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Large ranking games with diffusion control

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We consider a symmetric stochastic differential game where each player can control the diffusion intensity of an individual dynamic state process, and the players whose states at a deterministic finite time horizon are among the best $\alpha \in (0, 1)$ of all states receive a fixed prize. Within the mean field limit version of the game we compute an explicit equilibrium, a threshold strategy that consists in choosing the maximal fluctuation intensity when the state is below a given threshold, and the minimal intensity else. We show that for large n the symmetric n -tuple of the threshold strategy provides an approximate Nash equilibrium of the n -player game. We also derive the rate at which the approximate equilibrium reward and the best response reward converge to each other, as the number of players n tends to infinity. Finally, we compare the approximate equilibrium for large games with the equilibrium of the two player case.

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1 Introduction

We start by describing a game that models, in a stylized form, the competition among many agents who can each control the fluctuation intensity of an individual state until

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a fixed time horizon $T \in (0, \infty)$ and who receive some benefit if their final state is among the highest $\alpha \in (0, 1)$. We first consider the case where the players' states are martingales driven by independent Brownian motions. Later we also allow for a drift coefficient and added common noise (see Section 5).

Throughout let n be a large natural number representing the number of players. Let $0 < \sigma_1 < \sigma_2$. The set of strategies for each player i is denoted by \mathcal{A}_n and consists of the set of measurable functions $a : [0, T] \times \mathbb{R}^n \rightarrow [\sigma_1, \sigma_2]$. We write $\mathcal{A}_n^n := \mathcal{A}_n \times \dots \times \mathcal{A}_n$ for the set of all strategy tuples.

We denote the state process of player i by X^i . Given that each player chooses a strategy $a_j \in \mathcal{A}_n, j \in \{1, \dots, n\}$, we assume that the state process of player i satisfies

$$dX_t^{i,a} = a_i(t, X_t^{i,a}, X_t^{-i,a})dW_t^i, \quad X_0^{i,a} = 0, \quad (1)$$

where $X^{-i,a}$ denotes the $n - 1$ -dimensional process of the states of all players other than i . Moreover, $W = (W^1, \dots, W^n)$ denotes a n -dimensional Brownian motion on some filtered probability space. Note that the n -dimensional process $X^a = (X^{1,a}, \dots, X^{n,a})$ satisfies the n -dimensional SDE

$$dX_t = D(t, X_t)dW_t, \quad X_0 = 0, \quad (2)$$

where $D(t, x)$ is the diagonal matrix with the entries $a_1(t, x), \dots, a_n(t, x)$ on the diagonal. The existence of a weak solution of (2) follows from Theorem 2.6.1 in [22] since $D(t, x)$ is uniformly elliptic. Recall that a weak solution consists of a 6-tuple $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a)_{t \geq 0}, P^a, W^a, X^a)$, where the first four components build a filtered probability space, W^a is a Brownian motion with respect to the filtration (\mathcal{F}_t^a) , and X^a is a continuous (\mathcal{F}_t^a) -adapted process such that the pair (X^a, W^a) satisfies (2) (see e.g. [19], Section 5.3, for more details). We add the superscript a to the elements of the weak solution in order stress the dependence on the strategy tuple. Note that for different strategy tuples we may need to consider different probability spaces.

We suppose that each player aims at maximizing the probability of her own state to be greater than the empirical $(1 - \alpha)$ -quantile of all states at time T . More precisely, let

$$\mu^{n,a} = \frac{1}{n} \sum_{i=1}^n \delta_{X_T^{i,a}}$$

be the empirical distribution of the players' states at time T . We define $q(\mu^{n,a}, 1 - \alpha) = \inf\{r \in \mathbb{R} : \mu^{n,a}((-\infty, r]) \geq 1 - \alpha\}$. Note that $X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)$ if and only if the state of player i is among the best $\lfloor n\alpha \rfloor$ players at time T (see Remark 1.1 below).

We now briefly comment on the assumptions of the game. First notice that the expectation of any player's terminal state is always zero, whatever control is chosen. However, other distributional properties such as the variance, the skewness and the position of quantiles do depend on the strategy. For example, a larger a^i leads to a larger variance of the final outcome X_T^i . Moreover, if player i chooses large values for a^i when her state is small and small values when her state is large, then the distribution of X_T^i

has a negative skew. In particular, a state-dependent choice of the fluctuation intensity allows to influence the position of quantiles, and hence the probability of receiving the reward. So the game helps in understanding how a rank-based reward determines the distributional properties of the players' states and by which dynamic strategies they can be implemented.

In the following remark we relate the reward condition to the order statistics of the terminal states.

Remark 1.1. Let $a \in \mathcal{A}_n^n$. We can relate the quantile $q(\mu^{n,a}, 1 - \alpha)$ with the order statistics $X_T^{(1:n),a} \leq \dots \leq X_T^{(n:n),a}$ of the states $X_T^{1,a}, \dots, X_T^{n,a}$. We have $q(\mu^{n,a}, 1 - \alpha) = X_T^{([n(1-\alpha)]:n),a}$, and hence

1. if $n\alpha \in \mathbb{N}$, then $X_T^i > q(\mu^{n,a}, 1 - \alpha)$ is equivalent to being under the best $n\alpha$ players at time T , and
2. if $n\alpha \notin \mathbb{N}$, then $X_T^i > q(\mu^{n,a}, 1 - \alpha)$ is equivalent to being under the best $\lfloor n\alpha \rfloor$ players at time T .

Consequently, $X_T^i > q(\mu^{n,a}, 1 - \alpha)$ is always equivalent to being under the best $\lfloor n\alpha \rfloor$ players at time T .

The aim of the present article is to predict and analyze the players' behavior in the game. As usual, for this purpose we resort to Nash equilibria, which here are defined as follows.

Definition 1.2. A tuple $a = (a_1, \dots, a_n) \in \mathcal{A}_n^n$ is called *Nash equilibrium* of the n -player game if for all $i \in \{1, \dots, n\}$, $c \in \mathcal{A}_n$ and weak solutions $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$, $(\Omega^{(a-i,c)}, \mathcal{F}^{(a-i,c)}, (\mathcal{F}_t^{(a-i,c)}), P^{(a-i,c)}, W^{(a-i,c)}, X^{(a-i,c)})$ we have

$$P^a(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \geq P^{(a-i,c)}(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)), \quad (3)$$

where $(a-i, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$.

Note that we do not assume that the weak solutions are unique in law and hence that the probabilities appearing in (3) are independent of the weak solution chosen¹. Thus one can interpret (3) as follows: whatever weak solutions come up, with whatever distributions, no player has an incentive to deviate. However, we will see that for the Nash equilibria derived in this article uniqueness in law is always satisfied.

For the two player case with $\alpha = \frac{1}{2}$ one can show (see Section 6 below) that a Nash equilibrium is given when each player chooses, at any time $t < T$, the minimal volatility σ_1 if her current state value is larger than the opponent's state, and the maximal volatility σ_2 if her current state value is smaller.

We believe that for more than two players it is difficult to obtain a Nash equilibrium in closed form. To circumvent this difficulty, our idea is to compute an approximate

¹It is still an open problem to find conditions, sufficient and necessary at the same time, guaranteeing uniqueness in law of weak solutions of (2) in dimensions greater than 2 (see e.g. abstract of [23]).

Nash equilibrium for large games by considering the mean field limit of the game. The mean field game approach, introduced by [27], [28], [29] and [17], [16], is based on the observation that in a symmetric game with many similar players, the empirical distribution of the realized players' states can be approximated with the state distribution of a *single* player. Thus, the Nash equilibrium for the game with many players can be approximated by solving a single agent control problem where the reward depends on the distribution of the state.

We obtain that, within the mean field limit version of the game, there exists an equilibrium that consists in choosing the maximal fluctuation intensity when the state is below a given threshold, and the minimal intensity else (see Section 3). We refer to such a strategy as a threshold strategy.

The threshold of the mean field equilibrium strategy is chosen such that it coincides with the $(1 - \alpha)$ -quantile of the resulting final state distribution. The threshold can be identified as a fixed point of a mapping from the set of threshold strategies to itself. Fixed point arguments are a common tool for determining equilibria in mean field games (see, e.g., [7] and [8]).

Controlling the state with a threshold strategy entails that the state dynamics are a so-called oscillating Brownian motion (OBM), a process introduced in [20] and studied in [31] from a control theory perspective without interaction. The distribution of OBM is explicitly known at deterministic times. This allows us to compute the threshold of the mean field equilibrium in closed form.

We further show that the symmetric n -tuple of the mean field equilibrium is an approximate Nash equilibrium of the n -player game (see Section 4). More precisely, the difference of the expected payoff under the approximate equilibrium strategies and the best response payoff converges to zero as the number of players n tends to infinity. We show that the convergence is of the order $n^{-1/2}$.

In the approximate equilibrium the distribution of the final states is negatively skewed and the $(1 - \alpha)$ -quantile is further to the right than when using a strategy with constant fluctuations. Choosing the maximal fluctuation intensity below the threshold increases risk, measured, e.g., in terms of the variance of the final state. Thus, the results show that the rank-based reward incentivizes players to introduce a negative skew, and a high variance.

For implementing the threshold strategy of the approximate Nash equilibrium each player only needs to observe her own state process, but not the other players' states. This is a difference to the two player equilibrium, where each player chooses a high volatility if and only if her state is smaller than the opponent's state. In the two player equilibrium the *relative* position of the own state with respect to the opponent's state determines the choice of volatility. In contrast, in the mean field equilibrium, and hence in the approximate equilibrium for large games, the chosen volatility depends only on the *absolute* position of the own state, namely whether it is above or below the fixed equilibrium threshold (see Remark 4.3). The results indicate that in an exact equilibrium in games with $n \geq 3$ players, each player makes her control dependent on both the absolute position of the own state and the relative position with respect to other players' states.

We remark that for the game with $n \geq 3$ players it is already difficult to prove, by using abstract arguments, existence of an exact equilibrium in the first place, because of the discontinuity in the reward function. This applies to any game with discontinuous rank-based rewards. A workaround is to consider rewards depending continuously on the average state of all players. Espinosa and Touzi [13] consider a game where each player aims at maximizing the expected utility of her income depending on the deviation of the player's state from the average state. They characterize a Nash equilibrium in terms of a system of coupled backward stochastic differential equations (BSDEs). [10] establishes a Nash equilibrium in a market with N agents with the performance criteria of relative wealth level when the mean return rate is unobservable. Each investor has a heterogeneous prior belief on the return rate of the risky asset. By a separation result and a martingale argument, they show that the optimal investment strategy under this setting can be characterized by a fully-coupled linear FBSDE system. Two sets of deep neural networks are used for first computing each investor's estimate of the mean return rate and then numerically solving the FBSDEs. [14] uses a probabilistic approach to study an N -player exponential utility game and a mean field exponential utility game. Each player manages two stocks; one is driven by an individual shock and the other is driven by a common shock. Moreover, each player is concerned not only with her own terminal wealth but also with the relative performance of her competitors. A multi-dimensional FBSDE with quadratic growth (a mean field FBSDEs, respectively) is derived to characterize an equilibrium of the N -player game (of the mean field game, respectively). In [12], the authors solve a sequential game between a principal and several agents in competition with relative performance concerns. In particular, they connect the issue of existence of a Nash equilibrium between the competing agents to the existence of a solution to a multidimensional quadratic BSDE. In the present paper, we avoid BSDE arguments and solve our problem in a more direct way, by relying instead on the knowledge of the oscillating Brownian motion's marginal distributions.

