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# EXISTENCE OF SOLUTIONS FOR A SYSTEM INVOLVING THE (2,Q)-LAPLACIAN OPERATOR IN A BOUNDED DOMAIN

LAURE CARDOULIS\*

ABSTRACT. In this paper we study the existence of a non trivial weak solution for a system involving the Laplacian operator and the q-Laplacian operator in a bounded domain  $\Omega$  of  $\mathbb{R}^N$  with sufficiently smooth boundary.

## 1. INTRODUCTION

We consider in this paper the following system for  $i = 1, \dots, m$ ,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = g_i(\cdot, u_1, \dots, u_m) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \quad (S, q, g)$$

where  $\Omega$  is a bounded domain with sufficiently smooth boundary,  $\Omega \subset \mathbb{R}^N$ . We recall that the q-Laplacian operator is defined by  $\Delta_q \phi = \operatorname{div}(|\nabla \phi|^{q-2} \nabla \phi)$  and we suppose  $q > 2$  in the whole paper. We study the existence of a weak non-trivial solution  $u = (u_1, \dots, u_m) \in W$  for the system  $(S, q, g)$  where the variational space is  $W = (W_0^{1,q}(\Omega))^m$ ,  $W_0^{1,q}(\Omega)$  being the usual Sobolev space endowed with the norm  $\|\phi\|_0^{1,q}(\Omega) = (\int_{\Omega} |\nabla \phi|^q)^{1/q}$ . We also denote  $H = (W_0^{1,2}(\Omega))^m$  and  $\|\cdot\|_W, \|\cdot\|_H$ , the norms on  $W$  and  $H$  ( $\|u\|_W = (\sum_{i=1}^m \|u_i\|_{W_0^{1,q}(\Omega)}^q)^{1/q}$ ).

We assume throughout all the paper that the bounded functions  $a_{ij}, w_i$  (for  $i, j = 1, \dots, m$ ) satisfy the following hypothesis

- Assumption 1.1.**      **i):**  $a_{ij}, w_i \in L^\infty(\Omega)$ ,  $a_{ii} \geq 0$ ,  $w_i \geq 0$  a. e. on  $\Omega$ .  
**ii):** The matrix  $A = (a_{ij})$  is symmetric and satisfies  ${}^t \xi A \xi \geq 0$  for all  ${}^t \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ .

Note that the above Assumption 1.ii) is satisfied when the matrix  $A$  is a positive definite one. Introduce now the following functionals for  $u = (u_1, \dots, u_m) \in W$

$$H_1(u) = \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^2 + a_{ii} u_i^2 + \sum_{j=1, i \neq j}^m a_{ij} u_j u_i), \quad (1.1)$$

and

$$H_2(u) = \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^q + w_i |u_i|^q). \quad (1.2)$$

Since  $A$  is symmetric then  $H_1(u) = \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^2 + a_{ii} u_i^2 + 2 \sum_{j=1, i < j}^m a_{ij} u_j u_i)$ . Note that  $(H_1(u))^{1/2}$  and  $(H_2(u))^{1/q}$  define norms on  $H$  and  $W$  equivalent to the norms  $\|\cdot\|_H$  and  $\|\cdot\|_W$  respectively.

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We consider different cases for the functions  $g_i$  : in the second section we deal with  $g_i(\cdot, u_1, \dots, u_m) := h_i \in W^{-1, q'}(\Omega)$  the dual space of  $W_0^{1, q}(\Omega)$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ . In the third section, we define  $g_i(\cdot, u_1, \dots, u_m) := m_i |u_i|^{q-2} u_i$  where the functions  $m_i$  are bounded and indefinite. In the fourth section we consider the case  $g_i(\cdot, u_1, \dots, u_m) := \lambda f_i |u_i|^{\gamma-2} u_i$  where the functions  $f_i$  are still bounded and indefinite,  $\lambda$  is a positive real parameter and the coefficient  $\gamma$  satisfies some hypotheses in which  $\gamma < q$ .

In each of the precedent cases, the system  $(S, q, g)$  will be rewritten under a variational form with  $I(u)$  an adapted Euler functional defined in  $W$  and the existence of weak solutions for the system  $(S, q, g)$  will be equivalent to the existence of critical points for this functional  $I$ . In the second and third sections, we will minimize the Euler functional  $I$  using either standard arguments (cf. Theorem I.1.2 in [18]) or the Mountain-Pass Theorem. In the third section, we will use the principal eigenvalue  $\lambda_{1, q, \rho}$  of the q-Laplacian operator associated with a weight  $\rho$  whereas in the fourth section we will define a characteristic value  $\lambda_1^+$  (see (4.7)).

Equations and systems with the p-Laplacian have been widely studied for the existence of solutions or the maximum and antimaximum principles (see for examples [3, 9, 10, 11, 12, 13], see also [14] for the fibering procedure). These last few years, equations with the (p,q)-Laplacian have been studied (see for examples [4, 6, 15, 19, 21] in a bounded domain and [5] in  $\mathbb{R}^N$ ). Authors study the existence of solutions (sometimes the sign of these solutions and generalized eigenvalue problems) mainly by minimization of the energy functional either by standard arguments or the mountain-pass geometry, also by using the method of sub- and super-solutions. The case of the (2,q)-Laplacian arises in quantum physics (see [2]). A few systems with two equations have been studied (see for example [16] for a system with two equations, one with the p-Laplacian and the other one with the q-Laplacian ; see also [20] for a system of two equations with the (p,q)-Laplacian with critical nonlinearities) but as far as we know, there is no system with  $m$  equations for the (2,q)-Laplacian studied yet.

This paper is organised as follows: in section 2, we use standard arguments for minimizing the functional  $I$  when we consider the case where  $g_i(\cdot, u_1, \dots, u_m) := h_i \in W^{-1, q'}(\Omega)$ . In section 3 (in the case of  $g_i(\cdot, u_1, \dots, u_m) := m_i |u_i|^{q-2} u_i$  and  $q < 2^*$  where  $2^* = \frac{2N}{N-2}$  if  $N > 2$  and  $2^* = \infty$  if  $N \leq 2$ ), first we recall some results of the existence of the principal eigenvalue for the q-Laplacian operator associated with a bounded weight (and the existence of a positive eigenfunction associated with). Then we use the Mountain-Pass Theorem in order to get the existence of a non-trivial solution for our system. Finally in section 4 (when  $g_i(\cdot, u_1, \dots, u_m) := \lambda f_i |u_i|^{\gamma-2} u_i$  with  $2 < \gamma < q$  and  $\gamma < 2^*$  where  $2^* = \frac{2N}{N-2}$  if  $N > 2$  and  $2^* = \infty$  if  $N \leq 2$ ), first we follow a method introduced by Cherfils-Il'Yasov in [7] for one equation involving the (p-q)-Laplacian operator to define a characteristic value  $\lambda_1^+$ . Then we get the existence of a non-trivial solution by means of global minimization of the Euler functional.

2. FIRST CASE:  $g_i(\cdot, u_1, \dots, u_m) := h_i \in W^{-1, q'}(\Omega)$

In this case the system  $(S, q, g)$  is rewritten under the following form

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \quad (2.1)$$

with  $h_i \in W^{-1, q'}(\Omega)$  for each  $i = 1, \dots, m$ . Recall that  $-\Delta_q$  may be seen acting from  $W_0^{1, q}(\Omega)$  into  $W^{-1, q'}(\Omega)$  with  $\frac{1}{q} + \frac{1}{q'} = 1$  by

$$\langle -\Delta_q \phi, \psi \rangle_{q', q} = \int_{\Omega} |\nabla \phi|^{q-2} \nabla \phi \cdot \nabla \psi \text{ for all } \phi, \psi \in W_0^{1, q}(\Omega)$$

(see [8, 17]) where  $\langle \cdot, \cdot \rangle_{q', q}$  denotes the duality mapping between  $W^{-1, q'}(\Omega)$  and  $W_0^{1, q}(\Omega)$ . Therefore the Euler functional is, for  $u = (u_1, \dots, u_m) \in W$ ,

$$I(u) = \frac{1}{2} H_1(u) + \frac{1}{q} H_2(u) - \sum_{i=1}^m \langle h_i, u_i \rangle_{q', q}. \quad (2.2)$$

The result of the existence of solution for the system (2.1) is the following.

