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A NONABELIAN FOURIER TRANSFORM FOR TEMPERED UNIPOTENT REPRESENTATIONS

ANNE-MARIE AUBERT, DAN CIUBOTARU, AND BETH ROMANO

Abstract. We define an involution on the space of compact tempered unipotent representations of inner twists of a split simple p-adic group G and investigate its behaviour with respect to restrictions to reductive quotients of maximal compact open subgroups. In particular, we formulate a precise conjecture about the relation with a version of Lusztig's nonabelian Fourier transform on the space of unipotent representations of the (possibly disconnected) reductive quotients of maximal compact subgroups. We give evidence of the conjecture, including proofs for SL_n and PGL_n.

Contents

1.	Introduction	1
2.	Recollection on inner twists	4
3.	Generalized Springer correspondence for disconnected groups	6
4.	The Langlands parametrization	8
5.	Lusztig's nonabelian Fourier transform for finite groups	12
6.	Maximal compact subgroups	22
7.	Elliptic and compact pairs	24
8.	The dual nonabelian Fourier transform	28
9.	Elliptic unipotent representations	33
10.	$\mathsf{Sp}_4(F)$	41
11.	$\mathsf{SL}_n(F)$	43
12.	$\mathsf{PGL}_n(F)$	49
References		51

1. Introduction

The local Langlands correspondence predicts that the irreducible smooth representations of a reductive p-adic group G should be controlled by the geometry of the Langlands dual group $G^\vee$. Parahoric restriction allows us to pass from depth-zero representations of G to representations of certain finite reductive groups, and the local Langlands correspondence should reflect the rich structure in the representation theory of these finite groups [Lu1]. To understand the interplay between the representation theory of p-adic and finite groups, it is natural to start with the category of unipotent representations (or representations with unipotent reduction) of G defined by Lusztig in [Lu3]. By definition, unipotent representations of p-adic groups yield unipotent representations (in the sense of [DL, §7.8], [Lu1]) of related finite groups. In this paper, we define a nonabelian Fourier transform for inner forms of disconnected finite reductive groups, and for G simple and split, we formulate a conjecture relating this and the elliptic Fourier transform for the pure inner twists of G as

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defined in [Ci]. This work is partly motivated by the papers of Lusztig which proposed a theory of almost characters for p-adic groups [Lu6, Lu7] and of Mœglin and Waldspurger on the elliptic representations of the special orthogonal groups [MW, Wa2].

To describe the context of the conjecture, suppose G is an absolutely simple, split connected reductive group over a non-Archimedean local field F with finite residual field. The philosophy of the Langlands correspondence says that, in order to have a good geometric description, one should consider not just the representations of G, but rather the representations of all the pure inner twists $G' \in \mathrm{InnT}^p(G)$ of G, in the sense of Vogan [Vo]. This philosophy leads us to look at all inner forms of disconnected finite reductive groups: we shall see that these inner forms arise in parallel to the pure inner twists of G.

More specifically, for a finite connected reductive group $\mathbb{G}$, Lusztig [Lu1] defined a nonabelian Fourier transform $\mathrm{FT}_{\mathbb{G}}$ on the space of unipotent representations of $\mathbb{G}$, and previous work [CO2], [Ci] has shown that one can define an involution $\mathrm{FT}^{\vee}_{\mathrm{ell}}$ on the elliptic unipotent representation space for G in a way that lifts $\mathrm{FT}_{\mathbb{G}}$ in certain cases for reductive quotients $\mathbb{G}$ of maximal parahoric subgroups in G. Yet $\mathrm{FT}^{\vee}_{\mathrm{ell}}$ does not restrict to $\mathrm{FT}_{\mathbb{G}}$ under parahoric restriction for reductive quotients $\mathbb{G}$ of arbitrary parahoric subgroups of G. To adjust for this complication, we first find that we must look not at parahoric subgroups, but at maximal compact open subgroups of G, whose reductive quotients might not be connected. We extend Lusztig's definition to the case when $\mathbb{G}$ is disconnected by considering all inner forms of $\mathbb{G}$: we define an involution that mixes spaces of unipotent representations for these inner forms.

If K is a maximal compact open subgroup of G with reductive quotient $\overline{K}$, then all the inner forms of $\overline{K}$ appear as reductive quotients for maximal compact subgroups of pure inner twists of G. When we consider all maximal compact subgroups in all $G' \in \mathrm{InnT}^p(G)$, the elliptic Fourier transform has the potential to be compatible with the Fourier transform we define for disconnected finite reductive groups. We give a precise conjecture describing this compatibility. Our work generalizes that of [MW], [Wa2] who formulate and prove the conjecture in the case when $G = \mathsf{SO}_{2n+1}(F)$. Just as in [MW], we expect that the elliptic Fourier transform will prove useful in investigating the stability of tempered unipotent L-packets.

1.1. **Main results.** We now describe our work in more detail. As above, let us assume that G is a simple, split group over F. Then the Langlands correspondence, see Section 4, says that the L-packets of irreducible tempered unipotent representations of the groups $G' \in \mathrm{InnT}^p(G)$ are in one-to-one correspondence with $G^\vee$-conjugacy classes of elements $x = su \in G^\vee$ (Jordan decomposition) such that s is compact. The elements in the L-packet are parametrized by irreducible representations ϕ of the group of components $A_{G^\vee}(x)$ of the centralizer of x in $G^\vee$. Hence an L-packet is a collection $\{\pi(su, \phi) \mid \phi \in \widehat{A_{G^\vee}(su)}\}$. Let Γ_u denote the reductive part of the centralizer of u in $G^\vee$. In [Wa2, Ci], one considered the set of pairs $(s, h) \in \Gamma_u^2$ of commuting semisimple elements $\mathcal{Y}(\Gamma_u)$ and the subset of elliptic pairs $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, see Section 7.3. These will play a role below in the Langlands parametrization.

Each group $G' \in \mathrm{InnT}^p(G)$ has a finite collection of conjugacy classes of maximal compact open subgroups $\max(G')$. These are classified in terms the theory of [BT, IM], see Section 6. A compact group $K' \in \max(G')$ has a finite quotient $\overline{K}'$ which is the group of k-points of a (possibly disconnected) reductive group over a finite field k. Write $R_{\mathrm{un}}(\overline{K}')$ for the $\mathbb{C}$-vector space spanned by the irreducible unipotent representations of $\overline{K}'$. For connected finite groups, Lusztig [Lu1] defined the nonabelian Fourier transform, which is the change of bases matrix between the basis of irreducible unipotent characters and the basis of unipotent almost-characters. This is recalled in Section 5. We need to define an extension of this map to disconnected finite groups as in [Lu5]. To fit with our picture, we define a nonabelian Fourier transform for the representations of the inner forms of the finite

(possibly disconnected) reductive group $\overline{K}$, where $K \in \max(G)$. See Section 5.4. The point is that this transform gives an involution

$$\mathrm{FT}_{\mathrm{cpt,un}} \colon \mathcal{C}(G)_{\mathrm{cpt,un}} \to \mathcal{C}(G)_{\mathrm{cpt,un}}, \tag{1.1}$$

on the space

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{G' \in \mathrm{InnT}^p(G)} \bigoplus_{K' \in \max(G')} R_{\mathrm{un}}(\overline{K}'),$$

which we can think of as the sum over $K \in \max(G)$ of the unipotent representation spaces of the inner forms of $\overline{K}$. See (6.2) and Definition 6.1. It is important to notice that, in general, $\mathrm{FT}_{\mathrm{cpt,un}}$ mixes the inner forms of a given $\overline{K}$.

Since parabolic induction of characters is generally well understood, of particular interest is the space of elliptic (unipotent) tempered representations for all pure inner twists

$$\mathcal{R}^p_{\mathrm{un,ell}}(G) = \bigoplus_{G' \in \mathrm{InnT}^p(G)} \overline{R}_{\mathrm{un}}(G'),$$

see Section 8.1. The idea of elliptic tempered representations goes back to Arthur, and our approach have been influenced by the work of Reeder [Re3]. Generalizing [Re3], we prove in Theorem 9.1, under certain assumptions on G, that the local Langlands correspondence induces a isometric isomorphism

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}} : \bigoplus_{u \in G^\vee} \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u} \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G), \ (s,h) \mapsto \Pi(u,s,h), \tag{1.2}$$

where the left hand side has a natural elliptic inner product while the right hand side is endowed with the Euler-Poincaré product. The element u ranges over representatives of unipotent conjugacy classes in $G^\vee$. Since the left hand side has an obvious involution given by the flip $(s,h) \to (h,s)$, this defines an involution, the dual elliptic nonabelian transform

$$\mathrm{FT}^\vee_{\mathrm{ell}} \colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{R}^p_{\mathrm{un,ell}}(G). \tag{1.3}$$

We notice that $\mathrm{FT}^\vee_{\mathrm{ell}}$ mixes representations of the inner twists of G. We expect that there is a commutative diagram, Conjecture 8.5, up to certain roots for unity (see Remark 8.6):

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} & & \downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} \\ \mathcal{C}(G)_{\mathrm{cpt,un}} & \xrightarrow[\mathrm{FT}_{\mathrm{cpt,un}}]{} & \mathcal{C}(G)_{\mathrm{cpt,un}} \end{array} \tag{1.4}$$

where the vertical arrows are defined by taking invariants by the pro-unipotent radicals of maximal compact subgroups. This is a generalization of [Ci, Conjecture 1.3] with an important difference: we remark that the role of maximal compact subgroups (rather than maximal parahoric subgroups) and hence of a Fourier transform for inner forms of disconnected finite reductive groups in the conjecture is essential for treating all pure inner twists. We verify the conjecture when $\mathbf{G} = \mathsf{Sp}_4$ (Section 10), SL_n (Section 11), and PGL_n (Section 12). The results of Waldspurger [Wa2] show that this conjecture holds when $\mathbf{G} = \mathsf{SO}_{2n+1}$.

To extend beyond the case of elliptic representations, the right object to consider from the perspective of restrictions to compact subgroups, is the larger compact (unipotent) tempered representation space $\mathcal{R}^p_{\mathrm{un,cpt}}(G)$, see Section 8.4. This space was studied from the perspective of the trace Paley–Wiener Theorem in [CH2] (under the name "the rigid quotient"). To parametrize it, we introduce a space of "compact pairs" $\mathbb{C}[\mathcal{Y}(\Gamma_u)/_\sim]^{\Gamma_u}$, see Section 7.3 and Proposition 7.7. We expect that the same conjecture (1.4) holds with $\mathcal{R}^p_{\mathrm{un,cpt}}(G)$ replacing $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ above. In the case of $\mathbb{C}[\mathcal{Y}(\Gamma_u)/_\sim]^{\Gamma_u}$, the nonabelian Fourier transform should be closely related to that defined by Lusztig in [Lu7].

1.2. **Structure of the paper.** In Sections 2, 3, and 4, we review relevant background about inner twists of p-adic groups, the generalized Springer correspondence, and the local Langlands correspondence. In Section 5, we recall Lusztig's parametrization of unipotent representations of a connected reductive group over a finite field and the definition of the nonabelian Fourier transform on the space spaced by these representations. We then extend Lusztig's parametrization: for the (possibly disconnected) groups $\overline{K}$ that arise as reductive quotients of subgroups $K \in \max(G)$ as defined above, we parametrize the union over all inner forms $\overline{K}'$ of $\overline{K}$ of the set of unipotent representations of $\overline{K}'$, and we then define a nonabelian Fourier transform on the space spanned by these representations (see Section 5.4).

In Section 6, we return to the setting of p-adic groups. We review the parametrization of maximal compact open subgroups of $G' \in \mathrm{InnT}^p(G)$, under the assumption G is F-split. We define the space $\mathcal{C}(G)_{\mathrm{cpt,un}}$ in terms of these subgroups, and we use the Fourier transform of Section 5.4 to define an involution $\mathrm{FT}_{\mathrm{cpt,un}}$ on $\mathcal{C}(G)_{\mathrm{cpt,un}}$. In Section 7, we review the definitions of $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ and $\mathcal{Y}(\Gamma)$ for a complex reductive group Γ, and we define an equivalence relation on $\mathcal{Y}(\Gamma)$. We also review the definition of the elliptic pairing on the Grothendieck group of a finite group.

Section 8 contains the conjectures outlined above. We first review the Euler–Poincaré pairing and state Conjecture 8.1, which predicts that the local Langlands correspondence induces an isometric isomorphism at the level of elliptic spaces. We then define a restriction map $\mathrm{res}_{\mathrm{cpt,un}} \colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{C}(G)_{\mathrm{cpt,un}}$ and state Conjecture 8.5, which predicts that the elliptic nonabelian Fourier transform $\mathrm{FT}^\vee_{\mathrm{ell}}$ is compatible with $\mathrm{FT}_{\mathrm{cpt,un}}$ under $\mathrm{res}_{\mathrm{cpt,un}}$. We give evidence for this conjecture in Proposition 8.7, which considers linear combinations of twists of Steinberg representations.

We then extend this conjecture, considering not just $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ but also the space $\mathcal{R}^p_{\mathrm{un,cpt}}(G)$, which has a basis described in terms of elements of $\mathcal{Y}(\Gamma_u)$ under the equivalence relation defined in Section 7. We define a Fourier transform $\mathrm{FT}^\vee_{\mathrm{cpt,un}}$ on this space and conjecture that $\mathrm{FT}^\vee_{\mathrm{cpt,un}}$ is compatible with $\mathrm{FT}_{\mathrm{cpt,un}}$ under $\mathrm{res}_{\mathrm{cpt,un}}$.

In Section 9, we prove Conjecture 8.1 in the case when G is simple, split, and adjoint. In Section 9.4, we indicate how the proof can be extended to the non-adjoint case. In the final three sections, we verify the conjectures for explicit examples: in Section 10, we consider the group $\mathsf{Sp}_4(F)$; in Section 11, we consider $\mathsf{SL}_n(F)$, and in Section 12, we consider $\mathsf{PGL}_n(F)$.

1.3. **Notation and conventions.** Given a complex Lie group $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}$ for the center of $\mathcal{G}$. Given $x \in \mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(x)$ for the centralizer of x in $\mathcal{G}$. Similarly, if H is a subgroup of $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(H)$ for the centralizer of H in $\mathcal{G}$, and if φ is a homomorphism with image in $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(\varphi)$ for $\mathrm{Z}_{\mathcal{G}}(\mathrm{im}\ \varphi)$. We write $\mathcal{G}^\circ$ for the identity component of $\mathcal{G}$. If $x \in \mathcal{G}$, we write $A_{\mathcal{G}}(x) = \mathrm{Z}_{\mathcal{G}}(x)/\mathrm{Z}_{\mathcal{G}}(x)^\circ$ for the component group of $\mathrm{Z}_{\mathcal{G}}(x)$. If $u \in \mathcal{G}$ is unipotent, we write Γ_u for the reductive part of $\mathrm{Z}_{\mathcal{G}}(u)$. Given a torus $\mathcal{T}$, we write $X^*(\mathcal{T})$ for the character group of $\mathcal{T}$.

Given a finite group A, we write $\widehat{A}$ for the set of (isomorphism classes of) irreducible representations of A, and we write $R(A)$ for the $\mathbb{C}$-vector space with basis $\widehat{A}$. Given a finite set S, we write $\mathbb{C}[S]$ for the $\mathbb{C}$-vector space of functions $S \to \mathbb{C}$.

2. Recollection on inner twists

2.1. **Inner twists.** Let F be a non-Archimedean local field with finite residual field $k_F = \mathbb{F}_q$. We denote by $\mathfrak{o}_F$ the ring of integers of F. Let F_{s} be a fixed separable closure of F, and let Γ_F denote the Galois group of F_{s}/F. Let $F_{\mathrm{un}} \subset F_{\mathrm{s}}$ be the maximal unramified extension of F. Let Frob be the geometric Frobenius element of $\mathrm{Gal}(F_{\mathrm{un}}/F) \simeq \widehat{\mathbb{Z}}$, i.e., the topological generator which induces the inverse of the automorphism $x \mapsto x^q$ of k_F. We denote by $\mathrm{Fr}_{\mathbf{G}}$ the action of Frob on a connected reductive F-group $\mathbf{G}$. We now review definitions related

to inner twists and pure inner twists of a p-adic group. For details see, e.g., [Vo, Section 2], [Kal1, Section 2], [ABPS2, Section 1.3]. (Note that [Vo] uses the term "pure rational form" for what we call a pure inner twist.)

Let $G = \mathbf{G}(F)$. Write $\mathrm{Inn}(\mathbf{G})$ for the group of inner automorphisms of $\mathbf{G}$. Recall that an isomorphism $\alpha\colon \mathbf{H} \to \mathbf{G}$ of algebraic groups groups determines a 1-cocycle

$$\gamma_\alpha\colon \begin{array}{ccc} \Gamma_F & \to & \mathrm{Aut}(\mathbf{G}) \\ \sigma & \mapsto & \alpha\sigma\alpha^{-1}\sigma^{-1}. \end{array} \tag{2.1}$$

An *inner twist* of G consists of a pair (H, α), where $H = \mathbf{H}(F)$ for some connected reductive F-group $\mathbf{H}$, and $\alpha\colon \mathbf{H} \xrightarrow{\sim} \mathbf{G}$ is an isomorphism of algebraic groups such that $\mathrm{im}(\gamma_\alpha) \subset \mathrm{Inn}(\mathbf{G})$. Two inner twists $(H, \alpha), (H', \alpha')$ of G are *equivalent* if there exists $f \in \mathrm{Inn}(\mathbf{G})$ such that

$$\gamma_\alpha(\sigma) = f^{-1}\gamma_{\alpha'}(\sigma)\, \sigma f \sigma^{-1} \quad \forall \sigma \in \mathrm{Gal}(F_{\mathrm{s}}/F). \tag{2.2}$$

Denote the set of equivalence classes of inner twists of G by $\mathrm{InnT}(G)$.

An inner twist of G is the same thing as an inner twist of the unique quasi-split inner form $G^* = \mathbf{G}^*(F)$ of G. Thus the equivalence classes of inner twists of G are parametrized by the Galois cohomology group $H^1(F, \mathrm{Inn}(\mathbf{G}^*))$:

$$\mathrm{InnT}(G) \longleftrightarrow H^1(F, \mathrm{Inn}(\mathbf{G}^*)).$$

Example 2.1. For $G = \mathsf{SL}_n(F)$, there is a one-to-one correspondence

$$\mathrm{InnT}(\mathsf{SL}_n(F)) \longleftrightarrow \mathbb{Z}/n\mathbb{Z}. \tag{2.3}$$

This is given as follows. Let r be an integer mod n and let $m = \gcd(r, n)$. Then $n = dm$ and r/m is coprime to d. Therefore, there exists a division algebra $D_{d,r/m}$, central over F and of dimension $\dim_F D_{d,r/m} = d^2$. The corresponding inner twist is $\mathsf{SL}_m(D_{d,r/m})$.

The *pure inner twists* of G correspond by definition to cocycles $z \in Z^1(F, G)$ [Vo]. Such a cocycle is determined by the image $u := z(\mathrm{Frob}) \in G$. The corresponding inner twist of $\mathbf{G}$ is defined by the functorial image $z_{\mathrm{ad}} \in Z^1(F, \mathrm{Inn}(\mathbf{G}^*))$ of z. This pure inner twist is defined by the twisted Frobenius action Fr_u on $\mathbf{G}$ given by $\mathrm{Fr}_u = \mathrm{Ad}(u) \circ \mathrm{Fr}_{\mathbf{G}}$.

In cohomological terms, the short exact sequence

$$1 \longrightarrow \mathrm{Z}_{\mathbf{G}^*} \longrightarrow \mathbf{G}^* \longrightarrow \mathrm{Inn}(\mathbf{G}^*) \longrightarrow 1$$

induces a map in cohomology $\mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \to \mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$. An inner twist of $\mathbf{G}^*$ is a pure inner twist if and only if the corresponding element of $\mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$ is trivial [Vo, Lemma 2.10]. Denote by $\mathrm{InnT}^p(G^*)$ the set of equivalence classes of pure inner twists of $\mathbf{G}^*$. We have [Vo, Proposition 2.7]

$$\mathrm{InnT}^p(G^*) \longleftrightarrow \mathrm{H}^1(F, \mathbf{G}^*). \tag{2.4}$$

Example 2.2. If $\mathbf{G}$ is semisimple adjoint, every inner twist is pure, $\mathrm{InnT}^p(G) = \mathrm{InnT}(G)$. If $\mathbf{G}$ is semisimple and simply connected, $\mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \cong \mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$ and therefore, there is only one class of pure inner twists, the quasisplit form, $\mathrm{Inn}^p(G) = G^*$. When $G = \mathsf{SL}_n(F)$, the only pure inner twist is G itself, see [Vo, Example 2.12].

2.2. **The L-group.** Let $G^\vee$ denote the $\mathbb{C}$-points of the dual group of $\mathbf{G}$. It is endowed with an action of Γ_F. Let W_F be the Weil group of F (relative to F_{s}/F) and let ${}^LG := G^\vee \rtimes W_F$ denote the L-group of G.

Kottwitz proved in [Ko, Proposition 6.4] that there exists a natural isomorphism

$$\kappa_G\colon \mathrm{H}^1(F, \mathbf{G}) \xrightarrow{\sim} \mathrm{Irr}\Big(\pi_0\big(\mathrm{Z}_{G^\vee}^{W_F}\big)\Big). \tag{2.5}$$

Let $G^\vee_{\mathrm{sc}}$ denote the simply connected cover of the derived group $G^\vee_{\mathrm{der}}$ of $G^\vee$.We have $G^\vee_{\mathrm{sc}} = (G_{\mathrm{ad}})^\vee$, and

$$\kappa_{G^*_{\mathrm{ad}}}\colon \mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \xrightarrow{\sim} \mathrm{Irr}\big(\mathrm{Z}_{G^\vee_{\mathrm{sc}}}^{W_F}\big). \tag{2.6}$$

All the inner twists of a given group G share the same L-group, because the action of W_F on $G^\vee$ is only uniquely defined up to inner automorphisms. This also works the other way around: from the Langlands dual group ${}^LG$ one can recover the inner-form class of G.

Example 2.3. If $G = \mathsf{Sp}_{2n}(F)$, then we have $G^\vee = \mathsf{SO}_{2n+1}(\mathbb{C})$ and $G^\vee_{\mathrm{sc}} = \mathsf{Spin}_{2n+1}(\mathbb{C})$. It gives $\mathrm{Z}_{G^\vee_{\mathrm{sc}}} \simeq \mathbb{Z}/2\mathbb{Z}$. An inner twist of G is determined by its Tits index [Ti]. The group $G^* = G$ is split and its nontrivial inner twist is the group $\mathsf{SU}_{2n+1}(F) := \mathsf{SU}(n, h_r)$, where h_r is a non-degenerate Hermitian form of index $r = \lfloor n/2 \rfloor$ over the quaternion algebra Q over F (see for instance [Ar2, § 9]).

We will consider G as an inner twist of G^*, so endowed with an isomorphism $\mathbf{G} \to \mathbf{G}^*$. Via (2.6), G is labelled by a character ζ_G of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}^{W_F}$. We choose an extension ζ of ζ_G to $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}$.

3. Generalized Springer correspondence for disconnected groups

Let $\mathcal{G}$ be a possibly disconnected complex Lie group. We denote by $\mathcal{G}^\circ$ the neutral component of $\mathcal{G}$. Let u be a unipotent element in $\mathcal{G}^\circ$, and let $A_{\mathcal{G}^\circ}(u)$ denote the group of components of $\mathrm{Z}_{\mathcal{G}^\circ}(u)$.

Let ϕ° be an irreducible representation of $A_{\mathcal{G}^\circ}(u)$. The pair (u, ϕ°) is called *cuspidal* if it determines a $\mathcal{G}^\circ$-equivariant cuspidal local system on the $\mathcal{G}^\circ$-conjugacy class of u as defined in [Lu2]. In particular, if (u, ϕ°) is cuspidal, then u is a distinguished unipotent element in $\mathcal{G}^\circ$ (that is, u does not meet the unipotent variety of any proper Levi subgroup of $\mathcal{G}^\circ$), [Lu2, Proposition 2.8]. However, in general not every distinguished unipotent element supports a cuspidal representation.

Example 3.1. For $\mathcal{G} := \mathsf{SL}_n(\mathbb{C})$, the unipotent classes in $\mathcal{G}$ are in bijection with the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n: the corresponding $\mathcal{G}$-conjugacy class $\mathcal{O}_\lambda$ consists of unipotent matrices with Jordan blocks of sizes λ_1, λ_2, …,λ_r. We identify the center $\mathrm{Z}_{\mathcal{G}}$ with the group μ_n of complex n-roots of unity. For $u \in \mathcal{O}_\lambda$, the natural homomorphism $\mathrm{Z}_{\mathcal{G}} \to A_{\mathcal{G}}(u)$ is surjective with kernel $\mu_{n/\gcd(\lambda)}$, where $\gcd(\lambda) := \gcd(\lambda_1, \lambda_2, \dots, \lambda_r)$. Hence the irreducible $\mathcal{G}$-equivariant local systems on $\mathcal{O}_\lambda$ all have rank one, and they are distinguished by their central characters, which range over those $\chi \in \widehat{\mu_n}$ such $\gcd(\lambda)$ is a multiple of the order of χ. We denote these local systems by $\mathcal{E}_{\lambda,\chi}$. The unique distinguished unipotent class in $\mathcal{G}$ is the regular unipotent class $\mathcal{O}_{(n)}$, consisting of unipotent matrices with a single Jordan block. The cuspidal irreducible $\mathcal{G}$-equivariant local systems are supported on $\mathcal{O}_{(n)}$ and are of the form $\mathcal{E}_{(n),\chi}$, with $\chi \in \widehat{\mu_n}$ of order n (see [Lu2, (10.3.2)]).

The group $A_{\mathcal{G}^\circ}(u)$ may be viewed as a subgroup of the group $A_u := A_{\mathcal{G}}(u)$ of components of $\mathrm{Z}_{\mathcal{G}}(u)$. Let ϕ be an irreducible representation of $A_{\mathcal{G}}(u)$. We say that (u, ϕ) is a *cuspidal pair* if the restriction of ϕ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ϕ° such that one (or equivalently any) of the pairs (u, ϕ°) is cuspidal. Let

$$\mathbf{I}^{\mathcal{G}} := \{(U, \mathcal{E}) \mid U \text{ unipotent conjugacy class in } \mathcal{G},\ \mathcal{E} \text{ irred. } G\text{-eqvt. local system on } U\}.$$

This set can be identified with the set of $\mathcal{G}$-orbits of pairs (u, ϕ), where $u \in \mathcal{G}$ is unipotent and $\phi \in \widehat{A}_u$. If (ϕ, V_ϕ) is an irreducible A_u-representation, we can first regard it as an irreducible $\mathrm{Z}_{\mathcal{G}}(u)$-representation, and then the corresponding local system is $\mathcal{E} = (\mathcal{G} \times_{\mathrm{Z}_{\mathcal{G}}(u)} V_\phi \to \mathcal{G}/\mathrm{Z}_{\mathcal{G}}(u) \cong U)$. We denote by $\mathbf{I}^{\mathcal{G}}_{\mathrm{c}}$ the subset of $\mathbf{I}$ of cuspidal pairs. We write $\mathbf{I} := \mathbf{I}^{\mathcal{G}^\circ}$ and $\mathbf{I}_{\mathrm{c}} := \mathbf{I}^{\mathcal{G}^\circ}_{\mathrm{c}}$.

Let $\mathbf{J}^{\mathcal{G}}$ denote the set of $\mathcal{G}$-orbits of triples $j = (\mathcal{M}, U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}})$ such that $\mathcal{M}^\circ$ is a Levi subgroup of $\mathcal{G}^\circ$,

$$\mathcal{M} := \mathrm{Z}_{\mathcal{G}}(\mathrm{Z}^\circ_{\mathcal{M}^\circ}), \tag{3.1}$$

and $(U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}}) \in \mathbf{I}_{\mathrm{c}}(\mathcal{M}^\circ)$. We observe that $\mathcal{M}$ has identity component $\mathcal{M}^\circ$ and that $\mathrm{Z}^\circ_{\mathcal{M}} = \mathrm{Z}^\circ_{\mathcal{M}^\circ}$. We set $\mathbf{J} := \mathbf{J}^{\mathcal{G}^\circ}$. We notice that $\mathcal{M} = \mathcal{M}^\circ$ whenever $\mathcal{G} = \mathcal{G}^\circ$.

Let $\mathrm{Z}^\circ_{\mathcal{M}^\circ,\mathrm{reg}} = \{z \in Z^\circ_{\mathcal{M}^\circ} \mid \mathrm{Z}_{\mathcal{G}}(z) = \mathcal{M}^\circ\}$ and $Y_j(\mathcal{G}) = \bigcup_{x \in \mathcal{G}} x(\mathrm{Z}^\circ_{\mathcal{M}^\circ,\mathrm{reg}} U_{\mathrm{c}})x^{-1}$. Let $\overline{Y}_j(\mathcal{G})$ be the closure of $Y_j(\mathcal{G})$ in $\mathcal{G}$. We set $Y_j = Y_j(\mathcal{G}^\circ)$ and $\overline{Y}_j = \overline{Y}_j(\mathcal{G}^\circ)$. For example,

if $j_0 = (T, 1, \mathrm{triv})$ is the trivial cuspidal pair on the maximal torus T in $\mathcal{G}^\circ$, then Y_{j_0} is the variety of regular semisimple elements in $\mathcal{G}^\circ$, hence $\overline{Y}_{j_0} = \mathcal{G}^\circ$.

Set $W_j^\circ := \mathrm{N}_{\mathcal{G}^\circ}(\mathcal{M}^\circ)/\mathcal{M}^\circ$. This is a Coxeter group due to the particular nature of the Levi subgroups in $\mathcal{G}^\circ$ that support cuspidal local systems (see [Lu2, Theorem 9.2]).

One constructs a $\mathcal{G}^\circ$-equivariant semisimple perverse sheaf K_j supported on $\overline{Y}_j$ that has a W_j°-action and a decomposition [Lu2, Theorem 6.5]

$$K_j = \bigoplus_{\rho^\circ \in \widehat{W_j^\circ}} V_\rho \otimes A_{j,\rho^\circ},$$

where $(\rho^\circ, V_{\rho^\circ})$ ranges over the (equivalence classes of) irreducible W_j-representations and A_{j,ρ° is an irreducible $\mathcal{G}^\circ$-equivariant perverse sheaf. The perverse sheaf A_{j,ρ° has the property that there exists a (unique) pair $(U, \mathcal{E}^\circ) \in \mathbf{I}$ such that its restriction to the variety $\mathcal{G}^\circ_{\mathrm{un}}$ of unipotent elements in $\mathcal{G}^\circ$ is:

$$(A_{j,\rho^\circ})|_{\mathcal{G}^\circ_{\mathrm{un}}}[-\dim(\mathrm{Z}^\circ_{\mathcal{M}^\circ})] \cong \mathrm{IC}(\overline{U}, \mathcal{E}^\circ)[\dim(U)]. \tag{3.2}$$

In particular, the hypercohomology of A_{j°,ρ° on U is concentrated in one degree, namely

$$\mathcal{H}^{a_U}(A_{j,\rho^\circ})|_U \cong \mathcal{E}^\circ, \text{ where } a_U = -\dim(U) - \dim(\mathrm{Z}^\circ_{\mathcal{M}^\circ}).$$

If we set $\widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}^{\mathcal{G}^\circ} := \{(j, \rho^\circ) \,:\, j \in \mathbf{J}^{\mathcal{G}^\circ},\ \rho^\circ \in \widehat{W_j^\circ}\}$, the generalized Springer correspondence for $\mathcal{G}^\circ$ is the bijection

$$\nu^\circ \colon \mathbf{I}^{\mathcal{G}^\circ} \to \widetilde{\mathbf{J}}^{\mathcal{G}^\circ}, \quad (U, \mathcal{E}) \mapsto (j, \rho^\circ), \tag{3.3}$$

where the relation between (j, ρ°) and $(U, \mathcal{E})$ is given by (3.2). Let $\nu_{\mathrm{c}}^\circ \colon \mathbf{I} \to \mathbf{J}$ denote the composition of ν° with the projection from $\widetilde{\mathbf{J}}$ to $\mathbf{J}$.

We will now explain how, following [AMS1, § 4], one can extend the maps ν° and ν_{c}° to the case of disconnected groups. Let $j = (\mathcal{M}, U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}}) \in J^{\mathcal{G}}$. We set $W_j := \mathrm{N}_{\mathcal{G}}(j)/\mathcal{M}^\circ$. There exists a subgroup $\mathfrak{R}_j$ of W_j such that $W_j = W_j^\circ \rtimes \mathfrak{R}_j$ (see [AMS1, Lemma 4.2]). Suppose that $\sharp_j$ is a 2-cocycle

$$\sharp_j \colon \mathfrak{R}_j \times \mathfrak{R}_j \to \overline{\mathbb{Q}}_\ell^\times.$$

We view $\sharp_j$ as a 2-cocycle on W_j which is trivial on W_j°. Then the $\sharp_j$-twisted group algebra of W_j, denote by $\overline{\mathbb{Q}}_\ell[W_j, \sharp_j]$ is defined to be the $\overline{\mathbb{Q}}_\ell$-vector space $\overline{\mathbb{Q}}_\ell[W_j, \sharp_j]$ with basis $\{f_w \,:\, w \in W_j\}$ and multiplications rules

$$f_w f_{w'} = \sharp_j(w, w') f_{ww'}, \quad w, w' \in W_j.$$

One constructs a $\mathcal{G}$-equivariant semisimple perverse sheaf K_j supported on $\overline{Y}_j$ which has a W_j-action and a decomposition [Lu2, Theorem 6.5]

$$K_j = \bigoplus_{\rho \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_j, \sharp_j])} V_\rho \otimes A_{j,\rho},$$

where (ρ, V_ρ) ranges over the (equivalence classes of) simple modules of $\overline{\mathbb{Q}}_\ell[W_j, \sharp_j])$ and $A_{j,\rho}$ is an irreducible $\mathcal{G}$-equivariant perverse sheaf.

We set

$$\widetilde{\mathbf{J}}^{\mathcal{G}} := \{(j, \rho) \,:\, j \in \mathbf{J}^{\mathcal{G}},\ \rho \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_j, \sharp_j])\}. \tag{3.4}$$

The generalized Springer correspondence for $\mathcal{G}$ is a canonical bijection

$$\nu \colon \mathbf{I}^{\mathcal{G}} \to \widetilde{\mathbf{J}}^{\mathcal{G}}, \quad (U, \mathcal{E}) \mapsto (j, \rho) \tag{3.5}$$

defined in [AMS1, Theorem 5.5].

Definition 3.2. Let $\nu_{\mathrm{c}} = \nu_{\mathrm{c}}^{\mathcal{G}} \colon \mathbf{I}^{\mathcal{G}} \to \mathbf{J}^{\mathcal{G}}$ denote the composition of ν with the projection from $\widetilde{\mathbf{J}}^{\mathcal{G}}$ to $\mathbf{J}^{\mathcal{G}}$.

Suppose the $\mathcal{G}$-class U splits into $\mathcal{G}^\circ$-classes $U_1^\circ, \ldots, U_\ell^\circ$, for some $\ell \geq 1$. If we regard $\mathcal{E}$ as a $\mathcal{G}^\circ$-equivariant local system, then it restricts $\mathcal{E}|_{U_i^\circ} = \bigoplus_{t=1}^{k_i} \mathcal{E}_{i,t}^\circ$, $1 \leq i \leq \ell$, where $\nu^\circ(U_i^\circ, \mathcal{E}_{i,t}^\circ) = (j^\circ, \rho_{i,t}^\circ)$, with $j^\circ = \nu_{\mathrm{c}}^\circ(U_1^\circ, \mathcal{E}_{1,1}^\circ)$ and $\rho|_{W_j^\circ} = \bigoplus_{i,t} \rho_{i,t}^\circ$.

