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► **To cite this version:**

Imane Akjouj, Jamal Najim. Feasibility of sparse large Lotka-Volterra ecosystems. *Journal of Mathematical Biology*, 2022, 85 (66), pp.1-28. 10.1007/s00285-022-01830-3 . hal-03435163

HAL Id: hal-03435163

<https://hal.science/hal-03435163>

Submitted on 19 Nov 2021

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Feasibility of sparse large Lotka-Volterra ecosystems

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Abstract

Consider a large ecosystem (foodweb) with n species, where the abundances follow a Lotka-Volterra system of coupled differential equations. We assume that each species interacts with $d = d_n$ other species and that their interaction coefficients are independent random variables. This parameter d reflects the connectance of the foodweb and the sparsity of its interactions especially if d is much smaller than n . We address the question of feasibility of the foodweb, that is the existence of an equilibrium solution of the Lotka-Volterra system with no vanishing species. We establish that for a given range of d , namely $d \propto n$ or $d \geq \log(n)$ with an extra condition on the sparsity structure, there exists an explicit threshold depending on n and d and reflecting the strength of the interactions, which guarantees the existence of a positive equilibrium as the number of species n gets large. From a mathematical point of view, the study of feasibility is equivalent to the existence of a positive solution \mathbf{x}_n (component-wise) to the equilibrium linear equation:

$$\mathbf{x}_n = \mathbf{1}_n + \mathbf{M}_n \mathbf{x}_n,$$

where $\mathbf{1}_n$ is the $n \times 1$ vector with components 1 and \mathbf{M}_n is a large sparse random matrix, accounting for the interactions between

species. The analysis of such positive solutions essentially relies on large random matrix theory for sparse matrices and Gaussian concentration of measure. The stability of the equilibrium is established. The results in this article extend to a sparse setting the results obtained by Bizeul and Najim in [1].

Keywords: Theoretical ecology, Foodwebs, Feasibility and stability, Lotka-Volterra systems, Large random matrices, Gaussian concentration.

MSC Classification 2010: Primary 15B52 , 60G70 , Secondary 60B20 , 92D40.

1 Introduction

Lotka-Volterra system of coupled differential equations.

Large Lotka-Volterra (LV) systems are widely used in mathematical biology and ecology to model populations with interactions [2–4].

For a given foodweb, denote by $\mathbf{x}_n = (x_k(t))_{1 \leq k \leq n}$ the vector of abundances of the various species at time $t \geq 0$. In a LV system, the abundances are connected via the following coupled equations:

$$\frac{dx_k(t)}{dt} = x_k(t) \left(r_k - x_k(t) + \sum_{\ell=1}^n M_{k\ell} x_\ell(t) \right) \quad \text{for } k \in [n] := \{1, \dots, n\},$$

where $M_n = (M_{k\ell})$ stands for the interaction matrix, and r_k for the intrinsic growth of species k . At the equilibrium $\frac{d\mathbf{x}_n}{dt} = 0$, the abundance vector $\mathbf{x}_n = (x_k)_{k \in [n]}$ is solution of the system:

$$x_k \left(r_k - x_k + \sum_{\ell \in [n]} M_{k\ell} x_\ell \right) = 0 \quad \text{for } x_k \geq 0 \quad \text{and } k \in [n]. \quad (1)$$

An important question, which motivated recent developments [1, 5], is the existence of a *feasible* solution \mathbf{x}_n to (1), that is a solution where all the x_k 's are positive, corresponding to a scenario where no species disappears. Notice that in this latter case, the system (1) takes the much simpler form:

$$\mathbf{x}_n = \mathbf{r}_n + M_n \mathbf{x}_n,$$

where $\mathbf{r}_n = (r_k)$.

Aside from the question of feasibility arises the question of *stability* : for a complex system, how likely a perturbation of the solution \mathbf{x}_n at equilibrium will return to the equilibrium? Gardner and Ashby [6] considered stability issues of complex systems connected at random. Based on the circular law for

large random matrices with i.i.d. entries, May [7] provided a complexity/stability criterion and motivated the systematic use of large random matrix theory in the study of foodwebs, see for instance Allesina et al. [8]. Recently, Stone [9] and Gibbs et al. [10] revisited the relation between feasibility and stability.

In the spirit of May¹ and in the absence of any prior information, we shall model the interactions of matrix M_n as random and in order to simplify the analysis, we will consider intrinsic growths $(r_i)_{i \in [n]}$ equal to 1, and the equations under study will take the following form in the sequel:

$$\frac{dx_k(t)}{dt} = x_k(t) \left(1 - x_k(t) + \sum_{\ell \in [n]} M_{k\ell} x_\ell(t) \right) \quad \text{for } k \in [n]. \quad (2)$$

Sparse foodwebs

One of the most important parameters of the complexity of an ecosystem is its connectance, which is the proportion of interactions between species (see for instance [11]). This corresponds to the proportion of non-zero entries in the interaction matrix M_n . May's complexity/stability criterion asserts that the instability of an ecosystem increases with the connectance (i.e. the less sparse M_n is, the more unstable is the ecosystem equilibrium). More specifically, [12] specifies that the effect of the sparsity depends on the nature of the interactions (random, predator-prey, mutualistic or competitive). In the case of random interactions, [13] supports the idea that sparse ecosystems lead to a stable equilibrium. Based on ecological and biological data (see for instance [14]), recent studies [15] suggest that foodwebs can actually be very sparse. In a recent theoretical study, [16] study the properties of sparse ecological communities in relation with the strength of interactions.

To encode this sparsity in a simple parametric way, we first consider a directed d_n -regular graph with n vertices and its associated $n \times n$ adjacency matrix $\Delta_n = (\Delta_{ij})$:

$$\Delta_{ij} = \begin{cases} 1 & \text{if there is an edge pointing from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

In the considered graph, each vertex i has d_n edges pointing from a vertex $k \in [n]$ to i , and has d_n other edges pointing from i to a vertex $\ell \in [n]$. An edge pointing from i to i is called a loop. In particular, matrix Δ_n is deterministic, has exactly d_n non-null entries per row and per column, and $n \times d_n$ non-null entries overall.

Denote by A_n a $n \times n$ matrix with independent Gaussian $\mathcal{N}(0, 1)$ entries and consider the Hadamard product matrix $\Delta_n \circ A_n = (\Delta_{ij} A_{ij})$. Let $(\alpha_n)_{n \geq 1}$

¹Beware that May did not consider LV systems but rather used a random matrix model for the Jacobian at equilibrium of a generic system of coupled differential equations.

be a positive sequence. We assume that matrix M_n has the following form

$$M_n = \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}}. \quad (3)$$

Let us comment on the normalizing factor $1/(\alpha_n \sqrt{d_n})$. Theoretical results on sparse large random matrices [17] assert that asymptotically

$$\left\| \frac{\Delta_n \circ A_n}{\sqrt{d_n}} \right\| = \mathcal{O}(1), \quad (n \rightarrow \infty)$$

where $\|\cdot\|$ stands for the spectral norm, if the degree d_n of the graph satisfies $d_n \geq \log(n)$, a condition that we will assume in the remaining of the article. In particular, normalization $1/\sqrt{d_n}$ guarantees that matrix $\Delta_n \circ A_n/\sqrt{d_n}$ has a macroscopic effect in the LV system, even for large foodwebs (large n).

The extra normalization $1/\alpha_n$ is to be tuned to get a feasible solution.

Denote by $\mathbf{1}_n$ the $n \times 1$ vector of ones and by A^\top the transpose of matrix A . In the full matrix case $\Delta_n = \mathbf{1}_n \mathbf{1}_n^\top$, [5], based on [18], proved that a feasible solution is very unlikely to exist if $\alpha_n \equiv \alpha$ is a constant. We thus consider the regime where $\alpha_n \rightarrow \infty$ and will prove that there is a sharp threshold $\alpha_n \sim \sqrt{2 \log(n)}$ above which a feasible solution exists (with high probability) and below which does not. This phase transition has already been established in [1] for the full matrix case.

One can notice that, in sparse foodwebs ($d_n < n$), the interaction coefficients can be stronger than when the interaction matrix is full (i.e. when $d_n = n$) in the sense that $\frac{1}{\sqrt{d_n}} > \frac{1}{\sqrt{n}}$.

Models and feasibility results

The sparse random matrix model under investigation is given in (3). Specifying the range of d_n and the structure of Δ_n , we introduce hereafter two models amenable to analysis.

Model (A): Block permutation matrix.

Let $n = d \times m$. Denote by \mathcal{S}_m the group of permutations of $[m] = \{1, \dots, m\}$. Given $\sigma \in \mathcal{S}_m$, consider the associated permutation matrix

$$P_\sigma = (P_{ij})_{i,j \in [m]} \quad \text{where} \quad P_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{else.} \end{cases}$$

Denote by $J_d = \mathbf{1}_d \mathbf{1}_d^\top$ the $d \times d$ matrix of ones. Assume that

- matrix M_n is given by (3),
- $d = d_n \geq \log(n)$,

- matrix Δ_n introduced in (3) is a *block-permutation* adjacency matrix given by

$$\Delta_n = P_\sigma \otimes J_d = (P_{ij} J_d)_{i,j \in [m]}, \quad (4)$$

where \otimes is the Kronecker matrix product.

Notice that Δ_n still corresponds to the adjacency matrix of a d -regular graph.

Example 1 To illustrate these definitions, we provide an example. Let $n = m \times d$ with $m = 4$ and $\sigma \in \mathcal{S}_4$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

Matrices P_σ , Δ and $\Delta \circ A$ are respectively given by:

$$P_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} J_d & 0 & 0 & 0 \\ 0 & 0 & 0 & J_d \\ 0 & J_d & 0 & 0 \\ 0 & 0 & J_d & 0 \end{pmatrix}, \quad \Delta \circ A = \begin{pmatrix} A^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & A^{(2)} \\ 0 & A^{(3)} & 0 & 0 \\ 0 & 0 & A^{(4)} & 0 \end{pmatrix},$$

where $A^{(\mu)}$ ($\mu \in [4]$) is a $d \times d$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries.