**Theorem 2.1.** *Assume that Assumption 1.1 is satisfied and that  $h_i \in W^{-1, q'}(\Omega)$  for each  $i = 1, \dots, m$ . Then the system (2.1) has a unique solution.*

*Proof.* The functional  $I : W \rightarrow \mathbb{R}$  defined by (2.2) is weakly lower semi-continuous by the compactness of the embedding of  $W$  to  $(L^q(\Omega))^m$  and  $(L^2(\Omega))^m$  and of class  $C^1$  on  $W$ . Moreover this functional  $I$  is also coercive. Indeed by the Young's inequality we have

$$|\langle h_i, u_i \rangle_{q', q}| \leq \|h_i\|_{W^{-1, q'}(\Omega)} \|u_i\|_{W_0^{1, q}(\Omega)} \leq \frac{1}{2q} \|u_i\|_{W_0^{1, q}(\Omega)}^q + C \|h_i\|_{W^{-1, q'}(\Omega)}^{q'}$$

with  $C > 0$ ,  $C$  independent of  $u$ . And since  $H_1(u) \geq 0$  and  $H_2(u) \geq \|u\|_W$  we get that

$$I(u) \geq \frac{1}{2q} \|u\|_W - C \sum_{i=1}^m \|h_i\|_{W^{-1, q'}(\Omega)}^{q'}.$$

Therefore the functional  $I$  has a global minimizer (cf.[18, Theorem I.1.2]) and the system (2.1) has a solution.

Let us prove now the uniqueness of the solution. Suppose on the contrary that there exist two distinct solutions  $u = (u_1, \dots, u_m) \in W$  and  $v = (v_1, \dots, v_m) \in W$  for (2.1), so there exists  $k$  such that  $u_k \neq v_k$ . Since

$$(I'(u) - I'(v)) \cdot (u - v) = I'(u) \cdot u - I'(v) \cdot u - I'(u) \cdot v + I'(v) \cdot v = 0,$$

we have

$$\begin{aligned} & \sum_{i=1}^m \int_{\Omega} |\nabla u_i|^2 + \sum_{i,j=1}^m \int_{\Omega} a_{ij} u_j u_i + \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^q + w_i |u_i|^q) \\ & - \sum_{i=1}^m \int_{\Omega} \nabla v_i \cdot \nabla u_i - \sum_{i,j=1}^m \int_{\Omega} a_{ij} v_j u_i - \sum_{i=1}^m \int_{\Omega} (|\nabla v_i|^{q-2} \nabla v_i \cdot \nabla u_i + w_i |v_i|^{q-2} v_i u_i) \\ & - \sum_{i=1}^m \int_{\Omega} \nabla u_i \cdot \nabla v_i - \sum_{i,j=1}^m \int_{\Omega} a_{ij} u_j v_i - \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i \cdot \nabla v_i + w_i |u_i|^{q-2} u_i v_i) \end{aligned}$$

$$+ \sum_{i=1}^m \int_{\Omega} |\nabla v_i|^2 + \sum_{i,j=1}^m \int_{\Omega} a_{ij} v_j v_i + \sum_{i=1}^m \int_{\Omega} (|\nabla v_i|^q + w_i |v_i|^q) = 0.$$

So we get

$$\begin{aligned} & \sum_{i=1}^m \int_{\Omega} \nabla u_i \cdot (\nabla u_i - \nabla v_i) + \sum_{i,j=1}^m \int_{\Omega} a_{ij} u_j (u_i - v_i) + \sum_{i=1}^m \int_{\Omega} |\nabla u_i|^{q-2} \nabla u_i \cdot (\nabla u_i - \nabla v_i) \\ & + \sum_{i=1}^m \int_{\Omega} w_i |u_i|^{q-2} u_i (u_i - v_i) - \sum_{i=1}^m \int_{\Omega} \nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i,j=1}^m \int_{\Omega} a_{ij} v_j (u_i - v_i) \\ & - \sum_{i=1}^m \int_{\Omega} |\nabla v_i|^{q-2} \nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i=1}^m \int_{\Omega} w_i |v_i|^{q-2} v_i (u_i - v_i) = 0. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{i=1}^m \int_{\Omega} |\nabla u_i - \nabla v_i|^2 + \sum_{i=1}^m \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla v_i|^{q-2} \nabla v_i) \cdot (\nabla u_i - \nabla v_i) \\ & + \sum_{i,j=1}^m \int_{\Omega} a_{ij} (u_j - v_j) (u_i - v_i) + \sum_{i=1}^m \int_{\Omega} w_i (|u_i|^{q-2} u_i - |v_i|^{q-2} v_i) (u_i - v_i) = 0. \end{aligned}$$

The last equality can be rewritten under the following form with the duality product

$$\begin{aligned} & \sum_{i=1}^m \langle -\Delta u_i + \Delta v_i, u_i - v_i \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q u_i + \Delta_q v_i, u_i - v_i \rangle_{q',q} \\ & + \sum_{i,j=1}^m \langle a_{ij} (u_j - v_j), u_i - v_i \rangle_{2,2} + \sum_{i=1}^m \langle w_i (|u_i|^{q-2} u_i - |v_i|^{q-2} v_i), u_i - v_i \rangle_{q',q} = 0. \end{aligned}$$

Moreover a consequence of the strict convexity of the spaces  $W_0^{1,2}(\Omega)$  and  $W_0^{1,q}(\Omega)$  is that the duality mappings  $-\Delta$  and  $-\Delta_q$  are strictly monotone. So from  $u_k \neq v_k$  we get

$$\langle -\Delta u_k + \Delta v_k, u_k - v_k \rangle_{2,2} > 0,$$

and

$$\langle -\Delta_q u_k + \Delta_q v_k, u_k - v_k \rangle_{q',q} \geq (\|u_k\|_{W_0^{1,q}(\Omega)}^{q-1} - \|v_k\|_{W_0^{1,q}(\Omega)}^{q-1}) (\|u_k\|_{W_0^{1,q}(\Omega)} - \|v_k\|_{W_0^{1,q}(\Omega)}) \geq 0$$

since  $x \mapsto x^{q-1}$  is increasing on  $[0, \infty)$  (and even  $\langle -\Delta_q u_k + \Delta_q v_k, u_k - v_k \rangle_{q',q} > 0$  from [8, Proposition 1]).

Thus

$$\sum_{i=1}^m \langle -\Delta u_i + \Delta v_i, u_i - v_i \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q u_i + \Delta_q v_i, u_i - v_i \rangle_{q',q} > 0.$$

Furthermore, since the function  $x \mapsto |x|^{q-2} x$  is increasing and  $w_i \geq 0$ , we have

$$\sum_{i=1}^m \langle w_i (|u_i|^{q-2} u_i - |v_i|^{q-2} v_i), u_i - v_i \rangle_{q',q} \geq 0.$$

Finally from Assumption 1.1,

$$\sum_{i,j=1}^m \langle a_{ij} (u_j - v_j), u_i - v_i \rangle_{2,2} \geq 0.$$

Therefore we get a contradiction.  $\square$

**Remark:** We can generalize Theorem 2.1 replacing the 2-Laplacian operator by the p-Laplacian with  $2 < p < q$ , that for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = h_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and even for

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i |u_i|^{p-2} u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = h_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

under the additional hypothesis that the bounded functions  $b_i$ ,  $i = 1, \dots, m$  are non-negative.

### 3. SECOND CASE: $g_i(\cdot, u_1, \dots, u_m) := m_i |u_i|^{q-2} u_i$

In this section we assume that

**Assumption 3.1.**  $q < 2^*$  where  $2^* = \frac{2N}{N-2}$  if  $N > 2$  and  $2^* = \infty$  if  $N \leq 2$ ,

and we rewrite the system  $(S, q, g)$  under the following form:  
for  $i = 1, \dots, m$ ,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = m_i |u_i|^{q-2} u_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

Note that the decomposition with the weights  $c_i := m_i - w_i$  does not necessarily coincide with the decomposition  $c_i = c_{i+} - c_{i-}$  where  $c_{i+} = \max(c_i, 0)$  and  $c_{i-} = \max(-c_i, 0)$ . Define now for  $u = (u_1, \dots, u_m) \in W$  the functional

$$M(u) = \sum_{i=1}^m \int_{\Omega} m_i |u_i|^q. \quad (3.2)$$

The Euler functional associated with (3.1) is consequently for  $u = (u_1, \dots, u_m) \in W$ ,

$$I(u) = \frac{1}{2} H_1(u) + \frac{1}{q} H_2(u) - \frac{1}{q} M(u). \quad (3.3)$$

First let us recall the usual weighted eigenvalue problem for the q-Laplacian:

$$\begin{cases} -\Delta_q u = \lambda \rho |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

with a bounded weight function  $\rho$  and a real parameter  $\lambda$ . It is said that  $\lambda$  is an eigenvalue of the q-Laplacian associated with the weight  $\rho$  if (3.4) has a non-trivial solution  $u$  which is called an eigenfunction associated with  $\lambda$ . It is well known (see [1]) that if the Lebesgue measure of  $\{x \in \Omega, \rho(x) > 0\}$  is positive, then the first positive eigenvalue  $\lambda_{1,q,\rho}$  of  $-\Delta_q$  with weight function  $\rho$  is obtained by the Rayleigh quotient

$$\lambda_{1,q,\rho} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^q}{\int_{\Omega} \rho |u|^q}; u \in W_0^{1,q}(\Omega), \int_{\Omega} \rho |u|^q > 0 \right\}. \quad (3.5)$$

Moreover,  $\lambda_{1,q,\rho}$  has a positive eigenfunction  $\phi_{1,q,\rho} \in C_0^{1,\alpha_q}(\bar{\Omega})$  (for some  $\alpha_q \in (0, 1)$ ). Assume in this section that

**Assumption 3.2.** **i):** For all  $i = 1, \dots, m$ ,  $m_i \in L^\infty(\Omega)$ ,

**ii):** For all  $i = 1, \dots, m$ , the real 1 is not an eigenvalue of the q-Laplacian with the weight  $m_i - w_i$ .

Assume also in this section that either Assumption 3.3 or Assumption 3.4 holds

**Assumption 3.3.** *There exists  $k \in \{1, \dots, m\}$  such that:*

$$\text{meas}\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k-w_k} < 1.$$

**Assumption 3.4.** *There exist  $k, l \in \{1, \dots, m\}$ ,  $k \neq l$  such that:*

$$\text{meas}\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k-w_k} + \int_{\Omega} (w_l - m_l) |\phi_{1,q,m_k-w_k}|^q < 0$$

with  $\phi_{1,q,m_k-w_k}$  the normalized eigenfunction associated with  $\lambda_{1,q,m_k-w_k}$ .

