SUPPLEMENTARY MATERIALS: FINITE DIFFERENCE FORMULATION OF ANY LATTICE BOLTZMANN SCHEME

THOMAS BELLOTTI*, BENJAMIN GRAILLE†, AND MARC MASSOT*

Verification that corresponding Finite Difference schemes from Proposition 4.4 does not depend on the choice of s_1. In the case where we do not consider that the relaxation parameter of the conserved moment m_1 is equal to zero, one might ask if the result from Proposition 4.4 depends on the choice of s_1. It must be observed that in this case, it is of the foremost importance to enforce that $m_1^{\mathrm{eq}} = m_1$.

To this end, let us decompose the equilibrium matrix $\boldsymbol{B}$ under the form $\boldsymbol{B} = \boldsymbol{b}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{B}|_{s_1=0}$ where is the first column of $\boldsymbol{B}$ (thus depending on s_1). Observe that $\boldsymbol{B}|_{s_1=0}$ is independent of s_1. Inserting into the resulting Finite Difference scheme from Proposition 4.4 yields

$$\begin{aligned} m_1^{n+1} + \sum_{k=0}^{q-1} \left(\gamma_{q-1-k} - \left(\sum_{\ell=0}^{k} \gamma_{q+\ell-k} \boldsymbol{A}^\ell \boldsymbol{b}_1 \otimes \boldsymbol{e}_1 \right)_{11} \right) m_1^{n-k} \\ = \left(\sum_{k=0}^{q-1} \left(\sum_{\ell=0}^{k} \gamma_{q+\ell-k} \boldsymbol{A}^\ell \right) \boldsymbol{B}|_{s_1=0} \boldsymbol{m}^{\mathrm{eq}}|^{n-k} \right)_1 . \end{aligned} \tag{SM0.1}$$

The aim is to show that the left-hand side of this expression does not depend on s_1. *A priori*, the coefficients of the characteristic polynomial of $\boldsymbol{A}$ depend on s_1. Like in the proof of Proposition 6.3, we introduce $\Psi = \sum_{k=0}^{k=q} \tilde{\gamma}_k X^k$ where

$$\tilde{\gamma}_k = \begin{cases} 1, & \text{if} \quad k = q, \\ \gamma_k - \sum_{\ell=0}^{\ell=q-1-k} \gamma_{k+1+\ell} \boldsymbol{A}^\ell \boldsymbol{b}_1 \otimes \boldsymbol{e}_1|_{11}, & \text{if} \quad k \in [\![0, q-1]\!]. \end{cases}$$

Following exactly the same proof than Proposition 6.3 using the Lemma 8.1, we have that $\chi_{\boldsymbol{A}+\boldsymbol{b}_1\otimes\boldsymbol{e}_1} \equiv \Psi$. Observing that $\boldsymbol{A} + \boldsymbol{b}_1 \otimes \boldsymbol{e}_1 = \boldsymbol{A}|_{s_1=0}$, this means that Ψ does not depend on s_1, and thus the left-hand side of (SM0.1) does not depend on the choice of s_1. We are left to deal with the right-hand side of (SM0.1), where γ_k are $\boldsymbol{A}$ depend on s_1. However, the right-hand side is a continuous function of s_1, which equates a left-hand side which is constant in s_1, therefore we conclude that the right-hand side cannot depend on s_1. This shows that the choice of taking $s_1 = 0$ can be done without loss of generality.

Verification that equivalent Finite Difference schemes from Proposition 4.8 do not depend on the choice of $s_1, \ldots, s_N$. For any $i \in [\![1, N]\!]$, the independence of the result from s_i is a direct consequence of what we have proved in

*CMAP, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France. (thomas.bellotti@polytechnique.edu), (marc.massot@polytechnique.edu).

