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# Large ranking games with diffusion control

S. Ankirchner <sup>\*</sup>   N. Kazi-Tani <sup>†</sup>   J. Wendt <sup>‡</sup>   C. Zhou <sup>§</sup>

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We consider a symmetric stochastic differential game where each player can control the diffusion intensity of an individual dynamic state process, and the players whose states at a deterministic finite time horizon are among the best  $\alpha \in (0, 1)$  of all states receive a fixed prize. Within the mean field limit version of the game we compute an explicit equilibrium, a threshold strategy that consists in choosing the maximal fluctuation intensity when the state is below a given threshold, and the minimal intensity else. We show that for large  $n$  the symmetric  $n$ -tuple of the threshold strategy provides an approximate Nash equilibrium of the  $n$ -player game. We also derive the rate at which the approximate equilibrium reward and the best response reward converge to each other, as the number of players  $n$  tends to infinity. Finally, we compare the approximate equilibrium for large games with the equilibrium of the two player case.

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## 1 Introduction

We start by describing a game that models, in a stylized form, the competition among many agents who can each control the fluctuation intensity of an individual state until a

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<sup>\*</sup>Stefan Ankirchner, Institute for Mathematics, University of Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany. *Email*: s.ankirchner@uni-jena.de.

<sup>†</sup>Nabil Kazi-Tani, Institut Elie Cartan de Lorraine, Université de Lorraine, UFR MIM, 3 rue Augustin Fresnel. 57073 Metz Cedex 03, France. *Email*: nabil.kazi-tani@univ-lorraine.fr.

<sup>‡</sup>Julian Wendt, Institute for Mathematics, University of Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany. *Email*: julian.wendt@uni-jena.de.

<sup>§</sup>Chao Zhou, Department of Mathematics and Risk Management Institute, National University of Singapore, 10 Lower Kent Ridge Road, 119076 Singapore. *Email*: matzc@nus.edu.sg.

fixed time horizon  $T \in (0, \infty)$  and who receive some benefit if their final state is among the highest  $\alpha \in (0, 1)$ . Our game model is generic and allows for various interpretations. We refer to the end of this section for a detailed description of applications.

Throughout let  $n$  be a large natural number representing the number of players. Let  $0 < \sigma_1 < \sigma_2$ . The set of strategies for each player  $i$  is denoted by  $\mathcal{A}_n$  and consists of the set of measurable functions  $a : [0, T] \times \mathbb{R}^n \rightarrow [\sigma_1, \sigma_2]$ . We write  $\mathcal{A}_n^n := \mathcal{A}_n \times \dots \times \mathcal{A}_n$  for the set of all strategy tuples.

We denote the state process of player  $i$  by  $X^i$ . Given that each player chooses a strategy  $a_j \in \mathcal{A}_n, j \in \{1, \dots, n\}$ , we assume that the state process of player  $i$  satisfies

$$dX_t^{i,a} = a_i(t, X_t^{i,a}, X_t^{-i,a})dW_t^i, \quad (1)$$

where  $X^{-i,a}$  denotes the  $n - 1$ -dimensional process of the states of all players other than  $i$ . Moreover,  $W = (W^1, \dots, W^n)$  denotes a  $n$ -dimensional Brownian motion on some filtered probability space. Note that the  $n$ -dimensional process  $X^a = (X^{1,a}, \dots, X^{n,a})$  satisfies the  $n$ -dimensional SDE

$$dX_t = D(t, X_t)dW_t, \quad X_0 = 0, \quad (2)$$

where  $D(t, x)$  is the diagonal matrix with the entries  $a_1(t, x), \dots, a_n(t, x)$  on the diagonal. The existence of a weak solution of (2) follows from Theorem 2.6.1 in [23] since  $D(t, x)$  is positive definite. Recall that a weak solution consists of a 6-tuple  $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a)_{t \geq 0}, P^a, W^a, X^a)$ , where the first four components build a filtered probability space,  $W^a$  is a Brownian motion with respect to the filtration  $(\mathcal{F}_t^a)$ , and  $X^a$  is a continuous  $(\mathcal{F}_t^a)$ -adapted process such that the pair  $(X^a, W^a)$  satisfies (2) (see e.g. [20], Section 5.3, for more details). We add the superscript  $a$  to the elements of the weak solution in order to stress the dependence on the strategy tuple. Note that for different strategy tuples we may need to consider different probability spaces.

We suppose that each player aims at maximizing the probability of her own state to be greater than the empirical  $(1 - \alpha)$ -quantile of all states at time  $T$ . More precisely, let

$$\mu^{n,a} = \frac{1}{n} \sum_{i=1}^n \delta_{X_T^{i,a}}$$

be the empirical distribution of the players' states at time  $T$ . We define  $q(\mu^{n,a}, 1 - \alpha) = \inf\{r \in \mathbb{R} : \mu^{n,a}((-\infty, r]) \geq 1 - \alpha\}$ . Note that  $X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)$  if and only if the state of player  $i$  is among the best  $\lfloor n\alpha \rfloor$  players at time  $T$  (see Remark 1.1 below).

We now briefly comment on the assumptions of the game. First notice that the expectation of any player's terminal state is always zero, whatever control is chosen. However, other distributional properties such as the variance, the skewness and the position of quantiles do depend on the strategy. For example, a larger  $a^i$  leads to a larger variance of the final outcome  $X_T^i$ . Moreover, if player  $i$  chooses large values for  $a^i$  when her state is small and small values when her state is large, then the distribution of  $X_T^i$  has a negative skew. In particular, a state-dependent choice of the fluctuation intensity

allows to influence the position of quantiles, and hence the probability of receiving the reward. So the game helps in understanding how a rank-based reward determines the distributional properties of the players' states and by which dynamic strategies they can be implemented.

In the following remark we relate the reward condition to the order statistics of the terminal states.

