $\tilde{F}^N(x) \in \hat{\Delta}$. The latter image $\tilde{F}^N(x)$ lies in the definition domain of the initial function $\tilde{\phi}$ (which is defined on a neighborhood of $\hat{\Delta}$), and $\tilde{\phi}(x) = \tilde{\phi}_N(x) := \tilde{\phi}(\tilde{F}^N(x))$, by definition. This immediately implies C^∞ -smoothness of the extended function $\tilde{\phi}$ on W . Let us prove its ϕ -flatness.

This will automatically imply C^∞ -smoothness at points of the boundary interval J . To do this, we use the asymptotics

$$d\tilde{F}(\tau, \phi) = A + \text{flat}(\phi), \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad (2.15)$$

$$N(x) = N(\tau, \phi) = O\left(\frac{1}{\phi}\right). \quad (2.16)$$

Here the flat term in (2.15) is uniformly flat, see Convention 2.5. Formula (2.15) follows from (1.12). Formula (2.16) holds, since $N \leq N_+ + N_- = O(\frac{1}{\phi})$, which follows from Proposition 2.6, Statement d).

We study the derivatives of the functions $\tilde{\phi}_N - \phi$, $N = N(x)$, at the point $x = (\tau, \phi)$, as functions in x with fixed N chosen as above for this concrete x . To prove uniform flatness, we have to show that all its partial derivatives tend to zero uniformly in $\tau \in J$, as $\phi \rightarrow 0$. We prove this statement for the first derivatives (step 1) and then for the higher derivatives (step 2).

Without loss of generality everywhere below we consider that $N \geq 1$, i.e., x lies on the left from the sector Δ : for negative N the proof is analogous.

Step 1: the first derivatives. The initial function $\tilde{\phi}$ defined on a neighborhood of the set $\hat{\Delta}$ is already known to be ϕ -flat on $\hat{\Delta}$. The differential of the composition $\tilde{\phi}_N = \tilde{\phi} \circ \tilde{F}^N$ at the point x , $N = N(x)$, is equal to

$$d(\tilde{\phi} \circ \tilde{F}^N)(x) = d\tilde{\phi}(\tilde{F}^N(x))d\tilde{F}(\tilde{F}^{N-1}(x)) \dots d\tilde{F}(x). \quad (2.17)$$

Proposition 2.9 *For every sequence of points $x(k) = (\tau_{0k}, \phi_{0k}) \in W$ with $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$, and numbers $N_k = N(x(k)) \in \mathbb{N}$ with $\tilde{F}^{N_k}(x(k)) \in \hat{\Delta}$ the difference $d(\tilde{\phi} \circ \tilde{F}^{N_k})(x(k)) - d\phi$ tends to zero, as $k \rightarrow \infty$.*

Proposition 2.9 implies uniform convergence to zero of the first derivatives.

In its proof (given below) we use the following asymptotics of differential $d\tilde{F}(\tilde{F}^j(x))$ and technical proposition on matrix products. We denote

$$M(\tau, \phi) := \text{the Jacobian matrix of the differential } d\tilde{F}(\tau, \phi).$$

Proposition 2.10 *Let $x = (\tau_0, \phi_0) \in J \times (0, \frac{\eta}{4})$, $x_j = (\tau_j, \phi_j) := \tilde{F}^j(x)$, $j = 0, \dots, N(x)$. For every $m \in \mathbb{N}$ one has*

$$M(\tau_j, \phi_j) = A + o(\phi_0^m), \quad \text{as } \phi_0 \rightarrow 0; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (2.18)$$

uniformly in $j = 1, \dots, N(x)$ and in $\tau_0 \in J$ for each individual m .

Proof Formula (2.18) follows from (2.15) and Proposition 2.6, part c). \square

Proposition 2.11 *Consider arbitrary sequences of numbers $\phi_{0k} > 0$, $N_k \in \mathbb{N}$, $\phi_{0k} \rightarrow 0$, $N_k = O(\frac{1}{\phi_{0k}})$, as $k \rightarrow \infty$, and matrix collections*

$$\mathcal{M}_k = (M_{1;k}, \dots, M_{N_k;k}), \quad M_{j;k} \in \mathrm{GL}_2(\mathbb{R}),$$

$$M_{j;k} = A + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.19)$$

Here the latter asymptotics is uniform in $j = 1, \dots, N_k$ for each individual m , as $k \rightarrow \infty$. Then the products of the matrices $M_{j;k}$ have the asymptotics

$$\widehat{M}_k := M_{N_k;k} \dots M_{1;k} = \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}. \quad (2.20)$$