As already mentioned above, when the agents are numerous and homogeneous, it is tempting to assume that one particular agent does not affect the empirical distribution of players. The symmetry in the game allows to focus on only one representative agent, with a *fixed* distribution of other players' states. This idea has been made precise in the theory of stochastic differential games with mean field interactions, that have been introduced independently in [17] and [29] and then received a sustained attention in the stochastic control and differential games communities. For an overview of the theory, we refer the reader to the monographs [7, 8].

On the one hand, the fact that Nash equilibria in finite n -player games converge to the corresponding mean field limit justifies the use of mean field games (MFG). It has been proved under different sets of assumptions, using probabilistic weak formulations and compactness arguments [24, 25], or via a partial differential equation (PDE) approach [6]. In [6], the authors rely on the so called *master equation*, which is an infinite dimensional PDE describing the value function of the game, to prove that the empirical distribution of the players converges to the mean field equilibrium measure, which is unique given the monotonicity condition made in [6].

On the other hand, a MFG solution can be used to construct an approximate Nash

equilibrium for the corresponding finite n -player game: this is the approach that we follow in Section 4 of the present paper.

One particular feature of our framework is the control of the diffusion coefficient in (1). Specific examples of both standard and mean field games with volatility control include optimal energy consumption [11], green bonds investments [1], or optimal contracting between hierarchized players [18].

The idea to use mean field limits for analyzing differential games with rank-based rewards has been employed already in the articles [3],[4],[5] and [35]. All these articles consider versions of a rank-based mean field game, where agents influence the drift rate of the state, but not the diffusion rate.

The articles [3], [4], [35] provide conditions guaranteeing that the mean field games approximate the corresponding n -player game for large n ; and they also determine the rate at which the expected reward of the mean field approximation and the best response converge to each other, as $n \rightarrow \infty$. Motivated by these articles, in Section 4 we also compute a convergence rate for the game at hand. In the recent paper [2], the authors analyze a mean field game where the diffusion coefficient can be controlled and show that the solution of the problem is characterized by a McKean-Vlasov second order backward SDE. Note that it is not possible to directly apply the results of [2] to our setting, since the particular form of our criteria given as a probability in Definition 1.2 would make the terminal condition of the related second order BSDE discontinuous, which is not allowed in [2].

The rest of the paper is organized as follows: In Section 2 we provide some results on oscillating Brownian motion, that will be used in Section 3 to explicitly solve the game in the mean field regime. In Section 4, we construct approximate Nash equilibria for the finite n -player game, based on the mean field optimal strategies. In Section 5 we generalize these results to the case where the players' states are driven by an SDE with an additional drift component and with added common noise. In Section 6, we solve the 2-player game by explicitly solving the associated dynamic programming PDE, and give some comparison with the mean field equilibrium. The game considered in this paper is formulated in a generic way and thus allows for multiple interpretations. In Section 7 we illustrate our results within a specific application to the competition among mutual fund managers.

2 Oscillating Brownian motion

In this section we define the *oscillating Brownian motion* (OBM) introduced in [20]. We summarize important facts about OBMs that we will need in the course of this paper.

Let B be a Brownian motion on a complete probability space (Ω, \mathcal{F}, P) . Furthermore, let $(\mathcal{F}_t)_{t \geq 0}$ be the augmented Brownian filtration, generated by B and augmented by the P -null sets in \mathcal{F} . Let $0 < \sigma_1 < \sigma_2$ and define for any $b \in \mathbb{R}$ the threshold function

$m_b : \mathbb{R} \rightarrow [\sigma_1, \sigma_2]$ by

$$m_b(x) = \begin{cases} \sigma_1, & \text{if } x \geq b, \\ \sigma_2, & \text{if } x < b. \end{cases} \quad (4)$$

Definition 2.1. Let $b \in \mathbb{R}$. We call the solution $Y^{x,b}$ of the SDE

$$dY_t = m_b(Y_t)dB_t, \quad Y_0 = x, \quad (5)$$

oscillating Brownian motion (OBM) with threshold b .

Remark 2.2. Note that there exists a unique strong solution of (5), since there exists a weak solution according to Theorem 2.6.1 in [22] and pathwise uniqueness applies due to [34]. For further details see Remark 3.1 below.

For the reader's convenience we recall the following result.

Proposition 2.3. For all $x \in \mathbb{R}$ let Y^x be an OBM with threshold 0 and initial value x . Then, for all $t > 0$, the random variable Y_t^x has a density $p(t, x, y)$ with respect to the Lebesgue measure, given by

$$p(t, x, y) = \begin{cases} \frac{2\sigma_1}{\sigma_2(\sigma_1+\sigma_2)} \frac{1}{\sqrt{2\pi t}} e^{-\left(\frac{x}{\sigma_1} - \frac{y}{\sigma_2}\right)^2 \frac{1}{2t}}, & \text{if } x \geq 0, y < 0, \\ \frac{2\sigma_2}{\sigma_1(\sigma_1+\sigma_2)} \frac{1}{\sqrt{2\pi t}} e^{-\left(\frac{y}{\sigma_1} - \frac{x}{\sigma_2}\right)^2 \frac{1}{2t}}, & \text{if } x < 0, y \geq 0, \\ \frac{1}{\sigma_1\sqrt{2\pi t}} \left(e^{-\frac{(y-x)^2}{2\sigma_1^2 t}} + \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} e^{-\frac{(y+x)^2}{2\sigma_1^2 t}} \right), & \text{if } x \geq 0, y \geq 0, \\ \frac{1}{\sigma_2\sqrt{2\pi t}} \left(e^{-\frac{(y-x)^2}{2\sigma_2^2 t}} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} e^{-\frac{(y+x)^2}{2\sigma_2^2 t}} \right), & \text{if } x < 0, y < 0. \end{cases}$$

Proof. This follows from Theorem 1 in Keilson, Wellner [20]; see also Theorem 4 in [31] and Section 2 in Lejay, Pigato [30] (beware of the typo in the last case). \square

Corollary 2.4. For $x, b \in \mathbb{R}$ let $Y^{x,b}$ be an OBM with threshold b . Then, for all $t > 0$, the random variable $Y_t^{x,b}$ has the probability density function $p(t, x-b, y-b)$ with respect to the Lebesgue measure, where p is defined as in Proposition 2.3.

For the remainder of this section we denote by F^b the probability distribution function of the OBM $Y^{0,b}$ at time $T > 0$ for some $b \in \mathbb{R}$, i.e. $F^b(x) = P(Y_T^{0,b} \leq x)$, $x \in \mathbb{R}$. Note that using Corollary 2.4 we observe that F^b is given by

$$F^b(x) = \begin{cases} \Phi\left(\frac{x}{\sigma_2\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{x-2b}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, b \geq 0, \\ \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi\left(\frac{x-b(1-\frac{\sigma_1}{\sigma_2})}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}, & \text{if } x \geq b, b \geq 0, \\ \frac{2\sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{x-b(1-\frac{\sigma_2}{\sigma_1})}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, b < 0, \\ \Phi\left(\frac{x}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{2b-x}{\sigma_1\sqrt{T}}\right), & \text{if } x \geq b, b < 0. \end{cases} \quad (6)$$

Lemma 2.5. *Let $b \in \mathbb{R}$. Then F^b is Lipschitz continuous. Moreover, for $M > 0$ there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$ with $|x - b| < M$*

$$|F^b(x) - F^b(b)| \geq C|x - b|.$$

Proof. The Lipschitz continuity of F^b follows since F^b is absolutely continuous and its derivative is bounded. The second statement basically follows from the fact that the derivative is locally bounded from below.

We first consider the case $b \geq 0$. Note that F^b is given by

$$F^b(x) = \begin{cases} \Phi\left(\frac{x}{\sigma_2\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{x-2b}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, \\ \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi\left(\frac{x-b\left(1-\frac{\sigma_1}{\sigma_2}\right)}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}, & \text{if } x \geq b. \end{cases}$$

1. If $x \in (b - M, b)$ there exists a $\xi \in (x, b)$ such that

$$\begin{aligned} |F^b(x) - F^b(b)| &= F^b(b) - F^b(x) \\ &= \frac{1}{\sigma_2\sqrt{2\pi T}} e^{-\frac{\xi^2}{2\sigma_2^2 T}} \left(1 - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} e^{-\frac{4b(b-\xi)}{2\sigma_2^2 T}}\right) |x - b| \\ &\geq \frac{1}{\sigma_2\sqrt{2\pi T}} e^{-\frac{(b-M)^2 \vee b^2}{2\sigma_2^2 T}} \left(1 - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}\right) |x - b| \\ &\geq \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(b+M)^2}{2\sigma_1^2 T}} |x - b|. \end{aligned}$$

2. If $b < x < b + M$ we obtain by the mean-value theorem for some $\xi \in (b, x)$ that

$$\begin{aligned} |F^b(b) - F^b(x)| &= F^b(x) - F^b(b) \\ &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left(\Phi\left(\frac{x-b\left(1-\frac{\sigma_1}{\sigma_2}\right)}{\sigma_1\sqrt{T}}\right) - \Phi\left(\frac{b}{\sigma_2\sqrt{T}}\right) \right) \\ &= \frac{1}{\sigma_1\sqrt{2\pi T}} \frac{2\sigma_2}{\sigma_1 + \sigma_2} e^{-\frac{(\xi-b\left(1-\frac{\sigma_1}{\sigma_2}\right))^2}{2\sigma_1^2 T}} (x - b) \\ &\geq \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(b+M)^2}{2\sigma_1^2 T}} (x - b). \end{aligned}$$

The case $b < 0$ can be shown in the same way. Finally, for any $b \in \mathbb{R}$ the result follows with $C := \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(|b|+M)^2}{2\sigma_1^2 T}}$. □

Lemma 2.6. *There exists a $C > 0$ such that for all $b_1, b_2 \in \mathbb{R}$*

$$\sup_{x \in \mathbb{R}} |F^{b_1}(x) - F^{b_2}(x)| \leq C|b_1 - b_2|.$$

Proof. Let $0 \leq b_1 < b_2$. Then for $x < b_1$ we see that

$$\begin{aligned}
|F^{b_1}(x) - F^{b_2}(x)| &= \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \left| \Phi \left(\frac{x - 2b_2}{\sigma_2 \sqrt{T}} \right) - \Phi \left(\frac{x - 2b_1}{\sigma_2 \sqrt{T}} \right) \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \left| \frac{x - 2b_2}{\sigma_2 \sqrt{T}} - \frac{x - 2b_1}{\sigma_2 \sqrt{T}} \right| \\
&= \frac{2}{\sqrt{2\pi\sigma_2^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1| \\
&\leq \frac{2}{\sqrt{2\pi\sigma_1^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1|.
\end{aligned}$$

Similarly, for $x > b_2$ we have

$$\begin{aligned}
|F^{b_1}(x) - F^{b_2}(x)| &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left| \Phi \left(\frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left(\frac{x - b_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) \right| \\
&\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi}} \left| \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} - \frac{x - b_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right| \\
&= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi\sigma_1^2 T}} \left(1 - \frac{\sigma_1}{\sigma_2}\right) |b_2 - b_1| \\
&= \frac{2}{\sqrt{2\pi\sigma_1^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1|.
\end{aligned}$$

