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ABSTRACT

The value of an American option is the maximized value of the discounted cash flows from the option. At each time step, one needs to compare the immediate exercise value with the continuation value and decide to exercise as soon as the exercise value is strictly greater than the continuation value. We can formulate this problem as a dynamic programming equation, where the main difficulty comes from the computation of the conditional expectations representing the continuation values at each time step. In (Longstaff and Schwartz, 2001), these conditional expectations were estimated using regressions on a finite-dimensional vector space (typically a polynomial basis). In this paper, we follow the same algorithm; only the conditional expectations are estimated using Regression trees or Random forests. We discuss the convergence of the LS algorithm when the standard least squares regression is replaced with regression trees. Finally, we expose some numerical results with regression trees and random forests. The random forest algorithm gives excellent results in high dimensions.

***K*eywords** Regression trees, Random forests, Bermudan options, Optimal stopping

1 Introduction

Bermudan options are very widespread in financial markets. Their valuation adds a challenge of optimal stopping determination in comparison to European options. Bermudan options offer the investor the possibility to exercise his option at any date of his choice among a certain number of dates prior to the option expiry, called exercise dates. Naturally, the option holder will have to find the most optimal date to exercise. To do so, at each exercise date, he will compare the payoff of the immediate exercise to the expected value of continuation of the option and decide to exercise only if the immediate exercise value is the highest. We can formulate this problem as a dynamic programming equation, where the main difficulty comes from the computation of the conditional expectation representing the expected continuation value of the option. Many papers have discussed this issue, starting with regression-based algorithms; see for example (Tsitsiklis and Van Roy, 1999) and (Carriere, 1996). Also, in this category falls the most commonly used method for pricing Bermudan options which is the Least Squares Method (LSM) presented by Longstaff and Schwarz in (Longstaff and Schwartz, 2001) where the conditional expectation is estimated by a least squares regression of the post realized payoffs from continuation on some basis functions of the state variables (usually polynomial

functions). Another class of algorithms focuses on quantization approaches, see for example (Bally et al., 2005). The algorithm consists in computing the conditional expectations by projecting the diffusion on some optimal grid. We also have a class of duality based methods that give an upper bound on the option value for a given exercise policy by adding a quantity that penalizes the incorrect exercise decisions made by the sub-optimal policy, see for example (Rogers, 2002), (Andersen and Broadie, 2004) and (Lelong, 2018). The last class of algorithms is based on machine learning techniques. For example, using Neural networks to estimate the continuation values in (Kohler et al., 2010) or more recently in (Lapeyre and Lelong, 2021), or using Gaussian process regression as in (Ludkovski, 2018). Our solution falls in this last category of algorithms. We examine Bermudan options' prices when the continuation values' estimation is done using regression trees or random forests.

Let X, Y be two random variables with vales in $[0,1]^d$ and $\mathbb{R}$ respectively. A regression tree approximates the conditional expectation $\mathbb{E}[Y/X]$ with a piecewise constant function. The tree is built recursively, generating a sequence of partitions of $[0,1]^d$ that are finer and finer. The approximation value on each set in the partition can be seen as a terminal leaf of the tree. This algorithm is very simple and efficient. However, it can easily over-fit the data, which results in high generalization errors. To solve this issue, we use ensemble methods to aggregate multiple trees, which means that we create multiple trees and then combine them to produce improved results. We suggest using random forests (see (Breiman, 2001)). This method consists in averaging a combination of trees where each tree depends on a random vector sampled independently and identically for each tree in the forest. This vector will allow to differentiate the trees in the random forest and can be chosen in different ways. For example, one can draw for each tree a sub-sample of training from the global training data without replacement (this method is called bagging and is thoroughly studied in (Breiman, 1999)). A second method is random split selection, where at each node, the split is selected at random from among the K best splits, see (Dietterich, 2000). Other methods for aggregating regression trees into random forests can be found in the literature, see for example (Breiman, 2001) or (Ho, 1998).
The structure of the paper will be as follows. First, we present the regression trees algorithm and the algorithm of least squares using regression trees. Then, we proceed to present some convergence results for regression trees and study the convergence of the LS algorithm when regression trees are used to estimate the continuation values. Then, we briefly talk about Random Forests before we finally study some numerical examples.

2 Regression trees

Let X be a r.v with values in $[0,1]^d$ and Y a real-valued r.v. We want to approach the conditional expectation $\mathbb{E}[Y/X]$. Throughout this paper, we will consider for computational convenience that X has a density f_X in $[0,1]^d$ w.r.t the Lebesgue measure. So, $\forall a \in [0,1]^d, \mathbb{P}(X = a) = 0$. We assume given a training sample $D_M = \{(X_1, Y_1), \dots, (X_M, Y_M) \in [0,1]^d \times \mathbb{R}\}$ where the (X_i, Y_i)'s are i.i.d random variables following the law of (X, Y). An approximation using a regression tree consists in writing the conditional expectation as a piecewise constant function of X. Each domain where the function is constant can be seen as a terminal leaf of a tree. Formally, let us first consider the one-dimensional case $(d = 1)$ and let

$$\tilde{f}(x) = \begin{cases} Y_R, \ \forall x > x^* \\ Y_L, \ \forall x \leq x^* \end{cases}$$

where x^*, Y_R and Y_L are chosen as follows: with probability $0 < 1 - q < 1$ the parameters are chosen to minimize $\frac{1}{M}\sum_{i=1}^{M}(\tilde{f}(X_i) - Y_i)^2$ and with probability q, the threshold x^* is the midpoint and we only minimize over Y_L and Y_R. We made the choice of taking the midpoint from time to time only for technical reasons. in fact, this choice simplifies some mathematical demonstrations. Either we take the midpoint or optimise over x^*, we can express the optimal Y_L and Y_R as a function of x^* as follows:

$$\begin{aligned} Y_R &= \frac{\sum_{i=1}^{M} Y_i \mathbb{1}_{\{X_i > x^*\}}}{\sum_{i=1}^{M} \mathbb{1}_{\{X_i > x^*\}}} \\ Y_L &= \frac{\sum_{i=1}^{M} Y_i \mathbb{1}_{\{X_i \leq x^*\}}}{\sum_{i=1}^{M} \mathbb{1}_{\{X_i \leq x^*\}}} \end{aligned} \tag{1}$$

As a matter of fact, Y_R, Y_L are solution to the problem

$$\inf_{y,y'} \sum_{i=1}^{M} \left(y_i - (y\mathbb{1}_{\{X_i > x^*\}} + y'\mathbb{1}_{\{X_i \leq x^*\}})\right)^2.$$

Once the threshold x^* is determined, we split the samples into two groups following the sign of $X_i - x^*$ and repeat the process for each group. We stop the process if introducing a new leaf does not improve the MSE or when enough iterations have been made. In the end, we have a tree that approximates the conditional expectation with a piecewise constant function. The regression trees are an algorithmic tool to find an adapted partition and the corresponding weights of this piecewise constant function.
In the multi-dimensional case, we choose the direction (the index along which the optimization is performed) uniformly for each new split. Then, the process is iterated as in the one-dimensional case. We denote the resulting tree by $\hat{\mathcal{T}}_p^M : [0,1]^d \to \mathbb{R}$ where p represents the depth of the tree, i.e., the number of iterations done in the process of optimization. A tree of depth p has 2^p leaves
When the size of the training data is infinite, Equation (1) writes

$$\begin{cases} Y_R &= \mathbb{E}\left[Y/X > x^*\right] \\ Y_L &= \mathbb{E}\left[Y/X \leq x^*\right] \end{cases}$$

and the optimisation problem writes $\inf_{x^*} \mathbb{E}\left[(\tilde{f}(X) - Y)^2\right]$. In this case we obtain the regression tree $\mathcal{T}_p(X)$.

3 LS algorithm with regression trees

3.1 Notation

For $p \in \mathbb{N}$, let $\left(\prod_{j=1}^d \left[a_p^{i-1}(j), a_p^i(j)\right)\right)_{1\leq i\leq p}$ be a partition of $[0,1]^d$ with p elements. We write

$$\left[a_p^{i-1}, a_p^i\right) := \prod_{j=1}^d \left[a_p^{i-1}(j), a_p^i(j)\right)$$

and

$$\alpha_p^i = \mathbb{E}\left[Y/X \in \left[a_p^{i-1}, a_p^i\right)\right].$$

The regression tree $\mathcal{T}_p(X)$ can be written as follows

$$\mathcal{T}_p(X) = \sum_{i=1}^{2^p} \alpha_p^i \mathbb{1}_{\{X \in \prod_{j=1}^d [a_p^i(j) - a_p^{i-1}(j))\}}$$

with $\left([a_p^{i-1}, a_p^i)\right)_{1\leq i\leq 2^p}$ forming a partition of $[0,1]^d$.

Remark 3.1. *In the following, when there is no confusion we will continue to simply write $\mathcal{T}_p(X)$ respectively $\hat{\mathcal{T}}_p^M(X)$, otherwise we write $\mathcal{T}_p(X, \theta^p)$, respectively $\hat{\mathcal{T}}_p^M(X, \hat{\theta}^{p,M})$ where $\theta^p = (a_p^0, \ldots, a_p^{2^p}) \in ([0,1]^d)^{2^p+1}$ and $\hat{\theta}^{p,M} = (a_p^{0,M}, \ldots, a_p^{2^p,M}) \in ([0,1]^d)^{2^p+1}$.*

3.2 Description of the algorithm

Let T be a fixed maturity, and consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $\mathbb{P}$ is the risk neutral measure. Consider a Bermudan option that can be exercised at dates $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$. When exercised at time t_j, the option's discounted payoff is given by $Z_{t_j} = h_j(X_{t_j})$ with $(X_{t_j})_j$ being an adapted Markov process taking values in $\mathbb{R}^d$. The discounted value $(U_j)_{0\leq j\leq N}$ of this option is given by

$$U_{t_j} = \sup_{\tau \in \mathcal{T}_{t_j}, T} \mathbb{E}\left[Z_\tau / \mathcal{F}_{t_j}\right]. \tag{2}$$

Using the Snell envelope theory, we can know that U solves the dynamic programming equation

$$\begin{cases} U_{t_N} &= Z_{t_N} \\ U_{t_j} &= \max\left(Z_{t_j}, \mathbb{E}\left[U_{t_{j+1}} / \mathcal{F}_{t_j}\right]\right) \text{ for } 1 \leq j \leq N-1. \end{cases} \tag{3}$$

This equation can be rewritten in terms of optimal policy as follows

$$\left\{ \begin{array}{ll} \tau_N & = t_N = T \\ \tau_j & = t_j \mathbb{1}_{\left\{Z_{t_j} \geq \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}} + \tau_{j+1}\mathbb{1}_{\left\{Z_{t_j} < \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}} \text{ for } 1 \leq j \leq N-1 \end{array} \right. \tag{4}$$

where τ_j is the smallest optimal stopping time after t_j. As we are in a Markovian setting, we can write $\mathbb{E}\left[Z\tau_{j+1}/\mathcal{F}_{t_j}\right] = \mathbb{E}\left[Z_{\tau_{j+1}}/X_{t_j}\right]$. The main difficulty in solving this equation comes from the computation of the continuation value $\mathbb{E}\left[Z_{\tau_{j+1}}/X_{t_j}\right]$. In the Least Squares approach presented by (Longstaff and Schwartz, 2001), this conditional expectation is estimated by a linear regression on a countable set of basis functions of X_{t_j}. In our approach, we suggest to estimate it using a regression Tree of depth p, $\mathcal{T}_p^j$. The algorithm solves for the following policy

$$\left\{ \begin{array}{ll} \tau_N^p & = t_N = T \\ \tau_j^p & = t_j \mathbb{1}_{\left\{Z_{t_j} \geq \mathcal{T}_p^j(X_{t_j})\right\}} + \tau_{j+1}\mathbb{1}_{\left\{Z_{t_j} < \mathcal{T}_p^j(X_{t_j})\right\}} \text{ for } 1 \leq j \leq N-1. \end{array} \right. \tag{5}$$

