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About error bounds in metrizable topological vector spaces

Malek Abbasi · Michel Théra

Dedicated to Miguel Angel Goberna in recognition of his significant contribution to the development of optimization and semi infinite programming

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Abstract This paper aims to present some sufficient criteria under which a given function $f : \mathbb{X} \rightarrow \mathbb{Y}$ satisfies the error bound property, where \mathbb{X} and \mathbb{Y} are either topological vector spaces whose topologies are generated by metrics or metrizable subsets of some topological vector spaces. Then, we discuss the Hoffman estimation and obtain some results for the estimate of the distance to the set of solutions to a system of linear equalities. The advantage of our estimate is that it allows to calculate the coefficient of the error bound. The applications of this presentation are illustrated by some examples.

Keywords Error bound · Hoffman estimate · Hadamard directional derivative · Translation invariant metric · Strongly regular point · Homogeneously continuous functions

Mathematics Subject Classification (2010) 49J52 · 49J53 · 49J99

1 Introduction

The notion of error bound is a widely used notion in applied mathematics and thereby has received a lot of attention in the last years and decades. Indeed, it plays a key role in areas including variational analysis, mathematical programming, convergence properties of algorithms, sensitivity analysis, designing solution methods for non-convex quadratic problems, penalty functions, optimality conditions, weak sharp minima, stability and well-posedness of solutions, (sub)regularity and calmness of set-valued mappings, and subdifferential calculus (see in particular [1, 2] and the references therein). In this regard, Hoffman's estimation, as the starting point of the theory of error bounds, is very important and plays a considerable role in optimization and especially in iterative methods for solving linear systems and in sensitivity analysis of linear/integer programs [3–5]. Hoffman's estimation has been extended over the years to different contexts (see for instance [6–10] and the references therein for the fundamental role played by Hoffman's bounds).

In this work, we aim at providing some sufficient criteria under which the function f , acting either between metrizable topological vector spaces or between metrizable subsets of some topological vector spaces, satisfies the error bound property at a point $\bar{x} \in \mathbb{X}$. To

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characterize the error bound property of f , if $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ designate the metrics generating the topologies on X and Y , the following inequality is discussed:

$$d_{\mathbb{X}}(x, \bar{x}) \leq \kappa d_{\mathbb{Y}}(f(x), f(\bar{x})). \quad (1.1)$$

Indeed, we are looking for a neighborhood \mathbb{O} of \bar{x} and a constant $\kappa > 0$ such that the above inequality holds for all $x \in \mathbb{O}$. In this case, a point \bar{x} which satisfies relation (1.1) is called a *strongly regular point of f* (see [11]) and has a close relationship with the notion of *regularity* proposed by Ioffe (see, in [12, 13]). Let us recall this notion in the sequel.

Definition 1.1 (Ioffe) Let \mathbb{X} and \mathbb{Y} be normed spaces and $\bar{x} \in \mathbb{X}$. We say that \bar{x} is a *regular point* for the function $f : \mathbb{X} \rightarrow \mathbb{Y}$ with respect to the subset $\mathbb{U} \subset \mathbb{X}$ if there are $\tau > 0$ and a neighborhood \mathbb{V} of \bar{x} such that

$$\text{dist}(x, \mathbb{Q}) \leq \tau \|f(x) - f(\bar{x})\|, \quad (1.2)$$

for all $x \in \mathbb{V} \cap \mathbb{U}$, where $\mathbb{Q} := \{x \in \mathbb{U} : f(\bar{x}) = f(x)\}$ and $\text{dist}(x, \mathbb{Q}) := \inf_{q \in \mathbb{Q}} \|x - q\|$ is the distance from x to \mathbb{Q} .

Inequality (1.1) asserts that if \bar{x} is a strongly regular point of $f : \mathbb{X} \rightarrow \mathbb{Y}$, then it is a regular point for the function f with respect to any subset $\mathbb{U} \subset \mathbb{X}$ with $\bar{x} \in \mathbb{U}$ (note that $\text{dist}(x, \mathbb{Q}) \leq \|x - \bar{x}\|$ for all $x \in \mathbb{X}$). Thus inequality (1.1) helps to find an *upper error bound* for the distance from x to the solution set \mathbb{Q} (see [13–16]).

When X is a normed space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, we can also establish a relationship between strongly regular points and *sharp local minimizers* of f . This notion introduced by Ferris in [17] as a generalization of sharp minima due to Polyak [18] is well-studied for its usefulness in optimization (see [19, 20]). One says that $\bar{x} \in \text{dom } f := \{x \in \mathbb{X} : f(x) < +\infty\}$ is a *sharp local minimizer* of f with modulus $\tau > 0$ provided there exists some $r > 0$ such that the following inequality holds:

$$f(x) \geq f(\bar{x}) + \tau \|x - \bar{x}\| \quad \text{for all } x \in \mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, r), \quad (1.3)$$

where $\mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, r)$ denotes the open ball centered at \bar{x} with radius r . It is clear from the definition that any sharp minimizer is a local minimizer, and even more, it is a strict local minimizer, since clearly, $f(x)$ is strictly less than $f(\bar{x})$ when $x \in \mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, r) \setminus \{\bar{x}\}$. On the other hand, if \bar{x} is a sharp local minimizer of f with modulus $\tau > 0$, then

$$\|x - \bar{x}\| \leq \tau^{-1} |f(x) - f(\bar{x})| \quad \text{for all } x \in \mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, r),$$

implying that \bar{x} is a strongly regular point of f . The converse might not be true. Indeed, given the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) := x$ for all $x \in \mathbb{R}$ (identity function), clearly every $\bar{x} \in \mathbb{R}$ is a strongly regular point of f while this function has no local minimizer, and therefore \bar{x} fails to be a sharp local minimizer of f . Hence the notion of strongly regular point is weaker than the notion of sharp local minimizer. Note also that any smooth function cannot have a sharp local minimizer.

In the last section of this paper we prove that when \mathbb{X} is **finite dimensional**, if \bar{x} is both a strongly regular point and a local minimizer of f , then \bar{x} is a local weak ε -efficient solution of ∂f with respect to any pointed closed convex cone C .

This paper is intended to provide some sufficient conditions under which the inequality (1.1) is verified. We shall see that if the metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ are *translation invariant* and *positively homogeneous* and the function f satisfies certain continuity and differentiability properties (like the ones mentioned in [Theorem 3.1](#)), then finding a constant κ and a neighborhood of \bar{x} such that (1.1) holds is possible.

The paper is organized as follows. In [Section 1](#), we give a background information on the error bound property. In [Section 2](#), we introduce the notion of *homogeneously continuity* for functions acting between topological vector spaces and state some preliminaries. In [Section 3](#), using a notion of lower directional derivative, we obtain some sufficient conditions that contribute to characterize the error bound property of functions. In [Section 4](#), applying the results from [Section 3](#) and focusing our attention on Hoffman's estimate of approximate solutions of finite systems of linear inequalities, we prove some estimations for the approximate solutions of finite systems of linear inequalities and of linear equalities. Finally, in [Section 5](#), we establish a relationship between strongly regular points, local minimizer of f and local weak ε -efficient solution of ∂f (the subdifferential of f) with respect to any pointed closed convex cone C .

2 Preliminaries

Throughout this paper, unless specified otherwise, we assume that \mathbb{X} and \mathbb{Y} are arbitrary topological vector spaces whose topologies are induced by metrics or metrizable subsets of some topological vector spaces. The (continuous) dual of \mathbb{X} is denoted by \mathbb{X}^* . In the case that \mathbb{X} is a metric space, $\mathbb{B}_{\mathbb{X}}^{\circ}(x, r)$ denotes the open ball centered at x with radius r . If \mathbb{X} is a normed space, then the symbols $\mathbb{S}_{\mathbb{X}}$ and $\mathbb{B}_{\mathbb{X}}$ stand, respectively, for the unit sphere (i. e., the elements of \mathbb{X} of norm one) and the closed unit ball (i. e., the elements of \mathbb{X} of norm less than or equal to one). Some other notations are introduced as and when needed.