Proof Conjugating by the diagonal matrix $H_k := \mathrm{diag}(1, \phi_{0k}^{-1})$ transforms the matrices $M_{j;k}$ and their product respectively to the following matrices:

$$\widetilde{M}_{j;k} = B_k + o(\phi_{0k}^m), \quad B_k = \begin{pmatrix} 1 & \phi_{0k} \\ 0 & 1 \end{pmatrix}; \quad \widetilde{M}_k := \widetilde{M}_{N_k;k} \dots \widetilde{M}_{1;k}.$$

Claim 3. *One has*

$$\widetilde{M}_k = B_k^{N_k} + o(\phi_{0k}^m) = \begin{pmatrix} 1 & N_k \phi_{0k} \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m). \quad (2.21)$$

Proof Without loss of generality we can and will consider that $N_k \phi_{0k} \rightarrow C \in \mathbb{R}_{\geq 0}$, passing to a subsequence, since $N_k = O(\frac{1}{\phi_{0k}})$, by assumption. Let $\mathcal{UT} \subset \mathrm{GL}_2(\mathbb{R})$ denote the one-parametric subgroup of unipotent upper triangular matrices. Consider the tangent vector

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_1 \mathcal{UT} \subset T_1 \mathrm{GL}_2(\mathbb{R}).$$

Let us extend it to a left-invariant vector field on $\mathrm{GL}_2(\mathbb{R})$, which is tangent to the \mathcal{UT} -orbits under right multiplication action. Take a small transverse section $S \subset \mathrm{GL}_2(\mathbb{R})$ passing through the identity and consider the subset $U \subset \mathrm{GL}_2(\mathbb{R})$ foliated by arcs of phase curves of the field V starting in S and parametrized by time segment $[0, 2C]$. The subset U is a bordered domain (flowbox) diffeomorphic to the product $S \times [0, 2C]$ via the diffeomorphism sending a point $y \in U$ to the pair $(s(y), t(y))$ such that the orbit issued from

the point $s(y) \in S$ arrives to y in time $t(y)$. Fix an arbitrary $m \geq 3$. In the new chart (s, t) the multiplication by a matrix $\widetilde{M}_{j;k} = B_k + o(\phi_{0k}^m)$ from the right moves a point (s, t) to the point $(s, t + \phi_{0k})$ up to a small correction of order $o(\phi_{0k}^m)$. Therefore, the multiplication by $N_k \simeq \frac{C}{\phi_{0k}}$ similar matrices $\widetilde{M}_{j;k}$ with the $o(\phi_{0k}^m)$ in their asymptotics being uniform in j moves a point (s, t) to a point $(s, t + N_k \phi_{0k})$ up to a correction of order $N_k o(\phi_{0k}^m) = o(\phi_{0k}^{m-1})$. This implies (2.21) with m replaced by $m - 1$. Taking into account that m can be chosen arbitrary, this proves (2.21). \square

Conjugating formula (2.21) by the matrix H_k^{-1} and taking into account that $m \in \mathbb{N}$ is arbitrary yields (2.20). This proves Proposition 2.11. \square

Proof of Proposition 2.9. For $z \in \widehat{\Delta}$ set

$$St(z) := \left(\frac{\partial \widetilde{\phi}}{\partial \tau}, \frac{\partial \widetilde{\phi}}{\partial \phi} \right)(z).$$

The string of the first partial derivatives of the function $\widetilde{\phi}_N = \widetilde{\phi} \circ \widetilde{F}^N(x)$, $N = N(x)$, is equal to the product

$$St(\tau_N, \phi_N) M(\tau_{N-1}, \phi_{N-1}) \dots M(\tau_0, \phi_0), \quad (\tau_j, \phi_j) = \widetilde{F}^j(x), \quad j = 0, \dots, N-1,$$

$$St(\tau_N, \phi_N) = (0, 1) + o(\phi_0^m) \text{ for every } m \in \mathbb{N}, \quad (2.22)$$

by ϕ -flatness of the initial function $\widetilde{\phi}$ on $\widehat{\Delta}$ and by the uniform asymptotics $\phi_j = \phi_0(1 + o(1))$, $j = 1, \dots, N$ (Proposition 2.6, Statement c)).