We sample M paths of the model $X_{t_0}^{(m)}, \ldots, X_{t_N}^{(m)}$ along with the corresponding payoff paths $Z_{t_0}^{(m)}, \ldots, Z_{t_N}^{(m)}$, $m = 1, \ldots, M$. For each path we compute the conditional expectations $\mathbb{E}[Z_{\tau_{j+1}}/X_{t_j}]$ for $j = 1, \ldots, N-1$ and we deduce the τ_j's. The final approximation of the optimal policy, in which the truncated expansion is computed using a Monte Carlo approximation is given by the following equation

$$\left\{ \begin{array}{ll} \hat{\tau}_N^{p,(m)} & = t_N = T \\ \hat{\tau}_j^{p,(m)} & = t_j \mathbb{1}_{\left\{Z_{t_j}^{(m)} \geq \hat{\mathcal{T}}_p^{j,M}(X_{t_j}^{(m)})\right\}} + \hat{\tau}_{j+1}^{(m)}\mathbb{1}_{\left\{Z_{t_j}^{(m)} < \hat{\mathcal{T}}_p^{j,M}(X_{t_j}^{(m)})\right\}} \text{ for } 1 \leq j \leq N-1. \end{array} \right. \tag{6}$$

Finally, the time-0 price of the option is approximated by

$$U_0^{p,M} = \max\left(Z_0, \frac{1}{M}\sum_{m=1}^{M} Z_{\hat{\tau}_1^{p,(m)}}^{(m)}\right). \tag{7}$$

4 Convergence of the algorithm

4.1 Notation

Note that the paths $\tau_1^{p,(m)}, \ldots, \tau_N^{p,(m)}$ for $m = 1, \ldots, M$ are identically distributed but not independent. In fact, the estimation of $\mathcal{T}_p^j(X_{t_j})$ uses all the paths. For each time step j, let $\theta_j^p = (a_{0,j}^p, \ldots a_{2^p,j}^p)$ be the coefficients of the tree $\mathcal{T}_p^j$ and $\hat{\theta}_j^{p,M} = (\hat{a}_{0,j}^{p,M}, \ldots \hat{a}_{2^p,j}^{p,M})$ the coefficients of the tree $\hat{\mathcal{T}}_p^{j,M}$. Following the notation of (Clément et al., 2002), we introduce the vector ϑ of the coefficients of the successive expansions $\vartheta^p = (\theta_0^p, \ldots, \theta_{N-1}^p)$ and its Monte Carlo counterpart $\hat{\vartheta}^{p,M} = (\hat{\theta}_0^{p,M}, \ldots, \hat{\theta}_{N-1}^{p,M})$.
Let $t^p = (t_0^p, \ldots, t_{N-1}^p) \in ([0,1]^d)^{2^p+1}$ be a deterministic parameter, $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ and $x = (x_1, \ldots, x_N) \in ([0,1]^d)^N$ be deterministic vectors. We define the vector field $F = F_1, \ldots, F_N$ by

$$\left\{ \begin{array}{ll} F_N(t^p, z, x) & = z_N \\ F_j(t^p, z, x) & = z_j \mathbb{1}_{\{z_j \geq \mathcal{T}_p(x_j, t_j^p)\}} + F_{j+1}(t^p, z, x)\mathbb{1}_{\{z_j < \mathcal{T}(x_j, t_j^p)\}}, \text{ for } 1 \leq j \leq N-1. \end{array} \right.$$

$F_j(t^p, z, x)$ only depends on $t_j^p, \ldots, t_{N-1}^p$ and not on the first $j-1$ components of t^p. Moreover,

$$F_j(\vartheta^p, Z, X) = Z_{\tau_j^p},$$
$$F_j(\hat{\vartheta}^{p,M}, Z^{(m)}, X^{(m)} = Z_{\hat{\tau}_j^{p,(m)}}^{(m)}.$$

Moreover, we clearly have that for all $t^p \in ([0,1]^d)^{2^p}$:

$$|F_j(t^p, Z, X)| \leq \max_{k \geq j} |Z_{t_k}|. \tag{8}$$

4.2 Convergence of the conditional expectations

4.2.1 Some preliminary results

We define

$$J_p\left(a_p^0,\dots,a_p^p\right) := \mathbb{E}\left[\left|\sum_{i=1}^{p}\alpha_p^i \mathbb{1}_{\{X\in\left[a_p^{i-1}-a_p^i\right)\}} - \mathbb{E}\left[Y/X\right]\right|^2\right].$$

Lemma 4.1. *Let X be a r.v with a density h_X w.r.t to the Lebesgue measure on $[0,1]^d$ and Y be a real valued square integrable random variable. Let $\left(\left(\left[a_p^{i-1},a_p^i\right)\right)_{1\leq i\leq p}\right)_{p\in\mathbb{N}}$ be a sequence of partitions of $[0,1]^d$ such that $\lim_{p\to\infty}\max_{1\leq i\leq p}\max_{1\leq j\leq d}\left|a_p^i(j)-a_p^{i-1}(j)\right|=0$. Then,*

$$\lim_{p\to\infty} J_p\left(a_p^0,\dots,a_p^p\right)=0.$$

Proof. Consider the function

$$\begin{aligned} f:[0,1]^d &\to \mathbb{R}\\ x &\mapsto \mathbb{E}\left[Y/X=x\right]\end{aligned}$$

and define a piecewise constant approximation of f

$$\begin{aligned} x\mapsto f^{(p)}(x) &= \sum_{i=1}^{p}\left(\frac{1}{\mu\left(\left[a_p^{i-1},a_p^i\right]\right)}\int_{\left[a_p^{i-1},a_p^i\right]} f(s)ds\right)\mathbb{1}_{\{x\in\left[a_p^{i-1},a_p^i\right)\}}\\ &= \sum_{i=1}^{p}\alpha_p^i\mathbb{1}_{\{x\in\left[a_p^{i-1},a_p^i\right)\}}\end{aligned}$$

with μ the Lebesgue measure. First, we consider that f is continuous on $[0,1]^d$. Then, it is uniformly continuous on the compact set $[0,1]^d$. So,

$$\forall\epsilon>0,\exists\eta>0 \text{ s.t } \forall x,y \text{ s.t } |x-y|<\eta, |f(x)-f(y)|<\epsilon$$

Let $\epsilon>0$ and $\eta>0$ satisfying the above condition. For a large enough p

$$\max_{1\leq i\leq p}\max_{1\leq j\leq d}\left|a_p^i(j)-a_p^{i-1}(j)\right|<\sqrt{\frac{\eta}{d}}.$$

So,

$$\forall 1\leq i\leq p,\ \left|a_p^{i-1}-a_p^i\right| = \sum_{j=1}^{d}\left|a_p^i(j)-a_p^{i-1}(j)\right|^2<\eta.$$

Then,

$$\begin{aligned}\mathbb{E}\left[\left|f(X)-f^{(p)}(X)\right|^2\right] &= \int_{[0,1]^d}\left|f(x)-f^{(p)}(x)\right|^2 h_X(x)dx\\ &= \sum_{i=1}^{p}\int_{[a_p^{i-1},a_p^i]}\left|f(x)-\frac{1}{\mu([a_p^{i-1},a_p^i])}\int_{[a_p^{i-1},a_p^i]}f(s)ds\right|^2 h_X(x)dx\\ &\leq \sum_{i=1}^{p}\int_{[a_p^{i-1},a_p^i]}\frac{1}{\mu([a_p^{i-1},a_p^i])}\int_{[a_p^{i-1},a_p^i]}|f(x)-f(s)|^2 ds h_X(x)dx\\ &\leq \sum_{i=1}^{p}\int_{[a_p^{i-1},a_p^i]}\epsilon^2 h_X(x)dx\leq\epsilon^2.\end{aligned}$$

Finally,

$$\lim_{p\to\infty}\int_{[0,1]^d}\left|f(x)-f^{(p)}(x)\right|^2 dx=0.$$

The set of continuous functions on $[0,1]^d$ being dense in $\mathbb{L}^2\left([0,1]^d\right)$, the result still holds without the continuity assumption, which ends the proof. □

Theorem 4.2.

$$\lim_{p\to\infty}\mathbb{E}\left[\left|\mathcal{T}_p(X)-\mathbb{E}[Y/X]\right|^2\right]=0.$$

Proof. For all $1\leq j\leq d$

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|\right]\\
&\leq\frac{1}{d}\left[q\frac{1}{2}\mathbb{E}\left[\max_{1\leq i\leq 2^{p-1}}\left|a_{p-1}^i(j)-a_{p-1}^{i-1}(j)\right|\right]+(1-q)\mathbb{E}\left[\max_{1\leq i\leq 2^{p-1}}\left|a_{p-1}^i(j)-a_{p-1}^{i-1}(j)\right|\right]\right]\\
&+\frac{d-1}{d}\mathbb{E}\left[\max_{1\leq i\leq 2^{p-1}}\left|a_{p-1}^i(j)-a_{p-1}^{i-1}(j)\right|\right]\\
&\leq(1-\frac{q}{2d})\mathbb{E}\left[\max_{1\leq i\leq 2^{p-1}}\left|a_{p-1}^i(j)-a_{p-1}^{i-1}(j)\right|\right]\\
&\leq(1-\frac{q}{2d})^{2^p}
\end{aligned}$$

In fact, with a probability $\frac{1}{d}$, the index j is chosen for optimisation. In the other $d-1$ cases, we do not even cut along that direction, in which case the interval length is at most equal to the length of the largest interval at step $p-1$. When the index j is chosen: with probability q, the length of the interval is cut in two, and with probability $1-q$, it is cut to optimize the MSE. In that case, the interval length is at most equal to the length of the largest interval at step $p-1$.

The series $\sum_{p=0}^{\infty}(1-\frac{q}{2d})^{2^p}$ is finite. Thus, so is $\sum_{p=0}^{\infty}\mathbb{E}\left[\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|\right]$. Since $\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|$ is non negative for all p, using Tonelli's theorem we conclude that $\mathbb{E}\left[\sum_{p=0}^{\infty}\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|\right]=0$. As a result, the series $\sum_{p=0}^{\infty}\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|$ converges a.s. Then, $\lim_{p\to\infty}\max_{1\leq i\leq 2^p}\left|a_p^i(j)-a_p^{i-1}(j)\right|=0$ a.s for all j and $\lim_{p\to\infty}\max_{1\leq i\leq 2^p}\max_{1\leq j\leq d}\left|a_p^i(j)-a_p^{i-1}(j)\right|=0$.