We begin with the following definition in which \mathbb{X} and \mathbb{Y} are arbitrary topological vector spaces.

Definition 2.1 Let \mathbb{X} and \mathbb{Y} be topological vector spaces and $\mathbb{E} \subset \mathbb{X}$. A function $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be *homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{E}* if for every neighborhood \mathbb{V} of origin in \mathbb{Y} there exist a neighborhood \mathbb{O} of origin in \mathbb{X} and a $0 < \beta \leq 1$ such that

$$x - y \in \mathbb{O} \implies f(\bar{x} + tx) - f(\bar{x} + ty) \in t\mathbb{V},$$

for all $0 < t \leq \beta$ and all $x, y \in \mathbb{E}$.

Definition 2.2 If \mathbb{X} is a vector space and d is a metric on \mathbb{X} , we say that d is *translation invariant* (see [\[21\]](#)) if

$$d(x + u, y + u) = d(x, y) \text{ for all } x, y, u \text{ in } \mathbb{X}.$$

and we say that d is *positively homogeneous* if

$$d(tx, ty) = td(x, y) \text{ for all } x, y \text{ in } \mathbb{X} \text{ and all } t \geq 0.$$

Example 2.1

- (i) A *Fréchet space* is a complete locally convex vector space whose topology is metrizable and therefore is defined by a translation invariant metric d (see [\[21\]](#));
- (ii) If \mathbb{X} is a normed space, then the metric defined by the norm of \mathbb{X} (i.e., $d(x, y) := \|x - y\|$ for all $x, y \in \mathbb{X}$) is both translation invariant and positively homogeneous;
- (iii) The metric d defined in [Example 3.1](#) (and [Theorem 3.3](#)) below is both translation invariant and positively homogeneous;
- (iv) Note that there exist metrics that are positively homogeneous and not translation invariant. This is the case of the so-called “French railways distance” defined as follows: given two points A and B , the distance $D(A, B)$ is the usual distance Euclidean distance $d(A, B)$ if A and B lie on the same ray from the origin O and otherwise $D(A, B) := d(O, A) + d(O, B)$.

Proposition 2.1 Let \mathbb{X} and \mathbb{Y} be metrizable topological vector spaces whose topologies are respectively given by the metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ and let $\mathbb{E} \subset \mathbb{X}$. Suppose that $d_{\mathbb{X}}$ is translation invariant and $d_{\mathbb{Y}}$ is both translation invariant and positively homogeneous. Then, $f : \mathbb{X} \rightarrow \mathbb{Y}$ is homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{E} if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ and $0 < \beta \leq 1$ such that

$$d_{\mathbb{X}}(x, y) < \delta \implies d_{\mathbb{Y}}(f(\bar{x} + tx), f(\bar{x} + ty)) < t\varepsilon,$$

for all $0 < t \leq \beta$ and all $x, y \in \mathbb{E}$.

Proof This is a straightforward consequence of the above definitions. \square

Remark 2.1 As [Proposition 2.1](#) shows, the notion of homogeneously continuity given in [Definition 2.1](#) extends to all topological metrizable vector spaces [Definition 2.1](#) in [\[11\]](#) stated for normed spaces.

Example 2.2 Let $\mathbb{X} = \ell_1 := \{x := (x_n) \in \mathbb{R} : \sum_n |x_n| < \infty\}$ with the usual norm $\|x\| := \sum_n |x_n|$ for $x \in \ell_1$ and \mathbb{Y} be an arbitrary normed space. Let $f : \ell_1 \rightarrow \mathbb{Y}$ be an arbitrary continuous linear function. One may observe that f is homogeneously continuous at 0 on \mathbb{X} (norm \rightarrow norm). Now assume that \mathbb{U} is a weakly compact convex subset of ℓ_1 which contains the origin. Equip \mathbb{U} with the induced topology. We claim that $f|_{\mathbb{U}}$ (the restriction of f to \mathbb{U}) is homogeneously continuous at 0 on \mathbb{U} (weak \rightarrow norm). Since ℓ_1 is separable, thus \mathbb{U} is weakly metrizable. Indeed, if (x_n) is a dense sequence in \mathbb{S}_{ℓ_1} , then thanks to the Hahn-Banach Theorem, there exists a sequence $(x_n^*) \in \mathbb{X}^*$ such that $\|x_n^*\| = \langle x_n^*, x_n \rangle = \|x_n\| = 1$. Now define $d : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ as

$$d(x, y) := \sum_n 2^{-n} |\langle x_n^*, x - y \rangle|,$$

for all $x, y \in \mathbb{U}$. Then (\mathbb{U}, d) is a metric space and \mathbb{U} in its weak topology, is homeomorphic to this metric space, and hence is itself metrizable. One can easily check that d is translation invariant (and positively homogeneous). Now, by contradiction let us suppose that $f|_{\mathbb{U}}$ fails to be homogeneously continuous at 0 on \mathbb{U} (weak \rightarrow norm). Then, according to [Proposition 2.1](#), there would be $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exist $x_n, y_n \in \mathbb{U}$ with $d(x_n, y_n) < \frac{1}{n}$ and $0 < t_n \leq \frac{1}{n}$ such that $\|f|_{\mathbb{U}}(t_n x_n) - f|_{\mathbb{U}}(t_n y_n)\| \geq t_n \varepsilon$. Hence $\|f(x_n) - f(y_n)\| \geq \varepsilon$ for every $n \in \mathbb{N}$. But (x_n) and (y_n) , being contained in a weakly compact set, by the Eberlein-Smulian Theorem, they have subsequences, still denoted by (x_n) and (y_n) , converging weakly to some x_0 and y_0 , respectively, in \mathbb{U} . Letting $n \rightarrow \infty$ in $d(x_n, y_n) < \frac{1}{n}$, it follows that $x_0 = y_0$. Since ℓ_1 satisfies the Schur property (see [\[23\]](#)), (x_n) and (y_n) also converge strongly (in the norm topology) to x_0 and y_0 , respectively. Letting $n \rightarrow \infty$ in $\|f(x_n) - f(y_n)\| \geq \varepsilon$, we deduce that $\|f(x_0) - f(y_0)\|_{\ell_1} \geq \varepsilon$ which is absurd. This proves the above claim. \square

We now present various results providing some sufficient conditions under which a function f is homogeneously continuous. These results extend some similar conclusions stated in [\[11\]](#). Let us recall that a function $f : \mathbb{X} \rightarrow \mathbb{Y}$ ($d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ are metrics on \mathbb{X} and \mathbb{Y} , respectively) is said to be locally Lipschitz around $\bar{x} \in \mathbb{X}$ if there exist a neighborhood \mathbb{O} of \bar{x} and a $\lambda > 0$ such that

$$d_{\mathbb{Y}}(f(x), f(y)) \leq \lambda d_{\mathbb{X}}(x, y),$$

for all $x, y \in \mathbb{O}$.

Proposition 2.2 Suppose that \mathbb{X} and \mathbb{Y} are locally convex metrizable vector spaces and $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ are metrics on \mathbb{X} and \mathbb{Y} such that $d_{\mathbb{Y}}$ is translation invariant and $d_{\mathbb{X}}$ is both translation invariant and positively homogeneous. If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is locally Lipschitz around $\bar{x} \in \mathbb{X}$, then f is homogeneously continuous at \bar{x} on some convex neighborhood of origin $\mathbb{E} \subset \mathbb{X}$.