Note that Assumption 3.4 is satisfied when  $\lambda_{1,q,m_k-w_k}(m_k - w_k) + w_l - m_l < 0$  a. e. in  $\Omega$ . Our aim is to study the existence of a weak solution for the system (3.1) by minimizing the functional  $I$  defined by (3.3). As in section 2, the functional  $I$  is weakly lower semi-continuous on  $W$  but may be no more coercive so we cannot use standard arguments for minimizing  $I$ . First, we prove that any Palais-Smale sequence is bounded in  $W$  and has a strong convergent subsequence. Then we are able to apply the Mountain-Pass Lemma and Assumptions 3.3 or 3.4 allow us to get a non-trivial solution.

We say that  $(u_n) \subset W$ ,  $u_n = (u_{1n}, \dots, u_{mn})$ , is a Palais-Smale sequence if it satisfies the following conditions

$$|I(u_n)| \leq D \text{ for all } n \in \mathbb{N} \text{ and } \|I'(u_n)\|_{W^*} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.6)$$

with some constant  $D > 0$ ,  $W^*$  being the dual space of  $W$ .

**Lemma 3.1.** *Assume that Assumptions 1.1 and 3.2 are satisfied. If  $(u_n) \subset W$ ,  $u_n = (u_{1n}, \dots, u_{mn})$ , is a Palais-Smale sequence, then  $(u_n)$  is bounded in  $W$ .*

*Proof.* Let  $(u_n) \subset W$ ,  $u_n = (u_{1n}, \dots, u_{mn})$ , be a Palais-Smale sequence. We want to prove that  $(\|u_n\|_W)_n$  is bounded or equivalently that  $(H_2(u_n))_n$  is bounded. But

$$\frac{1}{q} H_2(u_n) = I(u_n) - \frac{1}{2} H_1(u_n) + \frac{1}{q} M(u_n) \leq D + \frac{1}{q} M(u_n) \leq D + C \|u_n\|_{(L^q(\Omega))^m}^q \quad (3.7)$$

with  $C$  a positive constant,  $C$  independent of  $u_n$  (since the functions  $m_i$  are bounded in the functional  $M(u)$  defined by (3.2)). So it is sufficient to show that  $(\|u_n\|_{(L^q(\Omega))^m})$  is bounded. We adapt ideas from [19]. Assume on the contrary that  $\alpha_n := \|u_n\|_{(L^q(\Omega))^m} \rightarrow_{n \rightarrow \infty} \infty$  (for a subsequence) and denote  $v_n = \frac{1}{\alpha_n} u_n = (v_{1n}, \dots, v_{mn})$ . From (3.7), we deduce that  $(\|v_n\|_W)$  is bounded and from the compact embedding of  $W$  into  $(L^q(\Omega))^m$  we get the existence of  $v_0 = (v_{01}, \dots, v_{0m}) \in W$  such that  $(v_n)$  converges to  $v_0$ , strongly in  $(L^q(\Omega))^m$  and weakly in  $W$  (for a subsequence).

• Now we prove that  $(v_n)$  converges strongly to  $v_0$  in  $W$ . Indeed by taking  $\phi_n := \frac{1}{\alpha_n^{q-1}} (v_n - v_0)$ , we obtain

$$\begin{aligned} I'(u_n) \cdot \phi_n &= \frac{1}{\alpha_n^{q-1}} \sum_{i=1}^m \int_{\Omega} (\nabla u_{in} \cdot \nabla (v_{in} - v_{0i}) + a_{ii} u_{in} (v_{in} - v_{0i})) \\ &+ \frac{1}{\alpha_n^{q-1}} \sum_{i=1}^m \int_{\Omega} (|\nabla u_{in}|^{q-2} \nabla u_{in} \cdot \nabla (v_{in} - v_{0i}) + w_i |u_{in}|^{q-2} u_{in} (v_{in} - v_{0i})) \end{aligned}$$

$$+ \frac{1}{\alpha_n^{q-1}} \sum_{i,j;i \neq j} \int_{\Omega} a_{ij} u_{jn} (v_{in} - v_{0i}) - \frac{1}{\alpha_n^{q-1}} \sum_{i=1}^m \int_{\Omega} m_i |u_{in}|^{q-2} u_{in} (v_{in} - v_{0i}). \quad (3.8)$$

But  $u_n = \alpha_n v_n$  so (3.8) becomes

$$\begin{aligned} I'(u_n) \cdot \phi_n &= \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \int_{\Omega} (\nabla v_{in} \cdot \nabla (v_{in} - v_{0i}) + a_{ii} v_{in} (v_{in} - v_{0i})) \\ &+ \sum_{i=1}^m \int_{\Omega} (|\nabla v_{in}|^{q-2} \nabla v_{in} \cdot \nabla (v_{in} - v_{0i}) + w_i |v_{in}|^{q-2} v_{in} (v_{in} - v_{0i})) \\ &+ \frac{1}{\alpha_n^{q-2}} \sum_{i,j;i \neq j} \int_{\Omega} a_{ij} v_{jn} (v_{in} - v_{0i}) - \sum_{i=1}^m \int_{\Omega} m_i |v_{in}|^{q-2} v_{in} (v_{in} - v_{0i}). \end{aligned} \quad (3.9)$$

Note that  $|I'(u_n) \cdot \phi_n| \leq \|I'(u_n)\|_{W^*} \|\phi_n\|_W = \|I'(u_n)\|_{W^*} \frac{1}{\alpha_n^{q-1}} \|v_n - v_0\|_W$  so  $I'(u_n) \cdot \phi_n \rightarrow_{n \rightarrow \infty} 0$  from (3.6),  $\alpha_n \rightarrow_{n \rightarrow \infty} \infty$  and  $(\|v_n\|_W)$  bounded. Moreover, since the functions  $a_{ij}, w_i, m_i$  are bounded there exists a positive constant, denoting  $C$  at each step, such that

$$|\int_{\Omega} a_{ij} v_{jn} (v_{in} - v_{0i})| \leq C \|v_{jn}\|_{L^2(\Omega)} \|v_{in} - v_{0i}\|_{L^2(\Omega)} \leq C \|v_n\|_W \|v_n - v_0\|_{(L^q(\Omega))^m}$$

and therefore

$$\int_{\Omega} a_{ij} v_{jn} (v_{in} - v_{0i}) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.10)$$

By the same way, for  $b_i = w_i$  or  $b_i = m_i$ ,

$$|\int_{\Omega} b_i |v_{in}|^{q-2} v_{in} (v_{in} - v_{0i})| \leq C (\int_{\Omega} |v_{in}|^q)^{\frac{q-1}{q}} (\int_{\Omega} |v_{in} - v_{0i}|^q)^{1/q} \leq C \|v_n\|_W^{q-1} \|v_n - v_0\|_{(L^q(\Omega))^m}$$

so

$$\int_{\Omega} b_i |v_{in}|^{q-2} v_{in} (v_{in} - v_{0i}) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.11)$$

Recall that  $\langle \cdot, \cdot \rangle_{q',q}$  is the duality product between  $W^{-1,q'}(\Omega)$  and  $W_0^{1,q}(\Omega)$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ . From (3.9), (3.10), (3.11), we deduce that

$$\frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \langle -\Delta v_{in}, v_{in} - v_{0i} \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q v_{in}, v_{in} - v_{0i} \rangle_{q',q} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.12)$$

Moreover we have (see also the proof of Theorem 2.1)

$$\langle -\Delta_q v_{in} + \Delta_q v_{0i}, v_{in} - v_{0i} \rangle_{q',q} \geq (\|v_{in}\|_{W_0^{1,q}(\Omega)}^{q-1} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}^{q-1}) (\|v_{in}\|_{W_0^{1,q}(\Omega)} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}) \geq 0 \quad (3.13)$$

and

$$\langle -\Delta v_{in} + \Delta v_{0i}, v_{in} - v_{0i} \rangle_{2,2} = \|v_{in} - v_{0i}\|_{W_0^{1,2}(\Omega)}^2 \geq (\|v_{in}\|_{W_0^{1,2}(\Omega)} - \|v_{0i}\|_{W_0^{1,2}(\Omega)})^2. \quad (3.14)$$