Let $\mathcal{G}$ be the σ-field generated by the splitting strategy (direction choice and threshold strategy).
Conditioning by $\mathcal{G}$ allows us to consider the partition $\left([a_p^{i-1}-a_p^i)\right)_{1\leq i\leq 2^p}$ deterministic and we can apply Lemma 4.1 to prove

$$\lim_{p\to\infty}\mathbb{E}\left[\left|\mathcal{T}_p-\mathbb{E}\left[Y/X\right]\right|^2/\mathcal{G}\right]=0\text{ a.s.}$$

Note that

$$\begin{aligned}
\mathbb{E}\left[\left|\mathcal{T}_p(X)\right|^2/\mathcal{G}\right]&\leq\mathbb{E}\left[\sum_{i=1}^{2^p}\mathbb{E}\left[Y^2/X\in[a_p^{i-1},a_p^i)\right]\mathbb{1}_{\{X\in[a_p^{i-1},a_p^i)\}}/\mathcal{G}\right]\\
&\leq\sum_{i=1}^{2^p}\mathbb{E}\left[\mathbb{E}\left[Y^2\mathbb{1}_{\{X\in[a_p^{i-1},a_p^i)\}}/X\in[a_p^{i-1},a_p^i)\right]/\mathcal{G}\right]\\
&\leq\sum_{i=1}^{2^p}\mathbb{E}\left[Y^2\mathbb{1}_{\{X\in[a_p^{i-1},a_p^i)\}}/\mathcal{G}\right]\\
&\leq\mathbb{E}\left[Y^2/\mathcal{G}\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\left[\left|\mathcal{T}_p-\mathbb{E}\left[Y/X\right]\right|^2/\mathcal{G}\right]&\leq 2\mathbb{E}\left[\left|\mathcal{T}_p(X)\right|^2/\mathcal{G}\right]+2\mathbb{E}\left[\mathbb{E}\left[Y/X\right]^2\right]\\
&\leq 2\left(\mathbb{E}\left[Y^2/\mathcal{G}\right]+\mathbb{E}\left[\mathbb{E}\left[Y/X\right]^2\right]\right)
\end{aligned}$$

Using Lebesgue's bounded convergence theorem,

$$\lim_{p\to\infty} \mathbb{E}\left[\left|\mathcal{T}_p - \mathbb{E}[Y/X]\right|^2\right] = \lim_{p\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left|\mathcal{T}_p - \mathbb{E}[Y/X]\right|^2/\mathcal{G}\right]\right] = 0. \qquad \square$$

4.2.2 Approximation of the conditional expectations with regression trees

Proposition 4.3.

$$\lim_{p\to\infty} \mathbb{E}\left[Z^p_{\tau_j}/\mathcal{F}_{T_j}\right] = \mathbb{E}\left[Z_{\tau_j}/\mathcal{F}_{T_j}\right] \ \textit{in}\ \mathbb{L}^2(\Omega) \text{ for } 1 \le j \le N \tag{9}$$

Proof. We proceed by induction.
For $j = N$, the proposition is true since $\tau^p_N = \tau_N = T$. Assume that the result holds for $j+1$, Let us prove that it still holds for j:

$$\begin{aligned}
\mathbb{E}\left[Z^p_{\tau_j} - Z_{\tau_j}/\mathcal{F}_{T_j}\right] &= Z_{t_j}\left(\mathbb{1}_{\left\{Z_{t_j} \ge \mathcal{T}^j_p(X_{t_j})\right\}} - \mathbb{1}_{\left\{Z_{t_j} \ge \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right) \\
&\quad + \mathbb{E}\left[Z_{\tau^p_{j+1}}\mathbb{1}_{\left\{Z_{t_j} < \mathcal{T}^j_p(X_{t_j})\right\}} - Z_{\tau_{j+1}}\mathbb{1}_{\left\{Z_{t_j} < \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right] \\
&= \left(Z_{t_j} - \mathbb{E}[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\right)\left(\mathbb{1}_{\left\{Z_{t_j} \ge \mathcal{T}^j_p(X_{t_j})\right\}} - \mathbb{1}_{\left\{Z_{t_j} \ge \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right) \\
&\quad + \mathbb{E}[Z^p_{\tau_{j+1}} - Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\mathbb{1}_{\left\{Z_{t_j} < \mathcal{T}^j_p(X_{t_j})\right\}} \\
&= A^p_j + \mathbb{E}[Z^p_{\tau_{j+1}} - Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\mathbb{1}_{\left\{Z_{t_j} < \mathcal{T}^j_p(X_{t_j})\right\}}
\end{aligned}$$

Where A^p_j is defined by

$$A^p_j = \left(Z_{t_j} - \mathbb{E}[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\right)\left(\mathbb{1}_{\left\{Z_{j_k} \ge \mathcal{T}^j_p(X_{t_j})\right\}} - \mathbb{1}_{\left\{Z_{t_j} \ge \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right)$$

On one hand, since the conditional expectation is an orthogonal projection, we have

$$\mathbb{E}\left[\left|\mathbb{E}\left[Z_{\tau^p_{j+1}} - Z_{\tau_{j+1}}/F_{t_j}\right]\right|^2\right] \le \mathbb{E}\left[\left|\mathbb{E}\left[Z_{\tau^p_{j+1}} - Z_{\tau_{j+1}}/F_{t_{j+1}}\right]\right|^2\right]$$

and using the induction assumption $\mathbb{E}\left[Z^p_{\tau_{j+1}} - Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right] \to 0$ in $\mathbb{L}^2(\Omega)$ when $p \to \infty$. On the other hand,

$$\begin{aligned}
\left|A^p_j\right| &= \left|Z_{t_j} - \mathbb{E}[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\right|\left|\mathbb{1}_{\left\{Z_{t_j} \ge \mathcal{T}^j_p(X_{t_j})\right\}} - \mathbb{1}_{\left\{Z_{t_j} \ge \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right| \\
&\le \left|Z_{t_j} - \mathbb{E}[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\right|\left|\mathbb{1}_{\left\{\mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right] > Z_{t_j} \ge \mathcal{T}^j_p(X_{t_j})\right\}} - \mathbb{1}_{\left\{\mathcal{T}^j_p(X_{t_j}) > Z_{t_j} \ge \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right\}}\right| \\
&\le \left|Z_{t_j} - \mathbb{E}[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}]\right|\left|\mathbb{1}_{\left\{|Z_{t_j} - \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]| \le |\mathcal{T}^j_p(X_{t_j}) - \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]|\right\}}\right| \\
&\le \left|\mathcal{T}^j_p(X_{t_j}) - \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right| \\
&\le \left|\mathcal{T}^j_p(X_{t_j}) - \mathbb{E}\left[Z_{\tau^p_{j+1}}/\mathcal{F}_{t_j}\right]\right| + \left|\mathbb{E}\left[Z_{\tau^p_{j+1}}/\mathcal{F}_{t_j}\right] - \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right|
\end{aligned}$$

Using the induction assumption, the second term goes to zero in $\mathbb{L}^2(\Omega)$ when $p \to \infty$. Let $\left([a_{i-1}(p), a_i(p))\right)_{1 \le i \le 2^p}$ be the partition generated by $\mathcal{T}^j_p$. We define

$$\bar{\mathcal{T}}^j_p(X_{t_j}) = \sum_{i=1}^{2^p} \mathbb{E}\left[Z_{\tau_{j+1}}/X_{t_j} \in [a_{i-1}(p), a_i(p))\right]\mathbb{1}_{\{X_{t_j} \in [a_{i-1}(p), a_i(p))\}}$$

Note that $\bar{\mathcal{T}}_p^j$ uses the partition given by $\mathcal{T}_p^j$ but the coefficients $\alpha_i(p)$ are given by the conditional expectations of $Z_{\tau_{j+1}}$ w.r.t X_{t_j} and not those of $Z_{\tau_{j+1}^p}$. Clearly,

$$\mathbb{E}\left[\left|\mathcal{T}_p^j(X_{t_j}) - \mathbb{E}\left[Z_{\tau_{j+1}^p}/\mathcal{F}_{t_j}\right]\right|^2\right] \leq \mathbb{E}\left[\left|\bar{\mathcal{T}}_p^j(X_{t_j}) - \mathbb{E}\left[Z_{\tau_{j+1}^p}/\mathcal{F}_{t_j}\right]\right|^2\right]$$
$$\leq 2\mathbb{E}\left[\left|\bar{\mathcal{T}}_p^j(X_{t_j}) - \mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right]\right|^2\right] + 2\mathbb{E}\left[\left|\mathbb{E}\left[Z_{\tau_{j+1}}/\mathcal{F}_{t_j}\right] - \mathbb{E}\left[Z_{\tau_{j+1}^p}/\mathcal{F}_{t_j}\right]\right|^2\right]$$

The second term goes to 0 using the induction assumption. As for the first term, note that the partition obtained with $\mathcal{T}_p^j$ verifies the conditions of Lemma 4.1. Then, using the same arguments as in the proof of Theorem 4.2, we can show that the first term also goes to 0. □

4.3 convergence of the Monte Carlo approximation

For this section, the depth p of the trees is fixed. We study the convergence with respect to the number of samples M.

4.3.1 Convergence of optimisation problems

We recall here two important results on the convergence of a sequence of optimization problems. Consider a sequence of real valued functions $(f_n)_n$ defined on a compact set $K \subset \mathbb{R}^d$. Define,

$$v_n = \inf_{x\in K} f_n(x)$$

and let x_n be a sequence of minimizers

$$f_n(x_n) = \inf_{x\in K} f_n(x).$$

From (Leake et al., 1994, chap. 2), we have the following result:

Lemma 4.4. *Assume that the sequence $(f_n)_n$ converges uniformly on K to a continuous function f. Let $v^* = \inf_{x\in K} f(x)$ and $\mathcal{S}^* = \{x \in K : f(x) = v^*\}$. Then $v_n \to v^*$ and $d(x_n, \mathcal{S}^*) \to 0$ a.s*

We will also use the following result which is a statement of the law of large numbers in Banach spaces. See (Leake et al., 1994, lemma. A1) or (Ledoux and Talagrand, 1991, Corollary 7.10, page 189)

Lemma 4.5. *Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d $\mathbb{R}^n$-valued random vectors and $h : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Assume that*

- *a.s, $\theta \in \mathbb{R}^d \mapsto h(\theta, \xi_1)$ is continuous,*
- *$\forall C > 0, \mathbb{E}\left[\sup_{|\theta|\leq C}|h(\theta, \xi_1)|\right] < \infty$.*

Then, a.s $\theta \in \mathbb{R}^d \mapsto \frac{1}{n}\sum_{i=1}^n h(\theta, \xi_i)$ converges locally uniformly to the continuous function $\theta \in \mathbb{R}^d \mapsto \mathbb{E}[h(\theta, \xi_1)]$, i.e

$$\lim_{n\to\infty} \sup_{|\theta|\leq C} \left|\frac{1}{n}\sum_{i=1}^n h(\theta, \xi_i) - \mathbb{E}[h(\theta, \xi_1)]\right| = 0 \text{ a.s.}$$

4.3.2 Strong law of large numbers

From (Clément et al., 2002), we have the following result

Lemma 4.6. *For every $j = 1, \ldots, N-1$,*

$$|F_j(a, Z, X) - F_j(b, Z, X)| \leq \left(\sum_{i=j}^N |Z_{t_i}|\right)\left(\sum_{i=j}^{N-1} \mathbb{1}_{\{|Z_{t_i} - \mathcal{T}_p^i(X_{t_i}, b_i)| \leq |\mathcal{T}_p^i(X_{t_i}, a_i) - \mathcal{T}_p^i(X_{t_i}, b_i)|\}}\right).$$

Proposition 4.7. *Assume that for all $p \in \mathbb{N}^*$, and all $1 \leq j \leq N-1$, $\mathbb{P}(Z_{t_j} = \mathcal{T}_p^j(X_{t_j}, \theta_j^p)) = 0$. Then, for all $j = 1, \ldots, N-1$, $\hat{\mathcal{T}}_p^{j,M}(X_{t_j}, \hat{\theta}_j^{p,M})$ converges to $\mathcal{T}_p^j(X_{t_j}, \theta_j^p)$ a.s as $M \to \infty$.*

Proof. We proceed by backward induction on j and forward induction on p .