**Remark 1.1.** Let  $a \in \mathcal{A}_n^n$ . We can relate the quantile  $q(\mu^{n,a}, 1 - \alpha)$  with the order statistics  $X_T^{(1:n),a} \leq \dots \leq X_T^{(n:n),a}$  of the states  $X_T^{1,a}, \dots, X_T^{n,a}$ . We have  $q(\mu^{n,a}, 1 - \alpha) = X_T^{(\lceil n(1-\alpha) \rceil : n), a}$ , and hence

1. if  $n\alpha \in \mathbb{N}$ , then  $X_T^i > q(\mu^{n,a}, 1 - \alpha)$  is equivalent to being under the best  $n\alpha$  players at time  $T$ , and
2. if  $n\alpha \notin \mathbb{N}$ , then  $X_T^i > q(\mu^{n,a}, 1 - \alpha)$  is equivalent to being under the best  $\lfloor n\alpha \rfloor$  players at time  $T$ .

Consequently,  $X_T^i > q(\mu^{n,a}, 1 - \alpha)$  is always equivalent to being under the best  $\lfloor n\alpha \rfloor$  players at time  $T$ .

The aim of the present article is to predict and analyze the players' behavior in the game. As usual, for this purpose we resort to Nash equilibria, which here are defined as follows.

**Definition 1.2.** A tuple  $a = (a_1, \dots, a_n) \in \mathcal{A}_n^n$  is called *Nash equilibrium* of the  $n$ -player game if for all  $i \in \{1, \dots, n\}$ ,  $c \in \mathcal{A}_n$  and weak solutions  $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$ ,  $(\Omega^{(a-i,c)}, \mathcal{F}^{(a-i,c)}, (\mathcal{F}_t^{(a-i,c)}), P^{(a-i,c)}, W^{(a-i,c)}, X^{(a-i,c)})$  we have

$$P^a(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \geq P^{(a-i,c)}(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)), \quad (3)$$

where  $(a-i, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$ .

Note that we do not assume that the weak solutions are unique in law and hence that the probabilities appearing in (3) are independent of the weak solution chosen<sup>1</sup>. Thus one can interpret (3) as follows: whatever weak solutions come up, with whatever distributions, no player has an incentive to deviate. However, we will see that for the Nash equilibria derived in this article uniqueness in law is always satisfied.

For the two player case with  $\alpha = \frac{1}{2}$  one can show (see Section 6 below) that a Nash equilibrium is given when each player chooses, at any time  $t < T$ , the minimal volatility  $\sigma_1$  if her current state value is larger than the opponent's state, and the maximal volatility  $\sigma_2$  if her current state value is smaller.

We believe that for more than two players it is difficult to obtain a Nash equilibrium in closed form. To circumvent this difficulty, our idea is to compute an approximate

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<sup>1</sup>It is still an open problem to find conditions, sufficient and necessary at the same time, guaranteeing uniqueness in law of weak solutions of (2) in dimensions greater than 2 (see e.g. abstract of [24]).

Nash equilibrium for large games by considering the mean field limit of the game. The mean field game approach, introduced by [28], [29], [30] and [18], [17], is based on the observation that in a symmetric game with many similar players, the empirical distribution of the realized players' states can be approximated with the state distribution of a *single* player. Thus, the Nash equilibrium for the game with many players can be approximated by solving a single agent control problem where the reward depends on the distribution of the state.

We obtain that, within the mean field limit version of the game, there exists an equilibrium that consists in choosing the maximal fluctuation intensity when the state is below a given threshold, and the minimal intensity else (see Section 3). We refer to such a strategy as a threshold strategy.

The threshold of the mean field equilibrium strategy is chosen such that it coincides with the  $(1 - \alpha)$ -quantile of the resulting final state distribution. The threshold can be identified as a fixed point of a mapping from the set of threshold strategies to itself. Fixed point arguments are a common tool for determining equilibria in mean field games (see, e.g., [8] and [9]).

Controlling the state with a threshold strategy entails that the state dynamics are a so-called oscillating Brownian motion (OBM), a process introduced in [21] and studied in [32] from a control theory perspective without interaction. The distribution of OBM is explicitly known at deterministic times. This allows us to compute the threshold of the mean field equilibrium in closed form.

We further show that the symmetric  $n$ -tuple of the mean field equilibrium is an approximate Nash equilibrium of the  $n$ -player game (see Section 4). More precisely, the difference of the expected payoff under the approximate equilibrium strategies and the best response payoff converges to zero as the number of players  $n$  tends to infinity. We show that the convergence is of the order  $\mathcal{O}(n^{-1/2})$ .

In the approximate equilibrium the distribution of the final states is negatively skewed and the  $(1 - \alpha)$ -quantile is further to the right than when using a strategy with constant fluctuations. Choosing the maximal fluctuation intensity below the threshold increases risk, measured, e.g., in terms of the variance of the final state. Thus, the results show that the rank-based reward incentivizes players to introduce a negative skew, and a high variance.

For implementing the threshold strategy of the approximate Nash equilibrium each player only needs to observe her own state process, but not the other players' states. This is a difference to the two player equilibrium, where each player chooses a high volatility if and only if her state is smaller than the opponent's state. In the two player equilibrium the *relative* position of the own state with respect to the opponent's state determines the choice of volatility. In contrast, in the mean field equilibrium, and hence in the approximate equilibrium for large games, the chosen volatility depends only on the *absolute* position of the own state, namely whether it is above or below the fixed equilibrium threshold (see Remark 4.3). The results indicate that in an exact equilibrium in games with  $n \geq 3$  players, each player makes her control dependent on both the absolute position of the own state and the relative position with respect to other players' states.