Model (B): d is proportional to n .

Assume that M_n is given by (3) and that $d = d_n$ satisfies

$$\lim_{n \rightarrow \infty} \frac{d_n}{n} = \beta > 0. \quad (5)$$

We can now state the main result of the article:

Theorem 1 *Let A_n be a $n \times n$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries and Δ_n given by Model (A) or (B). Assume that $\alpha_n \xrightarrow{n \rightarrow \infty} \infty$ and denote by*

$$\alpha_n^* = \sqrt{2 \log n}.$$

Let $\mathbf{x}_n = (x_k)_{k \in [n]}$ be the solution of

$$\mathbf{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{d_n}} (\Delta_n \circ A_n) \mathbf{x}_n. \quad (6)$$

Then

1. *If $\exists \varepsilon > 0$ such that eventually $\alpha_n \leq (1 - \varepsilon) \alpha_n^*$ then*

$$\mathbb{P} \left\{ \min_{k \in [n]} x_k > 0 \right\} \xrightarrow{n \rightarrow \infty} 0,$$

2. *If $\exists \varepsilon > 0$ such that eventually $\alpha_n \geq (1 + \varepsilon) \alpha_n^*$ then*

$$\mathbb{P} \left\{ \min_{k \in [n]} x_k > 0 \right\} \xrightarrow{n \rightarrow \infty} 1.$$

The results of Theorem 1 are illustrated in Fig. 1.

Remarks

1. By taking $d_n \geq \log(n)$, we guarantee that the spectral norm of matrix $\frac{\Delta_n \circ A_n}{\sqrt{d_n}}$ is of order $\mathcal{O}(1)$, see [17]. In particular, matrix $\left(I_n - \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}}\right)$ is invertible and the solution \mathbf{x}_n can be represented as:

$$\mathbf{x}_n = \left(I_n - \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}}\right)^{-1} \mathbf{1}_n .$$

2. An informal first-order expansion of the solution immediately explains this phase transition. If we expand the inverse matrix and neglect the remaining terms, we get

$$\mathbf{x}_n \simeq \mathbf{1}_n + \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}} \mathbf{1}_n = \mathbf{1} + \frac{\mathbf{z}_n}{\alpha_n}$$

where

$$\mathbf{z}_n = (z_i) \quad \text{and} \quad z_i = \sum_{j=1}^n \frac{(\Delta_n \circ A_n)_{ij}}{\sqrt{d_n}} .$$

Notice that the z_i 's remain i.i.d. $\mathcal{N}(0,1)$. Going one step further in the approximation yields

$$\min_{i \in [n]} x_i \simeq 1 + \frac{\min_{i \in [n]} z_i}{\alpha_n} .$$

By standard extreme value results, we have $\min_{i \in [n]} z_i \sim -\sqrt{2 \log(n)}$, hence the phase transition.

3. The component-wise positivity of the solution has been studied in the full matrix case, i.e. $\Delta_n = \mathbf{1}_n \mathbf{1}_n^\top$ and $d_n = n$, in [1] where the same phase transition phenomenon occurs. Proof of Theorem 1 can be handled as in [1] for Model (B) with non-trivial adaptations that will be specified.

In the case where $d_n \ll n$, a normalization issue occurs. To say it roughly, the Euclidian norm of vector $\mathbf{1}_n / \sqrt{d_n}$ is no longer of order $\mathcal{O}(1)$ but of order $\sqrt{n/d_n}$ and one needs to handle more carefully the sparsity of matrix Δ_n .

In this regard, the block-permutation structure of Model (A) is a technical and simplifying assumption. The problem of the component-wise positivity of \mathbf{x}_n for a general adjacency matrix Δ_n of a d -regular graph with $d \geq \log(n)$ remains open.

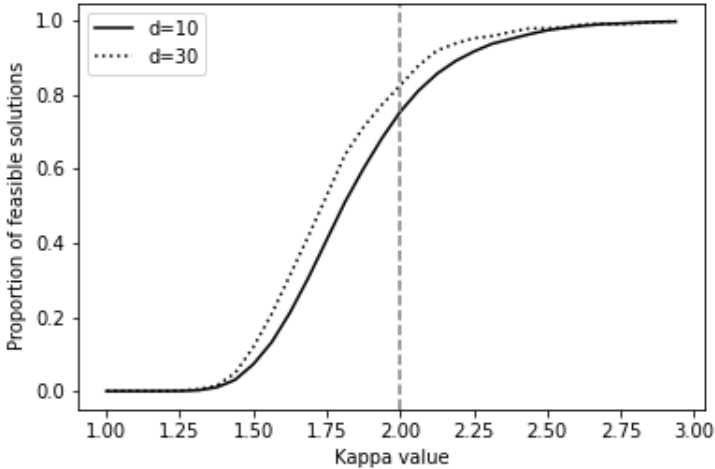


Fig. 1: Let $n = 15000$ with $\log(n) \simeq 9.61$. For $d = 10$ and $m = 1500$, we first draw at random a permutation $\sigma \in \mathcal{S}_m$ and fix $\Delta_n = P_\sigma \otimes \mathbf{1}_d \mathbf{1}_d^\top$ once for all. Each point of the solid line represents the proportion of feasible solutions \mathbf{x}_n of (6) over 2000 realizations of random matrices A_n for different values of κ , with $\alpha_n = \sqrt{\kappa \log(n)}$. The same simulation is realized with $d = 30$ over 500 realizations of A_n (dotted line).

Stability results

A classical property of (2) is the positivity of the orbits²: if $\mathbf{x}_n^0 \in (\mathbb{R}^{*+})^n$, then $\mathbf{x}_n^t \in (\mathbb{R}^{*+})^n$ as well ($t > 0$).

We first recall definitions related to stability from [19, Chapter 3]. An equilibrium \mathbf{x}_n is **stable** if for any given neighborhood W of \mathbf{x}_n , there exists a neighborhood V such that for any initial point $\mathbf{x}_n^0 \in V$, the orbit $\{\mathbf{x}_n^t; t \geq 0; \mathbf{x}_n^0 \in V\}$ stays in W . In addition, if the equilibrium is stable and the orbit converges to \mathbf{x}_n , the equilibrium is said **asymptotically stable**.

In the full matrix case ($\Delta_n = \mathbf{1}_n \mathbf{1}_n^\top$, $d_n = n$), it has been proved in [1] that in the regime where feasibility occurs, the system is asymptotically stable in the sense that the Jacobian matrix \mathcal{J} of the LV system (2) evaluated at \mathbf{x}_n :

$$\mathcal{J}(\mathbf{x}_n) = \text{diag}(\mathbf{x}_n) (-I_n + M_n) \quad (7)$$

has all its eigenvalues with negative real part.

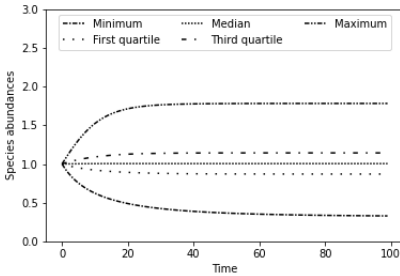
Finally, the equilibrium is **globally stable** when it is asymptotically stable and the neighborhood V can be taken as the whole state place $(\mathbb{R}^{*+})^n$.

²Beware that this property does not prevent some components $x_i(t)$ to converge to zero, hence does not enforce a feasible equilibrium.

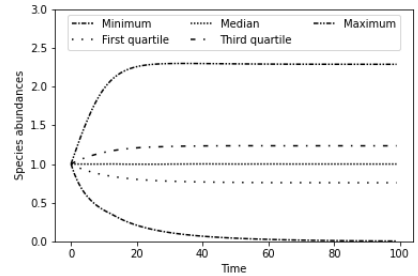
We complement Theorem 1 and prove that feasibility and global stability occur simultaneously.

Theorem 2 (Global stability, Takeuchi and Adachi [19, Theorem 3.2.1]) *Let $d_n \geq \log(n)$, $\alpha_n \xrightarrow{n \rightarrow \infty} \infty$, and Δ_n the adjacency matrix of a d_n -regular graph. Then, with probability going to one as $n \rightarrow \infty$, Eq. (1) admits a unique nonnegative solution \mathbf{x}_n . Moreover, this solution is a globally stable equilibrium.*

Beware that in this theorem, the solution, although unique, is no longer (component-wise) positive and may have zero components corresponding to vanishing species. Notice that the assumption over Δ_n covers Models (A) and (B) but is far less restrictive. We illustrate Theorem 2 in Fig. 2.



(a) Feasible equilibrium for $\alpha_n = \sqrt{3 \log(n)}$.



(b) Vanishing species for $\alpha_n = \sqrt{\log(n)}$.

Fig. 2: LV system with feasible equilibrium (left) and vanishing species (right): minimum, maximum and mean of the population dynamics (\mathbf{x}_n^t , $t > 0$) solution of (2) for $n = 5000$ ($\log(n) \simeq 8.51$), $d = 10$ and Δ_n follows Model (A). In the first figure, $\alpha_n > \sqrt{2 \log(n)}$, the minimum abundance remains positive. In the second one, $\alpha_n < \sqrt{2 \log(n)}$, the minimum abundance vanishes and the equilibrium is not feasible.

We now specify Theorem 2 in the case of feasibility.