Proof There exists a constant $\lambda > 0$ such that

$$d_{\mathbb{Y}}(f(x), f(y)) \leq \lambda d_{\mathbb{X}}(x, y),$$

for all x, y belonging to a neighborhood \mathbb{O} of \bar{x} in \mathbb{X} . Choose a convex neighborhood \mathbb{E} of 0 in \mathbb{X} such that $\{\bar{x}\} + \mathbb{E} \subset \mathbb{O}$. It follows that

$$d_{\mathbb{Y}}(f(\bar{x} + tx), f(\bar{x} + ty)) \leq \lambda d_{\mathbb{X}}(\bar{x} + tx, \bar{x} + ty) = t\lambda d_{\mathbb{X}}(x, y),$$

for all $x, y \in \mathbb{E}$ and all $0 \leq t \leq 1$ (since \mathbb{E} is convex and $0 \in \mathbb{E}$). Now let $\varepsilon > 0$ and take $0 < \delta < \varepsilon\lambda^{-1}$. It follows that

$$d_{\mathbb{X}}(x, y) < \delta \implies d_{\mathbb{Y}}(f(\bar{x} + tx), f(\bar{x} + ty)) < t\varepsilon$$

for all $x, y \in \mathbb{E}$ and all $0 < t \leq 1$. □

Proposition 2.3 *Let \mathbb{X} and \mathbb{Y} be metrizable topological vector spaces with corresponding metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ such that $d_{\mathbb{X}}$ is translation invariant and $d_{\mathbb{Y}}$ is both translation invariant and positively homogeneous. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a function, \mathbb{E} be a subset of \mathbb{X} equipped with the induced topology and $\bar{x} \in \mathbb{X}$. If the bifunction $f_{\mathbb{E}} : \mathbb{E} \times (0, 1] \rightarrow \mathbb{Y}$ defined by*

$$f_{\mathbb{E}}(x, t) := \frac{f(\bar{x} + tx) - f(\bar{x})}{t}$$

is uniformly continuous ($\mathbb{E} \times (0, 1]$ equipped with the product topology with the usual linear operations of vector additions and scalar multiplication), then f is homogeneously continuous at \bar{x} on \mathbb{E} .

Proof By hypothesis, there exist $\delta, \beta > 0$ such that for all $x, y \in \mathbb{E}$ with $d_{\mathbb{X}}(x, y) < \delta$ and all $|s - h| < \beta$ we have $d_{\mathbb{Y}}(f_{\mathbb{E}}(x, s), f_{\mathbb{E}}(y, h)) < \varepsilon$. It follows that

$$d_{\mathbb{Y}}\left(\frac{f(\bar{x} + tx) - f(\bar{x})}{t}, \frac{f(\bar{x} + ty) - f(\bar{x})}{t}\right) < \varepsilon,$$

for all $x, y \in \mathbb{E}$ with $d_{\mathbb{X}}(x, y) < \delta$ and all $0 < t \leq 1$. Since $d_{\mathbb{Y}}$ is both translation invariant and positively homogeneous, thus

$$d_{\mathbb{Y}}(f(\bar{x} + tx), f(\bar{x} + ty)) < t\varepsilon,$$

for all $d_{\mathbb{X}}(x, y) < \delta$ and all $0 < t \leq 1$. This completes the proof. □

To state the next result we need to recall the notion of Hadamard directional derivative.

Definition 2.3 *$f : \mathbb{X} \rightarrow \mathbb{Y}$ is Hadamard directionally differentiable at \bar{x} in the direction \mathbf{v} if there exists a function $f'_H(\bar{x}, \cdot) : \mathbb{X} \rightarrow \mathbb{Y}$ such that*

$$f'_H(\bar{x}, \mathbf{v}) = \lim_{\vartheta \rightarrow \mathbf{v}, t \downarrow 0} \frac{f(\bar{x} + t\vartheta) - f(\bar{x})}{t}$$

Corollary 2.1 *Let \mathbb{X} and \mathbb{Y} be metrizable topological vector spaces with corresponding metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ such that $d_{\mathbb{X}}$ is translation invariant and $d_{\mathbb{Y}}$ is both translation invariant and positively homogeneous. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous function, \mathbb{E} be a compact subset of \mathbb{X} (equipped with the induced topology) and $\bar{x} \in \mathbb{X}$. If f is Hadamard differentiable at \bar{x} , then f is homogeneously continuous at \bar{x} on \mathbb{E} .*

Proof Define the bifunction $\tilde{f}_{\mathbb{E}} : \mathbb{E} \times [0, 1] \rightarrow \mathbb{Y}$ as

$$\tilde{f}_{\mathbb{E}}(\mathbf{v}, t) := \begin{cases} \frac{f(\bar{x} + t\mathbf{v}) - f(\bar{x})}{t} & \text{if } 0 < t \leq 1, \\ f'_H(\bar{x}, \mathbf{v}) & \text{if } t = 0, \end{cases}$$

for all $(\mathbf{v}, t) \in \mathbb{E} \times [0, 1]$. Since f is Hadamard differentiable at \bar{x} , thus the bifunction $\tilde{f}_{\mathbb{E}}$ is continuous. Since $\mathbb{E} \times [0, 1]$ is compact, hence $\tilde{f}_{\mathbb{E}}$ is uniformly continuous. It follows that the bifunction $f_{\mathbb{E}} : \mathbb{E} \times (0, 1] \rightarrow \mathbb{Y}$ defined by

$$f_{\mathbb{E}}(x, t) := \frac{f(\bar{x} + tx) - f(\bar{x})}{t},$$

is uniformly continuous. Now apply [Proposition 2.3](#). \square

Proposition 2.4 *Let \mathbb{X} and \mathbb{Y} be metrizable topological vector spaces with corresponding metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ such that $d_{\mathbb{X}}$ is translation invariant and $d_{\mathbb{Y}}$ is both translation invariant and positively homogeneous. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a function, \mathbb{E} be a subset of \mathbb{X} and $\bar{x} \in \mathbb{X}$. If f is homogeneously continuous at \bar{x} on \mathbb{E} , then there exist $\delta, \beta > 0$ such that*

$$d_{\mathbb{Y}}\left(\frac{f(\bar{x} + tx) - f(\bar{x})}{t}, \frac{f(\bar{x} + ty) - f(\bar{x})}{t}\right) < \varepsilon,$$

for all $d_{\mathbb{X}}(x, y) < \delta$ and all $0 < t \leq \beta$.

Proof The proof is obvious; we therefore omit it. \square

3 Characterizing the error bound property

Definition 3.1 Given a function $f : \mathbb{X} \rightarrow \mathbb{Y}$ with \mathbb{Y} a normed space, we consider the following notion of *lower directional derivative* at $\bar{x} \in \mathbb{X}$ in direction $\mathbf{v} \in \mathbb{X}$ (see [11]) defined by

$$f'_l(\bar{x}, \mathbf{v}) := \liminf_{t \downarrow 0} \frac{\|f(\bar{x} + t\mathbf{v}) - f(\bar{x})\|}{t}.$$

Abbasi and Théra showed in [11] that if \mathbb{X} and \mathbb{Y} are both normed spaces, whose norms are denoted by the same symbol $\|\cdot\|$ and \mathbb{X} is finite dimensional, then it holds:

Theorem 3.1 *Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{S}_{α} for some positive scalar α ($\mathbb{S}_{\alpha} := \alpha\mathbb{S}$). If there exists some $\kappa > 0$ such that $\inf_{\mathbf{v} \in \mathbb{S}_{\alpha}} f'_l(\bar{x}, \mathbf{v}) > \kappa$, then there exists $\delta > 0$ such that*

$$\|x - \bar{x}\| \leq \frac{\alpha}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}(\bar{x}, \delta)$. In other words, \bar{x} is a strongly regular point of f .

We begin with the following result which is proved with the help of [Theorem 3.1](#).

Theorem 3.2 *Let \mathbb{X} and \mathbb{Y} be normed spaces, $\mathbb{U} \subset \mathbb{X}$ be a finite dimensional subspace of \mathbb{X} and $\bar{x} \in \mathbb{U}$. Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is homogeneously continuous at \bar{x} on the subset $\mathbb{E}_{\alpha} := \alpha(\mathbb{S} \cap \mathbb{U})$ (norm \rightarrow norm) with α an arbitrary positive number. If there exists some $\kappa > 0$ such that $\inf_{\mathbf{v} \in \mathbb{E}_{\alpha}} f'_l(\bar{x}, \mathbf{v}) > \kappa$, then there exists some $\delta > 0$ such that*

$$\|x - \bar{x}\| \leq \frac{\alpha}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, \delta) \cap \mathbb{U}$. In particular, there exists a constant κ' such that

$$\text{dist}(x, \mathbb{Q}) \leq \kappa' \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}_{\mathbb{X}}^{\circ}(\bar{x}, \delta) \cap \mathbb{U}$, where $\mathbb{Q} := \{x \in \mathbb{U} : f(\bar{x}) = f(x)\}$. In other words, \bar{x} is a regular point for the function f with respect to \mathbb{U} in the sense of Ioffe ([Definition 1.1](#)).