Now it remains to consider the case $x \in [b_1, b_2]$: We have

$$\begin{aligned}
&|F^{b_1}(x) - F^{b_2}(x)| \\
&= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left(\frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} - \Phi \left(\frac{x}{\sigma_2 \sqrt{T}} \right) + \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi \left(\frac{x - 2b_2}{\sigma_2 \sqrt{T}} \right) \right| \\
&= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left(\frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left(\frac{x}{\sigma_2 \sqrt{T}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi \left(\frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right| \\
&= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left(\frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \left(1 + \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}\right) \Phi \left(\frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right. \\
&\quad \left. + \Phi \left(\frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) - \Phi \left(\frac{x}{\sigma_2 \sqrt{T}} \right) \right| \\
&\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left| \Phi \left(\frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left(\frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right| + \left| \Phi \left(\frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) - \Phi \left(\frac{x}{\sigma_2 \sqrt{T}} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi}} \left| \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} - \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right| + \frac{1}{\sqrt{2\pi}} \left| \frac{2b_2 - x}{\sigma_2 \sqrt{T}} - \frac{x}{\sigma_2 \sqrt{T}} \right| \\
&\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi T}} \left| \frac{2}{\sigma_2} (x - b_2) + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) (x - b_1) \right| + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \\
&\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi T}} \left(\frac{2}{\sigma_2} |b_2 - b_1| + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) |b_2 - b_1| \right) + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \\
&\leq \frac{4\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi\sigma_1^2 T}} |b_2 - b_1| + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \\
&\leq \frac{6}{\sqrt{2\pi\sigma_1^2 T}} |b_2 - b_1|.
\end{aligned}$$

The case $b_1 < b_2 \leq 0$ can be shown similarly and all the remaining cases follow from an application of the triangle inequality. \square

3 The mean field approximation

Let B be a Brownian motion on a complete probability space (Ω, \mathcal{F}, P) . Furthermore, let $(\mathcal{F}_t)_{t \geq 0}$ be the augmented Brownian filtration, generated by B and augmented by the P -null sets in \mathcal{F} . We denote by \mathcal{M} the set of all progressively measurable control processes $(\beta_t)_{t \geq 0}$ taking values in $[\sigma_1, \sigma_2]$. Given that an agent chooses the control function $\beta \in \mathcal{M}$, her state process is given by

$$X_t^\beta := \int_0^t \beta_s dB_s.$$

Remark 3.1. All feedback controls with a feedback function $m : \mathbb{R} \rightarrow [\sigma_1, \sigma_2]$ of bounded variation are contained in \mathcal{M} . Indeed, since m is uniformly bounded away from zero there exists a weak solution to the SDE

$$dX_t = m(X_t) dB_t, \quad X_0 = 0, \quad (7)$$

according to Theorem 2.6.1 in [22], and pathwise uniqueness applies according to results in [34]. Hence, there exists a unique strong solution X^m to (7) (cf. Section 5.3 in [19]), and $(m(X_t^m))_{t \geq 0}$ belongs to \mathcal{M} .

Let $\alpha \in (0, 1)$ and denote by $q(X_T^\beta, 1 - \alpha)$ the $(1 - \alpha)$ -quantile of the random variable X_T^β , i.e. $q(X_T^\beta, 1 - \alpha) = \inf\{r \in \mathbb{R} : P(X_T^\beta \leq r) \geq 1 - \alpha\}$.

Definition 3.2. We call $\beta^* \in \mathcal{M}$ an *equilibrium strategy* of the mean field game if

$$P(X_T^{\beta^*} > q(X_T^{\beta^*}, 1 - \alpha)) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > q(X_T^{\beta^*}, 1 - \alpha)) \quad (8)$$

Remark 3.3. Notice that β^* is an equilibrium strategy if and only if it maximizes the quantile $q(X_T^\beta, 1 - \alpha)$ among all $\beta \in \mathcal{M}$. Therefore, the mean field game coincides with the control problem of maximizing the $(1 - \alpha)$ -quantile. We define β^* via the equation (8) since the equation can be interpreted as the mean field limit of (3).

Before proving that there exists an equilibrium strategy, we first show that it is enough to consider control processes of feedback type, that are constant equal to the minimal volatility σ_1 if the state is greater than or equal to a given threshold b , and that are constant equal to the maximal volatility σ_2 else. We refer to such controls as *threshold controls*. More precisely, the threshold control with barrier $b \in \mathbb{R}$ is the control with feedback function m_b , given in (4). In the following we simply write X^b for the state process X^{m_b} .

Lemma 3.4. *Let $b \in \mathbb{R}$. Then*

$$P(X_T^b > b) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > b). \quad (9)$$

Proof. Follows from [31], Remark 8. □

Lemma 3.4 shows that the strategy maximizing the probability of being above the quantile $q(X_T^m, 1 - \alpha)$ at time T is the threshold control with threshold $q(X_T^m, 1 - \alpha)$. The following theorem provides an explicit equilibrium strategy.

Theorem 3.5. *The threshold strategy m_{b^*} with threshold*

$$b^* = \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left(\frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left(\frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}, \end{cases} \quad (10)$$

is an equilibrium strategy for the mean field game. m_{b^} is the unique equilibrium strategy in the set of threshold strategies.*

The standard approach to solve mean field games is to consider mappings from probability distributions to the distributions of optimally controlled states and find their fixed points, the so-called *equilibrium measures* (see e.g. [7] and [8]). However, the setting of our paper allows to study the distributions of OBMs only, which can be parameterized by the real-valued threshold $b \in \mathbb{R}$. Hence it is enough to show that the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad b \mapsto q(X_T^b, 1 - \alpha)$$

has a unique fixed point in order to prove Theorem 3.5. Indeed, if $f(b) = b$, then $b = q(X_T^b, 1 - \alpha)$. Lemma 3.4 further implies that $P(X_T^b > q(X_T^b, 1 - \alpha)) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > q(X_T^b, 1 - \alpha))$, and hence that m_b is an equilibrium strategy.

Moreover, for any threshold control $m_b, b \in \mathbb{R}$, the controlled state process X^b is a so-called oscillating Brownian motion (OBM), introduced in Section 2. The transition densities of an OBM are explicitly known (see Proposition 2.3), and in particular, X_T^b possesses the probability density function $p(T, -b, x - b)$ (Corollary 2.4). As in Section 2 we denote the probability distribution function of X_T^b by F^b and emphasize that F^b is explicitly given by (6).

Proof of Theorem 3.5. We want to determine the unique fixed point $b \in \mathbb{R}$ of the function f , i.e. we solve the equation $P(X_T^b > b) = \alpha$. Suppose first that $\alpha \in (0, \frac{\sigma_2}{\sigma_1 + \sigma_2}]$. We observe by (6) that $P(X_T^b > b) = 1 - F^b(b) \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}$ if and only if $b \geq 0$, and in this case

$$P(X_T^b > b) = \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi\left(\frac{-b}{\sigma_2\sqrt{T}}\right).$$

Moreover, $P(X_T^b > b) = \alpha$ if and only if $b = -\sigma_2\sqrt{T}\Phi^{-1}\left(\frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2}\right)$. In this case, $q(X_T^b, 1 - \alpha) = b$; in other words b is the unique fixed point of the mapping f .

Now suppose that $\alpha \in (\frac{\sigma_2}{\sigma_1 + \sigma_2}, 1)$. Again with (6) we observe that $P(X_T^b > b) > \frac{\sigma_2}{\sigma_1 + \sigma_2}$ if and only if $b < 0$, and

$$P(X_T^b > b) = 1 - \frac{2\sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{b}{\sigma_1\sqrt{T}}\right).$$

As above $P(X_T^b > b) = \alpha$ if and only if $b = \sigma_1\sqrt{T}\Phi^{-1}\left(\frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1}\right)$. In this case, b is the unique fixed point of the function f and $b = q(X_T^b, 1 - \alpha)$. \square

Remark 3.6. Theorem 3.5 only provides uniqueness of the equilibrium m_{b^*} in the set of threshold strategies. It is not clear whether m_{b^*} is also unique in the larger class \mathcal{M} . Even for the control problem (9) there are no results on uniqueness of optimal controls to the best of our knowledge.

Remark 3.7. One can apply the results of McNamara [31] to generalize the terminal reward $\mathbb{1}_{(q(\mu, 1-\alpha), \infty)}(x)$ to functions $g(x, q(\mu, 1 - \alpha))$ that depend on the player's state and the population $(1 - \alpha)$ -quantile. Theorem 6 in [31] characterizes terminal reward functions for which threshold controls are optimal and it states necessary and sufficient conditions on these functions. This implies for the mean field game that if $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and satisfies

- (i) $g(\cdot, q)$ is continuous and has at most exponential growth for any $q \in \mathbb{R}$,
- (ii) $g(\cdot, q)$ is convex on $(-\infty, q]$ and concave on $[q, \infty)$ for any $q \in \mathbb{R}$,
- (iii) for any $x \geq 0$ and $q \in \mathbb{R}$ it holds

$$\sigma_2 g(\sigma_1 x + q, q) + \sigma_1 g(-\sigma_2 x + q, q) = (\sigma_1 + \sigma_2) g(q, q),$$

then one can show along the same lines as for the terminal reward $\mathbb{1}_{(q(\mu, 1-\alpha), \infty)}(x)$ that the threshold control m_{b^*} is an equilibrium strategy.

Note that the function $\mathbb{1}_{(q(\mu, 1-\alpha), \infty)}(x)$ does not satisfy the conditions on g above because it is not continuous. However, one can approximate the function $\mathbb{1}_{(q(\mu, 1-\alpha), \infty)}(x)$ by functions satisfying the assumptions (i)-(iii).

Comparative statics

In this section we analyze how the equilibrium strategy of the mean field game depends on the model parameters and how a change of the model parameters affects the player's behavior in the equilibrium. We start by studying the threshold b^* .

Lemma 3.8. *b^* is monotonically decreasing in α and σ_1 , and monotonically increasing in σ_2 . Moreover, $\lim_{\alpha \downarrow 0} b^* = \infty$ and $\lim_{\alpha \uparrow 1} b^* = -\infty$.*

Proof. Follows directly from the explicit representation of b^* in (10). \square

The dependence of the threshold b^* on the parameter α is illustrated in Figure 1a.

Corollary 3.9. *The variance of the terminal state $X_T^{b^*}$ is given by*

$$\text{Var}(X_T^{b^*}) = (\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2)T, \quad (11)$$

i.e. it is monotonically decreasing in α , and monotonically increasing in σ_1 and σ_2 .

Proof. It holds

$$\text{Var}(X_T^{b^*}) = E\left[(X_T^{b^*})^2\right] = \int_{-\infty}^{\infty} x^2 p(T, -b^*, x - b^*) dx,$$

with the probability density function p given by Proposition 2.3. The integral can be simplified to the right-hand side of (11) using the definition of b^* in (10). \square

To further analyze the behavior of the players in the equilibrium we introduce the quantity

$$R := \frac{1}{T} \int_0^T P(m_{b^*}(X_t^{b^*}) = \sigma_2) dt. \quad (12)$$

Note that R represents the average time where a single player, using the equilibrium strategy, chooses the maximal volatility σ_2 . One can show that R only depends on α and on σ_1 and σ_2 through their ratio $\frac{\sigma_1}{\sigma_2}$.

For the rest of the section we focus on the case where α is small, i.e. we assume that $\alpha < \frac{1}{2} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}$. The other case, namely $\alpha > \frac{1}{2}$ or $\alpha \geq \frac{\sigma_2}{\sigma_1 + \sigma_2}$, can be treated similarly. Note that since $\alpha < \frac{\sigma_2}{\sigma_1 + \sigma_2}$ the threshold b^* is positive.