Example 3.3. Let $\mathcal{G} = \mathsf{O}_{2n}(\mathbb{C})$, $\mathcal{G}^\circ = \mathsf{SO}_{2n}(\mathbb{C})$, $\mathcal{G}/\mathcal{G}^\circ \cong \mathbb{Z}/2\mathbb{Z}$. The unipotent classes in $\mathcal{G}$ are parametrized by partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ of $2n$ such that each even part appears with even multiplicity. If U_λ is the corresponding unipotent class, then U_λ is a single $\mathcal{G}^\circ$-class unless the partition λ is "very even"' [SpSt, CM], i.e., all parts λ_i are even, in which case U_λ splits into two $\mathcal{G}^\circ$-classes, U_λ^+ and U_λ^-.

Let $j = j_0$ correspond to the trivial cuspidal local system on the torus of $\mathcal{G}^\circ$. Then $W_{j_0}^\circ = W^\circ \cong W(D_n)$ and $W_j = W \cong W(B_n)$, hence $W/W^\circ \cong \mathcal{G}/\mathcal{G}^\circ = \mathbb{Z}/2\mathbb{Z}$. If λ is not a very even partition, $u \in U_\lambda$, then $A_u/A_{\mathcal{G}^\circ}(u) = \mathbb{Z}/2\mathbb{Z}$; if $(j_0, \rho^\circ) = \nu^\circ(U_\lambda^\circ, \phi^\circ)$, then there are two nonisomorphic ways ϕ, ϕ' in which one can extend ϕ° to A_u, and two nonisomorphic ways ρ, ρ' to extend ρ° to W, and these correspond under the disconnected Springer correspondence.

If, on the other hand, λ is a very even partition, let $u = u^+$ be a representative of U_λ^+ and u^- a representative of U_λ^-, then $A_u = A_{\mathcal{G}^\circ}(u^+) = A_{\mathcal{G}^\circ}(u^-) = \{1\}$. In this case, $\nu^\circ(U^\pm, \mathbf{1}) = \rho^\pm$ ($W(D_n)$-representations), where ρ is parametrized by a bipartition of n of the form $\alpha \times \alpha$ (necessarily n is even). Then $\nu(U, \mathbf{1}) = \rho$ ($W(B_n)$-representation), where $\rho|_{W(D_n)} = \rho^+ \oplus \rho^-$.

4. The Langlands parametrization

We use the notation of Section 2. In addition, we write I_F for the inertia subgroup of W_F, and we set $W_F' := W_F \times \mathsf{SL}_2(\mathbb{C})$. We have natural projections from $p_1 \colon W_F' \twoheadrightarrow W_F$ and $p_2 \colon {}^L G \twoheadrightarrow W_F$.

4.1. **Langlands parameters.** A *Langlands parameter* (or *L-parameter*) for G is a continuous morphism $\varphi \colon W_F' \to {}^L G$ such that $\varphi(w)$ is semisimple for each $w \in W_F$ (that is, $r(\varphi(w))$ is semisimple for every finite-dimensional representation r of $G^\vee$), the restriction of φ to $\mathsf{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups, and the diagram

$$\begin{tikzcd} W_F' \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & {}^L G \arrow[dl, "p_2"] \\ & W_F & \end{tikzcd}$$

commutes. Write $\Phi(G)$ for the set of $G^\vee$-conjugacy classes of Langlands parameters for G.

Let $\mathrm{Z}_{G^\vee}(\varphi)$ denote the centralizer in $G^\vee$ of $\varphi(W_F')$. We have

$$\mathrm{Z}_{G^\vee}(\varphi) \cap \mathrm{Z}_{G^\vee} = \mathrm{Z}_{G^\vee}^{W_F}, \tag{4.1}$$

and hence

$$\mathrm{Z}_{G^\vee}(\varphi)/\mathrm{Z}_{G^\vee}^{W_F} \simeq \mathrm{Z}_{G^\vee}(\varphi)\mathrm{Z}_{G^\vee}/\mathrm{Z}_{G^\vee}.$$

The group $\mathrm{Z}_{G^\vee}(\varphi)\mathrm{Z}_{G^\vee}/\mathrm{Z}_{G^\vee}$ can be considered as a subgroup of $G_{\mathrm{ad}}^\vee$ and we define $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}^1(\varphi)$ to be its inverse image under the canonical projection $p \colon G_{\mathrm{sc}}^\vee \to G_{\mathrm{ad}}^\vee$. The group $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}^1(\varphi)$ coincides with the one introduced by Arthur in [Ar1, p. 209] (denoted there by $S_{\phi,\mathrm{sc}}$). As observed in [Ar1], it is an extension of $\mathrm{Z}_{G^\vee}(\varphi)/\mathrm{Z}_{G^\vee}^{W_F}$ by $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}$. Let A_φ^1 denote the component group of $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}^1(\varphi)$.

Remark 4.1. The group A_φ^1 coincides with the group considered by both Arthur in [Ar1, (3.2)] (denoted there by $\widetilde{\mathcal{S}}_\phi$) and Kaletha in [Kal2, §4.6] in the parametrization of the L-packet of φ.

An *enhancement* of φ is an irreducible representation ϕ of A^1_φ. We denote by $\widehat{A}^1_\varphi$ the set of irreducible characters of A^1_φ. The pairs (φ,ϕ) are called *enhanced L-parameters*. Let $\phi \in \widehat{A}^1_\varphi$. Then ϕ determines a character ζ_ϕ of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}$. An enhanced L-parameter (ϕ,φ) is said to be *G-relevant* if $\zeta_\phi = \zeta$, where ζ is as defined in Section 2.2. The set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of G-relevant enhanced L-parameters is expected to parametrize the admissible dual of G.

The group $H^1(W_F, \mathrm{Z}_{G^\vee})$ acts on $\Phi(G)$ by

$$(z\varphi)(w,x) := z'(w)\,\varphi(w,z) \quad \varphi \in \Phi(G), w \in W_F, x \in \mathsf{SL}_2(\mathbb{C}), \tag{4.2}$$

where $z'\colon W_F \to \mathrm{Z}_{G^\vee}$ represents $z \in H^1(W_F, \mathrm{Z}_{G^\vee})$. This extends to an action of $H^1(W_F, \mathrm{Z}_{G^\vee})$ on $\Phi_{\mathrm{e}}(G)$ that does nothing to the enhancements.

A character of G is called *weakly unramified* if it is trivial on the kernel of the Kottwitz homomorphism. Let $X_{\mathrm{wur}}(G)$ denote the group of weakly unramified characters of G. There is a natural isomorphism

$$X_{\mathrm{wur}}(G) \simeq (\mathrm{Z}^{I_F}_{G^\vee})_{\mathrm{Frob}} \subset H^1(W_F, \mathrm{Z}_{G^\vee}), \tag{4.3}$$

see [Ha, §3.3.1]. Its identity component is the group $X_{\mathrm{un}}(G)$ of unramified characters of G. Via (4.2) and (4.3), the group $X_{\mathrm{wur}}(G)$ acts naturally on $\Phi_{\mathrm{e}}(G)$.

Let $\varphi\colon W_F \times \mathsf{SL}_2(\mathbb{C}) \to {}^LG$ be an L-parameter. We consider the (possibly disconnected) complex reductive group

$$\mathcal{G}_\varphi := \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}), \tag{4.4}$$

defined analogously to $\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi)$. Denote by $\mathcal{G}^\circ_\varphi$ its identity component.

We define elements $u_\varphi, s_\varphi \in G^\vee$ by

$$(u_\varphi, 1) = \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) \quad \text{and} \quad (s_\varphi, \mathrm{Frob}) = \varphi(\mathrm{Frob}, \mathrm{Id}_{\mathsf{SL}_2(\mathbb{C})}). \tag{4.5}$$

Then $u_\varphi \in \mathcal{G}^\circ_\varphi$.

We recall that by the Jacobson–Morozov theorem any unipotent element u of $\mathcal{G}^\circ_\varphi$ determines a homomorphisms of algebraic groups $\mathsf{SL}_2(\mathbb{C}) \to \mathcal{G}^\circ_\varphi$ taking the value u at $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence any enhanced L-parameter (φ,ϕ) is already determined by $\varphi|_{W_F}$, u_φ and ϕ. More precisely, the map

$$(\varphi,\phi) \mapsto (\varphi|_{W_F}, u_\varphi, \phi) \tag{4.6}$$

provides a bijection between $\Phi_{\mathrm{e}}(G)$ and the set of $G^\vee$-conjugacy classes of triples $(\varphi|_{W_F}, u_\varphi, \phi)$.

We define an action of $G^\vee_{\mathrm{sc}}$ on $G^\vee$ by setting

$$h \cdot g := h' g h'^{-1} \quad \text{for } h \in G^\vee_{\mathrm{sc}} \text{ and } g \in G^\vee, \text{ where } p(h) = h'\mathrm{Z}_{G^\vee}.$$

It induces an action of $G^\vee_{\mathrm{sc}}$ on ${}^LG$ and we denote by $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)$ the stabilizer in $G^\vee_{\mathrm{sc}}$ of $\varphi(W'_F)$ for this action.

On the other hand, the inclusion $\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi) \hookrightarrow \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}) \cap \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(u_\varphi)$ induces a group isomorphism

$$A^1_\varphi \xrightarrow{\sim} \pi_0\left(\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}) \cap \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(u_\varphi)\right). \tag{4.7}$$

As observed in [AMS1, (92)], another way to formulate (4.7) is

$$A^1_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi) := \mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)/\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)^\circ. \tag{4.8}$$

The L-parameter φ is is called

- *discrete* if there is no proper W_F-stable Levi subgroup $L^\vee \subset G^\vee$ such that $\phi(W'_F) \subset {}^LL$.
- *bounded* if s_φ belongs to a bounded subgroup of $G^\vee$.

We say that $(\varphi,\phi) \in \Phi_{\mathrm{e}}(G)$ is *cuspidal* if φ is discrete and (u_φ,ϕ) is a cuspidal pair for $\mathcal{G}_\varphi$ (as defined in Section 3). The set of G-relevant cuspidal (respectively discrete, bounded) enhanced L-parameters is expected to correspond to the set of *supercuspidal* (respectively, *square-integrable, tempered*) irreducible smooth G-representations [AMS1, § 6].

4.2. **Inertial classes.** For L a Levi subgroup of G and $g \in G^\vee$, the group $gL^\vee g^{-1}$ is not necessarily W_F-stable, so the group $G^\vee$ need not act on pairs of the form $({}^LL, (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$ with $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ a cuspidal enhanced L-parameter for L. In order to deal with this, as in [AMS1, Definition 7.1], we will have to consider all the pairs $(\mathrm{Z}_{{}^LG}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$ of the following form:

- $\mathcal{T}$ is a torus of $G^\vee$ such that the projection $\mathrm{Z}_{{}^LG}(\mathcal{T}) \to W_F$ is surjective.
- $\varphi_{\mathrm{c}} \colon W_F' \to \mathrm{Z}_{{}^LG}(\mathcal{T})$ satisfies the requirements in the definition of an L-parameter.
- Let $\mathcal{L} = G^\vee \cap \mathrm{Z}_{{}^LG}(\mathcal{T})$, and let $\mathcal{L}_{\mathrm{sc}}$ be the simply connected cover of the derived group of $\mathcal{L}$. Then ϕ_{c} is an irreducible representation of $\pi_0(\mathrm{Z}^1_{\mathcal{L}_{\mathrm{sc}}}(\varphi))$ such that $(u_{\varphi_{\mathrm{c}}}, \phi)$ is a cuspidal pair for $\mathrm{Z}^1_{\mathcal{L}_{\mathrm{sc}}}(\varphi_{\mathrm{c}}|_{W_F}))$ and ϕ_{c} is G-relevant as defined in [AMS1, Definition 7.2] (that is $\zeta_\phi = \zeta$ on $\mathcal{L}_{\mathrm{sc}} \cap \mathrm{Z}^{W_F}_{G^\vee_{\mathrm{sc}}}$ and $\phi = 1$ on $\mathcal{L}_{\mathrm{sc}} \cap \mathrm{Z}^\circ_{\mathcal{L}_c}$, where $\mathcal{L}_c$ denotes the preimage of $\mathcal{L}$ under $G^\vee_{\mathrm{sc}} \to G^\vee$).

Fix such a pair $(\mathrm{Z}_{{}^LG}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$. The group

$$X_{\mathrm{un}}(\mathrm{Z}_{{}^LG}(\mathcal{T})) := \Big(\mathrm{Z}_{(G^\vee \rtimes I_F) \cap \mathrm{Z}_{{}^LG}(\mathcal{T})}\Big)^\circ_{\mathrm{Frob}} \tag{4.9}$$

plays the role of unramified characters for $\mathrm{Z}_{{}^LG}(\mathcal{T})$. It acts on the enhanced L-parameters $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ (see [AMS1, (110) and (111)]) and we denote by $X_{\mathrm{un}}(\mathrm{Z}_{{}^LG}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ the orbit of $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$.

We denote by $\mathfrak{s}^\vee$ the $G^\vee$-conjugacy class of $(\mathrm{Z}_{{}^LG}(\mathcal{T}), X_{\mathrm{un}}(\mathrm{Z}_{{}^LG}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$. We write

$$\mathfrak{s}^\vee = \mathfrak{s}^\vee_G = [\mathrm{Z}_{{}^LG}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})]_{G^\vee},$$

and call $\mathfrak{s}^\vee$ an *inertial class* for $\Phi_{\mathrm{e}}(G)$. We denote by $\mathfrak{B}^\vee(G)$ the set of all such $\mathfrak{s}^\vee$.

Note that there exists a W_F-stable Levi subgroup $L^\vee$ of $G^\vee$ such that $\mathrm{Z}_{{}^LG}(\mathcal{T})$ is $G^\vee$-conjugate to $L^\vee \rtimes W_F$ and $\mathcal{L} = G^\vee \cap \mathrm{Z}_{{}^LG}(\mathcal{T})$ is $G^\vee$-conjugate to $L^\vee$. Conversely, every $G^\vee$-conjugate of this $L^\vee \rtimes W_F$ is of the form $\mathrm{Z}_{{}^LG}(\mathcal{T})$ for a torus $\mathcal{T}$ as above (see [AMS1, Lemma 6.2]).

We write

$$\mathfrak{s}^\vee_L = (\mathrm{Z}_{{}^LG}(\mathcal{T}), X_{\mathrm{un}}(\mathrm{Z}_{{}^LG}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})). \tag{4.10}$$

We will consider the groups

$$W_{\mathfrak{s}^\vee} = \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee_L)/L^\vee \quad \text{and} \quad J_{\varphi_{\mathrm{c}}} := \mathrm{Z}_{G^\vee}(\varphi_{\mathrm{c}}(I_F)). \tag{4.11}$$

The group $J_{\varphi_{\mathrm{c}}}$ is a complex (possibly disconnected) reductive group. Define $R(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$ as the set of $\alpha \in X^*(\mathcal{T}) \setminus \{0\}$ that appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of $J^\circ_{\varphi_{\mathrm{c}}}$. It is a root system (see [AMS2, Proposition 3.9]).

We set $W^\circ_{\mathfrak{s}^\vee} := \mathrm{N}_{J^\circ_{\varphi_{\mathrm{c}}}}(\mathcal{T})/\mathrm{Z}_{J^\circ_{\varphi_{\mathrm{c}}}}(\mathcal{T})$, where $W^\circ_{\mathfrak{s}^\vee}$ is the Weyl group of $R(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$. Let $R^+(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$ be the positive system defined by a parabolic subgroup $P^\circ_{\varphi_{\mathrm{c}}} \subset J^\circ_{\varphi_{\mathrm{c}}}$ with Levi factor $(L^\vee_{\varphi_{\mathrm{c}}})^\circ$. Two such parabolic subgroups $P^\circ_{\varphi_{\mathrm{c}}}$ are $J^\circ_{\varphi_{\mathrm{c}}}$-conjugate, so the choice is inessential.

Since $W^\circ_{\mathfrak{s}^\vee}$ acts simply transitively on the collection of positive systems for $R(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$, we obtain a semi-direct factorization

$$\begin{aligned} &W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^\vee}, \\ &\text{where } \mathfrak{R}_{\mathfrak{s}^\vee} = \{w \in W_{\mathfrak{s}^\vee} \,:\, w \cdot R^+(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T}) = R^+(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})\}. \end{aligned}$$

Definition 4.2. Let $\boldsymbol{\nu}_{\mathrm{c}} \colon \Phi_{\mathrm{e}}(G) \to \mathfrak{B}^\vee(G)$ be the map defined by

$$\boldsymbol{\nu}_{\mathrm{c}}(\varphi, \phi) = [\mathrm{Z}_{{}^LG}(\mathrm{Z}^\circ_{\mathcal{M}_\varphi}), \varphi|_{W_F}, u_{\mathrm{c}}, \phi_{\mathrm{c}}]_{G^\vee},$$

where $(\varphi|_{W_F}, u, \phi)_{G^\vee}$ is the image of $(\varphi, \phi)_{G^\vee}$ via the bijection (4.6), $(u_{\mathrm{c}}, \phi_{\mathrm{c}})$ corresponds to $(U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}}) \in \mathbf{I}_{\mathrm{c}}^{\mathcal{M}_\varphi}$ and $(\mathcal{M}_\varphi, U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}}) := \nu_{\mathrm{c}}^{\mathcal{G}_\varphi}(U, \mathcal{E})$ is the image under the map $\nu_{\mathrm{c}}^{\mathcal{G}_\varphi}$ from Definition 3.2 of the pair $(U, \mathcal{E}) \in \mathbf{I}^{\mathcal{G}_\varphi}$ associated with (u, ϕ).

We have the following decomposition (see [AMS1, (115)]):

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}, \quad \text{where} \quad \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee} := \boldsymbol{\nu}_{\mathrm{c}}^{-1}(\mathfrak{s}^\vee). \tag{4.12}$$

Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible smooth G-representations. For L a Levi subgroup of G, we denote by $\mathrm{Irr}_{\mathrm{cusp}}(L)$ the set of isomorphism classes of supercuspidal irreducible smooth L-representations.

Let $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$. We call (L, σ) a *supercuspidal pair*, and we consider such pairs up to *inertial equivalence*: this is the equivalence relation generated by

- unramified twists, $(L, \sigma) \sim (L, \sigma \otimes \chi)$ for $\chi \in X_{\mathrm{un}}(L)$,
- G-conjugation, $(L, \sigma) \sim (gLg^{-1}, g \cdot \sigma)$ for $g \in G$.

We denote the set of all inertial equivalence classes for G by $\mathfrak{B}(G)$ and a typical inertial equivalence class by $\mathfrak{s} := [L, \sigma]_G$.

In [Be], Bernstein attached to every $\mathfrak{s} \in \mathfrak{B}(G)$ a block $\mathfrak{R}(G)^{\mathfrak{s}}$ in the category $\mathfrak{R}(G)$ of smooth G-representations as follows. Denote by I_P^G the normalized parabolic induction functor, where P is a parabolic subgroup of G with Levi subgroup L. If $\pi \in \mathrm{Irr}(G)$ is a constituent of $I_P^G(\tau)$ for some $\sigma \in \mathrm{Irr}(L)$ such that $[L, \sigma]_G = \mathfrak{s}$, then $\mathfrak{s}$ is called the *inertial supercuspidal support* of π. We set

$$\mathrm{Irr}(G)^{\mathfrak{s}} := \{\pi \in \mathrm{Irr}(G) \,:\, \pi \text{ has inertial supercuspidal support } \mathfrak{s}\},$$
$$\mathfrak{R}(G)^{\mathfrak{s}} := \{\pi \in \mathfrak{R}(G) \,:\, \text{ every irreducible constituent of } \pi \text{ belongs to } \mathrm{Irr}(G)^{\mathfrak{s}}\}.$$

4.3. **Unipotent representations.** An irreducible smooth representation (π, V) of G is called *unipotent* if there exists a parahoric subgroup K of G such that the subspace V^{K^+} of the vectors in V that are invariant by the pro-unipotent radical K^+ of K contains an irreducible unipotent representation of the finite reductive group $\overline{K} := K/K^+$. We denote by $\mathrm{Irr}_{\mathrm{un}}(G)$ the set of isomorphism classes of irreducible unipotent G-representations.

Definition 4.3. An L-parameter $\varphi\colon W_F \times \mathsf{SL}_2(\mathbb{C}) \to {}^L G$ is called *unramified* if $\varphi(w, 1) = (1, w)$ for any element w of the inertia subgroup I_F of W_F. Denote by $\Phi_{\mathrm{un}}({}^L G)$ the set of $G^\vee$-conjugacy classes of unramified L-parameters φ and by $\Phi_{\mathrm{e,un}}(G)$ the set of unramified enhanced G-relevant parameters, i.e., the subset of the $(\varphi, \phi) \in \Phi_{\mathrm{e}}(G)$ such that φ is unramified. By definition

$$\Phi_{\mathrm{e,un}}({}^L G) = \bigsqcup_{G' \in \mathrm{InnT}(G)} \Phi_{\mathrm{e,un}}(G') = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unramified}, \phi \in \widehat{A_\varphi^1}\}.$$

Set also

$$\Phi^p_{\mathrm{e,un}}({}^L G) = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unramified}, \phi \in \widehat{A_\varphi}\},$$

so that

$$\Phi^p_{\mathrm{e,un}}({}^L G) = \bigsqcup_{G' \in \mathrm{Inn}^p(G)} \Phi_{\mathrm{e,un}}(G').$$

The set $\Phi_{\mathrm{e,un}}(G)$ is known to parametrize $\mathrm{Irr}_{\mathrm{un}}(G)$: such a parametrization was defined by Lusztig in [Lu3, Lu4] in the case when $\mathbf{G}$ is simple adjoint, and extended by Solleveld in [So1, So2] to the case when G is arbitrary. In the case when $G = \mathsf{GL}_n(F)$ or $\mathsf{SL}_n(F)$, it follows also from [HS] and [ABPS1]. Note that an unramified L-parameter φ is completely determined by the pair (s_φ, u_φ), as defined in (4.5). Thus we may phrase this parametrization as the following bijection:

$$\mathsf{LLC}\colon \Phi_{\mathrm{e,un}}({}^L G) \longleftrightarrow \bigsqcup_{G' \in \mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G'), \quad G^\vee \cdot (\varphi, \phi) \mapsto \pi(s_\varphi, u_\varphi, \phi). \tag{4.13}$$

This correspondence sends cuspidal (respectively, discrete, bounded) parameters to supercuspidal (respectively, square-integrable, tempered) irreducible unipotent representations.

Conversely, let $x \in G^\vee$ with Jordan decomposition $x = su$. There is an unramified L-parameter φ (unique up to $G^\vee$-conjugation) such that $u = u_\varphi$ and $s = s_\varphi$. We set

$$\mathcal{G}_s = \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}). \tag{4.14}$$

Notice that $\varphi|_{W_F}$ only depends on s, which explains the notation. By (4.8),

$$A^1_\varphi \cong A_{\mathcal{G}_s}(u).$$

Denote by

$$\Phi_{\mathrm{e,un}}({}^LG, s) = G^\vee \backslash \{(\varphi', \phi) \in \Phi_{\mathrm{e,un}}({}^LG) \mid \varphi'(\mathrm{Frob}, 1) = (s', \mathrm{Frob}),\ s' \in G^\vee \cdot s\}.$$

Then

$$\begin{aligned} \Phi_{\mathrm{e,un}}({}^LG, s) &= \mathrm{Z}_{G^\vee}(\varphi|_{W_F})\text{-orbits in } \{(u', \phi) \mid u' \in \mathcal{G}_s^\circ \text{ unipotent},\ \phi \in \widehat{A_{\mathcal{G}_s}(u')}\} \\ &= \mathcal{G}_s\text{-orbits in } \{(u', \phi) \mid u' \in \mathcal{G}_s^\circ \text{ unipotent},\ \phi \in \widehat{A_{\mathcal{G}_s}(u')}\}. \end{aligned} \tag{4.15}$$

The second equality follows from the fact that conjugation of unipotent elements is insensitive to isogenies. This allows us to rephrase the unramified local Langlands correspondence as follows. Let $\mathcal{C}(\mathcal{G})$, $\mathcal{C}(\mathcal{G})_{\mathsf{ss}}$, $\mathcal{C}(\mathcal{G})_{\mathrm{un}}$ denote the set of conjugacy classes, respectively semisimple, unipotent conjugacy classes in a complex group $\mathcal{G}$. Let $R_{\mathrm{un}}(G')$ be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}G'$. Then (4.13) can be written as the bijection

$$\mathsf{LLC}_{\mathrm{un}} : \bigsqcup_{s \in \mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigsqcup_{u \in \mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}} \widehat{A_{\mathcal{G}_s}(u)} \longleftrightarrow \bigsqcup_{G' \in \mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G'),$$

which induces a linear isomorphism

$$\mathsf{LLC}_{\mathrm{un}} \colon R(\Phi_{\mathrm{e,un}}({}^LG)) := \bigoplus_{s \in \mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u \in \mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}} R(A_{\mathcal{G}_s}(u)) \longrightarrow \bigoplus_{G' \in \mathrm{InnT}(G)} R_{\mathrm{un}}(G'), \tag{4.16}$$

mapping $\phi \in A_{\mathcal{G}_s}(u)$ to $\pi(s, u, \phi)$ as defined in (4.13). If we restrict to pure inner twists, then we need to replace the group $\mathcal{G}_s$ by the group

$$\mathcal{G}_s^p = \mathrm{Z}_{G^\vee}(\varphi|_{W_F}), \tag{4.17}$$

and the correspondence becomes

$$\mathsf{LLC}^p_{\mathrm{un}} \colon R(\Phi^p_{\mathrm{e,un}}({}^LG)) := \bigoplus_{s \in \mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u \in \mathcal{C}(\mathcal{G}_s^p)_{\mathrm{un}}} R(A_{\mathcal{G}_s^p}(u)) \longrightarrow \bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G'). \tag{4.18}$$

Example 4.4. For $G = \mathsf{SL}_n(F)$, recall that there is a one-to-one correspondence between $\mathrm{InnT}(\mathsf{SL}_n(F))$ and $\mathbb{Z}/n\mathbb{Z}$, where the inner twists are $\mathsf{SL}_m(D_{d,r/m})$, $m = \gcd(r, n)$, $r \in \mathbb{Z}/n\mathbb{Z}$.

The dual Langlands group is $G^\vee = \mathsf{PGL}_n(\mathbb{C})$. The correspondence (4.13) takes the form:

$$\bigsqcup_{r \in \mathbb{Z}/n\mathbb{Z}} \mathrm{Irr}_{\mathrm{un}}(\mathsf{SL}_m(D_{d,r/m})) \longleftrightarrow \mathsf{PGL}_n(\mathbb{C}) \backslash \{(x, \phi) \,|\, x \in \mathsf{PGL}_n(\mathbb{C}), \phi \in \widehat{A^1_x}\}, \tag{4.19}$$

in particular

$$\mathrm{Irr}_{\mathrm{un}}(\mathsf{SL}_n(F)) \longleftrightarrow \mathsf{PGL}_n(\mathbb{C}) \backslash \{(x, \phi) \,|\, x \in \mathsf{PGL}_n(\mathbb{C}), \phi \in \widehat{A_x}\}.$$

In this case, $G^\vee_{\mathrm{sc}} = \mathsf{SL}_n(\mathbb{C})$ and $\mathrm{Z}_{G^\vee_{\mathrm{sc}}} = C_n$. The irreducible central characters are therefore $\widehat{\mathrm{Z}_{\mathsf{SL}_n(\mathbb{C})}} = \{\zeta_r \mid r \in \mathbb{Z}/n\mathbb{Z}\}$. A Langlands parameter (x, ϕ) parametrizes an irreducible unipotent representation of $\mathsf{SL}_m(D_{d,r/m})$ if and only if $\zeta_\phi = \zeta_m$. In particular, the unipotent representations of $\mathsf{SL}_n(F)$ correspond to central characters $\zeta_0 = 1$.

Moreover, for $x \in \mathsf{PGL}_n(\mathbb{C})$, A^1_x is the group of components of $\mathrm{Z}^1_{\mathsf{SL}_n(\mathbb{C})}(x) = \{g \in \mathsf{SL}_n(\mathbb{C}) \mid gxg^{-1} = x\}$.

5. Lusztig's nonabelian Fourier transform for finite groups

We recall the definition of the nonabelian Fourier transform [Lu1]. For the background material, we follow [Lu1, GM, DM].

5.1. **Fourier transforms.** For a finite group Γ, define the set

$$\mathcal{M}(\Gamma) = \Gamma\text{-orbits on } \{(x,\sigma) \mid x \in \Gamma,\ \sigma \in \widehat{\mathrm{Z}_\Gamma(x)}\}, \tag{5.1}$$

where the action of Γ is: $g \cdot (x,\sigma) = (gxg^{-1}, \sigma^g)$, $\sigma^g(y) = (g^{-1}yg)$, $y \in \mathrm{Z}_\Gamma(gxg^{-1})$, $g \in \Gamma$. Define also

$$\mathcal{Y}(\Gamma) = \{(y,z) \in \Gamma \times \Gamma \mid yz = zy\}. \tag{5.2}$$

Write $\Gamma\backslash\mathcal{Y}(\Gamma)$ for the set of Γ-orbits on $\mathcal{Y}(\Gamma)$. Let $\mathsf{Sh}^\Gamma(\Gamma)$ be the category of Γ-equivariant coherent sheaves of Γ (Γ acting on itself by conjugation). The irreducible objects in $\mathsf{Sh}^\Gamma(\Gamma)$ are parametrized by $\mathcal{M}(\Gamma)$: for every pair $(x,\sigma) \in \mathcal{M}(\Gamma)$, let $\mathcal{V}((x,\sigma)) = \Gamma \times_{\mathrm{Z}_\Gamma(x)} \sigma$ be the corresponding irreducible Γ-equivariant sheaf. This means that there is a natural isomorphism $\mathbb{C}[\mathcal{M}(\Gamma)] \cong K(\mathsf{Sh}^\Gamma(\Gamma))$, where $K()$ is the K-group. Moreover, there is an isomorphism

$$\kappa : \mathbb{C}[\mathcal{M}(\Gamma)] \cong K(\mathsf{Sh}^\Gamma(\Gamma)) \to \mathbb{C}[\Gamma \setminus \mathcal{Y}(\Gamma)], \quad \mathcal{V} \mapsto ((y,z) \mapsto \mathrm{tr}(z, \mathcal{V}|_y)).$$

Lusztig [Lu1] defined a hermitian pairing on $\mathcal{M}(\Gamma)$:

$$\{(x,\sigma),(y,\tau)\} = \frac{1}{|\mathrm{Z}_\Gamma(x)||\mathrm{Z}_\Gamma(y)|} \sum_{\substack{g\in\Gamma \\ xgyg^{-1}=gyg^{-1}x}} \sigma(gyg^{-1})\tau(g^{-1}x^{-1}g), \tag{5.3}$$

and a linear map, the Fourier transform for Γ,

$$\mathrm{FT}_\Gamma \colon \mathbb{C}[\mathcal{M}(\Gamma)] \to \mathbb{C}[\mathcal{M}(\Gamma)],\ \mathrm{FT}_\Gamma(f)(x,\sigma) = \sum_{(y,\tau)\in\mathcal{M}(\Gamma)} \{(x,\sigma),(y,\tau)\} f(y,\tau). \tag{5.4}$$

See Lemma 5.1 for the interpretation of FT_Γ in terms of $\mathcal{Y}(\Gamma)$.

Now consider the following generalization. Suppose $\widetilde{\Gamma} = \Gamma \rtimes \langle\alpha\rangle$, where α has order c. Set $\Gamma' = \Gamma\alpha \subset \widetilde{\Gamma}$. As in [Lu1, §4.16], define two sets $\mathcal{M} = \mathcal{M}(\Gamma \trianglelefteq \widetilde{\Gamma})$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}}(\Gamma \trianglelefteq \widetilde{\Gamma})$ as follows:

$$\begin{aligned} \mathcal{M} &= \{(x,\sigma) \mid x \in \Gamma \text{ such that } \mathrm{Z}_{\widetilde{\Gamma}}(x) \cap \Gamma' \neq \emptyset,\ \sigma \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}}(x)} \text{ with } \sigma|_{\mathrm{Z}_\Gamma(x)} \text{ irreducible}\}, \\ \overline{\mathcal{M}} &= \{(x,\bar\sigma) \mid x \in \Gamma',\ \bar\sigma \in \widehat{\mathrm{Z}_\Gamma(x)}\}, \end{aligned} \tag{5.5}$$

modulo the equivalence relation given by conjugation by $\widetilde{\Gamma}$. In addition, the cyclic group $\langle\alpha\rangle$ acts on $\mathcal{M}$ by twists in the second entry of the pair (x,σ) and denote by $\sim_c$ the corresponding equivalence relation.

The set $\mathcal{M}$ is a subset of $\mathcal{M}(\widetilde{\Gamma})$. Given $(x,\bar\sigma) \in \overline{\mathcal{M}}$, we have that $(x,\sigma) \in \mathcal{M}(\widetilde{\Gamma})$ for any extension σ of $\bar\sigma$ to $\mathrm{Z}_{\widetilde{\Gamma}}(x)$. Thus the pairing $\{\ ,\ \}$ on $\mathcal{M}(\widetilde{\Gamma})$ induces a pairing

$$\{\ ,\ \} : \overline{\mathcal{M}} \times \mathcal{M} \to \mathbb{C}, \quad \{(x,\bar\sigma),(y,\tau)\} := c\{(x,\sigma),(y,\tau)\}, \tag{5.6}$$

for any fixed extension σ of $\bar\sigma$ to $\mathrm{Z}_{\widetilde{\Gamma}}(x)$.

Let $\mathcal{P} = \mathcal{P}(\Gamma \trianglelefteq \widetilde{\Gamma})$ and $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\Gamma \trianglelefteq \widetilde{\Gamma})$ be the spaces of functions on $\mathcal{M}(\widetilde{\Gamma})$ with support in $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively. The operator [Lu1, (4.16.1)] (see also [GM, §4.2.14])

$$\mathrm{FT}_{\Gamma\trianglelefteq\widetilde{\Gamma}} : \mathcal{P} \to \overline{\mathcal{P}},\ \mathrm{FT}_{\Gamma\trianglelefteq\widetilde{\Gamma}} f(x,\bar\sigma) = \sum_{(y,\tau)\in\mathcal{M}/\sim_c} \{(x,\bar\sigma),(y,\tau)\} f(y,\tau) \tag{5.7}$$

is an isomorphism with inverse $\mathrm{FT}^{-1}_{\Gamma\trianglelefteq\widetilde{\Gamma}} f(y,\tau) = \sum_{(x,\bar\sigma)\in\overline{\mathcal{M}}} \{(x,\bar\sigma),(y,\tau)\} f(x,\bar\sigma)$.