Proof Apply [Theorem 3.1](#) to the function $g := f|_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{Y}$ (f restricted to \mathbb{U}) and notice that \mathbb{E}_α is a nonempty compact subset of \mathbb{U} , g is homogeneously continuous at \bar{x} on \mathbb{E}_α and $\inf_{v \in \mathbb{E}_\alpha} g'_1(\bar{x}, v) > \kappa$. This completes the proof of the first claim. To prove the second one, note that $\bar{x} \in \mathbb{Q}$ and therefore $\text{dist}(x, \mathbb{Q}) \leq \|x - \bar{x}\|$ for all $x \in \mathbb{X}$. \square

Corollary 3.1 *Let \mathbb{X} and \mathbb{Y} be normed spaces, $\mathbb{U} \subset \mathbb{X}$ be a finite dimensional subspace of \mathbb{X} and $f : \mathbb{X} \rightarrow \mathbb{Y}$ be locally Lipschitz around $\bar{x} \in \mathbb{U}$. If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{E}} f'_1(\bar{x}, v) > \kappa$ where $\mathbb{E} := \mathbb{S} \cap \mathbb{U}$, then there exists some $\delta > 0$ such that*

$$\|x - \bar{x}\| \leq \frac{1}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}_{\mathbb{X}}^\circ(\bar{x}, \delta) \cap \mathbb{U}$.

Proof [Proposition 2.2](#) implies that, there exists some convex neighborhood of origin $\mathbb{V} \subset \mathbb{X}$ such that f is homogeneously continuous at \bar{x} on \mathbb{V} . Choose some $\alpha > 0$ such that $\mathbb{E}_\alpha := \alpha(\mathbb{S} \cap \mathbb{U}) \subset \mathbb{V}$. Hence, f is homogeneously continuous at \bar{x} on \mathbb{E}_α . One can easily check that $\inf_{v \in \mathbb{E}_\alpha} f'_1(\bar{x}, v) > \alpha\kappa$. Now apply [Theorem 3.2](#). \square

Let us recall that a subset \mathbb{D} of the dual space \mathbb{X}^* is said to be *total* whenever $\langle x^*, x \rangle = 0$ for all $x^* \in \mathbb{D}$ implies that $x = 0$ (see [\[23\]](#)). The following theorem yields a similar result when one considers the weak topology on \mathbb{X} . It subsumes the special case where \mathbb{X} is separable.

Theorem 3.3 *Let \mathbb{X} and \mathbb{Y} be Banach spaces with \mathbb{X}^* containing a countable total subset $\{x_1^*, x_2^*, \dots\}$ and $\mathbb{U} \subset \mathbb{X}$ be a weakly compact subset which contains the origin and a nonzero vector u . Define $d : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ as*

$$d(x, y) := \sum_n 2^{-n} |\langle x_n^*, x - y \rangle|,$$

for all $x, y \in \mathbb{U}$ and let $\mathbb{H} := \{x \in \mathbb{U} : d(x, 0) = d(u, 0)\}$. Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is homogeneously continuous at some $\bar{x} \in \mathbb{U}$ on \mathbb{H} (weak \rightarrow norm). If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{H}} f'_1(\bar{x}, v) > \kappa$, then there exists a weak neighborhood \mathbb{O} of \bar{x} in \mathbb{X} such that

$$d(x, \bar{x}) \leq \frac{d(u, 0)}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{O} \cap \mathbb{U}$.

Proof First note that $\mathbb{H} \neq \emptyset$ (since $u \in \mathbb{H}$) and the subset \mathbb{U} in its weak topology, is homeomorphic to the metric space (\mathbb{U}, d) , and hence is itself metrizable (see [\[23\]](#)). Hence the subset \mathbb{H} is weakly closed, and since $\mathbb{H} \subset \mathbb{U}$, is weakly compact. Let $\kappa < \rho < \inf_{v \in \mathbb{H}} f'_1(\bar{x}, v)$ and $\varepsilon := \rho - \kappa$. By hypothesis, for all $v \in \mathbb{H}$ there exists $0 < r_v \leq 1$ such that

$$\inf_{h \in (0, r_v]} \frac{\|f(\bar{x} + hv) - f(\bar{x})\|}{h} > \rho. \quad (3.1)$$

Since f is homogeneously continuous (weak \rightarrow norm) at \bar{x} on \mathbb{H} , thus there exist $\eta, \beta > 0$ such that

$$\left\| \frac{f(\bar{x} + tx) - f(\bar{x})}{t} - \frac{f(\bar{x} + ty) - f(\bar{x})}{t} \right\| < \varepsilon, \quad (3.2)$$

for all $x, y \in \mathbb{H}$ with $d(x, y) < \eta$ and all $0 < t \leq \beta$, by [Proposition 2.4](#). Let $\hat{r}_v := \min\{\eta, \beta, r_v\}$ for all $v \in \mathbb{H}$. Clearly we have

$$\mathbb{H} \subset \bigcup_{v \in \mathbb{H}} \mathbb{B}_{\mathbb{U}}^\circ(v, \hat{r}_v),$$

where by $\mathbb{B}_{\mathbb{U}}^\circ(v, \hat{r}_v)$ we mean the open ball centered at v with radius \hat{r}_v in the metric space (\mathbb{U}, d) . The (weak) compactness of \mathbb{H} implies that there exist $v_1, v_2, \dots, v_m \in \mathbb{E}$ such that

$$\mathbb{H} \subset \bigcup_{k=1}^m \mathbb{B}_{\mathbb{U}}^\circ(v_k, \hat{r}_{v_k}).$$

Now let $x \in \mathbb{B}_{\mathbb{U}}^{\circ}(\bar{x}, \delta d(u, 0)) \setminus \{\bar{x}\}$ and $\mathbf{v} := \frac{d(u, 0)}{d(x, \bar{x})}(x - \bar{x})$ where $\delta := \min\{\hat{r}_{\mathbf{v}_k} : 1 \leq k \leq m\}$. One can easily check that $\mathbf{v} \in \mathbb{H}$, and hence $\mathbf{v} \in \mathbb{B}_{\mathbb{U}}^{\circ}(\mathbf{v}_s, \hat{r}_{\mathbf{v}_s})$ for some $1 \leq s \leq m$. It follows that $d(\mathbf{v}, \mathbf{v}_s) < \eta$ and $\frac{d(x, \bar{x})}{d(u, 0)} < \beta$. By (3.2) we deduce that

$$\left\| \frac{f(\bar{x} + \frac{d(x, \bar{x})}{d(u, 0)} \mathbf{v}_s) - f(\bar{x})}{\frac{d(x, \bar{x})}{d(u, 0)}} - \frac{f(\bar{x} + \frac{d(x, \bar{x})}{d(u, 0)} \mathbf{v}) - f(\bar{x})}{\frac{d(x, \bar{x})}{d(u, 0)}} \right\| < \varepsilon.$$