Proposition 3 (Stability and convergence rate) *Let $d_n \geq \log(n)$, $\alpha_n \xrightarrow{n \rightarrow \infty} \infty$, and assume that Δ_n is given by Model (A) or (B). Denote by Σ_n the spectrum of the Jacobian matrix $\mathcal{J}(\mathbf{x}_n)$ given by (7).*

Assume that there exists $\varepsilon > 0$ such that eventually $\alpha_n \geq (1 + \varepsilon)\alpha_n^$. Then:*

1. *The probability that the equilibrium \mathbf{x}_n is feasible and globally stable converges to 1,*

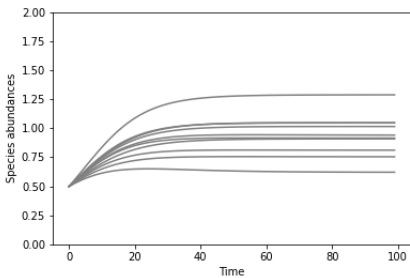
2. The spectrum Σ_n asymptotically coincides with $-\text{diag}(\mathbf{x}_n)$ in the sense that:

$$\max_{\lambda \in \Sigma_n} \min_{k \in [n]} |\lambda + x_k| \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0,$$

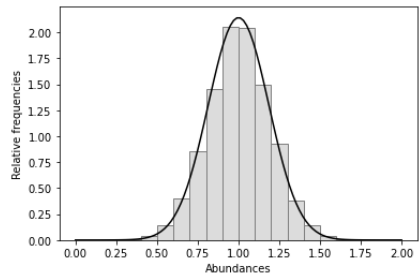
3. Moreover,

$$\max_{\lambda \in \Sigma_n} \text{Re}(\lambda) \leq -(1 - \ell^+) + o_P(1) \quad \text{where} \quad \ell^+ := \limsup_{n \rightarrow \infty} \frac{\alpha_n^*}{\alpha_n} < 1. \quad (8)$$

As a consequence of (8), for any $\mathbf{x}_n^0 \in (\mathbb{R}^{+*})^n$, the orbit \mathbf{x}_n^t converges to the equilibrium \mathbf{x}_n at an exponential convergence rate, see Fig. 3-(A).



(a) Population dynamics with starting abundances equals to $\frac{1}{2}$.



(b) Histogram of the equilibrium abundances.

Fig. 3: Consider the population dynamics $(\mathbf{x}_n^t, t > 0)$ solution of (2) where M is given by (3) and Δ_n follows Model (A) with $n = 15000$ species, $m = 1500$ blocks, $d = 10 > \log(n) \simeq 9.62$ and $\alpha_n = \sqrt{3 \log(n)}$. On the left, we plot 10 species randomly chosen over 15000 with starting abundances equals to $\frac{1}{2}$. On the right, the histogram of the abundances is represented, and the normal density with mean 1 and variance $\frac{1}{\alpha_n^2}$ is fitted. Notice the substantial spread of the abundances despite the high value of n .

Notations

If \mathbf{v} is a vector then $\|\mathbf{v}\|$ stands for its Euclidian norm; if A is a matrix then $\|A\|$ stands for its spectral norm and $\|A\|_F = \sqrt{\sum_{ij} |A_{ij}|^2}$ for its Frobenius norm. Let φ be a function from some space \mathcal{X} (usually \mathbb{R}) to \mathbb{R} then $\|\varphi\|_\infty = \sup_{x \in \mathcal{X}} |\varphi(x)|$. Convergence in probability is denoted by $\xrightarrow{\mathcal{P}}$. When no confusion can occur, we shall drop n and simply denote $A, \Delta, \alpha, d, \mathbf{x}$, etc. instead of $A_n, \Delta_n, \alpha_n, d_n, \mathbf{x}_n$, etc.

Organization of the paper

In Section 2, the spectral norm of a sparse matrix and the general strategy of proof are described. Proof of Theorem 1 is provided in Section 3 for Model (A), and in Section 4 for Model (B). Theorem 2 is proved in Section 5. In Section 6, we conclude and state an open question.

Acknowledgments

The authors thank Maxime Clénet, François Massol and Mylène Maïda for fruitful discussions and are grateful to Nick Cook for his insight on the singular values of a sparse random matrix (see Appendix A).

2 Spectral norm of the interaction matrix and strategy of proof

2.1 The spectral norm of $\Delta_n \circ A_n / \sqrt{d}$

In the following proposition which proof is based on [17], we provide an estimate of $\|\Delta \circ A / \sqrt{d}\|$. The fact that A 's entries are $\mathcal{N}(0, 1)$ and that $d_n \geq \log(n)$ is crucial.

Proposition 4 *Assume that A is a $n \times n$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries, that Δ is a $n \times n$ adjacency matrix of a d -regular graph, that $d \geq \log(n)$. Then there exists a constant $\kappa > 0$ independent from n (one can take for instance $\kappa = 22$) such that*

$$\mathbb{P} \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq \kappa \right) \xrightarrow{n \rightarrow \infty} 0.$$

In particular, let $\delta \in (0, 1)$ be fixed and $\alpha = \alpha(n) \xrightarrow{n \rightarrow \infty} \infty$. Then

$$\mathbb{P} \left(\left\| \frac{\Delta \circ A}{\alpha \sqrt{d}} \right\| \leq 1 - \delta \right) \xrightarrow{n \rightarrow \infty} 1.$$

Proof Applying [17, Corollary 3.11] to $\frac{\Delta \circ A}{\sqrt{d}}$ with $\epsilon = \frac{1}{2}$, we obtain

$$\mathbb{P} \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq 3 + \frac{15}{2\sqrt{\log \frac{3}{2}}} \times \frac{\sqrt{\log n}}{\sqrt{d}} + \frac{t}{\sqrt{d}} \right) \leq e^{-\frac{t^2}{2}}.$$

Fix $t = \sqrt{\log n}$, then $e^{-\frac{t^2}{2}} = \frac{1}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0$ and $\frac{t}{\sqrt{d}} \leq 1$ by assumption. Furthermore, there exists a rank n_1 such that for all $n \geq n_1$:

$$3 + \frac{15}{2\sqrt{\log(3/2)}} \times \frac{\sqrt{\log n}}{d} + \frac{t}{\sqrt{d}} \leq 4 + \frac{15}{2\sqrt{\log \frac{3}{2}}} < \kappa := 22.$$

Thus, $\mathbb{P} \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq \kappa \right) \rightarrow 0$. Since $\alpha \rightarrow \infty$, the last part of the proposition immediately follows. \square

2.2 Strategy of proof

Based on the previous control of the spectral norm in probability, we reduce the problem of feasibility to the control of the extreme values of high order terms of the resolvent, considered as a Neumann sum, see Lemma 5. This preliminary step is similar to [1, Section 2.1].

Going back to Eq.(6), we can write $\left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}}\right) \mathbf{x} = \mathbf{1}$. Introducing the resolvent $Q = \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}}\right)^{-1}$ which by Proposition 4 exists with probability tending to one, we obtain the representation

$$\mathbf{x} = (x_k)_k = \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}}\right)^{-1} \mathbf{1} = Q\mathbf{1}$$

which holds with growing probability. Denote by \mathbf{e}_k the $n \times 1$ k -th canonical vector, then $x_k = \mathbf{e}_k^\top \mathbf{x} = \mathbf{e}_k^\top Q\mathbf{1}$. Unfolding the resolvent as a Neumann sum, we obtain

$$x_k = \mathbf{e}_k^\top Q\mathbf{1} = \sum_{\ell=0}^{\infty} \mathbf{e}_k^\top \left(\frac{\Delta \circ A}{\alpha \sqrt{d}}\right)^\ell \mathbf{1} = 1 + \frac{Z_k}{\alpha} + \frac{R_k}{\alpha^2} \quad (9)$$

where

$$Z_k = \mathbf{e}_k^\top \left(\frac{\Delta \circ A}{\sqrt{d}}\right) \mathbf{1} \quad \text{and} \quad R_k = \mathbf{e}_k^\top \sum_{\ell=2}^{\infty} \frac{1}{\alpha^{\ell-2}} \left(\frac{\Delta \circ A}{\sqrt{d}}\right)^\ell \mathbf{1}.$$

Notice that the Z_k 's are i.i.d. $\mathcal{N}(0, 1)$ random variables and denote by $\check{M} = \min_{k \in [n]} Z_k$.

Eq. (9) immediatly yields

$$\begin{cases} \min_{k \in [n]} x_k \geq 1 + \frac{1}{\alpha} \check{M} + \frac{1}{\alpha^2} \min_{k \in [n]} R_k, \\ \min_{k \in [n]} x_k \leq 1 + \frac{1}{\alpha} \check{M} + \frac{1}{\alpha^2} \max_{k \in [n]} R_k. \end{cases} \quad (10)$$

Let $\alpha_n^* = \sqrt{2 \log n}$, $\beta_n^* = \alpha_n^* - \frac{1}{2\alpha_n^*} \log(4\pi \log n)$ and denote by $G(x) = e^{-e^{-x}}$ the cumulative distribution of a Gumbel distributed random variable. Then it is well-known, see for instance [20, Theorem 1.5.3], that

$$\mathbb{P}(\alpha_n^*(\check{M}_n + \beta_n^*) \geq x) \xrightarrow[n \rightarrow \infty]{} G(x). \quad (11)$$

By taking into account this convergence, we can rewrite (10) as

$$1 + \frac{\alpha_n^*}{\alpha_n} \left(-1 + o_P(1) + \frac{\min_{k \in [n]} R_k}{\alpha_n^* \alpha_n}\right) \leq \min_{k \in [n]} x_k \quad (12)$$

$$\leq 1 + \frac{\alpha_n^*}{\alpha_n} \left(-1 + o_P(1) + \frac{\max_{k \in [n]} R_k}{\alpha_n^* \alpha_n} \right).$$

where we used $(\alpha_n^*)^{-1}(\check{M} + \beta_n^*) = o_P(1)$. Theorem 1 will then follow from the following lemma.

Lemma 5 *Under the assumptions of Theorem 1, the following convergence holds*

$$\frac{\max_{k \in [n]} R_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad \text{and} \quad \frac{\min_{k \in [n]} R_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Proof of Lemma 5 relies on a careful analysis of the order of magnitude of the extreme values of the remaining term $(R_k)_{k \in [n]}$. The sparse structure of matrix $\Delta \circ A$ (either Model (A) or (B)) requires a specific analysis, substantially different from the one in [1].