Proposition 3.10. *Let $\alpha < \frac{1}{2}$. Then*

- (i) *R only depends on α and $\gamma := \frac{\sigma_1}{\sigma_2}$, and the definition of R is independent of T . Moreover,*

$$R(\alpha, \gamma) = 1 - \alpha - q(\alpha, \gamma) \left(\alpha q(\alpha, \gamma) + \frac{2}{\gamma + 1} \varphi(q(\alpha, \gamma)) \right), \quad (13)$$

where $q(\alpha, \gamma) := \Phi^{-1}\left(\frac{\alpha}{2}(\gamma + 1)\right)$.

- (ii) *R is monotonically decreasing in γ and α .*

Proof. (i): Using (6) we see that

$$\begin{aligned}
R &= \frac{1}{T} \int_0^T P(X_t^{b^*} < b^*) dt \\
&= \frac{1}{T} \int_0^T \left(\frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left(\frac{b^*}{\sigma_2 \sqrt{t}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \right) dt \\
&= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \int_0^1 \Phi \left(\frac{b^*}{\sigma_2 \sqrt{Ts}} \right) ds - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \\
&= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \int_0^1 \Phi \left(-\frac{1}{\sqrt{s}} \Phi^{-1} \left(\frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right) \right) ds - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \\
&= \frac{2}{\gamma + 1} \int_0^1 \Phi \left(-\frac{1}{\sqrt{s}} \Phi^{-1} \left(\frac{\alpha}{2} (\gamma + 1) \right) \right) ds - \frac{1 - \gamma}{1 + \gamma},
\end{aligned}$$

where $\gamma = \frac{\sigma_1}{\sigma_2}$. Hence, R depends on α and γ , but does not depend on T . To obtain an explicit formula for R note that a primitive function of $(0, \infty) \rightarrow \mathbb{R}$, $s \mapsto \Phi \left(\frac{1}{\sqrt{s}} \right)$ is given by

$$(0, \infty) \rightarrow \mathbb{R}, s \mapsto -\Phi \left(-\frac{1}{\sqrt{s}} \right) + s\Phi \left(\frac{1}{\sqrt{s}} \right) + \sqrt{s}\varphi \left(\frac{1}{\sqrt{s}} \right),$$

where $\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$. Moreover, $q = q(\alpha, \gamma) = \Phi^{-1} \left(\frac{\alpha}{2} (\gamma + 1) \right) < 0$ since $\frac{\alpha}{2} (\gamma + 1) < \frac{1}{2}$. With this we can further simplify the formula for R :

$$\begin{aligned}
R &= \frac{2}{\gamma + 1} \int_0^1 \Phi \left(-\frac{1}{\sqrt{s}} q \right) ds - \frac{1 - \gamma}{1 + \gamma} \\
&= \frac{2}{\gamma + 1} q^2 \int_0^{q^{-2}} \Phi \left(\frac{1}{\sqrt{t}} \right) dt - \frac{1 - \gamma}{1 + \gamma} \\
&= \frac{2}{\gamma + 1} q^2 \left(-\Phi(-|q|) + \frac{1}{q^2} \Phi(|q|) + \frac{1}{|q|} \varphi(|q|) \right) - \frac{1 - \gamma}{1 + \gamma} \\
&= \frac{2}{\gamma + 1} q^2 \left(-\Phi(q) + \frac{1}{q^2} \Phi(-q) + \frac{1}{|q|} \varphi(-q) \right) - \frac{1 - \gamma}{1 + \gamma} \\
&= \frac{2}{\gamma + 1} q^2 \left(-1 - \frac{1}{q^2} \right) \Phi(q) + \frac{2}{\gamma + 1} q^2 \left(\frac{1}{q^2} + \frac{1}{|q|} \varphi(q) \right) - \frac{1 - \gamma}{1 + \gamma} \\
&= -\alpha (q^2 + 1) + \frac{2}{\gamma + 1} (1 + |q| \varphi(q)) - \frac{1 - \gamma}{1 + \gamma} \\
&= -\alpha (q^2 + 1) + \frac{2}{\gamma + 1} |q| \varphi(q) + 1 \\
&= 1 - \alpha - q(\alpha, \gamma) \left(\alpha q(\alpha, \gamma) + \frac{2}{\gamma + 1} \varphi(q(\alpha, \gamma)) \right).
\end{aligned}$$

(ii): We prove the statement by examining the monotonicity of the integrand in (12). Note that $P(m_{b^*}(X_t^{b^*}) = \sigma_2) = P(X_t^{b^*} < b^*)$ and $P(X_t^b < b)$ is differentiable in $b > 0$

with

$$\begin{aligned}\partial_b P(X_t^b < b) &= \partial_b \left(\frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left(\frac{b}{\sigma_2 \sqrt{t}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \right) \\ &= \frac{2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{t}} \varphi \left(\frac{b}{\sigma_2 \sqrt{t}} \right) > 0,\end{aligned}$$

for any $b > 0$ and $t > 0$. Hence, $P(X_t^b < b)$ is monotonically increasing in b and consequently R is monotonically increasing in b^* . Then the monotonicity of b^* implies the result. In more detail, for $0 < \alpha_1 < \alpha_2$ we have $b^*(\alpha_1) > b^*(\alpha_2)$ and thus $R(\alpha_1, \gamma) > R(\alpha_2, \gamma)$ for any $\gamma \in (0, 1)$, i.e. R is monotonically decreasing in α . For the monotonicity in γ note that one can without loss of generality assume that σ_2 is fixed. We observe that $b^* = -\sigma_2 \sqrt{T} \Phi^{-1} \left(\frac{\alpha}{2}(\gamma + 1) \right)$ is monotonically decreasing in γ , and hence also $P(X_t^{b^*} < b^*)$ and R are monotonically decreasing in γ . \square

Corollary 3.9 and Proposition 3.10 show that if the proportion α of best players who receive a reward is reduced, then the players choose more often the maximal volatility σ_2 and the variance of the terminal state increases. In other words, intensifying competition leads to riskier strategies. This observation is in line with results in [36] and [37], obtained within models, where agents can control the time horizon of comparison. It is shown that as the number of agents increases and hence the proportion of agents receiving a reward decreases, the agents choose in expectation larger stopping rules and hence increase the standard deviation of the states at the comparison time.

Moreover, Proposition 3.10 implies that R increases as γ decreases, which is equivalent to decreasing σ_1 or increasing σ_2 (see Figure 1b). This means, the smaller γ the more time a generic player in the equilibrium chooses the maximal volatility σ_2 . The quantity $\frac{1}{\gamma}$ can be seen as a measure of a single player's leeway for choosing an action. A small value for γ implies a large leeway since it corresponds to either a small value for σ_1 or a large value for σ_2 (or both).

Proposition 3.10 implies that in the equilibrium agents play in average riskier when the leeway increases. This can be explained as follows: the larger the leeway, the bigger

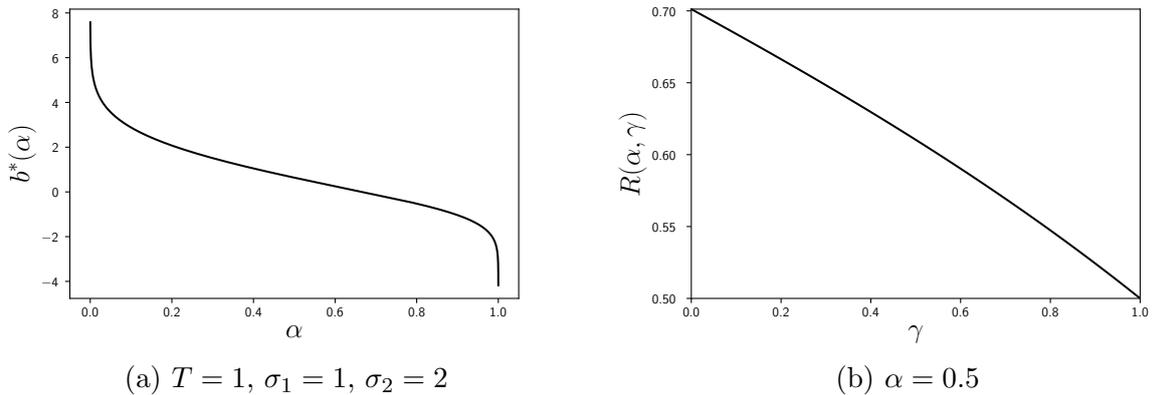


Figure 1: Dependence of b^* on α and R on γ .

is the opportunity for players choosing σ_2 to establish a lead over players choosing σ_1 . Thus, to preserve the chance to be among the best performing, the players need to choose a riskier strategy when the leeway of all players increases.

4 Approximate Nash equilibrium of the n -player game

We now come back to the n -player game introduced in Section 1. We show that the n -tuple consisting of the mean field equilibrium threshold strategies is an ε -Nash equilibrium in the n -player game, with ε converging to zero as n tends to infinity. Moreover, we determine a rate of convergence.

We first rigorously define what we mean by an ε -Nash equilibrium for the game at hand (cf., e.g., the similar definition in [4]).

Definition 4.1. Let $\varepsilon > 0$. A tuple $a = (a_1, \dots, a_n) \in \mathcal{A}_n^n$ is called ε -Nash equilibrium of the n -player game if for all $i \in \{1, \dots, n\}$, $c \in \mathcal{A}_n$ and weak solutions $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$, $(\Omega^{(a_{-i}, c)}, \mathcal{F}^{(a_{-i}, c)}, (\mathcal{F}_t^{(a_{-i}, c)})$, $P^{(a_{-i}, c)}$, $W^{(a_{-i}, c)}$, $X^{(a_{-i}, c)}$) we have

$$P^a(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) + \varepsilon \geq P^{(a_{-i}, c)}(X_T^{i,(a_{-i}, c)} > q(\mu^{n,(a_{-i}, c)}, 1 - \alpha)),$$

where $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$, and $\mu^{n,a}$, $\mu^{n,(a_{-i}, c)}$ are defined as in Section 1.

Our main result of this section is as follows:

Theorem 4.2. Let $n \geq 2$ and $a = (a_1, \dots, a_n)$ be the tuple of strategies in \mathcal{A}_n^n , defined by

$$a_i(x) := \begin{cases} \sigma_1, & \text{if } x_i \geq b^*, \\ \sigma_2, & \text{if } x_i < b^*, \end{cases} \quad (14)$$

for $x \in \mathbb{R}^n$ and

$$b^* := \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left(\frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left(\frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}. \end{cases}$$

Then there exists a sequence $\varepsilon_n \geq 0$ with $\lim_n \varepsilon_n = 0$ such that $a = (a_1, \dots, a_n)$ is an ε_n -Nash equilibrium of the n -player game. We can choose $\varepsilon_n \in \mathcal{O}(n^{-1/2})$.

Note that $\mathcal{O}(n^{-1/2})$ denotes the set of sequences $(u_n)_{n \in \mathbb{N}}$ such that there exists a constant $C > 0$ with $\limsup_{n \rightarrow +\infty} n^{1/2} |u_n| \leq C$.

Remark 4.3. Notice that for implementing the strategy (14) player i only needs to observe her own state. Therefore, the tuple of strategies (14) is also an approximate Nash equilibrium for the game version, where the players cannot observe each other.