†Institut de Mathématique d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France. (benjamin.graille@universite-paris-saclay.fr).

the previous section for $N = 1$. For the other moments, we observe that

$$\begin{aligned}
&\left(\sum_{k=0}^{q-N}\left(\sum_{\ell=0}^{k}\gamma_{i,q+1-N+\ell-k}\boldsymbol{A}_i^\ell\right)\boldsymbol{A}_i^\diamond\boldsymbol{m}^{n-k}\right)_i \\
&\quad+\left(\sum_{k=0}^{q-N}\left(\sum_{\ell=0}^{k}\gamma_{i,q+1-N+\ell-k}\boldsymbol{A}_i^\ell\right)\boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^{n-k}\right)_i \\
=&\left(\sum_{k=0}^{q-N}\left(\sum_{\ell=0}^{k}\gamma_{i,q+1-N+\ell-k}\boldsymbol{A}_i^\ell\right)\boldsymbol{A}_i^\diamond|_{s_t=0,\ t\in[\![1,N]\!]\setminus\{i\}}\boldsymbol{m}^{n-k}\right)_i \\
+&\left(\sum_{k=0}^{q-N}\left(\sum_{\ell=0}^{k}\gamma_{i,q+1-N+\ell-k}\boldsymbol{A}_i^\ell\right)\boldsymbol{B}|_{s_t=0,\ t\in[\![1,N]\!]\setminus\{i\}}\boldsymbol{m}^{\mathrm{eq}}|^{n-k}\right)_i,
\end{aligned}$$

then this quantity is independent on s_t for $t \in [\![1, N]\!] \setminus \{i\}$ because of the compensations between $\boldsymbol{A}_i^\diamond$ and $\boldsymbol{B}$. This concludes the proof.

Formalism with shift operators in time. In the main body of the paper, we have taken time shifts into account without introducing a specific time-shift operator, since most of work had to be done on the space variable. However, it can be interesting to also introduce a shift operator τ which acts on any function f by

$$\tau f^n = f^{n+1},$$

where the spatial variable is untouched, thus not listed. For the sake of presentation, we take $N = 1$. The lattice Boltzmann scheme under the form (3.4) can be restated as

$$(\tau\boldsymbol{I} - \boldsymbol{A})\boldsymbol{m}^n = \boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n. \tag{SM0.2}$$

Observe that now, what we call $\boldsymbol{A}_\tau := \tau\boldsymbol{I} - \boldsymbol{A} \in \mathcal{M}_q(\mathbb{R}[\tau] \times \mathcal{D}^d_{\Delta x})$. We can proceed in two equivalent ways:

- We multiply (SM0.2) by the adjugate of $\tau\boldsymbol{I} - \boldsymbol{A}$. Using the basic properties of the adjugate, see Chapter 0 in [SM8]

$$\begin{aligned}
\mathrm{adj}(\tau\boldsymbol{I} - \boldsymbol{A})(\tau\boldsymbol{I} - \boldsymbol{A})\boldsymbol{m}^n &= (\tau\boldsymbol{I} - \boldsymbol{A})\mathrm{adj}(\tau\boldsymbol{I} - \boldsymbol{A})\boldsymbol{m}^n, \\
&= \det(\tau\boldsymbol{I} - \boldsymbol{A})\boldsymbol{m}^n = \mathrm{adj}(\tau\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n, \\
&= \chi_{\boldsymbol{A}}(\tau)\boldsymbol{m}^n, \qquad \text{(SM0.3)}
\end{aligned}$$

  which is, as we shall see, exactly the Finite Difference scheme given by Proposition 4.4, upon selecting the first line.
- Another way of coming to the same result consists in applying the scheme (SM0.2) as many times as needed. We obtain

$$\begin{aligned}
\boldsymbol{m}^n &= \boldsymbol{m}^n, \\
\boldsymbol{A}_\tau\boldsymbol{m}^n &= \boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n, \\
\boldsymbol{A}_\tau^2\boldsymbol{m}^n &= \boldsymbol{A}_\tau\boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n, \\
&\vdots \\
\boldsymbol{A}_\tau^k\boldsymbol{m}^n &= \boldsymbol{A}_\tau^{k-1}\boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n,
\end{aligned} \tag{SM0.4}$$

for any $k \in \mathbb{N}^\star$. It is important to list the first equation (SM0.4), because it represents what happens at $k = 0$ and cannot be forgotten to yield the desired result. Therefore, consider the characteristic polynomial of $\boldsymbol{A}_\tau$ under the form $\chi_{\boldsymbol{A}_\tau} = \sum_{k=0}^{k=q} \gamma_k^\tau X^k$. By using the Cayley-Hamilton theorem for matrices with values on the commutative ring of space/time Finite Difference operators,[1] we end up with

$$\mathbf{0} = \left(\sum_{k=0}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^k \right) \boldsymbol{m}^n = \gamma_0^\tau \boldsymbol{m}^n + \left(\sum_{k=1}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^{k-1} \right) \boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n.$$