5.2. **Families of Weyl group representations.** Let W be a finite Weyl group with the set of simple generators S. The partition of $\widehat{W}$ into families is defined in [Lu1, §4.2] as follows. Let sgn denote the sign character of W. If $W = \{1\}$, there is only one family consisting of the trivial representation. Otherwise, assume that the families have been defined for all proper parabolic subgroups of W. Then $\mu, \mu' \in \widehat{W}$ belong to the same family of W if there exists a sequence $\mu = \mu_0, \mu_1, \dots, \mu_m = \mu'$, $\mu_i \in \widehat{W}$, such that for each i: there exists a parabolic subgroup $W_i \subsetneq W$ and $\mu_i', \mu_i'' \in \widehat{W_i}$ in the same family of W_i such that either

$$\langle \mu_i', \mu_{i-1}\rangle_{W_i} \neq 0,\ a_{\mu_i'} = a_{\mu_{i-1}}, \quad \langle \mu_i'', \mu_i\rangle_{W_i} \neq 0,\ a_{\mu_i''} = a_{\mu_i},$$

or

$$\langle \mu_i', \mu_{i-1} \otimes \mathrm{sgn}\rangle_{W_i} \neq 0,\ a_{\mu_i'} = a_{\mu_{i-1}\otimes \mathrm{sgn}}, \quad \langle \mu_i'', \mu_i \otimes \mathrm{sgn}\rangle_{W_i} \neq 0,\ a_{\mu_i''} = a_{\mu_i\otimes \mathrm{sgn}}.$$

Here a_μ is the a-invariant of μ defined in [Lu1, §4.1]. It follows from the definition that if $\mathcal{F} \subset \widehat{W}$ is a family, then so is $\mathcal{F} \otimes \mathrm{sgn}$ and the families for $W_1 \times W_2$ are $\mathcal{F}_1 \boxtimes \mathcal{F}_2$, where $\mathcal{F}_i$ is a family for W_i, $i = 1, 2$.

Suppose in addition that we have a Coxeter group automorphism $\sigma\colon W \to W$, i.e., $\sigma(S) = S$. Such an automorphism is called ordinary if, on each irreducible component of W, it is not the nontrivial graph automorphism of type B_2, G_2, or F_4. The automorphism σ acts on $\widehat{W}$ and it permutes the families $\mathcal{F}$. An important observation [Lu1, §4.17] is that if σ is ordinary and $\mathcal{F}$ is σ-stable, then every element of $\mathcal{F}$ is σ-stable.

5.3. **Families of unipotent representations.** Let $\mathbb{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$ with a Frobenius map $\mathrm{Fr}\colon \mathbb{G} \to \mathbb{G}$ such that there exists a maximal torus $\mathbb{T}_0$ with the property $\mathrm{Fr}(t) = t^q$, for all $t \in \mathbb{T}_0$. Let $W = N_{\mathbb{G}}(\mathbb{T}_0)/\mathbb{T}_0$ be the Weyl group. Recall that an irreducible representation $\rho \in \mathrm{Irr}\mathbb{G}^{\mathrm{Fr}}$ is called unipotent if $\langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(1)\rangle_{\mathbb{G}^{\mathrm{Fr}}} \neq 0$ for some Fr-stable maximal torus $\mathbb{T}$ of $\mathbb{G}$. Here $R_{\mathbb{T}}^{\mathbb{G}}$ is Deligne-Lusztig induction [DL, §7.8]. Let $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ denote the set of irreducible unipotent $\mathbb{G}^{\mathrm{Fr}}$-representations. By the results of Lusztig, the classification of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ is reduced to the case when $\mathbb{G}$ is adjoint simple, see for example the exposition in [GM, Remark 4.2.1]. More precisely, if $\pi\colon \mathbb{G} \to \mathbb{G}_{\mathrm{ad}}$ is the surjective homomorphism with central kernel ($\mathbb{G}_{\mathrm{ad}}$ is the semisimple adjoint group isogeneous to $\mathbb{G}/\mathrm{Z}_{\mathbb{G}}$), there exists a Frobenius map $\mathrm{Fr}_{\mathrm{ad}}$ such that $\mathrm{Fr}_{\mathrm{ad}} \circ \pi = \pi \circ \mathrm{Fr}$ such that the resulting group homomorphism $\pi\colon \mathbb{G}^{\mathrm{Fr}} \to \mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}}$ induces a bijection

$$\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}} \cong \mathrm{Irr}_{\mathrm{un}}\mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}}.$$

Furthermore, write $\mathbb{G}_{\mathrm{ad}} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_r$ is the decomposition into factors such that each $\mathbb{G}_i$ is semisimple adjoint, $\mathrm{Fr}_{\mathrm{ad}}$-stable and a direct product of simple algebraic groups that are cyclically permuted by $\mathrm{Fr}_{\mathrm{ad}}$. Let $\mathbb{H}_i$ be one of the simple factors in $\mathbb{G}_i$: if h_i is the number of copies of $\mathbb{H}_i$, then $\mathrm{Fr}_{\mathrm{ad}}^{h_i}$ preserves $\mathbb{H}_i$. Denote by Fr_i the restriction to $\mathbb{H}_i$. Then

$$\mathrm{Irr}_{\mathrm{un}}\mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}} \cong \prod_{i=1}^{r} \mathrm{Irr}_{\mathrm{un}}\mathbb{H}_i^{\mathrm{Fr}_i}.$$

The Frobenius map Fr induces a Coxeter group automorphism σ of W. Define a graph with vertices $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ as follows: $\rho_1, \rho_2 \in \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ are joined by an edge if and only if there is σ-stable $\mu \in \widehat{W}$ such that $\langle \rho_i, R_{\widetilde{\mu}}\rangle_{\mathbb{G}^{\mathrm{Fr}}} \neq 0$ for $i = 1, 2$ where $R_{\widetilde{\mu}}$ is the almost character associated to a fixed extension $\widetilde{\mu}$ of μ to $\widetilde{W} = W \rtimes \langle\sigma\rangle$ as defined in [Lu1, (3.7.1)]. Each connected component of this graph is called a *family* in $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$. One can define an equivalence relation on the set $\widehat{W}^\sigma$ of σ-stable irreducible W-representation: μ and μ' are equivalent if $R_{\widetilde{\mu}}$ and $R_{\widetilde{\mu}'}$ have unipotent constituents in the same family. By [Lu1], see also [GM, Proposition 4.2.3], the equivalence classes are the same as the σ-stable families in $\widehat{W}$, when σ is ordinary.

To each family $\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ corresponding to the σ-stable family $\mathcal{F} \subset \widehat{W}^{\sigma}$, Lusztig [Lu1, §4] attached finite groups $\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}$ such that $\widetilde{\Gamma}_{\mathcal{U}} = \Gamma_{\mathcal{U}}\langle\sigma\rangle$, a bijection

$$\mathcal{U} \longleftrightarrow \overline{\mathcal{M}}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}),\ \rho \mapsto \bar{x}_{\rho}, \tag{5.8}$$

scalars $\Delta(\bar{x}_{\rho}) \in \{\pm 1\}$ [Lu1, §6.7], and an injection

$$\mathcal{F} \longrightarrow \mathcal{M}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}),\ \mu \mapsto x_{\mu}, \tag{5.9}$$

such that, when σ is ordinary, [Lu1, Theorem 4.23] says that

$$\langle \rho, R_{\bar{\mu}} \rangle_{\mathbb{G}^F} = \Delta(\bar{x}_{\rho})\{\bar{x}_{\rho}, x_{\mu}\}. \tag{5.10}$$

Define the *unipotent almost characters* of $\mathbb{G}^{\mathrm{Fr}}$ to be the set of orthonormal class functions

$$R_x = \sum_{\rho \in \mathcal{U}} \Delta(\bar{x}_{\rho})\{\bar{x}_{\rho}, x\}\rho, \quad x \in \mathcal{M}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}). \tag{5.11}$$

Hence the unipotent nonabelian Fourier transform of $\mathbb{G}^{\mathrm{Fr}}$

$$\mathrm{FT}_{\mathbb{G}^{\mathrm{Fr}}} := \bigoplus_{\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}} \mathrm{FT}_{\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}} \tag{5.12}$$

gives the change of bases matrix, up to the signs $\Delta(\bar{x}_{\rho})$, between irreducible unipotent characters and almost characters.

For a family $\mathcal{U}$ parametrized by $\mathcal{M}(\Gamma_{\mathcal{U}})$, let $x \in \Gamma_{\mathcal{U}}$, and define the virtual combinations of unipotent characters

$$\begin{aligned} \Pi_{\mathcal{U}}(x, y) &= \sum_{\sigma \in \widehat{\mathrm{Z}_{\Gamma}(x)}} \sigma(y^{-1})\rho_{(x,\sigma)}, \text{ where } y \in \mathrm{Z}_{\Gamma}(x), \\ \Pi_{\mathcal{U}}(\sigma, \tau) &= \sum_{y \in \mathrm{Z}_{\Gamma}(x)} \tau(y)\rho_{(y,\sigma)}, \text{ if } \sigma, \tau \in \widehat{\mathrm{Z}_{\Gamma}(x)}, \end{aligned} \tag{5.13}$$

where $\rho_{(x,\sigma)}$ is the representation in $\mathcal{U}$ parametrized by $(x, \sigma) \in \mathcal{M}(\Gamma_{\mathcal{U}})$. The second linear combination only depends on $\mathrm{Z}_{\Gamma}(x)$, not on x.

Lemma 5.1 (cf. [DM]). *With the notation (5.13),* $\Gamma = \Gamma_{\mathcal{U}}$,

$$\mathrm{FT}_{\Gamma}(\Pi_{\mathcal{U}}(x, y)) = \Pi_{\mathcal{U}}(y, x), \quad \mathrm{FT}_{\Gamma}(\Pi_{\mathcal{U}}(\sigma, \tau)) = \Pi_{\mathcal{U}}(\tau, \sigma).$$

Proof. We verify the first formula. The second one is analogous (or it follows by change of bases.) Denote by C_y the conjugacy class of y in $\mathrm{Z}_{\Gamma}(x)$.

$$\begin{aligned} \mathrm{FT}_{\Gamma_{\mathcal{U}}}(\Pi_{\mathcal{U}}(x, y)) &= \sum_{\sigma} \sigma(y^{-1}) \sum_{(z,\tau)} \{(x, \sigma), (z, \tau)\}\rho_{(z,\tau)} \\ &= \sum_{\sigma} \sigma(y^{-1}) \sum_{(z,\tau)} \frac{1}{|\mathrm{Z}_{\Gamma}(x)||\mathrm{Z}_{\Gamma}(z)|} \sum_{g \in \Gamma,\ gzg^{-1} \in \mathrm{Z}_{\Gamma}(x)} \sigma(gzg^{-1})\tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \\ &= \sum_{(z,\tau)} \frac{1}{|\mathrm{Z}_{\Gamma}(z)|} \left(\sum_{g \in \Gamma,\ gzg^{-1} \in \mathrm{Z}_{\Gamma}(x)} \frac{1}{|\mathrm{Z}_{\Gamma}(x)|} \sum_{\sigma} \sigma(y^{-1})\sigma(gzg^{-1}) \right) \tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \\ &= \sum_{g \in \Gamma,\ gzg^{-1} \in C_y} \frac{1}{|C_y|} \frac{1}{|\mathrm{Z}_{\Gamma}(z)|} \sum_{\tau \in \widehat{\mathrm{Z}_{\Gamma}(z)}} \tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \\ &= \frac{1}{|C_y|} \sum_{y' \in C_y} \sum_{\tau \in \mathrm{Z}_{\Gamma}\widehat{(g^{-1}y'g)}} \tau(g^{-1}x^{-1}g)\rho_{(g^{-1}y'g,\tau)} \\ &= \frac{1}{|C_y|} \sum_{y' \in C_y} \Pi_{\mathcal{U}}(g^{-1}y'g, g^{-1}xg) = \Pi_{\mathcal{U}}(y, x), \end{aligned}$$

where we used column orthogonality of characters and that (y, x) is Γ-conjugate to $(g^{-1}y'g, g^{-1}xg)$ when $y' \in C_y$.

□

5.4. **Disconnected groups.** Now suppose that $\mathbb{G}$ is a disconnected reductive group with Frobenius map $\mathrm{Fr}\colon \mathbb{G} \to \mathbb{G}$ and identity component $\mathbb{G}^\circ$ such that $A = \mathbb{G}/\mathbb{G}^\circ$ is abelian. (In our applications, $\mathbb{G}/\mathbb{G}^\circ$ will almost always be a cyclic group.) By definition, the irreducible unipotent $\mathbb{G}^{\mathrm{Fr}}$-representations $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ are the constituents of all induced representations $\mathrm{Ind}_{\mathbb{G}^{\circ\mathrm{Fr}}}^{\mathbb{G}^{\mathrm{Fr}}}\rho$, where $\rho \in \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}}$. See [GM, Proposition 4.8.19] for the compatibility with the definition in terms of the appropriate version of $R_{\mathbb{T}}^{\mathbb{G}}(1)$. The parametrization of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ follows from that of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}}$ via Mackey induction using the explicit results for simple groups, e.g. [GM, Theorem 4.5.11 and 4.5.12].

We are interested in studying the irreducible unipotent representations for groups $\mathbb{G}^{\mathrm{Fr}}$ that are related via the structure theory of p-adic groups.

Definition 5.2. Let $\mathbb{G}$ be a reductive algebraic group over $\overline{\mathbb{F}}_p$ with identity component $\mathbb{G}^\circ$ and such that $\mathbb{G}/\mathbb{G}^\circ = A$ is a finite abelian group. Let Fr_0 be a Frobenius map on $\mathbb{G}$ and assume that $\mathbb{G}^{\mathrm{Fr}_0}$ is split. Given $a \in A$, conjugation by a defines an outer automorphism of $\mathbb{G}^\circ$, which induces an isomorphism, call it σ_a, of the based root datum of $\mathbb{G}^\circ$. For every a, define the Frobenius automorphism $\mathrm{Fr}_a = \mathrm{Fr}_0 \circ \sigma_a$. The set of *inner forms* of $\mathbb{G}$ is the collection of finite groups

$$\mathrm{Inn}\,\mathbb{G} = \{\mathbb{G}^{\mathrm{Fr}_a} \mid a \in A\}.$$

This definition agrees with the usual notion of inner forms. Indeed, the equivalence classes of forms of $\mathbb{G}$ inner to the split form are in one-to-one correspondence with the first Galois cohomology group

$$\mathrm{Inn}\,\mathbb{G} \leftrightarrow H^1(\mathbb{F}_q, \mathbb{G}) \cong H^1(\mathbb{F}_q, \mathbb{G}/\mathbb{G}^\circ) = H^1(\mathbb{F}_q, A) \cong A,$$

using Lang's theorem $H^1(\mathbb{F}_q, \mathbb{G}^\circ) = 0$, see [Se, III.§2, Corollary 3] for example.

By (5.8), every unipotent family $\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}_0}$ has an associated finite group $\Gamma_{\mathcal{U}} = \widetilde{\Gamma}_{\mathcal{U}}$ (since $\mathbb{G}^{\circ\mathrm{Fr}_0}$ is split). The group A acts on the set of families $\mathcal{U}$. For every orbit $\mathcal{O}_A = A \cdot \mathcal{U}$ with representative $\mathcal{U}$, let $\mathrm{Z}_A(\mathcal{U})$ be the corresponding isotropy group. Then $\mathrm{Z}_A(\mathcal{U})$ permutes the elements of $\mathcal{U}$, hence the corresponding parameters $\mathcal{M}(\Gamma_u)$. If $\Gamma_{\mathcal{U}}$ is abelian, which turns out to be the case in all of the examples of interest to us when $A \neq \{1\}$, this defines automatically an action of $\mathrm{Z}_A(\mathcal{U})$ on $\Gamma_{\mathcal{U}}$, hence a group

$$\widetilde{\Gamma}_{\mathcal{U}}^A = \Gamma_{\mathcal{U}} \rtimes \mathrm{Z}_A(\mathcal{U}). \tag{5.14}$$

See [Lu5, §17] for more details.

Proposition 5.3. *Let $\mathbb{G}$ be as in Examples 5.4-5.9. The parametrization (5.8) induces a bijection*

$$\bigsqcup_{a \in A} \mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{F_a}) \longleftrightarrow \bigsqcup_{\mathcal{U} \subset A \backslash \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ F_0}} \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^A),$$

where $\mathcal{U}$ in the right hand side ranges over a set of representatives of the A-orbits of families in $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ F_0}$.

Proof. The proof is presented in each case in the examples. □

Let $R_{\mathrm{un}}(\mathbb{G}^{F_a})$ be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{F_a})$. The bijection of Proposition 5.3 induces a linear isomorphism

$$\bigoplus_{a \in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}) \to \bigoplus_{\mathcal{U} \subset A \backslash \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}_0}} \mathbb{C}[\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^A)]. \tag{5.15}$$

The right-hand side of (5.15) has the involution given by (5.4). Define

$$\mathrm{FT}_{\mathbb{G}}\colon \bigoplus_{a\in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}) \to \bigoplus_{a\in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}), \tag{5.16}$$

to be the corresponding involution on the left-hand side.

In the examples below, when A is clear from the context, we may write $\widetilde{\Gamma}_{\mathcal{U}}$ in place of $\widetilde{\Gamma}^A_{\mathcal{U}}$ for simplicity of notation.

Example 5.4. Let $\mathbb{H}$ be a connected almost simple $\mathbb{F}_q$-split group and $\mathbb{G} = (\mathbb{H}\times\mathbb{H})\rtimes\mathbb{Z}/2\mathbb{Z}$, where the nontrivial element δ of $A = \mathbb{Z}/2\mathbb{Z}$ acts by flipping the two copies of $\mathbb{H}$. There are two inner forms:

$$\operatorname{Inn}\mathbb{G} = \{\mathbb{H}(\mathbb{F}_q)^2 \rtimes \mathbb{Z}/2\mathbb{Z}, \mathbb{H}(\mathbb{F}_{q^2})\rtimes\mathbb{Z}/2\mathbb{Z}\},$$

the second one for the Frobenius map $\mathrm{Fr}_1(h_1,h_2) = (\mathrm{Fr}_0(h_2), \mathrm{Fr}_0(h_1))$, $h_1, h_2 \in \mathbb{H}$. A family of $\mathbb{G}^\circ(\mathbb{F}_q) = H(\mathbb{F}_q)^2$ is $\mathcal{U}_1\boxtimes\mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are unipotent families of $\mathbb{H}$. The A-orbits are either $\{\mathcal{U}_1\boxtimes\mathcal{U}_2, \mathcal{U}_2\boxtimes\mathcal{U}_1\}$ for $\mathcal{U}_1\neq\mathcal{U}_2$ or $\{\mathcal{U}\boxtimes\mathcal{U}\}$. Assume that all $\Gamma_{\mathcal{U}}$ are abelian. Set

$$\widetilde{\Gamma}^{\mathbb{Z}/2\mathbb{Z}}_{\mathcal{U}_1\boxtimes\mathcal{U}_2} = \Gamma_{\mathcal{U}_1}\times\Gamma_{\mathcal{U}_2},\ \mathcal{U}_1\neq\mathcal{U}_2, \quad \widetilde{\Gamma}^{\mathbb{Z}/2\mathbb{Z}}_{\mathcal{U}\boxtimes\mathcal{U}} = \Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z},$$

with the flip action of δ. There are $\frac{\ell(\ell+3)}{2}$ conjugacy classes in $\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}$, $\ell = |\Gamma_{\mathcal{U}}|$, and they are represented by

- $(x,x')\sim(x',x)$ if $x\neq x'\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x')) = \Gamma^2_{\mathcal{U}}$;
- (x,x), $x\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x)) = \Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z}$;
- $(x,1)\delta$, $x\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1)\delta) = \langle\Gamma^{\Delta}_{\mathcal{U}}, (x,1)\delta\rangle$, where $\Gamma^{\Delta}_{\mathcal{U}}$ is the diagonal copy of $\Gamma_{\mathcal{U}}$.

When $\mathcal{U}_1\neq\mathcal{U}_2$, if $\rho_1\in\mathcal{U}_1$, $\rho_2\in\mathcal{U}_2$, then $\rho_1\times\rho_2 := \mathrm{Ind}^{\mathbb{G}(\mathbb{F}_q)}_{\mathbb{G}^\circ(\mathbb{F}_q)}(\rho_1\boxtimes\rho_2)$ is parametrized by $(\bar{x}_{\rho_1}, \bar{x}_{\rho_2})\in\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}_1\boxtimes\mathcal{U}_2})$.

In the second case, let $\rho,\rho'\in\mathcal{U}$. If $\rho\neq\rho'$, then $\rho\times\rho'\cong\rho'\times\rho$ is an irreducible representation of $\mathbb{G}(\mathbb{F}_q)$. If $\bar{x}_\rho = (x,\sigma)$, $\bar{x}_{\rho'} = (x',\sigma')$ are the corresponding parameters of ρ, ρ' in $\mathcal{M}(\Gamma_u)$, then the parameter for $\rho\times\rho'$ in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ is $((x,x'),\sigma\boxtimes\sigma')$, if $x\neq x'$, or $((x,x),\sigma\times\sigma')$, where $\sigma\times\sigma' = \mathrm{Ind}^{\Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z}}_{\Gamma_{\mathcal{U}}}(\sigma\boxtimes\sigma')$, if $\sigma\neq\sigma'$.

If $\rho=\rho'$, then we can extend $\rho\boxtimes\rho$ in two different ways to $\mathbb{G}(\mathbb{F}_q)$, denoted by $(\rho\times\rho)^\pm$ relative to the character of $\mathbb{Z}/2\mathbb{Z}$. The corresponding parameters in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ are $((x,x),(\sigma\times\sigma)^\pm)$, with the obvious notation.

For the second inner form, the irreducible unipotent representations of $\mathbb{H}(\mathbb{F}_{q^2})$ are given by the same families $\mathcal{U}$ as for $\mathbb{H}(\mathbb{F}_q)$ and δ fixes each unipotent representation ρ of $\mathbb{H}(\mathbb{F}_{q^2})$. Let ρ be an irreducible $\mathbb{H}(\mathbb{F}_{q^2})$-representation in $\mathcal{U}$ parametrized by $\bar{x}_\rho = (x,\sigma)$, $x\in\Gamma_{\mathcal{U}}$, $\sigma\in\widehat{\Gamma}_{\mathcal{U}}$. Then it can be extended in two different ways $\rho^\pm$ to $\mathbb{H}(\mathbb{F}_{q^2})\rtimes\mathbb{Z}/2\mathbb{Z}$. The centralizer $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1)\delta) = \langle\Gamma^{\Delta}_{\mathcal{U}}, (x,1)\delta\rangle$ is isomorphic to the direct product $C_x := \langle(y,y)\mid y\neq x\in\Gamma_{\mathcal{U}}\rangle\times\langle(x,1)\delta\rangle$, since $((x,1)\delta)^2 = (x,x)$. Regard σ as a representation of the subgroup $\Gamma^{\Delta}_{\mathcal{U}}$ and there are two ways $\sigma^\pm$ to extend it to C_x coming from the short exact sequence $1\to\langle(x,x)\rangle\to\langle(x,1)\delta\rangle\to\mathbb{Z}/2\mathbb{Z}\to 1$. We attach $((x,1)\delta,\sigma^\pm)\in\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ to $\rho^\pm$. To fix a choice of $\pm$, we fix a choice of primitive ℓ-th root of unity ζ_ℓ for each ℓ. Then, if $\sigma((x,x)) = \zeta_k^j$, for some j, where k is the order of x, then $\sigma^+((x,1)\delta) = \zeta^j_{2k}$. For our applications, $\mathbb{H}$ will be a classical group and therefore, $\Gamma_{\mathcal{U}}$ a 2-group, hence x will have order $k\leq 2$.

Example 5.5. Let $\mathbb{G}^\circ = \mathsf{GL}^m_k$, $\mathbb{G}^{\circ\mathrm{Fr}_0} = \mathsf{GL}_k(\mathbb{F}_q)^m$, and $A = \mathbb{Z}/m\mathbb{Z}$ acting by cyclic permutations on the factors of $\mathbb{G}^\circ$. Then

$$\operatorname{Inn}\mathbb{G} = \{\mathbb{G}^{\circ\mathrm{Fr}_r}\rtimes\mathbb{Z}/m\mathbb{Z} = \mathsf{GL}_k(\mathbb{F}_{q^{m/d}})^d\rtimes\mathbb{Z}/m\mathbb{Z}\mid r\in\mathbb{Z}/m\mathbb{Z},\ d=\gcd(r,m)\}.$$

Each unipotent family of $\mathsf{GL}_k(\mathbb{F}_{q^{m/d}})^d$ is a singleton $\{\rho_1\boxtimes\cdots\boxtimes\rho_d\}$ where $\rho_i\in\widehat{S}_k$, $1\leq i\leq d$. Hence, we can ignore the difference between unipotent families and irreducible

representations of symmetric groups. The irreducible representations of $S_k^d \rtimes \mathbb{Z}/m\mathbb{Z}$ are constructed by Mackey theory.

Start with a unipotent representation $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_m$ of $\mathsf{GL}_k(\mathbb{F}_q)^m \rtimes \mathbb{Z}/m\mathbb{Z}$ with stabilizer $\mathbb{Z}/c\mathbb{Z}$, $c|m$. This means that $\rho_i = \rho_{i+m/c}$ for all i (viewed mod m) and that $\mathbb{Z}/(m/c)\mathbb{Z}$ has no fixed points under the cyclic action on $\rho_1 \boxtimes \cdots \boxtimes \rho_{m/c}$. The corresponding unipotent family $\widetilde{\mathcal{U}}$ that we construct for $\operatorname{Inn}\mathbb{G}$ has

$$\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/m\mathbb{Z}} = \mathbb{Z}/c\mathbb{Z}.$$

The irreducible representations $\widetilde{\rho}$ of $\mathbb{G}^{\mathrm{Fr}_0}$ whose restriction to $\mathbb{G}^{\circ\mathrm{Fr}_0}$ contain ρ are in one-to-one correspondence to the characters of $\mathbb{Z}/c\mathbb{Z}$, hence they are parametrized un $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}})$ by the pairs $(0, \sigma)$, $\sigma \in \widehat{\mathbb{Z}/c\mathbb{Z}}$.

For every $r \in \mathbb{Z}/m\mathbb{Z}$ such that m/c divides $d = \gcd(r, m)$, consider the representation of $\mathbb{G}^{\circ\mathrm{Fr}_r}$ given by $\rho^r = \rho_1 \boxtimes \cdots \boxtimes \rho_d$. The stabilizer of this representation in $\mathbb{Z}/m\mathbb{Z}$ is also $\mathbb{Z}/c\mathbb{Z}$. The irreducible representations $\widetilde{\rho}^r$ of $\mathbb{G}^{\mathrm{Fr}_r}$ whose restriction to $\mathbb{G}^{\circ\mathrm{Fr}_r}$ contain ρ^r are again in one-to-one correspondence to the characters of $\mathbb{Z}/c\mathbb{Z}$, and we parametrize them in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}})$ by the pairs $(\frac{rc}{m}, \sigma)$, $\sigma \in \widehat{\mathbb{Z}/c\mathbb{Z}}$. This completes the parametrization via $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/m\mathbb{Z}})$ of the unipotent representations for $\operatorname{Inn}\mathbb{G}$ corresponding to the family $\mathcal{U} = \{\rho\}$ in $\mathbb{G}^{\circ\mathrm{Fr}_0}$.

Example 5.6. Let $\mathbb{G}^\circ = \mathsf{SO}_{2n}$, $n \geq 2$, $A = \mathbb{Z}/2\mathbb{Z} = \langle\delta\rangle$. There are two inner forms $\operatorname{Inn}\mathbb{G} = \{\mathrm{O}_{2n}^+(\mathbb{F}_q), \mathrm{O}_{2n}^-(\mathbb{F}_q)\}$. In this case, we use the parametrizations of [Lu1, §4.6,§4.18]. Recall that symbol for type D_n is an array $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_b \\ \mu_1 & \mu_2 & \dots & \mu_{b'} \end{pmatrix}$, $b + b' = 2m$, $0 \leq \lambda_1 < \cdots < \lambda_{b'}$, $0 \leq \mu_1 < \cdots < \mu_{b'}$, which is considered the same as the array where the rows are flipped. A symbol where $b = b' = m$ and $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots$, $\sum \lambda_i^2 + \lambda\mu_i^2 = n + m^2 - m$ is called special.

Let Z be a special symbol. In the case when $\lambda_i = \mu_i$ for all $1 \leq i \leq m$, one attaches to Z two unipotent $\mathbb{G}^{\circ\mathrm{Fr}_0}$-families, $\mathcal{U}' = \{\rho\}$ and $\mathcal{U}'' = \{\rho'\}$, each consisting of a single unipotent representation and with $\Gamma_{\mathcal{U}'} = \Gamma_{\mathcal{U}''} = \{1\}$. In this case, the action of δ flips the two families. Hence they give rise to a single family $\{\widetilde{\rho}\}$ for $\mathbb{G}^{F_0}$, $\widetilde{\rho}|_{\mathbb{G}^{\circ\mathrm{Fr}_0}} = \rho \oplus \rho'$, and $\widetilde{\Gamma}_{\mathcal{U}'} = \{1\}$.

Assume now that the two rows of the symbol Z are not identical, i.e., Z is nondegenerate in the sense of *loc. cit.*. Then Z defines one unipotent family for $\mathbb{G}^{\circ\mathrm{Fr}_0}$ and one for $\mathbb{G}^{\circ\mathrm{Fr}_1}$. Each element of these families is stable under δ so it can be lifted to two different $\mathbb{G}^{\mathrm{Fr}_0}$, respectively $\mathbb{G}^{\mathrm{Fr}_1}$, representations.

The unipotent representations in the $\mathbb{G}^{\circ\mathrm{Fr}_0}$-family $\mathcal{U}_Z$ corresponding to Z are indexed by the set $\mathcal{M}_Z$ of symbols Λ such that $b - b' \equiv 0 \bmod 4$, $b + b' = 2m$. The unipotent representations in the $\mathbb{G}^{\circ\mathrm{Fr}_1}$-family $\mathcal{U}_Z^-$ corresponding to Z are indexed by the set $\mathcal{M}_Z^-$ of symbols Λ such that $b - b' \equiv 2 \bmod 4$, $b + b' = 2m$. Let Z_1 be the set of elements that appear as entries of Z only once. Let $2d = |Z_1|$. Define

- V_{Z_1}: the set of subsets $X \subseteq Z_1$ of even cardinality, with the structure of an $\mathbb{F}_2$-vector space with the sum given by the symmetric difference;
- V'_{Z_1}: the set of subsets $X \subseteq Z_1$ with the structure of an $\mathbb{F}_2$-vector space with the sum given by the symmetric difference, modulo the line spanned by Z_1 itself;
- $(V'_{Z_1})^+$: the subspace of V'_{Z_1} where the elements are the subsets X of even cardinality;
- $(V'_{Z_1})^-$: the subspace of V'_{Z_1} where the elements are the subsets X of odd cardinality.

Notice that $(V'_{Z_1})^+$ is also the image of the projection of V_{Z_1} to V'_{Z_1}. The dimensions of V'_{Z_1} and V_{Z_1} over $\mathbb{F}_2$ are $2d - 1$, while the dimension of (V'_{Z_1}) is $2d - 2$. There is a nonsingular pairing

$$(\ ,\ ) \colon V'_{Z_1} \times V_{Z_1} \to \mathbb{F}_2, \quad (X_1, X_2) \mapsto |X_1 \cap X_2| \bmod 2. \tag{5.17}$$

This pairing restricts to a nonsingular symplectic $\mathbb{F}_2$-form of $(V'_{Z_1})^+$. If V_{Z_1} has the basis $e_1, e_2, \dots, e_{2d-1}$ as in [Lu1], then $(V'_{Z_1})^+$ is spanned by the images $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2d-1}$ modulo

the relation $\bar{e}_1 + \bar{e}_3 + \cdots + \bar{e}_{2d-1} = 0$. Let

$$\bar{I}' = \text{subspace of } (V'_{Z_1})^+ \text{ spanned by } \bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{2d-1},$$
$$\bar{I}'' = \text{subspace of } (V'_{Z_1})^+ \text{ spanned by } \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{2d-2};$$

they are maximal isotropic subspaces of $(V'_{Z_1})^+$ and $(V'_{Z_1})^+ = \bar{I}' \oplus \bar{I}''$. Then

$$\Gamma_{\mathcal{U}_Z} = \bar{I}'' \cong (\mathbb{Z}/2\mathbb{Z})^{d-1}.$$

As shown in [Lu1, §4.6], there is a natural bijection

$$\mathcal{M}_Z \leftrightarrow \mathcal{M}(\Gamma_{\mathcal{U}_Z}) \cong (V'_{Z_1})^+ = \bar{I}' \oplus \bar{I}''$$

(where $\bar{I}'$ is identified with the group of characters of $\Gamma_{\mathcal{U}_Z}$). Denote

$$\widetilde{\Gamma}_{\mathcal{U}_Z} = \{v \in V'_{Z_1} \mid (v, e_{2i}) = 0,\ 1 \le i \le d-1\} \cong (\mathbb{Z}/2\mathbb{Z})^d. \tag{5.18}$$

Clearly, $\Gamma_{\mathcal{U}_Z} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}_Z}$. As shown in [Lu1, §4.18], there is a natural bijection

$$\mathcal{M}_Z^- \leftrightarrow \overline{\mathcal{M}}(\Gamma_{\mathcal{U}_Z} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}_Z}) \cong (V'_{Z_1})^- \cong (\widetilde{\Gamma}_{\mathcal{U}_Z} \setminus \Gamma_{\mathcal{U}_Z}) \times \bar{I}'.$$

Let $\widetilde{\mathcal{U}}_Z$ be the set (family) of irreducible representations in $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}_0} \sqcup \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}_1}$ whose restrictions to $\mathbb{G}^{\circ\mathrm{Fr}_0}$ (resp. $\mathbb{G}^{\circ\mathrm{Fr}_1}$) are in $\mathcal{U}_Z$ (resp. $\mathcal{U}_Z^-$). Since each unipotent representation in $\mathcal{U}_Z$ and $\mathcal{U}_Z^-$ extends in two different ways to the corresponding disconnected group, the parametrization above implies easily that there is natural bijection

$$\widetilde{\mathcal{U}}_Z \longleftrightarrow \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}_Z}). \tag{5.19}$$

Explicitly, let $\{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_d\}$ be the spanning set of V'_{Z_1} subject to $\sum_{i=1}^{2d} \bar{f}_i = 0$, such that $\bar{e}_i = \bar{f}_i + \bar{f}_{i+1}$, $1 \le i \le 2d-1$. Then an $\mathbb{F}_2$-basis of $\widetilde{\Gamma}_{\mathcal{U}_Z}$ is given by $\{\bar{f}_1, \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{2d}\}$. An irreducible character of $\Gamma_{\mathcal{U}_Z} = \langle \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{2d}\rangle$ can be extended in two different ways to $\widetilde{\Gamma}_{\mathcal{U}_Z}$ by setting the character value on $\bar{f}_1$ to 1 or -1. The value 1 corresponds to the representations of the identity components of $\mathbb{G}^{\mathrm{Fr}_0}$, $\mathbb{G}^{\mathrm{Fr}_1}$ extended by letting δ act trivially, while the -1 value to the ones where δ acts by -1.