We remark that for the game with  $n \geq 3$  players it is already difficult to prove, by using abstract arguments, existence of an exact equilibrium in the first place, because of the discontinuity in the reward function. This applies to any game with discontinuous rank-based rewards. A workaround is to consider rewards depending continuously on the average state of all players. Espinosa and Touzi [13] consider a game where each player aims at maximizing the expected utility of her income depending on the deviation of the player's state from the average state. They characterize a Nash equilibrium in terms of a system of coupled backward stochastic differential equations (BSDEs). [10] establishes a Nash equilibrium in a market with  $N$  agents with the performance criteria of relative wealth level when the mean return rate is unobservable. Each investor has a heterogeneous prior belief on the return rate of the risky asset. By a separation result and a martingale argument, they show that the optimal investment strategy under this setting can be characterized by a fully-coupled linear FBSDE system. Two sets of deep neural networks are used for first computing each investor's estimate of the mean return rate and then numerically solving the FBSDEs. [14] uses a probabilistic approach to study an  $N$ -player exponential utility game and a mean field exponential utility game. Each player manages two stocks; one is driven by an individual shock and the other is driven by a common shock. Moreover, each player is concerned not only with her own terminal wealth but also with the relative performance of her competitors. A multi-dimensional FBSDE with quadratic growth (a mean field FBSDEs, respectively) is derived to characterize an equilibrium of the  $N$ -player game (of the mean field game, respectively). In [12], the authors solve a sequential game between a principal and several agents in competition with relative performance concerns. In particular, they connect the issue of existence of a Nash equilibrium between the competing agents to the existence of a solution to a multidimensional quadratic BSDE. In the present paper, we avoid BSDE arguments and solve our problem in a more direct way, by relying instead on the knowledge of the oscillating Brownian motion's marginal distributions.

As already mentioned above, when the agents are numerous and homogeneous, it is tempting to assume that one particular agent does not affect the empirical distribution of players. The symmetry in the game allows to focus on only one representative agent, with a *fixed* distribution of other players' states. This idea has been made precise in the theory of stochastic differential games with mean field interactions, that have been introduced independently in [18] and [30] and then received a sustained attention in the stochastic control and differential games communities. For an overview of the theory, we refer the reader to the monographs [8, 9].

On the one hand, the fact that Nash equilibria in finite  $n$ -player games converge to the corresponding mean field limit justifies the use of mean field games (MFG). It has been proved under different sets of assumptions, using probabilistic weak formulations and compactness arguments [25, 26], or via a partial differential equation (PDE) approach [7]. In [7], the authors rely on the so called *master equation*, which is an infinite dimensional PDE describing the value function of the game, to prove that the empirical distribution of the players converges to the mean field equilibrium measure, which is unique given the monotonicity condition made in [7].

On the other hand, a MFG solution can be used to construct an approximate Nash

equilibrium for the corresponding finite  $n$ -player game: this is the approach that we follow in Section 4 of the present paper.

One particular feature of our framework is the control of the diffusion coefficient in (1). Specific examples of both standard and mean field games with volatility control include optimal energy consumption [11], green bonds investments [1], or optimal contracting between hierarchized players [19].

The idea to use mean field limits for analyzing differential games with rank-based rewards has been employed already in the articles [3],[4], [37] and [5]. All these articles consider versions of a rank-based mean field game, where agents influence the drift rate of the state, but not the diffusion rate.

The articles [3], [4], [37] provide conditions guaranteeing that the mean field games approximate the corresponding  $n$ -player game for large  $n$ ; and they also determine the rate at which the expected reward of the mean field approximation and the best response converge to each other, as  $n \rightarrow \infty$ . Motivated by these articles, in Section 4 we also compute a convergence rate for the game at hand. In the recent paper [2], the authors analyze a mean field game where the diffusion coefficient can be controlled and show that the solution of the problem is characterized by a McKean-Vlasov second order backward SDE. Note that it is not possible to directly apply the results of [2] to our setting, since the particular form of our criteria given as a probability in Definition 1.2 would make the terminal condition of the related second order BSDE discontinuous, which is not allowed in [2].

## Applications

The game considered in this paper is formulated in a generic way and thus allows for multiple interpretations. To illustrate this fact, we provide some example interpretations in risk management, animal reproduction behavior, or political science.

**Risk management.** We consider first the competition among managers receiving a bonus if their firm is among the best performing. In this example the state  $X^i$  represents the value increment process of firm  $i$ , led by manager  $i$ . Each manager can take decisions determining the volatility of the value process. Risk constraints imposed by a regulator, or the firms' owners, entail that the manager must not take decisions leading to a volatility higher than  $\sigma_2$ . Also the most prudent decisions imply a positive volatility, given by  $\sigma_1$ . The reward of the game represents the bonus that manager  $i$  receives at  $T$ , e.g. at the end of the year, if the value increment of firm  $i$  until  $T$  is among the highest  $\alpha$ .

**Animal mating.** As a second application consider an animal social group, where the fittest  $\alpha$  have the opportunity to reproduce at a mating date  $T$ . Here the state  $X^i$  can be interpreted as the level of fitness of animal  $i$ . Suppose that each animal controls its diffusion intensity by choosing among a range of food search strategies and that the quantity of food found determines the level of fitness. Dynamics (1) means that each

animal has the choice between strategies with a possible high yield, but also high energy consumption, and strategies with a low expected yield and energy consumption. The Nash equilibrium of the game predicts how animals will control their level of fitness in order to maximize the probability of reproduction. We refer to [35] for an overview on game theoretical concepts for analyzing animal behavior.

**Political science.** Finally, consider the situation where the dynamics (1) models the states of political candidates, competing for an election taking place at time  $T$ . More precisely,  $X^i$  represents here a level of favourable opinions, or votes, according to polls that are updated dynamically. It is standard to use Brownian diffusions in order to model outcomes in political science [6, 15]. In our case, candidate  $i$  determines the level of risk that she takes by controlling the diffusion coefficient:  $\sigma_2$  represents a high, or maximal level of risk, in the sense that a political announcement or decision leads to highly unpredictable outcomes on the public opinion, while  $\sigma_1$  is associated with a minimal risk level, i.e. with political announcements for which the outcome on opinions is more predictable, at least on a large part of the voters. Each candidate observes both the behavior and poll results of her opponents and determines a level of risk taken so as to maximize the probability of being ranked first.

The rest of the paper is organized as follows: In Section 2, we provide some results on the oscillating Brownian motion, that will be used in Section 3 to explicitly solve the game in the mean field regime. In Section 4, we construct approximate Nash equilibria for the finite  $n$ -player game, based on the mean field optimal strategies. In Section 5, we generalize these results to the case where the players' state processes have an additional purely time-dependent drift. Finally, in Section 6, we solve the 2-player game by explicitly solving the associated dynamic programming PDE, and give some comparison with the mean field equilibrium.