The zero-order term of the characteristic polynomial of a matrix is given by the determinant of such matrix (up to a sign), see [SM2], thus $\gamma_0^\tau = (-1)^q \det(\boldsymbol{A}_\tau) = (-1)^q \det(\tau \boldsymbol{I} - \boldsymbol{A}) = (-1)^q \chi_{\boldsymbol{A}}(\tau) = (-1)^q \sum_{k=0}^{k=q} \gamma_k \tau^k$, which corresponds to the characteristic polynomial of the matrix $\boldsymbol{A}$, containing only spatial Finite Difference operators, evaluated on the time-shift operator τ. On the other hand, using the Cayley-Hamilton theorem once more

$$(\tau \boldsymbol{I} - \boldsymbol{A}) \left(\sum_{k=1}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^{k-1} \right) = \sum_{k=1}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^k = -\gamma_0^\tau \boldsymbol{I} = -(-1)^q \det(\tau \boldsymbol{I} - \boldsymbol{A}) \boldsymbol{I}.$$

By using the basic property of adjugate matrices, namely that for any square matrix $\boldsymbol{C}$, whe have that $\boldsymbol{C}\mathrm{adj}(\boldsymbol{C}) = \det(\boldsymbol{C})\boldsymbol{I}$, we deduce from the previous equality that

$$\left(\sum_{k=1}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^{k-1} \right) = -(-1)^q \mathrm{adj}(\tau \boldsymbol{I} - \boldsymbol{A}).$$

This gives the same Finite Difference scheme than (SM0.3).

We want to make the link with the result from Proposition 4.4. Utilizing (8.1) to deduce an explicit expression for the previous adjugate, we gain

$$\left(\sum_{k=1}^{q} \gamma_k^\tau \boldsymbol{A}_\tau^{k-1} \right) = -(-1)^q \sum_{k=1}^{q} \gamma_k \sum_{\ell=0}^{k-1} \boldsymbol{A}^\ell \tau^{k-1-\ell} = -(-1)^q \sum_{k=0}^{q-1} \left(\sum_{\ell=0}^{q-1-k} \gamma_{k+1+\ell} \boldsymbol{A}^\ell \right) \tau^k,$$

where we have performed the usual change of indices. At the very end, the Finite Difference scheme coming from these computations is

$$(-1)^q \left(\sum_{k=0}^{q} \gamma_k \tau^k \right) \boldsymbol{m}^n - (-1)^q \left(\sum_{k=0}^{q-1} \left(\sum_{\ell=0}^{q-1-k} \gamma_{k+1+\ell} \boldsymbol{A}^\ell \right) \tau^k \right) \boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n = \mathbf{0}.$$

This is the final scheme we want to find from Proposition 4.4 plus a shift. Thus, we multiply it by τ^{1-q} and take advantage of the monicity of the characteristic polynomial

$$\tau \boldsymbol{m}^n = -\left(\sum_{k=0}^{q-1} \gamma_k \tau^{k+1-q} \right) \boldsymbol{m}^n + \left(\sum_{k=0}^{q-1} \left(\sum_{\ell=0}^{q-1-k} \gamma_{k+1+\ell} \boldsymbol{A}^\ell \right) \tau^{k+1-q} \right) \boldsymbol{B}\boldsymbol{m}^{\mathrm{eq}}|^n.$$

[1] Which are the polynomials with real coefficients in the variables τ and those to account for the space, namely x, y and z.

This can be rewritten without the shift operators in time by

$$\boldsymbol{m}^{n+1} = -\sum_{k=0}^{q-1} \gamma_k \boldsymbol{m}^{n+1-q+k} + \sum_{k=0}^{q-1} \left(\sum_{\ell=0}^{k} \gamma_{q+\ell-k} \boldsymbol{A}^\ell \right) \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}|^{n-k},$$

where another change of indices in the last sums has been made.