Example 5.7. Let $\mathbb{G}^\circ$ be of type A_{k-1}, $k \ge 3$, or E_6, and $A = \mathbb{Z}/2\mathbb{Z} = \langle\delta\rangle$ acting by the nontrivial automorphism of the Dynkin diagram. The nonsplit form has $\mathbb{G}^{\circ\mathrm{Fr}_1}$ of type ${}^2A_{k-1}$ or ${}^2E_6$, respectively. By [Lu1, §4.19], every unipotent family $\mathcal{U}$ of $\mathbb{G}^\circ$ is fixed pointwise by A. Hence

$$\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/2\mathbb{Z}} = \Gamma_U \times \mathbb{Z}/2\mathbb{Z}, \text{ for all } \mathcal{U}.$$

Each irreducible representation ρ of the split form $\mathbb{G}^{\circ\mathrm{Fr}_0}$ can be extended to $\mathbb{G}^{\mathrm{Fr}_0}$ in two different ways $\rho^\pm$ corresponding to the two characters of $\mathbb{Z}/2\mathbb{Z}$. If the parameter for ρ is $\bar{x}_\rho = (x, \sigma) \in \mathcal{M}(\Gamma_{\mathcal{U}})$, then the parameters for $\rho^\pm$ are $((x, 1), \sigma^\pm)$ with the obvious notation.

Similarly, an irreducible representation ρ' of the nonsplit form $\mathbb{G}^{\circ\mathrm{Fr}_1}$ can be extended to $\mathbb{G}^{\mathrm{Fr}_1}$ in two different ways $\rho'^\pm$. If the parameter for ρ' is $\bar{x}_\rho = (x', \sigma') \in \mathcal{M}(\Gamma_{\mathcal{U}})$, then the parameters for $\rho'^\pm$ are $((x', \delta), \sigma'^\pm)$.

Example 5.8. Let $\mathbb{H}$ be of type D_k, $k \ge 2$ and $\mathbb{G}^\circ = \mathbb{H} \times \mathbb{H}$. Let $A = \langle\delta_1\rangle \times \langle\delta_2\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where δ_1 acts by the nontrivial outer automorphism of the Dynkin diagram of type D_k, and δ_2 flips the $\mathbb{H}$-factors. This case is therefore a combination of Example 5.6 and Example 5.4 and the parametrization of families for the inner forms of $\mathbb{G}$ follows from these examples, i.e., the same parametrizations as in Example 5.4 but constructed from the orthogonal families $\widetilde{U}_Z$ from Example 5.6.

Now for the same $\mathbb{H}$ and $\mathbb{G}^\circ$, suppose $A = \langle\delta\rangle = \mathbb{Z}/4$. If s'_1, s''_1 are the two commuting extremal reflections of the first $\mathbb{H} = D_k$ and s'_2, s''_2 are the similar reflections for the second $\mathbb{H} = D_k$, then δ acts by the cyclic permutation:

$$\delta\colon \quad s'_1 \mapsto s'_2 \mapsto s''_1 \mapsto s''_2 \mapsto s'_1.$$

On all the other simple reflections of the two components of type D_k, δ acts by the obvious diagram flip (of order 2). To describe the inner forms, let Fr denote the Frobenius map of $\mathbb{H}$ whose fixed points is the nonsplit group of type ${}^2D_k$. Then

$$\mathrm{Fr}_1 \colon \mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\mathbb{H}, \quad \mathrm{Fr}_1(h_1,h_2)=(\mathrm{Fr}(h_2),h_1)$$

is a Frobenius automorphism and $\mathrm{Fr}_r=\mathrm{Fr}_1^r$, $r\in\mathbb{Z}/4$, e.g., [GM, Example 1.4.23]. The identity components of the inner forms are the finite reductive groups of types:

$$\mathbb{G}^{\circ\mathrm{Fr}_0}:\ D_k\times D_k,\quad \mathbb{G}^{\circ\mathrm{Fr}_1}:\ {}^2D_k,\quad \mathbb{G}^{\circ\mathrm{Fr}_2}:\ {}^2D_k\times\ {}^2D_k,\quad \mathbb{G}^{\circ\mathrm{Fr}_3}:\ {}^2D_k.$$

If ρ_1,ρ_2 are two unipotent representations of D_k, the action of δ is

$$\delta(\rho_1,\rho_2)=(\rho_2',\rho_1),\ \text{where } \rho_2'=\begin{cases}\rho_2, & \text{if the symbol of } \rho_2 \text{ is nondegenerate,}\\ \rho_2^-, & \text{otherwise,}\end{cases}$$

where ρ_2^- is unipotent representation parametrized by the other degenerate symbol with the same rows. See Example 5.6.

We start with a family $\mathcal{U}_1\times\mathcal{U}_2$ of $\mathbb{G}^{\circ\mathrm{Fr}_0}=D_k\times D_k$. If either $\mathcal{U}_2$ consists of a degenerate symbol, then the stabilizer in A is always 1, regardless of what $\mathcal{U}_1$ is. (Similarly if $\mathcal{U}_1$ is degenerate.) In this case,

$$\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}_1\times\mathcal{U}_2}=\Gamma_{\mathcal{U}_1}.$$

(Recall that $\Gamma_{\mathcal{U}_2}=1$ necessarily.) Since the stabilizer in $\mathbb{Z}/4$ of each representation $\rho_1\boxtimes\rho_2$, $\rho_1\in\mathcal{U}_1$, $\rho_2\in\mathcal{U}_2$ is also trivial in this case, it follows that there is a one-to-one correspondence between the representations $\mathrm{Ind}_{\mathbb{G}^{\circ\mathrm{Fr}_0}}^{\mathbb{G}^{\mathrm{Fr}_0}}(\rho_1\boxtimes\rho_2)$ and $\rho_1\in\mathcal{U}_1$, hence a parametrization by $\mathcal{M}(\Gamma_{\mathcal{U}_1})$ as expected.

For the rest of the example, assume that all families correspond to nondegenerate symbols. Let Z_1, Z_2 be two nondegenerate symbols of type D_k. Let $\mathcal{U}_1,\mathcal{U}_2$ be the corresponding families for D_k and $\mathcal{U}_1^-$, $\mathcal{U}_2^-$ the families for ${}^2D_k$. Suppose first that $Z_1\neq Z_2$, then the stabilizer in A is $\mathbb{Z}/2\mathbb{Z}=\langle\delta^2\rangle$, hence the group for the inner forms is

$$\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}_1\boxtimes\mathcal{U}_2}=\Gamma_{\mathcal{U}_1}\times\Gamma_{\mathcal{U}_2}\times\mathbb{Z}/2\mathbb{Z}.$$

If $\rho_1\in\mathcal{U}_1$ and $\rho_2\in\mathcal{U}_2$, the stabilizer in A of $\rho_1\boxtimes\rho_2$ is also $\mathbb{Z}/2\mathbb{Z}=\langle\delta^2\rangle$. By Mackey theory, we get two irreducible representations of $\mathbb{G}^{\mathrm{Fr}_0}$ by inducing $\rho_1\boxtimes\rho_2$ twisted by the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$. If $\bar{x}_{\rho_i}=(x_i,\sigma_i)\in\mathcal{M}(\Gamma_{\mathcal{U}_i})$, $i=1,2$, then the two induced representations are parametrized by $((x_1,x_2,1),\sigma_1\boxtimes\sigma_2\boxtimes\tau)\in\mathcal{M}(\Gamma_{\mathcal{U}_1}\times\Gamma_{\mathcal{U}_2}\times\mathbb{Z}/2\mathbb{Z})$, where τ is the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$.

If $\rho_1'\in\mathcal{U}_1^-$ and $\rho_2'\in\mathcal{U}_2^-$, the analysis is analogous. The difference is that $\bar{x}_{\rho_i'}=(x_i',\sigma_i')\in\overline{\mathcal{M}}(\Gamma_{\mathcal{U}_i}\trianglelefteq\widetilde{\Gamma}_{\mathcal{U}_i})$, $i=1,2$, where $x_i'\in\widetilde{\Gamma}_{\mathcal{U}_i}\setminus\Gamma_{\mathcal{U}_i}$, $\sigma_i\in\widehat{\Gamma}_{\mathcal{U}_i}$. Write $x_i=y_i\alpha$, $i=1,2$, $y_i\in\Gamma_{\mathcal{U}_i}$, where α is the nontrivial automorphism of the D_k diagram. Then the two unipotent representations of $\mathbb{G}^{\mathrm{Fr}_2}$ whose restriction to $\mathbb{G}^{\circ\mathrm{Fr}_2}$ contain $\rho_1'\boxtimes\rho_2'$ are parametrized by $((y_1,y_2,\delta^2),\sigma_1\boxtimes\sigma_2\boxtimes\tau)\in\mathcal{M}(\Gamma_{\mathcal{U}_1}\times\Gamma_{\mathcal{U}_2}\times\mathbb{Z}/2\mathbb{Z})$, where τ is the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$.

Finally, if $Z_1=Z_2=Z$ with the families $\mathcal{U}$ of D_k and $\mathcal{U}^-$ of ${}^2D_k$, then the stabilizer of $\mathcal{U}\boxtimes\mathcal{U}$ is $A=\mathbb{Z}/4$. In this case, set

$$\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}=(\Gamma_{\mathcal{U}}\times\Gamma_{\mathcal{U}})\rtimes\mathbb{Z}/4. \tag{5.20}$$

All four inner forms $\mathbb{G}^{\mathrm{Fr}_r}$ contribute in this case. Each conjugacy class in $\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}$ is represented by an element (x,y,r), $x,y\in\Gamma_{\mathcal{U}}$, and some $r\in\mathbb{Z}/4$ and it will correspond to a unipotent representation of $\mathbb{G}^{\mathrm{Fr}_r}$, for the same r.

If $r=0$, then the conjugacy classes are given by $(x,x',0)\sim(x',x,0)$ and its stabilizer in $\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}$ is $\Gamma^2_{\mathcal{U}}\times\langle\delta^2\rangle$ if $x\neq x'$ or all $\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}$ if $x=x'$. If $\rho,\rho'\in\mathcal{U}$ with parameters $\bar{x}_\rho=(x,\sigma)$, $\bar{x}_{\rho'}=(x',\sigma')$, it is clear that there is a perfect matching between the Mackey

construction of induced representations and the parameters $((x,x',\bar{0}),\widetilde{\sigma}) \in \mathcal{M}(\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}})$, where $\widetilde{\sigma} \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x',\bar{0}))}$.

If $r=2$, the conjugacy classes are given by $(x,x',\delta^2) \sim (x',x,\delta^2)$ and its stabilizer in $\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}$ is $\Gamma^2_{\mathcal{U}} \times \langle \delta^2 \rangle$ if $x \neq x'$ or all $\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}$ if $x = x'$. If $\rho, \rho' \in \mathcal{U}^-$ with parameters $\bar{x}^-_\rho = (x\alpha,\sigma)$, $\bar{x}^-_{\rho'} = (x'\alpha,\sigma')$, again there is a perfect matching between the Mackey construction of induced representations and the parameters $((x,x',\delta^2),\widetilde{\sigma}) \in \mathcal{M}(\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}})$, where $\widetilde{\sigma} \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x',\bar{2}))}$.

If $r=1$ or 3, the conjugacy classes are given by $(x,1,\delta^r)$, cf. Example 5.4. The stabilizer in this case is $\langle \Gamma^\Delta_{\mathcal{U}}, \delta^2, (x,1,\delta)\rangle$. Let ρ be a representation in the unipotent family $\mathcal{U}^-$ with parameter $\bar{x}_\rho = (x\alpha,\sigma)$, $\sigma \in \widehat{\Gamma}_{\mathcal{U}}$. It can be extended in four different ways to ${}^2D_k \rtimes \mathbb{Z}/4$ corresponding to the characters of $\mathbb{Z}/4$. Since x has order 2, the cyclic group

$$\langle (x,1,\delta)\rangle = \langle (1,1,1),(x,1,\delta),(x,x,\delta^2),(1,x,\delta^3)\rangle \cong \mathbb{Z}/4.$$

Notice that there is a short exact sequence (which does not split)

$$1 \longrightarrow \Gamma^\Delta_{\mathcal{U}} \longrightarrow \mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1,\delta^r)) \longrightarrow \mathbb{Z}/4 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/4$ is generated by the image of $(x,1,\delta)$. This means that $\sigma \in \widehat{\Gamma}_{\mathcal{U}}$, viewed as a representation of $\Gamma^\Delta_{\mathcal{U}}$ can be lifted in four different ways to $\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1,\delta^r))$: first one lifts σ in two different ways to $\sigma^\pm$, representations of $\Gamma^\Delta_{\mathcal{U}} \times \langle \delta^2\rangle$ corresponding to the trivial and the sign character of $\langle \delta^2\rangle$. Then, fixing a square roots $\zeta^\pm$ of $\sigma^\pm(x,x,\delta^2)$, one constructs lifts $\widetilde{\sigma}^i$ of σ, $0 \leq i \leq 3$, by setting

$$\widetilde{\sigma}^0((x,1,\delta)) = \zeta^+,\ \widetilde{\sigma}^1((x,1,\delta)) = -\zeta^+,\ \widetilde{\sigma}^2((x,1,\delta)) = \zeta^-,\ \widetilde{\sigma}^3((x,1,\delta)) = -\zeta^-.$$

Notice that $\{\pm\zeta^\pm\}$ is the set of 4-th roots of 1, and this gives the desired parametrization.

Example 5.9. Let $\mathbb{G}^\circ$ be of type D_4 and $A = \mathbb{Z}/3\mathbb{Z} = \langle \delta\rangle$ acting on the Dynkin diagram by cyclically permuting the extremal nodes. The Weyl group $W(D_4)$ has 13 irreducible representations which we denote by bipartitions of 4, $\alpha \times \beta$ up to swapping α and β, except where $\alpha = \beta$, there are two non-isomorphic representations $\alpha \times \alpha^\pm$. There is one cuspidal unipotent representation ρ_c, and in total 14 unipotent representations of $\mathbb{G}^{\circ\mathrm{Fr}_0}$.

All families are singletons with associated finite group $\Gamma_{\mathcal{U}} = \{1\}$, except the family

$$\{(12)\times(1),(22)\times\emptyset,(11)\times(2),\rho_c\}$$

for which the finite group is $\Gamma_{\mathcal{U}} = \mathbb{Z}/2\mathbb{Z}$. This family and the following four singleton families:

$$\{(4)\times\emptyset\},\quad \{(1111)\times\emptyset\},\quad \{(3)\times(1)\},\quad \{(111)\times(1)\}$$

are A-stable and in fact each element in the family is A-stable. The remaining 6 unipotent (singleton) families form two A-orbits:

$$\{(13)\times\emptyset,(2)\times(2)^+,(2)\times(2)^-\} \text{ and } \{(112)\times\emptyset,(11)\times(11)^+,(11)\times(11)^-\}.$$

According to our recipe, the groups $\widetilde{\Gamma}^{\mathbb{Z}/3\mathbb{Z}}_{\mathcal{U}}$ are:

- $\mathbb{Z}/3\mathbb{Z}$ corresponding to each of the four A-stable singleton families $\mathcal{U}$;
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for the unique family with 4 elements;
- $\{1\}$ for each one of the two nontrivial A-orbits.

Hence the right hand side of Proposition 5.3 is $\mathcal{M}(\mathbb{Z}/3\mathbb{Z})^4 \sqcup \mathcal{M}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}) \sqcup \mathcal{M}(\{1\})^2$ which has $3^2\times 4 + 6^2 + 1^2 \times 2 = 74$ elements.

The irreducible unipotent representations of the disconnected group $D_4 \rtimes \mathbb{Z}/3\mathbb{Z}$ are parametrized, via Mackey theory, by the elements $(x,\sigma) \in \mathcal{M}(\widetilde{\Gamma}^{\mathbb{Z}/3\mathbb{Z}}_{\mathcal{U}})$, where $x \in \Gamma_{\mathcal{U}}$ and $\mathcal{U}$ ranges over a set of representatives of the A-orbits of families of D_4. There are 26 such irreducible representations.

The other two A-forms corresponding to δ and δ^{-1} are both isomorphic to the finite group of type ${}^3D_4$. There are 8 irreducible unipotent representations of ${}^3D_4$ each coming from one of γ-stable irreducible unipotent representations of D_4. By induction, there are $8 \times 3 = 24$ irreducible unipotent representations of ${}^3D_4 \rtimes \mathbb{Z}/3\mathbb{Z}$. The irreducible representations of the ${}^3D_4$ corresponding to δ are parametrized by $(x\delta, \sigma) \in \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/3\mathbb{Z}})$, where $x \in \Gamma_{\mathcal{U}}$ and $\mathcal{U}$ ranges over the set of A-stable families. Similarly for δ^{-1}.

6. Maximal compact subgroups

We return to the setting of Section 2, so $\mathbf{G}$ is a connected reductive group over F and $G = \mathbf{G}(F)$. In this section, we assume in addition that $\mathbf{G}$ is simple and F-split with maximal F-split torus $\mathbf{S}$. Let Π_G be a set of simple roots for $\mathbf{G}$ with respect to $\mathbf{S}$, and extend Π_G to a set of simple affine roots $\Pi_G^a = \Pi_G \cup \{\alpha_0\}$. Let $I \subset \mathbf{G}(\mathfrak{o}_F)$ be the corresponding Iwahori subgroup of G, with $\mathbf{S}(\mathfrak{o}_F) = \mathbf{S}(F) \cap I$. Let $\widetilde{W}_G = \mathrm{N}_G(\mathbf{S}(F))/\mathbf{S}(\mathfrak{o}_F)$ be the Iwahori–Weyl group:

$$G = \bigsqcup_{w \in \widetilde{W}_G} I\dot{w}I,$$

where $\dot{w}$ denotes a choice of a lift in $N_G(\mathbf{S}(F))$ of $w \in \widetilde{W}_G$. The finite Weyl group is $W_G = \mathrm{N}_G(\mathbf{S}(F))/\mathbf{S}(F)$. Let W_G^a be the affine Weyl group generated by the simple reflections $\{s_i \mid i \in \Pi_G^a\}$. Then

$$\widetilde{W}_G = W_G^a \rtimes \Omega_G,$$

where Ω_G is a finite abelian group, the stabilizer in $\widetilde{W}_G$ of I.

6.1. Let $\max(G)$ denote the set of conjugacy classes of maximal compact open subgroups in G. To parametrize $\max(G)$, we define $S_{\max}(G)$ to be the set of Ω_G-orbits of pairs $(A, \mathcal{O})$, where A is a subgroup of Ω_G and $\mathcal{O}$ is an A-orbit in Π_G^a satisfying

$$\mathrm{Stab}_{\Omega_G}(\mathcal{O}) = A.$$

By [IM] and [BT], $\max(G)$ is parametrized by $S_{\max}(G)$. Explicitly, given $(A, \mathcal{O}) \in S_{\max}(G)$, we construct an element $K_{\mathcal{O}} \in \max(G)$ as follows: let $\widetilde{W}_{\mathcal{O}}$ be the finite subgroup of $\widetilde{W}_G$, generated by A and $\{s_i,\ i \in \Pi_G^a \setminus \mathcal{O}\}$. Set

$$K_{\mathcal{O}} = \bigsqcup_{w \in \widetilde{W}_{\mathcal{O}}} I\dot{w}I.$$

The map $(A, \mathcal{O}) \mapsto K_{\mathcal{O}}$ defines a bijection between $S_{\max}(G)$ and $\max(G)$. (Note that a pair $(A, \mathcal{O}) \in S_{\max}(G)$ is completely determined by $\mathcal{O}$, so the notation $K_{\mathcal{O}}$ is unambiguous.) In this notation, the maximal hyperspecial subgroup $\mathbf{G}(\mathfrak{o}_F)$ is $K_{\{\alpha_0\}}$, where α_0, as defined above, is the unique simple affine root in $\Pi_G^a \setminus \Pi_G$.

Given $(A, \mathcal{O}) \in S_{\max}(G)$, let $\widetilde{W}_{\mathcal{O}}^{\circ}$ be the normal subgroup of $\widetilde{W}_{\mathcal{O}}$ generated by $\{s_i,\ i \in \Pi_a \setminus \mathcal{O}\}$. Then $K_{\mathcal{O}}^{\circ} := \bigsqcup_{w \in \widetilde{W}_{\mathcal{O}}^{\circ}} I\dot{w}I$ is a parahoric subgroup of G, and we denote by $K_{\mathcal{O}}^{+}$ its pro-unipotent radical. There is a short exact sequence

$$1 \longrightarrow K_{\mathcal{O}}^{\circ} \longrightarrow K_{\mathcal{O}} \longrightarrow A \longrightarrow 1.$$

Set $\overline{K}_{\mathcal{O}}^{\circ} = K_{\mathcal{O}}^{\circ}/K_{\mathcal{O}}^{+}$ and $\overline{K}_{\mathcal{O}} = K_{\mathcal{O}}/K_{\mathcal{O}}^{+}$. Then $\overline{K}_{\mathcal{O}} = M_{\mathcal{O}}(k_F)$, $\overline{K}_{\mathcal{O}}^{\circ} = M_{\mathcal{O}}^{\circ}(k_F)$, for a reductive k_F-split group $M_{\mathcal{O}}$ with identity component $M_{\mathcal{O}}^{\circ}$ and $M_{\mathcal{O}}/M_{\mathcal{O}}^{\circ} \cong A$. Let $\mathrm{Inn}\,\overline{K}_{\mathcal{O}} \leftrightarrow A$ be the collection of inner forms of $M_{\mathcal{O}}$.

Now let G' be a pure inner twist of G. By Section 2, $\mathrm{InnT}^p G \cong \mathrm{H}^1(F, \mathbf{G})$. If $\mathbf{G}_{\mathrm{sc}}$ is the simply connected cover of $\mathbf{G}$ and identifying Ω_G with the kernel of the surjection $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$, then by [Kn, Satz 2], $\mathrm{H}^1(F, \mathbf{G}) \cong \mathrm{H}^2(F, \Omega_G) \cong \Omega_G$, with the last equivalence because $\mathbf{G}$ is F-split. Given $x \in \Omega_G$, let $G_x \in \mathrm{InnT}^p G$ be the corresponding pure inner twist. If we

denote the set of conjugacy classes of maximal compact open subgroups of G_x by $\max(G_x)$, then there is a one-to-one correspondence

$$\{(A,\mathcal{O}) \in S_{\max}(G) \mid x \in A\} \longleftrightarrow \max(G_x), \tag{6.1}$$

which we write as $(A,\mathcal{O}) \mapsto K_{x,\mathcal{O}}$. Note that $\overline{K}_{x,\mathcal{O}} \in \operatorname{Inn}\overline{K}_{\mathcal{O}}$ is the inner form given by $x \in A \cong \mathrm{H}^1(k_F, M_{\mathcal{O}}) \leftrightarrow \operatorname{Inn}\overline{K}_{\mathcal{O}}$ (see Definition 5.2). For fixed $(A,\mathcal{O}) \in S_{\max}(G)$, we have

$$\bigcup_{x\in A} \overline{K}_{x,\mathcal{O}} = \operatorname{Inn}\overline{K}_{\mathcal{O}}. \tag{6.2}$$

For every maximal compact open subgroup $K' \in \max(G')$, define $R_{\mathrm{un}}(\overline{K}'_{\mathcal{O}})$ to be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}\overline{K}'$, and let

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{G' \in \operatorname{Inn}\mathrm{T}^p G} \bigoplus_{K' \in \max(G')} R_{\mathrm{un}}(\overline{K}'). \tag{6.3}$$

By the discussion above, we have

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{x\in\Omega} \bigoplus_{\substack{(A,\mathcal{O})\in S_{\max}(G)\\ \text{with } x\in A}} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}}) \tag{6.4}$$

$$= \bigoplus_{(A,\mathcal{O})\in S_{\max}(G)} \bigoplus_{x\in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}}). \tag{6.5}$$

Note that for each $(A,\mathcal{O}) \in S_{\max}(G)$, by (6.2), $\oplus_{x\in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ has the involution $\mathrm{FT}_{\overline{K}_{\mathcal{O}}}$ given by (5.16). Putting together these involutions for all choices of $(A,\mathcal{O})$ gives the following definition.

Definition 6.1. Let $\mathrm{FT}_{\mathrm{cpt,un}} = \bigoplus_{(A,\mathcal{O})\in S_{\max}(G)} \mathrm{FT}_{\overline{K}_{\mathcal{O}}}$ be the involution on $\mathcal{C}(G)_{\mathrm{cpt,un}}$ defined by using (5.16) and (6.2).

Notice that $\mathrm{FT}_{\mathrm{cpt,un}}$ always preserves the space $R_{\mathrm{un}}(\overline{\mathbf{G}}(\mathfrak{o}_F))$, since $\mathbf{G}(\mathfrak{o}_F)$ corresponds to the pair $(A,\{\alpha_0\})$ with A trivial. In the case when $\mathbf{G}$ is simply connected, we have $\Omega_G = 1$ and $K_{\mathcal{O}} = K_{\mathcal{O}}^{\circ}$ for all $\mathcal{O}$, so $\mathrm{FT}_{\mathrm{cpt,un}}$ preserves the space $R_{\mathrm{un}}(\overline{K}_{\mathcal{O}})$ for all $K_{\mathcal{O}} \in \max(G)$. But in general it does not preserve $R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ for every maximal compact open subgroup $K_{x,\mathcal{O}}$, which can be seen even in the case when $\mathbf{G} = \mathsf{PGL}_2$ (see Example 12.3).

Example 6.2. We list the type of groups $\overline{K}_{\mathcal{O}}$ in the case when $\mathbf{G}$ is adjoint. Since are only interested in the unipotent representations, only the Lie type of $\overline{K}_{\mathcal{O}}^{\circ}$ is important.

(1) If $\mathbf{G}$ is also simply connected, which is the case for types G_2, F_4, E_8, the set $\max(G)$ is in one-to-one correspondence with the maximal subsets of Π_a (equivalently, $\mathcal{O}$ is a single vertex in Π_G^a).

(2) If $\mathbf{G} = \mathsf{PGL}_n$, $\Omega_G = \mathbb{Z}/n$ acting by cyclically permuting Π_G^a, then for every divisor m of n, we have an orbit $\mathcal{O}_m$ in Π_G^a with stabilizer $A = \mathbb{Z}/m\mathbb{Z}$ and

$$\overline{K}_{\mathcal{O}_m} = P(\mathsf{GL}_{n/m}^m) \rtimes \mathbb{Z}/m\mathbb{Z},$$

where the semidirect product is given by the permutation action. This case corresponds to Example 5.5.

(3) If $\mathbf{G} = \mathsf{SO}_{2n+1}$, $\Omega_G = \mathbb{Z}/2\mathbb{Z}$, then $\overline{K}_{\mathcal{O}}$ is either SO_{2n+1} ($A = 1$) or $\mathsf{SO}_{2m+1}\times \mathrm{O}_{2(n-m)}$ ($A = \mathbb{Z}/2\mathbb{Z}$), for $0 \le m < n$. This case corresponds to Example 5.6.

(4) If $\mathbf{G} = \mathrm{PSp}_{2n}$, $\Omega = \mathbb{Z}/2\mathbb{Z}$, then $\overline{K}_{\mathcal{O}}$ is type
- $C_k \times C_\ell$ ($A = 1$) for $0 \le k < \ell$, $k + \ell = n$;
- $(C_k \times C_k) \rtimes \mathbb{Z}/2\mathbb{Z}$ ($A = \mathbb{Z}/2\mathbb{Z}$), if $n = 2k$, or
- $(C_i \times C_i \times A_{n-1-2i}) \rtimes \mathbb{Z}/2$ ($A = \mathbb{Z}/2\mathbb{Z}$), for $0 \le i < \frac{n}{2}$.

Here $\mathbb{Z}/2\mathbb{Z}$ acts by flipping the two type C factors and by the nontrivial diagram automorphism on the type A factor. These cases are covered by Examples 5.4,5.7.

(5) If $\mathbf{G} = \mathrm{PSO}_{4m}$, $m \geq 2$, $\Omega_G = \langle \delta_1 \rangle \times \langle \delta_2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then δ_1 acts by flipping Π_a horizontally and δ_2 by the vertical flip. For each subgroup $A \leq \Omega_G$, we give the possible $\overline{K}^\circ_{\mathcal{O}}$.
   (a) If $A = \Omega$, $\overline{K}^\circ_{\mathcal{O}}$ is of type: $D_k \times D_k \times A_{2m-2k-1}$, $2 \leq k < m$, $D_m \times D_m$, or A_{2m-3}, where A acts on $D_\ell \times D_\ell$ as in Example 5.8, while on $A_{2m-2k-1}$ δ_1 acts trivially, and δ_2 by the nontrivial diagram automorphism.
   (b) If $A = \langle \delta_1 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type: $D_k \times D_{2m-k}$, $2 \leq k < m$, or D_{2m-1}, where δ_1 acts by the nontrivial automorphism of type D_ℓ. These cases are covered by Example 5.6.
   (c) If $A = \langle \delta_2 \rangle$ or $A = \langle \delta_1\delta_2 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type A_{2m-1} with the diagram automorphism action as in Example 5.7.
   (d) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is the hyperspecial subgroup of type D_{2m-1}.

(6) If $\mathbf{G} = \mathrm{PSO}_{4m+2}$, $m \geq 2$, $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/4$:
   (a) If $A = \Omega_G$, $\overline{K}^\circ_{\mathcal{O}}$ is of type: $D_k \times D_k \times A_{2m-2k}$, $2 \leq k \leq m$, or A_{2m-2}, where δ acts on $D_\ell \times D_\ell$ as in Example 5.8, while on A_{2m-2k}, δ acts by the nontrivial diagram automorphism.
   (b) If $A = \langle \delta^2 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type: D_{2m} or $D_k \times D_{2m-k+1}$, $2 \leq k \leq m$, where δ^2 acts on each factor by the nontrivial automorphism of type D_ℓ, as in Example 5.6.
   (c) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is the hyperspecial subgroup of type D_{2m}.

(7) Let $\mathbf{G}$ be of type E_6 with $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/3\mathbb{Z}$.
   (a) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is either of type E_6 or $A_5 \times A_1$.
   (b) If $A = \Omega_G$, then $\overline{K}^\circ_{\mathcal{O}}$ is of type A_2^3 with δ acting by permutation, as in Example 5.5, $A_1^3 \times A_1$, where δ permutes the first 3 factors and it fixes the last one, or D_4, where δ acts as in Example 5.9.

(8) Let $\mathbf{G}$ be of type E_7 with $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/2\mathbb{Z}$.
   (a) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is either of type E_7, $D_6 \times A_1$, or $A_5 \times A_2$.
   (b) If $A = \Omega_G$, then $\overline{K}^\circ_{\mathcal{O}}$ is of type E_6 with δ acting by the nontrivial diagram automorphism as in Example 5.7, $D_4 \times A_1^2$, with δ acting by an order 2 diagram automorphism of D_4 (Example 5.6) and by flipping the two A_1's, $A_2^2 \times A_2$, flipping the first A_2's and trivially on the third A_2, $A_3^2 \times A_1$, with the flip on A_3^2 and trivial action on A_1, or A_7 with the nontrivial diagram automorphism, see Examples 5.4,5.7.

7. Elliptic and compact pairs

7.1. **Finite groups.** Suppose H is a finite group. If (δ, V_δ) is a finite-dimensional H-representation over $\mathbb{C}$, define

$$(f, f')^\delta_{\mathrm{ell}} = \frac{1}{|H|} \sum_{h \in H} \det_{V_\delta}(1 - \delta(h)) f(h^{-1}) f'(h), \tag{7.1}$$

for every two functions $f, f' \colon H \to \mathbb{C}$. For $\rho, \rho' \in R(H)$, set

$$(\rho, \rho')^\delta_{\mathrm{ell}} = (\chi_\rho, \chi_{\rho'})^\delta_{\mathrm{ell}},$$

where χ_ρ, $\chi_{\rho'}$ denote the corresponding characters. The basic facts about $(\ ,\ )^\delta_H$ can be found in [Re3, §2]. An element $h \in H$ is called (δ-)elliptic if $V_\delta^{\delta(h)} = 0$. The set H_{ell} of elliptic elements of H is obviously closed under conjugation by H and let $H \backslash H_{\mathrm{ell}}$ denote the set of elliptic conjugacy classes. Let $\overline{R}(H)$ be the quotient of $R(H)$ by the radical of the form $(\ ,\ )^\delta_{\mathrm{ell}}$. As in *loc. cit.*, there is a natural identification of $\overline{R}(H)$ with the space of class functions of H supported on H_{ell}.

For every $h \in H$, let $\mathbf{1}_h$ denote the characteristic function of the conjugacy class of H. Clearly, $\{\mathbf{1}_h \mid h \in H \backslash H_{\mathrm{ell}}\}$ is an orthogonal basis of $\overline{R}(H)$ with respect to the elliptic pairing $(\ ,\ )^\delta_{\mathrm{ell}}$.

Suppose that, in addition, we are given $\theta : H \to H$ an automorphism. Let $\langle\theta\rangle$ denote the cyclic group generated by θ and $H' = H \rtimes \langle\theta\rangle$. If (δ, V_δ) is a finite-dimensional complex H'-representation, define

$$(f, f')^\delta_{\theta-\mathrm{ell}} = \frac{1}{|H|}\sum_{h\in H} \det_{V_\delta}(1-\delta(\theta h))f((\theta h)^{-1})f'(\theta h), \tag{7.2}$$

for every two functions $f, f' : H' \to \mathbb{C}$. For $\rho, \rho' \in R(H')$, set

$$(\rho, \rho')^\delta_{\theta-\mathrm{ell}} = (\chi_\rho, \chi_{\rho'})^\delta_{\theta-\mathrm{ell}},$$

where χ_ρ, $\chi_{\rho'}$ denote the corresponding characters.

7.2. **Complex reductive groups.** Let $\mathcal{G}$ be a possibly disconnected complex reductive group with identity component $\mathcal{G}^\circ$. If $x \in \mathcal{G}^\circ$ is given, fix a Borel subgroup B_x of $\mathrm{Z}_{\mathcal{G}}(x)^\circ$ and a maximal torus T_x in B_x. Let $\mathfrak{t}_x$ be the complex Lie algebra of T_x. As in [Re3, Wa2], we define a complex representation $(\delta_x, \mathfrak{t}_x)$ of $A_{\mathcal{G}}(x)$ as follows. Since $A_{\mathcal{G}}(x)$ acts on $\mathrm{Z}_{\mathcal{G}}(x)^\circ$ by the adjoint action, every element $z \in A_{\mathcal{G}}(x)$ acts on $\mathrm{Z}_{\mathcal{G}}(x)^\circ$ via an automorphism α_z. There exists $y \in \mathrm{Z}_{\mathcal{G}}(x)^\circ$ such that $\alpha_z \circ \mathrm{Ad}(y)$ preserves B_x and T_x. This means that $\alpha_z \circ \mathrm{Ad}(y)$ defines an automorphism of the cocharacter lattice $X_*(T_x)$ in $\mathrm{Z}_{\mathcal{G}}(x)^\circ$, and therefore a linear isomorphism of $\mathfrak{t}_x$. This is $\delta_x(z)$, and this construction gives a representation of $A_{\mathcal{G}}(x)$. We consider the elliptic theory of the finite group $A_{\mathcal{G}}(x)$ with respect to the representation δ_x.

An element $g \in \mathcal{G}$ is called *elliptic* if the centralizer $\mathrm{Z}_{\mathcal{G}}(g)$ contains no nontrivial torus.

7.3. **Definitions.** Suppose Γ is a (possibly disconnected) complex reductive group with identity component Γ°. Extending the definition in Section 5.1, we define the sets (cf. [Ci, Def. 1.1])

$$\begin{aligned} \mathcal{Y}(\Gamma) &= \{(s,h) \in \Gamma\times\Gamma \mid s,h \text{ semisimple},\ sh = hs\},\\ \mathcal{Y}(\Gamma)_{\mathrm{ell}} &= \{(s,h) \in \Gamma\times\Gamma \mid s,h \text{ semisimple},\ sh = hs,\ \mathrm{Z}_\Gamma(s,h) \text{ is finite}\}. \end{aligned} \tag{7.3}$$

Here $\mathrm{Z}_\Gamma(s,h) = \mathrm{Z}_\Gamma(s)\cap\mathrm{Z}_\Gamma(h)$ and the finiteness condition is equivalent to saying that no nontrivial torus in Γ centralizes both s and h. We refer to elements of $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ as *elliptic pairs.* Notice that the condition in $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ is equivalent to saying that h is elliptic in $\mathrm{Z}_\Gamma(s)$ or equivalently s is elliptic in $\mathrm{Z}_\Gamma(h)$.