Hence

$$\begin{aligned} \left\| \frac{f(\bar{x} + \frac{d(x, \bar{x})}{d(u, 0)} \mathbf{v}) - f(\bar{x})}{\frac{d(x, \bar{x})}{d(u, 0)}} \right\| &> \left\| \frac{f(\bar{x} + \frac{d(x, \bar{x})}{d(u, 0)} \mathbf{v}_s) - f(\bar{x})}{\frac{d(x, \bar{x})}{d(u, 0)}} \right\| - \varepsilon \\ &> \rho - \varepsilon = \kappa \quad \text{by (3.1),} \end{aligned}$$

since $\frac{d(x, \bar{x})}{d(u, 0)} < r_{\mathbf{v}_s}$. It follows that

$$\frac{\|f(x) - f(\bar{x})\|}{d(x, \bar{x})} > \frac{\kappa}{d(u, 0)}.$$

Thus

$$d(x, \bar{x}) \leq \frac{d(u, 0)}{\kappa} \|f(x) - f(\bar{x})\|,$$

which holds for every $x \in \mathbb{B}_{\mathbb{U}}^{\circ}(\bar{x}, \delta d(u, 0))$. Notice that, by the argument stated at the beginning of the proof, $\mathbb{B}_{\mathbb{U}}^{\circ}(\bar{x}, \delta d(u, 0))$ is a weak neighborhood of \bar{x} in \mathbb{U} . Thus there must exist some weak neighborhood \mathbb{O} of \bar{x} in \mathbb{X} such that $\mathbb{O} \cap \mathbb{U} = \mathbb{B}_{\mathbb{U}}^{\circ}(\bar{x}, \delta d(u, 0))$. This completes the proof. \square

Remark 3.1 Note that [Theorem 3.3](#) applies when \mathbb{X} is a separable Banach space. Indeed if (x_n) is a countable dense set in the unit sphere of \mathbb{X} , and if we choose $x_n^* \in \mathbb{X}^*$ such that $x_n^*(x_n) = 1$, then (x_n^*) is total in \mathbb{X}^* .

Example 3.1 let $\mathbb{X} = \ell_1 := \{x := (x_n) \subset \mathbb{R} : \sum_n |x_n| < \infty\}$ with the usual norm $\|x\| := \sum_n |x_n|$ for $x \in \ell_1$ and \mathbb{Y} be an arbitrary normed space with the norm $\|\cdot\|$. For each $n \in \mathbb{N}$ let e_n denote the k th unit vector in ℓ_1 (i.e., e_k is just the sequence whose only nonzero entry is a “1” in the k th coordinate). Then, $\mathbb{X}^* = \ell_1^* = \ell_\infty$ contains the countable total subset $\{e_1, e_2, \dots\}$. Assume that \mathbb{U} is a weakly compact convex subset of ℓ_1 which contains the subset $\{0, e_1\}$. Define $d : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ as

$$d(x, y) := \sum_n 2^{-n} |\langle e_n, x - y \rangle|,$$

for all $x, y \in \mathbb{U}$ and let $\mathbb{H} := \{x \in \mathbb{U} : d(x, 0) = d(e_1, 0) = \frac{1}{2}\}$. Then (\mathbb{U}, d) is a metric space and \mathbb{U} in its weak topology, is homeomorphic to this metric space, and hence is itself metrizable. Now let $f : \ell_1 \rightarrow \mathbb{Y}$ be an arbitrary injective continuous linear function (norm \rightarrow norm). In [Example 3.1](#), we observed that $f|_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{Y}$ (the restriction of f to \mathbb{U}) with \mathbb{U} equipped with the subspace topology (weak), is homogeneously continuous at 0 on \mathbb{U} (and so on \mathbb{H}). On the other hand, $\inf_{\mathbf{v} \in \mathbb{H}} f'_i(0, \mathbf{v}) > 0$. Indeed, for each $\mathbf{v}_0 \in \mathbb{H}$ one has

$$f'_i(0, \mathbf{v}_0) = \liminf_{t \downarrow 0} \frac{\|f(t\mathbf{v}_0) - f(0)\|}{t} = \|f(\mathbf{v}_0)\|.$$

Since $\mathbf{v}_0 \in \mathbb{H}$, thus $\mathbf{v}_0 \neq 0$ and since f is injective thus $\|f(\mathbf{v}_0)\| > 0$. By Schur's Theorem, the function $\mathbf{v} \mapsto \|f|_{\mathbb{H}}(\mathbf{v})\|$ is continuous (weak \rightarrow norm), and \mathbb{H} is weakly compact, thus the function $\mathbf{v} \mapsto \|f|_{\mathbb{H}}(\mathbf{v})\|$ attains its minimum at some $\mathbf{v}_1 \in \mathbb{H}$. It follows that

$$\inf_{\mathbf{v} \in \mathbb{H}} f'_i(0, \mathbf{v}) = \inf_{\mathbf{v} \in \mathbb{H}} \|f(\mathbf{v})\| = \inf_{\mathbf{v} \in \mathbb{H}} \|f|_{\mathbb{H}}(\mathbf{v})\| = \|f|_{\mathbb{H}}(\mathbf{v}_1)\| > 0,$$

proving the above claim. Thus the function f satisfies the conditions of [Theorem 3.3](#). Hence, if $\inf_{v \in \mathbb{H}} \|f(v)\| > \kappa$, then there exists a weak neighborhood \mathbb{O} of 0 in ℓ_1 such that

$$d(x, 0) \leq \frac{d(e_1, 0)}{\kappa} \|f(x) - f(0)\|,$$

for all $x \in \mathbb{O} \cap \mathbb{U}$. Hence

$$\sum_n 2^{-n} |\langle e_n, x \rangle| \leq \frac{1}{2\kappa} \|f(x)\|,$$

for all $x \in \mathbb{O} \cap \mathbb{U}$. In particular, by letting $\mathbb{Y} := \mathbb{R}$ we deduce that for every functional $x^* \in \ell_1^* = \ell_\infty$ and every $0 < \beta < 2 \inf_{v \in \mathbb{H}} |\langle x^*, v \rangle|$ there exists a weak neighborhood \mathbb{O} of 0 in ℓ_1 such that

$$\beta \sum_n 2^{-n} |\langle e_n, x \rangle| \leq |\langle x^*, x \rangle|,$$

for all $x \in \mathbb{O} \cap \mathbb{U}$.

When \mathbb{X} is a Hilbert space having a countable basis ($\dim \mathbb{X} = \aleph_0$), we get the following result.

Corollary 3.2 *Let \mathbb{X} be a Hilbert space having a countable basis, \mathbb{Y} be a Banach space and $\bar{x} \in \mathbb{B}_{\mathbb{X}}$. Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is homogeneously continuous at \bar{x} on $\mathbb{B}_{\mathbb{X}}$ (weak \rightarrow norm). If there exists some $\kappa > 0$ such that*

$$\inf \left\{ f'_l(\bar{x}, v) : \frac{1}{2} \leq \|v\| \leq 1 \right\} > \kappa,$$

then there exist a sequence $(\hbar_n)_n \in \mathbb{S}_{\mathbb{X}}$ and a weak neighborhood \mathbb{O} of \bar{x} in \mathbb{X} such that

$$\sum_n 2^{-n} |\langle \hbar_n, x - \bar{x} \rangle| \leq \frac{1}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{O} \cap \mathbb{B}_{\mathbb{X}}$.

Proof Since \mathbb{X} is a Hilbert space with a countable basis, thus \mathbb{X} is separable (see [\[21\]](#)). Suppose that $(x_n)_n$ be a dense sequence in the unit sphere of \mathbb{X} . Thanks to the Hahn-Banach theorem (see [\[23\]](#)), there must exist a sequence $(\hbar_n)_n \in \mathbb{X}^* = \mathbb{X}$ such that $\|\hbar_n\| = \langle \hbar_n, x_n \rangle = \|x_n\| = 1$; not only is the family $(\hbar_n)_n$ total in \mathbb{X} , but given $x \in \mathbb{X}$ we have

$$\|x\| = \sup\{|\langle \hbar_n, x \rangle| : n \in \mathbb{N}\}. \quad (3.3)$$

Define $d : \mathbb{B}_{\mathbb{X}} \times \mathbb{B}_{\mathbb{X}} \rightarrow \mathbb{R}$ as

$$d(x, y) := \sum_n 2^{-n} |\langle \hbar_n, x - y \rangle|,$$

for all $x, y \in \mathbb{B}_{\mathbb{X}}$. Let $\alpha := d(x_1, 0)$ and $\mathbb{H} := \{x \in \mathbb{B}_{\mathbb{X}} : d(x, 0) = \alpha\}$. Note that $\frac{1}{2} \leq \alpha \leq 1$. Indeed

$$\begin{aligned} \alpha &= d(x_1, 0) \\ &= \sum_n 2^{-n} |\langle \hbar_n, x_1 \rangle| \\ &\leq \sum_n 2^{-n} \|\hbar_n\| \|x_1\| \\ &= \sum_n 2^{-n} = 1. \end{aligned}$$