Within the game version without observability the strategies can be assumed to be of open-loop type. In the game with observability closed-loop strategies are more appropriate, and therefore in Definition 4.1 we allow only for closed-loop equilibria. Notice, however, that the approximate equilibrium strategies (14) can be interpreted also as open-loop controls. We refer to Chapter 2 of [7] for more details on open- and closed-loop Nash equilibria.

In order to prove Theorem 4.2 we first state two lemmas. The first is due to results of Hoeffding and we state it in a version suitable for our setting. This result can be found, e.g., in Theorem 1 and Theorem 2 of [15] in a slightly more general form.

Lemma 4.4. *Let X_1, \dots, X_n be independent random variables with $0 \leq X_i \leq 1$ for $i = 1, \dots, n$, defined on a probability space (Ω, \mathcal{F}, P) . Then we have for $t > 0$*

$$P\left(\sum_{i=1}^n (X_i - EX_i) \geq nt\right) \leq e^{-2nt^2}.$$

Lemma 4.4 yields the next result for empirical quantiles.

Lemma 4.5. *Let X_1, \dots, X_n be i.i.d. random variables on a probability space (Ω, \mathcal{F}, P) and $\mu := P_{X_1}$. Define $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ and let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$. Then for all $\varepsilon > 0$ we have*

$$P(|q(\mu^n, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2e^{-2n\delta_\varepsilon^2},$$

where $\delta_\varepsilon := \min\{(1 - \alpha_n) - F_\mu(q(\mu, 1 - \alpha_n) - \varepsilon), F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon) - (1 - \alpha_n)\}$ and where F_μ denotes the distribution function of μ .

Proof. Let $\varepsilon > 0$. Note that

$$\begin{aligned} & P(q(\mu^n, 1 - \alpha_n) > q(\mu, 1 - \alpha_n) + \varepsilon) \\ &= P\left(\sum_{j=1}^n (\mathbb{1}_{(q(\mu, 1 - \alpha_n) + \varepsilon, \infty)}(X_j) - (1 - F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon)))\right. \\ &\qquad\qquad\qquad \left. > n(F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon) - (1 - \alpha_n))\right), \\ & P(q(\mu^n, 1 - \alpha_n) < q(\mu, 1 - \alpha_n) - \varepsilon) \\ &= P\left(\sum_{j=1}^n (\mathbb{1}_{(-\infty, q(\mu, 1 - \alpha_n) - \varepsilon]}(X_j) - F_\mu(q(\mu, 1 - \alpha_n) - \varepsilon))\right. \\ &\qquad\qquad\qquad \left. > n((1 - \alpha_n) - F_\mu(q(\mu, 1 - \alpha_n) - \varepsilon))\right). \end{aligned}$$

Now using Lemma 4.4 we obtain that

$$P(|q(\mu^n, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2 \exp(-2n\delta_\varepsilon^2).$$

□

Corollary 4.6. *Let $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$ be a weak solution of (1) for the control a defined in Theorem 4.2, and let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$. Moreover, let $\mu^{n,a} = \frac{1}{n} \sum_{i=1}^n \delta_{X_T^{i,a}}$ and $\mu := P_{X_T^{1,a}}^a$. If $0 < \varepsilon < M$, then there exists a constant $C > 0$ such that*

$$P^a (|q(\mu^{n,a}, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2e^{-2nC\varepsilon^2}.$$

Proof. Note that $X^{i,a}$ has the same distribution as the OBM Y^{0,b^*} with threshold b^* . Hence, if $0 < \varepsilon < M$, we can apply Lemma 4.5 and Lemma 2.5, and thus there exists a constant $C > 0$ such that $\delta_\varepsilon \geq C\varepsilon$. We obtain

$$P^a (|q(\mu^{n,a}, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2 \exp(-2nC^2\varepsilon^2).$$

□

Proof of Theorem 4.2. Let $i \in \{1, \dots, n\}$. We compare $a = (a_1, \dots, a_n)$ with the tuple where player i deviates from m_{b^*} by choosing a strategy $c \in \mathcal{A}_n$. To this end we write $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$. Note that for the control (a_{-i}, c) there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, X^{(a_{-i}, c)})$ of (2). For simplicity we omit the dependence on the strategy in the notation. Moreover, we can find a strong solution X^a of (2) for the control a on the same probability space and with the same Brownian motion: The controls a_j only depend on the j -th state and are uniformly bounded away from zero, which implies weak solvability. In addition, they are of bounded variation implying pathwise uniqueness. Hence, there is a unique strong solution (see also Remark 3.1). We emphasize that the distribution of $X^{j,a}$ is same as the distribution of the OBM Y^{0,b^*} with threshold b^* . We set $\mu := P_{X_T^{1,a}}$.

We need to show that

$$P(X_T^{i,(a_{-i},c)} > q(\mu^{n,(a_{-i},c)}, 1 - \alpha)) - P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \leq \frac{C}{\sqrt{n}}, \quad (15)$$

for some constant $C > 0$ independent of the control c . We denote by F^b the distribution function of an OBM with threshold level b at time T .

1. We first estimate $P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha))$ from below. Notice that $X_T^{i,a}$ and the empirical quantile $q(\mu^{n,a}, 1 - \alpha)$ are not independent. We therefore consider the empirical measure $\mu^{n-1,a} = \frac{1}{n-1} \sum_{j \neq i} \delta_{X_T^{j,a}}$ and define

$$A(n) := q\left(\mu^{n-1,a}, \frac{n}{n-1}(1 - \alpha)\right). \quad (16)$$

Notice that $A(n)$ is independent of $X_T^{i,a}$ and

$$A(n) \geq q(\mu^{n,a}, 1 - \alpha). \quad (17)$$

Thus,

$$P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \geq P(X_T^{i,a} > A(n)) = E[1 - F^{b^*}(A(n))], \quad (18)$$

2. Next we estimate the first term in (15) from above: We replace in (15) the quantile $q(\mu^{n,(a-i,c)}, 1 - \alpha)$ with a quantile that does not depend on c . To this end observe that

$$D(n) := q\left(\mu^{n-1,a}, 1 - \frac{n}{n-1}\alpha\right) \leq q(\mu^{n,(a-i,c)}, 1 - \alpha). \quad (19)$$

Indeed, we have since $(n-1)(1 - \frac{n}{n-1}\alpha) = n(1 - \alpha) - 1 < n(1 - \alpha)$

$$\begin{aligned} q\left(\mu^{n-1,a}, 1 - \frac{n}{n-1}\alpha\right) &= X_T^{(\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor; n-1), a-i} \\ &\leq X_T^{(\lfloor n(1-\alpha) \rfloor; n), (a-i, c)} = q(\mu^{n,(a-i,c)}, 1 - \alpha), \end{aligned}$$

where $X_T^{(\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor; n-1), a-i}$ denotes the $\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor$ -th order statistics of the $(n-1)$ states $X_T^{j,a}, j \neq i$, and $X_T^{(\lfloor n(1-\alpha) \rfloor; n), (a-i, c)}$ denotes the $\lfloor n(1-\alpha) \rfloor$ -th order statistics of the n states $X_T^{1,(a-i,c)}, \dots, X_T^{n,(a-i,c)}$. In other words, suppose that r is greater than or equal to the right-hand side of (19). Then at most $\lfloor n\alpha \rfloor$ players are better than r . In the $n-1$ game without player i then also at most $\lfloor n\alpha \rfloor$ players are better than r . This means $\mu^{n-1,a-i}((-\infty, r]) \geq 1 - \frac{n}{n-1}\alpha$. Hence we have shown (19).

From (19) we obtain

$$\begin{aligned} P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) &\leq P\left(X_T^{i,(a-i,c)} > D(n)\right) \\ &= E\left[P\left(X_T^{i,(a-i,c)} > D(n) \mid D(n)\right)\right]. \end{aligned} \quad (20)$$

If player i knew from the very beginning the value $D(n)$, then $m_{D(n)}$ would be the control maximizing the probability for player i 's state to be greater than $D(n)$ at time T : Note that we can assume, without loss of generality, that Ω is a complete, separable metric space and \mathcal{F} is its Borel- σ -algebra. Then Theorem 5.3.18 in [19] implies that there exists a regular conditional probability $Q : \Omega \times \mathcal{F} \rightarrow [0, 1]$ for \mathcal{F} given $\sigma(D(n))$ and Q satisfies

$$Q\left(\omega, \left\{X_T^{i,(a-i,c)} > D(n)\right\}\right) = Q\left(\omega, \left\{X_T^{i,(a-i,c)} > b\right\}\right) \Big|_{b=D(\omega,n)}, \quad (21)$$

for a.e. $\omega \in \Omega$. Since W^i is also a Brownian motion under $Q(\omega, \cdot)$, because W^i and $D(n)$ are independent, Lemma 3.4 implies that for any $b \in \mathbb{R}$ we have

$$Q\left(\omega, \left\{X_T^{i,(a-i,c)} > b\right\}\right) \leq Q\left(\omega, \left\{X_T^{i,(a-i,m_b)} > b\right\}\right), \text{ for } P\text{-a.e. } \omega \in \Omega. \quad (22)$$

Hence, with (21) and (22)

$$\begin{aligned} P\left(X_T^{i,(a-i,c)} > D(n) \mid D(n)\right)(\omega) &= Q\left(\omega, \left\{X_T^{i,(a-i,c)} > D(n)\right\}\right) \\ &\leq Q\left(\omega, \left\{X_T^{i,(a-i,m_b)} > b\right\}\right) \Big|_{b=D(\omega,n)}, \end{aligned} \quad (23)$$

for a.e. $\omega \in \Omega$. The random variable $X_T^{i,(a-i,m_b)}$ has the same distribution as the OBM with threshold level b at time T , and therefore with (23) we have

$$P\left(X_T^{i,(a-i,c)} > D(n) | D(n)\right) \leq 1 - F^b(b)|_{b=D(n)}, \text{ a.s.}$$

This implies using (20)

$$P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) \leq E[1 - F^b(b)|_{b=D(n)}]. \quad (24)$$

3. We can now combine the estimates in (18) and (24) above leading to

$$\begin{aligned} & P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) - P\left(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)\right) \\ & \leq E[F^{b^*}(A(n)) - F^{D(n)}(D(n))] \\ & \leq E\left|F^{b^*}(A(n)) - F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right)\right| \end{aligned} \quad (25)$$

$$+ \left|F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right) - F^{b^*}\left(q\left(\mu, 1 - \frac{n}{n-1}\alpha\right)\right)\right| \quad (26)$$

$$+ E\left|F^{b^*}\left(q\left(\mu, 1 - \frac{n}{n-1}\alpha\right)\right) - F^{b^*}(D(n))\right| \quad (27)$$

$$+ E\left|F^{b^*}(D(n)) - F^{D(n)}(D(n))\right|. \quad (28)$$

Using the Lipschitz continuity of F^{b^*} (see Lemma 2.5) we can estimate the term (25) as follows

$$\begin{aligned} & E\left|F^{b^*}(A(n)) - F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right)\right| \\ & \leq C_1 E\left[\left|A(n) - q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right| \wedge 2\right] \\ & = C_1 \int_0^\infty P\left(\left|A(n) - q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right| \wedge 2 > \varepsilon\right) d\varepsilon \\ & = C_1 \int_0^2 P\left(\left|A(n) - q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right| > \varepsilon\right) d\varepsilon \end{aligned}$$

for some constant $C_1 > 0$. Moreover, Corollary 4.6 yields that for some $C_2 > 0$ we have

$$P\left(\left|A(n) - q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right| > \varepsilon\right) \leq 2e^{-2(n-1)C_2\varepsilon^2}, \quad 0 < \varepsilon \leq 2,$$

and thus we obtain

$$\begin{aligned} & E\left|F^{b^*}(A(n)) - F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right)\right| \\ & \leq C_1 \int_0^2 2e^{-2(n-1)C_2\varepsilon^2} d\varepsilon \leq \frac{\sqrt{2\pi}C_1}{\sqrt{C_2}} \frac{1}{\sqrt{(n-1)}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \quad (29)$$