## 2 Oscillating Brownian motion

In this section we define the *oscillating Brownian motion* (OBM) introduced in [21]. We summarize important facts about OBMs that we will need in the course of this paper.

Let  $B$  be a Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Furthermore, let  $(\mathcal{F}_t)_{t \geq 0}$  be the augmented Brownian filtration, generated by  $B$  and augmented by the  $P$ -null sets in  $\mathcal{F}$ . Let  $0 < \sigma_1 < \sigma_2$  and define for any  $b \in \mathbb{R}$  the threshold function  $m_b : \mathbb{R} \rightarrow [\sigma_1, \sigma_2]$  by

$$m_b(x) = \begin{cases} \sigma_1, & \text{if } x \geq b, \\ \sigma_2, & \text{if } x < b. \end{cases} \quad (4)$$

**Definition 2.1.** Let  $b \in \mathbb{R}$ . We call the solution  $Y^{x,b}$  of the SDE

$$dY_t = m_b(Y_t)dB_t, \quad Y_0 = x, \quad (5)$$

*oscillating Brownian motion* (OBM) with threshold  $b$ .



**Remark 2.2.** Note that there exists a unique strong solution of (5), since there exists a weak solution according to Theorem 2.6.1 in [23] and pathwise uniqueness applies due to [36]. For further details see Remark 3.1 below.

For the reader's convenience we recall the following result.

**Proposition 2.3.** For all  $x \in \mathbb{R}$  let  $Y^x$  be an OBM with threshold 0 and initial value  $x$ . Then, for all  $t > 0$ , the random variable  $Y_t^x$  has a density  $p(t, x, y)$  with respect to the Lebesgue measure, given by

$$p(t, x, y) = \begin{cases} \frac{2\sigma_1}{\sigma_2(\sigma_1+\sigma_2)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} \frac{1}{\sigma_1-\sigma_2}}, & \text{if } x \geq 0, y < 0, \\ \frac{2\sigma_2}{\sigma_1(\sigma_1+\sigma_2)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t} \frac{1}{\sigma_1-\sigma_2}}, & \text{if } x < 0, y \geq 0, \\ \frac{1}{\sigma_1\sqrt{2\pi t}} \left( e^{-\frac{(y-x)^2}{2\sigma_1^2 t}} + \frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2} e^{-\frac{(y+x)^2}{2\sigma_1^2 t}} \right), & \text{if } x \geq 0, y \geq 0, \\ \frac{1}{\sigma_2\sqrt{2\pi t}} \left( e^{-\frac{(y-x)^2}{2\sigma_2^2 t}} + \frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2} e^{-\frac{(y+x)^2}{2\sigma_2^2 t}} \right), & \text{if } x < 0, y < 0. \end{cases}$$

*Proof.* This follows from Theorem 1 in Keilson, Wellner [21]; see also Theorem 4 in [32] and Section 2 in Lejay, Pigato [31] (beware of the typo in the last case).  $\square$

**Corollary 2.4.** For  $x, b \in \mathbb{R}$  let  $Y^{x,b}$  be an OBM with threshold  $b$ . Then, for all  $t > 0$ , the random variable  $Y_t^{x,b}$  has the probability density function  $p(t, x-b, y-b)$  with respect to the Lebesgue measure, where  $p$  is defined as in Proposition 2.3.

For the remainder of this section we denote by  $F^b$  the probability distribution function of the OBM  $Y^{0,b}$  at time  $T > 0$  for some  $b \in \mathbb{R}$ , i.e.  $F^b(x) = P(Y_T^{0,b} \leq x)$ ,  $x \in \mathbb{R}$ . Note that using Corollary 2.4 we observe that  $F^b$  is given by

$$F^b(x) = \begin{cases} \Phi\left(\frac{x}{\sigma_2\sqrt{T}}\right) - \frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2} \Phi\left(\frac{x-2b}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, b \geq 0, \\ \frac{2\sigma_2}{\sigma_1+\sigma_2} \Phi\left(\frac{x-b(1-\frac{\sigma_1}{\sigma_2})}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}, & \text{if } x \geq b, b \geq 0, \\ \frac{2\sigma_1}{\sigma_1+\sigma_2} \Phi\left(\frac{x-b(1-\frac{\sigma_2}{\sigma_1})}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, b < 0, \\ \Phi\left(\frac{x}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2} \Phi\left(\frac{2b-x}{\sigma_1\sqrt{T}}\right), & \text{if } x \geq b, b < 0. \end{cases} \quad (6)$$

**Lemma 2.5.** Let  $b \in \mathbb{R}$ . Then  $F^b$  is Lipschitz continuous. Moreover, for  $M > 0$  there exists a constant  $C > 0$  such that for all  $x \in \mathbb{R}$  with  $|x-b| < M$

$$|F^b(x) - F^b(b)| \geq C|x-b|.$$

*Proof.* The Lipschitz continuity of  $F^b$  follows since  $F^b$  is absolutely continuous and its derivative is bounded. The second statement basically follows from the fact that the derivative is locally bounded from below.

We first consider the case  $b \geq 0$ . Note that  $F^b$  is given by

$$F^b(x) = \begin{cases} \Phi\left(\frac{x}{\sigma_2\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{x - 2b}{\sigma_2\sqrt{T}}\right), & \text{if } x < b, \\ \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi\left(\frac{x - b\left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1\sqrt{T}}\right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}, & \text{if } x \geq b. \end{cases}$$

1. If  $x \in (b - M, b)$  there exists a  $\xi \in (x, b)$  such that

$$\begin{aligned} |F^b(x) - F^b(b)| &= F^b(b) - F^b(x) \\ &= \frac{1}{\sigma_2\sqrt{2\pi T}} e^{-\frac{\xi^2}{2\sigma_2^2 T}} \left(1 - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} e^{-\frac{4b(b-\xi)}{2\sigma_2^2 T}}\right) |x - b| \\ &\geq \frac{1}{\sigma_2\sqrt{2\pi T}} e^{-\frac{(b-M)^2 \vee b^2}{2\sigma_2^2 T}} \left(1 - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}\right) |x - b| \\ &\geq \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(b+M)^2}{2\sigma_1^2 T}} |x - b|. \end{aligned}$$