And

$$\alpha = d(x_1, 0)$$

$$\begin{aligned}
&= \sum_n 2^{-n} |\langle \tilde{h}_n, x_1 \rangle| \\
&\geq 2^{-1} |\langle \tilde{h}_1, x_1 \rangle| \\
&= \frac{1}{2}.
\end{aligned}$$

On the other hand, for every $x \in \mathbb{H}$ we have

$$\|x\| = \sup\{|\langle \tilde{h}_n, x \rangle| : n \in \mathbb{N}\},$$

by (3.3). Hence $\|x\| \geq |\langle \tilde{h}_n, x \rangle|$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\|x\| &= \sum_n 2^{-n} \|x\| \\
&\geq \sum_n 2^{-n} |\langle \tilde{h}_n, x \rangle| \\
&= d(x, 0) = \alpha.
\end{aligned}$$

Hence $\mathbb{H} \subset \{x \in \mathbb{B}_{\mathbb{X}} : \|x\| \geq \frac{1}{2}\}$. Thus $\inf_{v \in \mathbb{H}} f'_i(\bar{x}, v) > \kappa$, by hypothesis. We now apply [Theorem 3.3](#). It follows that there exists a weak neighborhood \mathbb{O} of \bar{x} in \mathbb{X} such that

$$\sum_n 2^{-n} |\langle \tilde{h}_n, x - \bar{x} \rangle| \leq \frac{\alpha}{\kappa} \|f(x) - f(\bar{x})\| \leq \frac{1}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{O} \cap \mathbb{B}_{\mathbb{X}}$. This completes the proof. \square

4 Applications: On Hoffman's estimation

Theorem 4.1 (Hoffman, 1952) [24, 25] *Let $(x_i^*)_{i=1,2,\dots,k}$ be a finite family of linear forms on \mathbb{R}^n . Set*

$$\mathbb{C}_0 := \{x \in \mathbb{R}^n \text{ such that } \langle x_i^*, x \rangle \leq 0, \quad (i = 1, 2, \dots, k)\}, \quad (4.1)$$

$\Phi(x) := \max\{\langle x_i^*, x \rangle, \quad i = 1, 2, \dots, k\}$ and $[\Phi(x)]_+ := \max(\Phi(x), 0)$. Then, there exists $\kappa > 0$ such that

$$\text{dist}(x, \mathbb{C}_0) \leq \kappa [\Phi(x)]_+. \quad (4.2)$$

Now let \mathbb{X} and \mathbb{Y} be arbitrary normed spaces, $A : \mathbb{X} \rightarrow \mathbb{Y}$ a continuous linear function and $x_i^* \in X^*$, $i = 1, 2, \dots, k$ some given functionals. Let

$$\mathbb{C} := \{x \in \mathbb{X} : A(x) = 0, \langle x_i^*, x \rangle \leq 0, \quad i = 1, 2, \dots, k\}.$$

We have the following estimation due to Ioffe.

Theorem 4.2 (Ioffe, 1979) [26] *If $\text{Ran} A$, the range of A , is closed, then there exists some $\kappa > 0$ such that*

$$\text{dist}(x, \mathbb{C}) \leq \kappa \left(\|A(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right), \quad (4.3)$$

for all $x \in \mathbb{X}$.

Now set $\mathbb{G} := \text{Ker} A \cap (\cap_{i=1}^k \text{Ker} x_i^*)$. Ioffe's Theorem yields the following estimation.

Corollary 4.1 *If $\text{Ran}(A)$ is closed, then there exists some $\kappa > 0$ such that*

$$\text{dist}(x, \mathbb{G}) \leq \kappa \left(\|A(x)\| + \sum_{i=1}^k |\langle x_i^*, x \rangle| \right), \quad (4.4)$$

for all $x \in \mathbb{X}$.

Proof Define the functionals $y_i^*, i = 1, \dots, 2k$ as

$$y_i^* := \begin{cases} x_i^* & \text{if } 1 \leq i \leq k, \\ -x_{i-k}^* & \text{if } k < i \leq 2k. \end{cases} \quad (4.5)$$

Obviously $\mathbb{G} = \mathbb{C} = \{x \in \mathbb{X} : A(x) = 0, \langle y_i^*, x \rangle \leq 0, i = 1, \dots, 2k\}$. One has

$$\begin{aligned} \sum_{i=1}^{2k} [\langle y_i^*, x \rangle]_+ &= \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ + \sum_{i=k+1}^{2k} [\langle -x_{i-k}^*, x \rangle]_+ \\ &= \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ + \sum_{i=1}^k [\langle -x_i^*, x \rangle]_+ \\ &= \sum_{i=1}^k ([\langle x_i^*, x \rangle]_+ + [\langle -x_i^*, x \rangle]_+) = \sum_{i=1}^k |\langle x_i^*, x \rangle|. \end{aligned}$$

Thus, applying [Theorem 4.2](#) to the functionals $y_i^*, i = 1, \dots, 2k$ and to the function A , ensures the existence of some $\kappa > 0$ such that

$$\text{dist}(x, \mathbb{G}) \leq \kappa \left(\|A(x)\| + \sum_{i=1}^k |\langle x_i^*, x \rangle| \right),$$

for all $x \in \mathbb{X}$. □

In the above results, the image of the function A is supposed to be closed. There are many examples of continuous linear functions which fail to have a closed image. This is the case of the function $A : L_1(\mathbb{R}) \rightarrow L_1(\mathbb{R})$ defined as $A(f) := gf$ where $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$g(t) := \frac{1}{1+t^2}.$$

Functions with compact support are dense in $L_1(\mathbb{R})$, hence $A(L^1(\mathbb{R}))$ is dense in $L_1(\mathbb{R})$. One can easily show that $A(L_1(\mathbb{R})) \neq L^1(\mathbb{R})$; hence $A(L_1(\mathbb{R}))$ is not closed in $L_1(\mathbb{R})$. As another example see [Example 4.1](#) below.

In the sequel, we aim to achieve some estimations similar to the one obtained in (4.4) without using the closedness of the range of A .

Theorem 4.3 *Let \mathbb{X} and \mathbb{Y} be normed spaces, $A : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous linear function and $x_i^* \in \mathbb{X}^*, i = 1, 2, \dots, k$ be given. Suppose that $L : \mathbb{X} \rightarrow \mathbb{X}$ is a continuous linear function such that $\text{Ker } L = \mathbb{G}$. Then, for every finite dimensional subspace $\mathbb{U} \subset \mathbb{X}$ one has*

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma_{\mathbb{U}}} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right),$$

for every $x \in \mathbb{U}$ where $\gamma_{\mathbb{U}}$ is a positive real number given as

$$\gamma_{\mathbb{U}} := \inf \left\{ \|L(u)\| + \sum_{i=1}^n [\langle x_i^*, u \rangle]_+ : u \in \mathbb{U}, \text{dist}(u, \mathbb{G}) = 1 \right\}.$$

Proof Let us consider the quotient spaces $\mathcal{M} := \frac{\mathbb{X}}{\mathbb{G}}$ and $\mathcal{U} := \frac{\mathbb{U}}{\mathbb{G}}$. Denote by $[x]$ the equivalence class containing x in \mathcal{M} (and \mathcal{U}), that is $[x] := x + \mathbb{G}$. We note $\|[x]\| := \inf\{\|x+y\| : y \in \mathbb{G}\}$. Obviously \mathcal{U} is a finite dimensional subspace of \mathcal{M} . Let $\mathbb{S}_{\mathcal{M}} := \{[x] \in \mathcal{M} : \|[x]\| = 1\}$. Consider the continuous linear function $\bar{L} : \mathcal{M} \rightarrow \mathbb{X}$ defined as $\bar{L}([x]) := L(x)$ for all $[x] \in \mathcal{M}$. Also for each $1 \leq i \leq k$ define $\langle [x]_i^*, [x] \rangle := \langle x_i^*, x \rangle$ for all $[x] \in \mathcal{M}$. Obviously each $[x]_i^*$

belongs to \mathbb{G}^\perp and hence, belongs to the dual of \mathcal{M} (which is isometrically isomorphic to \mathbb{G}^\perp [21]). Set

$$\bar{\mathcal{C}} := \{[x] \in \mathcal{M} : \bar{L}([x]) = 0, \langle [x_i]^*, [x] \rangle \leq 0, \quad i = 1, 2, \dots, k\}.$$