The sets $\mathcal{Y}(\Gamma), \mathcal{Y}(\Gamma)_{\mathrm{ell}}$ have Γ-actions via conjugation $g\cdot(s,h) = (gsg^{-1}, ghg^{-1})$. They also have a natural Γ-equivariant involution given by the flip

$$(s,h) \mapsto (h,s).$$

Let $\Gamma\backslash\mathcal{Y}(\Gamma)$, $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ be the sets of Γ-orbits. Then we get an involution

$$\mathsf{flip}\colon \Gamma\backslash\mathcal{Y}(\Gamma) \to \Gamma\backslash\mathcal{Y}(\Gamma),\ \mathsf{flip}([(s,h)]) = [(h,s)],$$

which preserves $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$.

Lemma 7.1. $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ *is a finite set.*

Proof. If $(s,h) \in \mathcal{Y}(\Gamma)_{\mathrm{ell}}$. The cyclic group $\langle s\rangle$ is in $\mathrm{Z}_\Gamma(s,h)$, hence s has finite order. Moreover, s must be isolated in Γ in the sense that $\mathrm{Z}_\Gamma(s)$ is semisimple. The classification of isolated semisimple automorphisms of Γ is well known, in particular, there are finitely many automorphisms up to inner conjugation. □

Define the relations on $\mathcal{Y}(\Gamma)$:

$$\begin{aligned} &(s,h) \sim_L (s,h') \text{ if } \gamma h\gamma^{-1} \in h'T \text{ for some } \gamma \in \mathrm{Z}_\Gamma(s) \text{ and } T \text{ a maximal torus in } \mathrm{Z}_\Gamma(s,h);\\ &(s,h) \sim_R (s',h) \text{ if } \gamma_1 s\gamma_1^{-1} \in s'T \text{ for some } \gamma_1 \in \mathrm{Z}_\Gamma(h) \text{ and } T \text{ as before.} \end{aligned} \tag{7.4}$$

Let $\sim$ be the equivalence relation on $\mathcal{Y}(\Gamma)$ generated by $\sim_L$ and $\sim_R$. Denote also by $\sim$ the equivalence relation induced on $\Gamma\backslash\mathcal{Y}(\Gamma)$. The subsets $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{cpt}}$ and $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ are closed under $\sim$-equivalence and notice that

$$\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}/_{\sim} = \Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}.$$

Lemma 7.2. *Fix $s \in \Gamma$ semisimple. The projection map $\mathrm{Z}_\Gamma(s) \to A_\Gamma(s)$, $h \mapsto \bar{h}$, induces:*

(1) a bijection between $\sim_L$-classes of pairs $(s,h) \in \mathcal{Y}(\Gamma)$ and conjugacy classes in $A_\Gamma(s)$;
(2) a bijection between $\mathrm{Z}_\Gamma(s)$-orbits of elliptic pairs (s,h) and the elliptic conjugacy classes in $A_\Gamma(s)$.

Proof. We need a result from the theory of semisimple automorphisms of reductive groups, e.g. [Som, Proposition 9]: if x, y are semisimple elements in a reductive group $\mathcal{G}$ such that their images in the group of components $\mathcal{G}/\mathcal{G}^\circ$ are in the same conjugacy class, and S is a maximal torus in $\mathrm{Z}_{\mathcal{G}}(x)$, then there exist $g \in \mathcal{G}$ and $s \in S$ such that $gyg^{-1} = xs$.

Apply this to $\mathcal{G} = \mathrm{Z}_\Gamma(s)$ (a reductive group), then (1) follows immediately (this is in fact the motivation for the definition $\sim$).

Now suppose h, h' semisimple elements such that (s,h) and (s,h') are elliptic pairs. The elliptic condition implies that the maximal torus in $\mathrm{Z}_{\mathcal{G}}(h)$ is trivial, hence $s = 1$ in the relation above, and h and h' are $\mathcal{G}$-conjugate. This implies that if $\overline{h} = \overline{h'}$ then $[(s,h)] = [(s,h')]$.

To prove (2), it remains to show that (s,h) is an elliptic pair if and only if $\bar{h}$ is elliptic in $A_\Gamma(s)$. This is just a matter of checking the definitions in the case $\mathcal{G} = \mathrm{Z}_\Gamma(s)$. Given the semisimple element $h \in \mathcal{G}$, choose a maximal torus T_s in $\mathcal{G}$ which is normalized by h. Then h is not elliptic if and only if there exists a nontrivial torus $S \subset T_s$ which centralizes h, equivalently if and only if $\delta_s(\bar{h})$ fixes a nonzero element of $\mathfrak{t}_s$, i.e., if $\overline{h}$ is not elliptic in $A_\Gamma(s)$.

□

Remark 7.3. In every $\sim$-equivalence class of $\mathcal{Y}(\Gamma_u)$, we may choose (not uniquely) a representative (s,h) such that both s, h have finite order. We call these representatives *compact pairs*.

For every $(s,h) \in \mathcal{Y}(\Gamma)$, define

$$\Pi(s,h) = \sum_{\phi \in \widehat{A_\Gamma(s)}} \phi(h)\phi \in R(A_\Gamma(s)), \tag{7.5}$$

and let $\overline{\Pi}(s,h)$ denote the image in $\overline{R}(A_\Gamma(s))$. Here $\phi(h)$ is interpreted as $\phi(\bar{h})$ where $\bar{h}$ is the image of h in $A_\Gamma(s)$. Let $\mathbb{C}[\mathcal{Y}(\Gamma)_{\mathrm{ell}}]^\Gamma$ denote the Γ-equivariant functions on $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$; this space can be identified with $\mathbb{C}[\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}]$. Let $\mathbf{1}_{[(s,h)]}$ denote the characteristic function of the Γ-orbit of (s,h).

Proposition 7.4. *The correspondence $\mathbf{1}_{[(s,h)]} \mapsto \overline{\Pi}(s,h)$ induce an isomorphism*

$$\mathbb{C}[\mathcal{Y}(\Gamma)_{\mathrm{ell}}]^\Gamma \cong \bigoplus_{s \in \mathcal{C}(\Gamma)_{\mathrm{ss}}} \overline{R}(A_\Gamma(s)).$$

Proof. In light of Lemma 7.2, the only thing left is to remark that $\overline{\Pi}(s,h)$ forms a basis of $\overline{R}(A_\Gamma(s))$ as h ranges over a set of representatives of $\mathrm{Z}_\Gamma(s)$-conjugacy classes such that (s,h) is an elliptic pair. It is elementary that in $R(A_\Gamma(s))$,

$$\overline{\Pi}(s,h) = |\mathrm{Z}_{A_\Gamma(s)}(\overline{h})|\ \mathbf{1}_{\overline{h}^{-1}},$$

and the claim follows. □

We say that $\Gamma_M \subset \Gamma$ is a *Levi subgroup* if there exists a torus $S \subset \Gamma^\circ$ such that $\Gamma_M = \mathrm{Z}_\Gamma(S)$. If a pair (s,h) is in $\mathcal{Y}(\Gamma_M)$, denote by $\Pi^{\Gamma_M}(s,h)$ the combination defined analogous to (7.5).

Lemma 7.5. *Suppose $s \in \Gamma_M$ is semisimple.*

(1) The inclusion $\mathrm{Z}_{\Gamma_M}(s) \to \mathrm{Z}_\Gamma(s)$ induces an inclusion $A_{\Gamma_M}(s) \to A_\Gamma(s)$.

(2) For every $(s,h) \in \mathcal{Y}(\Gamma_M)$, $\mathrm{Ind}_{A_{\Gamma_M}(s)}^{A_\Gamma(s)} \Pi^{\Gamma_M}(s,h) = \Pi(s,h)$.

Proof. (1) This is a well-known argument. We need to show that $\mathrm{Z}_{\Gamma_M}(s) \cap \mathrm{Z}_\Gamma(s)^\circ$ is connected, and hence in $\mathrm{Z}_{\Gamma_M}(s)^\circ$. But $\mathrm{Z}_{\Gamma_M}(s) \cap \mathrm{Z}_\Gamma(s)^\circ = \Gamma_M \cap \mathrm{Z}_\Gamma(s)^\circ = \mathrm{Z}_\Gamma(S) \cap \mathrm{Z}_\Gamma(s)^\circ = \mathrm{Z}_{\mathrm{Z}_\Gamma(s)^\circ}(S)$ which is connected since the centralizer of any torus in a connected reductive group is connected.

(2) This is elementary using that $\phi(\overline{h}) = \sum_{\psi \in \widehat{A_{\Gamma_M}(s)}} \langle \phi, \psi \rangle_{A_{\Gamma_M}(s)} \psi(\overline{h})$ for every $\phi \in \widehat{A_\Gamma(s)}$, by restriction of characters. □

Lemma 7.6. *Let $(s,h) \in \mathcal{Y}(\Gamma)$ be given and suppose S is a maximal torus in $\mathrm{Z}_\Gamma(s,h)^\circ$. Set $\Gamma_M = \mathrm{Z}_\Gamma(S)$. Then $\mathrm{Z}_{\Gamma_M}(s,h)^\circ = \mathrm{Z}^\circ_{\Gamma_M}$, i.e., (s,h) is an elliptic pair in $\Gamma_M/\mathrm{Z}^\circ_{\Gamma_M}$.*

Proof. Let S_1 be a torus in $\mathrm{Z}_{\Gamma_M}(s,h)^\circ$. Then $S_1 \subset \mathrm{Z}_\Gamma(s,h)^\circ$ and since it commutes with S which is maximal in $\mathrm{Z}_\Gamma(s,h)^\circ$, it follows that $S_1 \subset S \subset \mathrm{Z}^\circ_{\Gamma_M}$. □

Let $\mathcal{M}$ denote a set of representatives for the Γ-conjugacy classes of Levi subgroups Γ_M in Γ. Let κ_Γ be the assignment

$$(s,h) \in \mathcal{Y}(\Gamma) \mapsto (s,h) \in \mathcal{Y}(\Gamma_M/\mathrm{Z}^\circ_{\Gamma_M})_{\mathrm{ell}}$$

from Lemma 7.6.

Proposition 7.7. *The map κ_Γ induces a linear isomorphism*

$$\kappa_\Gamma : \mathbb{C}[\mathcal{Y}(\Gamma)/_\sim]^\Gamma \longrightarrow \bigoplus_{\Gamma_M \in \mathcal{M}} \mathbb{C}[\mathcal{Y}(\Gamma_M/\mathrm{Z}^\circ_{\Gamma_M})_{\mathrm{ell}}]^{N_\Gamma(\Gamma_M)}.$$

In particular, $\Gamma \backslash \mathcal{Y}(\Gamma)/_\sim$ is finite.

Proof. Suppose $(s,h),(s',h') \in \mathcal{Y}(\Gamma)$ are Γ-conjugate, then the corresponding Γ_M, $\Gamma_{M'}$ in Lemma 7.5 are conjugate. In addition, if $t \in S$ the maximal torus in $\mathrm{Z}_\Gamma(s,h)$, so that $(s,h) \sim (s,ht)$, then (s,ht) is also a pair in Γ_M and $(s,h) \equiv (s,ht) \bmod \mathrm{Z}^\circ_{\Gamma_M} \supset S$. This shows that the linear map κ_Γ is well defined.

The map is clearly surjective. It is injective because firstly, if $(s',h') = \gamma(s,h)\gamma^{-1}$ such that $\Gamma_{M'} = \Gamma_M$, then $\gamma \in N_\Gamma(\Gamma_M)$ (which explains the $N_\Gamma(\Gamma_M)$-invariants in the codomain of κ_Γ), and secondly, if $(s,h) \equiv (s',h') \bmod \mathrm{Z}^\circ_{\Gamma_M}$, we have $(s',h') = (sz_1, hz_2)$ for some z_1, z_2 in the torus $\mathrm{Z}^\circ_{\Gamma_M} \subset \mathrm{Z}_\Gamma(s,h)^\circ$, which means that $(s,h) \sim (s',h')$. □

Remark 7.8. Our main application will be to consider $\Gamma = \Gamma_u$, the reductive part of the centralizer of a unipotent element u in the Langlands dual group $G^\vee$, while Γ_M will be the centralizer of u in a Levi subgroup $M^\vee$.

7.4. **Elliptic pairs in Γ°.** In applications, we will often encounter the situation where the group Γ is connected. For this reason, it is useful to have a precise description of the elliptic pairs in Γ°. Suppose $s \in \Gamma^\circ$ a semisimple element. Let T be a maximal torus of Γ containing s and let Φ be the system of roots of T in Γ° and $W(\Gamma^\circ)$ the Weyl group of T in Γ°. If $\alpha \in \Phi$, let X_α be the corresponding one-parameter unipotent subgroup in Γ°. For each $w \in W(\Gamma^\circ)$, we fix a representative $\dot{w}$ of w in $\mathrm{N}_{\Gamma^\circ}(T)$. Recall [Car, Theorem 3.5.3]

$$\begin{aligned} Z_{\Gamma^\circ}(s)^\circ &= \langle T, X_\alpha \mid \alpha(s) = 1,\ \alpha \in \Phi \rangle \\ Z_{\Gamma^\circ}(s) &= \langle T, X_\alpha, \dot{w} \mid \alpha(s) = 1,\ \alpha \in \Phi,\ wsw^{-1} = s,\ w \in W(\Gamma^\circ) \rangle. \end{aligned} \tag{7.6}$$

We say that $w \in W(\Gamma^\circ)$ is *elliptic* if T^w is finite, equivalently if $\mathfrak{t}^w = 0$, where $\mathfrak{t}$ is the Lie algebra of T.

Proposition 7.9. *With the notation as above,*

$$\mathcal{Y}(\Gamma^\circ)_{\mathrm{ell}} = \Gamma^\circ \cdot \{(s,\dot{w}) \mid s \text{ is regular},\ w \in W(\Gamma^\circ) \text{ is elliptic},\ s \in T^w\}.$$

Proof. Since we are considering Γ°-orbits of pairs $(s,h) \in \mathcal{Y}(\Gamma^\circ)_{\mathrm{ell}}$, we may assume that $s \in T$ and h is in a semisimple conjugacy class of $Z_{\Gamma^\circ}(s)$. If $h \in Z_{\Gamma^\circ}(s)^\circ$, since $Z_{\Gamma^\circ}(s)^\circ$ is reductive ([Car, Theorem 3.5.4]), h is contained in a maximal torus of $Z_{\Gamma^\circ}(s)^\circ$, hence (s,h) is not an elliptic pair. This means that h must be in $Z_{\Gamma^\circ}(s) \setminus Z_{\Gamma^\circ}(s)^\circ$. By (7.6), we can assume that $h = \dot{w}$ for some $w \in W(\Gamma^\circ)$ such that $s \in T^w$. It is clear that $Z_{\Gamma^\circ}(s,\dot{w}) \supseteq T^w$, which means that w is necessarily elliptic if $(s,\dot{w})$ is an elliptic pair. Suppose s is not regular. Then there exists $\alpha \in \Phi$ such that $\alpha(s) = 1$. Let $O_w = \{\alpha, w(\alpha), w^2(\alpha), \dots, w^{n-1}(\alpha)\}$, where n is the order of w. Then in the Lie algebra of Γ, there exists an appropriate sum of root vectors $e = \sum_{\beta \in O_w} e_\beta$ that is invariant under $\mathrm{Ad}(\dot{w})$ and therefore $Z_{\Gamma^\circ}(s,\dot{w})$ contains the one-parameter subgroup for e and it is infinite.

Conversely, suppose $(s,\dot{w})$ is such that s is regular and w is elliptic. By (7.6), $Z_{\Gamma^\circ}(s) = W(\Gamma^\circ)_s T$, where $W(\Gamma^\circ)_s = \{w_1 \in W(\Gamma^\circ) \mid w_1 s w_1^{-1} = s\}$. Then $Z_{\Gamma^\circ}(s,\dot{w})$ is finite if and only if $\mathrm{Ad}(\dot{w})$ has no nonzero fixed points on the Lie algebra $\mathfrak{z}_{\Gamma^\circ}(s)$. But $\mathfrak{z}_{\Gamma^\circ}(s) = \mathfrak{t}$, so this is equivalent with w being elliptic. □

Remark 7.10. If Γ is connected and simply connected, then $\mathcal{Y}(\Gamma)_{\mathrm{ell}} = \emptyset$. This is because in that case, for every regular semisimple s, $Z_\Gamma(s) = Z_\Gamma(s)^\circ = T$, a maximal torus.

8. The dual nonabelian Fourier transform

Let $\mathbf{G}$ be a connected semisimple algebraic F-group and $G = \mathbf{G}(F)$. Let $\mathfrak{R}_{\mathrm{un}}(G)$ denote the category of smooth unipotent representations of G. If $V, V' \in \mathrm{Irr}_{\mathrm{un}} G$, let

$$\mathrm{EP}_G(V,V') = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(V,V'), \tag{8.1}$$

where $\mathrm{Ext}^i(V,V')$ are calculated in the category $\mathfrak{R}(G)$ of all smooth G-representations ([SS]), or equivalently, since $\mathfrak{R}_{\mathrm{un}}(G)$ is a direct summand of $\mathfrak{R}(G)$, in the category $\mathfrak{R}_{\mathrm{un}}(G)$. We remark that this is a finite sum by Bernstein's result on the finiteness of the cohomological dimension of G. Extend, as we may, $\mathrm{EP}_G(\ ,\ )$ as a hermitian pairing on $R_{\mathrm{un}}(G)$ (as defined in Section 4.3). Let $\overline{R}_{\mathrm{un}}(G)$ denote the quotient of $R_{\mathrm{un}}(G)$ by the radical of EP_G.

Let $R_{\mathrm{un}}^{\mathrm{temp}}(G)$ be the subspace spanned by the irreducible unipotent tempered representations and let $\overline{R}_{\mathrm{un}}^{\mathrm{temp}}(G)$ be the image of $R_{\mathrm{un}}^{\mathrm{temp}}(G)$ in $\overline{R}_{\mathrm{un}}(G)$. As it is well-known ([SS, Re1]), as a consequence of the (parabolic induction) Langlands classification:

$$\overline{R}_{\mathrm{un}}^{\mathrm{temp}}(G) = \overline{R}_{\mathrm{un}}(G). \tag{8.2}$$

Let $\mathfrak{B}_{\mathrm{un}}(G)$ denote the unipotent Bernstein center so that $R_{\mathrm{un}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G)} R(G)^{\mathfrak{s}}$, where $R(G)^{\mathfrak{s}}$ is the $\mathbb{C}$-span of irreducible objects in the subcategory $\mathfrak{R}(G)^{\mathfrak{s}}$ (defined in Section 4.2). Since there are no nontrivial extensions between objects in different Bernstein components, we have an EP-orthogonal decomposition:

$$\overline{R}_{\mathrm{un}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G)} \overline{R}(G)^{\mathfrak{s}}.$$

With the same notation for an inner twist G' of G, we get

$$\bigoplus_{G' \in \mathrm{InnT}(G)} \overline{R}_{\mathrm{un}}(G') = \bigoplus_{G' \in \mathrm{InnT}(G)} \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G')} \overline{R}(G')^{\mathfrak{s}}. \tag{8.3}$$

Recall the unramified Langlands correspondence in the form (4.16). Apply the definitions above to $u \in \mathcal{G}_s$ unipotent to obtain a representation δ_u^s of $A_{\mathcal{G}_s}(u)$ on the Cartan subalgebra $\mathfrak{t}_u^s$ in the Lie algebra of $\mathrm{Z}_{\mathcal{G}_s}(u)$. Let $(\ ,\ )_{\mathrm{ell}}^{\delta_u^s}$ be the elliptic inner product on $R(A_{\mathcal{G}_s}(u))$ and let $\overline{R}(A_{\mathcal{G}_s}(u))$ be the elliptic quotient by the radical of the form. One expects the following correspondence to hold.

Conjecture 8.1. *The unramified Langlands correspondence (4.16) induces isometric isomorphisms:*

$$\overline{\mathsf{LLC}}_{\mathrm{un}}\colon \bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigoplus_{u\in\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}}\overline{R}(A_{\mathcal{G}_s}(u))\longrightarrow\bigoplus_{G'\in\mathrm{InnT}(G)}\overline{R}_{\mathrm{un}}(G'),\tag{8.4}$$

and

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}}:\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigoplus_{u\in\mathcal{C}(\mathcal{G}_s^p)_{\mathrm{un}}}\overline{R}(A_{\mathcal{G}_s^p}(u))\longrightarrow\bigoplus_{G'\in\mathrm{Inn}^p(G)}\overline{R}_{\mathrm{un}}(G'),\tag{8.5}$$

where the spaces on the left are endowed with the elliptic inner products $(\ ,\ )_{\mathrm{ell}}^{\delta_u^s}$*, while the spaces on the right have the Euler-Poincaré pairings* $\mathrm{EP}_{G'}$*.*

Here $\mathcal{C}(G^\vee)_{\mathsf{ss}}$ and $\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}$ refer to conjugacy classes of semisimple and unipotent elements, as defined in Section 4.3.

Remark 8.2. In [Re1], Reeder proves that this elliptic correspondence holds in the case of irreducible representations with Iwahori-fixed vectors of a split adjoint group. In section 9, we prove Conjecture 8.1 for a simple F-split group $\mathbf{G}$ under the assumption that $\mathrm{Z}_{\mathbf{G}}$ is cyclic, in particular, when $\mathbf{G}$ is adjoint. In Proposition 11.6 below, we also illustrate the result with a direct proof for $\mathbf{G}=\mathsf{SL}_n$.

8.1. **The elliptic Fourier transform: the split case.** Suppose G is the split F-form. In order to apply the ideas in Section 7.3, we rephrase the left hand side of (8.5). Since Frob acts trivially on $G^\vee$, in this situation we have $\mathcal{G}_s^p=\mathrm{Z}_{G^\vee}(s)$, and the hence

$$\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigoplus_{u\in\mathcal{C}(\mathcal{G}_s^p)_{\mathrm{un}}}\overline{R}(A_{\mathcal{G}_s^p}(u))=\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigoplus_{u\in\mathcal{C}(\mathrm{Z}_{G^\vee}(s))_{\mathrm{un}}}\overline{R}(A_{G^\vee}(su)),$$

which can be written as

$$\bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}}\bigoplus_{s\in\mathbb{C}(\Gamma_u)_{\mathsf{ss}}}\overline{R}(A_{\Gamma_u}(s))=\bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}}\mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u},$$

via Proposition 7.4, where Γ_u is the reductive part of $\mathrm{Z}_{G^\vee}(u)$, as before. Denote for simplicity

$$\mathcal{R}^p_{\mathrm{un,ell}}(G)=\bigoplus_{G'\in\mathrm{InnT}^p(G)}\overline{R}_{\mathrm{un}}(G'),\tag{8.6}$$

endowed with the Euler-Poincaré pairing $\mathrm{EP}=\bigoplus_{G'}\mathrm{EP}_{G'}$. Hence the elliptic unramified local Langlands correspondence for pure inner forms of a split group can be viewed as the isomorphism

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}}:\bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}}\mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u}\longrightarrow\mathcal{R}^p_{\mathrm{un,ell}}(G).\tag{8.7}$$

For every class of elliptic pairs $[(s,h)]\in\Gamma_u\backslash\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, define the virtual combination (cf. [Wa2, Ci]):

$$\Pi(u,s,h)=\sum_{\phi\in\widehat{A_{\Gamma_u}(s)}}\phi(h)\,\pi(s,u,\phi).\tag{8.8}$$

Regard $\Pi(u,s,h)$ (or rather its image) as an element in $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. As before $\phi(h)=\phi(\bar{h})$, where $\bar{h}$ is the image of h in $A_{\Gamma_u}(s)$.

Lemma 8.3. *With the notation as above and* $A_x=A_{G^\vee}(x)$*,*

(1) $\mathrm{EP}(\Pi(u,s,h),\Pi(u',s',h'))=0$ *if* $x=su$ *and* $x'=s'u'$ *are not* $G^\vee$*-conjugate;*
(2)

$$\begin{aligned}\mathrm{EP}(\Pi(u,s,h),\Pi(u,s,h'))&=(|\mathrm{Z}_{A_x}(\bar{h})|\mathbf{1}_{\bar{h}^{-1}},|\mathrm{Z}_{A_x}(\bar{h}')|\mathbf{1}_{\bar{h}'^{-1}})_{\mathrm{ell}}^{A_x}\\&=\begin{cases}|Z_{A_x}(\bar{h})|\det_{\mathfrak{t}_x}(1-\bar{h}^{-1}), & \text{if } h,h' \text{ are conjugate.}\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

Hence, the combinations $\{\Pi(u,s,h)\}$ *define an orthogonal basis of* $\mathcal{R}^p_{\mathrm{un,ell}}(G)$.

Proof. This is a straight-forward consequence of Theorem 9.1 (and an elementary calculation for the last equality in (2)). □

Definition 8.4. (cf. [Ci], [Wa1]). The (dual) *elliptic nonabelian Fourier transform* is the involutive linear map $\mathrm{FT}^\vee_{\mathrm{ell}}\colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{R}^p_{\mathrm{un,ell}}(G)$, defined by

$$\mathrm{FT}^\vee_{\mathrm{ell}}(\Pi(u,s,h)) = \Pi(u,h,s), \quad (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent}.$$

For every $G' \in \mathrm{InnT}^p(G)$, and $K'_{\mathcal{O}} \in \max(G')$ consider the restriction map

$$\mathrm{res}_{K'_{\mathcal{O}}}\colon \mathrm{Irr}_{\mathrm{un}} G' \to R_{\mathrm{un}}(\overline{K}_{\mathcal{O}}),\ V \mapsto V^{K'^{+}_{\mathcal{O}}}. \tag{8.9}$$

We define a linear map $\mathrm{res}_{\mathrm{cpt,un}}\colon \oplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G') \to \mathcal{C}(G)_{\mathrm{cpt,un}}$ by setting

$$\mathrm{res}_{\mathrm{cpt,un}}(V) = \sum_{K'_{\mathcal{O}} \in \max(G')} \mathrm{res}_{K'_{\mathcal{O}}}(V)$$

for all $G' \in \mathrm{InnT}^p(G)$ and $V \in \mathrm{Irr}_{\mathrm{un}}(G')$. With notation as in Section 6, for each $(A,\mathcal{O}) \in S_{\max}(G)$, we let $\mathrm{proj}_{\mathcal{O}}$ be the projection map $\mathcal{C}(G)_{\mathrm{cpt,un}} \to \oplus_{x \in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ with respect to the decomposition (6.5), and let $\mathrm{res}_{\mathcal{O}} = \mathrm{res}_{\mathrm{cpt,un}} \circ \mathrm{proj}_{\mathcal{O}}$. We have

$$\mathrm{res}_{\mathrm{cpt,un}} = \bigoplus_{(A,\mathcal{O}) \in S_{\max}(G)} \mathrm{res}_{\mathcal{O}}\,. \tag{8.10}$$

We can now formulate the conjecture for elliptic representations.

Conjecture 8.5. *Let G be a simple F-split group. The following diagram commutes*

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ {\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} \\ \mathcal{C}(G)_{\mathrm{cpt,un}} & \xrightarrow[\mathrm{FT}_{\mathrm{cpt,un}}]{} & \mathcal{C}(G)_{\mathrm{cpt,un}} \end{array}$$

up to roots of unity. More precisely, for every unipotent element $u \in G^\vee$, elliptic pair $(s,h) \in \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, and maximal compact open subgroup $K_{\mathcal{O}}$ of G, there exists a root of unity $\zeta = \zeta(u,s,h,\mathcal{O})$ such that

$$\mathrm{res}_{\mathcal{O}}(\Pi(u,h,s)) = \zeta \cdot (\mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{\mathcal{O}})(\Pi(u,s,h)).$$

Remark 8.6. If $K_{\mathcal{O}}$ is the maximal hyperspecial compact subgroup of G, so that in particular $\mathrm{res}_{\mathcal{O}} = \mathrm{res}_{K_{\mathcal{O}}}$, we expect that the only roots of unity ζ that appear are the well-known $\Delta(\bar{x}_\rho) \in \{\pm 1\}$, see [Lu1, §6.7], for certain families of unipotent representations of the finite groups of types E_7 and E_8. But for other maximal compact subgroups, Proposition 11.8 shows that new roots of unity can appear.

8.2. **Example** $G = \mathsf{SL}_2(F)$**.** As an example of the correspondence for elliptic unipotent representations, consider $G = \mathsf{SL}_2(F)$ and $G^\vee = \mathsf{PGL}_2(\mathbb{C})$. Let $X(\nu)$ denote the unramified principal series of G, where $\nu \in \mathbb{C}/(2\pi i/\log q)$, as in [Cas, Appendix]. The elliptic tempered representations occur at $\nu = \pm 1$ and $\nu = \pi i/\log q$ (the unramified quadratic character).

At $\nu = \pi i/\log q$, there is a decomposition $X(\nu) = X^+ \oplus X^-$, where $X^\pm$ are irreducible tempered representations. The virtual elliptic combination, in the sense of Arthur, is $X^+ - X^-$.

At $\nu = \pm 1$, one of the subquotients of the principal series is the Steinberg represetation St (the other being the trivial representation). The Steinberg representation is a square integrable representation.

In terms of the local Langlands correspondence for unipotent representations, we have

$$\mathrm{Irr}_{\mathrm{un}}(\mathsf{SL}_2(F)) \sqcup \mathrm{Irr}_{\mathrm{un}}(D^\times) \longleftrightarrow \mathsf{PGL}_2(\mathbb{C}) \backslash \{(x,\phi) \mid x \in \mathsf{PGL}_2(\mathbb{C}),\ \phi \in \widehat{A^1_x}\}, \tag{8.11}$$

where D is the 4-dimensional division algebra over F, hence $\mathrm{Irr}_{\mathrm{un}} D^\times = \{\mathrm{triv}_{D^\times}\}$.

The elements x parameterizing elliptic representations are

$$x = u_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z \text{ and } x = s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z.$$

If $x = u_1$, then $A^1_{u_1} = \mathrm{Z}_{\mathsf{SL}_2(\mathbb{C})} = C_2$ with $\widehat{C}_2 = \{\mathrm{triv}, \mathrm{sgn}\}$, and the parameterization is

$$\mathrm{St} \longleftrightarrow (u_1, \mathrm{triv}), \quad \mathrm{triv}_{D^\times} \longleftrightarrow (u_1, \mathrm{sgn}).$$

Since $\Gamma_{u_1} = \{1\}$, there is one associated elliptic virtual combination

$$\Pi(u_1, 1, 1) = \mathrm{St} + \mathrm{triv}_{D^\times}. \tag{8.12}$$

If $x = s$, $Z_{\mathsf{PGL}_2(\mathbb{C})}(s) = \mathsf{O}_2/Z$ and then $Z^1_{\mathsf{SL}_2(\mathbb{C})}(s) = \mathsf{O}_2$. Hence $A^1_s = C_2$. The Langlands parameterization is

$$X^+ \longleftrightarrow (s, \mathrm{triv}), \quad X^- \longleftrightarrow (s, \mathrm{sgn}).$$

The unipotent part of x is $u_0 = 1$, and $\Gamma_{u_0} = \mathsf{PGL}_2(\mathbb{C})$, which means that

$$\mathcal{Y}(\Gamma_{u_0})_{\mathrm{ell}} = \Gamma_{u_0} \cdot (s, \dot{w}), \text{ where } \dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Z.$$

There is only one associated elliptic tempered virtual combination (up to sign):

$$\Pi(u_0, s, \dot{w}) = X^+ - X^-. \tag{8.13}$$

8.3. **Regular unipotent elements.** In Section 11, we will verify this conjecture completely when $G = \mathsf{SL}_n$ and PGL_n, but here we illustrate it in the case when u is a regular unipotent element.

Proposition 8.7. *Let $u_r \in G^\vee$ be a regular unipotent element. Then*

$$\mathrm{res}_{\mathrm{cpt,un}}(\Pi(u_r, h, s)) = \mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{\mathrm{cpt,un}}(\Pi(u_r, s, h)).$$

for all $(s, h) \in \mathcal{Y}(\Gamma_{u_r})$. In particular, Conjecture 8.5 holds true with trivial roots of unity.

Proof. In this case $\Gamma_{u_r} = \mathrm{Z}_{G^\vee}$ and every pair (s, h) in $\mathcal{Y}(\mathrm{Z}_{G^\vee})$ is elliptic. Write the natural identification

$$\Omega_G \xrightarrow{\sim} \widehat{\mathrm{Z}_{G^\vee}}$$

as $x \mapsto \phi_x$. Then for $(s, h) \in \mathcal{Y}(\Gamma_{u_r})$, we have $\Pi(u_r, s, h) = \sum_{x \in \Omega} \phi_x(h) \pi(s, u_r, \phi_x)$. Note that $\pi(1, u_r, \phi_x)$ is the Steinberg representation St_{G_x} of G_x, so $\pi(s, u_r, \phi_x) \simeq \mathrm{St}_{G_x} \otimes \chi_s$, where χ_s is the weakly unramified character corresponding to s under (4.3).

For the rest of the proof, we fix $(A, \mathcal{O}) \in S_{\max}(G)$. Then given $s \in \mathrm{Z}_{G^\vee}$, the character χ_s is trivial on the parahoric $K^\circ_{x,\mathcal{O}}$ so defines a character, call it σ_s, of A under the natural isomorphism $K_{x,\mathcal{O}}/K^\circ_{x,\mathcal{O}} \simeq A$. We have

$$\mathrm{res}_{\mathcal{O}}\, \pi(s, u_r, \phi_x) = \begin{cases} 0, & \text{if } x \notin A, \\ \mathrm{St}_{K^\circ_{\mathcal{O},x}} \rtimes \sigma_s & \text{if } x \in A, \end{cases}$$

where $\mathrm{St}_{K^\circ_{\mathcal{O},x}}$ is the Steinberg character of the finite group $\overline{K}^\circ_{\mathcal{O},x}$ inflated to $K^\circ_{\mathcal{O},x}$. Note that that for every $x \in A$

$$\phi_x(h) = \sigma_h(x), \text{ for all } h \in \mathrm{Z}_{G^\vee}.$$

Thus

$$\mathrm{res}_{\mathcal{O}}\, \Pi(u_r, s, h) = \sum_{x \in A} \sigma_h(x) \mathrm{St}_{K^\circ_{\mathcal{O},x}} \rtimes \sigma_s. \tag{8.14}$$

With notation as in Section 5, let $\mathcal{U}_{\mathcal{O},\mathrm{St}} = \{\mathrm{St}_{K^\circ_{\mathcal{O}}}\}$ be the Steinberg family in $\mathrm{Irr}_{\mathrm{un}}(\overline{K}_{\mathcal{O}})$, and let $\widetilde{\mathcal{U}}_{\mathcal{O},\mathrm{St}} \subset \cup_{x \in A} \mathrm{Irr}_{\mathrm{un}}(\overline{K_{x,\mathcal{O}}})$ be the family parametrized by $\widetilde{\Gamma}^A_{\mathcal{U}_{\mathcal{O},\mathrm{St}}} = A$ under the bijection of Proposition 5.3. Then by (8.14), $\mathrm{res}_{\mathcal{O}}\, \Pi(u_r, s, h)$ corresponds to $\Pi_{\widetilde{\mathcal{U}}_{\mathcal{O},\mathrm{St}}}(\sigma_s, \sigma_h)$ defined as in (5.13). The claim then follows from Lemma 5.1. □

8.4. **The compact Fourier transform: the split case.** We extend the conjecture from the elliptic case. Recall the spaces $\mathbb{C}[\mathcal{Y}(\Gamma_u)/_\sim]^\Gamma_u$ defined in Section 7.3, particularly Proposition 7.7. For every compact pair $(s,h) \in \Gamma_u\backslash\mathcal{Y}(\Gamma_u)/_\sim$, define $\Pi(u,s,h) \in$ as in (8.8). Let

$$\mathcal{R}^p_{\mathrm{un,cpt}}(G) = \mathrm{span}\langle \Pi(u,s,h) \mid u \in \mathcal{C}(G^\vee)_{\mathrm{un}},\ (s,h) \in \Gamma_u\backslash\mathcal{Y}(\Gamma_u)/_\sim\rangle. \tag{8.15}$$

Since s,h have finite order, all of the irreducible $G'(F)$-representations, $G' \in \mathrm{InnT}^p(G)$, that occur in the virtual linear combination $\Pi(u,s,h)$ are tempered. Hence $\mathcal{R}^p_{\mathrm{un,cpt}}(G)$ is a subspace of $\bigoplus_{G'} R^{\mathrm{temp}}_{\mathrm{un}}(G')$.