2. If  $b < x < b + M$  we obtain by the mean-value theorem for some  $\xi \in (b, x)$  that

$$\begin{aligned} |F^b(b) - F^b(x)| &= F^b(x) - F^b(b) \\ &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left( \Phi\left(\frac{x - b\left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1\sqrt{T}}\right) - \Phi\left(\frac{b}{\sigma_2\sqrt{T}}\right) \right) \\ &= \frac{1}{\sigma_1\sqrt{2\pi T}} \frac{2\sigma_2}{\sigma_1 + \sigma_2} e^{-\frac{(\xi - b\left(1 - \frac{\sigma_1}{\sigma_2}\right))^2}{2\sigma_1^2 T}} (x - b) \\ &\geq \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(b+M)^2}{2\sigma_1^2 T}} (x - b). \end{aligned}$$

The case  $b < 0$  can be shown in the same way. Finally, for any  $b \in \mathbb{R}$  the result follows with  $C := \frac{1}{\sqrt{2\pi T}} \frac{2\sigma_1}{\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{(|b|+M)^2}{2\sigma_1^2 T}}$ . □

**Lemma 2.6.** *There exists a  $C > 0$  such that for all  $b_1, b_2 \in \mathbb{R}$*

$$\sup_{x \in \mathbb{R}} |F^{b_1}(x) - F^{b_2}(x)| \leq C |b_1 - b_2|.$$

*Proof.* Let  $0 \leq b_1 < b_2$ . Then for  $x < b_1$  we see that

$$\begin{aligned} |F^{b_1}(x) - F^{b_2}(x)| &= \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \left| \Phi\left(\frac{x - 2b_2}{\sigma_2\sqrt{T}}\right) - \Phi\left(\frac{x - 2b_1}{\sigma_2\sqrt{T}}\right) \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \left| \frac{x - 2b_2}{\sigma_2\sqrt{T}} - \frac{x - 2b_1}{\sigma_2\sqrt{T}} \right| \\ &= \frac{2}{\sqrt{2\pi\sigma_2^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1| \end{aligned}$$

$$\leq \frac{2}{\sqrt{2\pi\sigma_1^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1|.$$

Similarly, for  $x > b_2$  we have

$$\begin{aligned} |F^{b_1}(x) - F^{b_2}(x)| &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left| \Phi \left( \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left( \frac{x - b_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) \right| \\ &\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi}} \left| \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} - \frac{x - b_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right| \\ &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi\sigma_1^2 T}} \left(1 - \frac{\sigma_1}{\sigma_2}\right) |b_2 - b_1| \\ &= \frac{2}{\sqrt{2\pi\sigma_1^2 T}} \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} |b_2 - b_1|. \end{aligned}$$

Now it remains to consider the case  $x \in [b_1, b_2]$ : We have

$$\begin{aligned} &|F^{b_1}(x) - F^{b_2}(x)| \\ &= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left( \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} - \Phi \left( \frac{x}{\sigma_2 \sqrt{T}} \right) + \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi \left( \frac{x - 2b_2}{\sigma_2 \sqrt{T}} \right) \right| \\ &= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left( \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left( \frac{x}{\sigma_2 \sqrt{T}} \right) - \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \Phi \left( \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right| \\ &= \left| \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left( \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \left(1 + \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}\right) \Phi \left( \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right. \\ &\quad \left. + \Phi \left( \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) - \Phi \left( \frac{x}{\sigma_2 \sqrt{T}} \right) \right| \\ &\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \left| \Phi \left( \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} \right) - \Phi \left( \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) \right| + \left| \Phi \left( \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right) - \Phi \left( \frac{x}{\sigma_2 \sqrt{T}} \right) \right| \\ &\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi}} \left| \frac{x - b_1 \left(1 - \frac{\sigma_1}{\sigma_2}\right)}{\sigma_1 \sqrt{T}} - \frac{2b_2 - x}{\sigma_2 \sqrt{T}} \right| + \frac{1}{\sqrt{2\pi}} \left| \frac{2b_2 - x}{\sigma_2 \sqrt{T}} - \frac{x}{\sigma_2 \sqrt{T}} \right| \\ &\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi T}} \left| \frac{2}{\sigma_2} (x - b_2) + \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) (x - b_1) \right| + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \\ &\leq \frac{2\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi T}} \left( \frac{2}{\sigma_2} |b_2 - b_1| + \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) |b_2 - b_1| \right) + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{4\sigma_2}{\sigma_1 + \sigma_2} \frac{1}{\sqrt{2\pi\sigma_1^2 T}} |b_2 - b_1| + \frac{2}{\sqrt{2\pi\sigma_2^2 T}} |b_2 - b_1| \\
&\leq \frac{6}{\sqrt{2\pi\sigma_1^2 T}} |b_2 - b_1|.
\end{aligned}$$

The case  $b_1 < b_2 \leq 0$  can be shown similarly and all the remaining cases follow from an application of the triangle inequality.  $\square$

### 3 The mean field approximation

Let  $B$  be a Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Furthermore, let  $(\mathcal{F}_t)_{t \geq 0}$  be the augmented Brownian filtration, generated by  $B$  and augmented by the  $P$ -null sets in  $\mathcal{F}$ . We denote by  $\mathcal{M}$  the set of all progressively measurable control processes  $(\beta_t)_{t \geq 0}$  taking values in  $[\sigma_1, \sigma_2]$ . Given that an agent chooses the control function  $\beta \in \mathcal{M}$ , her state process is given by

$$X_t^\beta := \int_0^t \beta_s dB_s.$$

**Remark 3.1.** All feedback controls with a feedback function  $m : \mathbb{R} \rightarrow [\sigma_1, \sigma_2]$  of bounded variation are contained in  $\mathcal{M}$ . Indeed, since  $m$  is uniformly bounded away from zero there exists a weak solution to the SDE

$$dX_t = m(X_t)dB_t, \quad X_0 = 0, \quad (7)$$

according to Theorem 2.6.1 in [23], and pathwise uniqueness applies according to results in [36]. Hence, there exists a unique strong solution  $X^m$  to (7) (cf. Section 5.3 in [20]), and  $(m(X_t^m))_{t \geq 0}$  belongs to  $\mathcal{M}$ .