We have $\text{Ker } \bar{L} = \text{Ker } L = \mathbb{G}$ and therefore $\bar{\mathcal{C}} = \{[0]\}$. Now define the function $f : \mathcal{M} \rightarrow \mathbb{R}$ as

$$f([x]) := \|\bar{L}([x])\| + \sum_{i=1}^k [\langle [x_i]^*, [x] \rangle]_+.$$

We now verify the conditions of [Theorem 3.2](#) for the function f at $[\bar{x}] = [0]$. For the sake of convenience let $\mathcal{E} := \mathbb{S}_{\mathcal{M}} \cap \mathcal{U}$. Obviously

$$\mathcal{E} = \{u \in \mathbb{U} : \inf\{\|u + y\| : y \in \mathbb{G}\} = 1\} = \{u \in \mathbb{U} : \text{dist}(u, \mathbb{G}) = 1\}. \quad (4.6)$$

Then, for all $[v] \in \mathcal{E}$ one has

$$\lim_{t \downarrow 0, \vartheta \rightarrow v} \frac{f([t\vartheta]) - f([0])}{t} = \|\bar{L}([v])\| + \sum_{i=1}^k [\langle [x_i]^*, [v] \rangle]_+.$$

Thus f is homogeneously continuous at $[0]$ on \mathcal{E} , by [Corollary 2.1](#). We also have

$$f'([0], [v]) = \|\bar{L}([v])\| + \sum_{i=1}^k [\langle [x_i]^*, [v] \rangle]_+ = f([v]) = \|L(v)\| + \sum_{i=1}^k [\langle x_i^*, v \rangle]_+. \quad (4.7)$$

If $[v] \in \mathcal{E}$, then $[v] \notin \bar{\mathcal{C}}$ and therefore

$$\|\bar{L}([v])\| + \sum_{i=1}^k [\langle [x_i]^*, [v] \rangle]_+ > 0.$$

The continuity of f and the compactness of \mathcal{E} imply that $f|_{\mathcal{E}}$ attains its minimum at some $[\bar{v}] \in \mathcal{E}$. Then, $f([\bar{v}]) > 0$, by the above discussion. Hence by (4.6) and (4.7) we obtain

$$\begin{aligned} \gamma_{\mathbb{U}} &= \inf \left\{ \|L(u)\| + \sum_{i=1}^n [\langle x_i^*, u \rangle]_+ : u \in \mathbb{U}, \text{dist}(u, \mathbb{G}) = 1 \right\} \\ &= \inf_{[u] \in \mathcal{E}} f'([0], [u]) \\ &= f([\bar{v}]) > 0. \end{aligned}$$

Let $0 < \kappa < \gamma_{\mathbb{U}}$. [Theorem 3.2](#) yields that, there exists some $\delta > 0$ such that

$$\|[x] - [0]\| = \|[x]\| \leq \frac{1}{\kappa} \left(\|\bar{L}([x])\| + \sum_{i=1}^k [\langle [x_i]^*, [x] \rangle]_+ \right),$$

for all $[x] \in \mathbb{B}_{\mathcal{M}}^\circ([0], \delta) \cap \mathcal{U}$. Since f is sublinear, thus

$$\|[x]\| \leq \frac{1}{\kappa} \left(\|\bar{L}([x])\| + \sum_{i=1}^k [\langle [x_i]^*, [x] \rangle]_+ \right),$$

for all $[x] \in \mathcal{U}$. It follows that

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\kappa} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right),$$

for all $x \in \mathbb{U}$. Letting $\kappa \rightarrow \gamma_{\mathbb{U}}$ in the above inequality proves the claim. \square

Remark 4.1 When \mathbb{X} is separable, the existence of the linear continuous function $L : \mathbb{X} \rightarrow \mathbb{X}$ discussed in [Theorem 4.3](#) is straightforward. Indeed, \mathbb{G} is a closed subspace of \mathbb{X} and \mathbb{X} is separable, thus there exists a continuous linear function $L : \mathbb{X} \rightarrow \mathbb{X}$ with $\text{Ker } L = \mathbb{G}$ (see [27]).

Corollary 4.2 *Let \mathbb{X} and \mathbb{Y} be normed spaces, $A : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous linear function and $x_i^* \in \mathbb{X}^*$, $i = 1, 2, \dots, k$ be given. Suppose that $L : \mathbb{X} \rightarrow \mathbb{X}$ is a continuous linear function such that $\text{Ker } L = \mathbb{G}$. Set*

$$\gamma := \inf \left\{ \|L(u)\| + \sum_{i=1}^n [\langle x_i^*, u \rangle]_+ : u \in \mathbb{X}, \text{dist}(u, \mathbb{G}) = 1 \right\}.$$

If $\gamma > 0$, then

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right),$$

for every $x \in \mathbb{X}$.

Proof Let $x \in \mathbb{X}$ and \mathbb{U} be the subspace generated by the singleton $\{x\}$. By [Theorem 4.3](#) we have

$$\text{dist}(t, \mathbb{G}) \leq \frac{1}{\gamma_{\mathbb{U}}} \left(\|L(t)\| + \sum_{i=1}^k [\langle x_i^*, t \rangle]_+ \right),$$

for all $t \in \mathbb{U}$ where

$$\gamma_{\mathbb{U}} := \inf \left\{ \|L(u)\| + \sum_{i=1}^n [\langle x_i^*, u \rangle]_+ : u \in \mathbb{U}, \text{dist}(u, \mathbb{G}) = 1 \right\}.$$

Note that $\gamma_{\mathbb{U}} \geq \gamma > 0$. Hence

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right).$$

This completes the proof. \square

Corollary 4.3 *Let \mathbb{X} be a normed space and $A : \mathbb{X} \rightarrow \mathbb{X}$ be a continuous linear function. Then, for every finite dimensional subspace $\mathbb{U} \subset \mathbb{X}$ one has*

$$\text{dist}(x, \text{Ker } A) \leq \frac{\|A(x)\|}{\inf \{\|A(u)\| : u \in \mathbb{U}, \text{dist}(u, \text{Ker } A) = 1\}},$$

for all $x \in \mathbb{U}$.

Proof In [Theorem 4.3](#), let $\mathbb{X} = \mathbb{Y}$ and $x_i^* \equiv 0$ for all $1 \leq i \leq k$. Then, $\mathbb{G} = \text{Ker } A$. Now take $L := A$ and use [Theorem 4.3](#). \square

Corollary 4.4 *Let \mathbb{X} be a normed space and $A : \mathbb{X} \rightarrow \mathbb{X}$ be a continuous linear function. Set*

$$\sigma := \inf \{\|A(u)\| : u \in \mathbb{X}, \text{dist}(u, \text{Ker } A) = 1\}$$

If $\sigma > 0$, then

$$\text{dist}(x, \text{Ker } A) \leq \frac{1}{\sigma} \|A(x)\|, \quad (4.8)$$

for all $x \in \mathbb{X}$. Consequently, if A is injective and $\sigma > 0$, then $\text{Ran } A$ is closed.