- Step 1: $j = N-1$
  - For $p = 1$, let

$$h : \mathbb{R} \times \mathbb{R} \times [0,1]^d \times \mathbb{R} \to \mathbb{R}$$
$$\alpha, \beta, a, x, z \mapsto \left| z - \alpha \mathbb{1}_{\{x \in [0,a)\}} - \beta \mathbb{1}_{\{x \in [a,1]\}} \right|^2$$

We recall that we use the notation of Section 3.1, meaning here that 0 and 1 are d-dimensional. The random function $\alpha, \beta, a \mapsto h(\alpha, \beta, a, X_{t_{N-1}}, Z_{t_N})$ is a.s continuous on $\mathbb{R} \times \mathbb{R} \times [0,1]^d$ (since $X_{t_{N-1}}$ has a density, $\mathbb{P}(X_{t_{N-1}} = a) = 0$). Let $C > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{a \in [0,1]^d, |\alpha| < C, |\beta| < C} \left| h(\alpha, \beta, a, X_{T_{N-1}}, Z_{t_N}) \right| \right] \\
&= \mathbb{E}\left[\sup_{a \in [0,1]^d, |\alpha| < C, |\beta| < C} \left| Z_{t_N} - \alpha \mathbb{1}_{\{X_{t_{N-1}} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}} \in [a,1]\}} \right|^2 \right] \\
&\leq 2\mathbb{E}\left[Z_{t_N}^2 \right] + 2\mathbb{E}\left[\sup_{a \in [0,1]^d, |\alpha| < C, |\beta| < C} \left| \alpha \mathbb{1}_{\{X_{t_{N-1}} \in [0,a)\}} + \beta \mathbb{1}_{\{X_{t_{N-1}} \in [a,1]\}} \right|^2 \right] \\
&\leq 2\mathbb{E}\left[Z_{t_N}^2 \right] + 2C^2 \\
&< \infty \text{ a.s.}
\end{aligned}$$

Using Lemma 4.5, The random function $\alpha, \beta, a \mapsto \frac{1}{M} \sum_{m=1}^{M} \left| Z_{t_N}^{(m)} - \alpha \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a,1]\}} \right|^2$ converges uniformly to the function $\alpha, \beta, a \mapsto \mathbb{E}\left[\left| Z_{t_N} - \alpha \mathbb{1}_{\{X_{t_{N-1}} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}} \in [a,1]\}} \right|^2 \right]$. Since $\hat{\mathcal{T}}_1^{N-1,M}(X_{t_{N-1}}, \hat{\theta}_{N-1}^{1,M}) = \inf_{\alpha,\beta,a} \frac{1}{M} \sum_{m=1}^{M} \left| Z_{t_N}^{(m)} - \alpha \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a,1]\}} \right|^2$ and $\mathcal{T}_1^{N-1}(X_{t_{N-1}}, \theta_{N-1}^{1,M}) = \inf_{\alpha,\beta,a} \mathbb{E}\left[\left| Z_{t_N} - \alpha \mathbb{1}_{\{X_{t_{N-1}} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}} \in [a,1]\}} \right|^2 \right]$, we conclude using Lemma 4.4 that $\hat{\mathcal{T}}_1^{N-1,M}(X_{t_{N-1}}, \hat{\theta}_{N-1}^{1,M})$ converges to $\mathcal{T}_1^{N-1}(X_{t_{N-1}}, \theta_{N-1}^{1})$ a.s as $M \to \infty$.

  - Suppose that the result holds for p and we will prove it for $p+1$.

We write $\hat{\alpha}_p^M = \left(\hat{\alpha}_p^{0,M}, \ldots, \hat{\alpha}_p^{2^p,M}\right) \in \mathbb{R}^{2^p}$, $\hat{a}_p^M = \left(\hat{a}_p^{0,M}, \ldots, \hat{a}_p^{2^p,M}\right) \in ([0,1]^d)^{2^p+1}$, $\alpha_p = \left(\alpha_p^0, \ldots, \alpha_p^{2^p}\right) \in \mathbb{R}^{2^p}$ and $a_p = \left(a_p^0, \ldots, a_p^{2^p}\right) \in ([0,1]^d)^{2^p+1}$. Let $i \in \{1, \ldots, 2^p\}$ and consider

$$\hat{\nu}_{p,N-1}^M(\alpha, \beta, a) = \frac{1}{M} \sum_{m=1}^{M} \left| Z_{t_N}^{(m)} - \alpha \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a_p^{i-1,M}, a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a, a_p^{i,M})\}} \right|^2.$$
$$\nu_{p,N-1}^M(\alpha, \beta, a) = \frac{1}{M} \sum_{m=1}^{M} \left| Z_{t_N}^{(m)} - \alpha \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a_p^{i-1}, a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}}^{(m)} \in [a, a_p^{i})\}} \right|^2.$$

Using the same arguments as in the case $p = 1$, it is easy to see that the random function $\alpha, \beta, a \mapsto \nu_{p,N-1}^M(\alpha, \beta, a)$ converges a.s uniformly to the function $\alpha, \beta, a \mapsto \mathbb{E}\left[\left| Z_{t_N} - \alpha \mathbb{1}_{\{X_{t_{N-1}} \in [a_p^{i-1}, a)\}} - \beta \mathbb{1}_{\{X_{t_{N-1}} \in [a, a_p^{i})\}} \right|^2 \right]$.

Now, it suffices to show that $\sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left| \hat{\nu}_{p,N-1}^M(\alpha, \beta, a) - \nu_{p,N-1}^M(\alpha, \beta, a) \right| \to 0$ a.s

when $M \to \infty$

$$\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C} \left|\hat{\nu}^M_{p,N-1}(\alpha,\beta,a) - \nu^M_{p,N-1}(\alpha,\beta,a)\right|$$

$$\leq \sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C} \frac{1}{M}\sum_{m=1}^{M}\left|\alpha \mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[a_p^{i-1},a_p^{i-1,M}]\}} + \beta \mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[a_p^{i,M},a_p^{i}]\}}\right|$$

$$\left|2Z^{(m)}_{t_N} - \alpha\left(\mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[\min(a_p^{i-1,M},a_p^{i-1}),\max(a_p^{i-1,M},a_p^{i-1}))\}} + 2\mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[\max(a_p^{i-1,M},a_p^{i-1}),a]\}}\right)\right.$$

$$\left. - \beta\left(\mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[\min(a_p^{i,M},a_p^{i}),\max(a_p^{i,M},a_p^{i})]\}} + 2\mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[a,\min(a_p^{i,M},a_p^{i}))\}}\right)\right|$$

$$\leq \sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C} \frac{1}{M}\sum_{m=1}^{M}\left[\left|2Z^{(m)}_{t_N}\right| + 6C\right] C\left(\mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[a_p^{i-1},a_p^{i-1,M}]\}} + \mathbb{1}_{\{X^{(m)}_{t_{N-1}}\in[a_p^{i,M},a_p^{i}]\}}\right).$$

Let $\epsilon > 0$, using the induction assumption on p

$$\limsup_M \sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C} \left|\hat{\nu}^M_{p,N-1}(\alpha,\beta,a) - \nu^M_{p,N-1}(\alpha,\beta,a)\right|$$

$$\leq \limsup_M \sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C} \frac{1}{M}\sum_{m=1}^{M}\left[\left|2Z^{(m)}_{t_N}\right| + 6C\right] C\left(\left|\mathbb{1}_{\{\left|X^{(m)}_{t_{N-1}} - a_p^i\right|\leq\epsilon\}}\right| + \left|\mathbb{1}_{\{\left|X^{(m)}_{t_{N-1}} - a_p^{i+1}\right|\leq\epsilon\}}\right|\right)$$

$$\leq C(6C + 2\mathbb{E}\left[|2Z_{t_N}|\right])\left(\mathbb{P}(\left|X_{t_{N-1}} - a_p^i\right| \leq \epsilon) + \mathbb{P}(\left|X_{t_{N-1}} - a_p^{i+1}\right|) \leq \epsilon\right).$$

Since $\lim_{\epsilon\to 0}\mathbb{P}(\left|X_{t_{N-1}} - a_p^i\right| \leq \epsilon) = \mathbb{P}(X_{t_{N-1}} = a_p^i) = 0$ and $\lim_{\epsilon\to 0}\mathbb{P}(\left|X_{t_{N-1}} - a_p^{i+1}\right| \leq \epsilon) = \mathbb{P}(X_{t_{N-1}} = a_p^{i+1}) = 0$. As a result, $\left|\hat{\nu}^M_{p,N-1}(\alpha,\beta,a) - \nu^M_{p,N-1}(\alpha,\beta,a)\right| \to 0$ uniformly when $M \to \infty$. Thus, the random function $\alpha,\beta,a \mapsto \hat{\nu}^M_{p,N-1}(\alpha,\beta,a)$ converges uniformly to the function $\alpha,\beta,a \mapsto \mathbb{E}\left[\left|Z_{t_N} - \alpha\mathbb{1}_{\{X_{t_{N-1}}\in[a_p^{i-1},a)\}} - \beta\mathbb{1}_{\{X_{t_{N-1}}\in[a,a_p^i)\}}\right|^2\right]$ and using the same arguments as in the step $p = 1$, we conclude that $\hat{\mathcal{T}}^{N-1}_{p+1}(X_{t_{N-1}},\hat{\theta}^{p,M}_{N-1})$ converges to $\mathcal{T}^{N-1}_{p+1}(X_{t_{N-1}},\theta^p_{N-1})$ a.s as $M \to \infty$.

- So far, we have proved that for all p, $\hat{\mathcal{T}}^{N-1}_p(X_{t_{N-1}},\hat{\theta}^{p,M}_{N-1})$ converges to $\mathcal{T}^{N-1}_p(X_{t_{N-1}},\theta^p_j)$ a.s as $M \to \infty$. Now, Suppose that $\hat{\mathcal{T}}^k_p(X_{t_k},\hat{\theta}^{p,M}_k)$ converges to $\mathcal{T}^k_p(X_{t_k},\theta^p_k)$ a.s as $M \to \infty$ for all p and for $k = N-1,\dots,j+1$. We should prove that the result still holds for j
  - For $p = 1$, consider

$$\hat{\nu}^M_{1,j}(\alpha,\beta,a) = \frac{1}{M}\sum_{m=1}^{M}\left|F_{j+1}\left(\hat{\vartheta}^{1,M},Z^{(m)},X^{(m)}\right) - \alpha\mathbb{1}_{\{X^{(m)}_{t_j}\in[0,a)\}} - \beta\mathbb{1}_{\{X^{(m)}_{t_j}\in[a,1]\}}\right|^2$$

$$\nu^M_{1,j}(\alpha,\beta,a) = \frac{1}{M}\sum_{m=1}^{M}\left|F_{j+1}\left(\vartheta^{1},Z^{(m)},X^{(m)}\right) - \alpha\mathbb{1}_{\{X^{(m)}_{t_j}\in[0,a)\}} - \beta\mathbb{1}_{\{X^{(m)}_{t_j}\in[a,1]\}}\right|^2.$$

The function $\nu^M_{1,j}$ writes as the sum of i.i.d random variables. Let $C \geq 0$, using Equation (8)

$$\mathbb{E}\left[\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\left|F_{j+1}\left(\vartheta^1,Z,X\right) - \alpha\mathbb{1}_{\{X_{t_j}\in[0,a)\}} - \beta\mathbb{1}_{\{X_{t_j}\in[a,1]\}}\right|^2\right]$$

$$\leq 2\mathbb{E}\left[\left|F_{j+1}\left(\vartheta^1,Z,X\right)\right|^2\right] + 2\mathbb{E}\left[\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\left|\alpha\mathbb{1}_{\{X_{t_j}\in[0,a)\}} + \beta\mathbb{1}_{\{X_{t_j}\in[a,1]\}}\right|^2\right]$$

$$\leq 2\mathbb{E}\left[\max_{l\geq j+1}(Z_{t_l})^2\right] + 2C^2$$

$$< \infty.$$

Using Lemma 4.5, $\alpha, \beta, a \mapsto \nu_{1,j}^M(\alpha, \beta, a)$ converges a.s uniformly to the function $\alpha, \beta, a \mapsto \mathbb{E}\left[\left|F_{j+1}(\vartheta^1, Z, X) - \alpha \mathbb{1}_{\{X_{t_j} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_j} \in [a,1]\}}\right|^2\right]$.