Let  $\alpha \in (0, 1)$  and denote by  $q(X_T^\beta, 1 - \alpha)$  the  $(1 - \alpha)$ -quantile of the random variable  $X_T^\beta$ , i.e.  $q(X_T^\beta, 1 - \alpha) = \inf\{r \in \mathbb{R} : P(X_T^\beta \leq r) \geq 1 - \alpha\}$ .

**Definition 3.2.** We call  $\beta^* \in \mathcal{M}$  an *equilibrium strategy* of the mean field game if

$$P(X_T^{\beta^*} > q(X_T^{\beta^*}, 1 - \alpha)) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > q(X_T^{\beta^*}, 1 - \alpha)) \quad (8)$$

**Remark 3.3.** Notice that  $\beta^*$  is an equilibrium strategy if and only if it maximizes the quantile  $q(X_T^\beta, 1 - \alpha)$  among all  $\beta \in \mathcal{M}$ . Therefore, the mean field game coincides with the control problem of maximizing the  $(1 - \alpha)$ -quantile. We define  $\beta^*$  via the equation (8) since the equation can be interpreted as the mean field limit of (3).

Before proving that there exists an equilibrium strategy, we first show that it is enough to consider control processes of feedback type, that are constant equal to the minimal volatility  $\sigma_1$  if the state is greater than or equal to a given threshold  $b$ , and that are constant equal to the maximal volatility  $\sigma_2$  else. We refer to such controls as *threshold controls*. More precisely, the threshold control with barrier  $b \in \mathbb{R}$  is the control with feedback function  $m_b$ , given in (4). In the following we simply write  $X^b$  for the state process  $X^{m_b}$ .

**Lemma 3.4.** *Let  $b \in \mathbb{R}$ . Then*

$$P(X_T^b > b) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > b). \quad (9)$$

*Proof.* Follows from [32], Remark 8.  $\square$

Lemma 3.4 shows that the strategy maximizing the probability of being above the quantile  $q(X_T^m, 1 - \alpha)$  at time  $T$  is the threshold control with threshold  $q(X_T^m, 1 - \alpha)$ . The following theorem provides an explicit equilibrium strategy.

**Theorem 3.5.** *The threshold strategy  $m_{b^*}$  with threshold*

$$b^* = \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left( \frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left( \frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}, \end{cases} \quad (10)$$

*is an equilibrium strategy for the mean field game.  $m_{b^*}$  is the unique equilibrium strategy in the set of threshold strategies.*

The standard approach to solve mean field games is to consider mappings from probability distributions to the distributions of optimally controlled states and find their fixed points, the so-called *equilibrium measures* (see e.g. [8] and [9]). However, the setting of our paper allows to study the distributions of OBMs only, which can be parameterized by the real-valued threshold  $b \in \mathbb{R}$ . Hence it is enough to show that the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad b \mapsto q(X_T^b, 1 - \alpha)$$

has a unique fixed point in order to prove Theorem 3.5. Indeed, if  $f(b) = b$ , then  $b = q(X_T^b, 1 - \alpha)$ . Lemma 3.4 further implies that  $P(X_T^b > q(X_T^b, 1 - \alpha)) = \max_{\beta \in \mathcal{M}} P(X_T^\beta > q(X_T^b, 1 - \alpha))$ , and hence that  $m_b$  is an equilibrium strategy.

Moreover, for any threshold control  $m_b, b \in \mathbb{R}$ , the controlled state process  $X^b$  is a so-called oscillating Brownian motion (OBM), introduced in Section 2. The transition densities of an OBM are explicitly known (see Proposition 2.3), and in particular,  $X_T^b$  possesses the probability density function  $p(T, -b, x - b)$  (Corollary 2.4). As in Section 2 we denote the probability distribution function of  $X_T^b$  by  $F^b$  and emphasize that  $F^b$  is explicitly given by (6).

*Proof of Theorem 3.5.* We want to determine the unique fixed point  $b \in \mathbb{R}$  of the function  $f$ , i.e. we solve the equation  $P(X_T^b > b) = \alpha$ . Suppose first that  $\alpha \in (0, \frac{\sigma_2}{\sigma_1 + \sigma_2}]$ . We observe by (6) that  $P(X_T^b > b) = 1 - F^b(b) \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}$  if and only if  $b \geq 0$ , and in this case

$$P(X_T^b > b) = \frac{2\sigma_2}{\sigma_1 + \sigma_2} \Phi \left( \frac{-b}{\sigma_2 \sqrt{T}} \right).$$

Moreover,  $P(X_T^b > b) = \alpha$  if and only if  $b = -\sigma_2 \sqrt{T} \Phi^{-1} \left( \frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right)$ . In this case,  $q(X_T^b, 1 - \alpha) = b$ ; in other words  $b$  is the unique fixed point of the mapping  $f$ .

Now suppose that  $\alpha \in (\frac{\sigma_2}{\sigma_1 + \sigma_2}, 1)$ . Again with (6) we observe that  $P(X_T^b > b) > \frac{\sigma_2}{\sigma_1 + \sigma_2}$  if and only if  $b < 0$ , and

$$P(X_T^b > b) = 1 - \frac{\sigma_1}{\sigma_1 + \sigma_2} \Phi\left(\frac{b}{\sigma_1 \sqrt{T}}\right).$$

As above  $P(X_T^b > b) = \alpha$  if and only if  $b = \sigma_1 \sqrt{T} \Phi^{-1}\left(\frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1}\right)$ . In this case,  $b$  is the unique fixed point of the function  $f$  and  $b = q(X_T^b, 1 - \alpha)$ .  $\square$

**Remark 3.6.** We briefly analyze how the threshold (10) of the equilibrium strategy depends on the model parameters.

(a) First notice that  $b^*$  increases as  $\alpha$  decreases, and

$$\lim_{\alpha \downarrow 0} b^* = \infty, \quad \lim_{\alpha \uparrow 1} b^* = -\infty.$$

Thus, reducing the proportion  $\alpha$  of best players who receive a reward implies that the players choose a higher threshold up to which they control their process with the maximal diffusion rate. In other words, intensifying competition leads to riskier strategies.