To recap, the Finite Difference schemes can be written as

$$\underbrace{\det(\tau \boldsymbol{I} - \boldsymbol{A}) \boldsymbol{m}^n - \mathrm{adj}(\tau \boldsymbol{I} - \boldsymbol{A}) \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}|^n}_{\text{(FD)}} = \boldsymbol{0}.$$

Multiplying by $(\tau \boldsymbol{I} - \boldsymbol{A})$ (the inverse of the resolvent) provides

$$\underbrace{\det(\tau \boldsymbol{I} - \boldsymbol{A})}_{\chi_{\boldsymbol{A}}(\tau)} \underbrace{((\tau \boldsymbol{I} - \boldsymbol{A}) \boldsymbol{m}^n - \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}|^n)}_{\substack{\text{lattice Boltzmann scheme} \\ \text{(LBM)}}} = \boldsymbol{0}.$$

Formally this means that

$$(\tau \boldsymbol{I} - \boldsymbol{A})(\mathrm{FD}) = \det(\tau \boldsymbol{I} - \boldsymbol{A})(\mathrm{LBM}) = \boldsymbol{0}.$$

We see that in the Finite Difference formulation, the simplest part of the scheme to deal with concerns the linear part on the moments, since it is diagonal, whereas for the original lattice Boltzmann scheme, it was the hardest one due to the presence of the non-conserved moments multiplied by $\tau \boldsymbol{I} - \boldsymbol{A}$. On the other hand, the equilibrium part was easy to deal with in the original lattice Boltzmann scheme, since involving only $\boldsymbol{B}$, whereas in the corresponding Finite Difference schemes, it is more involved due to the presence of the adjugate matrix $\mathrm{adj}(\tau \boldsymbol{I} - \boldsymbol{A})$.

Finally, observe that the same procedure works for the ODEs introduced at the very beginning of the paper, where the place of the time shift operator τ would be taken by the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$.

Proof of Proposition 5.7.

Proof. By the choice of polynomial, we have that

$$\left(\sum_{k=0}^{\deg(\nu_{\boldsymbol{A}})} \psi_k \boldsymbol{A}^k \right)_{1\cdot} = \left(\psi_0 + \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k (\boldsymbol{A}^k)_{11}, 0, \dots, 0 \right).$$

Restarting from the proof of Proposition 4.4, we have

$$\begin{aligned}
\sum_{k=0}^{\deg(\nu_{\boldsymbol{A}})} \psi_k m_1^{\tilde{n}+k} &= m_1^{\tilde{n}+\deg(\nu_{\boldsymbol{A}})} + \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})-1} \psi_k m_1^{\tilde{n}+k} + \psi_0 m_1^{\tilde{n}}, \\
&= \left(\left(\sum_{k=0}^{\deg(\nu_{\boldsymbol{A}})} \psi_k \boldsymbol{A}^k \right) \boldsymbol{m}^{\tilde{n}} \right)_1 + \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k \left(\sum_{\ell=0}^{k-1} \boldsymbol{A}^\ell \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}{}^{|\tilde{n}+k-1-\ell} \right)_1, \\
&= \psi_0 m^{\tilde{n}} + \left(\sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k (\boldsymbol{A}^k)_{11} \right) m_1^{\tilde{n}} \\
&+ \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k \left(\sum_{\ell=0}^{k-1} \boldsymbol{A}^\ell \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}{}^{|\tilde{n}+k-1-\ell} \right)_1,
\end{aligned}$$

therefore

$$\begin{aligned}
m_1^{\tilde{n}+\deg(\nu_{\boldsymbol{A}})} = &- \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})-1} \psi_k m_1^{\tilde{n}+k} + \left(\sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k (\boldsymbol{A}^k)_{11} \right) m_1^{\tilde{n}} \\
&+ \sum_{k=1}^{\deg(\nu_{\boldsymbol{A}})} \psi_k \left(\sum_{\ell=0}^{k-1} \boldsymbol{A}^\ell \boldsymbol{B} \boldsymbol{m}^{\mathrm{eq}}{}^{|\tilde{n}+k-1-\ell} \right)_1 .
\end{aligned} \tag{SM0.5}$$

Performing the usual change of variable yields the result. ∎

Proof on Lemma 5.10.