Definition 8.8. The *dual compact (nonabelian) Fourier transform* is

$$\mathrm{FT}^\vee_{\mathrm{un,cpt}}\colon \mathcal{R}^p_{\mathrm{un,cpt}}(G) \to \mathcal{R}^p_{\mathrm{un,cpt}}(G), \quad \mathrm{FT}^\vee_{\mathrm{un,cpt}}(\Pi(u,s,h)) = \Pi(u,h,s). \tag{8.16}$$

Conjecture 8.9. *Let G be a simple F-split group. The following diagram commutes*

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,cpt}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{cpt,un}}} & \mathcal{R}^p_{\mathrm{un,cpt}}(G) \\ \downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} & & \downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} \\ \mathcal{C}(G)_{\mathrm{cpt,un}} & \xrightarrow[\mathrm{FT}_{\mathrm{cpt,un}}]{} & \mathcal{C}(G)_{\mathrm{cpt,un}} \end{array}$$

Let $\mathcal{L}^\vee$ be the set of $G^\vee$-conjugacy classes of Levi subgroups $M^\vee$ in $G^\vee$. If the unipotent $G^\vee$-class of u meets the Levi subgroup $M^\vee$, assume that $u \in M^\vee$, and denote $\Gamma_u^{M^\vee} = \Gamma_u \cap M^\vee$. Proposition 7.7 implies that we have a natural isomorphism

$$\begin{aligned} \kappa_u\colon \bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)/_\sim]^{\Gamma_u} &\longrightarrow \bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \bigoplus_{M^\vee\in\mathcal{L}^\vee, u\in M^\vee} \mathbb{C}[\mathcal{Y}(\Gamma_u^{M^\vee}/\mathrm{Z}^\circ_{\Gamma_u^{M^\vee}})_{\mathrm{ell}}]^{\mathrm{N}_{\Gamma_u}(\Gamma_u^{M^\vee})} \\ &\cong \left(\bigoplus_{(M^\vee,u)} \mathbb{C}[\mathcal{Y}(\Gamma_u^{M^\vee}/\mathrm{Z}^\circ_{\Gamma_u^{M^\vee}})_{\mathrm{ell}}]\right)/_{\sim G^\vee}, \end{aligned} \tag{8.17}$$

where the pairs $(M^\vee, u)$ range over Levi subgroups $M^\vee$ in $G^\vee$ and $u \in M^\vee$ unipotent. Let M be the Levi subgroup of G whose dual is $M^\vee$. Conjecture 8.1 (proved in some cases in Section 9) in the interpretation (8.7) says that there is natural isomorphism

$$\overline{\mathrm{LLC}^p}^M_{\mathrm{un}}: \left(\bigoplus_{u\in M^\vee} \mathbb{C}[\mathcal{Y}(\Gamma_u^{M^\vee}/\mathrm{Z}^\circ_{\Gamma_u^{M^\vee}})_{\mathrm{ell}}]\right)/_{\sim M^\vee} \to \mathcal{R}^p_{\mathrm{un,ell}}(M/\mathrm{Z}^\circ_M).$$

On the left hand side u ranges over the unipotent elements of $M^\vee$. Composing with κ_u, we get a natural isomorphism

$$\mathrm{LLC}^p_{\mathrm{cpt,un}}\colon \bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)/_\sim]^{\Gamma_u} \longrightarrow \left(\bigoplus_M \mathcal{R}^p_{\mathrm{un,ell}}(M/\mathrm{Z}^\circ_M)\right)/_{\sim G}, \tag{8.18}$$

where M ranges over the set of Levi subgroups of G. In particular, if $\mathcal{L}$ is the set of conjugacy classes of Levi subgroups of G, this implies a natural isomorphism

$$\mathcal{R}^p_{\mathrm{un,cpt}}(G) \cong \bigoplus_{M\in\mathcal{L}} \mathcal{R}^p_{\mathrm{un,ell}}(M/\mathrm{Z}^\circ_M). \tag{8.19}$$

We make this more precise when G is simply connected, in which case G is the only pure inner form. We recall several known results from the K-theory of G, see [BDK, Da, CH2, Kaz]. Since the category of unipotent representations of G is a direct summand of the full category of smooth representations of G, we state the results directly in the unipotent setting. Following [CH2, §6.7], define the rigid quotient

$$\overline{R}(G)_{\mathrm{un,rig}} = R_{\mathrm{un}}(G)/R(G)_{\mathrm{un,diff}}; \tag{8.20}$$

here $R(G)_{\mathrm{un,diff}} \subset R_{\mathrm{un}}(G)$ is the $\mathbb{C}$-span by $i_M(\sigma) - i_M(\sigma \otimes \chi)$, where M ranges over the set of standard Levi subgroups of G, $\sigma \in R_{\mathbb{C}}(M)_{\mathrm{un}}$, and χ is an unramified character of M. For every M, i_M denote the functor of (normalized) parabolic induction: since we only work at the level of Grothendieck groups, the choice of parabolic containing M is not important. The parabolic form of the Langlands classification implies that $R(G)_{\mathrm{un,rig}}$ is spanned by (the images of) tempered representations, and moreover by $\bigoplus_{M \in \mathcal{L}} i_M(\overline{R}(M/\mathrm{Z}_M^\circ)_{\mathrm{un}})$. Hence the map

$$\mathcal{R}^p_{\mathrm{un,cpt}}(G) \hookrightarrow R(G)_{\mathrm{un}} \twoheadrightarrow \overline{R}(G)_{\mathrm{un,rig}} \tag{8.21}$$

is surjective. Let $\mathcal{H}(G) = C_c^\infty(G)$ be the Hecke algebra of G, $\overline{\mathcal{H}}(G)$ the cocenter of $\mathcal{H}(G)$ (the quotient by the space of commutators). Let $\overline{\mathcal{H}}(G)^{\mathrm{rig}}$ be the rigid cocenter, the image in $\overline{\mathcal{H}}(G)$ of $\sum_{K \in \max(G)} C_c^\infty(K)$. Let $\overline{\mathcal{H}}(G)^{\mathrm{rig}}_{\mathrm{un}}$ be the rigid unipotent cocenter, the image of $\sum_{K \in \max(G)} C_c^\infty(K)_{\mathrm{un}}$. The rigid trace Paley-Wiener Theorem [BDK, CH2] and Kazhdan's Density Theorem [Kaz] imply that the trace map

$$\mathrm{tr} : \mathcal{H}(G)^{\mathrm{rig}}_{\mathrm{un}} \to (R(G)_{\mathrm{un,rig}})^*, \ f \mapsto (\pi \mapsto \mathrm{tr}(f)) \tag{8.22}$$

is an isomorphism. By [Da, Theorem 4.25]

$$\dim \mathcal{H}(G)^{\mathrm{rig}}_{\mathrm{un}} = \sum_{M \in \mathcal{L}} \dim \overline{R}(M/\mathrm{Z}_M^\circ)_{\mathrm{un}},$$

and therefore by (8.19), $\dim \overline{R}(G)_{\mathrm{un,rig}} = \dim \mathcal{R}^p_{\mathrm{un,cpt}}(G)$. Comparing to (8.21), we arrive at

Corollary 8.10. *Suppose G is split and simply connected. There exist natural isomorphisms $\overline{R}(G)_{\mathrm{un,rig}} \cong \mathcal{R}^p_{\mathrm{un,cpt}}(G) \cong \bigoplus_{M \in \mathcal{L}} \mathcal{R}^p_{\mathrm{un,ell}}(M/\mathrm{Z}_M^\circ) \cong \bigoplus_{u \in \mathcal{C}(G^\vee)_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)/\sim]^{\Gamma_u}$.*

We expect that a similar result holds for arbitrary isogenies.

Remark 8.11. The Fourier transform $\mathrm{FT}_{\mathrm{cpt,un}}$ should be compatible with parabolic induction. This is deduced in [MW, §6.4] from the works of Lusztig and Shoji. In this situation, in particular when G is simply connected, Conjecture 8.5 implies Conjecture 8.9.

9. Elliptic unipotent representations

The main result of this section is:

Theorem 9.1. *Suppose G is a simple split F-group of adjoint type. Then Conjecture 8.1 holds true for all pure inner forms of G, in the sense of (8.5).*

Remark 9.2. In Section 9.4, we explain how Theorem 9.1 can be extended to other isogenies, in particular, to the situation when all R-groups that appear in the disconnected generalized Springer correspondence (Section 3) are cyclic. See Remark 9.12 and Corollary 9.13.

The strategy of the proof is as follows. As explained in Section 4, the set of unramified enhanced Langlands parameters $\Phi_{\mathrm{e,un}}(G')$, where G' is an inner twist of G, decomposes into a disjoint union $\Phi_{\mathrm{e,un}}(G') = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')} \Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}$. Consequently, there is a decomposition

$$R(\Phi_{\mathrm{e,un}}(G')) = \bigoplus_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')} R(\Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}).$$

In [AMS2], an affine Hecke algebra with possibly unequal parameters $\mathcal{H}(\mathfrak{s}^\vee)$ is constructed such that there is a natural bijection

$$\mathrm{Irr}\ \mathcal{H}(\mathfrak{s}^\vee) \longleftrightarrow \Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee} \tag{9.1}$$

which induces a natural linear isomorphism

$$R(\mathcal{H}(\mathfrak{s}^\vee)) \cong R(\Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}).$$

We need to study the elliptic space $\overline{R}(\mathcal{H}(\mathfrak{s}^\vee))$. The important fact for elliptic theory is that $\mathcal{H}(\mathfrak{s}^\vee)$ is a deformation of an extended affine Weyl group $\widetilde{W}_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee} \ltimes X^*(T_{\mathfrak{s}^\vee})$, where $T_{\mathfrak{s}^\vee} = \Phi_{\mathrm{e}}(L')^{\mathfrak{s}^\vee_{L'}}$ for L' a Levi subgroup of G' that corresponds to $\mathfrak{s}^\vee$. This allows us to use the results of [OS], in order to further reduce to $\overline{R}(\mathcal{H}(\mathfrak{s}^\vee)) \cong \overline{R}(\widetilde{W}_{\mathfrak{s}^\vee})$. Moreover, the latter space, is equivalent with a direct sum of elliptic spaces for certain finite groups

$$\overline{R}(\widetilde{W}_{\mathfrak{s}^\vee}) \cong \bigoplus_{s \in W_{\mathfrak{s}^\vee} \backslash T_{\mathfrak{s}^\vee}} \overline{R}(\mathrm{Z}_{W_{\mathfrak{s}^\vee}}(s)).$$

We then use results of [Wa2] and the generalized Springer correspondence in order to relate the spaces $\overline{R}(\mathrm{Z}_{W_{\mathfrak{s}^\vee}})(s)$ to the relevant spaces of Langlands parameters (for the various unipotent elements) in $\Phi_{\mathrm{e,un}}(G')^{\mathfrak{s}^\vee}$.

Finally, by [Lu3, Lu4, So1] for each $\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')$, there exists $\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G')$ such that the Hecke algebra $\mathcal{H}(\mathfrak{s})$ for $\mathfrak{s}$ is naturally isomorphic to $\mathcal{H}(\mathfrak{s}^\vee)$. The fact that the elliptic space for the representations in the block $\mathfrak{R}(G')^{\mathfrak{s}}$ is naturally isomorphic to $\overline{R}(\mathcal{H}(\mathfrak{s}))$ is immediate by the exactness of the equivalence of categories between $\mathfrak{R}(G')^{\mathfrak{s}}$ and $\mathcal{H}(\mathfrak{s})$-modules.

9.1. **Euler-Poincaré pairings for affine Hecke algebras.** We begin by recalling several known facts about the elliptic theory for affine Weyl groups and affine Hecke algebras. The main reference is [OS], see also [CO1]. The notation in this section is self contained and independent of the previous sections. For applications, the root datum in this section will be specialized to the root datum of the Langlands dual group $G\vee$, as well as to the root data for the affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee)$ which occur on the dual side of the local Langlands correspondence.

Let $\mathcal{R} = (X, R, X^\vee, R^\vee, \Pi)$ be a based root datum. Here $X, X^\vee$ are lattices in perfect duality $\langle\ ,\ \rangle\colon X \times X^\vee \to \mathbb{Z}$, $R \subset X \setminus \{0\}$ and $R^\vee \subset X^\vee \setminus \{0\}$ are the finite sets of roots and coroots respectively, and $\Pi \subset R$ is a basis of simple roots. Let W be the finite Weyl group with set of generators $S = \{s_\alpha : \alpha \in \Pi\}$. Set $\widetilde{W} = W \ltimes X$, the extended affine Weyl group), and $W^a = W \ltimes Q$, the affine Weyl group, where Q is the root lattice of R. Then W^a is normal in $\widetilde{W}$ and $\Omega := \widetilde{W}/W^a \cong X/Q$ is an abelian group. We assume that $\mathcal{R}$ is semisimple, which means that Ω is a finite group.

The set $R^a = R^\vee \times \mathbb{Z} \subset X^\vee \times \mathbb{Z}$ is the set of affine roots. A basis of simple affine roots is given by $\Pi^a = (\Pi^\vee \times \{0\}) \cup \{(\gamma^\vee, 1) : \gamma^\vee \in R^\vee \text{ minimal}\}$. For every affine root $\mathbf{a} = (\alpha^\vee, n)$, let $s_{\mathbf{a}}\colon X \to X$ denote the reflection $s_{\mathbf{a}}(x) = x - ((x, \alpha^\vee) + n)\alpha$. The affine Weyl group W^a has a set of generators $S^a = \{s_{\mathbf{a}} : \mathbf{a} \in \Pi^a\}$. Let $l\colon \widetilde{W} \to \mathbb{Z}$ be the length function.

Set $E = X \otimes_{\mathbb{Z}} \mathbb{C}$, so the discussion regarding elliptic theory of W and E from the previous sections applies. We denote a typical element of $\widetilde{W}$ by wt_x, where $w \in W$ and $x \in X$. The extended affine Weyl group $\widetilde{W}$ acts on E via $(wt_x) \cdot v = w \cdot v + x$, $v \in E$.

An element $wt_x \in \widetilde{W}$ is called elliptic if $w \in W$ is elliptic (with respect to the action on E). For basic facts about elliptic theory for $\widetilde{W}$, see [OS, sections 3.1, 3.2]. There are finitely many elliptic conjugacy classes in $\widetilde{W}$ (and in W^a). The following fact is well known, see for example [CO1, Lemma 5.4].

Lemma 9.3. *Suppose C is an elliptic conjugacy class in W^a. Then there exists one and only one maximal $J \subsetneq S^a$ such that $C \cap W_J \neq \emptyset$, and in this case $C \cap W_J$ forms a single elliptic W_J-conjugacy class.*

Define the Euler-Poincaré pairing of $\widetilde{W}$ by:

$$\langle U, V\rangle^{\widetilde{W}}_{\mathrm{EP}} = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i_{\widetilde{W}}(U, V), \quad U, V \text{ finite-dimensional } \widetilde{W}\text{-modules.} \tag{9.2}$$

Let $R(\widetilde{W})$ be the Grothendieck group of $\widetilde{W}$-mod (the category of finite-dimensional modules), and set $\overline{R}(\widetilde{W}) = R(\widetilde{W})/\mathrm{rad}\langle\ ,\ \rangle^{\widetilde{W}}_{\mathrm{EP}}$. By [OS, Theorem 3.3], the Euler-Poincaré pairing

for $\widetilde{W}$ can also be expressed as an elliptic integral. More precisely, define the conjugation-invariant elliptic measure μ_{ell} on $\widetilde{W}$ by setting $\mu_{\mathrm{ell}} = 0$ on nonelliptic conjugacy classes, and for an elliptic conjugacy class C such that $v \in E$ is an isolated fixed point for some element of C, set

$$\mu_{\mathrm{ell}}(C) = \frac{|Z_{\widetilde{W}}(v) \cap C|}{|Z_{\widetilde{W}}(v)|};$$

here $Z_{\widetilde{W}}(v)$ is the isotropy group of v in $\widetilde{W}$. Then

$$\langle U, V\rangle_{\mathrm{EP}}^{\widetilde{W}} = (\chi_U, \chi_V)_{\mathrm{ell}}^{\widetilde{W}} := \int_{\widetilde{W}} \chi_U \chi_V \; d\mu_{\mathrm{ell}}, \; U, V \in \widetilde{W}\text{-mod}, \tag{9.3}$$

where χ_U, χ_V are the characters of U and V.

Set $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. Then W acts on T. For every $s \in T$, set $W_s = \{w \in W : w{\cdot}s = s\}$. One considers the elliptic theory of the finite group W_s acting on the cotangent space of T at s. By Clifford theory, the induction map

$$\mathrm{Ind}_s \colon W_s\text{-mod} \to \widetilde{W}\text{-mod}, \quad \mathrm{Ind}_s(U) := \mathrm{Ind}_{W_s \ltimes X}^{\widetilde{W}}(U \otimes s)$$

maps irreducible modules to irreducible modules. By [OS, Theorem 3.2], the map

$$\bigoplus_{s \in T/W} \mathrm{Ind}_s \colon \bigoplus_{s \in T/W} \overline{R}(W_s)_{\mathbb{C}} \to \overline{R}(\widetilde{W})_{\mathbb{C}} \tag{9.4}$$

is an isomorphism of metric spaces, in particular,

$$\langle \mathrm{Ind}_s U, \mathrm{Ind}_s V\rangle_{\mathrm{EP}}^{\widetilde{W}} = (U, V)_{\mathrm{ell}}^{W_s}, \; U, V \in W_s\text{-mod}. \tag{9.5}$$

A space $\overline{R}(W_s)_{\mathbb{C}}$ in the left hand side of (9.4) is nonzero if and only if s is an isolated element of T, more precisely $s \in T_{\mathrm{iso}}$, where

$$T_{\mathrm{iso}} = \{s \in T^\vee : w \cdot s = s \text{ for some elliptic } w \in W\}.$$

Example 9.4. Let $\mathcal{R}$ be the root datum of $\mathsf{PGL}_n(\mathbb{C})$. In other words, T is the maximal diagonal torus of $\mathsf{PGL}_n(\mathbb{C})$, $X = X^*(T)$ is the group of characters, and $X^\vee = X_*(T)$ the group of cocharacters. In this case, $Q = X$ and

$$\widetilde{W} = W^a = \langle s_i, \; 0 \le i \le n-1 \mid (s_i s_j)^{m(i,j)} = 1, \; 0 \le i, j \le n-1\rangle,$$

where $m(i,i) = 1$, $m(i,j) = 2$ if $1 < |i-j| < n-1$, and $m(i,j) = 3$ if $|i-j| = 1$ or $|i-j| = n-1$, when $n \ge 3$. If $n = 2$, then $\widetilde{W} = W^a = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1\rangle$.

With this notation, the finite Weyl group is $W = \langle s_1, \dots, s_{n-1}\rangle \subset W^a$. For every $0 \le i \le n-1$, denote $W_i = \langle s_0, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1}\rangle \subset W^a$. These are the maximal (finite) parahoric subgroups of W^a. In particular, $W_0 = W$ and $W_i \cong S_n$ for all i. The space $E = \mathfrak{t}^* \cong \mathbb{C}^{n-1}$ and each W_i acts on E by the reflection representation. Therefore, there exists a unique elliptic W_i-conjugacy class represented by the Coxeter element $w_i = s_0 s_1 \cdots s_{i-1} s_{i+1} \cdots s_{n-1}$. Thus by Lemma 9.3, there are exactly n elliptic conjugacy classes in W^a each determined by the condition that it meets W_i in the conjugacy class of w_i, $0 \le i \le n-1$. In particular, $\dim \overline{R}(\widetilde{W})_{\mathbb{C}} = n$ in this case.

On the other hand, by (9.4), we need to consider W-orbits in T_{iso}. Since there is only one elliptic conjugacy class in W, every W-orbit in T_{iso} is represented by an element of

$$T^{s_1 s_2 \cdots s_{n-1}} = \{\Delta_n(z) \mid z \in \mu_n\}, \quad \Delta_n(z) = \mathrm{diag}(1, z, z^2, \dots, z^{n-1}) \in T, \; \mu_n = \{z \mid z^n = 1\},$$

as noticed before. Two elements $\Delta_n(z)$ and $\Delta_n(z')$ of $T^{s_1 s_2 \cdots s_{n-1}}$ are W-conjugate if and only if z and z' have the same order. Fix a primitive n-th root ζ of 1. This means that (9.4) becomes in this case:

$$\bigoplus_{d|n} \overline{R}(W_{\Delta_n(\zeta^{n/d})})_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}}.$$

If $z = \zeta^m$, where $n = dm$, then $\Delta_n(z)$ is W-conjugate to $(\underbrace{1,\dots,1}_{m},\underbrace{z,\dots,z}_{m},\dots,\underbrace{z^{d-1},\dots,z^{d-1}}_{m})$.

Hence, one calculates

$$W_{\Delta_n(z)} \cong S_m^d \rtimes C_d \text{ and } \mathfrak{t}^*_{\Delta_n(z)} = \{(a_1,\dots,a_n) \mid \sum_{i=1}^n a_i = 0\}. \tag{9.6}$$

The action is the natural permutation action. There are $\varphi(d)$ elliptic conjugacy classes in represented by $(w_m, 1, \dots, 1)\cdot x_d$, where w_m is a fixed m-cycle (Coxeter element) of S_m and x_d is one of the $\varphi(d)$ generators of C_d, see Lemma 11.5. Again $\sum_{d|n}\varphi(d) = n$.

Let $\vec{\mathbf{q}} = \{\mathbf{q}(s) : s \in S^a\}$ be a set of invertible, commuting indeterminates such that $\mathbf{q}(s) = \mathbf{q}(s')$ whenever s, s' are W^a-conjugate. Let $\Lambda = \mathbb{C}[\mathbf{q}(s), \mathbf{q}(s)^{-1} : s \in S^a]$.

The generic affine Hecke algebra $\mathcal{H}(\mathcal{R}, \vec{\mathbf{q}})$ associated to the root datum $\mathcal{R}$ and the set of indeterminates $\vec{\mathbf{q}}$ is the unique associative, unital Λ-algebra with basis $\{T_w : w \in \widetilde{W}\}$ and relations

(i) $T_w T_{w'} = T_{ww'}$, for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$;
(ii) $(T_s - \mathbf{q}(s)^2)(T_s + 1) = 0$ for all $s \in S^a$.

Fix an indeterminate $\mathbf{q}$. Given a W^a-invariant function $m\colon S^a \to \mathbb{R}$, we may define a homomorphism $\lambda_m\colon \Lambda \to \mathbb{C}[\mathbf{q}]$, $\mathbf{q}(s) = \mathbf{q}^{m(s)}$. Consider the specialized affine Hecke algebra

$$\mathcal{H}(\mathcal{R}, \mathbf{q}, m) = \mathcal{H}(\mathcal{R}, \vec{\mathbf{q}}) \otimes_\Lambda \mathbb{C}_{\lambda_m}. \tag{9.7}$$

Example 9.5. Let $\mathcal{R}$ be the root datum of $\mathsf{PGL}_n(\mathbb{C})$. If $n = 2$, the generic affine Hecke algebra has two indeterminates $\mathbf{q}(s_0)$ and $\mathbf{q}(s_1)$ and it is generated by $T_0 = T_{s_0}$, $T_1 = T_{s_1}$ subject only to the quadratic relations

$$(T_i - \mathbf{q}(s_i)^2)(T_i + 1) = 0,\ i = 0, 1.$$

If $n \geq 3$, all the simple reflections are W^a-conjugate. There is only one indeterminate $\mathbf{q}$ such that the affine Hecke algebra is generated by $\{T_i = T_{s_i}, 0 \leq i \leq n-1\}$ subject to the relations:

(i) $T_i T_j = T_j T_i$, $1 < |i - j| < n - 1$;
(ii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $0 \leq i \leq n-1$; $\quad T_0 T_{n-1} T_0 = T_{n-1} T_0 T_{n-1}$;
(iii) $(T_i - \mathbf{q}^2)(T_i + 1) = 0$.

Specialize further the indeterminate $\mathbf{q}$ to $q > 1$. Let $\mathcal{H} = \mathcal{H}(\mathcal{R}, q, m)$ be the resulting affine Hecke algebra over $\mathbb{C}$. If U, V are two finite-dimensional $\mathcal{H}$-modules, define the Euler-Poincaré pairing [OS, §3.4]:

$$\mathrm{EP}_{\mathcal{H}}(U, V) = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i_{\mathcal{H}}(U, V). \tag{9.8}$$

This is a finite sum since $\mathcal{H}$ has finite cohomological dimension [OS, Proposition 2.4]. The pairing $\mathrm{EP}_{\mathcal{H}}$ is symmetric and positive semidefinite. It extends to a hermitian positive-semidefinite pairing on the complexified Grothendieck group $R(\mathcal{H})_{\mathbb{C}}$ of finite-dimensional $\mathcal{H}$-modules. We wish to compare the Euler-Poincaré pairings for the $\mathcal{H}(\mathcal{R}, q, m)$ and $\mathcal{H}(\mathcal{R}, q^\epsilon, m)$, where $\epsilon \in [0, 1]$. Suppose we have a family of maps

$$\sigma_\epsilon\colon \mathcal{H}(\mathcal{R}, q, m)\text{-mod} \to \mathcal{H}(\mathcal{R}, q^\epsilon, m)\text{-mod}, \quad \sigma_\epsilon(\pi, V) = (\pi_\epsilon, V), \tag{9.9}$$

such that

(a) for every $w \in \widetilde{W}$ and every (π, V), then assignment $\epsilon \mapsto \pi(\epsilon)(T_w)$ is a continuous map $[0, 1] \to \mathrm{End}(V)$.

Then [OS, Theorem 3.5] shows that

$$\mathrm{EP}_{\mathcal{H}(\mathcal{R},q,m)}(U, V) = \mathrm{EP}_{\mathcal{H}(\mathcal{R},q^\epsilon,m)}(\sigma_\epsilon(U), \sigma_\epsilon(V)), \quad \text{for all } \epsilon \in [0, 1].$$

In particular, notice that $\mathcal{H}(\mathcal{R}, q^0, m) = \mathbb{C}[\widetilde{W}]$, meaning that

$$\mathrm{EP}_{\mathcal{H}(\mathcal{R},q,m)}(U,V) = \langle \sigma_0(U), \sigma_0(V)\rangle_{\mathrm{EP}}^{\widetilde{W}}. \tag{9.10}$$

Using [OS, Theorem 1.7] or equivalently, for the affine Hecke algebras that occur for unipotent representations of p-adic groups, via the geometric constructions of [KL, Lu2], we know that scaling maps σ_ϵ as above exist and in addition, they also behave well with respect to harmonic analysis:

(b) for every $\epsilon \in [0,1]$, V is unitary (resp., tempered) if and only if $\sigma_\epsilon(V)$ is unitary (resp., tempered);
(c) for every $\epsilon \in (0,1]$, V is a discrete series if and only if $\sigma_\epsilon(V)$ is a discrete series.

Denoting by $\overline{R}(\mathcal{H})_{\mathbb{C}}$ the quotient of $R(\mathcal{H})_{\mathbb{C}}$ by the radical of $\mathrm{EP}_{\mathcal{H}}$, it follows [OS, Proposition 3.9] that the scaling map σ_0 induces an injective isometric map

$$\sigma_0 \colon \overline{R}(\mathcal{H})_{\mathbb{C}} \to \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{s\in T/W} \overline{R}(W_s)_{\mathbb{C}}. \tag{9.11}$$

In fact this map is an isometric isomorphism, for example by the results of [CH1].

9.2. **Elliptic inner products for Weyl groups (after Waldspurger** [Wa2]**).** Let $\mathcal{G} = \mathcal{G}^\circ$ be a complex connected reductive group and $\theta \colon \mathcal{G} \to \mathcal{G}$ a quasi-semisimple automorphism of $\mathcal{G}$ of finite order.

The automorphism θ acts on $\mathbf{I}$ via: $(U,\mathcal{E}) \mapsto (\theta(U), (\theta^{-1})^*(\mathcal{E}))$. Let $\mathbf{I}^\theta$ denote the fixed points of this action and suppose $(U,\mathcal{E}) \in \mathbf{I}^\theta$. If we fix $u \in U$, there exists $g \in \mathcal{G}$ such that $\mathrm{Ad}(x)\circ\theta(u) = u$, hence $\mathrm{Ad}(x)\circ\theta$ preserves $\mathrm{Z}_{\mathcal{G}}(u)$ and hence it defines an automorphism of A_u, denoted θ_u. As explained in [Wa2, p. 612], if $\phi \in \widehat{A}_u$ corresponds to the local system $\mathcal{E}$, the fact that $(\theta^{-1})^*(\mathcal{E}) \cong \mathcal{E}$ is equivalent to the condition that ϕ extends to a representation $\widetilde{\phi}$ of $A_u \rtimes \langle\theta_u\rangle$.

Fix a Borel subgroup B_u of $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ and a maximal torus T_u in B_u. Let $\mathfrak{t}_u$ be the complex Lie algebra of T_u. We define a complex representation $(\delta_u, \mathfrak{t}_u)$ of $A_u \rtimes \langle\theta_u\rangle$, extending the previous definition for the action of A_u. Since A_u acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ by the adjoint action and θ_u acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ as above, every element $z \in A_u \rtimes \langle\theta_u\rangle$ acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ via an automorphism α_z. There exists $y \in \mathrm{Z}_{\mathcal{G}}(u)^\circ$ such that $\alpha_z \circ \mathrm{Ad}(y)$ preserves B_u and T_u. This means that $\alpha_z \circ \mathrm{Ad}(y)$ defines an automorphism of the cocharacter lattice $X_*(T_u)$ which also preserves the sublattice $X_*(Z_{\mathcal{G}}^\circ)$, and therefore a linear isomorphism of $\mathfrak{t}_u$. This is $\delta_u(z)$, and this construction gives a representation of $A_u \rtimes \langle\theta_u\rangle$.

Suppose $(U,\mathcal{E})$ and $(U',\mathcal{E}')$ are two elements of $\mathbf{I}^\theta$ represented by (u,ϕ) and (u',ϕ'), respectively. Define

$$(\widetilde{\phi}, \widetilde{\phi}')_{\theta-\mathrm{ell}} = \begin{cases} (\widetilde{\phi}, \widetilde{\phi}')^{\delta_u}_{\theta-\mathrm{ell}}, & \text{if } U = U', \\ 0, & \text{if } U \neq U'. \end{cases} \tag{9.12}$$

This is θ-elliptic pairing on $\bigoplus_U R(A_u \rtimes \langle\theta_u\rangle)$.

The relation between this elliptic pairing and the generalized Springer correspondence [Lu2] is explained in [Wa2, §3]. The automorphism θ acts naturally on all of the objects involved in the definition of the Springer correspondence. As discussed in [Wa2, §3], this leads to an action of $W_j \rtimes \langle\theta\rangle$ on $\mathfrak{z}_{\mathcal{M}}$, the Lie algebra of the $\mathrm{Z}_{\mathcal{M}}$, and to a θ-generalized Springer correspondence $\nu \colon \mathbf{I}^\theta \to \widetilde{\mathbf{J}}^\theta$. For every $(j,\rho) \in \widetilde{\mathbf{J}}^\theta$, let $\widetilde{\rho}$ denote the extension of ρ to a representation of $W_j \rtimes \langle\theta\rangle$ as in *loc. cit.*.

Let $i = (U,\mathcal{E})$, $i' = (U',\mathcal{E}')$ be two elements of $\mathbf{I}^\theta$, and $\nu(i) = (j,\rho)$, $\nu(i') = (j',\rho')$. For every $m \in \mathbb{Z}$, the constructible sheaf $\mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U$ decomposes as a direct sum of G-equivariant local systems on U. As in [Wa2], setting

$$H^m_{i,i'} = \mathrm{Hom}(\mathcal{E}, \mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U),$$

the automorphism θ defines a linear map $\theta^m_{i,i'} : H^m_{i,i'} \to H^m_{i,i'}$. In particular,

$$H^m_{i,i} = 0 \text{ if } m \neq 0 \text{ and } \dim H^\circ_{i,i} = 1.$$

We may arrange the construction so that $\theta^\circ_{i,i}$ is the identity map. Moreover, it is clear that $H^m_{i,i'} \neq 0$ for some m only if $U \subset \overline{U'}$. (Recall that the restriction of $A_{j',\rho'}$ to $\mathcal{G}_{\mathrm{un}}$) is supported on $\overline{U'}$.)

Define the virtual representation of $W_j \rtimes \langle\theta\rangle$

$$\widetilde{\boldsymbol{\rho}} = \sum_{\rho' \in \widehat{W}_j^\theta} P_{j,\rho,\rho'} \widetilde{\rho}', \text{ where } P_{j,\rho,\rho'} = \sum_{m\in\mathbb{Z}} \mathrm{tr}(\theta^m_{i,i'}). \tag{9.13}$$

In this virtual combination, $P_{j,\rho,\rho} = 1$ and $P_{j,\rho,\rho'} \neq 0$ implies that $U \subset \overline{U'}$ if $(U,\mathcal{E}) = \nu^{-1}(j,\rho)$ and $(U',\mathcal{E}') = \nu^{-1}(j,\rho')$.

Example 9.6. When θ is the trivial automorphism of $\mathcal{G}$ and $j = j_0$ (the case of the classical Springer correspondence), $\widetilde{\boldsymbol{\rho}}$ can be identified with the reducible W-representation on the ϕ-isotypic component ($\phi \in \widehat{A}_u$ corresponding to $\mathcal{E}$) of the total cohomology of the Springer fiber of u.

Consider the θ-elliptic pairing $(\ ,\ )^{W_j}_{\theta-\mathrm{ell}}$ on $\bigoplus_{j\in\mathbf{J}^\theta} R(W_j \rtimes \langle\theta\rangle)$, defined on each summand via the action of $W_j \rtimes \langle\theta\rangle$ on $\mathfrak{z}_{\mathcal{M}}$ and extended orthogonally to the direct sum.