Take arbitrary sequence of points $x(k) := (\tau_{0k}, \phi_{0k})$, $\tau_{0k} \in J$, $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$. Set

$$(\tau_{jk}, \phi_{jk}) := \widetilde{F}^j(x(k)), \quad N_k := N(x(k)).$$

The sequence of collections of Jacobian matrices $M_{j+1;k} := M(\tau_{jk}, \phi_{jk})$, $j = 0, \dots, N_k - 1$, satisfy the conditions of Proposition 2.11, by (2.15) and Convention 2.5. Therefore, their product \widehat{M}_k , which is the Jacobian matrix of the differential $d\widetilde{F}^{N_k}(x(k))$, has asymptotics (2.20):

$$\widehat{M}_k := \text{the Jacobian matrix of } d\widetilde{F}^{N_k}(x(k)) = \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m). \quad (2.23)$$

Thus, the matrix-string of the differential $d\widetilde{\phi}_{N_k}(\tau_{0k}, \phi_{0k})$ is the product

$$St(\tau_{N_k}, \phi_{N_k}) \widehat{M}_k = ((0, 1) + o(\phi_{0k}^m)) \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m) = (0, 1) + o(\phi_{0k}^{m-1}),$$

since $N_k = O(\frac{1}{\phi_{0k}})$, see (2.16). For $m = 2$ we get that the differential $d(\tilde{\phi}_{N_k}(\tau, \phi) - \phi)$ taken at the point $x(k)$ tends to zero, as $k \rightarrow \infty$. This proves Proposition 2.9. \square

Step 2: the higher derivatives. For a smooth function f defined on a neighborhood of a point x by $j_x^\ell(f)$ we will denote its ℓ -jet at x . Below we prove the following proposition.

Proposition 2.12 *In the conditions of Proposition 2.9 for every $\ell \in \mathbb{N}$ the ℓ -jet at $x(k)$ of the difference $\tilde{\phi} \circ F^{N_k} - \phi$ tends to zero, as $k \rightarrow \infty$.*

Proposition 2.12 will imply C^∞ -smoothness and ϕ -flatness of the extended function $\tilde{\phi}$ at the points of the boundary interval $J \times \{0\}$.

For every $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^2$ let J_x^ℓ denote the space of ℓ -jets of functions at the point x . The map \tilde{F} induces a transformation of functions, $g \mapsto g \circ \tilde{F}$. This induces linear operators in the jet spaces, $D_\ell \tilde{F}(x) : J_{\tilde{F}(x)}^\ell \rightarrow J_x^\ell$. We identify the space of ℓ -jets at each point in \mathbb{R}^2 with the ℓ -jet space at the origin, which in its turn is identified with the space $\mathcal{P}_{\leq \ell}$ of polynomials of degrees no greater than ℓ . Thus, we consider the operator $D_\ell \tilde{F}(x)$ as acting on the above space $\mathcal{P}_{\leq \ell}$. One has

$$D_\ell \tilde{F}^{N_k}(x) = D_\ell \tilde{F}(F^{N_k-1}(x)) \dots D_\ell \tilde{F}(x). \quad (2.24)$$

Linear changes of variables (τ, ϕ) act on the space $\mathcal{P}_{\leq \ell}$ and induce an injective linear representation $\rho : \text{GL}_2(\mathbb{R}) \rightarrow \text{GL}(\mathcal{P}_{\leq \ell})$. Let A denote the unipotent Jordan cell, see (2.19).

Proposition 2.13 *For every sequence of points $x(k) = (\tau_{0k}, \phi_{0k}) \in W$ with $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$, set $N_k := N(x(k))$, one has*

$$D_\ell \tilde{F}^{N_k}(x(k)) = \rho(A^{N_k}) + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}. \quad (2.25)$$

Proof One has

$$D_\ell \tilde{F}(\tau, \phi) = \rho(A) + \text{flat}(\phi), \quad (2.26)$$

by (2.15). Set $x_j(k) = (\tau_{jk}, \phi_{jk}) = \tilde{F}^j(x(k))$, $j = 0, \dots, N_k - 1$. One has

$$D_\ell \tilde{F}(x_j(k)) = \rho(A) + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}, \quad (2.27)$$

by (2.26) and Proposition 2.6, Statement c). We use (2.24) and the following multidimensional version of Proposition 2.11.