It remains to prove that $\forall C > 0 \quad \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left|\hat{\nu}_{1,j}^M(a, \alpha, \beta) - \nu_{1,j}^M(a, \alpha, \beta)\right| \to 0$ a.s when $M \to \infty$.

Then, using Equation (8) and Lemma 4.6

$$
\begin{aligned}
&\sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left|\hat{\nu}_{1,j}^M(a, \alpha, \beta) - \nu_{1,j}^M(a, \alpha, \beta)\right| \\
&\leq \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \frac{1}{M} \sum_{m=1}^{M} \left|F_{j+1}\left(\hat{\vartheta}^{1,M}, Z^{(m)}, X^{(m)}\right) - F_{j+1}\left(\vartheta^1, Z^{(m)}, X^{(m)}\right)\right| \\
&\left|F_{j+1}\left(\hat{\vartheta}^{1,M}, Z^{(m)}, X^{(m)}\right) + F_{j+1}\left(\vartheta^1, Z^{(m)}, X^{(m)}\right) - 2\alpha \mathbb{1}_{\{X_{t_j}^{(m)} \in [0,a)\}} - 2\beta \mathbb{1}_{\{X_{t_j}^{(m)} \in [a,1]\}}\right| \\
&\leq \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \frac{1}{M} \sum_{m=1}^{M} 2\left(\max_{l \geq j+1} \left|Z_{t_l}^{(m)}\right| + 2C\right) \left|F_{j+1}\left(\hat{\vartheta}^{1,M}, Z^{(m)}, X^{(m)}\right) - F_{j+1}\left(\vartheta^1, Z^{(m)}, X^{(m)}\right)\right| \\
&\leq \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \frac{1}{M} \sum_{m=1}^{M} 2\left(\max_{l \geq j+1} \left|Z_{t_l}^{(m)}\right| + 2C\right) \left(\sum_{i=j+1}^{N} \left|Z_{t_i}^{m)}\right| \sum_{i=j+1}^{N-1} \mathbb{1}_{\{\left|Z_{t_i}^{(m)} - \mathcal{T}_1^i(X_{t_i}^{(m)})\right| \leq \left|\hat{\mathcal{T}}_1^i(X_{t_i}^{(m)}) - \mathcal{T}_1^i(X_{t_i}^{(m)})\right|\}}\right)
\end{aligned}
$$

Let $\epsilon > 0$, using the induction assumption on j,

$$
\begin{aligned}
&\limsup_M \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left|\hat{\nu}_{1,j}^M(a, \alpha, \beta) - \nu_{1,j}^M(a, \alpha, \beta)\right| \\
&\leq \limsup_M \sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \frac{1}{M} \sum_{m=1}^{M} 2\left(\max_{l \geq j+1} \left|Z_{t_l}^{(m)}\right| + 2C\right) \left(\sum_{i=j+1}^{N} \left|Z_{t_i}^{(m)}\right| \sum_{i=j+1}^{N-1} \mathbb{1}_{\{\left|Z_{t_i}^{(m)} - \mathcal{T}_1^i(X_{t_i}^{(m)})\right| \leq \epsilon\}}\right).
\end{aligned}
$$

Since $\mathbb{P}(Z_{t_j}^{(m)} = \hat{\mathcal{T}}_p^j(X_{t_j}^{(m)}, \hat{\theta}_j^p)) = 0$, then $\lim_{\epsilon \to 0} \mathbb{1}_{\{\left|Z_{t_i}^{(m)} - \mathcal{T}_1^i(X_{t_i}^{(m)})\right| \leq \epsilon\}} = 0$ a.s and we conclude that a.s. $\left|\hat{\nu}_{1,j}^M(a, \alpha, \beta) - \nu_{1,j}^M(a, \alpha, \beta)\right|$ converges to zero uniformly. Thus, $a, \alpha, \beta \mapsto \hat{\nu}_{1,j}^M(a, \alpha, \beta)$ converges a.s uniformly to the function $a, \alpha, \beta \mapsto \mathbb{E}\left[\left|F_{j+1}(\vartheta^1, Z, X) - \alpha \mathbb{1}_{\{X_{t_j} \in [0,a)\}} - \beta \mathbb{1}_{\{X_{t_j} \in [a,1]\}}\right|^2\right]$

– We suppose the result is true for p, and let us verify that it still holds for $p+1$. We write $\hat{\alpha}_p^M = \left(\hat{\alpha}_p^{0,M}, \ldots, \hat{\alpha}_p^{2^p,M}\right) \in \mathbb{R}^{2^p}$, $\hat{a}_p^M = \left(\hat{a}_p^{0,M}, \ldots, \hat{a}_p^{2^p,M}\right) \in ([0,1]^d)^{2^p+1}$, $\alpha_p = \left(\alpha_p^0, \ldots, \alpha_p^{2^p}\right) \in \mathbb{R}^{2^p}$ and $a_p = \left(a_p^0, \ldots, a_p^{2^p}\right) \in ([0,1]^d)^{2^p+1}$. Let $i \in \{1, \ldots, 2^p\}$ and consider

$$
\begin{aligned}
\hat{\nu}_{p,j}^M(\alpha, \beta, a) &= \frac{1}{M} \sum_{m=1}^{M} \left|F_{j+1}(\hat{\vartheta}^{p,M}, Z^{(m)}, X^{(m)}) - \alpha \mathbb{1}_{\{X_{t_j}^{(m)} \in [a_p^{i-1,M}, a)\}} - \beta \mathbb{1}_{\{X_{t_j}^{(m)} \in [a, a_p^{i,M})\}}\right|^2. \\
\nu_{p,j}^M(\alpha, \beta, a) &= \frac{1}{M} \sum_{m=1}^{M} \left|F_{j+1}(\vartheta^p, Z^{(m)}, X^{(m)}) - \alpha \mathbb{1}_{\{X_{t_j}^{(m)} \in [a_p^{i-1}, a)\}} - \beta \mathbb{1}_{\{X_{t_j}^{(m)} \in [a, a_p^i)\}}\right|^2
\end{aligned}
$$

The function $\nu_{p,j}^M$ writes as the sum of i.i.d random variables. Let $C \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left|F_{j+1}\left(\vartheta^p, Z^{(m)}, X^{(m)}\right) - \alpha \mathbb{1}_{\{X_{t_j}^{(m)} \in [a_p^{i-1}, a)\}} - \beta \mathbb{1}_{\{X_{t_j}^{(m)} \in [a, a_p^i)\}}\right|^2\right] \\
&\leq 2\mathbb{E}\left[\left|F_{j+1}\left(\vartheta^p, Z^{(m)}, X^{(m)}\right)\right|^2\right] + 2\mathbb{E}\left[\sup_{a \in [0,1]^d, |\alpha| \leq C, |\beta| \leq C} \left|\alpha \mathbb{1}_{\{X_{t_j}^{(m)} \in [a_p^{i-1}, a)\}} + \beta \mathbb{1}_{\{X_{t_j}^{(m)} \in [a, a_p^i)\}}\right|^2\right] \\
&\leq 2\mathbb{E}\left[\max_{l \geq j+1} (Z_{t_l})^2\right] + 2C^2 \\
&< \infty.
\end{aligned}
$$

We conclude that a.s $\alpha,\beta,a \mapsto \nu^M_{p,j}(\alpha,\beta,a)$ converges uniformly to the function $\alpha,\beta,a \mapsto \mathbb{E}\left[\left|F_{j+1}(\vartheta^p,Z,X)-\alpha\mathbb{1}_{\{X_{t_j}\in[a_p^{i-1},a)\}}-\beta\mathbb{1}_{\{X_{t_j}\in(a,a_p^i)\}}\right|^2\right]$. Let $C>0$

$$
\begin{aligned}
&\left|\hat{\nu}^M_{p,j}(\alpha,\beta,a)-\nu^M_{p,j}(\alpha,\beta,a)\right| \\
&\leq \frac{1}{M}\sum_{m=1}^{M}\left|F_{j+1}(\hat{\vartheta}^{p,M},Z^{(m)},X^{(m)})-F_{j+1}(\vartheta^p,Z^{(m)},X^{(m)})+\alpha\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i-1},a_p^{i-1,M})\}}+\beta\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i,M},a_p^i]\}}\right| \\
&\Bigg|F_{j+1}(\hat{\vartheta}^{p,M},Z^{(m)},X^{(m)})+F_{j+1}(\vartheta^p,Z^{(m)},X^{(m)}) \\
&-\alpha\left(\mathbb{1}_{\{X^{(m)}_{t_j}\in[\min(a_p^{i-1,M},a_p^{i-1}),\max(a_p^{i-1,M},a_p^{i-1}))\}}+2\mathbb{1}_{\{X^{(m)}_{t_j}\in[\max(a_p^{i-1,M},a_p^{i-1}),a]\}}\right) \\
&-\beta\left(\mathbb{1}_{\{X^{(m)}_{t_j}\in[\min(a_p^{i,M},a_p^{i}),\max(a_p^{i,M},a_p^{i})]\}}+2\mathbb{1}_{\{X^{(m)}_{t_j}\in[a,\min(a_p^{i,M},a_p^{i}))\}}\right)\Bigg| \\
&\leq \frac{1}{M}\sum_{m=1}^{M}\left[\left|F_{j+1}(\hat{\vartheta}^{p,M},Z^{(m)},X^{(m)})-F_{j+1}(\vartheta^p,Z^{(m)},X^{(m)})\right|+\left|\alpha\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i-1},a_p^{i-1,M})\}}+\beta\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i,M},a_p^{i}]\}}\right|\right] \\
&\Bigg[\left|F_{j+1}(\hat{\vartheta}^{p,M},Z^{(m)},X^{(m)})+F_{j+1}(\vartheta^p,Z^{(m)},X^{(m)})\right| \\
&+\left|\alpha\left(\mathbb{1}_{\{X^{(m)}_{t_j}\in[\min(a_p^{i-1,M},a_p^{i-1}),\max(a_p^{i-1,M},a_p^{i-1}))\}}+2\mathbb{1}_{\{X^{(m)}_{t_j}\in[\max(a_p^{i-1,M},a_p^{i-1}),a]\}}\right)\right. \\
&\left.+\beta\left(\mathbb{1}_{\{X^{(m)}_{t_j}\in[\min(a_p^{i,M},a_p^{i}),\max(a_p^{i,M},a_p^{i})]\}}+2\mathbb{1}_{\{X^{(m)}_{t_j}\in[a,\min(a_p^{i,M},a_p^{i}))\}}\right)\right|\Bigg]
\end{aligned}
$$