In exactly the same way one can estimate the term (27) and hence we have for some constant $C_3 > 0$

$$E\left|F^{b^*}\left(q\left(\mu, 1 - \frac{n}{n-1}\alpha\right)\right) - F^{b^*}(D(n))\right| \leq \frac{C_3}{\sqrt{n-1}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (30)$$

For the term (28) we obtain, using Lemma 2.6,

$$\begin{aligned} & E |F^{b^*}(D(n)) - F^{D(n)}(D(n))| \\ & \leq E [C_4 |b^* - D(n)| \wedge 2] \\ & \leq C_4 |b^* - q(\mu, 1 - \frac{n}{n-1}\alpha)| + C_4 E [|q(\mu, 1 - \frac{n}{n-1}\alpha) - D(n)| \wedge 2] \end{aligned}$$

for some constants $C_4 > 0$. The second term can be estimated using Corollary 4.6

$$\begin{aligned} E [|q(\mu, 1 - \frac{n}{n-1}\alpha) - D(n)| \wedge 2] &= \int_0^2 P(|q(\mu, 1 - \frac{n}{n-1}\alpha) - D(n)| > \varepsilon) d\varepsilon \\ &\leq \int_0^2 2e^{-2(n-1)C_5\varepsilon^2} d\varepsilon \leq \frac{\sqrt{2\pi}}{\sqrt{C_5}} \frac{1}{\sqrt{n-1}}, \end{aligned}$$

for some constant $C_5 > 0$. For the first term it holds that

$$\begin{aligned} |b^* - q(\mu, 1 - \frac{n}{n-1}\alpha)| &= \left| (F^{b^*})^{-1}(1 - \alpha) - (F^{b^*})^{-1}(1 - \frac{n}{n-1}\alpha) \right| \\ &\leq C_6 \left| (1 - \alpha) - (1 - \frac{n}{n-1}\alpha) \right| = \frac{C_6\alpha}{n-1}, \end{aligned}$$

for some constant $C_6 > 0$, because $\lim_{n \rightarrow \infty} q(\mu, 1 - \frac{n}{n-1}\alpha) = q(\mu, 1 - \alpha) = b^*$, and because the probability density function of the OBM with threshold b^* is locally uniformly bounded away from zero, e.g., on the interval $(b^* - \varepsilon, b^* + \varepsilon)$ for some fixed $\varepsilon > 0$. Therefore,

$$E |F^{b^*}(D(n)) - F^{D(n)}(D(n))| \leq \left(C_4 C_6 \alpha + \frac{\sqrt{2\pi} C_4}{\sqrt{C_5}} \right) \frac{1}{\sqrt{n-1}}. \quad (31)$$

Recall that $F^{b^*}(x) = \mu((-\infty, x])$, $x \in \mathbb{R}$, and hence the term (26) can be rewritten as follows

$$\begin{aligned} & |F^{b^*}(q(\mu, \frac{n}{n-1}(1 - \alpha))) - F^{b^*}(q(\mu, 1 - \frac{n}{n-1}\alpha))| \\ &= \left| \frac{n}{n-1}(1 - \alpha) - (1 - \frac{n}{n-1}\alpha) \right| = \frac{1}{n-1} \in \mathcal{O}\left(\frac{1}{n}\right). \end{aligned} \quad (32)$$

Finally, equations (29)-(32) imply that

$$P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) - P\left(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)\right) \leq \frac{C}{\sqrt{n-1}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

for an appropriate constant $C > 0$ independent of c . Thus, the feedback strategy $a = (m_{b^*}, \dots, m_{b^*})$ is an $\mathcal{O}(n^{-\frac{1}{2}})$ -Nash equilibrium. \square

Remark 4.7. Remark 3.7 presents a generalization of the mean field game to a more general reward function g . Under additional assumptions on g one can show that the strategy a , defined in (14), yields also an approximate Nash equilibrium in the n -player game with reward function g . Sufficient conditions for this result to hold true are, e.g., that g satisfies all the assumptions (i)-(iii) from Remark 3.7, g is uniformly bounded and $g(x, \cdot)$ is monotonically decreasing and continuous for all $x \in \mathbb{R}$.

5 Large ranking games with drift and common noise

In this section we consider an extension of our n -player game model described in Section 1. We make the same assumptions as for the n -player game, but we add a drift component and a common noise to the dynamics of each players' state process. The drift and common diffusion coefficient are allowed to depend on all the players' states. Moreover, in order to take into account the effect of the common noise, we allow the control of each player to depend on an additional factor, which is a one-dimensional process observable by all the players. More precisely, for $i = 1, \dots, n$, we assume that the state process of player i satisfies

$$\begin{aligned} dX_t^{i,a} &= b_0(t, X_t^{i,a}, X_t^{-i,a}) dt + a_i(t, X_t^{i,a}, X_t^{-i,a}, q_t^a) dW_t^i + \sigma_0(t, X_t^{i,a}, X_t^{-i,a}) dB_t, \\ X_0^{i,a} &= 0, \end{aligned} \quad (33)$$

with

$$dq_t^a = b_0(t, X_t^a) dt + \sigma_0(t, X_t^a) dB_t, \quad q_0^a = 0, \quad (34)$$

and where $b_0, \sigma_0 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable functions, $W = (W^1, \dots, W^n)$ is a Brownian motion on \mathbb{R}^n and B is a one-dimensional Brownian motion. We assume that, for any $i = 1, \dots, n$, the distribution of the bracket $\langle W^i, B \rangle$ is fixed. In particular, note that we do not necessarily assume that B is independent from W . The players' controls are now measurable functions $a_i : [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightarrow [\sigma_1, \sigma_2]$, $i = 1, \dots, n$. Slightly abusing notations, we still denote \mathcal{A}_n the set of such controls, and write $\mathcal{A}_n^n = \mathcal{A}_n \times \dots \times \mathcal{A}_n$ for the set of all strategy tuples.

Remark 5.1. A particular case covered by our analysis is when b_0 and σ_0 are functions of the empirical distribution of all the players' states. Even when b_0 and σ_0 are only functions of time (independent of the variable $x \in \mathbb{R}^n$), the dynamics given by (33) and (34) substantially generalizes the no common noise case, and allows to model stronger interactions (see Section 7 for an application).

Remark 5.2. For simplicity, b_0 and σ_0 appearing in (33) and (34) above do not depend on q_t^a . Adding this dependence would not entail additional difficulties in the proofs of Theorem 5.3, whose statement would remain the same.

As before we suppose that each player aims at maximizing the probability of her own state to be greater than the empirical $(1 - \alpha)$ -quantile of all states at time T , i.e. player i wants to maximize the quantity

$$P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha))$$

over all admissible controls a . In the case without drift we have shown that an approximate Nash equilibrium is given when each player chooses the threshold control m_{b^*} (see Theorem 4.2). In order to adapt this result to the case with drift and common noise, we need to allow the players' controls to depend on an additional factor q , that itself depends only on the common noise and drift coefficients. A weak solution to (33)-(34) is

now a tuple $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a)_{t \in [0, T]}, P^a, W^a, B^a, X^a, q^a)$ where $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a)$ is a filtered probability space supporting an n -dimensional $(\mathcal{F}_t^a)_{t \in [0, T]}$ -Brownian motion W and a one-dimensional $(\mathcal{F}_t^a)_{t \in [0, T]}$ -Brownian motion B , such that (X^a, q^a, W^a, B^a) solves (33)-(34), with both processes X^a and q^a being $(\mathcal{F}_t^a)_{t \in [0, T]}$ -adapted and continuous. Recall that we assume that the correlation between W^a and B^a is fixed, but that we do not necessarily assume that these processes are independent. An approximate Nash equilibrium is then defined in a way analogous to Definition 4.1 in this new context.

Finally, we assume that the coefficients b_0 and σ_0 are such that, for any $a \in \mathcal{A}_n^n$, there exists at least one weak solution to the system (33)-(34). An example of sufficient condition is the following: W^a and B^a are independent, b_0 and σ_0 are measurable and bounded, and σ_0 is additionally bounded away from 0.

Theorem 5.3. *Let $n \geq 2$ and $a = (a_1, \dots, a_n)$ be the tuple of strategies in \mathcal{A}_n^n , defined by*

$$a_i(t, x_1, \dots, x_n, q) := \begin{cases} \sigma_1, & \text{if } x_i \geq b^* + q, \\ \sigma_2, & \text{if } x_i < b^* + q, \end{cases} \quad (35)$$

for $(t, x, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$ and

$$b^* := \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left(\frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left(\frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}. \end{cases}$$

Then there exists a sequence $\varepsilon_n \geq 0$ with $\lim_n \varepsilon_n = 0$ such that $a = (a_1, \dots, a_n)$ is an ε_n -Nash equilibrium of the n -player game. We can choose $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$.

Remark 5.4. To be able to execute the optimal strategy a_i , player i needs to observe not only her own state X_i , but also the process q^a defined by (34). So in presence of a state-dependent drift and common noise coefficient, it is no longer the case that the tuple of closed-loop approximate optimal strategies provide also an approximate Nash equilibrium for the version of the game where the players cannot observe each other.

Remark 5.5. The fact that B and W in (33) and (34) need not be independent allows for different interpretations of the game. Indeed, consider for instance the particular case where $B = W^1$: this can be interpreted as a *break in symmetry* between the players, since the idiosyncratic noise of player 1 affects all the other players.

To prove Theorem 5.3, we use the method introduced in [26] and exploit the *translation invariance* of our reward functional, given by $g(x, \mu) = \mathbb{1}_{(1-\alpha, \infty)}(\mu((-\infty, x]))$ for $x \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the set of all probability measures on \mathbb{R} . Recall (see e.g. Definition 2.1 in [26]) that g is translation invariant if for any $x, y \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$ we have $g(x + y, \mu) = g(x, \mu(\cdot + y))$. Note that we do not directly use the results of [26] for the mean field version of the game, but adapt the results of [26] to the finite player case to show that from the ε_n -Nash equilibrium obtained in Theorem

4.2 for the game without common noise, we can construct an ε_n -Nash equilibrium of the finite player game, with common noise and drift.

First we show that for the candidate optimal strategy a defined above, pathwise uniqueness of the state equation holds true. This result is not used in the proof of Theorem 5.3, but it shows that for the equilibrium strategy derived in this section, the players' behavior is such that the associated processes are well-defined, and in particular that their law is unique.