3 Proof of Theorem 1 for Model (A)

We assume that Δ_n follows Model (A).

In order to prove Lemma 5, we first take advantage of the fact that $\|\Delta \circ A / \sqrt{d}\|$ is typically lower than κ (see Proposition 4) and replace R_k by a truncated version \tilde{R}_k (step 1). We then prove that $A \mapsto \tilde{R}_k(A)$ is Lipschitz (step 2). The quantity \tilde{R}_k being Lipschitz, its centered version is sub-Gaussian if the matrix entries are Gaussian i.i.d. We finally prove that $\tilde{R}_k(A)$ is uniformly integrable (step 3). The conclusion easily follows. Although the general strategy is similar to the one developed in [1], the proofs are substantially different. In particular, proofs of step 2 and 3 heavily rely on the block permutation structure of the matrices.

3.1 Step 1: Truncation

Toward proving Lemma 5, sub-Gaussianness is an important property, which follows from Lipschitz properties by standard concentration of measure arguments. Unfortunately $A \mapsto R_k(A)$ fails to be Lipschitz (simply notice that $R_k(A)$ has quadratic and higher order terms). In order to circumvent this issue, we provide a truncated version of R_k .

Let $\kappa > 0$ as in Prop. 4 (one can take $\kappa = 22$), $\eta \in (0, 1)$ and $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$ a smooth function:

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [0, \kappa + 1 - \eta], \\ 0 & \text{if } x \geq \kappa + 1 \end{cases}, \quad (13)$$

strictly decreasing from 1 to 0 for $x \in (\kappa + 1 - \eta, \kappa + 1)$. According to Prop. 4,

$$\varphi_d(A) := \varphi \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right)$$

is equal to one with high probability. We introduce the truncated value:

$$\tilde{R}_k(A) = \varphi_d(A)R_k(A) .$$

We have

$$\begin{aligned} \mathbb{P} \left(\max_k R_k(A) \neq \max_k \tilde{R}_k(A) \right) &\leq \mathbb{P} \left(\exists k \in [n], R_k(A) \neq \tilde{R}_k(A) \right) \\ &\leq \mathbb{P}(\varphi_d(A) < 1) \\ &\leq \mathbb{P} \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq \kappa \right) \xrightarrow{n \rightarrow \infty} 0 , \end{aligned}$$

from which we deduce

$$\frac{\max_{k \in [n]} R_k - \max_{k \in [n]} \tilde{R}_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 . \quad (14)$$

It is therefore sufficient to prove

$$\frac{\max_{k \in [n]} \tilde{R}_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad (15)$$

to establish the first part of Lemma 5. The property of the minimum can be proved similarly.

3.2 Step 2: Lipschitz property for $\tilde{R}_k(A)$

For $\ell \geq 2$, we introduce the following summand terms:

$$\rho_{k,\ell}(A) = \mathbf{e}_k^\top \frac{1}{\alpha^{\ell-2}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \mathbf{1} \quad \text{and} \quad \tilde{\rho}_{k,\ell}(A) = \varphi_d(A) \rho_{k,\ell}(A) , \quad (16)$$

so that $R_k(A) = \sum_{\ell=2}^{\infty} \rho_{k,\ell}(A)$ and $\tilde{R}_k(A) = \sum_{\ell=2}^{\infty} \tilde{\rho}_{k,\ell}(A)$.

The following lemma is the main result of this section.

Lemma 6 *Let $\kappa > 0$ as in Proposition 4, $\delta \in (0, 1)$ and n_0 such that for all $n \geq n_0$,*

$$\frac{\kappa + 1}{\alpha_n} \leq 1 - \delta .$$

For $\ell \geq 2$ and $n \geq n_0$, the function $\tilde{\rho}_{k,\ell} : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathbb{R}$ is K_ℓ -Lipschitz, i.e.

$$|\tilde{\rho}_{k,\ell}(A) - \tilde{\rho}_{k,\ell}(B)| \leq K_\ell \|A - B\|_F , \quad (17)$$

where $K_\ell = K_\ell(\kappa, n_0, \delta) > 0$ is a constant independent from k , d and $n \geq n_0$. Moreover, $K := \sum_{\ell \geq 2} K_\ell < \infty$. In particular, the function \tilde{R}_k is K -Lipschitz :

$$|\tilde{R}_k(A) - \tilde{R}_k(B)| \leq K \|A - B\|_F . \quad (18)$$

Given a $n \times n$ matrix C , we define its hermitization matrix $\mathcal{H}(C)$ by:

$$\mathcal{H}(C) = \begin{pmatrix} 0 & C \\ C^\top & 0 \end{pmatrix}.$$

A well-known property of $\mathcal{H}(C)$ is its symmetric spectrum and the fact that the singular values of C are the non-negatives eigenvalues of $\mathcal{H}(C)$. In particular, $\|C\|$ corresponds to the largest eigenvalue of $\mathcal{H}(C)$.

In order to prove Lemma 6, we first consider the case where $\mathcal{H}(\Delta \circ A)$ has a simple spectrum, a sufficient condition for the differentiability of $\|\Delta \circ A\|$, we then prove that the Euclidian norm of the gradient of $\tilde{\rho}_{k,\ell}(A)$ is bounded : $\|\nabla \tilde{\rho}_{k,\ell}(A)\| \leq K_\ell$ and finally proceed by approximation to get the general Lipschitz property.

Proof We first consider the case where $\mathcal{H}(\Delta \circ A)$ has a simple spectrum. In this case, $\|\Delta \circ A\|$ is equal to the largest eigenvalue of $\mathcal{H}(\Delta \circ A)$ which has multiplicity 1 and is thus differentiable. Denote by $\partial_{ij} = \frac{\partial}{\partial A_{ij}}$. Notice that if $\Delta_{ij} = 0$, then for any smooth function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, $\partial_{ij} f(\Delta \circ A) = 0$. If needed, we will take advantage of this property.

We have :

$$\|\nabla \tilde{\rho}_{k,\ell}(A)\| = \sqrt{\sum_{i,j=1}^n |\partial_{ij} \tilde{\rho}_{k,\ell}(A)|^2},$$

and

$$\begin{aligned} \partial_{ij} \tilde{\rho}_{k,\ell}(A) &= \partial_{ij} (\varphi_d(A) \rho_{k,\ell}(A)) \\ &= (\partial_{ij} \varphi_d(A)) \rho_{k,\ell}(A) + \varphi_d(A) \partial_{ij} \rho_{k,\ell}(A) =: S_{1,ij} + S_{2,ij}. \end{aligned}$$

In particular,

$$\sum_{i,j=1}^n |\partial_{ij} \tilde{\rho}_{k,\ell}(A)|^2 \leq 2 \sum_{i,j=1}^n |S_{1,ij}|^2 + 2 \sum_{i,j=1}^n |S_{2,ij}|^2.$$

We first evaluate $\sum_{i,j} |S_{1,ij}|^2$.

Recall that $\|\Delta \circ A\|$ being the maximum eigenvalue of $\mathcal{H}(\Delta \circ A)$ which by assumption is simple, it is differentiable by [21, Theorem 6.3.12]. Let \mathbf{u} and \mathbf{v} be respectively the left and right normalized singular vectors associated to the largest singular value $\|\Delta \circ A\|$ of $\Delta \circ A$. Then

$$\mathcal{H}(\Delta \circ A) \bar{\mathbf{w}} = \|\Delta \circ A\| \bar{\mathbf{w}}, \quad \text{where } \bar{\mathbf{w}} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

Notice that $\|\bar{\mathbf{w}}\|^2 = 2$. We have

$$\partial_{ij} \varphi_d(A) = \frac{1}{\sqrt{d}} \varphi' \left(\frac{\|\Delta \circ A\|}{\sqrt{d}} \right) \partial_{ij} \|\Delta \circ A\|$$

and

$$\partial_{ij} \|\Delta \circ A\| = \begin{cases} \frac{1}{\|\bar{\mathbf{w}}\|} (\mathbf{u}^\top \mathbf{e}_i \mathbf{e}_j^\top \mathbf{v} + \mathbf{v}^\top \mathbf{e}_j \mathbf{e}_i^\top \mathbf{u}) = \mathbf{u}^\top \mathbf{e}_i \mathbf{e}_j^\top \mathbf{v} & \text{if } \Delta_{ij} \neq 0, \\ 0 & \text{else.} \end{cases} \quad (19)$$

Let $i \in [n]$. Denote by

$$\mathcal{I}_i = \{j \in [n], \Delta_{ij} = 1\}; \quad (20)$$

notice that $\text{card}(\mathcal{I}_i) = d$. We have

$$\begin{aligned} \sum_{i,j \in [n]} |S_{1,ij}|^2 &= \sum_{i \in [n]} \sum_{j \in \mathcal{I}_i} \left| \mathbf{u}^\top \mathbf{e}_i \mathbf{e}_j^\top \mathbf{v} \varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2, \\ &\leq \left| \varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2 \sum_{i \in [n]} \left| \mathbf{u}^\top \mathbf{e}_i \right|^2 \sum_{j \in [n]} \left| \mathbf{e}_j^\top \mathbf{v} \right|^2, \\ &= \left| \varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2. \end{aligned}$$

We now focus on

$$\left| \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2 = \left| \mathbf{e}_k^\top \frac{\mathbf{1}}{\alpha^{\ell-2}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \frac{\mathbf{1}}{\sqrt{d}} \right|^2.$$

Notice that $\|\mathbf{1}/\sqrt{d}\| = \sqrt{n/d}$. Since matrix $\Delta \circ A$ follows Model (A), one can notice that $(\Delta \circ A)^\ell$ remains a block matrix with only d nonzero terms per row (and per column as well). This property is fundamental for the remaining estimates and fully relies on the Model (A) assumption.