Let $C>0$,

$$
\begin{aligned}
&\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\left|\hat{\nu}^M_{p,j}(\alpha,\beta,a)-\nu^M_{p,j}(\alpha,\beta,a)\right| \\
&\leq \sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\frac{1}{M}\sum_{m=1}^{M}2\left(\max_{l\geq j+1}\left|Z^{(m)}_{t_l}\right|+3C\right) \\
&\left[\left(\sum_{i=j+1}^{N}\left|Z^{(m)}_{t_i}\right|\sum_{i=j+1}^{N-1}\mathbb{1}_{\{\left|Z_{t_i}-\mathcal{T}^i_p(X^{(m)}_{t_i})\right|\leq\left|\hat{\mathcal{T}}^{i,M}_p(X^{(m)}_{t_i})-\mathcal{T}^i_p(X^{(m)}_{t_i})\right|\}}\right)+\left|\alpha\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i-1},a_p^{i-1,M})\}}+\beta\mathbb{1}_{\{X^{(m)}_{t_j}\in[a_p^{i,M},a_p^{i}]\}}\right|\right]
\end{aligned}
$$

Let $\epsilon>0$, using the induction assumption on p, $\lim_{M\to\infty}a_p^{i-1,M}=a_p^{i-1}$ and $\lim_{M\to\infty}a_p^{i,M}=a_p^i$ a.s and using the induction assumption on j, $\lim_M\to\infty\hat{\mathcal{T}}^{i,M}_p(X^{(m)}_{t_i})=\mathcal{T}^i_p(X^{(m)}_{t_i})\ \forall i$. Then

$$
\begin{aligned}
&\limsup_M\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\left|\hat{\nu}^M_{p,j}(\alpha,\beta,a)-\nu^M_{p,j}(\alpha,\beta,a)\right| \\
&\leq \limsup_M\sup_{a\in[0,1]^d,|\alpha|\leq C,|\beta|\leq C}\frac{1}{M}\sum_{m=1}^{M}2\left(\max_{l\geq j+1}\left|Z^{(m)}_{t_l}\right|+3C\right) \\
&\left[C\left(\mathbb{1}_{\{\left|X^{(m)}_{t_j}-a_p^{i-1}\right|\leq\epsilon\}}+\beta\mathbb{1}_{\{\left|X^{(m)}_{t_j}-a_p^{i}\right|\leq\epsilon\}}\right)+\left(\sum_{k=j+1}^{N}\left|Z^{(m)}_{t_k}\right|\sum_{k=j+1}^{N-1}\mathbb{1}_{\{\left|Z^{(m)}_{t_k}-\mathcal{T}^k_p(X^{(m)}_{t_k})\right|\leq\epsilon\}}\right)\right]
\end{aligned}
$$

since $\lim_{\epsilon\to0}\mathbb{1}_{\{\left|Z_{t_i}-\mathcal{T}^i_p(X^{(m)}_{t_i})\right|\leq\epsilon\}}=\lim_{\epsilon\to0}\left|\alpha\mathbb{1}_{\{\left|X^{(m)}_{t_j}-a_p^{i-1}\right|\leq\epsilon\}}+\beta\mathbb{1}_{\{\left|X^{(m)}_{t_j}-a_p^{i}\right|\leq\epsilon\}}\right|=0$, we conclude that a.s $\left|\hat{\nu}^M_{p,j}(\alpha,\beta,a)-\nu^M_{p,j}(\alpha,\beta,a)\right|\to0$ uniformly when $M\to\infty$, and thus the random function $\alpha,\beta,a\mapsto\hat{\nu}^M_{p,j}(\alpha,\beta,a)$ converges a.s uniformly to the function $\alpha,\beta,a\mapsto\mathbb{E}\left[\left|F_{j+1}(\vartheta^p,Z,X)-\alpha\mathbb{1}_{\{X_{t_j}\in[a_p^{i-1},a)\}}-\beta\mathbb{1}_{\{X_{t_j}\in(a,a_p^i)\}}\right|^2\right]$ which concludes the induction.

□

Theorem 4.8. *Assume that for all $p \in \mathbb{N}^*$, and all $1 \leq j \leq N-1$, $\mathbb{P}(Z_{t_j} = \mathcal{T}_p^j(X_{t_j}, \theta_j^p)) = 0$. Then, for $\alpha = 1, 2$ and for every $j = 1, \ldots, N$,*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \left(Z^{(m)}_{\tau_j^{p,(m)}} \right)^{\alpha} = \mathbb{E}\left[(Z_{\tau_j^p})^{\alpha} \right] \text{ a.s.}$$

Proof. Note that $\mathbb{E}\left[(Z_{\tau_j^p})^{\alpha} \right] = \mathbb{E}\left[F_j(\vartheta^p, Z, X)^{\alpha}) \right]$ and by the strong law of large numbers

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} F_j\left((\vartheta^p, Z^{(m)}, X^{(m)})^{\alpha} \right) = \mathbb{E}\left[F_j(\vartheta^p, Z, X)^{\alpha} \right] \text{ a.s.}$$

It remains to prove that

$$\Delta F_M = \frac{1}{M} \sum_{m=1}^{M} \left(F_j(\hat{\vartheta}^{p,M}, Z^{(m)}, X^{(m)})^{\alpha} - F_j((\vartheta^p, Z^{(m)}, X^{(m)})^{\alpha}) \right) \xrightarrow[M\to\infty]{a.s} 0.$$

For any $x, y \in \mathbb{R}$, and $\alpha = 1, 2$, $|x^{\alpha} - y^{\alpha}| \leq |x - y| \left| x^{\alpha-1} + y^{\alpha-1} \right|$. Using Lemma 4.6 and Equation (8), we have

$$\begin{aligned}
|\Delta F_M| &\leq \frac{1}{M} \sum_{m=1}^{M} \left| F_j(\hat{\vartheta}^{p,M}, Z^{(m)}, X^{(m)})^{\alpha} - F_j(\vartheta^p, Z^{(m)}, X^{(m)})^{\alpha} \right| \\
&\leq 2 \frac{1}{M} \sum_{m=1}^{M} \sum_{i=j}^{N} \max_{k\geq j} \left| Z^{(m)}_{t_k} \right|^{\alpha-1} \left| Z^{(m)}_{t_i} \right| \sum_{i=j}^{N-1} \mathbb{1}_{\{ \left| Z^{(m)}_{t_i} - \mathcal{T}_p^i(X^{(m)}_{t_i}) \right| \leq \left| \hat{\mathcal{T}}_p^{i,M}(X^{(m)}_{t_i}) - \mathcal{T}_p^i(X^{(m)}_{t_i}) \right| \}}.
\end{aligned}$$

Using Proposition 4.7, for all $i = j, \ldots, N-1$, $\left| \hat{\mathcal{T}}_p^{i,M}(X_{t_i}) - \mathcal{T}_p^i(X_{t_i}) \right| \to 0$ a.s when $M \to \infty$. Then for any $\epsilon > 0$,

$$\begin{aligned}
&\limsup_M |\Delta F_M| \\
&\leq 2 \limsup_M \frac{1}{M} \sum_{m=1}^{M} \sum_{i=j}^{N} \max_{k\geq j} \left| Z^{(m)}_{t_k} \right|^{\alpha-1} \left| Z^{(m)}_{t_i} \right| \sum_{i=j}^{N-1} \mathbb{1}_{\{ \left| Z^{(m)}_{t_i} - \mathcal{T}_p^i(X^{(m)}_{t_i}) \right| \leq \epsilon \}} \\
&\leq 2\mathbb{E}\left[\sum_{i=j}^{N} \max_{k\geq j} |Z_{t_k}|^{\alpha-1} |Z_{t_i}| \sum_{i=j}^{N-1} \mathbb{1}_{\{ \left| Z_{t_i} - \mathcal{T}_p^i(X_{t_i}) \right| \leq \epsilon \}} \right].
\end{aligned}$$

We conclude that $\limsup_M |\Delta F_M| = 0$ by letting ϵ go to 0 which ends the proof. □

5 Random forests

Definition 5.1. *A Random Forest is a collection of regression trees $\{\mathcal{T}_{p,\Theta_k}, k = 1, \ldots\}$ where the $\{\Theta_k\}$ are i.i.d random vectors. We denote the resulting forest by $\mathcal{H}_{B,p}(X) = \sum_{k=1}^{B} \frac{1}{B} \mathcal{T}_{p,\Theta_k}(X)$ where B is the number of trees in the forest and p the depth of the trees, and $\mathcal{H}_p = \mathbb{E}_\Theta\left[\mathcal{T}_{p,\Theta}\right] = \lim_{B\to\infty} \mathcal{H}_{B,p}(X)$*

Theorem 5.2.

$$\lim_{B\to\infty} \mathbb{E}\left[|Y - \mathcal{H}_{B,p}(X)|^2 \right] = \mathbb{E}\left[|Y - \mathcal{H}_p(X)|^2 \right]$$

See Theorem 11.1 in (Breiman, 2001).

Theorem 5.3.

$$\mathbb{E}\left[|Y - \mathcal{H}_p(X)|^2 \right] \leq \bar{\rho} \mathbb{E}_\Theta\left[\mathbb{E}\left[|Y - \mathcal{T}_{p,\Theta}(X)|^2 \right] \right]$$

where $\bar{\rho}$ is the weighted correlation between the residuals $Y - \mathcal{T}_{p,\Theta}(X)$ and $Y - \mathcal{T}_{p,\Theta'}(X)$ and Θ and Θ' are independent. See Theorem 11.2 in (Breiman, 2001)

Theorem 5.3 says that to have a good generalization error in the random forest, one should have small generalization errors in the basis trees, and the basis trees should not be highly correlated.

6 Numerical results

6.1 Description

This section studies the price of some Bermudan options using regression trees or random forests to approximate the conditional expectations. We compare the results to some reference prices and those given by the standard Longstaff Schwarz method with regression on polynomial functions. We use the Scikit-Learn library in Python, (Pedregosa et al., 2011). For regression trees, this library offers two methods of splitting: "best" to choose the best split, meaning that the split threshold is the one that minimizes the MSE and the direction for splitting is the one that gives the lowest MSE among all directions. "random" to choose the best random split, meaning that the split threshold is the one that minimizes the MSE and the direction for splitting is chosen randomly. For the following tests, we will use the latter method, which is just slightly different from what we presented in Section 2 in the way that no mid-point cuts will be considered. We also use the feature `min_samples_leaf` which allows us to set a minimum number of samples in each node. This will allow us to avoid over-fitting. For random forests, we will use the bootstrapping method (`Bootstrap=True`), meaning that for each tree in the forest, we will use a sub-sample drawn randomly and with replacement from the training data. We will also use the feature `max_samples` which allows having a specific number of data points or a percentage of the training data attributed to each tree. Having the trees trained on different data as much as possible allows us to have a low correlation between the trees which, using Theorem 5.3, should make the random forest more robust.
Following the work of (Longstaff and Schwartz, 2001), we only use the in-the-money paths to learn the continuations values, which significantly improves the numerical computations. All the prices that we show are obtained after resimulation, meaning that the paths used in the estimation of the conditional expectations are not the same ones used by the Monte Carlo which means that the prices we show are unbiased.