From (3.13) and (3.14) we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^m (\|v_{in}\|_{W_0^{1,q}(\Omega)}^{q-1} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}^{q-1}) (\|v_{in}\|_{W_0^{1,q}(\Omega)} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}) \\ &+ \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m (\|v_{in}\|_{W_0^{1,2}(\Omega)} - \|v_{0i}\|_{W_0^{1,2}(\Omega)})^2 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \langle -\Delta v_{in}, v_{in} - v_{0i} \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q v_{in}, v_{in} - v_{0i} \rangle_{q',q} \\ &\quad + \sum_{i=1}^m \langle \Delta_q v_{0i}, v_{in} - v_{0i} \rangle_{q',q} + \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \langle \Delta v_{0i}, v_{in} - v_{0i} \rangle_{2,2}. \end{aligned}$$

Because the right-hand side of the above estimate tends to 0 as  $n$  tends to infinity (from (3.12) and the weak convergence of  $(v_n)$  to  $v_0$  in  $W$ ) we obtain that for  $i = 1, \dots, m$ ,  $\|v_{in}\|_{W_0^{1,q}(\Omega)} \rightarrow \|v_{0i}\|_{W_0^{1,q}(\Omega)}$  as  $n \rightarrow \infty$  and therefore  $(v_n)$  strongly converges to  $v_0$  in  $W$ .

• Finally, we prove that  $v_{0i}$  is a non-trivial solution of the eigenvalue problem of the  $q$ -Laplacian with weight  $m_i - w_i$  for at least one  $i$ .

Let  $\phi = (\phi_1, \dots, \phi_m) \in W$ . Taking  $\frac{1}{\alpha_n^{q-1}}\phi$  as a test function, since  $u_n = \alpha_n v_n$ , we have

$$\begin{aligned} I'(u_n) \cdot \frac{1}{\alpha_n^{q-1}}\phi &= \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \int_{\Omega} (\nabla v_{in} \cdot \nabla \phi_i + \sum_{j=1}^m \int_{\Omega} a_{ij} v_{jn} \phi_i) \\ &+ \sum_{i=1}^m \int_{\Omega} (|\nabla v_{in}|^{q-2} \nabla v_{in} \cdot \nabla \phi_i + w_i |v_{in}|^{q-2} v_{in} \phi_i - m_i |v_{in}|^{q-2} v_{in} \phi_i). \end{aligned}$$

Letting  $n \rightarrow \infty$ , we see that for each  $i = 1, \dots, m$ ,  $v_{0i}$  is a weak solution to

$$\begin{cases} -\Delta_q v_{0i} + w_i |v_{0i}|^{q-2} v_{0i} = m_i |v_{0i}|^{q-2} v_{0i} & \text{in } \Omega \\ v_{0i} = 0 & \text{on } \partial\Omega \end{cases}. \quad (3.15)$$

Since  $\|v_n\|_{(L^q(\Omega))^m} = 1$  and  $(v_n)$  converges strongly to  $v_0$  in  $W$  we get that  $\|v_0\|_W \geq 1$ . Therefore there exists  $i$  such that  $v_{0i}$  is a non-trivial solution of (3.15). This contradicts Assumption 3.2.  $\square$

**Lemma 3.2.** *Assume that Assumptions 1.1 and 3.2 are satisfied. If  $(u_n) \subset W$ ,  $u_n = (u_{1n}, \dots, u_{mn})$ , is a Palais-Smale sequence, then  $(u_n)$  has a strong convergent subsequence in  $W$ .*

*Proof.* Let  $(u_n)$  be a Palais-Smale sequence in  $W$ ,  $u_n = (u_{1n}, \dots, u_{mn})$ . By Lemma 3.1, the sequence  $(u_n)$  is bounded in  $W$ . From the compact embedding of  $W^{1,q}(\Omega)$  into  $L^q(\Omega)$  we get the existence of  $u_0 = (u_{01}, \dots, u_{0m}) \in W$  such that  $(u_n)$  converges to  $u_0$  strongly in  $(L^q(\Omega))^m$  and weakly in  $W$  (for a subsequence still denoted by  $(u_n)$ ). We want to prove that  $\|u_n\|_W \rightarrow \|u_0\|_W$  as  $n \rightarrow \infty$  and we proceed as in the proof of Lemma 3.1.

Since  $|I'(u_n) \cdot (u_n - u_0)| \leq \|I'(u_n)\|_{W^*} (\|u_n\|_W + \|u_0\|_W)$  we deduce that

$$I'(u_n) \cdot (u_n - u_0) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.16)$$

But

$$\begin{aligned} I'(u_n) \cdot (u_n - u_0) &= \sum_{i=1}^m \int_{\Omega} (\nabla u_{in} \cdot \nabla (u_{in} - u_{0i}) + \sum_{j=1}^m a_{ij} u_{jn} (u_{in} - u_{0i})) \\ &+ \sum_{i=1}^m \int_{\Omega} (|\nabla u_{in}|^{q-2} \nabla u_{in} \cdot \nabla (u_{in} - u_{0i}) + (w_i - m_i) |u_{in}|^{q-2} u_{in} (u_{in} - u_{0i})). \end{aligned}$$

As in Lemma 3.1, denoting  $b_i$  either  $w_i$  or  $m_i$ , we have for  $i, j = 1, \dots, m$ ,

$$\int_{\Omega} b_i |u_{in}|^{q-2} u_{in} (u_{in} - u_{0i}) \rightarrow 0 \text{ as } n \rightarrow \infty, \int_{\Omega} a_{ij} u_{jn} (u_{in} - u_{0i}) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.17)$$

From (3.16) and (3.17), we get that

$$\sum_{i=1}^m \langle -\Delta u_{in}, u_{in} - u_{0i} \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q u_{in}, u_{in} - u_{0i} \rangle_{q',q} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^m (\|u_{in}\|_{W_0^{1,q}(\Omega)}^{q-1} - \|u_{0i}\|_{W_0^{1,q}(\Omega)}^{q-1}) (\|u_{in}\|_{W_0^{1,q}(\Omega)} - \|u_{0i}\|_{W_0^{1,q}(\Omega)}) \\ &\quad + \sum_{i=1}^m (\|u_{in}\|_{W_0^{1,2}(\Omega)} - \|u_{0i}\|_{W_0^{1,2}(\Omega)})^2 \\ &\leq \sum_{i=1}^m \langle -\Delta u_{in}, u_{in} - u_{0i} \rangle_{2,2} + \sum_{i=1}^m \langle -\Delta_q u_{in}, u_{in} - u_{0i} \rangle_{q',q} \\ &\quad + \sum_{i=1}^m \langle \Delta_q u_{0i}, u_{in} - u_{0i} \rangle_{q',q} + \sum_{i=1}^m \langle \Delta u_i, u_{in} - u_{0i} \rangle_{2,2}. \end{aligned}$$

As in Lemma 3.1 we deduce that for  $i = 1, \dots, m$ ,  $\|u_{in}\|_{W_0^{1,q}(\Omega)} \rightarrow \|u_{0i}\|_{W_0^{1,q}(\Omega)}$  as  $n \rightarrow \infty$  and therefore  $(u_n)$  strongly converges to  $u_0$  in  $W$ .  $\square$

So we can state the main result of this section

**Theorem 3.1.** *Assume that Assumptions 1.1, 3.1 and 3.2 are satisfied. Assume also that either Assumption 3.3 or 3.4 holds. Then the system (3.1) has a non-trivial solution in  $W$ .*

*Proof.* The  $C^1$ -functional  $I$  satisfies the Palais-Smale conditions and  $I(0) = 0$ .

• First, we claim that there exist positive constants  $\rho^* > 0$  and  $\delta > 0$  such that  $I(u) \geq \delta$  for any  $u = (u_1, \dots, u_m) \in W$  satisfying  $\|u\|_W = \rho^*$ .

Let  $u = (u_1, \dots, u_m) \in W$ . Put  $\rho = \|u\|_W$  and note that  $H_1(u) \geq \|u\|_H^2$  and  $H_2(u) \geq \rho^q$ .