Denote by

$$\mathcal{J}_{k,\ell} = \left\{ p \in [n], \left[(\Delta \circ A)^\ell \right]_{kp} \neq 0 \right\} \quad (21)$$

and by $\mathbf{1}^{\mathcal{J}_{k,\ell}}$ the $n \times 1$ vector with zero coordinates except those belonging to $\mathcal{J}_{k,\ell}$, set to 1. In particular, $\|\mathbf{1}^{\mathcal{J}_{k,\ell}}\| = \sqrt{d}$. Then

$$\mathbf{e}_k^\top (\Delta \circ A)^\ell \mathbf{1} = \mathbf{e}_k^\top (\Delta \circ A)^\ell \mathbf{1}^{\mathcal{J}_{k,\ell}}.$$

We have

$$\begin{aligned} \left| \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2 &= \left| \mathbf{e}_k^\top \left(\frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{\ell-2} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \frac{\mathbf{1}}{\sqrt{d}} \right|^2, \\ &= \left| \mathbf{e}_k^\top \left(\frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{\ell-2} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \frac{\mathbf{1}^{\mathcal{J}_{k,\ell}}}{\sqrt{d}} \right|^2, \\ &\leq \|\mathbf{e}_k^\top\|^2 \left\| \left(\frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{\ell-2} \right\|^2 \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^4 \left\| \frac{\mathbf{1}^{\mathcal{J}_{k,\ell}}}{\sqrt{d}} \right\|^2, \\ &\leq \left\| \frac{\Delta \circ A}{\alpha \sqrt{d}} \right\|^{2(\ell-2)} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^4. \end{aligned}$$

Using the fact that $\varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) = 0$ if $\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq \kappa + 1$, we have

$$\begin{aligned} \left| \varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \frac{\mathbf{1}}{\sqrt{d}} \rho_{k,\ell}(A) \right|^2 &\leq \left| \varphi' \left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \right|^2 \left\| \frac{\Delta \circ A}{\alpha \sqrt{d}} \right\|^{2(\ell-2)} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^4, \\ &\leq \|\varphi'\|_\infty^2 (1-\delta)^{2(\ell-2)} (1+\kappa)^4, \end{aligned}$$

and finally

$$\sum_{i,j=1}^n |S_{1,ij}|^2 \leq \|\varphi'\|_\infty^2 (1-\delta)^{2(\ell-2)} \times (1+\kappa)^4. \quad (22)$$

We now evaluate $\sum_{i,j=1}^n |S_{2,ij}|^2 = \sum_{i,j \in [n]} |\varphi_d(A) \partial_{ij} \rho_{k,\ell}(A)|^2$.

Recall the definitions of \mathcal{I}_i and $\mathcal{J}_{k\ell}$ introduced in (20), (21). We have

$$\partial_{ij}\rho_{k,\ell}(A) = \frac{1}{\alpha^{\ell-2}(\sqrt{d})^\ell} \sum_{p=0}^{\ell-1} \mathbf{e}_k^\top (\Delta \circ A)^p \mathbf{e}_i \mathbf{e}_j^\top (\Delta \circ A)^{\ell-1-p} \mathbf{1} \quad \text{if } j \in \mathcal{I}_i$$

and zero else. Then

$$\begin{aligned} \sum_{i \in [n]} \sum_{j \in \mathcal{I}_i} |\partial_{ij}\rho_{k,\ell}(A)|^2 &\leq \frac{\ell}{\alpha^{2(\ell-2)}d^\ell} \left(\sum_{i \in [n]} \sum_{j \in \mathcal{I}_i} \left| \mathbf{e}_k^\top (\Delta \circ A)^{\ell-1} \mathbf{e}_i \mathbf{e}_j^\top \mathbf{1} \right|^2 \right. \\ &\quad \left. + \sum_{p=0}^{\ell-2} \sum_{i=1}^n \sum_{j \in \mathcal{I}_i} \left| \mathbf{e}_k^\top (\Delta \circ A)^p \mathbf{e}_i \mathbf{e}_j^\top (\Delta \circ A)^{\ell-1-p} \mathbf{1} \right|^2 \right), \\ &= \frac{\ell}{\alpha^{2(\ell-2)}d^\ell} \left(d \sum_{i \in [n]} \left| [(\Delta \circ A)^{\ell-1}]_{k,i} \right|^2 \right. \\ &\quad \left. + d \sum_{p=0}^{\ell-2} \sum_{i \in [n]} \sum_{j \in \mathcal{I}_i} \left| [(\Delta \circ A)^p]_{k,i} \mathbf{e}_j^\top (\Delta \circ A)^{\ell-1-p} \frac{\mathbf{1}}{\sqrt{d}} \right|^2 \right), \\ &\leq \frac{\ell}{\alpha^{2(\ell-2)}d^{\ell-1}} \left(\left[(\Delta \circ A)^{\ell-1} \left((\Delta \circ A)^{\ell-1} \right)^\top \right]_{k,k} \right. \\ &\quad \left. + \sum_{p=0}^{\ell-2} \sum_{i \in [n]} \left| [(\Delta \circ A)^p]_{k,i} \right|^2 \sum_{j \in \mathcal{I}_i} \left| \mathbf{e}_j^\top (\Delta \circ A)^{\ell-1-p} \frac{\mathbf{1}}{\sqrt{d}} \right|^2 \right). \end{aligned} \quad (23)$$

We concentrate on the term $T = \sum_{j \in \mathcal{I}_i} \left| \mathbf{e}_j^\top (\Delta \circ A)^{\ell-1-p} \frac{\mathbf{1}}{\sqrt{d}} \right|^2$ and prove that

$$T \leq \|\Delta \circ A\|^{2(\ell-1-p)}. \quad (24)$$

Let $I_{\mathcal{I}_i} = \text{diag}(\mathbf{1}^{\mathcal{I}_i}(k); k \in [n])$, where $\mathbf{1}^{\mathcal{I}_i}$ is the $n \times 1$ vector with component 1 if it belongs to \mathcal{I}_i and zero else, then

$$T = \frac{\mathbf{1}^\top}{\sqrt{d}} \left[(\Delta \circ A)^{\ell-1-p} \right]^\top I_{\mathcal{I}_i} (\Delta \circ A)^{\ell-1-p} \frac{\mathbf{1}}{\sqrt{d}}.$$

Notice that $(\Delta \circ A)^{\ell-1-p}$ has the form $(P_\tau \otimes \mathbf{1}_d \mathbf{1}_d^\top) \circ B$ for some $\tau \in \mathcal{S}_m$ and some $n \times n$ matrix B . In particular, taking into account the matching between the indices of $I_{\mathcal{I}_i}$ and $(\Delta \circ A)^{\ell-1-p}$'s blocs, there exists a $d \times d$ bloc of matrix $(\Delta \circ A)^{\ell-1-p}$ say B_i such that matrix

$$\left[(\Delta \circ A)^{\ell-1-p} \right]^\top I_{\mathcal{I}_i} (\Delta \circ A)^{\ell-1-p}$$

is zero except a $d \times d$ bloc $B_i^\top B_i$ on the diagonal and

$$T = \frac{\mathbf{1}^\top}{\sqrt{d}} B_i^\top B_i \frac{\mathbf{1}}{\sqrt{d}} \leq \|B_i^\top B_i\| \leq \|B_i\|^2 \leq \left\| (\Delta \circ A)^{\ell-1-p} \right\|^2 \leq \|\Delta \circ A\|^{2(\ell-1-p)}.$$

Eq.(24) is established. Notice in particular that the estimate does not depend on the index i . Plugging this estimate into (23) yields

$$\sum_{i \in [n]} \sum_{j \in \mathcal{I}_i} |\partial_{ij}\rho_{k,\ell}(A)|^2$$

$$\begin{aligned}
 &\leq \frac{\ell}{\alpha^{2(\ell-2)}d^{\ell-1}} \left(\left\| (\Delta \circ A)^{\ell-1} \right\|^2 \right. \\
 &\quad \left. + \sum_{p=0}^{\ell-2} \left[((\Delta \circ A)^p)^* (\Delta \circ A)^p \right]_{kk} \|\Delta \circ A\|^{2(\ell-p-1)} \right), \\
 &\leq \frac{\ell}{\alpha^{2(\ell-2)}d^{\ell-1}} \left(\|\Delta \circ A\|^{2(\ell-1)} + \sum_{p=0}^{\ell-2} \|\Delta \circ A\|^{2p} \|\Delta \circ A\|^{2(\ell-p-1)} \right), \\
 &= \frac{\ell^2}{\alpha^{2(\ell-2)}d^{\ell-1}} \|\Delta \circ A\|^{2(\ell-1)} = \ell^2 \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{2(\ell-2)} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2.
 \end{aligned}$$

Multiplying by $|\varphi_d(A)|^2$ finally yields the appropriate estimates:

$$\begin{aligned}
 \sum_{i,j \in [n]} |S_{2,i,j}|^2 &\leq \ell^2 |\varphi_{d,\sigma}(A)|^2 \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{2(\ell-2)} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2, \\
 &\leq \ell^2 (1-\delta)^{2(\ell-2)} (1+\kappa)^2.
 \end{aligned} \tag{25}$$

Combining (22) and (25), we obtain :

$$\begin{aligned}
 \|\nabla \tilde{\rho}_{k,\ell}(A)\| &\leq \sqrt{2 \sum_{i,j=1}^n |S_{1,i,j}|^2 + 2 \sum_{i,j=1}^n |S_{2,i,j}|^2}, \\
 &\leq 2(1-\delta)^{\ell-2} (\kappa+1)^2 (\|\varphi'\|_\infty + \ell) =: K_\ell.
 \end{aligned} \tag{26}$$

where K_ℓ does not depend upon k, n, d and is summable.