Proof. The proof goes like the standard one of Lemma 5.3. Consider $\mu_{\boldsymbol{A}} = X^{\deg(\mu_{\boldsymbol{A}})} + \omega_{\deg(\mu_{\boldsymbol{A}})-1} X^{\deg(\mu_{\boldsymbol{A}})-1} + \cdots + \omega_1 X + \omega_0$. Consider the Euclidian division between $\mu_{\boldsymbol{A}}$ and $\tilde{\nu}_{\boldsymbol{A}}$: there exist $Q, R \in \mathcal{D}^d_{\Delta x}[X]$ such that

$$\mu_{\boldsymbol{A}} = \tilde{\nu}_{\boldsymbol{A}} Q + R,$$

with either $0 < \deg(R) < \deg(\tilde{\nu}_{\boldsymbol{A}})$ or $\deg(R) = 0$ (constant reminder polynomial). Let us indeed write

$$\begin{aligned}
Q &= q_{\deg(\mu_{\boldsymbol{A}})-\deg(\tilde{\nu}_{\boldsymbol{A}})} X^{\deg(\mu_{\boldsymbol{A}})-\deg(\tilde{\nu}_{\boldsymbol{A}})} + \cdots + q_1 X + q_0, \\
R &= r_{\deg(R)} X^{\deg(R)} + \cdots + r_1 X + r_0,
\end{aligned}$$

Suppose that $R \not\equiv 0$, then we have for every $j \in [\![1, q]\!]$

$$\begin{aligned}
&\overbrace{(\boldsymbol{A}^{\deg(\mu_{\boldsymbol{A}})})_{1j} + \omega_{\deg(\mu_{\boldsymbol{A}})-1} (\boldsymbol{A}^{\deg(\mu_{\boldsymbol{A}})-1})_{1j} + \cdots + \omega_1 (\boldsymbol{A})_{1j} + \omega_0 \delta_{1j}}^{=0} \\
&\quad = r_{\deg(R)} (\boldsymbol{A}^{\deg(R)})_{1j} + \cdots + r_1 (\boldsymbol{A})_{1j} + r_0 \delta_{1j} + \\
&\underbrace{\left((\boldsymbol{A}^{\deg(\tilde{\nu}_{\boldsymbol{A}})})_{1j} + \psi_{\deg(\tilde{\nu}_{\boldsymbol{A}})-1} (\boldsymbol{A}^{\deg(\tilde{\nu}_{\boldsymbol{A}})-1})_{1j} + \cdots + \psi_1 (\boldsymbol{A})_{1j} + \psi_0 \delta_{1j} \right)}_{=0} \\
&\times \left(q_{\deg(\mu_{\boldsymbol{A}})-\deg(\nu_{\boldsymbol{A}})} (\boldsymbol{A}^{\deg(\mu_{\boldsymbol{A}})-\deg(\nu_{\boldsymbol{A}})})_{1j} + \cdots + q_1 (\boldsymbol{A})_{1j} + q_0 \delta_{1j} \right),
\end{aligned}$$

thus

$$r_{\deg(R)}(\boldsymbol{A}^{\deg(R)})_{1j} + \cdots + r_1(\boldsymbol{A})_{1j} + r_0\delta_{1j} = 0, \qquad j \in [\![1, q]\!],$$

with $0 < \deg(R) < \deg(\tilde{\nu}_{\boldsymbol{A}})$, which contradicts the minimality of $\tilde{\nu}_{\boldsymbol{A}}$. Thus necessarily $\deg(R) = 0$ so the polynomial is constant, but to have the previous property, the constant must be zero, thus $R \equiv 0$. ∎

Proof of Lemma 6.1.

Proof. Let $f : \mathcal{L} \to \mathbb{R}$ with $f \in \ell^2(\mathcal{L}) \cap \ell^1(\mathcal{L})$. We have, for every wave number $\boldsymbol{\xi} \in [-\pi/\Delta x, \pi/\Delta x]^d$

$$\begin{aligned} T^{\hat{\boldsymbol{z}}}_{\Delta x} f(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{d/2}} \sum_{\boldsymbol{x}\in\mathcal{L}} e^{-\imath \boldsymbol{x}\cdot\boldsymbol{\xi}} f(\boldsymbol{x} - \boldsymbol{z}\Delta x), \\ &= \frac{1}{(2\pi)^{d/2}} \sum_{\boldsymbol{y}\in\mathcal{L}} e^{-\imath(\boldsymbol{y}+\boldsymbol{z}\Delta x)\cdot\boldsymbol{\xi}} f(\boldsymbol{y}) = e^{-\imath\Delta x \boldsymbol{z}\cdot\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi}). \end{aligned}$$ ∎

Proof of Lemma 8.1.