Moreover, since  $q < 2^*$ , for  $i = 1, \dots, m$ ,

$$\left| \int_{\Omega} m_i |u_i|^q \right| \leq \left( \int_{\Omega} |m_i|^r \right)^{1/r} \left( \int_{\Omega} |u_i|^{qt} \right)^{1/t} \text{ with } \frac{1}{r} + \frac{1}{t} = 1 \text{ and } s := qt < 2^*.$$

From the continuous embedding of  $W^{1,2}(\Omega) \subset L^s(\Omega)$  we deduce the existence of a positive constant  $C_1$  such that  $\left| \int_{\Omega} m_i |u_i|^q \right| \leq C_1 \|u_i\|_{W^{1,2}(\Omega)}^q$ . Thus

$$|M(u)| \leq C_1 \|u\|_H^q$$

and

$$I(u) \geq \frac{1}{q} \rho^q + \frac{1}{2} \|u\|_H^2 \left( 1 - \frac{2C_1}{q} \|u\|_H^{q-2} \right).$$

Recall also that there exists a positive constant  $C_2 > 0$  such that  $\|u\|_H \leq C_2 \|u\|_W$  for all  $u \in W$ .

Therefore if  $\rho \leq \rho^* := \frac{1}{C_2} \left( \frac{q}{2C_1} \right)^{\frac{1}{q-2}}$ , then  $1 - \frac{2C_1}{q} \|u\|_H^{q-2} \geq 1 - \frac{2C_1}{q} (C_2 \rho)^{q-2} \geq 0$  and

$$I(u) \geq \frac{1}{q} \rho^q := \delta.$$

• Assume here that Assumption 3.3 is satisfied with  $k = 1$  for simplicity.

Let  $\phi_{1,q,m_1-w_1}$  be the normalized eigenfunction associated with  $\lambda_{1,q,m_1-w_1}$  (i. e. be such that  $\int_{\Omega} (m_1 - w_1) |\phi_{1,q,m_1-w_1}|^q = 1$ , we may choose such  $\phi_{1,q,m_1-w_1}$  because

the equation (3.4) is homogeneous). Denote  $\Phi_q = (\phi_{1,q,m_1-w_1}, 0, \dots, 0)$  and take  $R$  sufficiently large such that  $\|R\Phi_q\|_W > \rho^*$ . We have from (3.4) and (3.5)

$$\begin{aligned} I(R\Phi_q) &= \frac{R^2}{2} H_1(\Phi_q) + \frac{R^q}{q} \int_{\Omega} (|\nabla \phi_{1,q,m_1-w_1}|^q + (w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q) \\ &= \frac{R^2}{2} H_1(\Phi_q) + \frac{R^q}{q} \int_{\Omega} (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q. \end{aligned}$$

So, since  $\lambda_{1,q,m_1-w_1} < 1$ ,

$$I(R\Phi_q) = \frac{R^2}{2} H_1(\Phi_q) + \frac{R^q}{q} (\lambda_{1,q,m_1-w_1} - 1) < 0$$

for  $R$  sufficiently large. Therefore we can apply the mountain-pass theorem to deduce that  $I$  has a non-trivial critical point which is a non-trivial weak solution of the system (3.1).

• Assume now that Assumption 3.4 is satisfied with  $k = 1$  and  $l = 2$  for simplicity. Denote again  $\phi_{1,q,m_1-w_1}$  the normalized eigenfunction associated with  $\lambda_{1,q,m_1-w_1}$  such that  $\int_{\Omega} (m_1 - w_1)|\phi_{1,q,m_1-w_1}|^q = 1$  and denote here  $\Psi_q = (0, \phi_{1,q,m_1-w_1}, 0, \dots, 0)$ . Take  $R$  sufficiently large such that  $\|R\Psi_q\|_W > \rho^*$ . We have here

$$\begin{aligned} I(R\Psi_q) &= \frac{R^2}{2} H_1(\Psi_q) + \frac{R^q}{q} \int_{\Omega} (|\nabla \phi_{1,q,m_1-w_1}|^q + (w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q) \\ &= \frac{R^2}{2} H_1(\Psi_q) + \frac{R^q}{q} \int_{\Omega} (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q. \end{aligned}$$

From Assumption 3.4, we get that  $I(R\Psi_q) < 0$  for  $R$  sufficiently large. Therefore, as in the precedent case, we apply the mountain-pass theorem and deduce that  $I$  has a non-trivial critical point.  $\square$

**Remark:** As in section 2, we can generalize Theorem 3.1 replacing the 2-Laplacian operator by the  $p$ -Laplacian with  $2 \leq p < q$  for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i |u_i|^{p-2} u_i + \lambda w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = \lambda m_i |u_i|^{q-2} u_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

under the additional hypotheses that the bounded functions  $b_i$ ,  $i = 1, \dots, m$  are non-negative and  $\lambda$  is a real parameter. Then the hypothesis ii) in Assumption 3.2 is replaced by  $\lambda$  is not an eigenvalue of  $-\Delta_q$  associated with  $m_i - w_i$  for each  $i$ . Moreover the hypothesis  $\lambda_{1,q,m_k-w_k} < 1$  in Assumption 3.3 is replaced by  $\lambda_{1,q,m_k-w_k} < \lambda$ .

#### 4. THIRD CASE: $g_i(\cdot, u_1, \dots, u_m) := \lambda f_i |u_i|^{\gamma-2} u_i$

In this section we rewrite the system  $(S, q, g)$  under the following form: for  $i = 1, \dots, m$ ,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = \lambda f_i |u_i|^{\gamma-2} u_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

We assume throughout all this section that the indefinite bounded functions  $f_i$  and the coefficients  $\gamma$  and  $q$  satisfy the following hypotheses

- Assumption 4.1.**
- i):**  $2 < \gamma < q$ ,
  - ii):**  $\gamma < 2^*$  where  $2^* = \frac{2N}{N-2}$  if  $2 < N$  and  $2^* = \infty$  if  $2 \geq N$ ,
  - iii):** For each  $i = 1, \dots, m$ ,  $f_i \in L^\infty(\Omega)$  and  $\text{meas}\{x \in \Omega, f_i(x) > 0\} \neq \emptyset$ .

We also define the functionals

$$F(u) = \sum_{i=1}^m \int_{\Omega} f_i |u_i|^{\gamma} \quad (4.2)$$

and

$$I_{\lambda}(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \frac{\lambda}{\gamma}F(u) \quad (4.3)$$

where  $H_1$  and  $H_2$  are respectively defined by (1.1) and (1.2). We recall that we study here the existence of a weak non-trivial solution  $u = (u_1, \dots, u_m) \in W$  for the system (4.1) with respect to the real positive parameter  $\lambda$  and that the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for the Euler functional  $I_{\lambda}$ . The main result is the existence of a weak non-trivial solution for the system (4.1) associated with  $\lambda > \lambda_1^+$  where  $\lambda_1^+$  is defined by (4.7). For the first part of this section we follow a method developed by Cherfils-Il'Yasov in [7] for one equation with the (p,q)-Laplacian operator. This method is based on proving the existence of solution for  $\lambda = \lambda_1^+$  then on applying the mountain-pass theorem for  $\lambda > \lambda_1^+$ . Although we also could apply the mountain-pass theorem for our case, we will use in fact standard arguments to minimize the functional  $I_{\lambda}$ .

In section 4.1 we present some preliminary results: we define  $\lambda_1^+$  and we prove the existence of a solution for the system (4.1) for  $\lambda = \lambda_1^+$ . The section 4.2 is devoted to the main theorem of the existence of a solution for the system (4.1) associated with  $\lambda > \lambda_1^+$ .

**4.1. Some preliminaries results.** As in [7] we define for  $\lambda > 0$ ,  $t > 0$  and  $u \in W$ ,  $\tilde{I}_{\lambda}(t, u) = I_{\lambda}(tu)$ .

**Lemma 4.1.** *Assume that Assumptions 1.1, 4.1 i), 4.1 iii) are satisfied. For given  $u$  in  $W$ ,  $u \neq 0$  such that  $F(u) \neq 0$ , the unique solution  $(t(u), \lambda(u))$  of the system*

$$\begin{cases} \frac{\partial}{\partial t} \tilde{I}_{\lambda}(t, u) = 0 \\ \frac{\partial^2}{\partial t^2} \tilde{I}_{\lambda}(t, u) = 0 \end{cases} \text{ is given by}$$

$$t(u) = \left( \frac{\gamma - 2}{q - \gamma} \right)^{\frac{1}{q-2}} \left( \frac{H_1(u)}{H_2(u)} \right)^{\frac{1}{q-2}} > 0, \quad \lambda(u) = C_{q,\gamma} \frac{H_1(u)^{\alpha} H_2(u)^{1-\alpha}}{F(u)} \quad (4.4)$$

with

$$\alpha = \frac{q - \gamma}{q - 2}, \quad C_{q,\gamma} = \frac{q - 2}{(q - \gamma)^{\alpha} (\gamma - 2)^{1-\alpha}}. \quad (4.5)$$

*Proof.* The system (S)  $\begin{cases} \frac{\partial}{\partial t} \tilde{I}_{\lambda}(t, u) = 0 \\ \frac{\partial^2}{\partial t^2} \tilde{I}_{\lambda}(t, u) = 0 \end{cases}$  is equivalent to the system

$$\begin{cases} tH_1(u) + t^{q-1}H_2(u) - \lambda t^{\gamma-1}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases}$$

and to the following system

$$\begin{cases} H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases} .$$

Therefore

$$(q-2)t^{q-2}H_2(u) - \lambda(\gamma-2)t^{\gamma-2}F(u) = 0. \quad (4.6)$$

Note that the system (S) is not solvable in the case where  $u \in W$ ,  $u \neq 0$  satisfies  $F(u) = 0$  (since if  $u \neq 0$ , then  $H_2(u) \neq 0$  and from (4.6) we deduce  $F(u) \neq 0$ ).