So far, we have established a local estimate over $\|\nabla \tilde{\rho}_{k,\ell}(A)\|$ for any matrix A such that $\mathcal{H}(\Delta \circ A)$ has a simple spectrum. We first establish the Lipschitz estimate (17) for two such matrices A and B .

Let A, B such that $\mathcal{H}(\Delta \circ A)$ and $\mathcal{H}(\Delta \circ B)$ have simple spectrum and consider the interpolation matrix

$$A_t = (1-t)A + tB$$

for $t \in [0; 1]$. The continuity of the eigenvalues implies that there exists $\epsilon > 0$ sufficiently small such that $\mathcal{H}(\Delta \circ A_t)$ has a simple spectrum for $t \leq \epsilon$ and $t \geq 1 - \epsilon$. By an argument in [22, Chapter 2.1], the number of eigenvalues of $\mathcal{H}(\Delta \circ A_t)$ remains constant for $t \in [0, 1]$, except maybe for a finite number of points $(t_l; 1 \leq l \leq L)$: $t_0 = 0 < t_1 < \dots < t_L < t_{L+1} = 1$. Since $\mathcal{H}(\Delta \circ A_t)$ has simple spectrum for $t \in [0; \epsilon) \cup (1 - \epsilon; 1]$, it has simple spectrum for all $t \notin \{t_l, l \in [L]\}$. We can now proceed:

$$\begin{aligned}
 |\tilde{\rho}_{k,\ell}(A_{t_1}) - \tilde{\rho}_{k,\ell}(A)| &= \left| \lim_{\tau \nearrow t_1} \int_0^\tau \frac{d}{dt} \tilde{\rho}_{k,\ell}(A_t) dt \right| \\
 &= \left| \lim_{\tau \nearrow t_1} \int_0^\tau \nabla \tilde{\rho}_{k,\ell}(A_t) \circ \frac{d}{dt}(A_t) dt \right|, \\
 &\leq \lim_{\tau \nearrow t_1} \int_0^\tau \|\nabla \tilde{\rho}_{k,\ell}(A_t)\| \times \|B - A\|_F dt \leq K_\ell t_1 \|B - A\|_F.
 \end{aligned}$$

By iterating the process over the intervals (t_{l-1}, t_l) , we get

$$|\tilde{\rho}_{k,\ell}(B) - \tilde{\rho}_{k,\ell}(A)| \leq \sum_{l=1}^{L+1} |\tilde{\rho}_{k,\ell}(A_{t_l}) - \tilde{\rho}_{k,\ell}(A_{t_{l-1}})|,$$

$$\leq \sum_{l=1}^{L+1} K_\ell (t_l - t_{l-1}) \|B - A\|_F = K_\ell \|B - A\|_F.$$

Hence the Lipschitz property along the segment $[A, B]$.

To go beyond, we proceed by density and prove that for a given matrix Δ as in Model (A), the set of matrices $(\Delta \circ A)$ such that $\mathcal{H}(\Delta \circ A)$ has a simple spectrum is dense in the set of matrices $(\Delta \circ A, A \in \mathbb{R}^{n \times n})$.

Let P_σ be the permutation matrix used to define Δ in (4) and I_d the identity matrix of size d . We define the following $n \times n$ matrices

$$\Pi = P_\sigma \otimes I_d \quad \text{and} \quad D_A = (\Delta \circ A)\Pi^\top. \quad (27)$$

Notice that Π is a $n \times n$ permutation matrix and that D_A is a block diagonal matrix with $d \times d$ blocks on the diagonal. Since $\Pi\Pi^\top = \Pi^\top\Pi = I_n$, we also have

$$D_A \Pi = \Delta \circ A.$$

In the framework of Example 1, matrices Π and D_A are given by:

$$\Pi = \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & I_d \\ 0 & I_d & 0 & 0 \\ 0 & 0 & I_d & 0 \end{pmatrix} \quad \text{and} \quad D_A = \begin{pmatrix} A^{(1)} & 0 & 0 & 0 \\ 0 & A^{(2)} & 0 & 0 \\ 0 & 0 & A^{(3)} & 0 \\ 0 & 0 & 0 & A^{(4)} \end{pmatrix}.$$

An important feature of D_A is that $\Delta \circ A$ and D_A have the same singular values:

$$D_A D_A^\top = (\Delta \circ A)\Pi^\top \Pi (\Delta \circ A)^\top = (\Delta \circ A)(\Delta \circ A)^\top,$$

hence $\mathcal{H}(\Delta \circ A)$ and $\mathcal{H}(D_A)$ have the same eigenvalues and their spectrum, if simple, is simultaneously simple. Denote by $(A_{(\mu)})_{\mu \in [m]}$ the m diagonal $d \times d$ blocks of matrix D_A and consider their SVD

$$A_{(\mu)} = U_{(\mu)} \Lambda_{(\mu)} V_{(\mu)}.$$

Consider a simultaneous ε -perturbation of the $\Lambda_{(\mu)}$'s into $\Lambda_{(\mu)}^\varepsilon$ so that all the $\Lambda_{(\mu)}^\varepsilon$'s have distinct diagonal elements, ε -close to the $\Lambda_{(\mu)}$'s. Denote by

$$A_{(\mu)}^\varepsilon = U_{(\mu)} \Lambda_{(\mu)}^\varepsilon V_{(\mu)}.$$

and let D_A^ε be the block diagonal matrix with blocks $(A_{(\mu)}^\varepsilon)_{\mu \in [m]}$. Then $\mathcal{H}(D_A^\varepsilon)$ is arbitrarily close to $\mathcal{H}(D_A)$ and has a simple spectrum. Note that $D_A^\varepsilon \Pi$ is ε -close to $\Delta \circ A$, is such that $\mathcal{H}(D_A^\varepsilon \Pi)$ has a simple spectrum and has the same pattern as $\Delta \circ A$ in the sense that:

$$\Delta_{ij} = 0 \quad \Rightarrow \quad (D_A^\varepsilon \Pi)_{ij} = 0.$$

To emphasize this property, we introduce the $n \times n$ matrix A^ε defined as

$$[A^\varepsilon]_{ij} = \begin{cases} [D_A^\varepsilon \Pi]_{ij} & \text{if } \Delta_{ij} = 1, \\ A_{ij} & \text{else} \end{cases}$$

so that

$$\|\Delta \circ A^\varepsilon - \Delta \circ A\|_F = \|A^\varepsilon - A\|_F \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We can now conclude. Let $\Delta \circ A, \Delta \circ B$ be given and $D_A^\varepsilon \Pi = \Delta \circ A^\varepsilon$ and $D_B^\varepsilon \Pi = \Delta \circ B^\varepsilon$ constructed as previously; notice that $C \mapsto \tilde{\rho}_{k,\ell}(C)$ is continuous. Then

$$\begin{aligned} |\tilde{\rho}_{k,\ell}(B) - \tilde{\rho}_{k,\ell}(A)| &\leq |\tilde{\rho}_{k,\ell}(B^\varepsilon) - \tilde{\rho}_{k,\ell}(B)| + K_\ell \|B_\varepsilon - A_\varepsilon\|_F + |\tilde{\rho}_{k,\ell}(A_\varepsilon) - \tilde{\rho}_{k,\ell}(A)|, \\ &\xrightarrow{\varepsilon \rightarrow 0} K_\ell \|B - A\|_F. \end{aligned}$$

This concludes the proof of the Lipschitz property. \square

3.3 Step 3: uniform estimate for $\mathbb{E}\tilde{R}_k(A)$

As a consequence of the Lipschitz property of \tilde{R}_k , $\tilde{R}_k(A)$ if centered is sub-Gaussian if A is a $n \times n$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries. The following estimate easily follows using Tsirelson-Ibragimov-Sudakov inequality ([23, Theorem 5.5]).

Proposition 7 *Under the assumptions of Lemma 6, the following estimate holds true:*

$$\mathbb{E} \max_{k \in [n]} \left(\tilde{R}_k - \mathbb{E}\tilde{R}_k \right) \leq K \sqrt{2 \log n}.$$

For the proof, see [1, Proposition 2.3].

The rest of the section is devoted to the control of $\mathbb{E}\tilde{R}_k(A)$.

Proposition 8 *Under the assumptions of Theorem 1, there exists $n_1 \in \mathbb{N}$ and a constant $C > 0$ such that for all $n \geq n_1$,*

$$\sup_{k \in [n]} \left| \mathbb{E}\tilde{R}_k(A) \right| \leq C.$$

Proof Recall that $n = d \times m$ and that $\Delta \circ A$ is a block permutation matrix with m blocks $(A^{(\mu)})_{\mu \in [m]}$ of size $d \times d$. We choose a given block $A^{(\mu)}$ and denote by μ_1, \dots, μ_d the d indices corresponding to the rows of block $A^{(\mu)}$ in $\Delta \circ A$. By exchangeability, we have

$$\forall k \in [d], \quad \mathbb{E}\tilde{R}_{\mu_k}(A) = \mathbb{E}\tilde{R}_{\mu_1}(A).$$

Denote by $\mathbf{1}^{(\mu)}$ the $n \times 1$ vector with ones for the indices $(\mu_i)_{i \in [d]}$ and zeros elsewhere. We have

$$\begin{aligned} \left| \mathbb{E}\tilde{R}_{\mu_k}(A) \right| &= \left| \frac{1}{d} \sum_{i=1}^d \mathbb{E}\tilde{R}_{\mu_i}(A) \right| \\ &= \left| \frac{1}{d} \sum_{i=1}^d \mathbb{E} \left(\varphi_d(A) e_{\mu_i}^\top \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{-1} \mathbf{1} \right) \right|, \\ &= \left| \frac{1}{d} \mathbb{E} \left(\varphi_d(A) \mathbf{1}^{(\mu)\top} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{-1} \mathbf{1} \right) \right|, \\ &\leq \mathbb{E} \left| \varphi_d(A) \frac{\mathbf{1}^{(\mu)\top}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{-1} \frac{\mathbf{1}}{\sqrt{d}} \right|. \end{aligned} \quad (28)$$