This observation is in line with results in [38] and [39], obtained within models, where agents can control the time horizon of comparison. It is shown that as the number of agents increases and hence the proportion of agents receiving a reward decreases, the agents choose in expectation larger stopping rules and hence increase the standard deviation of the states at the comparison time.

(b) Note that  $b^*$  is proportional to  $\sqrt{T}$ . Hence, if  $\alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}$ , then increasing the run time of the game encourages players to choose higher thresholds since  $b^* \geq 0$ . If  $\alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}$ , however, increasing the run time results in a lower threshold  $b^*$  since  $b^* < 0$ , i.e. players are incentivized to choose lower thresholds.

## 4 Approximate Nash equilibrium of the $n$ -player game

We now come back to the  $n$ -player game introduced in Section 1. We show that the  $n$ -tuple consisting of the mean field equilibrium threshold strategies is an  $\varepsilon$ -Nash equilibrium in the  $n$ -player game, with  $\varepsilon$  converging to zero as  $n$  tends to infinity. Moreover, we determine a rate of convergence.

We first rigorously define what we mean by an  $\varepsilon$ -Nash equilibrium for the game at hand (cf., e.g., the similar definition in [4]).

**Definition 4.1.** Let  $\varepsilon > 0$ . A tuple  $a = (a_1, \dots, a_n) \in \mathcal{A}_n^n$  is called  $\varepsilon$ -Nash equilibrium of the  $n$ -player game if for all  $i \in \{1, \dots, n\}$ ,  $c \in \mathcal{A}_n$  and weak solutions  $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$ ,  $(\Omega^{(a-i,c)}, \mathcal{F}^{(a-i,c)}, (\mathcal{F}_t^{(a-i,c)})$ ,  $P^{(a-i,c)}, W^{(a-i,c)}, X^{(a-i,c)})$  we have

$$P^a(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) + \varepsilon \geq P^{(a-i,c)}(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)),$$

where  $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$ , and  $\mu^{n,a}$ ,  $\mu^{n,(a_{-i},c)}$  are defined as in Section 1.

Our main result of this section is as follows:

**Theorem 4.2.** *Let  $n \geq 2$  and  $a = (a_1, \dots, a_n)$  be the tuple of strategies in  $\mathcal{A}_n^n$ , defined by*

$$a_i(x) := \begin{cases} \sigma_1, & \text{if } x_i \geq b^*, \\ \sigma_2, & \text{if } x_i < b^*, \end{cases} \quad (11)$$

for  $x \in \mathbb{R}^n$  and

$$b^* := \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left( \frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left( \frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}. \end{cases}$$

Then there exists a sequence  $\varepsilon_n \geq 0$  with  $\lim_n \varepsilon_n = 0$  such that  $a = (a_1, \dots, a_n)$  is an  $\varepsilon_n$ -Nash equilibrium of the  $n$ -player game. We can choose  $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$ .

**Remark 4.3.** Notice that in Definition 4.1 we allow for closed-loop equilibria, while the approximate optimal strategies given in (11) are also open-loop. Indeed, in the approximate Nash equilibrium of the previous theorem, player  $i$  only needs to observe her own state, and pathwise uniqueness implies that it is enough to observe its associated Brownian motion. We refer to Chapter 2 of [8] for more details on open- and closed-loop Nash equilibria.

In order to prove Theorem 4.2 we first state two lemmas. The first is due to results of Hoeffding and we state it in a version suitable for our setting. This result can be found, e.g., in Theorem 1 and Theorem 2 of [16] in a slightly more general form.

**Lemma 4.4.** *Let  $X_1, \dots, X_n$  be independent random variables with  $0 \leq X_i \leq 1$  for  $i = 1, \dots, n$ , defined on a probability space  $(\Omega, \mathcal{F}, P)$ . Then we have for  $t > 0$*

$$P \left( \sum_{i=1}^n (X_i - EX_i) \geq nt \right) \leq e^{-2nt^2}.$$

Lemma 4.4 yields the next result for empirical quantiles.

**Lemma 4.5.** *Let  $X_1, \dots, X_n$  be i.i.d. random variables on a probability space  $(\Omega, \mathcal{F}, P)$  and  $\mu := P_{X_1}$ . Define  $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$  and let  $(\alpha_n)_{n \in \mathbb{N}}$  be a sequence in  $(0, 1)$ . Then for all  $\varepsilon > 0$  we have*

$$P(|q(\mu^n, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2e^{-2n\delta_\varepsilon^2},$$

where  $\delta_\varepsilon := \min\{(1 - \alpha_n) - F_\mu(q(\mu, 1 - \alpha_n) - \varepsilon), F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon) - (1 - \alpha_n)\}$  and where  $F_\mu$  denotes the distribution function of  $\mu$ .

*Proof.* Let  $\varepsilon > 0$ . Note that

$$\begin{aligned}
& P(q(\mu^n, 1 - \alpha_n) > q(\mu, 1 - \alpha_n) + \varepsilon) \\
&= P\left(\sum_{j=1}^n (\mathbb{1}_{(q(\mu, 1 - \alpha_n) + \varepsilon, \infty)}(X_j) - (1 - F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon)))\right. \\
&\qquad\qquad\qquad \left. > n(F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon) - (1 - \alpha_n))\right), \\
& P(q(\mu^n, 1 - \alpha_n) < q(\mu, 1 - \alpha_n) - \varepsilon) \\
&= P\left(\sum_{j=1}^n (\mathbb{1}_{(-\infty, q(\mu, 1 - \alpha_n) + \varepsilon]}(X_j) - F_\mu(q(\mu, 1 - \alpha_n) + \varepsilon))\right. \\
&\qquad\qquad\qquad \left. > n((1 - \alpha_n) - F_\mu(q(\mu, 1 - \alpha_n) - \varepsilon))\right).
\end{aligned}$$