Theorem 9.7 ([Wa2, Théorème p. 616])**.** *Let $i = (U,\mathcal{E})$, $i' = (U',\mathcal{E}')$ be two elements of $\mathbf{I}^\theta$, and $\nu(i) = (j,\rho)$, $\nu(i') = (j',\rho')$. Let (u,ϕ), (u',ϕ'), $\phi \in \widehat{A}_u$, $\phi' \in \widehat{A}_{u'}$, be representatives for i, i', respectively. Then*

$$(\widetilde{\phi},\widetilde{\phi}')_{\theta-\mathrm{ell}} = (\widetilde{\boldsymbol{\rho}},\widetilde{\boldsymbol{\rho}}')^{W_j}_{\theta-\mathrm{ell}}.$$

9.3. **The proof of Theorem 9.1: the case of adjoint groups.** In this subsection, suppose that G is a simple F-split group of adjoint type. This means that $G^\vee$ is simply connected, hence, for every $s \in T^\vee$, $\mathrm{Z}_{G^\vee}(s)$ is connected. We may apply Theorem 9.7 to

$$\mathcal{G} = \mathrm{Z}_{G^\vee}(s) \text{ and } \theta \text{ the trivial automorphism.}$$

Denote $\mathbf{I}^s = \mathbf{I}^{\mathrm{Z}_{G^\vee}(s)}$, $\mathbf{J}^s = \mathbf{J}^{\mathrm{Z}_{G^\vee}(s)}$, and $\widetilde{\mathbf{J}}^s = \widetilde{\mathbf{J}}^{\mathrm{Z}_{G^\vee}(s)}$, so that the generalized Springer correspondence for $\mathrm{Z}_{G^\vee}(s)$ is the map

$$\nu_s \colon \mathbf{I}^s \to \widetilde{\mathbf{J}}^s,\ (U,\mathcal{E}) \mapsto (j,\rho),$$

and

$$\widetilde{\boldsymbol{\rho}} = \sum_{\rho' \in \widehat{W}_j} P_{j,\rho,\rho'} \rho', \text{ where } P_{j,\rho,\rho'} = \sum_{m\in\mathbb{Z}} \dim \mathrm{Hom}(\mathcal{E}, \mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U).$$

For convenience, let us also denote

$$\widetilde{\nu}_s \colon \mathbf{I}^s \to \widetilde{\mathbf{J}}^s,\ (U,\mathcal{E}) \mapsto (j,\widetilde{\boldsymbol{\rho}}). \tag{9.14}$$

Recall that for every semisimple element $s \in G^\vee$, $\mathcal{G}^p_s = \mathrm{Z}_{G^\vee}(s)$.

Proposition 9.8. *The maps $\widetilde{\nu}_s$ from (9.14) induce an isometric isomorphism*

$$\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathrm{ss}}} \overline{R}(A_{\mathcal{G}^p_s}(u)) \cong \bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathrm{ss}}} \bigoplus_{j\in\mathbf{J}^s} \overline{R}(W_j), \quad (\phi,\phi')^{\delta^s_u}_{\mathrm{ell}} = (\widetilde{\nu}_s(\phi),\widetilde{\nu}_s(\phi'))^{W_j}_{\mathrm{ell}}.$$

Proof. This is immediate from Theorem 9.7 applied to each $\mathcal{G}^p_s$. □

9.4. **Extending Theorem 9.1.** In order to extend the results to the case when G is simple F-split, but not adjoint, we need first some results about Mackey induction. We follow a construction from [CH1, §4.2]. Suppose H' is a finite group, H a normal subgroup of H', and $H'/H = \mathfrak{R}$ is abelian. The groups $H', \mathfrak{R}$ act on $\widehat{H}$. For every H-character χ, and $\gamma \in \mathfrak{R}$, denote by ${}^{\gamma}\chi$ the H-character ${}^{\gamma}\chi(h) = \chi(\gamma^{-1}h\gamma)$ (it doesn't depend on the choice of coset representative γ).

If $\sigma \in \widehat{H}$, let $\mathfrak{R}_\sigma$ and H_σ denote the corresponding isotropy groups of σ. For each $\gamma \in \mathfrak{R}_\sigma$, fix an isomorphism $\phi_\gamma \colon {}^{\gamma}\sigma \to \sigma$ and define the twisted trace as $\mathrm{tr}_\gamma(\sigma)(h) = \mathrm{tr}(\sigma(h) \circ \phi_\gamma)$, $h \in H$. The choices of ϕ_γ (each unique up to scalar) define a factor set, or a 2-cocyle, $\beta_\sigma : \mathfrak{R}_\sigma \times \mathfrak{R}_\sigma \to \mathbb{C}^\times$.

Remark 9.9. We assume that the action of $\mathfrak{R}$ can be normalized so that β_σ is trivial. This is the case for example when $\mathfrak{R}$ is cyclic.

If τ is a (virtual) $\mathfrak{R}_\sigma$-representation, we may form the Mackey induced (virtual) H'-representation

$$\sigma \rtimes \tau = \mathrm{Ind}_{H'_\sigma}^{H'}(\sigma \otimes \tau).$$

If τ is an irreducible $\mathfrak{R}_\sigma$-representation, then $\sigma \rtimes \tau$ is an irreducible H'-representation. In fact, $\widehat{H'} = \{\sigma \rtimes \tau \mid \sigma \in \mathfrak{R}\backslash\widehat{H},\ \tau \in \widehat{\mathfrak{R}}_\sigma\}$.

Given $\gamma \in \mathfrak{R}$, if $\gamma \in \mathfrak{R}_\sigma$, define $\tau_{\sigma,\gamma}$ to be the virtual $\mathfrak{R}_\sigma$-representation whose character is the delta function on γ. Then $\{\sigma \rtimes \tau_{\sigma,\gamma} \mid \sigma \in \mathfrak{R}\backslash\widehat{H},\ \gamma \in \mathfrak{R}\}$ is a basis of $R(H')$. As in [CH1, Lemma 4.2.2]

$$\chi_{\sigma \rtimes \tau_{\sigma,\gamma}}(h) = \begin{cases} 0, & \text{if } h \notin H\gamma, \\ \sum_{\gamma' \in \mathfrak{R}/\mathfrak{R}_\sigma} {}^{\gamma'}(\mathrm{tr}_\gamma(\sigma))(h), & \text{if } h \in H\gamma. \end{cases} \tag{9.15}$$

Notice that

$$H'/H'_\sigma \cong \mathfrak{R}/\mathfrak{R}_\sigma$$

indexes the $\mathfrak{R}$-orbit (equivalently, the H'-orbit) of $\sigma \in \widehat{H}$. Suppose H' is endowed with a representation δ and we define the corresponding elliptic pairings $(\ ,\ )_{\mathrm{ell}}^{H'}$ and $(\ ,\ )_{\gamma-\mathrm{ell}}^{H}$ (for each $\gamma \in \mathfrak{R}$).

Lemma 9.10. *For every $\gamma_1, \gamma_2 \in \mathfrak{R}$ and every $\sigma_1, \sigma_2 \in \widehat{H}$, the H'-elliptic pairing is given by*

$$(\sigma_1 \rtimes \tau_{\sigma_1,\gamma_1}, \sigma_2 \rtimes \tau_{\sigma_2,\gamma_2})_{\mathrm{ell}}^{H'} = \begin{cases} \frac{1}{|\mathfrak{R}|} \sum_{\gamma' \in \mathfrak{R}/\mathfrak{R}_{\sigma_1}, \gamma'' \in \mathfrak{R}/\mathfrak{R}_{\sigma_2}} ({}^{\gamma'}\sigma_1, {}^{\gamma''}\sigma_2)_{\gamma_1-\mathrm{ell}}^{H}, & \text{if } \gamma_1 = \gamma_2, \\ 0, & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

Proof. The orthogonality of the two characters when $\gamma_1 \neq \gamma_2$ follows at once since the first is supported on $\gamma_1 H$ and the second on $\gamma_2 H$. The first formula follows from (9.15) by explicitating the definition of the elliptic pairing. □

Lemma 9.10 allows thus to extend the proof of Theorem 9.7 to the case when $\mathcal{G}'$ is disconnected as long as:

($\star$): the cocycles $\sharp_j$ that occur in the disconnected Springer correspondence (3.5) can be trivialized.

Proposition 9.11. *Retain the notation from Section 3 and suppose that ($\star$) holds. Let $\nu : \mathbf{I}^{\mathcal{G}} \to \widetilde{\mathbf{J}}^{\mathcal{G}}$ be the generalized Springer correspondence (3.5). Let $i = (U, \mathcal{E})$, $i' = (U', \mathcal{E}')$ be two elements of $\mathbf{I}^{\mathcal{G}}$, and $\nu(i) = (j, \rho)$, $\nu(i') = (j', \rho')$. Let (u, ϕ), (u', ϕ'), $\phi \in \widehat{A}_u$, $\phi' \in \widehat{A}_{u'}$ be representatives for i, i', respectively. Then*

$$(\phi, \phi')_{\mathrm{ell}} = \begin{cases} 0, & \text{if } (j, U) \neq (j', U'), \\ (\phi, \phi')_{\mathrm{ell}}^{A_u} = (\boldsymbol{\rho}, \boldsymbol{\rho}')_{\mathrm{ell}}^{W_j}, & \text{if } (j, U) = (j', U') \text{ and } \mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u) = \mathrm{Z}_{\mathcal{G}}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u). \end{cases}$$

Proof. Suppose $j = j'$, otherwise the claim is true by definition. Let $\rho_1^\circ, \rho_2^\circ \in \widehat{W_j^\circ}$ and suppose that they have unipotent supports U_1°, U_2°, respectively, in the connected generalized Springer correspondence such that $\mathcal{G} \cdot U_1^\circ \neq \mathcal{G} \cdot U_2^\circ$. Let $\boldsymbol{\rho_i^\circ}$, $i = 1, 2$ be the corresponding reducible Springer representations, as in Section 9.2. We assume that they all have appropriate twisted extensions as in Section 9.2 and drop $\widetilde{}$ from the notation. Using Lemma 9.10 applied to $H = W_j^\circ$, $H' = W_j$, $\mathfrak{R} = \mathfrak{R}_j$, for every $\gamma \in (\mathfrak{R}_j)_{\rho_1^\circ} \cap (\mathfrak{R}_j)_{\rho_2^\circ}$,

$$(\boldsymbol{\rho_1^\circ} \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho_2^\circ} \rtimes \tau_{\rho_1^\circ,\gamma})_{\mathrm{ell}}^{W_j} = \frac{1}{|\mathfrak{R}_j|} \sum_{\gamma',\gamma''} ({}^{\gamma'}\boldsymbol{\rho_1^\circ}, {}^{\gamma''}\boldsymbol{\rho_2^\circ})_{\gamma-\mathrm{ell}}^{W_j} = 0,$$

by Theorem 9.7. We used implicitly here that the stabilizer in $\mathfrak{R}_j$ of ρ_i° and $\boldsymbol{\rho_i^\circ}$ are the same. In conjunction with the second claim in Lemma 9.10, this implies that $(\boldsymbol{\rho_1^\circ} \rtimes \tau_1, \boldsymbol{\rho_2^\circ} \rtimes \tau_2)_{\mathrm{ell}}^{W_j} = 0$ for all $\tau_i \in \widehat{(\mathfrak{R}_j)}_{\rho_i^\circ}$, $i = 1, 2$. Hence $(\boldsymbol{\rho_1}, \boldsymbol{\rho_2})_{\mathrm{ell}}^{W_j} = 0$ whenever ρ_1, ρ_2 have distinct unipotent (disconnected) Springer support, which proves the first part of the claim.

Now assume that $u = u'$ and $\phi, \phi' \in \widehat{A}_u$. Suppose ρ_i° occurs in the restriction of ρ_i to W_j° and that $\phi_i^\circ \in \widehat{A}_{\mathcal{G}^\circ}(u)$ (which necessarily occurs in the restriction of ϕ_i) corresponds to ρ_i° in the connected generalized Springer correspondence. We observe that there is a natural injection

$$\mathfrak{R}_j = W_j/W_j^\circ \simeq \mathrm{N}_{\mathcal{G}}(j)/\mathrm{N}_{\mathcal{G}^\circ}(\mathcal{M}^\circ) \hookrightarrow \mathcal{G}/\mathcal{G}^\circ. \tag{9.16}$$

Hence every $\gamma \in \mathfrak{R}_j$ can be regarded as au automorphism of $\mathcal{G}^\circ$ and in particular, Theorem 9.7 can be applied with γ in place of θ. We wish to compare $(\boldsymbol{\rho_1^\circ} \rtimes \tau_{\rho_1^\circ,\gamma_1}, \boldsymbol{\rho_2^\circ} \rtimes \tau_{\rho_2^\circ,\gamma_2})_{\mathrm{ell}}^{W_j}$ and $(\phi_1^\circ \rtimes \tau_{\phi_1^\circ,\gamma_1}, \phi_2^\circ \rtimes \tau_{\phi_2^\circ,\gamma_2})_{\mathrm{ell}}^{A_u}$. By [AMS1, Lemma 4.4],

$$(\mathfrak{R}_j)_{\rho_i^\circ} \cong (W_j)_{\rho_i^\circ}/W_j \cong (A_u)_{\phi_i^\circ}/A_{\mathcal{G}^\circ}(u), \quad i = 1, 2,$$

which implies that there is an identification between γ_i, $i = 1, 2$ for the ρ_i°'s and for the ϕ_i°'s in the setting of Lemma 9.10. Hence if $\gamma_1 \neq \gamma_2$, both elliptic products are zero.

Suppose $\gamma_1 = \gamma_2 = \gamma \in (\mathfrak{R}_j)_{\rho_1^\circ} \cap (\mathfrak{R}_j)_{\rho_2^\circ} = ((A_u)_{\phi_1^\circ} \cap (A_u)_{\phi_2^\circ})/A_{\mathcal{G}^\circ}(u)$. To simplify the formulas, set $n_i = |(\mathfrak{R}_j)_{\rho_i^\circ}| = |(A_u)_{\phi_1^\circ}/A_{\mathcal{G}^\circ}(u)$, $i = 1, 2$. Then, firstly by Lemma 9.10 and secondly by Theorem 9.7,

$$\begin{aligned} n_1 n_2 |\mathfrak{R}_j| (\boldsymbol{\rho_1^\circ} \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho_2^\circ} \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} &= \sum_{\gamma',\gamma'' \in \mathfrak{R}_j} ({}^{\gamma'}\boldsymbol{\rho_1^\circ}, {}^{\gamma''}\boldsymbol{\rho_2^\circ})_{\gamma-\mathrm{ell}}^{W_j^\circ} \\ &= \sum_{u_0 \in \mathfrak{R}_j \mathcal{G}^\circ \cdot u/\mathcal{G}^\circ} \sum_{\substack{\gamma',\gamma'' \in \mathfrak{R}_j \\ \mathcal{G}^\circ \cdot ({}^{\gamma'}u_0) = \mathcal{G}^\circ \cdot ({}^{\gamma''}u_0) = \mathcal{G}^\circ \cdot u_0}} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u_0)}. \end{aligned}$$

The first sum is over the representatives of the $\mathcal{G}^\circ$-orbits that are conjugate to $\mathcal{G}^\circ \cdot u$ via $\mathfrak{R}_j$. Since the corresponding summand for two different $\mathcal{G}^\circ$-orbits $u_0, u_0' \in \mathcal{G} \cdot u$ are equal (as they are related by an outer automorphism of $\mathcal{G}^\circ$), it follows that

$$n_1 n_2 \frac{|\mathfrak{R}_j|}{N_u} (\boldsymbol{\rho_1^\circ} \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho_2^\circ} \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} = \sum_{\substack{\gamma',\gamma'' \in \mathfrak{R}_j \\ \mathcal{G}^\circ \cdot ({}^{\gamma'}u) = \mathcal{G}^\circ \cdot ({}^{\gamma''}u) = \mathcal{G}^\circ \cdot u}} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}, \tag{9.17}$$

where N_u is the number of $\mathcal{G}^\circ$-conjugacy classes in $\mathfrak{R}_j \mathcal{G}^\circ \cdot u$. It is easy to see (using orbit-stabilizer counting) that $\frac{|\mathfrak{R}_j|}{N_u} = |\mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)|$. Moreover, an element $\gamma \in \mathfrak{R}_j$ has the property that $\mathcal{G}^\circ \cdot ({}^\gamma u) = \mathcal{G}^\circ \cdot u$ if and only if $\gamma \in \mathrm{Z}_{\mathcal{G}}(u) \bmod \mathcal{G}^\circ$. Hence (9.17) becomes:

$$n_1 n_2 |\mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)| (\boldsymbol{\rho_1^\circ} \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho_2^\circ} \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} = \sum_{\gamma',\gamma'' \in \mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}.$$

On the other hand, applying Lemma 9.10 to A_u, we get

$$n_1 n_2 |A_u/A_{\mathcal{G}^\circ}(u)|(\phi_1^\circ \rtimes \tau_{\phi_1^\circ,\gamma_1}, \phi_2^\circ \rtimes \tau_{\phi_2^\circ,\gamma_2})_{\mathrm{ell}}^{A_u} = \sum_{r',r''\in A_u/A_{\mathcal{G}^\circ}(u)} ({}^{r'}\phi_1^\circ, {}^{r''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}. \tag{9.18}$$

Notice that

$$A_u/A_{\mathcal{G}^\circ}(u) \cong \mathrm{Z}_{\mathcal{G}}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u) \hookrightarrow \mathcal{G}/\mathcal{G}^\circ.$$

The claim follows.

□

Remark 9.12. (1) As remarked in the proof of Proposition 9.11, there is an injection $\mathfrak{R}_j \to \mathcal{G}/\mathcal{G}^\circ$. In our case, $\mathcal{G} = \mathrm{Z}_{G^\vee}(s)$ for a semisimple element s. Then using a well-known result [SpSt], $\mathcal{G}/\mathcal{G}^\circ$ is a subgroup of C, the kernel of the covering $G^\vee_{\mathrm{sc}} \to G^\vee$, in particular, $\mathcal{G}/\mathcal{G}^\circ$ is a subgroup of $\mathrm{Z}(G^\vee_{\mathrm{sc}})$. This implies that if $G^\vee$ is simple, the only case when $\mathcal{G}/\mathcal{G}^\circ$ might not be cyclic is if $G^\vee = \mathsf{PSO}_{4m}(\mathbb{C})$. Hence, by Remark 9.9, assumption $(\star)$ holds for all simple groups $G^\vee$ with the possible exception of $\mathsf{PSO}_{4m}(\mathbb{C})$.

(2) The condition $\mathrm{Z}_{\mathfrak{R}_j\mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u) = \mathrm{Z}_{\mathcal{G}}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)$ in Proposition 9.11 should not be needed and we do not know if it is always satisfied. When $j = j_0$ is the cuspidal datum associated to the trivial local system on the maximal torus of $\mathcal{G}^\circ$, the $\mathfrak{R}_{j_0} = \mathcal{G}/\mathcal{G}^\circ$, hence this condition holds automatically.

As a consequence of Proposition 9.11 and Remark 9.12, we have:

Corollary 9.13. *Let G is a simple split F-group other than $\mathsf{Spin}_{4m}(F)$. Then Conjecture 8.1 holds true for all Iwahori-spherical representations of the pure inner twists of G, in the sense of (8.5).*

10. $\mathsf{Sp}_4(F)$

As a useful example, we present the case $G = \mathsf{Sp}_4(F)$. Firstly, there are 6 unipotent representations of the finite group $\mathsf{Sp}_4(\mathbb{F}_q)$: 5 in bijection with the finite Weyl group of type C_2 and one cuspidal representation θ. Using Lusztig's notation for the irreducible representations of the Weyl group of type B/C, there are 3 families $\mathcal{F}$ of unipotent representations with associated finite groups Γ as follows:

- $\Gamma = \{1\}$, $\mathcal{F} = \{2 \times \emptyset\}$;
- $\Gamma = \{1\}$, $\mathcal{F} = \{0 \times 11\}$;
- $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{F} = \{1 \times 1, 11 \times \emptyset, \emptyset \times 2, \theta\}$ with associated parameters, in order, $M(\Gamma) = \{(1, \mathbf{1}), (-1, \mathbf{1}), (1, \epsilon), (-1, \epsilon)\}$.

For the $\mathbb{Z}/2\mathbb{Z}$-family, the stable combinations are:

$$\begin{aligned} \sigma(1,1) &= 1 \times 1 + \emptyset \times 2, & \sigma(-1,1) &= 11 \times \emptyset + \theta, \\ \sigma(1,-1) &= 1 \times 1 - \emptyset \times 2, & \sigma(-1,-1) &= 11 \times \emptyset - \theta, \end{aligned} \tag{10.1}$$

and Lusztig's Fourier transform acts by the flip $\sigma(x,y) \mapsto \sigma(y,x)$. For the singleton families, the Fourier transform is the identity.

Next, to the p-adic group $\mathsf{Sp}_4(F)$: the unipotent representations are parameterized by data in the dual group $G^\vee = \mathsf{SO}_5(\mathbb{C})$. In particular, the list of unipotent classes u and their attached groups Γ_u is:

u	Γ_u
(5)	1
(311)	$\mathsf{S}(\mathsf{O}_1 \times \mathsf{O}_2) \cong \mathsf{O}_2$
(221)	Sp_2
(1^5)	SO_5

$\pi(u,s,\phi)$	$K_0 \to \mathsf{Sp}_4(\mathbb{F}_q)$	$K_1 \to \mathsf{Sp}_2(\mathbb{F}_q)^2$	$K_2 \to \mathsf{Sp}_4(\mathbb{F}_q)$
$(s_0, \mathbf{1})$	$1\times 1 + \emptyset \times 11$	$\mathbf{1}\boxtimes\epsilon + \epsilon\boxtimes\mathbf{1} + \epsilon\boxtimes\epsilon$	$1\times 1 + \emptyset\times 11$
(s_0, ϵ)	$\emptyset \times 2$	$\epsilon\boxtimes\epsilon$	$\emptyset\times 2$
$(s_1, \mathbf{1})$	$\emptyset\times 2 + \emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon + \epsilon\boxtimes\epsilon$	1×1
(s_1, ϵ)	1×1	$\epsilon\boxtimes\mathbf{1} + \epsilon\boxtimes\epsilon$	$\emptyset\times 2 + \emptyset\times 11$
$(s_2, \mathbf{1}\boxtimes\mathbf{1})$	$\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon$	$11\times\emptyset$
$(s_2, \epsilon\boxtimes\mathbf{1})$	θ_{K_0}	0	0
$(s_2, \mathbf{1}\boxtimes\epsilon)$	0	0	θ_{K_2}
$(s_2, \epsilon\boxtimes\epsilon)$	$11\times\emptyset$	$\epsilon\boxtimes\mathbf{1}$	$\emptyset\times 11$

TABLE 1. Elliptic $\mathsf{Sp}_4(F)$-representations attached to $u = (311) \in \mathsf{SO}_5$

The interesting case is $u = (311)$. Write

$$\Gamma_u = \langle z, \delta \mid z \in \mathbb{C}^\times,\ \delta^2 = 1,\ \delta z \delta^{-1} = z^{-1}\rangle.$$

Then $\mathrm{Z}_{\Gamma_u}(\pm\delta) = A_{\Gamma_u}(\pm\delta) = \{\pm 1, \pm\delta\} \cong C_2 \times C_2$ and $A_{\Gamma_u}(\pm 1) = \{1, \delta\} \cong C_2$. There are 6 conjugacy classes of elliptic pairs:

$$[(\pm 1, \delta)], \quad [(\delta, \pm 1)], \quad [(\delta, \pm\delta)], \tag{10.2}$$

and the flip acts as

$$\mathsf{flip}([(\pm 1, \delta)]) = [(\delta, \pm 1)], \quad \mathsf{flip}([(\delta, \pm\delta)]) = [(\delta, \pm\delta)]. \tag{10.3}$$

There are three conjugacy classes of isolated semisimple elements in $T^\vee = \{(a, b) \mid a, b \in \mathbb{C}^\times\}$ in $\mathsf{SO}_5(\mathbb{C})$. In this notation, the Weyl group $W(B_2)$ acts on T by flips and inverses. The representatives of the three classes are:

- $s_0 = (1, 1)$, $Z_{G^\vee}(s_0) = \mathsf{SO}_5$;
- $s_1 = (-1, 1)$, $Z_{G^\vee}(s_1) = \mathsf{S}(\mathsf{O}_2 \times \mathsf{O}_3)$;
- $s_2 = (-1, -1)$, $Z_{G^\vee}(s_2) = \mathsf{S}(\mathsf{O}_1 \times \mathsf{O}_4) \cong \mathsf{O}_4$.

All three s_0, s_1, s_2 occur in $\Gamma_u = \mathsf{O}_2$ and in the notation above for $\mathsf{O}_2 = \langle z, \delta\rangle$, they are

$$s_0 \leftrightarrow 1 \in \mathsf{O}_2, \quad s_1 \leftrightarrow -1 \in \mathsf{O}_2, \quad s_0 \leftrightarrow \delta \in \mathsf{O}_2.$$

Consequently, there are 8 elliptic tempered representations of the form $\pi(s_i, u, \phi)$, $i = 0, 1, 2$, $u = (311)$: 6 are Iwahori-spherical, and 2 are supercuspidal. Out of these, 4 are discrete series representations, all those for $s_2 = \delta$. The parahoric restrictions are given in Table 1. We computed them using the same method as in [Re2, (6.2)], but since in our case $G^\vee$ is not simply connected, we also need to involve the Mackey induction for graded affine Hecke algebras attached to disconnected groups.

The corresponding stable combinations are

$$\begin{aligned}
\Pi(u, 1, \delta) &= \pi(u, s_0, \mathbf{1}) - \pi(u, s_0, \epsilon),\\
\Pi(u, -1, \delta) &= \pi(u, s_1, \mathbf{1}) - \pi(u, s_1, \epsilon),\\
\Pi(u, \delta, 1) &= \pi(u, s_2, \mathbf{1}\boxtimes\mathbf{1}) + \pi(u, s_2, \mathbf{1}\boxtimes\epsilon) + \pi(u, s_2, \epsilon\boxtimes\mathbf{1}) + \pi(u, s_2, \epsilon\boxtimes\epsilon),\\
\Pi(u, \delta, -1) &= \pi(u, s_2, \mathbf{1}\boxtimes\mathbf{1}) + \pi(u, s_2, \mathbf{1}\boxtimes\epsilon) - \pi(u, s_2, \epsilon\boxtimes\mathbf{1}) - \pi(u, s_2, \epsilon\boxtimes\epsilon),\\
\Pi(u, \delta, \delta) &= \pi(u, s_2, \mathbf{1}\boxtimes\mathbf{1}) - \pi(u, s_2, \mathbf{1}\boxtimes\epsilon) + \pi(u, s_2, \epsilon\boxtimes\mathbf{1}) - \pi(u, s_2, \epsilon\boxtimes\epsilon),\\
\Pi(u, \delta, -\delta) &= \pi(u, s_2, \mathbf{1}\boxtimes\mathbf{1}) - \pi(u, s_2, \mathbf{1}\boxtimes\epsilon) - \pi(u, s_2, \epsilon\boxtimes\mathbf{1}) + \pi(u, s_2, \epsilon\boxtimes\epsilon).
\end{aligned}$$

The corresponding parahoric restrictions are in Table 2.

One can easily verify by inspection using Table 2 that the conjecture holds in this case.

$\Pi(u,s,h)$	K_0	K_1	K_2
$(1,\delta)$	$(1\times 1-\emptyset\times 2)+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon+\epsilon\boxtimes\mathbf{1}$	$(1\times 1-\emptyset\times 2)+\emptyset\times 11$
$(\delta,1)$	$(11\times\emptyset+\theta_{K_0})+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon+\epsilon\boxtimes\mathbf{1}$	$(11\times\emptyset+\theta_{K_2})+\emptyset\times 11$
$(-1,\delta)$	$(-1\times 1+\emptyset\times 2)+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon-\epsilon\boxtimes\mathbf{1}$	$(1\times 1-\emptyset\times 2)-\emptyset\times 11$
$(\delta,-1)$	$(-11\times\emptyset-\theta_{K_0})+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon-\epsilon\boxtimes\mathbf{1}$	$(11\times\emptyset+\theta_{K_2})-\emptyset\times 11$
(δ,δ)	$(-11\times\emptyset+\theta_{K_0})+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon-\epsilon\boxtimes\mathbf{1}$	$(11\times\emptyset-\theta_{K_2})-\emptyset\times 11$
$(\delta,-\delta)$	$(11\times\emptyset-\theta_{K_0})+\emptyset\times 11$	$\mathbf{1}\boxtimes\epsilon+\epsilon\boxtimes\mathbf{1}$	$(11\times\emptyset-\theta_{K_2})+\emptyset\times 11$

TABLE 2. Elliptic $\mathsf{Sp}_4(F)$ stable combinations attached to $u=(311)\in\mathsf{SO}_5$

11. $\mathsf{SL}_n(F)$

11.1. **Elliptic pairs for $G^\vee=\mathsf{PGL}_n(\mathbb{C})$.** Consider the case $G^\vee=\mathsf{PGL}_n(\mathbb{C})$. Let Z denote the centre of $\mathsf{GL}_n(\mathbb{C})$. In the Weyl group of type A_{n-1} ($W=S_n$), denote by $\dot{w}_n$ the permutation matrix corresponding to the n-cycle $(1,2,3,\dots,n)$. For every n-th root z of 1, let

$$\Delta_n(z)=\mathrm{diag}(1,z,z^2,\dots,z^{n-1})Z\in\mathsf{PGL}_n(\mathbb{C}).$$

Fix ζ_n a primitive n-th root of 1 and set $s_n=\Delta_n(\zeta_n)$. Notice that $\dot{w}_n$ and s_n commute in $\mathsf{PGL}_n(\mathbb{C})$.

Lemma 11.1. *Suppose $\Gamma=\mathsf{PGL}_n(\mathbb{C})$. Then*

$$\mathcal{Y}(\Gamma)_{\mathrm{ell}}=\bigsqcup_{k\in(\mathbb{Z}/n\mathbb{Z})^\times}\Gamma\cdot(s_n,\dot{w}_n^k).$$

In particular, there are $\varphi(n)$ Γ-orbits in $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$. The flip $(s,h)\to(h,s)$ induces the following map on Γ-orbits in $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$:

$$\mathsf{flip}\colon([(s_n,\dot{w}_n^k)])\to[(s_n,\dot{w}_n^{-k})],\quad k\in(\mathbb{Z}/n\mathbb{Z})^\times.$$

Proof. Let T be the diagonal torus in Γ. By Proposition 7.9, the only possible elliptic pairs are conjugate to $(s,\dot{w})$ where w is elliptic and s is regular such that $s\in T^w$. If the group is semisimple of type A_{n-1}, then the only elliptic elements of the Weyl group are the n-cycles. We may assume that $\dot{w}=\dot{w}_n$. It is easy to see that

$$T^{\dot{w}_n}=\{\Delta_n(z)\mid z^n=1\}.$$

Since $s\in T^{\dot{w}_n}$ needs to be regular, it follows that the corresponding z must be a primitive root of 1.

Now fix $s=s_n$. Every other $\Delta(\zeta')$ with ζ' a primitive n-th root is conjugate in Γ to s_n. The centralizer is $\mathrm{Z}_\Gamma(s)=\langle T,\dot{w}_n^k\mid k\in\mathbb{Z}/n\mathbb{Z}\rangle\cong T\rtimes\mathbb{Z}/n\mathbb{Z}$. This means that $\dot{w}_n^i$ is conjugate to $\dot{w}_n^j$ in $\mathrm{Z}_\Gamma(s)$ if and only if $i=j$. On the other hand, $\dot{w}_n^k$ is elliptic if and only if $k\in(\mathbb{Z}/n\mathbb{Z})^\times$, hence the claim follows.

For the claim about the Fourier transform, let $x\in\mathsf{GL}_n(\mathbb{C})$ be such that $x^{-1}\dot{w}_nx=s_n$, where x is the matrix corresponding to a basis of eigenvectors of $\dot{w}_n$. Then a calculation shows that

$$x^{-1}s_nx=\dot{w}_n^{-1}\text{ in }\mathsf{PGL}_n(\mathbb{C}).$$

From this:

$$\mathsf{flip}([(s_n,\dot{w}_n^k)])=[(w_n^k,s_n)]=[(xs_n^kx^{-1},s_n)]=[(s_n^k,x^{-1}s_nx)]=[(s_n^k,\dot{w}_n^{-1})].$$

Finally, let p be a permutation matrix such that $p^{-1}s_n^kp=s_n$ (this exists since k is coprime to n). This has the effect $p^{-1}\dot{w}_np=\dot{w}_n^k$, hence $[(s_n^k,\dot{w}_n^{-1})]=[(s_n,\dot{w}_n^{-k})]$.

□

Now let u be a unipotent element in $\mathsf{PGL}_n(\mathbb{C})$. Via the Jordan canonical form, u is parameterized by a partition λ of n, where we write $\lambda = (\underbrace{1,\dots,1}_{r_1}, \underbrace{2,\dots,2}_{r_2}, \dots, \underbrace{\ell,\dots,\ell}_{r_\ell})$. As it well known (see for example [CM, Theorem 6.1.3])

$$\Gamma_u = \left(\prod_{i=1}^{\ell} \mathsf{GL}_{r_i}(\mathbb{C})^i_\Delta\right)/Z, \tag{11.1}$$

where H^i_Δ means H embedded diagonally into the product of i copies of H. In particular, Γ_u is connected. Let T_r denote the diagonal torus in GL_r, Z_r the center of GL_r, $\bar{T}_r = T_r/Z_r$, and $W_r \cong S_r$ the Weyl group. A maximal torus in Γ_u is $T_u = \prod_{i=1}^{\ell}(T_{r_i})^i_\Delta/Z$ and the Weyl group is $W_u = \prod_{i=1}^{\ell}(W_{r_i})^i_\Delta$.

Let $w = \prod_{i=1}^{\ell}(w_i)^i_\Delta \in W_u$, $w_i \in W_{r_i}$ be given. We need T_u^w to be finite. The morphism

$$\pi\colon T_u \twoheadrightarrow \prod_{i=1}^{\ell}(\bar{T}_{r_i})^i_\Delta,\ (t_i)_i \bmod Z \mapsto (t_i \bmod Z_i)$$

is surjective and W_u-equivariant. Since $(\bar{1},\dots,\bar{1},(\bar{T}^{w_i}_{r_i})^i_\Delta,\bar{1},\dots,\bar{1}) \subset T_u^w$ for each i, it follows that w_i is elliptic for PGL_{r_i}, hence, each w_i is an r_i-cycle.

Proposition 11.2. *For $u \in \mathsf{PGL}_n(\mathbb{C})$, $\mathcal{Y}(\Gamma)_{\mathrm{ell}} \neq \emptyset$ if and only if the corresponding partition λ is a rectangular, i.e., $\lambda = (\underbrace{i,\dots,i}_{r_i})$ for some i. In this case,*

$$\Gamma_u = \mathsf{GL}_{r_i}(\mathbb{C})^i_\Delta/Z \cong \mathsf{PGL}_{r_i}(\mathbb{C}),$$

so $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$ is as in Lemma 11.1.