We start by expanding $\left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{-1}$:

$$\left| \varphi_d(A) \frac{\mathbf{1}^{(\mu)\top}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \left(I - \frac{\Delta \circ A}{\alpha \sqrt{d}} \right)^{-1} \frac{\mathbf{1}}{\sqrt{d}} \right| = \left| \varphi_d(A) \sum_{\ell=2}^{\infty} \frac{\mathbf{1}^{(\mu)\top}}{\alpha^{\ell-2} \sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \frac{\mathbf{1}}{\sqrt{d}} \right|.$$

Notice that $(\Delta \circ A)^\ell$ is a block matrix constituted of m blocks of size $d \times d$. In particular, among the d row

$$\left(\left[(\Delta \circ A)^\ell \right]_{ij} \right)_{i \in \{\mu_1, \dots, \mu_d\}, j \in [n]},$$

there exist ν_1, \dots, ν_d (consecutive) indices such that the only non-null entries are

$$\left(\left[(\Delta \circ A)^\ell \right]_{ij} \right)_{i \in \{\mu_1, \dots, \mu_d\}, j \in \{\nu_1, \dots, \nu_d\}}.$$

Denote by $\mathbf{1}^{(\nu)}$ the $n \times 1$ vector of ones for the indices $(\nu_i)_{i \in [d]}$ and zeroes elsewhere. As a consequence of the previous remark,

$$\mathbf{1}^{(\mu)\top} (\Delta \circ A)^\ell \mathbf{1} = \mathbf{1}^{(\mu)\top} (\Delta \circ A)^\ell \mathbf{1}^{(\nu)}$$

and

$$\begin{aligned} \frac{1}{\alpha^{\ell-2}} \left| \frac{\mathbf{1}^{(\mu)\top}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \frac{\mathbf{1}}{\sqrt{d}} \right| &= \frac{1}{\alpha^{\ell-2}} \left| \frac{\mathbf{1}^{(\mu)\top}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \frac{\mathbf{1}^{(\nu)}}{\sqrt{d}} \right| \\ &\leq \frac{1}{\alpha^{\ell-2}} \left\| \frac{\mathbf{1}^{(\mu)}}{\sqrt{d}} \right\| \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^\ell \left\| \frac{\mathbf{1}^{(\nu)}}{\sqrt{d}} \right\|, \\ &\leq \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{\ell-2} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2. \end{aligned}$$

Let $\kappa > 0$ as in Proposition 4, $\delta \in (0, 1)$, $n_0 \in \mathbb{N}$ as in Lemma 6, then

$$\begin{aligned} \sum_{\ell=2}^{\infty} \frac{1}{\alpha^{\ell-2}} \left| \varphi_d(A) \frac{\mathbf{1}^{(\mu)*}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^\ell \frac{\mathbf{1}}{\sqrt{d}} \right| &\leq \varphi_d(A) \sum_{\ell=2}^{\infty} \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{\ell-2} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2, \\ &= \varphi_d(A) \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2 \sum_{\ell=0}^{\infty} \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^\ell \\ &\leq (1 + \kappa)^2 \sum_{\ell=2}^{\infty} (1 - \delta)^\ell, \\ &\leq \frac{(1 + \kappa)^2}{\delta}. \end{aligned}$$

Plugging this estimate into (28) concludes the proof of the estimation of $|\mathbb{E} \tilde{R}_{\mu_k}(A)|$. This estimate being uniform over μ_1, \dots, μ_d and over all the blocks $(A^{(\mu)})$, the proposition is proved. \square

3.4 Proof of lemma 5

Combining Lemma 6, Propositions 7 and 8 one can prove Lemma 5 as in [1, Section 2.3] with minor adaptations.

4 Proof of Theorem 1 for Model (B)

We assume that Δ_n follows Model (B).

The strategy of proof closely follows the one in [1], with one specific issue to handle: the uniform bound on $\mathbb{E}\tilde{R}_k$. An important property exploited in [1] to establish a uniform bound over $\mathbb{E}\tilde{R}_k$ was the exchangeability of the \tilde{R}_k 's (or block exchangeability in the case of Model (A)). There is not enough structure in Model (B) to guarantee this exchangeability (which might not hold).

We carefully address this issue hereafter.

4.1 A uniform bound over $\mathbb{E}\tilde{R}_k$ for Model (B)

Proposition 9 *Under the assumptions of Theorem 1, uniformly in $k \in [n]$,*

$$\mathbb{E}\tilde{R}_k = \mathcal{O}\left(\frac{\alpha}{\sqrt{d}}\right).$$

Proof of Proposition 9 relies on two important facts.

- The fact that almost surely $\mathcal{H}(\Delta \circ A)$ has a simple spectrum, hence the Lipschitz function $\|\Delta \circ A\|$ is almost surely differentiable with an explicit formula for the partial derivatives, see (19). Details are provided in Appendix A.
- The Gaussian integration by parts (i.b.p.) formula: If $Z \sim \mathcal{N}(0, 1)$ then $\mathbb{E}Zf(Z) = \mathbb{E}f'(Z)$. Interestingly, this formula holds for f Lipschitz. In this case, f is absolutely continuous hence almost surely differentiable (see for instance [24, Chap. 7, Thm. 4]) with linear growth at infinity.

Recall that $\varphi_d(A) = \varphi\left(\frac{\|\Delta \circ A\|}{\sqrt{d}}\right)$, where φ is defined in (13).

Proof In order to get an asymptotic bound over $\mathbb{E}\tilde{R}_k(A)$, we expand its expression:

$$\begin{aligned} \mathbb{E}\tilde{R}_k(A) &= \mathbb{E}\left[\varphi_d(A)e_k^\top \left(\frac{\Delta \circ A}{\sqrt{d}}\right)^2 Q\mathbf{1}\right], \\ &= \frac{1}{d} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}\left[\varphi_d(A)(\Delta \circ A)_{ki} ((\Delta \circ A)Q)_{ij}\right], \\ &= \frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}\left[\varphi_d(A)(\Delta \circ A)_{ki} (-\delta_{ij} + Q_{ij})\right], \\ &= -\frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \mathbb{E}[\varphi_d(A)(\Delta \circ A)_{ki}] + \frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}[\varphi_d(A)(\Delta \circ A)_{ki} Q_{ij}]. \end{aligned}$$

At this point, we use the Gaussian i.b.p. formula applied to $A \mapsto \varphi_d(A)$ which is Lipschitz and a.s. differentiable with explicit derivative (see (19)).

$$\begin{aligned} \mathbb{E}\tilde{R}_k(A) &= -\frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \mathbb{E}[\partial_{ki}\varphi_d(A)] + \frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}[\partial_{ki}(\varphi_d(A)Q_{ij})], \\ &= -\frac{\alpha}{d} \sum_{i \in \mathcal{I}_k} \mathbb{E}\left[u_k v_i \varphi'\left(\frac{\|\Delta \circ A\|}{\sqrt{d}}\right)\right] + \frac{\alpha}{d} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}\left[u_k v_i \varphi'\left(\frac{\|\Delta \circ A\|}{\sqrt{d}}\right) Q_{ij}\right] \\ &\quad + \frac{\alpha}{\sqrt{d}} \sum_{i \in \mathcal{I}_k} \sum_{j \in [n]} \mathbb{E}[\varphi_d(A) (\partial_{ki} Q_{ij})], \end{aligned}$$

$$=: T_1 + T_2 + T_3.$$

We first handle the term T_1 by Cauchy-Schwarz inequality:

$$\begin{aligned} |T_1| &\leq \frac{\alpha}{d} \mathbb{E} \left| u_k \sum_{i \in \mathcal{I}_k} v_i \varphi' \left(\frac{\|\Delta \circ A\|}{\sqrt{d}} \right) \right|, \\ &\leq \frac{\alpha}{d} \mathbb{E} \left[\sqrt{d} \|\mathbf{v}\| \left| \varphi' \left(\frac{\|\Delta \circ A\|}{\sqrt{d}} \right) \right| \right] = \mathcal{O} \left(\frac{\alpha}{\sqrt{d}} \right). \end{aligned}$$

We now handle the term T_2 :

$$\begin{aligned} |T_2| &\leq \frac{\alpha}{d} \mathbb{E} \left[|\varphi'| \left| \sum_i \sum_j v_i Q_{ij} \right| \right], \\ &\leq \frac{\alpha}{d} \mathbb{E} [|\varphi'| \cdot |\mathbf{v}^* Q \mathbf{1}|] \leq \frac{\alpha}{\sqrt{d}} \sqrt{\frac{n}{d}} \mathbb{E} [|\varphi'| \|Q\|] = \mathcal{O} \left(\frac{\alpha}{\sqrt{d}} \right). \end{aligned}$$

We finally handle the term T_3 . Notice that $\partial_{ki} Q_{ij} = \frac{1}{\alpha \sqrt{d}} Q_{ik} Q_{ij}$ and denote by $\boldsymbol{\omega} := (Q_{ik} \mathbf{1}_{\mathcal{I}_k})_{i \in [n]}$. Notice that $\|\boldsymbol{\omega}\|^2 \leq \mathbf{e}_k^* Q^* Q \mathbf{e}_k$ hence $\|\boldsymbol{\omega}\| \leq \|Q\|$ and

$$\begin{aligned} |T_3| &= \frac{1}{d} \left| \mathbb{E} [\varphi_d(A) \boldsymbol{\omega}^* Q \mathbf{1}] \right| \leq \frac{1}{d} \mathbb{E} [\varphi_d(A) \|\boldsymbol{\omega}\| \|Q\| \|\mathbf{1}\|], \\ &\leq \frac{\sqrt{n}}{d} \mathbb{E} [\varphi_d(A) \|Q\|^2] = \mathcal{O} \left(\frac{1}{\sqrt{d}} \right). \end{aligned}$$

Combining these asymptotic notations finally yields :

$$\mathbb{E} \tilde{R}_k(A) = \mathcal{O} \left(\frac{\alpha}{\sqrt{d}} \right).$$

□

Notice that even if the bound obtained in Proposition 9 is weaker than the one obtained in Proposition 8 or in [1, Prop. 2.4], it is still sufficient to establish the feasibility under Model (B).