Proposition 2.14 Consider arbitrary sequences of numbers $\phi_{0k} > 0$, $N_k \in \mathbb{N}$, $\phi_{0k} \rightarrow 0$, $N_k = O(\frac{1}{\phi_{0k}})$, as $k \rightarrow \infty$, and matrix collections

$$\mathcal{M}_k = (M_{1;k}, \dots, M_{N_k;k}), \quad M_{j;k} \in \text{GL}(\mathcal{P}_{\leq \ell}),$$

$$M_{j;k} = \rho(A) + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.28)$$

Here the latter asymptotics is uniform in $j = 1, \dots, N_k$ for each individual m , as $k \rightarrow \infty$. Then the product of the matrices $M_{j;k}$ has the asymptotics

$$\widehat{M}_k := M_{N_k;k} \dots M_{1;k} = \rho(A^{N_k}) + o(\phi_{0k}^m) \text{ for every } m \in \mathbb{N}. \quad (2.29)$$

Proof Conjugating the matrices $M_{j;k}$ by $\rho(H_k)$, $H_k := \text{diag}(1, \phi_{0k}^{-1})$, transforms them to matrices

$$\widetilde{M}_{j;k} = \rho(B_k) + o(\phi_{0k}^{m'}), \quad B_k = \begin{pmatrix} 1 & \phi_{0k} \\ 0 & 1 \end{pmatrix}, \quad m' = m - \dim \mathcal{P}_{\leq \ell}.$$

It suffices to show that the product of the matrices $\widetilde{M}_{j;k}$ has asymptotics $\rho(B_k^{N_k}) + o(\phi_{0k}^m)$ for every $m \in \mathbb{N}$, as in the proof of Claim 3. This is done by considering the left-invariant vector field on $\text{GL}(\mathcal{P}_{\leq \ell})$ tangent to the orbits of the subgroup $\rho(\text{GL}_2(\mathbb{R}))$ acting on $\text{GL}(\mathcal{P}_{\leq \ell})$ by right multiplication and repeating the arguments from the proof of Claim 3. \square

Formula (2.25) is deduced from Proposition 2.14 and formulas (2.24), (2.27), as formula (2.23). \square

Proof of Proposition 2.12. The polynomial representing the ℓ -jet of the initial function $\widetilde{\phi}$ at a point $z \in \widehat{\Delta}$ tends to the linear polynomial $P(\tau, \phi) = \phi$, as $z \rightarrow 0$, so that its distance to $P(\tau, \phi)$ is $o(\phi^m)$ for every $m \in \mathbb{N}$, by flatness of $\widetilde{\phi}$ on $\widehat{\Delta}$. This together with Proposition 2.6, Statement c) implies that the distance of its ℓ -jet at the point $\widetilde{F}^{N_k}(x(k))$ to the polynomial ϕ is asymptotic to $o(\phi_{0k}^m)$. The image of the latter ℓ -jet under the operator $D_\ell \widetilde{F}^{N_k}(x(k))$ is also $o(\phi_{0k}^m)$ -close to ϕ for every $m \in \mathbb{N}$. This follows from the previous statement, formula (2.25), the fact that $\rho(A)$ fixes ϕ and the asymptotics $N_k = O(\phi_{0k}^{-1})$. Finally we get that the difference of the ℓ -jet of the function ϕ at $x(k)$ and the ℓ -jet $j_{x(k)}^\ell(\widetilde{\phi} \circ \widetilde{F}^{N_k})$ of the extended function tends to zero, as $k \rightarrow \infty$. Proposition 2.12 is proved. \square

Lemma 2.8 follows from Proposition 2.12. It implies Theorem 1.18. \square

2.5 Proof of existence in Theorem 1.15 and its addendum

Let F be a C^∞ -lifted strongly billiard-like map. Let (τ, h) be the coordinates from Theorem 2.1. Set $\phi = \sqrt{h}$. Let \tilde{F} denote the map F written in the coordinates (τ, ϕ) , which is C^∞ -smooth and takes the form $(\tau, \phi) \mapsto (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi))$ (Theorem 2.1). There exists a \tilde{F} -invariant function $\tilde{\phi} = \phi + \text{flat}(\phi)$ (Theorem 1.18). The function $\tilde{h} := \tilde{\phi}^2$ is F -invariant, C^∞ -smooth, and $\tilde{h} = h + \text{flat}(h)$; hence $\frac{\partial \tilde{h}}{\partial h} \neq 0$ on J and on some domain adjacent to J . The existence in Theorem 1.15 is proved.

Let us now prove the addendum to Theorem 1.15. Let us fix a function \tilde{h} constructed above. Let θ denote the time function of the Hamiltonian vector field with the Hamiltonian function \tilde{h} , normalized to vanishes on the vertical axis $\{\tau = 0\}$. The coordinates (θ, \tilde{h}) are symplectic, and in these coordinates $F(\theta, \tilde{h}) = (\theta + \xi(\tilde{h}), \tilde{h})$ for some function $\xi(\tilde{h}) = \sqrt{\tilde{h}}\psi(\tilde{h})$ in one variable, $\psi(0) > 0$, as in Subsection 2.1, Claim 1. Afterwards modifying the functions \tilde{h} and θ , as at the end of Subsection 2.1, we get new coordinates (τ, \tilde{h}) (with new τ) in which F takes the form (1.11). The addendum is proved.