We deduce that

$$\lambda = \frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}.$$

Replacing  $\lambda$  by  $\frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}$  in  $H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0$ , we get that  $t^{q-2} = \frac{(\gamma-2)H_1(u)}{q-2}$ . And we obtain (4.4) associated with (4.5).  $\square$

Thus we can define the following characteristic points (recall that  $F$  is defined by (4.2))

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\} \text{ and } \lambda_1^+ = \frac{\gamma}{2^\alpha q^{1-\alpha}} \Lambda_1^+. \quad (4.7)$$

**Lemma 4.2.** *Assume that Assumptions 1.1 and 4.1 are satisfied.*

*We have  $0 < \Lambda_1^+ < \lambda_1^+$ .*

*Proof.* Let  $u = (u_1, \dots, u_m) \in W$  be such that  $F(u) > 0$ .

First from  $\gamma < 2^*$ , let  $(t, l)$  be such that  $\gamma < t < 2^*$  and  $\frac{1}{l} + \frac{\gamma}{t} = 1$ . Since  $W_0^{1,2}(\Omega) \subset L^t(\Omega)$  with a continuous embedding and since the functions  $f_i$  are bounded, there exist positive constants still denoting  $C$  at each step and depending on some Sobolev constants, such that for  $i = 1, \dots, m$

$$|\int_{\Omega} f_i |u_i|^\gamma| \leq (\int_{\Omega} |f_i|^l)^{1/l} (\int_{\Omega} |u_i|^t)^{\gamma/t} \leq C \|u_i\|_{L^t(\Omega)}^\gamma \leq C \|u_i\|_{W_0^{1,2}(\Omega)}^\gamma.$$

Then

$$F(u) \leq CH_1(u)^{\gamma/2}.$$

By the same way, from  $\gamma < q$ , let  $s = \frac{q}{\gamma}$  and  $r$  be such that  $\frac{1}{s} + \frac{1}{r} = 1$ .

Then we have

$$|\int_{\Omega} f_i |u_i|^\gamma| \leq m (\int_{\Omega} |f_i|^r)^{1/r} (\int_{\Omega} |u_i|^s)^{1/s} \leq C \|u_i\|_{L^q(\Omega)}^\gamma \leq C \|u_i\|_{W_0^{1,q}(\Omega)}^\gamma$$

and

$$F(u) \leq CH_2(u)^{\gamma/q}.$$

Therefore there exists a positive constant  $C'$ , independent of  $u$ , such that

$$\lambda(u) = C_{q,\gamma} \frac{H_1(u)^\alpha H_2(u)^{1-\alpha}}{F(u)} \geq C' C_{q,\gamma} \frac{F(u)^{\frac{2\alpha}{\gamma}} F(u)^{\frac{q(1-\alpha)}{\gamma}}}{F(u)} = C' C_{q,\gamma}$$

since  $\frac{2\alpha}{\gamma} + \frac{q(1-\alpha)}{\gamma} = 1$ . Thus  $\Lambda_1^+ > 0$ .

Finally we prove that  $\Lambda_1^+ < \lambda_1^+$ .

Indeed note that  $\lambda_1^+ > \Lambda_1^+ \Leftrightarrow \frac{\gamma}{2^\alpha q^{1-\alpha}} \Lambda_1^+ > \Lambda_1^+ \Leftrightarrow (\frac{\gamma}{2})^{q-2} > (\frac{q}{2})^{\gamma-2}$ .

Denote  $\mu = \frac{q-2}{2} > 0$  and  $\eta = \frac{\gamma-2}{2} > 0$ . Since  $2 < \gamma < q$  we have  $\mu > \eta$ . Moreover the function  $f$  defined by  $f(x) = (1+x)^{1/x}$ , is strictly decreasing on  $(0, \infty)$ . Then  $(1+\mu)^{1/\mu} < (1+\eta)^{1/\eta}$ . And we get that  $(\frac{q}{2})^{\gamma-2} < (\frac{\gamma}{2})^{q-2}$ . So  $\Lambda_1^+ < \lambda_1^+$ .  $\square$

We obtain now the following result that will enable us to get the existence of a non-trivial solution for the system (4.1) associated with  $\lambda_1^+$ .

**Proposition 4.1.** *Assume that Assumptions 1.1 and 4.1 are satisfied. Assume that  $u = (u_1, \dots, u_m) \in W$  satisfies  $F(u) \neq 0$  and  $\lambda'(u) = 0$  (i.e.  $u$  is a critical point of  $\lambda(u)$ ). Then  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m) \in W$  is a non-trivial solution of the system (4.1) associated with  $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}} \lambda(u)$  where for all  $i = 1, \dots, m$ ,  $\tilde{u}_i = \frac{1}{s} u_i$  and  $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}} t(u) > 0$ . Moreover  $I_{\tilde{\lambda}}(\tilde{u}) = 0$ .*

*Proof.* Let  $u = (u_1, \dots, u_m) \in W$  which satisfies  $F(u) \neq 0$  and  $\lambda'(u) = 0$ . For all test function  $\phi$ , we have

$$\frac{\partial \lambda}{\partial u_1}(u) \cdot \phi = 0.$$

So

$$\begin{aligned} & 2C_{q,\gamma}\alpha(H_1(u))^{\alpha-1}(H_2(u))^{1-\alpha}(F(u))^{-1} \int_{\Omega} (\nabla u_1 \cdot \nabla \phi + a_{11}u_1\phi + \sum_{j=2}^m a_{1j}u_j\phi) \\ & + qC_{q,\gamma}(1-\alpha)(H_1(u))^\alpha(H_2(u))^{-\alpha}(F(u))^{-1} \int_{\Omega} (|\nabla u_1|^{q-2}\nabla u_1 \cdot \nabla \phi + w_1|u_1|^{q-2}u_1\phi) \\ & - \gamma C_{q,\gamma}\alpha(H_1(u))^\alpha(H_2(u))^{1-\alpha}(F(u))^{-2} \int_{\Omega} f_1|u_1|^{\gamma-2}u_1\phi = 0. \end{aligned}$$

And

$$\begin{aligned} & 2C_{q,\gamma}\alpha \left( \frac{H_1(u)}{H_2(u)} \right)^{\alpha-1} \int_{\Omega} (\nabla u_1 \cdot \nabla \phi + a_{11}u_1\phi + \sum_{j=2}^m a_{1j}u_j\phi) \\ & + qC_{q,\gamma}(1-\alpha) \left( \frac{H_1(u)}{H_2(u)} \right)^\alpha \int_{\Omega} (|\nabla u_1|^{q-2}\nabla u_1 \cdot \nabla \phi + w_1|u_1|^{q-2}u_1\phi) \\ & - \lambda(u)\gamma \int_{\Omega} f_1|u_1|^{\gamma-2}u_1\phi = 0. \end{aligned}$$

Define  $\tilde{u}_i = \frac{1}{s} u_i$  for  $i = 1, \dots, m$ ,  $s > 0$  and  $H(u) = \frac{H_1(u)}{H_2(u)}$ . Then

$$\begin{aligned} & 2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s \int_{\Omega} (\nabla \tilde{u}_1 \cdot \nabla \phi + a_{11}\tilde{u}_1\phi + \sum_{j=2}^m a_{1j}\tilde{u}_j\phi) \\ & + C_{q,\gamma}(1-\alpha)(H(u))^\alpha qs^{q-1} \int_{\Omega} (|\nabla \tilde{u}_1|^{q-2}\nabla \tilde{u}_1 \cdot \nabla \phi + w_1|\tilde{u}_1|^{q-2}\tilde{u}_1\phi) \\ & - \lambda(u)\gamma s^{\gamma-1} \int_{\Omega} f_1|\tilde{u}_1|^{\gamma-2}\tilde{u}_1\phi = 0. \end{aligned}$$

And equivalently

$$\begin{aligned} & 2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s^{2-\gamma} \int_{\Omega} (\nabla \tilde{u}_1 \cdot \nabla \phi + a_{11}\tilde{u}_1\phi + \sum_{j=2}^m a_{1j}\tilde{u}_j\phi) \\ & + C_{q,\gamma}(1-\alpha)(H(u))^\alpha qs^{q-\gamma} \int_{\Omega} (|\nabla \tilde{u}_1|^{q-2}\nabla \tilde{u}_1 \cdot \nabla \phi + w_1|\tilde{u}_1|^{q-2}\tilde{u}_1\phi) \\ & - \lambda(u)\gamma \int_{\Omega} f_1|\tilde{u}_1|^{\gamma-2}\tilde{u}_1\phi = 0. \end{aligned}$$