5 Proofs of Theorem 2 and Proposition 3

5.1 Proof of Theorem 2

The proof is a combination of Takeuchi and Adachi's theorem [19, Theorem 3.2.1] and Proposition 4. We first recall the definition of Volterra-Liapunov stability, see for instance [19, Section 3.2]: Let B be a $n \times n$ real matrix. B is **Volterra-Liapunov stable** if there exists a $n \times n$ positive definite diagonal matrix D such that $DB + B^\top D$ is negative definite.

Going back to Eq. (2), according to Takeuchi and Adachi's theorem [19, Th. 3.2.1], this LV system has a unique nonnegative and globally stable equilibrium if $M_n - I_n$ is Volterra-Liapunov stable.

We now rely on the asymptotic spectral properties of M_n to study the Volterra-Liapunov stability of $M_n - I_n$. We drop the subscript n in the sequel. Take $D = I$ then

$$D(M - I) + (M - I)^\top D = M + M^\top - 2I$$

is an hermitian matrix. This matrix is negative definite if all its eigenvalues are negative. Given that $M + M^T$ is also hermitian, we just have to check that the spectral radius $\rho(M + M^T) < 2$. According to Proposition 4:

$$\mathbb{P}(\rho(M + M^T) < 2) \geq \mathbb{P}(\|M\| < 1) \xrightarrow[n \rightarrow +\infty]{} 1.$$

Thus, the probability that $M - I$ is Volterra-Liapunov stable converges to 1 as $n \rightarrow \infty$. By [19, Th. 3.2.1], this implies that the probability that the LV system (2) has a unique nonnegative and globally stable equilibrium converges to 1 as $n \rightarrow \infty$.

5.2 Proof of Proposition 3

We first prove the first part of the proposition. By Theorem 2, there exists a unique nonnegative globally stable equilibrium to (2). If there exists $\epsilon > 0$ such that eventually $\alpha_n \geq (1 + \epsilon)\alpha_n^*$ where $\alpha_n^* = \sqrt{2 \log n}$, then this equilibrium \mathbf{x}_n is positive by Theorem 1 with overwhelming probability as $n \rightarrow \infty$.

The rest of the proof closely follows the proof of [1, Corollary 1.4] and is omitted.

6 Conclusion

In this article we study the feasibility and stability of sparse large ecosystems modelled by a large Lotka-Volterra system of coupled differential equations:

$$\frac{d\mathbf{x}_n}{dt} = \mathbf{x}_n(\mathbf{1}_n - \mathbf{x}_n + M_n \mathbf{x}_n).$$

Our work is motivated by recent research [15] which suggests that in the light of many ecological and biological datasets living networks are often sparse. It also illustrates the interest to study feasibility in relation with the normalization of the interaction matrix's entries beyond the non-sparse full i.i.d. models, and opens perspectives to study models with more structure such as elliptic interactions or patch models.

In the model under investigation, the interaction matrix M_n is a sparse random matrix, where the sparsity is encoded by a patterned matrix Δ_n based on an underlying d_n -regular graph, and the randomness by i.i.d. random variables (matrix A_n) for non-null entries. The single parameter d_n of the regular graph provides an easy one-dimensional parametrization of the connectance of the foodweb.

Our main conclusion is that beyond the standard normalization $1/\sqrt{d_n}$ of the interaction matrix $\Delta \circ A$, which guarantees a bounded norm

$$\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| = \mathcal{O}_P(1),$$

an extra factor $1/\alpha_n$ with $\alpha_n \rightarrow \infty$ is needed to reach feasibility. The interaction matrix finally writes

$$M_n = \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}}$$

and a sharp phase transition occurs at $\alpha_n^* = \sqrt{2 \log(n)}$. Interestingly, the same phase transition as in the non-sparse case occurs.

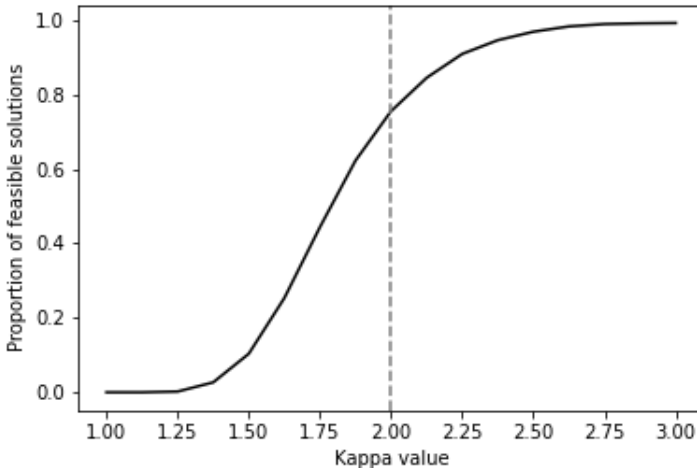


Fig. 4: Let $n = 15000$, $d = 10$ (notice that $d \geq \log(n) \simeq 9.61$). Matrix Δ_n is drawn at random once for all among the adjacency matrices of d -regular graphs (and a priori does not follow Model (A)). Each point of the curve represents the proportion of feasible solutions \mathbf{x}_n of Eq. (6) over 1500 realizations of random matrices A_n for different values of κ , with $\alpha_n = \sqrt{\kappa \log(n)}$. The phase transition resemble those of Figure 1.

In the sparse setting $\log(n) \leq d_n \ll n$, we rely on an extra block-structure assumption over matrix Δ_n , namely Model (A), to establish the feasibility and the phase transition. Our method of proof crucially relies on this technical assumption which somehow concentrates the non-null entries of the sparse interaction matrix (and its powers) into localized blocks.

However simulations (cf. Fig 4) suggest that this block structure assumption is not necessary and could be relaxed. Hence the following:

Open question 10 *Let Δ_n the adjacency matrix of a deterministic d_n regular graph, with $d_n \geq \log(n)$, and A_n a random matrix with i.i.d. $\mathcal{N}(0, 1)$ entries. Consider the*

equation

$$\mathbf{x}_n = \mathbf{1}_n + \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}} \mathbf{x}_n, \quad \alpha_n \rightarrow \infty.$$

Is it true that the same phase transition as in Theorem 1 holds?

Appendix A With probability one, the singular values of a sparse random matrix are distinct

We establish hereafter that with probability one the singular values of matrix $\Delta \circ A$ are distinct, a key argument in the proof of Proposition 9 to compute the partial derivatives of $A \mapsto \|\Delta \circ A\|$.

The lemma below and its proof are inspired by Nick Cook [25], whom we thank for his help.

Lemma 11 (Cook [25]) *Let $n \geq 1$, A_n a $n \times n$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries and Δ_n the adjacency matrix of a d -regular graph. Then with probability one, all the singular values of $\Delta_n \circ A_n$ are distinct.*

Remark

The original statement of Cook is slightly more general: matrix A_n entries only need a distribution with positive density, and the deterministic matrix Δ_n only needs a generalized diagonal, i.e. $(\Delta_{i\sigma(i)}; i \in [n])$ for some $\sigma \in \mathcal{S}_n$, with $n - 1$ non null entries.

Proof Let \mathcal{E}_Δ be the set of matrices with entries supported on the nonzero entries of Δ ,

$$\mathcal{E}_\Delta = \left\{ \Delta \circ X; X = (X_{ij}) \in \mathbb{R}^{n \times n} \right\}.$$

Thus, \mathcal{E}_Δ is the support of the law of $\Delta \circ A$. Besides, \mathcal{E}_Δ is a variety as a subspace of $\mathbb{R}^{n \times n}$.

Let \mathcal{R} denote the set of matrices with a repeated singular value. It is the set of $n \times n$ matrices X for which the characteristic polynomial p of $X^\top X$ has zero discriminant (ρ), see for instance [26, Section 3.3.2].

$$\mathcal{R} = \left\{ X \in \mathbb{R}^{n \times n}; \rho \left(p \left(X^\top X \right) \right) = 0 \right\} = \left\{ X \in \mathbb{R}^{n \times n}; P(X) = 0 \right\},$$

where $P : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by $P(X) = \rho(p(X^\top X))$ is a polynomial in the entries of X . It follows that \mathcal{R} is an algebraic variety in $\mathbb{R}^{n \times n}$.

Hence, $\mathcal{E}_\Delta \cap \mathcal{R}$ is either equal to \mathcal{E}_Δ , or a subvariety of \mathcal{E}_Δ of zero Lebesgue measure (under the product measure on \mathcal{E}_Δ).

For the claim, it suffices to show that $\mathcal{E}_\Delta \not\subset \mathcal{R}$ hence to exhibit $Y \in \mathcal{E}_\Delta$ with distinct singular values. By Birkhoff's theorem [21, Theorem 8.7.2], the doubly stochastic matrix Δ/d writes

$$\frac{\Delta}{d} = \sum_{\sigma \in \mathcal{S}_n} a_\sigma P_\sigma, \quad a_\sigma \geq 0 \quad \text{and} \quad \sum_{\sigma \in \mathcal{S}_n} a_\sigma = 1,$$

where P_σ is the permutation matrix associated to $\sigma \in \mathcal{S}_n$. There exists in particular σ^* with $a_{\sigma^*} > 0$ and $P_{\sigma^*} \in \mathcal{E}_\Delta$. Let $P_{\sigma^*} = (P_{ij})_{i,j \in [n]}$ then matrix $Y = (iP_{ij})_{i,j \in [n]}$ has distinct singular values $(1, \dots, n)$. This completes the proof. \square

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