2.6 Proof of Proposition 1.19 and non-uniqueness in Theorems 1.18, 1.15

Let us prove the statement of Proposition 1.19 for a map \tilde{F} of type (1.12). We prove it for line fields: for other objects the proof is analogous. Without loss of generality we can and will consider that asymptotics (1.12) is uniform in $\tau \in J$ (replacing J by a relatively compact subinterval $J' \Subset J$), as in Convention 2.5. Let G_1 and G_2 be two line fields on W with distinct germs at J . This means that there exists a sequence of points $x(k) = (\tau(k), \phi(k))$ with $\phi(k) \rightarrow 0$ and $\tau(k)$ lying in a compact subset in J such that the lines $G_1(x), G_2(x) \subset T_x \mathbb{R}^2$ are distinct. Taking a subsequence, we can and will consider that $x(k) \rightarrow x = (\tau_0, 0)$, as $k \rightarrow \infty$. The two-sided orbit of a point $x(k)$ with big k consists of points whose ϕ -coordinates are $\phi(k)(1 + o(1))$ and whose τ -coordinates form an asymptotic arithmetic progression with step $\phi(k)(1 + o(1))$, the $o(1)$ are uniform; if k is so big that $|o(1)| < \frac{1}{2}$, then the orbit forms a $2\phi(k)$ -net on the $2\phi(k)$ -neighborgood of the interval J . See Proposition 2.6. At each point of the orbit the lines of the fields G_1 and G_2 are distinct, since this holds at $x(k)$ and by \tilde{F} -invariance. Therefore, passing to limit, as $k \rightarrow \infty$, we get that for every point $z \in J$ there exist points z' arbitrarily close to z with $G_1(z') \neq G_2(z')$. Hence, the germs at z of the line fields G_1 and G_2 are distinct. The first statement of Proposition

1.19, for a map \tilde{F} of type (1.12), is proved. Its second statement, for a strongly billiard-like map $F : V \cup J \rightarrow F(V \cup J) \subset \mathbb{R}^2$ follows from its first statement and the fact that each C^∞ -lifted strongly billiard-like map is conjugated to a map \tilde{F} of type (1.12) by a homeomorphism that is smooth on the complement to the boundary interval J . The latter homeomorphism is the composition of a diffeomorphism and the map $(\tau, h) \mapsto (\tau, \phi)$, $\phi = \sqrt{h}$, see the discussion in the previous subsection. Proposition 1.19 is proved.

Non-uniqueness in Theorem 1.18 (its addendum) follows from non-uniqueness of germ at $(0, 0)$ of the foliation $\tilde{\phi} = \text{const}$ on $S_{\chi, \eta} \supset \hat{\Delta}$ (Proposition 2.3), Lemma 2.8 and Proposition 1.19.

Let us prove non-uniqueness statement of Theorem 1.15. Given a C^∞ -lifted strongly billiard map F written in the above-mentioned coordinates (τ, h) , the corresponding lifting \tilde{F} in coordinates (τ, ϕ) has a continuum of foliations $\tilde{\phi} = \text{const}$ by level curves of a \tilde{F} -invariant function $\tilde{\phi} = \phi + \text{flat}(\phi)$; foliations with pairwise distinct germs at each point in J (the addendum to Theorem 1.18). Each of them projects via the map $(\tau, \phi) \mapsto (\tau, h)$, $h = \phi^2$, to a foliation by level curves of an F -invariant C^∞ -smooth function $\tilde{h} = (\tilde{\phi})^{\frac{1}{2}} = h + \text{flat}(h)$. The F -invariant foliations thus obtained have pairwise distinct germs at each point in J , by construction. This proves non-uniqueness in Theorem 1.15.

2.7 Proof of Theorem 1.2, Theorem 1.4 and its addendum

Proof of Theorem 1.2. The billiard map acting on oriented lines by reflection from the curve γ is a C^∞ -lifted strongly billiard-like map (Example 1.13 and Proposition 1.10). The corresponding interval J is identified with the family of oriented lines tangent to γ and defining the same orientation of the curve γ , as its parametrization. Each invariant curve of a strongly billiard-like map that is close to J corresponds to a caustic of the billiard in γ . This together with Theorem 1.15 implies the statements of Theorem 1.2. \square

The proof of Theorem 1.4 and its addendum repeats the above proof of Theorem 1.2 with obvious changes.

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