Proof. The proof is basically given in [SM4]. First assume that $\boldsymbol{D} \in \mathcal{M}_r(\mathcal{R})$ is invertible, therefore we have $\det(\boldsymbol{D} + \boldsymbol{u}\otimes\boldsymbol{v}) = \det(\boldsymbol{D})\det(\boldsymbol{I} + \boldsymbol{D}^{-1}\boldsymbol{u}\otimes\boldsymbol{v})$. Construct

$$\begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{v}^\intercal & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} + \boldsymbol{D}^{-1}\boldsymbol{u}\boldsymbol{v}^\intercal & \boldsymbol{D}^{-1}\boldsymbol{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{v}^\intercal & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{D}^{-1}\boldsymbol{u} \\ 0 & 1 + \boldsymbol{v}^\intercal\boldsymbol{D}^{-1}\boldsymbol{u} \end{pmatrix}.$$

The determinant of the first and third matrices on the left hand side is one. The determinant of the second one is exactly the quantity we want to compute. Thus

$$\det(\boldsymbol{D} + \boldsymbol{u}\otimes\boldsymbol{v}) = \det(\boldsymbol{D})\left(1 + \boldsymbol{v}^\intercal\boldsymbol{D}^{-1}\boldsymbol{u}\right) = \det(\boldsymbol{D}) + \boldsymbol{v}^\intercal(\det(\boldsymbol{D})\boldsymbol{D}^{-1})\boldsymbol{u}.$$

Take $\boldsymbol{D} = \boldsymbol{C} - x\boldsymbol{I}$ for any $x \in \mathcal{R}$. Its inverse is a rational function in x, thus exists. We have

$$\begin{aligned} \det(\boldsymbol{C} - x\boldsymbol{I} + \boldsymbol{u}\otimes\boldsymbol{v}) &= \det(\boldsymbol{C} - x\boldsymbol{I}) + \boldsymbol{v}^\intercal(\det(\boldsymbol{C} - x\boldsymbol{I})(\boldsymbol{C} - x\boldsymbol{I})^{-1})\boldsymbol{u}, \\ &= \det(\boldsymbol{C} - x\boldsymbol{I}) + \boldsymbol{v}^\intercal\mathrm{adj}(\boldsymbol{C} - x\boldsymbol{I})\boldsymbol{u}, \end{aligned}$$

by the basic properties of the adjugate matrix. Taking $x = 0$ yields the thesis. ∎

Additional examples. In this section, we gather more examples concerning the application of our theory to lattice Boltzmann schemes which can be found in the literature.

$\mathbf{D_1Q_2}$ with one conservation law. Consider the scheme by [SM3, SM7] taking $d = 1$ and $q = 2$ with $c_1 = 1$ and $c_2 = -1$ and

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix}, \qquad \boldsymbol{S} = \mathrm{diag}(0, s), \quad \text{with} \quad s \neq 1. \tag{SM0.6}$$

The scheme can be used to simulate a non-linear scalar conservation law (advection, Burgers, *etc.*) using an acoustic scaling and a non-linear diffusion equation with a parabolic scaling. However, the scheme is not rich enough to simulate more complex equations. As already pointed out in the introduction , the Finite Difference equivalent

of this scheme has already been studied by [SM3] in the case where the equilibria are linear functions.

It can be easily seen, even by hand since dealing with a 2×2 matrix, that

$$\chi_{\boldsymbol{A}} = X^2 - \frac{1}{2}(2-s)(\mathsf{x}+\bar{\mathsf{x}})X + (1-s).$$

The minimal polynomial coincides with the characteristic polynomial. This can be seen, as usual, by trying to consider α_0 and α_1 such that

$$\alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{A} = \begin{pmatrix} \alpha_0 + \frac{(\mathsf{x}+\bar{\mathsf{x}})}{2}\alpha_1 & \frac{(1-s)(\mathsf{x}-\bar{\mathsf{x}})}{2\lambda}\alpha_1 \\ \frac{\lambda(\mathsf{x}-\bar{\mathsf{x}})}{2}\alpha_1 & \alpha_0 + \frac{(1-s)(\mathsf{x}+\bar{\mathsf{x}})}{2}\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The only way of annihilating the first entry is to take $\alpha_0 = 0$, which is trivial. Thus the minimal polynomial is of degree 2 and then coincides with the characteristic polynomial. The equivalent Finite Difference scheme is

$$m_1^{n+1} = \frac{1}{2}(2-s)(\mathsf{x}+\bar{\mathsf{x}})m_1^n - (1-s)m_1^{n-1} + \frac{s(\mathsf{x}-\bar{\mathsf{x}})}{2\lambda} m_2^{\mathrm{eq}}|^n.$$

The scheme is a θ-scheme between a Lax-Friedrichs scheme (for $s = 1$) and a leap-frog scheme (for $s = 2$).