Proof The proof of the first assertion is similar to that of [Corollary 4.2](#); we therefore omit it. To prove the second claim just note that by (4.8), we have $\|x\| \leq \frac{1}{\sigma} \|A(x)\|$ for every $x \in \mathbb{X}$. Hence \mathbb{X} and $\text{Ran } A$ are isomorphic. This completes the proof. \square

Example 4.1 Let $\mathbb{X} = \mathbb{Y} = \ell_2 := \{x := (x_n) \in \mathbb{R} : \sum_n x_n^2 < \infty\}$ with the usual norm $\|x\|_{\ell_2}^2 := \sum_n x_n^2$ for all $x \in \ell_2$. Define $A : \ell_2 \rightarrow \ell_2$ as

$$A(x = (x_1, x_2, \dots)) := \sum_n \frac{\langle e_n, x \rangle}{n} e_n = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots),$$

for all $x \in \ell_2$. One can easily check that A is linear and injective. We have

$$\|A(x)\|_{\ell_2}^2 = \sum_n \left(\frac{x_n}{n}\right)^2 \leq \sum_n x_n^2 = \|x\|^2.$$

Hence A is a bounded linear function ($A \in \mathcal{B}(\ell_2)$) with $\|A\| \leq 1$ (note that $\|A(e_1)\|_{\ell_2} = 1$ and therefore $\|A\| = 1$). One can also check that A fails to have a closed range. To see this let $y = (y_1, y_2, \dots) \in \text{Ran}A$. Thus $A(x = (x_1, x_2, \dots)) = y$ for some $x \in \ell_2$. It follows that $x_n = ny_n$ for all $n \in \mathbb{N}$. Hence $y \in \text{Ran}A$ if and only if $\sum_n n^2 y_n^2 < \infty$. Thus for each $i \in \mathbb{N}$ the vector $y_i = (1, 2^{-(\frac{3}{2} + \frac{1}{i})}, 3^{-(\frac{3}{2} + \frac{1}{i})}, \dots) \in \text{Ran}A$ and the vector $y = (1, 2^{-\frac{3}{2}}, 3^{-\frac{3}{2}}, \dots) \notin \text{Ran}A$ while $y_i \rightarrow y$. This implies that $\text{Ran}A$ is not closed. By [Corollary 4.4](#) we obtain $\sigma := \inf\{\|A(u)\| : u \in \mathbb{S}_{\mathbb{X}}\} = 0$. [Corollary 4.3](#) yields

$$\text{dist}(x, \text{Ker}A) = \|x\|_{\ell_2} \leq \frac{\|A(x)\|_{\ell_2}}{\inf\{\|A(u)\|_{\ell_2} : u \in \mathbb{U} \cap \mathbb{S}_{\mathbb{X}}\}},$$

for every finite dimensional subspace $\mathbb{U} \subset \mathbb{X}$ and for all $x \in \mathbb{U}$.

5 Sharp minimizer, strongly regular points and local efficient solutions

As we said at the beginning of this paper, if \bar{x} is a sharp local minimizer of the extended real-valued function f with constant $\tau > 0$, then \bar{x} is a strongly regular point of f . As the above discussion shows, the converse may fail to hold. But, a strongly regular point \bar{x} which is a local minimizer of f is a sharp local minimizer of the extended real-valued function f .

In this section, we claim that a strongly regular point \bar{x} which is a local minimizer of f is a local efficient solution of the subdifferential mapping ∂f at \bar{x} .

Recall that the Moreau-Rockafellar subdifferential $\partial f : \mathbb{X} \rightrightarrows \mathbb{X}^*$ of an extended-real-valued convex function f is defined by

$$\partial f(x) := \{x^* \in \mathbb{X}^* : \langle x^*, y \rangle \leq f(x+y) - f(x) \quad \forall y \in \mathbb{X}\}.$$

To prove this claim, we need to recall the notion of efficient solution (see [\[28\]](#) and references therein).

Definition 5.1 Let $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping and $\bar{x} \in \mathbb{X}$. For any subset of \mathbb{X} , note $F(\mathbb{U}) := \cup_{x \in \mathbb{U}} F(x)$. Given $\varepsilon > 0$ and \mathbb{C} a closed convex cone supposed to be pointed, i.e. $\mathbb{C} \cap -\mathbb{C} = \{0\}$, we say that $\bar{y} \in F(\bar{x})$ is a *local weak ε -efficient element* of F with respect to \mathbb{C} if there exist a neighborhood $\mathbb{O} \subset \mathbb{X}$ of \bar{x} and an element $y_\varepsilon \in \mathbb{Y}$ such that $\|y_\varepsilon\| < \varepsilon$ and

$$F(\mathbb{O}) \cap \{\bar{y} - y_\varepsilon - \text{int}C\} = \emptyset.$$

When this case occurs we say that \bar{x} is a *local weak ε -efficient solution* of F with respect to \mathbb{C} .

In the next corollary, we assume that \mathbb{X} is a finite-dimensional space partially ordered by a pointed closed convex cone \mathbb{C} with nonempty interior ($\text{int}C \neq \emptyset$).

Corollary 5.1 ([28] Corollary 5) *Suppose that $f : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous proper function on a finite-dimensional vector space \mathbb{X} . If $0 \in \text{int} \partial f(\bar{x})$ for some $\bar{x} \in \text{int} \text{dom} f$, then \bar{x} is a local weak ε -efficient solution of ∂f with respect to \mathbb{C} .*

We now are completely ready to prove the above claim.

Proposition 5.1 *Let $f : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued convex lower semicontinuous proper function on a finite-dimensional vector space \mathbb{X} and $\varepsilon > 0$. If $\bar{x} \in \text{int dom } f$ is a strongly regular point of f and if $0 \in \partial f(\bar{x})$, then \bar{x} is a local weak ε -efficient solution of ∂f with respect to \mathbb{C} .*

Proof By hypothesis there exist a neighborhood \mathbb{U} of \bar{x} and a constant $\kappa > 0$ such that

$$\|x - \bar{x}\| \leq \kappa |f(x) - f(\bar{x})|,$$

for all $x \in \mathbb{U}$. Since $0 \in \partial f(\bar{x})$, thus \bar{x} is a local minimizer of f . Hence, in view of the above inequality, there is some $\delta > 0$ such that

$$\kappa^{-1} \|y\| \leq f(\bar{x} + y) - f(\bar{x}),$$

for all $y \in \mathbb{B}_{\mathbb{X}}^{\circ}(0, \delta)$. This last inequality implies that $\kappa^{-1} \partial \|\cdot\|(0) \subset \partial f(\bar{x})$ (where $\partial \|\cdot\|(0)$ represents the subdifferential of the function $x \mapsto \|x\|$ at zero). Hence $\kappa^{-1} \mathbb{B} \subset \partial f(\bar{x})$ (see Example 2.23 in [29]). This implies that $0 \in \text{int } \partial f(\bar{x})$. Now apply Corollary 5.1. \square

In the proof of Proposition 5.1 we saw that, for an extended-real-valued convex lower semicontinuous proper function f , if $\bar{x} \in \text{int dom } f$ is a strongly regular point of f and $0 \in \partial f(\bar{x})$, then one has $\kappa^{-1} \mathbb{B} \subset \partial f(\bar{x})$. As remarked by Thibault in his forthcoming book, sharp minimizers of convex functions enjoy a similar property.

Proposition 5.2 (Thibault) [30] *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued function defined on a normed space X and let $\bar{x} \in \text{dom } f$. If \bar{x} is a sharp minimizer of f with constant $\tau > 0$, then*

$$\tau \mathbb{B}_{X^*} \subset \partial f(\bar{x}).$$

6 Conclusion

In this work we studied the notion of strongly regular points for functions acting between topological vector spaces whose topologies are generated by metrics. The following observations are the main achievements of this study:

- ✓ By using a notion of lower directional derivative, we have obtained sufficient conditions guaranteeing that a given point \bar{x} be a strongly regular point;
- ✓ Some estimations for the approximate solutions of finite systems of linear inequalities and linear equalities, were proven;
- ✓ A relationship between the notions of strongly regular points, local minimizer and local weak ε -efficient solution of the subdifferential of an extended-real-valued lower semicontinuous convex function with respect to any pointed closed convex cone was obtained.

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8 Declarations

Competing interests:

The authors declare that they have no competing interests.

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