Proof. Let $w \in W_u$ be elliptic as above. We pass to the Lie algebra $\mathfrak{t}_u = \mathfrak{s}(\bigoplus(\mathfrak{t}_{r_i})^i_\Delta)$. Since $\mathfrak{t}^{w_i}_{r_i} = \mathbb{C}\cdot \mathrm{Id}_{r_i}$, we see that

$$\mathfrak{t}_u^w = \{(a_1\mathrm{Id}_{r_1}, a_2\mathrm{Id}_{r_2}, a_2\mathrm{Id}_{r_2}, \dots, \underbrace{a_i\mathrm{Id}_{r_i},\dots,a_i\mathrm{Id}_{r_i}}_{i}, \dots) \mid \sum_{i=1}^{\ell} ia_i = 0\}.$$

The element w is elliptic if and only if $\mathfrak{t}_u^w = 0$. From the condition $\sum_{i=1}^{\ell} ia_i = 0$, we that this can only happen if there exists a unique i such that $r_i \neq 0$. □

Corollary 11.3. *The number of orbits of elliptic pairs for $\mathsf{PGL}_n(\mathbb{C})$ is*

$$\sum_{u \text{ unipotent class}} |\Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}| = n.$$

Proof. From Proposition 11.2, the only unipotent classes that contribute are the rectangular ones, which are in one-to-one correspondence with divisors d of n. For the unipotent class $u = (\underbrace{n/d,\dots,n/d}_{d})$, Lemma 11.1 says that there are $\varphi(d)$ orbits of elliptic pairs. Hence the total number is $\sum_{d|n}\varphi(d) = n$.

□

11.2. **Elliptic unipotent representations of $\mathsf{SL}_n(F)$.** It is instructive to make explicit the elliptic correspondence Conjecture 8.1 for $G = \mathsf{SL}_n(F)$. Let $K_0 = \mathsf{SL}_n(\mathfrak{o}_F)$ and let $I \subset K_0$ be an Iwahori subgroup. Let $\mathcal{H}(G,I) = \{f \in C_c^\infty(G) \mid f(i_1gi_2) = f(g),\ \text{for all } i_1, i_2 \in I\}$ be the Iwahori-Hecke algebra (under convolution with respect to a fixed Haar measure). The algebra $\mathcal{H}(g,I)$ is naturally isomorphic to the affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, \sqrt{q}, 1)$, where q is the order of the residual field of F and $\mathcal{R}$ is the root datum for $\mathsf{PGL}_n(\mathbb{C})$.

Every irreducible unipotent G-representation has nonzero fixed vectors under I, in other words, $\mathfrak{R}_{\mathrm{un}}(G) = \mathfrak{R}_I(G)$, where $\mathfrak{R}_I(G)$ is the category of smooth representations generated by their I-fixed vectors. The functor

$$m_I \colon \mathfrak{R}_{\mathrm{un}}(G) = \mathfrak{R}_I(G) \to \mathcal{H}(G,I) - \mathrm{Mod}, \quad V \mapsto V^I,$$

is an equivalence of categories. The Langlands parameterization in this case can be read off the Kazhdan-Lusztig classification of irreducible modules for $\mathcal{H}(G,I)$ extended to this setting in [Re1]:

$$\mathrm{Irr}_{\mathrm{un}}\mathsf{SL}_n(F) \leftrightarrow \mathrm{Irr}\ \mathcal{H}(G,I) \leftrightarrow \mathsf{PGL}_n(\mathbb{C})\backslash\{(x,\phi) \mid x \in \mathsf{PGL}_n(\mathbb{C}), \phi \in \widehat{A}_x\}. \tag{11.2}$$

The exact functor m_I induces an isomorphism

$$\mathrm{Ext}^i_G(V,V') = \mathrm{Ext}^i_{\mathcal{H}(G,I)}(V^I, V'^I), \text{ for all } i,$$

and therefore $\mathrm{EP}_G(V,V') = \mathrm{EP}_{\mathcal{H}(G,I)}(V^I, V'^I)$ for all $V, V' \in \mathrm{Irr}\ \mathfrak{R}_I(G)$. Since EP_G and $\mathrm{EP}_{\mathcal{H}(G,I)}$ are additive, they extend to pairings on the Grothendieck groups of finite-length representations. Let $R_I(G)_{\mathbb{C}}$ be the $\mathbb{C}$-span of $\mathrm{Irr}\ \mathfrak{R}_I(G)$ and $\overline{R}_I(G)_{\mathbb{C}}$ the quotient by the radical of EP_G. Thus:

Lemma 11.4. *The equivalence of categories m_I gives an isomorphism $\overline{m}_I \colon \overline{R}_I(G)_{\mathbb{C}} \to \overline{R}(\mathcal{H}(G,I))_{\mathbb{C}}$ which is isometric with respect to EP_G and $\mathrm{EP}_{\mathcal{H}(G,I)}$.*

The elliptic theory of affine Hecke algebras is well understood [OS], and we will review the basic facts in Section 9.1. In particular, via (9.10) and (9.4)), we get that

$$\overline{R}_I(G)_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{s \in W \backslash T^\vee_{\mathrm{iso}}} \overline{R}(W_s)_{\mathbb{C}}, \tag{11.3}$$

where $\widetilde{W} = W^a$, W, $T^\vee$, W_s are as in Example 9.4. Recall that if $n = dm$, then we consider $s_{d\times m} := \Delta_n(z)$, where z is a primitive d-th root of 1. In that case,

$$\mathrm{Z}_{G^\vee}(s_{d\times m}) = \mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d) \rtimes \mathbb{Z}/d\mathbb{Z},$$

which has component group $A_{G^\vee}(s_{d\times m}) = \mathbb{Z}/d\mathbb{Z}$. The Lie algebra of the maximal (diagonal) torus in $\mathrm{Z}_{G^\vee}(s_{d\times m})$ is

$$\mathfrak{t}^\vee_{d\times m} = \{\underline{x} = (x_1, \dots, x_n) \mid \sum_i x_i = 0\},$$

on which $W_{s_{d\times m}}$ acts in the standard way: break $(x_1, \dots, x_n)$ into m-tuples

$$\underline{y}_i = (x_{(i-1)m+1}, x_{(i-1)m+2}, \dots, x_{im}), \quad 1 \le i \le d.$$

Then the i-th S_m acts by the natural permutation action on $\underline{y}_i$, whereas $\mathbb{Z}/d\mathbb{Z}$ permutes cyclically $(\underline{y}_1, \dots, \underline{y}_d)$. We consider the elliptic theory of $W_{s_{d\times m}}$ on $\mathfrak{t}^\vee_{d\times m}$ with respect to this action.

Lemma 11.5. *There are $\varphi(d)$ elliptic conjugacy classes of $W_{s_{d\times m}}$ acting on $\mathfrak{t}^\vee_{d\times m}$ with representatives $g_\xi = (w_m, 1, 1, \dots, 1)\xi$, where ξ ranges over the elements of order d in $\mathbb{Z}/d\mathbb{Z}$, and w_m is a fixed m-cycle in S_m.*

Proof. We first show that each such element is elliptic. Without loss of generality, suppose that ξ acts as the standard cycle $(1, 2, \dots, d)$ and $w_m = (1, 2, \dots, m)$. Then $g_\xi \cdot \underline{x} = \underline{x}$ implies $x_1 = x_{(d-1)m+1} = x_{(d-2)m+1} = \cdots = x_{m+1}$ which then equals to x_m (because of the effect of w_m), then with $x_{dm} = x_{(d-1)m} = \cdots = x_{2m}$, which then equals x_{m-1} etc. It follows that all coordinates x_i are the same and since the sum of the coordinates is 0, we get that there are no nonzero fixed points.

Secondly, no two g_ξ's are conjugate. This is clear because if ξ, ξ' are distinct in $\mathbb{Z}/d\mathbb{Z}$, then $x\xi$ and $x'\xi'$ are in different conjugacy classes for all x, x' in S_m^d.

It remains to show that these are all the elliptic conjugacy classes. Let $x\xi$ be an element with $x = (\sigma_1, \dots, \sigma_d) \in S_m^d$ and $\xi \in \mathbb{Z}/d\mathbb{Z}$. If ξ does not have order d, then there exists points $\underline{x} = (\underline{y}_1, \dots, \underline{y}_d) \in \mathfrak{t}^\vee_{d\times m}$ (here, as above, each $\underline{y}_i$ is an m-tuple) fixed under the action of ξ such that not all y_i's are equal. This means in particular, that there exists j such that $\underline{y}_j = (x_{(j-1)m+1}, \dots, x_{jm})$ is arbitrary and $\sum_{l=(j-1)m+1}^{jm} x_l \neq 0$. But then every $\sigma_j \in S_m$ has a nonzero fixed point $\underline{y}_j$, for example taking all of the entries of $\underline{y}_j$ to be equal, and therefore $x\xi$ is not elliptic.

This means that necessarily ξ has order d. We claim that the conjugacy classes of $x\xi$ are in one-to-one correspondence with conjugacy classes of S_m via the correspondence

$$w \in S_m \mapsto (w, 1, \dots, 1)\xi \in S_m^d \rtimes \mathbb{Z}/d\mathbb{Z}.$$

Without loss of generality, suppose ξ acts by shifting the indices $i \to i+1 \bmod d$. We show that every element $x\xi$, $x = (\sigma_1, \dots, \sigma_d)$, is conjugate to an element of the form $(w, 1, \dots, 1)\xi$. This is equivalent to the existence of permutations $z_1, \dots, z_d \in S_m$ such that

$$\sigma_1 = z_1 w z_2^{-1},\ \sigma_2 = z_2 z_3^{-1}, \dots, \sigma_{d-1} = z_{d-1} z_d^{-1},\ \sigma_d = z_d z_1^{-1}.$$

This can be solved easily, by taking $z_1 = 1$, then $z_d = \sigma_d$, $z_{d-1} = \sigma_{d-1}\sigma_d, \dots, z_2 = \sigma_2\sigma_3 \dots \sigma_d$, $w = \sigma_1\sigma_2 \dots \sigma_d$.

A similar calculation shows that $(w, 1, \dots, 1)\xi$ and $(w', 1, \dots, 1)\xi$ are conjugate if and only if w, w' are conjugate in S_m. (If $w' = zwz^{-1}$, then $(w', 1, \dots, 1)\xi$ and $(w, 1, \dots, 1)\xi$ are conjugate via $(z, z, \dots, z)$.)

Finally, if element $(w, 1, \dots, 1)\xi$, ξ of order d, is elliptic then w is elliptic in S_m, otherwise if $\underline{y}$ is a fixed point of w, $(\underline{y}, \dots, \underline{y})$ is a fixed point of $(w, 1, \dots, 1)\xi$. This concludes the proof. □

On the other hand, we have unipotent classes u in $\mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d)$ and we need to look at the elliptic theory of $A_{G^\vee}(s_{d\times m}u)$ on the Lie algebra of the maximal torus in $\mathrm{Z}_{\mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d \rtimes C_d}(u)$. Let $u = u_{d\times m}$ be the unipotent element given by the principal Jordan normal form on each of the GL_m-blocks. Then the reductive part of the centralizer is

$$\mathrm{Z}_{G^\vee}(s_{d\times m}u_{d\times m})^{\mathrm{red}} = \mathrm{P}(\mathrm{Z}_{\mathsf{GL}_m(\mathbb{C})^d} \rtimes \mathbb{Z}/d\mathbb{Z},$$

hence $A_{G^\vee}(s_{d\times m}u_{d\times m}) = \mathbb{Z}/d\mathbb{Z}$ and this acts on the Cartan subalgebra

$$t^\vee(s_{d\times m}u_{d\times m}) = \left\{ (z_1\mathrm{Id}_m, \dots, z_d\mathrm{Id}_m) \in \mathbb{C}^n \mid \sum_i z_i = 0 \right\}.$$

In particular, $\overline{R}(A_{s_{d\times m}u_{d\times m}})_{\mathbb{C}}$ has dimension $\varphi(d)$ and can be identified with the class functions on the elements of order d in $\mathbb{Z}/d\mathbb{Z}$. Thus, in the case of $\mathsf{SL}_n(F)$, the elliptic correspondence for unipotent representations takes the following very concrete form.

Proposition 11.6. *Let $G = \mathsf{SL}_n(F)$. The local Langlands correspondence for unipotent representations induces an isometric isomorphism*

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}} \colon \bigoplus_{d|n} \overline{R}(A_{x_{d\times m}}) \longrightarrow \overline{R}_{\mathrm{un}}(\mathsf{SL}_n(F)), \quad \phi \mapsto \pi(x_{d\times m}, \phi)$$

where $x_{d\times m} = s_{d\times m}u_{d\times m} \in \mathsf{PGL}_n(\mathbb{C})$ is as above and $A_{x_{d\times m}} = \mathbb{Z}/d\mathbb{Z}$.

The connection with the elliptic pairs for $G^\vee = \mathsf{PGL}_n(\mathbb{C})$ from Proposition 11.2 is:

$$\bigoplus_{u\in\mathcal{C}(\mathsf{PGL}_n(\mathbb{C}))_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u} = \bigoplus_{d|n} \mathbb{C}[\mathcal{Y}(\Gamma_{u_{d\times m}})_{\mathrm{ell}}]^{\Gamma_{u_{d\times m}}} \cong \bigoplus_{d|n} \overline{R}(A_{x_{d\times m}}).$$

11.3. **The elliptic Fourier transform for $\mathsf{SL}_n(F)$.** The results so far imply that we have an equivalence

$$\overline{R}_{\mathrm{un}}(\mathsf{SL}_n(F))_{\mathbb{C}} \cong \overline{R}(\mathcal{H})_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{u\in\mathcal{C}(\mathsf{PGL}_n(\mathbb{C}))_{\mathrm{un}}} \Gamma_u\backslash\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}.$$

The spaces involved are all n-dimensional and we describe the basis of $\overline{R}_{\mathrm{un}}(G)_{\mathbb{C}}$ given by the virtual characters $\Pi(u,s,h)$.

First consider the two extremes. At one extreme, we have the regular unipotent class u_{reg}. Then $\Gamma_{u_{\mathrm{reg}}}=\{1\}$ and $\pi(u_{\mathrm{reg}},1,1)=\mathrm{St}$. At the other, end, $u=1$, $\Gamma_1=\mathsf{PGL}_n(\mathbb{C})$, and there are $\varphi(n)$ orbits of elliptic pairs $(s_n,\dot{w}_n^k)$, $k\in(\mathbb{Z}/n\mathbb{Z})^\times$, as in Lemma 11.1. The component group is $A_{s_n}=\langle\dot{w}_n\rangle\cong\mathbb{Z}/n\mathbb{Z}$. Let $\pi(s_n)$ denote the tempered unramified principal series of G with Satake parameter $s_n\in W\backslash T^\vee$. Since $W_{s_n}=A_{s_n}=\mathbb{Z}/n\mathbb{Z}$, as it is well known via the theory of the (analytic) R-groups, there is a decomposition

$$\pi(1,s_n)=\bigoplus_{\phi\in\widehat{A}_{s_n}}\pi(1,s_n,\phi),$$

where each $\pi(s_n,\phi)$ is an irreducible tempered G-representation. Identifying $\widehat{\mathbb{Z}/n\mathbb{Z}}$ with $\mathbb{Z}/n\mathbb{Z}$ (via the primitive n-th root ζ_n of 1), we get

$$\Pi(1,s_n,\dot{w}_n^k)=\sum_{\ell\in\mathbb{Z}/n\mathbb{Z}}\zeta_n^{\ell k}\,\pi(1,s_n,\phi_\ell),\qquad k\in(\mathbb{Z}/n\mathbb{Z})^\times, \tag{11.4}$$

where $\phi_\ell(\zeta_n)=\zeta_n^\ell$. Moreover, as an $\mathcal{H}$-module, $\pi(1,s_n,\phi_\ell)^I$ is the (unique) irreducible tempered $\mathcal{H}$-module with central character $W\cdot s_n$ such that

$$\sigma_0(\pi(1,s_n,\phi_\ell)^I)=\mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z}\ltimes X}^{S_n\ltimes X}(\phi_\ell\otimes s_n).$$

Now, more generally, by Proposition 11.2, $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\neq\emptyset$ if and only if $u_{d\times m}$ is labelled by a rectangular partition $(\underbrace{m,\dots,m}_{d})$ of n. In this case $\Gamma_u=\mathsf{PGL}_d(\mathbb{C})$. Recall $s_{d\times m}=(\underbrace{1,\dots,1}_{m},\underbrace{\zeta_d,\dots,\zeta_d}_{m},\dots,\underbrace{\zeta_d^{d-1},\dots,\zeta_d^{d-1}}_{m})$ and $x_{d\times m}=s_{d\times m}u_{d\times m}$. Consider the parabolically induced tempered G-representation

$$\pi(u_{d\times m},s_{d\times m})=\mathrm{Ind}_{P_{d\times m}}^{\mathsf{SL}_n(F)}((\mathrm{St}_m\otimes\mathbb{C}_1)\boxtimes(\mathrm{St}_m\otimes\mathbb{C}_{\zeta_d})\boxtimes\cdots\boxtimes(\mathrm{St}_m\otimes\mathbb{C}_{\zeta_d^{d-1}})),$$

where $P_{d\times m}$ is the block-upper-triangular parabolic subgroup with Levi subgroup $M_{d\times m}=S(\mathsf{GL}_m(F)^d)$, St_m is the Steinberg representation of $\mathsf{GL}_m(F)$, and $\mathbb{C}_z$ is the unramified character of $\mathsf{GL}_m(F)$ corresponding to the semisimple element $z\mathrm{Id}_m$ in the dual complex group $\mathsf{GL}_m(\mathbb{C})$. The R-group in this case is $\mathbb{Z}/d\mathbb{Z}$ which coincides with $A_{x_{d\times m}}$. We have a decomposition into irreducible tempered G-representations:

$$\pi(u_{d\times m},s_{d\times m})=\bigoplus_{\phi\in\widehat{A}_{x_{d\times m}}}\pi(u_{d\times m},s_{d\times m},\phi).$$

Taking I-fixed vectors and the deformation σ_0, we have

$$\sigma_0(\pi_{d\times m},s_{d\times m},\phi_\ell)^I)=\mathrm{Ind}_{W_{s_{d\times m}}\ltimes X}^{S_n\ltimes X}(\mathrm{sgn}_{d\times m}\phi_\ell\otimes s_{d\times m}),$$

where recall that $W_{s_{d\times m}}=S_m^d\rtimes\mathbb{Z}/d\mathbb{Z}$ and $\mathrm{sgn}_{d\times m}$ is the sign character of S_m^d.

Define

$$\Pi(u_{d\times m},s_{d\times m},\dot{w}_{d\times m}^k)=\sum_{\ell\in\mathbb{Z}/d\mathbb{Z}}\zeta_d^{\ell k}\,\pi(u_{d\times m},s_{d\times m},\phi_\ell),\qquad k\in(\mathbb{Z}/d\mathbb{Z})^\times. \tag{11.5}$$

The elliptic Fourier transform in this case is

$$\mathrm{FT}_{\mathrm{ell}}^\vee(\Pi(u_{d\times m},s_{d\times m},\dot{w}_{d\times m}^k))=\Pi(u_{d\times m},s_{d\times m},\dot{w}_{d\times m}^{-k}),\qquad k\in(\mathbb{Z}/d\mathbb{Z})^\times. \tag{11.6}$$

The maximal compact open subgroups of $\mathsf{SL}_n(F)$ are maximal parahoric subgroups K_i, one for each vertex i of the affine Dynkin diagram. With this notation, $K_0 = \mathsf{SL}_n(\mathfrak{o}_F)$. Moreover, $\operatorname{Inn}\overline{K}_i = \{K_i\}$. All K_i are isomorphic to K_0 (conjugate in $\mathsf{GL}_n(F)$), hence for all i, the nonabelian Fourier transform of $\overline{K}_i$ is the identity. Let $W_i \cong S_n$ denote the finite parahoric subgroup of W^a corresponding to K_i, so that $W_0 = W$. The isomorphism $W_i \cong W$ is given by the map $s_j \mapsto s_{(j-i) \bmod n}$. By Mackey induction/restriction

$$\operatorname{Ind}_{W_{s_{d\times m}} \ltimes X}^{S_n \ltimes X}(\operatorname{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m})|_{W_i} \cong \operatorname{Ind}_{(W_{s_{d\times m}})_i}^{W_i}(\operatorname{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m}), \tag{11.7}$$

where $(W_{s_{d\times m}})_i = (W_{s_{d\times m}} \ltimes X) \cap W_i$. Let $\gamma = \epsilon_1 - \epsilon_n$ be the highest root of type A_{n-1}, in the standard coordinates, so that $s_0 = s_\gamma t_\gamma$, denoting by $t_\gamma \in X \subset W^a$ the corresponding translation. Then one can see that $W_{s_{d\times m}} \cong (W_{s_{d\times m}})_i$ is given by sending

$$s_j \mapsto \begin{cases} s_j, & \text{if } j \neq i \\ s_i t_{\epsilon_i - \epsilon_{i+1}}, & \text{if } j = i, \end{cases} \quad 1 \leq j < n.$$

Lemma 11.7. *For every $0 \leq i < n$,*

$$\operatorname{Ind}_{(W_{s_{d\times m}})_i}^{W_i}(\operatorname{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m}) \cong \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_{\ell+\lfloor \frac{i}{m}\rfloor}).$$

Proof. In light of the observation before the statement of the Lemma, we only need to trace how the inducing character changes on the generator corresponding to i. Denote by $(S_m^d)_i$ the image of S_m^d inside $(W_{s_{d\times m}})_i$, and similarly for $(\mathbb{Z}/d\mathbb{Z})_i$. If s_i is a generator of $(S_m^d)_i$, equivalently $k \neq 0$, then the value of the character $s_{d\times m}$ on $t_{\epsilon_i - \epsilon_{i+1}}$ is 1. On the other hand, if s_i is not a generator of $(S_m^d)_i$, then there is also no change. This means that the inducing character on the $(S_m^d)_i$ is still $\operatorname{sgn}_{d\times m}$.

The generator ξ of $\mathbb{Z}/d\mathbb{Z}$ is, in cycle notation, a product of the disjoint cycles $(l, m+l, 2m+l, \ldots, (d-1)m+l)$, where l ranges from 0 to $m-1$ (by $l = 0$, we understand the cycle $(m, 2m, \ldots, dm)$). Then the simple reflection i contributes to the cycle l for $i = jm + l$, $j = \lfloor \frac{i}{m} \rfloor$. In $(\mathbb{Z}/d\mathbb{Z})_i$, we then get a $\theta_{\epsilon_i - \epsilon_{i+1}}$, which we need to move to the end of the product of cycles, and we get that the image $(\xi)_i$ in $(\mathbb{Z}/d\mathbb{Z})_i$ is $(\xi)_i = \xi t_{\epsilon_i - \epsilon_{(d-1)m+l}}$. The character $s_{d\times m}$ acts on $t_{\epsilon_i - \epsilon_{(d-1)m+l}}$ by ζ_d^j, which means that $\phi_\ell s_{d\times m}$ acts on $(\xi)_i$ by $\zeta_d^{\ell+j}$, which proves the claim. □

Proposition 11.8. *Conjecture 8.5 holds true for $G = \mathsf{SL}_n(F)$. More precisely, for each $0 \leq i < n$,*

$$\operatorname{res}_{K_i} \circ \mathrm{FT}^\vee_{\mathrm{ell,un}}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^k_{d\times m})) = \zeta_d^{2k\lfloor \frac{i}{m}\rfloor} \mathrm{FT}_{\mathrm{cpt,un}} \circ \operatorname{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^k_{d\times m})).$$

Proof. To verify Conjecture 8.5, given (11.6), it is sufficient to compare the restrictions to W_i of $\sigma_0(\operatorname{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^k_{d\times m})^I))$ and $\sigma_0(\operatorname{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^{-k}_{d\times m})^I)$ as virtual W_i-characters. For this, we apply (11.7) and Lemma 11.7 and get with $j = \lfloor \frac{i}{m} \rfloor$:

$$\begin{aligned}\sigma_0(\operatorname{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^k_{d\times m})^I)) &\cong \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{kl} \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_{\ell+j}) \\ &= \zeta_d^{-kj} \sum_\ell \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_\ell).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_0(\operatorname{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}^{-k}_{d\times m})^I)) &\cong \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{-kl} \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_{\ell+j}) \\ &= \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{kl} \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_{-\ell+j}) = \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{kl} \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_{\ell-j}) \\ &= \zeta_d^{kj} \sum_\ell \operatorname{Ind}_{W_{s_{d\times m}}}^{S_n}(\operatorname{sgn}_{d\times m}\phi_\ell),\end{aligned}$$

where we have used that

$$\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_\ell) \cong \mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{-\ell}),$$

because $\phi_{-\ell}$ is the $\mathbb{Z}/d\mathbb{Z}$-representation contragredient to ϕ_ℓ, which implies that the two sides are contragredient to each other, but all irreducible S_n-representations are self-contragredient. □

12. $\mathsf{PGL}_n(F)$

Now suppose $\mathbf{G} = \mathsf{PGL}_n$. The dual group $G^\vee$ is $\mathsf{SL}_n(\mathbb{C})$. Each unipotent element $u \in G^\vee$ corresponds to a partition λ_u of n, and if $\lambda_u = \lambda = (\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2},\dots,\underbrace{\ell,\dots,\ell}_{r_\ell})$, then

$$\Gamma_u \simeq \left\{(x_1,\dots,x_\ell) \in \prod_{i=1}^{\ell} \mathsf{GL}_{r_i}(\mathbb{C}) \mid \prod_{i=1}^{\ell} \det(x_i)^i = 1\right\}. \tag{12.1}$$

Lemma 12.1. *The group Γ_u contains elliptic pairs if and only if u is regular unipotent. In this case $\mathcal{Y}(\Gamma_u) = \mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s,h) \mid s,h \in \mathrm{Z}_{\mathsf{SL}_n(\mathbb{C})}\}$.*

Proof. The proof is very similar to that of Proposition 11.2. Note that if u is not rectangular, then Γ_u has infinite center: for example, with notation as in (12.1), given $t \in \mathbb{C}^\times$, the element $(t\mathrm{Id}_{r_1}, t\mathrm{Id}_{r_2},\dots,t\mathrm{Id}_{r_{\ell-1}}, t^{r_\ell - \frac{n}{\ell}}\mathrm{Id}_{r_\ell}) \in \mathrm{Z}_{\Gamma_u}$. So if Γ_u contains an elliptic pair, then λ_u is of the form $[k,k,\dots,k]$ for some k dividing n, and $\Gamma_u \simeq \{x \in \mathsf{GL}_{\frac{n}{k}}(\mathbb{C}) \mid \det(x)^k = 1\}$. Explicitly, we can think of Γ_u as a split extension

$$1 \to \mathsf{SL}_{\frac{n}{k}}(\mathbb{C}) \to \Gamma_u \to \mu_{\frac{n}{k}} \to 1$$

where the first inclusion is the natural one, and the map to $\mu_{\frac{n}{k}}$ is given by the determinant. Now, given semisimple elements $s,h \in \Gamma_u$ such that $sh = hs$, there exists $g \in \mathsf{SL}_{\frac{n}{k}}(\mathbb{C})$ such that gsg^{-1}, ghg^{-1} are both diagonal in $\mathsf{GL}_{\frac{n}{k}}(\mathbb{C})$. So a maximal torus of $\mathsf{SL}_{\frac{n}{k}}(\mathbb{C})$ centralizes both s and h, and if (s,h) is an elliptic pair, we must have $k = n$. □

We can now easily prove Conjecture 8.5 in this case.

Theorem 12.2. *Conjecture 8.5 holds when $\mathbf{G} = \mathsf{PGL}_n$. More precisely, when $\mathbf{G} = \mathsf{PGL}_n$, we have*

$$\mathrm{res}_{\mathrm{cpt,un}} \circ \mathrm{FT}_{\mathrm{ell}}^\vee = \mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{\mathrm{cpt,un}}.$$

Proof. Using Lemma 12.1, the proof of the theorem reduces to Proposition 8.7. □

To illustrate the theorem, we explicitly describe the case when $\mathbf{G} = \mathsf{PGL}_2$. Note that even this low-rank example shows that certain choices were necessary in our set up: to relate $\mathrm{FT}_{\mathrm{ell}}^\vee$ to a finite Fourier transform for the non-split inner twist of G, first, we must consider maximal compact subgroups instead of just parahorics (otherwise the restrictions of $\Pi(u,1,1)$ and $\Pi(u,-1,1)$ would be the same, though $\mathrm{FT}_{\mathrm{ell}}^\vee$ fixes the first but not the second). Second, $\mathrm{FT}_{\mathrm{cpt,un}}$ must mix subspaces corresponding to distinct inner twists to give a well-defined linear map.

Example 12.3. Now let $\mathbf{G} = \mathsf{PGL}_2$. Then G has a unique non-split inner twist G', which we can describe explicitly as follows: Let

$$D = \left\{\begin{pmatrix} a & \varpi b \\ \overline{b} & \overline{a}\end{pmatrix} \mid a,b,c,d \in F_{(2)}\right\},$$

where $F_{(2)}$ is the degree-2 unramified extension of F. Then D is a 4-dimensional division algebra over F, and we can take $G' := D^\times/F^\times$.

Let χ_0 be the unramified character of $F^\times$ given by $\varpi \mapsto -1$. Then the nontrivial weakly unramified character of G (resp. G') is given by $\chi := \chi_0 \circ \det$ (resp. $\chi' := \chi_0 \circ \det$). Let St_G denote the Steinberg representation of G (and similarly for $\mathrm{St}_{G'}$, which is the

trivial representation of G'), and let $u \in \mathsf{SL}_2(\mathbb{C})$ be regular unipotent. Then the virtual representations corresponding to our 4 elliptic pairs are

$$\begin{aligned}
\Pi(u,1,1) &= \mathrm{St}_G + \mathrm{St}_{G'} \\
\Pi(u,1,-1) &= \mathrm{St}_G - \mathrm{St}_{G'} \\
\Pi(u,-1,1) &= (\mathrm{St}_G \otimes \chi) + (\mathrm{St}_{G'} \otimes \chi') \\
\Pi(u,-1,-1) &= (\mathrm{St}_G \otimes \chi) - (\mathrm{St}_{G'} \otimes \chi').
\end{aligned}$$

The involution $\mathrm{FT}^\vee_{\mathrm{ell}}$ switches $\Pi(u,1,-1)$ and $\Pi(u,-1,1)$, and fixes the other two sums.

Let I be the Iwahori subgroup of G given by

$$I = \left\{ \begin{pmatrix} a & \varpi b \\ c & d \end{pmatrix} \mid a,d \in \mathfrak{o}_F^\times, b,c \in \mathfrak{o}_F \right\}.$$

With notation as in Section 6, we have $\Omega_G \simeq \mathbb{Z}/2\mathbb{Z}$. The set $S_{\max}(G)$ contains two elements $(A,\mathcal{O})$: one corresponding to $A = \Omega_G$, and one corresponding to A trivial. Thus the group G has two conjugacy classes of maximal compact open subgroups: the maximal parahoric subgroup $K_0 := \mathsf{PGL}_2(\mathfrak{o}_F)$ (which corresponds to A trivial) and $K_1 := \mathrm{N}_G(I)$ (which corresponds to $A = \Omega_G$). Note that K_1 contains I with index 2: it is generated by I and $\sigma := \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right)$. The reductive quotients are given by $\overline{K}_0 \simeq \mathsf{PGL}_2(\mathbb{F}_q)$ and $\overline{K}_1 \simeq \mathbb{F}_q^\times \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that χ is trivial on K_0 and on I, but $\chi(\sigma) = -1$, so χ induces the sign character on the component group of $\overline{K}_1$.

Note that, in the notation of Section 6, we have $G' = G_x$, where x is the nontrivial element of Ω_G. Thus G' has a unique conjugacy class of maximal compact subgroups, corresponding to the element $(A,\mathcal{O})$ of $S_{\max}(G)$ with $A = \Omega_G$. Explicitly, the only parahoric subgroup of G' is $I' := \mathfrak{o}_D^\times/\mathfrak{o}_F^\times$, where $\mathfrak{o}_D$ is the ring of integers of D. The normalizer $K_1' = N_{G'}(I')$ is maximal compact, generated by I' and σ (defined as above), which again has order 2 in G'. The reductive quotient $\overline{K'}_1 \simeq (\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times) \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that as above χ' is trivial on I' but takes the value -1 on σ, so χ' factors through the sign character of the component group of $\overline{K}_1'$.

The space $\mathcal{C}(G)_{\mathrm{cpt,un}}$ is given by

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = R_{\mathrm{un}}(\overline{K}_0) \oplus R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{K}_1'),$$

and the map $\mathrm{res}_{\mathrm{cpt,un}}$ is defined on the virtual representations above by

$$\begin{aligned}
\Pi(u,1,1) &\mapsto \mathrm{St}_{K_0} + \mathrm{St}_I + \mathrm{St}_{I'} \\
\Pi(u,1,-1) &\mapsto \mathrm{St}_{K_0} + \mathrm{St}_I - \mathrm{St}_{I'} \\
\Pi(u,-1,1) &\mapsto \mathrm{St}_{K_0} + (\mathrm{St}_I \otimes \mathrm{sgn}) + (\mathrm{St}_{I'} \otimes \mathrm{sgn}) \\
\Pi(u,-1,-1) &\mapsto \mathrm{St}_{K_0} + (\mathrm{St}_I \otimes \mathrm{sign}) - (\mathrm{St}_{I'} \otimes \mathrm{sgn}),
\end{aligned}$$

where, as in the proof of Proposition 8.7, St_K denotes the Steinberg representation of $\overline{K}$, and sgn denotes the sign representation of the relevant component group.

If $(A,\mathcal{O}) \in S_{\max}(G)$ with A trivial, then $\mathrm{res}_{\mathcal{O}}(\Pi(u,s,h)) = \mathrm{St}_{K_0}$ for all elliptic pairs (s,h). In this case, $\mathrm{FT}_{\mathrm{cpt,un}}$ restricts to the identity map on $R_{\mathrm{un}}(\overline{K}_0)$.

Now suppose $(A,\mathcal{O}) \in S_{\max}(G)$ with $A = \Omega_G$. Then $\mathrm{res}_{\mathcal{O}}$ is given by projection onto $R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{K}_1')$. In the notation of Section 5.4, and with $\mathcal{U}$ the (one-element) family consisting of the Steinberg representation of $\overline{K}_1$, we have $\widetilde{\Gamma}^A_{\mathcal{U}} = A$, so $\mathcal{M}(\widetilde{\Gamma}^A_{\mathcal{U}})$ consists of four elements: $(1,\mathrm{triv}), (1,\mathrm{sign}), (x,\mathrm{triv}), (x,\mathrm{sgn})$ (where, as above, x is the nontrivial element of Ω_G). These correspond to following elements of $R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{K}_1')$:

$$\begin{aligned}
(1,\mathrm{triv}) &\longleftrightarrow \mathrm{St}_I \\
(1,\mathrm{sgn}) &\longleftrightarrow \mathrm{St}_I \otimes \mathrm{sgn} \\
(x,\mathrm{triv}) &\longleftrightarrow \mathrm{St}_{I'} \\
(x,\mathrm{sgn}) &\longleftrightarrow \mathrm{St}_{I'} \otimes \mathrm{sgn}.
\end{aligned}$$

With notation as in (5.13), we have

$$\begin{aligned}
\mathrm{res}_{\mathcal{O}}(\Pi(u,1,1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{triv},\mathrm{triv}) \\
\mathrm{res}_{\mathcal{O}}(\Pi(u,1,-1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{triv},\mathrm{sgn}) \\
\mathrm{res}_{\mathcal{O}}(\Pi(u,-1,1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{sgn},\mathrm{triv}) \\
\mathrm{res}_{\mathcal{O}}(\Pi(u,-1,-1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{sgn},\mathrm{sgn}),
\end{aligned}$$

where $\widetilde{\mathcal{U}}$ is the family indexed by $\Gamma^A_{\mathcal{U}}$. Thus the proof of Proposition 8.7, and Conjecture 8.5, may be easily verified.

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