Proposition 5.6. *Let $a \in \mathcal{A}_n^n$ be defined as in Theorem 5.3. Assume furthermore that the maps $x \mapsto b_0(\cdot, x)$ and $x \mapsto \sigma_0(\cdot, x)$ are Lipschitz continuous on \mathbb{R}^n , uniformly in t and have linear growth. Then pathwise uniqueness for the system (33)-(34) holds true and there exists a strong solution.*

Proof. To shorten the notations, we will write $X_t^a - q_t := (X_t^{1,a} - q_t, \dots, X_t^{n,a} - q_t)$. Define for $i = 1, \dots, n$, $Y_t^{i,a} := X_t^{i,a} - q_t^a$. Then this process Y^a satisfies,

$$dY_t^{i,a} = a_i(t, Y_t^{1,a} + q_t^a, \dots, Y_t^{n,a} + q_t^a, q_t^a) dW_t^{i,a}, i = 1, \dots, n, \quad (36)$$

$$dq_t^a = b_0(t, Y_t^{1,a} + q_t^a, \dots, Y_t^{n,a} + q_t^a) dt + \sigma_0(t, Y_t^{1,a} + q_t^a, \dots, Y_t^{n,a} + q_t^a) dB_t^a. \quad (37)$$

By definition of a_i , $a_i(t, y_i + q, q) = m_{b^*}(y_i)$ for any $t \geq 0$ and $y \in \mathbb{R}^n$. This implies that the n equations in (36) are decoupled and independent of q^a . Hence by Remark 3.1, each of these n equations enjoys pathwise uniqueness and has a strong solution. Now by our assumptions on b_0 and σ_0 , (37) also has a unique strong solution. The existence of a unique strong solution for the dynamics of $(X_t^a - q_t^a, q_t^a)$ entails the existence of a unique strong solution for the dynamics of (X_t^a, q_t^a) . \square

Proof of Theorem 5.3. Let $c \in \mathcal{A}_n$ and fix $i \in \{1, \dots, n\}$. As in the proof of Theorem 4.2, we compare a with the tuple where player i deviates from a_i by choosing a strategy c . To this end, we write $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$. Let $(\Omega^c, \mathcal{F}^c, (\mathcal{F}_t^c), P^c, W^c, B^c, X^c, q^c)$ be a weak solution to (33)-(34) for the strategy (a_{-i}, c) . We define the processes Y^a and Y^c by setting for $j = 1, \dots, n$ and $t \in [0, T]$

$$\begin{aligned} Y_t^{j,a} &:= X_t^{j,a} - q_t^a, \\ Y_t^{j,c} &:= X_t^{j,c} - q_t^c. \end{aligned}$$

Using again that $a_j(t, y_j + q, q) = m_{b^*}(y_j)$ for any $t \geq 0$ and $y \in \mathbb{R}^n$, we get that $(\Omega^c, \mathcal{F}^c, (\mathcal{F}_t^c), P^c, W^c, B^c, Y^c, q^c)$ is a weak solution to

$$dY_t^{j,c} = m_{b^*}(Y_t^{j,c}) dW_t^{j,c}, i = 1, \dots, n, j \neq i, \quad (38)$$

$$dY_t^{i,c} = c(Y_t^{1,c} + q_t^c, \dots, Y_t^{n,c} + q_t^c, q_t^c) dW_t^{i,c}, \quad (39)$$

$$dq_t^c = b_0(t, Y_t^{1,c} + q_t^c, \dots, Y_t^{n,c} + q_t^c) dt + \sigma_0(t, Y_t^{1,c} + q_t^c, \dots, Y_t^{n,c} + q_t^c) dB_t^c. \quad (40)$$

On the other hand, $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, B^a, Y^a, q^a)$ is a weak solution to

$$dY_t^{j,a} = m_{b^*}(Y_t^{j,a}) dW_t^{j,a}, i = 1, \dots, n, \quad (41)$$

$$dq_t^a = b_0(t, Y_t^{1,a} + q_t^a, \dots, Y_t^{n,a} + q_t^a)dt + \sigma_0(t, Y_t^{1,a} + q_t^a, \dots, Y_t^{n,a} + q_t^a)dB_t^a. \quad (42)$$

The states Y^a and Y^c correspond to the n -player game without drift and without common noise. Y^a are the states controlled with the approximate Nash equilibrium, given in Theorem 4.2, and Y^c represents the states where one player deviates from the approximate Nash equilibrium strategy by choosing the control \tilde{c} , given by $\tilde{c}(t, y, q) = c(t, y_1 + q, \dots, y_n + q, q)$, $t \in [0, T]$, $x \in \mathbb{R}^n$. Therefore, an immediate extension of Theorem 4.2 implies that

$$P^c(Y_T^{i,c} > q(\nu^{n,(a-i,c)}, 1 - \alpha)) - P^a(Y_T^{i,a} > q(\nu^{n,a}, 1 - \alpha)) \leq \varepsilon_n, \quad (43)$$

for some sequence $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$ independent of i , c and the choice of the weak solutions. Now, the translation invariance of our gain functional implies that for $d \in \{a, c\}$

$$P^d(X_T^{i,d} > q(\mu^{n,d}, 1 - \alpha)) = P^d(Y_T^{i,d} > q(\nu^{n,d}, 1 - \alpha)), \quad (44)$$

where $\mu^{n,d} = \frac{1}{n} \sum_{j=1}^n \delta_{X_T^{j,d}}$ and $\nu^{n,d} = \frac{1}{n} \sum_{j=1}^n \delta_{Y_T^{j,d}}$. Finally, (43) and (44) imply that

$$\begin{aligned} & P^c(X_T^{i,c} > q(\mu^{n,c}, 1 - \alpha)) - P^a(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \\ &= P^c(Y_T^{i,c} > q(\nu^{n,c}, 1 - \alpha)) - P^a(Y_T^{i,a} > q(\nu^{n,a}, 1 - \alpha)) \leq \varepsilon_n. \end{aligned}$$

□

6 Comparison with the two player case

In this section we consider our game with two players. We show that there exists a Nash equilibrium in an explicit form by using classical verification techniques for the Isaacs equations of the game. For simplicity we consider only time-homogeneous control functions. To this end, define $\tilde{\mathcal{A}}_2$ as the set of all measurable functions $a : \mathbb{R}^2 \rightarrow [\sigma_1, \sigma_2]$. Uniqueness in law holds true for the SDE (46) for all controls in $\tilde{\mathcal{A}}_2$ (see [21], Theorem 3). For controls in the larger control set \mathcal{A}_2 there are no results on uniqueness in law for (46) to the best of our knowledge. This means that the payoff of the players might depend on the chosen weak solution. Nevertheless, one can show that the results of this section also apply for the set \mathcal{A}_2 , because the controls a_1^* and a_2^* , defined in Theorem 6.1 below, are mutually best responses irrespective of the chosen weak solution. Our main result is as follows:

Theorem 6.1. *Let $\alpha = \frac{1}{2}$ and $n = 2$. Then the strategy $(a_1^*, a_2^*) \in \tilde{\mathcal{A}}_2^2$ is a Nash equilibrium of the two player game, where (a_1^*, a_2^*) is defined by*

$$a_1^*(x, y) = \begin{cases} \sigma_1, & \text{if } x \geq y, \\ \sigma_2, & \text{if } x < y, \end{cases} \quad (45)$$

and $a_2^*(x, y) = a_1^*(y, x)$ for $(x, y) \in \mathbb{R}^2$.

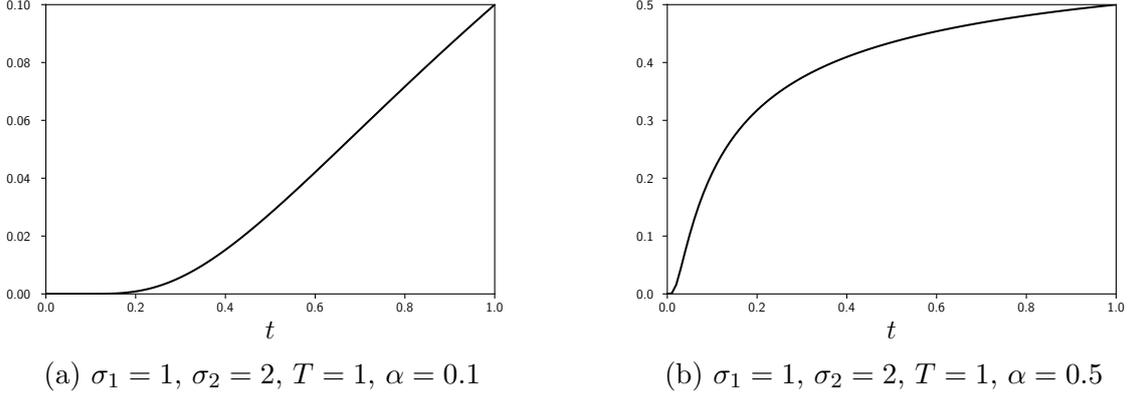


Figure 2: Part of the players choosing the small volatility σ_1 in dependence of time.

One can interpret Theorem 6.1 and 3.5 as providing Nash equilibria for the extreme cases $n = 2$ and $n = \infty$. Observe that the equilibrium strategies of the two extreme cases differ considerably. In the two player equilibrium the *relative* position of the own state with respect to the opponent's state determines the choice of volatility. In contrast, the strategy of the mean field equilibrium only depends on the *absolute* position of the own state. For implementing (45) a player needs to observe her own state process and the one of the opponent, whereas the mean field equilibrium only requires observation of the own state. The results thus indicate that any information about the other players' states becomes less useful as the number of players increases. This is plausible since in a symmetric equilibrium the empirical distribution of all players' states converges to the distribution of the own state as $n \rightarrow \infty$.

A comparison of Theorem 6.1 and 3.5 further reveals that a larger number of players incentivizes players to choose a high volatility more often. Indeed, in the two player case with $\alpha = \frac{1}{2}$ the player with the higher state always chooses the small volatility. In the large game with $\alpha = \frac{1}{2}$ the percentage of players choosing the smaller volatility increases over time and reaches 50% at T . One can show that all players choose σ_2 at the beginning (see Figure 2b).

A related two player diffusion control game has been studied in McNamara [33]. In [33], however, the two players control the same single state, whereas in our game each player controls her own process. As in our two player game only the relative position matters, one can derive the equilibrium from Theorem 6.1 by studying the difference of the state processes, which has a similar dynamics as the state in the game of [33].

We first describe the two player game in more detail before proving Theorem 6.1. We denote by X the state of player 1 and by Y the state of player 2. If the players choose the strategy $a \in \tilde{\mathcal{A}}_2^2$, their dynamics are governed by the SDE

$$\begin{aligned} dX_s^{t,x,y,a} &= a_1(X_s^{t,x,y,a}, Y_s^{t,x,y,a})dW_s^1, \\ dY_s^{t,x,y,a} &= a_2(X_s^{t,x,y,a}, Y_s^{t,x,y,a})dW_s^2, \quad s \in [t, T], \end{aligned} \quad (46)$$

with initial data $(X_t, Y_t) = (x, y) \in \mathbb{R}^2, t \in [0, T]$. As mentioned already in Section 1,

there exists a weak solution of (46) for any control $a \in \tilde{\mathcal{A}}_2^2$. Moreover, for any control $a \in \tilde{\mathcal{A}}_2^2$ uniqueness in law of the SDE (46) holds true because Theorem 3 in [21] applies for two-dimensional equations. Although for any weak solution the filtered probability space and the Brownian motion depend on the chosen control, we omit this dependence in this section to simplify notation.