$\mathrm{D_1Q_3}$ SRT for one conservation law. Consider the $\mathrm{D_1Q_3}$ SRT scheme by [SM6], also corresponding to that of [SM9] which reads with our notations $d = 1$, $q = 3$ and $c_1 = 0$, $c_2 = 1$ and $c_3 = -1$ and

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \lambda & -\lambda \\ 0 & \lambda^2 & \lambda^2 \end{pmatrix}, \qquad \boldsymbol{S} = \mathrm{diag}(0, \omega, \omega), \quad \text{with} \quad \omega \neq 1,$$

The characteristic polynomial, corresponding to the minimal polynomial is

$$\chi_{\boldsymbol{A}} = X^3 + (\omega(\mathsf{x}+\bar{\mathsf{x}}) - (\mathsf{x}+1+\bar{\mathsf{x}}))X^2 + (1-\omega)((\mathsf{x}+\bar{\mathsf{x}}) + (1-\omega))X - (1-\omega)^2.$$

Hence the equivalent Finite Difference scheme is

$$\begin{aligned} m_1^{n+1} = {} & (1-\omega)(\mathsf{x}+\bar{\mathsf{x}})m_1^n + m_1^n - (1-\omega)(\mathsf{x}+\bar{\mathsf{x}})m_1^{n-1} - (1-\omega)^2 m_1^{n-1} \\ & + (1-\omega)^2 m_1^{n-2} + \frac{\omega(\mathsf{x}-\bar{\mathsf{x}})}{2\lambda} m_2^{\mathrm{eq}}|^n - \frac{\omega(1-\omega)(\mathsf{x}-\bar{\mathsf{x}})}{2\lambda} m_2^{\mathrm{eq}}|^{n-1} \\ & + \frac{\omega(\mathsf{x}-2+\bar{\mathsf{x}})}{2\lambda^2} m_3^{\mathrm{eq}}|^n + \frac{\omega(1-\omega)(\mathsf{x}-2+\bar{\mathsf{x}})}{2\lambda^2} m_3^{\mathrm{eq}}|^{n-1}, \end{aligned}$$

coinciding with the one found by [SM6].

$\mathrm{D_1Q_3}$ MRT for one conservation law. Consider the $\mathrm{D_1Q_3}$ MRT scheme by [SM6], which is constructed in the same way than the previous one except for $\boldsymbol{S} = \mathrm{diag}(0, \omega_2, \omega_3)$ with $\omega_2, \omega_3 \neq 1$. The characteristic and minimal polynomial coincide and are given by

$$\begin{aligned} \chi_{\boldsymbol{A}} = {} & X^3 + (-1 + (\mathsf{x}+\bar{\mathsf{x}})(\omega_2/2 + \omega_3/2 - 1))X^2 \\ & + (1 + \omega_2\omega_3 - \omega_2 - \omega_3 + (1 - \omega_2/2 - \omega_3/2)(\mathsf{x}+\bar{\mathsf{x}}))X \\ & - (1-\omega_2)(1-\omega_3). \end{aligned}$$

Then the equivalent Finite Difference scheme is

$$\begin{aligned} m_1^{n+1} = &(1-\omega_2/2-\omega_3/2)(\mathsf{x}+\overline{\mathsf{x}})m_1^n + m_1^n - (1-\omega_2/2-\omega_3/2)(\mathsf{x}+\overline{\mathsf{x}})m_1^{n-1} \\ &- (1-\omega_2-\omega_3+\omega_2\omega_3)m_1^{n-1} + (1-\omega_2)(1-\omega_3)m_1^{n-2} \\ &+ \frac{\omega_2(\mathsf{x}-\overline{\mathsf{x}})}{2\lambda} m_2^{\mathrm{eq}}|^n - \frac{\omega_2(1-\omega_3)(\mathsf{x}-\overline{\mathsf{x}})}{2\lambda} m_2^{\mathrm{eq}}|^{n-1} \\ &+ \frac{\omega_3(\mathsf{x}-2+\overline{\mathsf{x}})}{2\lambda^2} m_3^{\mathrm{eq}}|^n + \frac{\omega_3(1-\omega_2)(\mathsf{x}-2+\overline{\mathsf{x}})}{2\lambda^2} m_3^{\mathrm{eq}}|^{n-1}, \end{aligned}$$

corresponding to the one found by [SM6].