6.2 Black and Scholes

Consider the Black and Scholes model

$$\begin{cases} dS_t^i & = rS_t^i dt + \sigma_i S_t^i dB_t^i, \\ d < B^i, B^j >_t & = \rho_{ij} dt. \end{cases}$$

where σ_i is the volatility of the underlying S^i, assumed to be deterministic, r is the interest rate, assumed constant, and ρ_{ij}, represents the correlation between the underlyings S^i and S^j, assumed constant.

6.2.1 One-dimensional put option

We consider the Bermudan put option with payoff $(K - S_\tau)^+$ with maturity $T = 1$ year, $K = 110$, $S_0 = 100$, $\sigma = 0.25$, exercisable at $N = 10$ different dates. We consider $r = 0.1$. We have a reference price for this option of 11.987 computed by a convolution method in (Lord et al., 2007). The LSM algorithm converges to the correct price with only a polynomial of degree 3. Figure 1, shows the price of the option when we use regression trees with a random split strategy (continuous line) or a best split strategy (dotted line) to estimate the conditional expectations. With the random strategy, the best price we get is 11.89. The case `min_samples_leaf=1` and `max_depth=20` gives a price of 10.5, which is far from the reference price. This result is due to over-fitting. In fact, for this case, the number of degrees of freedom is too big. The tree fits the training data too well, but it cannot generalize when confronted with new data. For the best split strategy, we obtain a slightly better price of 11.94. However, depending on the tree parameters, the price fluctuates, and we can see that the best split strategy is not necessarily better than the random split strategy. Thus, for the following, we will keep using the random split strategy. Random forests with basis trees of maximum depth 5 and minimum 100 samples in each leaf converge to the correct price with only ten trees.

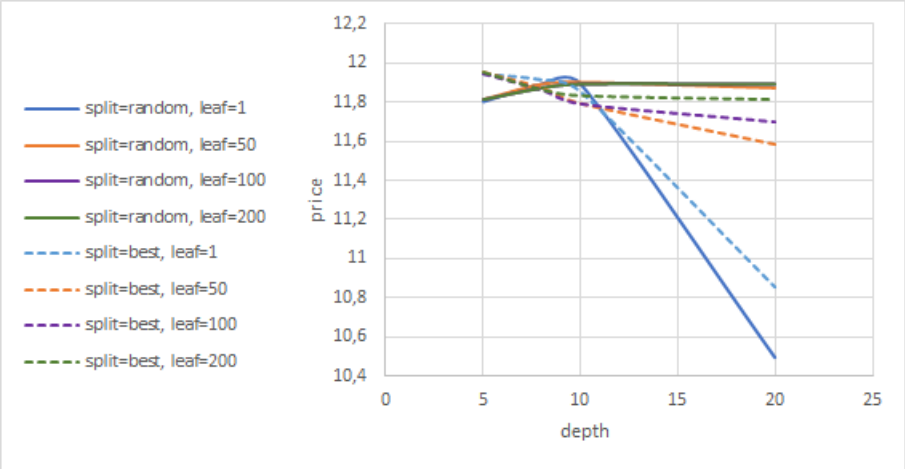

Figure 1: one dimensional put with regression trees, true price=11.987

6.2.2 Call option on the maximum of two assets

We consider a call option on the maximum of 2 assets with payoff $(\max(S_\tau^1, S_\tau^2) - K)^+$, we use the same set of parameters as in (Glasserman, 2004), for which we have reference prices of 13.90, 8.08 and 21.34 for $S_0^i = 100, 90$ and 110 respectively. The LSM algorithm using a polynomial of degree 5 converges to a price of 13.90, 8.06, 21.34 for the cases $K = 100, 90, 110$ respectively. This is a small dimensional problem, so the convergence of the LSM is expected. With regressions trees we have slightly less satisfying results as shown in Figure 2. We can still see the case of over-fitting when giving the regression trees too many degrees of freedom. Aggregating the regression trees into random forests immediately improves the results as shown in Figure 3. Note that the lower the percentage of data in each basis tree, the better the results. This confirms the results of Theorem 5.3 .

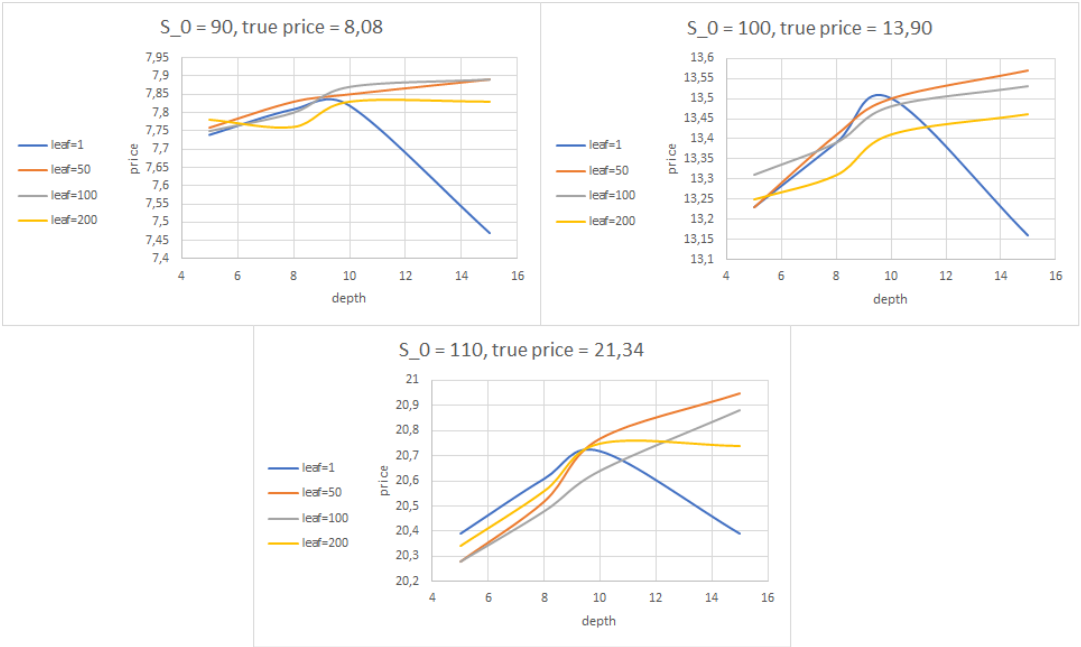

Figure 2: Call on the maximum of two assets with regression trees, $K = 100, T = 3$ years, $\sigma^i = 0.2, r = 0.05, \rho_{ij} = 0, \delta_i = 0.1, N = 9, M = 100,000$

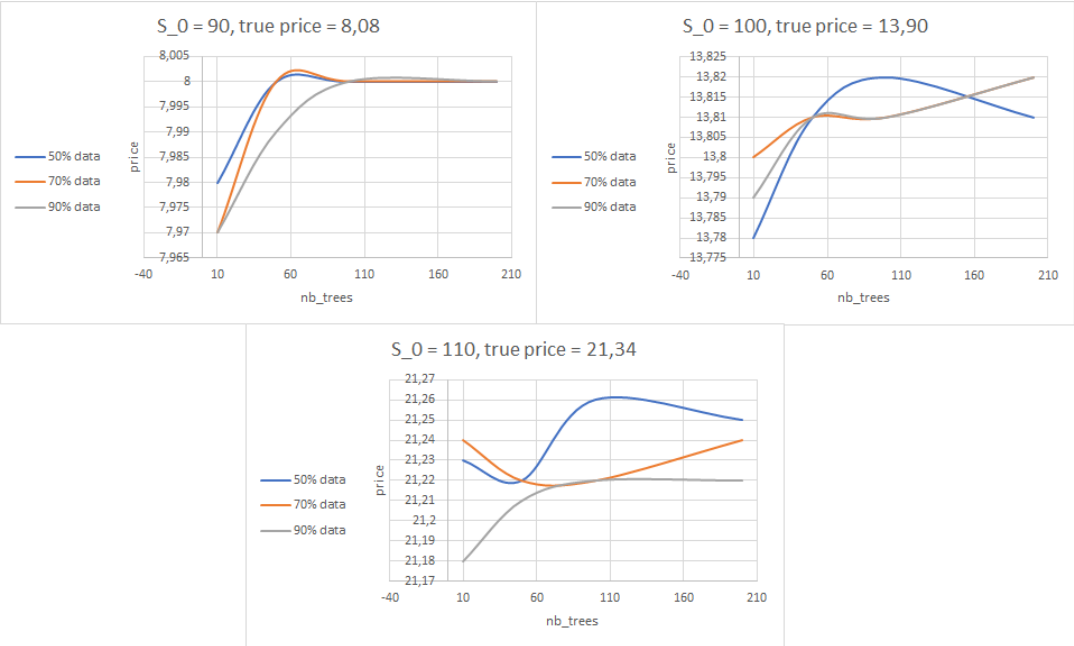

Figure 3: Call on the maximum of two assets with random forests, $K = 100, T = 3$ years, $\sigma^i = 0.2, r = 0.05, \rho_{ij} = 0, \delta_i = 0.1, N = 9, M = 100,000$

6.2.3 Geometric basket option

We consider a Bermudan Put option on a geometric basket of d underlying with payoff $\left(K - (\prod_{i=1}^{d} S_\tau^i)^{\frac{1}{d}}\right)^+$. We test the following option for $d = 2, 10, 40$ for which we have reference prices from (Cox et al., 1979) using the CRR tree method. With the LSM algorithm, we converge to the correct price 4.57 for the case $d = 2$, using only a polynomial of degree 3. For the case $d = 10$, we can at most use a polynomial of degree 3 due to the curse of dimensionality. With this parametrization, we obtain a price of 2.90 for a true price of 2.92. For the case $d = 40$, we cannot go further than a polynomial of degree 1, which yields a price of 2.48 for a reference price of 2.52. Figure 4 shows the results obtained with regression trees. For the case $d = 2$, the best price we get is 4.47 and, as expected, the LSM algorithm has a better performance. This is also the case for the cases $d = 10$ and $d = 40$ where the best prices we obtain are 2.84 and 2.46 respectively. Notice that even though these are high dimensional cases, the trees converge with only a depth of 5 or 8. We also notice the importance of the parameter `min_samples_leaf`. In fact, letting the trees grow without managing this parameter (case leaf1) leads to a problem of over-fitting. The results get better when we use random forests as shown in Figure 5. For these random forests we used basis trees of `max_depth=8` and `min_samples_leaf=100`. Notice for the case $d = 2$, the curve where only 50% of the data is used gives much better results as in this case the basis trees are the less correlated. For the cases $d = 10$ and $d = 40$, the best choice is not necessarily to use 50% of the data in each tree. As these are larger dimensions, having the trees trained on a small percentage of the training data maybe not enough. One may consider extending the size of the training data itself. Furthermore, we notice that once the percentage of data to use in each tree is chosen, the price of the option converges as the number of trees in the forest grows.