Lemma 6.2. *For all $a \in \tilde{\mathcal{A}}_2^2$, $t \in [0, T)$ and $(x, y) \in \mathbb{R}^2$ we have*

$$P(X_T^{t,x,y,a} = Y_T^{t,x,y,a}) = 0.$$

Proof. Let $Z_s = X_s^{t,x,y,a} - Y_s^{t,x,y,a}$ for all $s \in [t, T]$. Moreover, we use the short hand notation $a_{i,s} = a_i(X_s^{t,x,y,a}, Y_s^{t,x,y,a})$. Notice that \tilde{W} , given by

$$\tilde{W}_s = \int_t^s \frac{1}{\sqrt{a_{1,r}^2 + a_{2,r}^2}} (a_{1,r} dW_r^1 - a_{2,r} dW_r^2), \quad s \geq t,$$

is a Brownian motion, and

$$dZ_s = \sqrt{a_{1,s}^2 + a_{2,s}^2} d\tilde{W}_s, \quad s \in [t, T]. \quad (47)$$

Observe that $\sqrt{a_{1,s}^2 + a_{2,s}^2}$ is bounded and bounded away from zero. Theorem 1 in [32] entails that $P(Z_T = 0) = 0$, and hence we obtain the result. \square

Note that for $\alpha = \frac{1}{2}$ and $n = 2$ the game described in Section 1 is for each player equivalent to the task of maximizing the probability of being ahead at time T . Moreover, for each player maximizing the probability of being ahead is equivalent to minimizing the probability for the opponent to be ahead at time T . Hence, it is enough to consider the payoff of player 1, which is given by

$$J(t, x, y, a) := P(X_T^{t,x,y,a} \geq Y_T^{t,x,y,a}) = P(X_T^{t,x,y,a} > Y_T^{t,x,y,a}).$$

The last equality follows from Lemma 6.2.

For $(t, x, y) \in [0, T) \times \mathbb{R}^2$ the lower value of the game is then defined by

$$\underline{V}(t, x, y) = \sup_{a_1 \in \tilde{\mathcal{A}}} \inf_{a_2 \in \tilde{\mathcal{A}}} J(t, x, y, a_1, a_2),$$

and the upper value of the game is defined by

$$\overline{V}(t, x, y) = \inf_{a_2 \in \tilde{\mathcal{A}}} \sup_{a_1 \in \tilde{\mathcal{A}}} J(t, x, y, a_1, a_2).$$

Notice that the definitions imply $\underline{V} \leq \overline{V}$.

We now prove Theorem 6.1 by constructing a solution to the Isaacs equations of the game and then show, by classical verification techniques, that they agree with the upper

and lower value function of the game. Moreover, we show that (a_1^*, a_2^*) is a saddle point of J , i.e.

$$\inf_{a_2 \in \tilde{\mathcal{A}}} J(t, x, y, a_1^*, a_2) = \sup_{a_1 \in \tilde{\mathcal{A}}} J(t, x, y, a_1, a_2^*) = V(t, x, y),$$

where the value function V is given by $V = \bar{V} = \underline{V}$. Thus we obtain that (a_1^*, a_2^*) is a Nash equilibrium.

The upper and lower Isaacs equation are given by

$$-w_t(t, x, y) - \frac{1}{2} \inf_{b \in [\sigma_1, \sigma_2]} \sup_{a \in [\sigma_1, \sigma_2]} (a^2 w_{xx} + b^2 w_{yy})(t, x, y) = 0, \quad (48)$$

$$-v_t(t, x, y) - \frac{1}{2} \sup_{a \in [\sigma_1, \sigma_2]} \inf_{b \in [\sigma_1, \sigma_2]} (a^2 v_{xx} + b^2 v_{yy})(t, x, y) = 0, \quad (49)$$

with $w(T, x, y) = \mathbb{1}_{(0, \infty)}(x - y)$ and $v(T, x, y) = \mathbb{1}_{(0, \infty)}(x - y)$, respectively. Note that (48) and (49) coincide and hence it is enough to consider either one of those equations. Moreover, one can easily verify that the function G , defined by

$$G(t, x, y) := \Phi \left(\frac{x - y}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}} \right), \quad (t, x, y) \in [0, T) \times \mathbb{R}^2,$$

and $G(T, x, y) := \mathbb{1}_{(0, \infty)}(x - y)$, solves both (48) and (49). We show by a classical verification that G coincides with \underline{V} and \bar{V} .

Lemma 6.3. $\underline{V}(t, x, y) = \bar{V}(t, x, y) = G(t, x, y)$ for $(t, x, y) \in [0, T) \times \mathbb{R}^2$ and hence the value function of the problem is given by

$$V(t, x, y) = \Phi \left(\frac{x - y}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}} \right), \quad (t, x, y) \in [0, T) \times \mathbb{R}^2.$$

Proof. We show that $G = \bar{V}$ and $G = \underline{V}$ on $[0, T) \times \mathbb{R}^2$. Hence we obtain all the statements of Lemma 6.3. Note that for $t \in [0, T)$ and $(x, y) \in \mathbb{R}^2$ we have

$$\partial_{xx} G(t, x, y) = \partial_{yy} G(t, x, y) \begin{cases} \leq 0, & \text{if } x \geq y, \\ > 0, & \text{if } x < y, \end{cases} \quad (50)$$

and

$$\partial_t G(t, x, y) + \frac{1}{2} \partial_{xx} G(t, x, y) (\sigma_1^2 + \sigma_2^2) = 0 \quad (51)$$

since G satisfies (48) and (49).

We start with showing that $G \leq \bar{V}$. To this end fix $a_2 \in \tilde{\mathcal{A}}$. By (50) and (51) we have

$$0 \leq \partial_t G(t, x, y) + \frac{1}{2} \partial_{xx} G(t, x, y) (a_1^*(x, y)^2 + a_2(x, y)^2) \quad (52)$$

for all $(t, x, y) \in [0, T) \times \mathbb{R}^2$. Now fix $(t, x, y) \in [0, T) \times \mathbb{R}^2$, set $\hat{a} := (a_1^*, a_2)$ and let $(X, Y) = (X^{t, x, y, \hat{a}}, Y^{t, x, y, \hat{a}})$ be a weak solution of the state SDE (46) on the time interval

$[t, T]$ with initial condition (x, y) . Recall that the solution (X, Y) is unique in law. By Itô's formula and (52), for all $\delta \in (0, T - t)$ we have

$$\begin{aligned} & G(T - \delta, X_{T-\delta}, Y_{T-\delta}) \\ &= G(t, x, y) + M_{T-\delta} + \int_t^{T-\delta} \left(\partial_t G + \frac{1}{2} \partial_{xx} G ((a_1^*)^2 + a_2^2) \right) (s, X_s, Y_s) ds \\ &\geq G(t, x, y) + M_{T-\delta}, \end{aligned}$$

where M_s , $s \in [t, T)$, is a martingale with $M_t = 0$. By first taking expectations, and then letting $\delta \downarrow 0$, we arrive by dominating convergence at

$$G(t, x, y) \leq \lim_{\delta \downarrow 0} E[G(T - \delta, X_{T-\delta}, Y_{T-\delta})] = P(X_T \geq Y_T) = J(t, x, y, a_1^*, a_2).$$

Note that we can apply dominated convergence since by Lemma 6.2 the discontinuity points of $G(T, \cdot, \cdot)$ are attained with probability zero. Since a_2 is arbitrary, we also have $G(t, x, y) \leq \inf_{a_2 \in \tilde{\mathcal{A}}} J(t, x, y, a_1^*, a_2) \leq \bar{V}(t, x, y)$.

We next show that $G \geq \bar{V}$. To this end fix $a_1 \in \tilde{\mathcal{A}}$. By (50) and (51) we have

$$0 \geq \partial_t G(t, x, y) + \frac{1}{2} \partial_{xx} G(t, x, y) (a_1(x, y)^2 + a_2^*(x, y)^2) \quad (53)$$

for all $(t, x, y) \in [0, T) \times \mathbb{R}^2$. Now fix $(t, x, y) \in [0, T) \times \mathbb{R}^2$, set $\hat{a} := (a_1, a_2^*)$ and let $(X, Y) = (X^{t, x, y, \hat{a}}, Y^{t, x, y, \hat{a}})$ be a weak solution of the state SDE (46) on the time interval $[t, T]$ with initial condition (x, y) . With Itô's formula and (53) one can derive similar to the reasoning above that

$$E[G(T - \delta, X_{T-\delta}, Y_{T-\delta})] \leq G(t, x, y)$$

for any $\delta \in (0, T - t)$. Letting $\delta \downarrow 0$ we obtain $G(t, x, y) \geq J(t, x, y, a_1, a_2^*)$, and hence also $G(t, x, y) \geq \bar{V}(t, x, y)$. Thus, we have shown $G = \bar{V}$ and

$$\sup_{a_1 \in \tilde{\mathcal{A}}} J(t, x, y, a_1, a_2^*) = J(t, x, y, a_1^*, a_2^*) = \inf_{a_2 \in \tilde{\mathcal{A}}} J(t, x, y, a_1^*, a_2). \quad (54)$$

The proof of $G = \underline{V}$ is similar and therefore omitted. \square

Proof of Theorem 6.1. The statement follows immediately from the proof of Lemma 6.3: (54) implies that (a_1^*, a_2^*) is a saddle point of J and thus (a_1^*, a_2^*) is a Nash equilibrium of the two player game. \square

7 Application: Risk taking by mutual fund managers

The game considered in this paper is formulated in a generic way and thus allows for multiple interpretations. In the following we illustrate an interpretation in terms of a competition among mutual funds.

Suppose that the players are managers of mutual funds, with state X^i representing the return process of the fund managed by player i . Assume further that the return dynamics satisfy (33). The dB -term in (33) then represents the systematic risk of all funds, and the dW^i -term the idiosyncratic risk of fund i . The process q^a can be viewed as the market return that is common to all states. Note that we allow the market return process to depend on all the players' returns. By shifting the return process by q^a one obtains the idiosyncratic return dynamics (1).

Each manager can control the volatility of her mutual fund by holding risky or less risky assets in her portfolio. A maximal implementable volatility σ_2 reflects risk constraints that are imposed on the mutual funds, e.g. by a regulator. A minimal positive volatility σ_1 means that the managers cannot completely eliminate idiosyncratic risk. The reward of the game can be interpreted as an additional income that manager i receives at T , e.g. at the end of the calendar year, if the return of fund i until T is among the highest α .

The additional income can be, e.g., a bonus that the manager receives from the fund's owner if the fund's performance is among the best of the year. Alternatively, the additional income can be generated by an increase of the size of the fund. The size of inflow into a fund and its outflow depends on the performance of the fund. There is empirical evidence that the flow does not depend linearly on the performance. Indeed, the results from [9] indicate that net inflow into a fund due to a good performance is larger than the net outflow due to a bad performance (see Section III in [9]).

Our results show that the fund managers have an incentive to alter the riskiness of the portfolio once the portfolio performance, corrected by the market return q^a , has attained the winning level, the equilibrium threshold b^* , that only α funds will have attained by the end of the year. Below the winning level the managers are willing to increase the volatility of their portfolios even without additional expected return. In other words, the managers are willing to gamble in order to maximize the probability of being among the best. Thus, our results reveal that the winner-take-all bonus scheme of the game leads to risk incentives that are not in the interest of risk-averse fund participants. Indeed, a risk averse participant always prefers the smallest volatility, given that the expected returns are always the same.

The implication that mutual funds reduce their riskiness once they have attained the winning level goes in line with the observation reported in [9], Section V, that mutual funds tend to lock in gains in the final quarter of the year if they have well performed in the first three quarters.

Note that if α is not too large, then b^* is positive and hence the percentage of funds having attained the threshold increases over the course of the year. This entails that also the number of funds choosing the minimal volatility σ_1 increases over the year (see Figure 2a).

The comparative statics results at the end of Section 3 imply that the managers will opt for the maximal volatility more often if α decreases or if $1/\gamma$, the range of implementable volatilities, increases. Moreover, the variance of the returns increases as α decreases and $1/\gamma$ increases.

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