$\mathbf{D_2Q_4}$ for one conservation law. Consider $d = 2$ and $q = 4$ with $\boldsymbol{c}_1 = (1,0)^\intercal$, $\boldsymbol{c}_2 = (0,1)^\intercal$, $\boldsymbol{c}_3 = (-1,0)^\intercal$ and $\boldsymbol{c}_4 = (0,-1)^\intercal$ and

$$\text{(SM0.7)} \qquad \boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & 0 & -\lambda \\ \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 \end{pmatrix}, \qquad \boldsymbol{S} = \mathrm{diag}(0,s,s,1), \quad \text{with} \quad s \neq 1.$$

Therefore $N = 1$ and $Q = 2$. This can be used, for example, coupled with other schemes of the same nature (building what we call a "vectorial scheme" [SM5]) to easily simulate systems of non-linear conservation laws for $d = 2$, see [SM1]. After some computation, the characteristic polynomial of $\boldsymbol{A}$ reads

$$\begin{aligned} \chi_{\boldsymbol{A}} = &X^3 + (2s-3)\frac{(\mathsf{x}+\overline{\mathsf{x}}+\mathsf{y}+\overline{\mathsf{y}})}{4}X^2 + (1-s)\left((2-s)\frac{(\mathsf{xy}+\overline{\mathsf{x}}\mathsf{y}+\mathsf{x}\overline{\mathsf{y}}+\overline{\mathsf{x}}\overline{\mathsf{y}})}{4}+1\right)X \\ &- (1-s)^2\frac{(\mathsf{x}+\overline{\mathsf{x}}+\mathsf{y}+\overline{\mathsf{y}})}{4}. \end{aligned}$$

One can check as usual that it coincides with the minimal polynomial. The equivalent Finite Difference scheme taking $m_4^{\mathrm{eq}} \equiv 0$ for simplicity is

$$\begin{aligned} m_1^{n+1} = &-(2s-3)\mathsf{A}_{\mathrm{a}}m_1^n - (1-s)m_1^{n-1} - (1-s)(2-s)\mathsf{A}_{\mathrm{d}}m_1^{n-1} + (1-s)^2\mathsf{A}_{\mathrm{a}}m_1^{n-2} \\ &+ \frac{s}{2\lambda}(\mathsf{x}-\overline{\mathsf{x}})m_2^{\mathrm{eq}}|^n + \frac{s}{2\lambda}(\mathsf{y}-\overline{\mathsf{y}})m_3^{\mathrm{eq}}|^n \\ &- \frac{s(1-s)}{\lambda}\frac{1}{2}\left(\mathsf{y}\frac{(\mathsf{x}-\overline{\mathsf{x}})}{2} + \overline{\mathsf{y}}\frac{(\mathsf{x}-\overline{\mathsf{x}})}{2}\right) m_2^{\mathrm{eq}}|^{n-1} \\ &- \frac{s(1-s)}{\lambda}\frac{1}{2}\left(\mathsf{x}\frac{(\mathsf{y}-\overline{\mathsf{y}})}{2} + \overline{\mathsf{x}}\frac{(\mathsf{y}-\overline{\mathsf{y}})}{2}\right) m_3^{\mathrm{eq}}|^{n-1}, \end{aligned}$$

where we have introduced the short-hands $\mathsf{A}_{\mathrm{a}} := (\mathsf{x}+\overline{\mathsf{x}}+\mathsf{y}+\overline{\mathsf{y}})/4 \in \mathcal{D}^d_{\Delta x}$ and $\mathsf{A}_{\mathrm{d}} := (\mathsf{xy}+\mathsf{x}\overline{\mathsf{y}}+\overline{\mathsf{x}}\mathsf{y}+\overline{\mathsf{x}}\overline{\mathsf{y}})/4 \in \mathcal{D}^d_{\Delta x}$, yielding respectively the average between neighbors along the axis and along the diagonals.

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