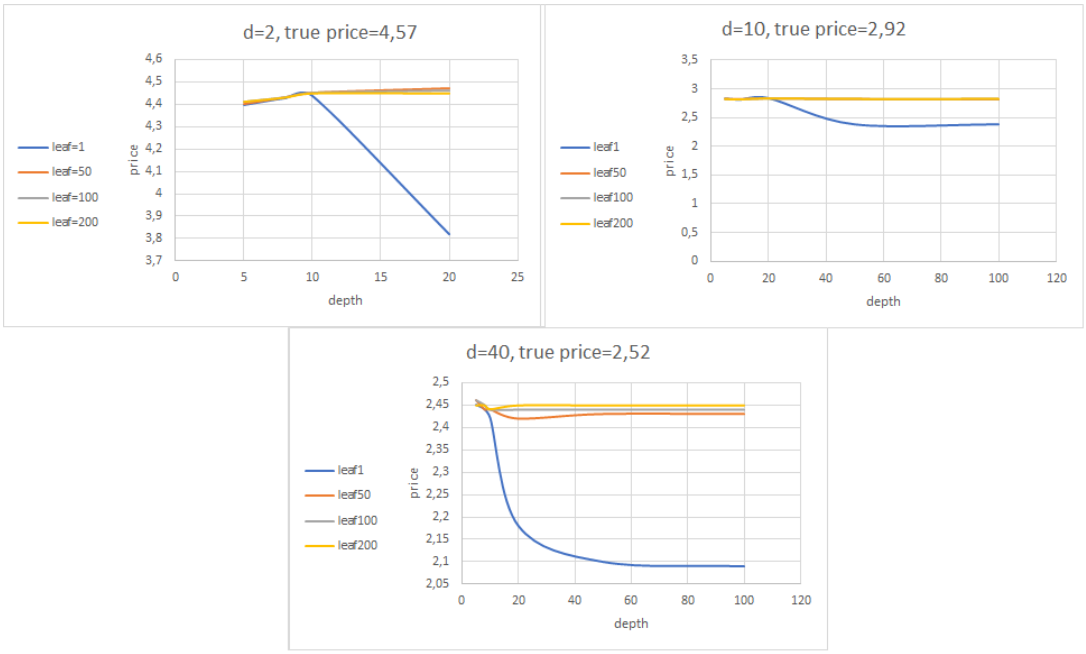

Figure 4: Geometric put option with regression trees

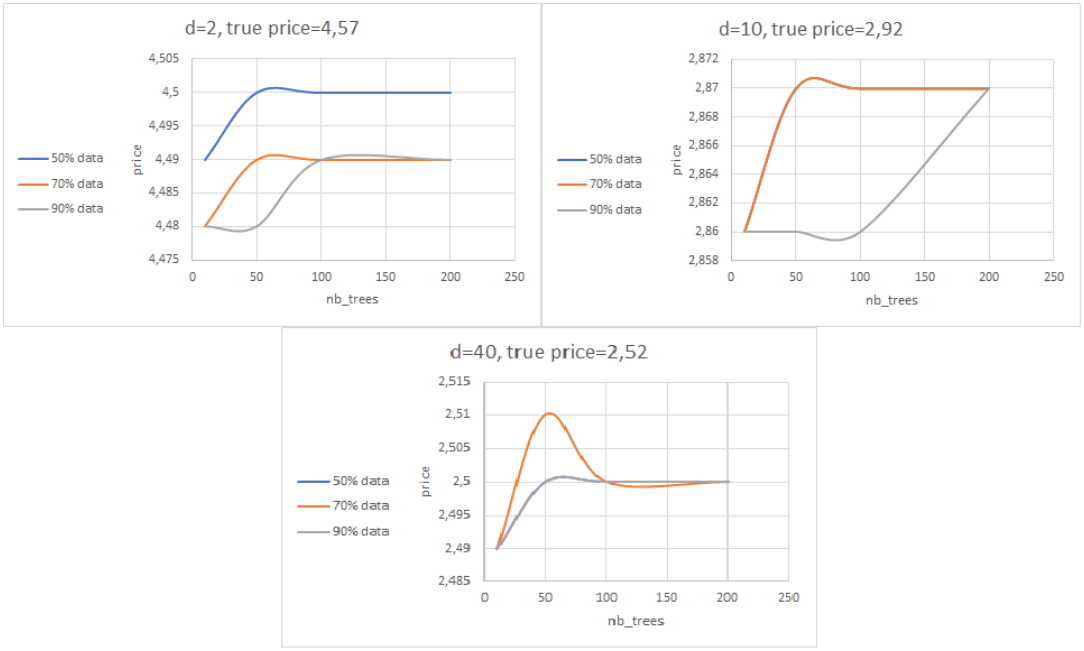

Figure 5: Geometric put option with random forests

6.2.4 A put basket option

We consider a put option on the basket of $d = 40$ asset with payoff $\left(K - \sum_{i=1}^{d} \omega_i S_T^i\right)^+$. We test this payoff for $d = 40$ for which we have a reference price from (Goudenège et al., 2019) between 2.15 and 2.22 using the following set of parameters: $T = 1, S_i = 100, K = 100, r = 0.05, \sigma_i = 0.2, \rho_{ij} = 0.2, \omega_i = \frac{1}{d}$ and $N = 10$. With a polynomial of degree 1, we obtain a price of 2.15 using the LSM algorithm. The results obtained with regression trees are shown in Figure 6

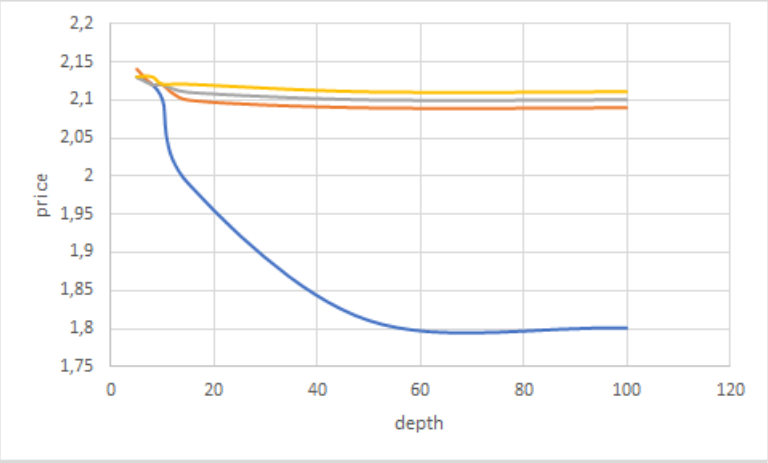

Figure 6: Put on a basket of 40 asset with regression trees

Even though this example is high dimensional, we do not need a lot of parameters to estimate the conditional expectations (the trees converge for very small depths). This will not be the case for the next example which is very non linear. The aggregation into random forests leads to a price of 2.16 using only 50 trees.

6.2.5 A call on the max of 50 asset

We consider a call option on the maximum of $d = 50$ asset with payoff $\left(\max_{1\leq i\leq d}(S_T^i) - K\right)^+$ with the following characteristics: $K = 100, T = 3$ years, $S_0^i = 100, \sigma_i = 0.2, \delta_i = 0.1, \rho_{ij} = 0\ \forall i, j, r = 0.05, N = 9, M = 100000$. (Becker et al., 2019) report $[69.56, 69.95]$ as the 95% confidence interval for the option price. With the LSM algorithm we find a price of 67.88 with a polynomial of degree 1. This a difficult example and we need to use bigger trees to approach the conditional expectations. At maturity, the payoff depends only on one direction (corresponding to the best performance), if the cuts in the tree never consider that direction, the estimation will not be correct. As a result, we consider a number of cuts big enough to ensure that each direction is taken into consideration. We allow the depth to grow while monitoring the `min_samples_leaf` in order to have a significant number of samples in each leaf. Table 1 shows the results obtained with regression trees. As the best price we obtain is given by depth=100 and `min_samples_leaf` = 100, we use this set of parameters for the random forest part. Table 2 shows the results that we obtain with this method.

depth	`min_samples_leaf`	price
50	50	66,89
50	100	66.88
100	50	67.13
100	100	67.31
200	50	67.16
200	100	67.28

Table 1: A call option on the maximum of 50 asset with regression trees

`nb_trees`	`max_samples`	price
10	50%	68,32
10	70%	68,32
10	90%	68,29

Table 2: A call option on the maximum of 50 asset with random forests

Using only regression trees is not enough to have acceptable results. However, as soon as we aggregate the regressor into random forests, we obtain very satisfying results and with just 10 trees we converge to a good price. We can also

notice in this example that using uncorrelated trees leads to better results (see the case `max_samples`= 50% or 70% against the case `max_samples` = 90%).

6.3 A put in the Heston model

We consider the Heston model defined by

$$dS_t = S_t(r_t dt + \sqrt{\sigma_t}(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2))$$
$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \xi\sqrt{\sigma_t} dW_t^1$$

and we consider a put option with payoff $(K - S_T)^+$. we have no reference price for this option, so we will just compare the results of regression trees and random forests to the LSM method. We use the following set of parameters: $K = 100, S_0 = 100, T = 1, \sigma_0 = 0.01, \xi = 0.2, \kappa = 2, \rho = -0.3, r = 0.1, N = 10$ and $M = 100,000$. The LSM method yields a price of 1.70. Figures 7 and 8 show the results obtained with regression trees and random forests. Both methods converge to the same price of LSM. We notice for this example the occurrence of the over-fitting phenomenon for regression trees with `max_depth=15` and `min_sample_leaf=1`. We also have the same behavior for random forests in function of the percentage of data given to each basis tree.

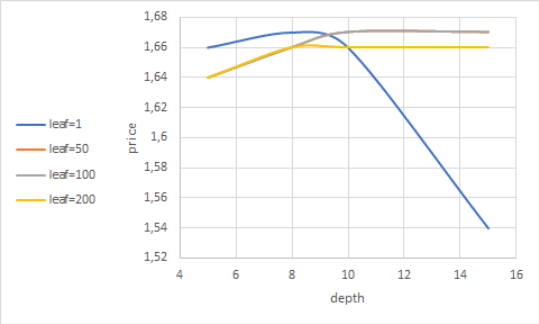

Figure 7: A put option in the Heston model with regression trees

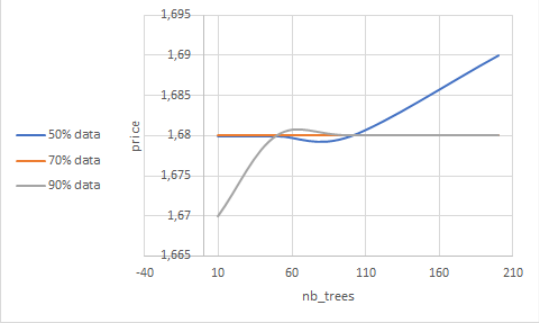

Figure 8: A put option in the Heston model with random forests

7 Conclusion

Pricing Bermudan options comes down to solving a dynamic programming equation where the main trouble comes from the computation of the conditional expectations representing the conditional expectations. We have explored the usage of regression trees and random forests for the computations of these quantities. We have proved in two steps the convergence of the algorithm when regression trees are used: first, the convergence of the conditional expectations; Then, the convergence of the Monte Carlo approximation. This problem was particularly hard to solve given that the regression trees do not solve a global optimization problem as does the functional regression used in the LSM algorithm. We have shown through numerical experiments that we obtain good prices for some classical examples using regression trees. The aggregation of regression trees into random forests yields even better results. We came to the conclusion that for small dimensional problems, a simpler algorithm like the LSM is efficient enough. However,

for high dimensional problems, the usage of polynomial regressions becomes impossible as this technique suffers from the curse of dimensionality. In this case, it is interesting to consider using random forests. Instead of using all the features of the problem, the basis trees in the forest only use a subset of the features which can help combat the problem of the curse of dimensionality.

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