Multiplying this last equation by  $\frac{1}{2^\alpha q^{1-\alpha}}$  and denoting  $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}} \lambda(u)$  we get

$$\left( \frac{2(q-\gamma)}{q(\gamma-2)H(u)} \right)^{\frac{\gamma-2}{q-2}} s^{2-\gamma} \int_{\Omega} (\nabla \tilde{u}_1 \cdot \nabla \phi + a_{11}\tilde{u}_1\phi + \sum_{j=2}^m a_{1j}\tilde{u}_j\phi)$$

$$\begin{aligned}
& + \left( \frac{q(\gamma-2)H(u)}{2(q-\gamma)} \right)^{\frac{q-\gamma}{q-2}} s^{q-\gamma} \int_{\Omega} (|\nabla \tilde{u}_1|^{q-2} \nabla \tilde{u}_1 \cdot \nabla \phi + w_1 |\tilde{u}_1|^{q-2} \tilde{u}_1 \phi) \\
& \quad - \tilde{\lambda} \int_{\Omega} f_1 |\tilde{u}_1|^{\gamma-2} \tilde{u}_1 \phi = 0.
\end{aligned}$$

Choosing  $s = \left( \frac{2(q-\gamma)}{q(\gamma-2)H(u)} \right)^{\frac{1}{q-2}}$  we obtain

$$\begin{aligned}
& \int_{\Omega} (\nabla \tilde{u}_1 \cdot \nabla \phi + a_{11} \tilde{u}_1 \phi + \sum_{j=2}^m a_{1j} \tilde{u}_j \phi) + \int_{\Omega} (|\nabla \tilde{u}_1|^{q-2} \nabla \tilde{u}_1 \cdot \nabla \phi + w_1 |\tilde{u}_1|^{q-2} \tilde{u}_1 \phi) \\
& \quad - \tilde{\lambda} \int_{\Omega} f_1 |\tilde{u}_1|^{\gamma-2} \tilde{u}_1 \phi = 0.
\end{aligned}$$

Doing the same for  $\tilde{u}_i$ ,  $i = 2, \dots, m$ , we get that  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)$  is a weak solution of (4.1) associated with  $\tilde{\lambda}$ .

Now we prove that  $I_{\tilde{\lambda}}(\tilde{u}) = 0$ .

Recall that  $\tilde{u} = \frac{1}{s}u$  and  $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}} \lambda(u)$ . Then we have

$$I_{\tilde{\lambda}}(\tilde{u}) = \frac{H_1(u)}{2s^2} + \frac{H_2(u)}{qs^q} - \frac{C_{q,\gamma} H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha} s^\gamma}.$$

Denoting  $r = \frac{C_{q,\gamma} H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha} s^\gamma}$ , since  $\frac{1}{s} = \left(\frac{q}{2}\right)^{\frac{1}{q-2}} t(u) > 0$  and  $\alpha = \frac{q-\gamma}{q-2}$  we get

$$I_{\tilde{\lambda}}(\tilde{u}) = r \left[ \frac{1}{C_{q,\gamma}} \left( \frac{qH_1(u)}{2H_2(u)} \right)^{1-\alpha} \left( \frac{q}{2} \right)^{\alpha-1} (t(u))^{2-\gamma} + \frac{1}{C_{q,\gamma}} \left( \frac{2H_2(u)}{qH_1(u)} \right)^\alpha \left( \frac{q}{2} \right)^\alpha (t(u))^{q-\gamma} - 1 \right].$$

But  $t(u) = \left( \frac{\gamma-2}{q-\gamma} \right)^{\frac{1}{q-2}} \left( \frac{H_1(u)}{H_2(u)} \right)^{\frac{1}{q-2}} > 0$  and  $C_{q,\gamma} = \frac{q-2}{(q-\gamma)^\alpha (\gamma-2)^{1-\alpha}}$  so

$$I_{\tilde{\lambda}}(\tilde{u}) = r \left[ \frac{q-\gamma}{q-2} + \frac{\gamma-2}{q-2} - 1 \right] = 0.$$

□

**Proposition 4.2.** *Assume that Assumptions 1.1 and 4.1 are satisfied and  $0 < \lambda < \Lambda_1^+$ . Then the system (4.1) has no non-trivial solution in  $W$  associated with  $\lambda$ .*

*Proof.* Assume that  $0 < \lambda < \Lambda_1^+$ . Assume also that the system (4.1) has a non-trivial solution  $u = (u_1, \dots, u_m) \in W$  associated with  $\lambda$ . Then we have  $H_1(u) + H_2(u) = \lambda F(u)$ . Note that this is impossible if  $F(u) \leq 0$ .

Therefore assume now that  $F(u) > 0$ .

Recall that  $\tilde{I}_\lambda(t, v) = I_\lambda(tv) = \frac{t^2}{2} H_1(v) + \frac{t^q}{q} H_2(v) - \frac{\lambda t^\gamma}{\gamma} F(v)$  for all  $t > 0$  and  $v \in W$ . We have  $\frac{\partial}{\partial t} \tilde{I}_\lambda(t, v) = t H_1(v) + t^{q-1} H_2(v) - \lambda t^{\gamma-1} F(v)$  and in particular, since  $u$  is a weak solution of (4.1), note that

$$\frac{\partial}{\partial t} \tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|} u) = I'_\lambda(u) \cdot \frac{1}{\|u\|} u = 0.$$

Moreover we have  $\frac{\partial}{\partial t} \tilde{I}_\lambda(t, v) = t^{\gamma-1} R_\lambda(t, v)$  with

$$R_\lambda(t, v) = t^{2-\gamma} H_1(v) + t^{q-\gamma} H_2(v) - \lambda F(v).$$

Let  $v \in W$  be such that  $v \neq 0$  and  $F(v) > 0$ . Note that from Lemma 4.1 we have  $R_{\lambda(v)}(t(v), v) = 0$ .

Moreover we can prove that  $R_\lambda(t, v) \geq R_{\lambda(v)}(t, v)$  for all  $t > 0$ .

Indeed let  $f(t) = t^{2-\gamma}H_1(v) + t^{q-\gamma}H_2(v)$ . The function  $f$  admits a global minimum on  $t(v)$  on  $(0, \infty)$  so  $f(t) \geq f(t(v)) = \left(\frac{H_1(v)}{\alpha}\right)^\alpha \left(\frac{H_2(v)}{1-\alpha}\right)^{1-\alpha} > 0$ . Therefore  $R_\lambda(t, v) \geq R_{\lambda(v)}(t, v)$  for all  $t > 0$ .

Finally since  $\lambda < \Lambda_1^+ \leq \lambda(v)$ , we get that  $R_\lambda(t, v) > R_{\lambda(v)}(t, v)$  for all  $t > 0$ . Thus  $R_\lambda(t, v) \geq R_\lambda(t(v), v) > R_{\lambda(v)}(t(v), v) = 0$  and  $\frac{\partial}{\partial t} \tilde{I}_\lambda(t, v) = \frac{\partial}{\partial t} I_\lambda(t, v) > 0$  for all  $t > 0$ .

So, choosing  $t = \|u\|$  and  $v = \frac{1}{\|u\|}u$ , we get a contradiction since  $\frac{\partial}{\partial t} \tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|}u) = 0$ .  $\square$

Now we obtain a minimizer for  $\Lambda_1^+$ .

**Proposition 4.3.** *Assume that Assumptions 1.1 and 4.1 are satisfied. There exists  $v = (v_1, \dots, v_m) \in W$  such that  $\lambda(v) = \Lambda_1^+$ .*

*Proof.* First note that  $\lambda(tu) = \lambda(u)$  for all  $t > 0$  and  $u \in W$ .

Define  $\tilde{t}(u) = \frac{1}{((H_1(u))^\alpha (H_2(u))^{1-\alpha})^{\frac{1}{\gamma}}}$  for  $u \in W \setminus \{0\}$  and note that

$$(H_1(\tilde{t}(u)u)^\alpha (H_2(\tilde{t}(u)u))^{1-\alpha} = 1.$$

Therefore we can derive that

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W \text{ such that } F(u) > 0 \text{ and } H_1(u)^\alpha H_2(u)^{1-\alpha} = 1\}.$$

Then we consider a minimizing sequence  $(v_n)$  of  $\Lambda_1^+$ .

We have  $\gamma = 2\alpha + q(1-\alpha)$ , so

$$\|v_n\|_H^\gamma = \|v_n\|_H^{2\alpha} \|v_n\|_H^{q(1-\alpha)}$$

and since  $W \subset H$  with a continuous embedding, there exists a positive constant  $C$  such that

$$\|v_n\|_H^\gamma \leq C \|v_n\|_H^{2\alpha} \|v_n\|_W^{q(1-\alpha)}.$$

But  $H_1$  and  $H_2$  are equivalent norms respectively in  $H$  and  $W$  so we get that

$$\|v_n\|_H^\gamma \leq C (H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha} = C$$

(for a positive constant  $C$ ). We deduce that  $(v_n)$  is a bounded sequence in  $H$ . By the compact embedding  $W_0^{1,2}(\Omega) \subset L^\gamma(\Omega)$  (for  $\gamma < 2^*$ ), we get the existence of  $v = (v_1, \dots, v_m) \in H$  such that  $(v_n)$  converges to  $v$ , strongly in  $(L^\gamma(\Omega))^m$  and weakly in  $H$  (for a subsequence).

Afterwards we prove that  $F(v) > 0$ ,  $v \in W$  and since  $H_1$  and  $H_2$  are weakly lower semi-continuous in  $H$  and  $W$  respectively, we get that  $\lambda(v) = \Lambda_1^+$ .

Indeed, since  $F$  is a continuous function and  $F(v_n) > 0$ ,  $F(v_n) \rightarrow_{n \rightarrow \infty} F(v)$ , we have  $F(v) \geq 0$ . Moreover, if  $F(v) = 0$ , then  $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} \rightarrow_{n \rightarrow \infty} \infty$ . This contradicts  $\lambda(v_n) \rightarrow_{n \rightarrow \infty} \Lambda_1^+$ . So  $F(v) > 0$  and  $v \neq 0$ .

Now we prove that  $v \in W$ . Recall that  $(v_n)$  is a bounded sequence in  $H$  and that  $(v_n)$  converges to  $v \neq 0$  strongly in  $(L^\gamma(\Omega))^m$ . So there exists a positive constant  $C'$

such that  $\|v_n\|_{(L^\gamma(\Omega))^m} \geq C' > 0$  for  $n$  large enough. Therefore, from the continuous embedding  $H \subset (L^\gamma(\Omega))^m$ , we get that  $\|v_n\|_H \geq C' > 0$  for  $n$  large enough. Finally from  $\|v_n\|_H \geq C' > 0$  and  $\|v_n\|_H^{2\alpha} \|v_n\|_W^{(1-\alpha)q} \leq C$  we obtain that  $(v_n)$  is a bounded sequence in  $W$ . Therefore  $(v_n)$  admits a subsequence, still denoted  $(v_n)$  such that  $(v_n)$  converges to  $v$  strongly in  $(L^\gamma(\Omega))^2$  and weakly in  $W$ . Thus  $v \in W$ .

Finally we prove that  $\lambda(v) = \Lambda_1^+$ .

From the weakly semi-continuousness of  $H_1$  and  $H_2$  respectively on  $H$  and  $W$  we have

$$H_1(v) \leq \liminf H_1(v_n) \text{ and } H_2(v) \leq \liminf H_2(v_n).$$

But  $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} = \frac{C_{q,\gamma}(H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha}}{F(v_n)} \rightarrow_{n \rightarrow \infty} \Lambda_1^+$ . Passing to the limit inf as  $n$  tends to  $\infty$  we get that  $\Lambda_1^+ \geq \frac{C_{q,\gamma}(H_1(v))^\alpha (H_2(v))^{1-\alpha}}{F(v)} = \lambda(v)$ . We deduce that

$$\lambda(v) = \Lambda_1^+.$$

This concludes the proof.  $\square$

Contrary to [7], we are not able to prove that the minimizer  $v$  is non-negative because of the coupling terms  $a_{ij}v_jv_i$  in  $H_1(v)$ . Finally combining Propositions 4.1 and 4.3, since  $v$  (defined by Proposition 4.3) is a critical point of  $\lambda(u)$ , we derive the existence of a non-trivial weak solution  $u^+ = (u_1^+, \dots, u_m^+)$  for the system (4.1) associated with  $\lambda_1^+$ . This is the following result

**Proposition 4.4.** *Assume that Assumptions 1.1 and 4.1 are satisfied.*

*There exists  $u^+ = (u_1^+, \dots, u_m^+) \in W$  a non-trivial solution for the system (4.1) associated with  $\lambda_1^+$ . Moreover  $I_{\lambda_1^+}(u^+) = 0$  and  $F(u^+) > 0$ .*

*Proof.* From Proposition 4.3 we have  $\lambda(v) = \Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\}$ . Thus  $v$  is a critical point of the function  $\lambda$  on  $W$ . From Proposition 4.1 we derive that there exists a non-trivial solution  $u^+ = (u_1^+, \dots, u_m^+)$  of system (4.1) associated with  $\frac{\gamma}{2^\alpha q^{1-\alpha}} \lambda(v) = \lambda_1^+$  where for all  $i = 1, \dots, m$ ,  $u_i^+ = \frac{1}{s} v_i$  and  $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}} t(v) > 0$ . Moreover from Proposition 4.1,  $I_{\lambda_1^+}(u^+) = 0$  and from Proposition 4.3,  $F(u^+) = \frac{1}{s^\gamma} F(v) > 0$ .  $\square$

## 4.2. Main result.

**Theorem 4.1.** *Assume that Assumptions 1.1 and 4.1 are satisfied. If  $\lambda > \lambda_1^+$ , then the system (4.1), associated with  $\lambda$ , admits a non-trivial solution in  $W$ .*

*Proof.* Even if we could follow [7] for proving this result using the mountain-pass theorem, we use here standard arguments by global minimization of the  $C^1$ -functional  $I_\lambda$ . Note that  $I_\lambda$  is weakly lower semi-continuous by the compact embedding of  $W$  into  $(L^q(\Omega))^m$  and  $(L^2(\Omega))^m$ . Moreover  $I_\lambda$  is coercive: indeed for any  $u \in W$ ,

$$I_\lambda(u) \geq \frac{1}{q} H_2(u) - \frac{\lambda}{\gamma} F(u).$$

Since  $|F(u)| \leq C \|u\|_W^\gamma$  with  $C$  a positive constant, we get that

$$I_\lambda(u) \geq \frac{1}{q} \|u\|_W^q \left( 1 - \frac{\lambda C q}{\gamma} \|u\|_W^{\gamma-q} \right).$$

Thus  $I_\lambda$  is coercive. Furthermore from Proposition 4.4, we have  $I_{\lambda_1^+}(u^+) = 0$  and  $F(u^+) > 0$ . Finally from the hypothesis  $\lambda > \lambda_1^+$ , we get that  $I_\lambda(u^+) < I_{\lambda_1^+}(u^+) = 0$ . Therefore we deduce that  $I_\lambda$  has a non-trivial critical point which is a non-trivial weak solution of the system (4.1) associated with  $\lambda$ .  $\square$

**Remarks:** We can get the same results for a larger class of coefficients, assuming that  $a_{ij}, w_i, f_i \in L^r(\Omega)$  for some  $r > 1$  as in [7]. But we have not been able to adapt this method for a system with a (p,q)-Laplacian operator (with  $p \neq 2$ ) and even for a non-symmetric system with a (2,q)-Laplacian operator. However in the particular case where the matrix  $A$  is not symmetric and has the following form:  $A = (a_{ij})$  with  $a_{j1} = Ka_{1j}$  for  $j = 2, \dots, m$  for some positive constant  $K > 0$  ( $K$  independent of  $j$ ) and  $a_{ij} = a_{ji}$  for  $i, j \geq 2$ , we can generalize all the above results. Indeed we introduce the diagonal matrix  $D = (d_{ij})$  with  $d_{11} = K$ ,  $d_{ii} = 1$  for  $i = 2, \dots, m$  and  $d_{ij} = 0$  if  $i \neq j$ . We replace the functionals  $H_1, H_2$  and  $F$  (defined before by (1.1),(1.2),(4.2)) by

$$H_2(u) = \sum_{i=1}^m d_{ii} \int_{\Omega} (|\nabla u_i|^q + w_i |u_i|^q), \quad F(u) = \sum_{i=1}^m d_{ii} \int_{\Omega} f_i |u_i|^\gamma,$$

$$H_1(u) = \sum_{i=1}^m d_{ii} \int_{\Omega} (|\nabla u_i|^2 + a_{ii} u_i^2 + \sum_{j=1, i \neq j}^m a_{ij} u_j u_i),$$

$$H_1(u) = \sum_{i=1}^m d_{ii} \int_{\Omega} |\nabla u_i|^2 + \int_{\Omega} {}^t U D A U \text{ with } {}^t U = (u_1, \dots, u_m).$$

Therefore if we assume that the matrix  $DA$  satisfies the following hypothesis  ${}^t \xi D A \xi \geq 0$  for all  ${}^t \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ , we still derive that the Euler functional  $I_\lambda$  defined by (4.3) (with the new functionals  $H_1, H_2$  and  $F$ ) is associated with the system (4.1) and the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for  $I_\lambda$ . Finally note that we obtain the existence of a non-trivial solution for (4.1) in Theorem 4.1 but contrary to [7], due to the coupling terms, we are not able to get the existence of a non-negative solution.

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