Now using Lemma 4.4 we obtain that

$$P(|q(\mu^n, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2 \exp(-2n\delta_\varepsilon^2).$$

□

**Corollary 4.6.** *Let  $(\Omega^a, \mathcal{F}^a, (\mathcal{F}_t^a), P^a, W^a, X^a)$  be a weak solution of (1) for the control  $a$  defined in Theorem 4.2, and let  $(\alpha_n)_{n \in \mathbb{N}}$  be a sequence in  $(0, 1)$ . Moreover, let  $\mu^{n,a} = \frac{1}{n} \sum_{i=1}^n \delta_{X_T^{i,a}}$  and  $\mu := P_{X_T^{1,a}}^a$ . If  $0 < \varepsilon < M$ , then there exists a constant  $C > 0$  such that*

$$P^a(|q(\mu^{n,a}, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2e^{-2nC\varepsilon^2}.$$

*Proof.* Note that  $X^{i,a}$  has the same distribution as the OBM  $Y^{0,b^*}$  with threshold  $b^*$ . Hence, if  $0 < \varepsilon < M$ , we can apply Lemma 4.5 and Lemma 2.5, and thus there exists a constant  $C > 0$  such that  $\delta_\varepsilon \geq C\varepsilon$ . We obtain

$$P^a(|q(\mu^{n,a}, 1 - \alpha_n) - q(\mu, 1 - \alpha_n)| > \varepsilon) \leq 2 \exp(-2nC^2\varepsilon^2).$$

□

*Proof of Theorem 4.2.* Let  $i \in \{1, \dots, n\}$ . We compare  $a = (a_1, \dots, a_n)$  with the tuple where player  $i$  deviates from  $m_{b^*}$  by choosing a strategy  $c \in \mathcal{A}_n$ . To this end we write  $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$ . Note that for the control  $(a_{-i}, c)$  there exists a weak solution  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, X^{(a_{-i}, c)})$  of (2). For simplicity we omit the dependence on the strategy in the notation. Moreover, we can find a strong solution  $X^a$  of (2) for the control  $a$  on the same probability space and with the same Brownian motion: The controls  $a_j$  only depend on the  $j$ -th state and are uniformly bounded away from zero, which implies weak solvability. In addition, they are of bounded variation implying pathwise uniqueness. Hence, there is a unique strong solution (see also Remark 3.1).



We emphasize that the distribution of  $X^{j,a}$  is same as the distribution of the OBM  $Y^{0,b^*}$  with threshold  $b^*$ . We set  $\mu := P_{X_T^{1,a}}$ .

We need to show that

$$P(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)) - P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \leq \frac{C}{\sqrt{n}}, \quad (12)$$

for some constant  $C > 0$  independent of the control  $c$ . We denote by  $F^b$  the distribution function of an OBM with threshold level  $b$  at time  $T$ .

1. We first estimate  $P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha))$  from below. Notice that  $X_T^{i,a}$  and the empirical quantile  $q(\mu^{n,a}, 1 - \alpha)$  are not independent. We therefore consider the empirical measure  $\mu^{n-1,a} = \frac{1}{n-1} \sum_{j \neq i} \delta_{X_T^{j,a}}$  and define

$$A(n) := q\left(\mu^{n-1,a}, \frac{n}{n-1}(1 - \alpha)\right). \quad (13)$$

Notice that  $A(n)$  is independent of  $X_T^{i,a}$  and

$$A(n) \geq q(\mu^{n,a}, 1 - \alpha). \quad (14)$$

Thus,

$$P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \geq P(X_T^{i,a} > A(n)) = E[1 - F^{b^*}(A(n))], \quad (15)$$

2. Next we estimate the first term in (12) from above: We replace in (12) the quantile  $q(\mu^{n,(a-i,c)}, 1 - \alpha)$  with a quantile that does not depend on  $c$ . To this end observe that

$$D(n) := q\left(\mu^{n-1,a}, 1 - \frac{n}{n-1}\alpha\right) \leq q(\mu^{n,(a-i,c)}, 1 - \alpha). \quad (16)$$

Indeed, we have since  $(n-1)(1 - \frac{n}{n-1}\alpha) = n(1 - \alpha) - 1 < n(1 - \alpha)$

$$\begin{aligned} q\left(\mu^{n-1,a}, 1 - \frac{n}{n-1}\alpha\right) &= X_T^{(\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor; n-1), a-i} \\ &\leq X_T^{(\lfloor n(1-\alpha) \rfloor; n), (a-i, c)} = q(\mu^{n,(a-i,c)}, 1 - \alpha), \end{aligned}$$

where  $X_T^{(\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor; n-1), a-i}$  denotes the  $\lfloor (n-1)(1 - \frac{n}{n-1}\alpha) \rfloor$ -th order statistics of the  $(n-1)$  states  $X_T^{j,a}, j \neq i$ , and  $X_T^{(\lfloor n(1-\alpha) \rfloor; n), (a-i, c)}$  denotes the  $\lfloor n(1-\alpha) \rfloor$ -th order statistics of the  $n$  states  $X_T^{1,(a-i,c)}, \dots, X_T^{n,(a-i,c)}$ . In other words, suppose that  $r$  is greater than or equal to the right-hand side of (16). Then at most  $\lfloor n\alpha \rfloor$  players are better than  $r$ . In the  $n-1$  game without player  $i$  then also at most  $\lfloor n\alpha \rfloor$  players are better than  $r$ . This means  $\mu^{n-1,a-i}((-\infty, r]) \geq 1 - \frac{n}{n-1}\alpha$ . Hence we have shown (16).

From (16) we obtain

$$\begin{aligned} P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) &\leq P\left(X_T^{i,(a-i,c)} > D(n)\right) \\ &= E\left[P\left(X_T^{i,(a-i,c)} > D(n) \mid D(n)\right)\right]. \end{aligned} \quad (17)$$

If player  $i$  knew from the very beginning the value  $D(n)$ , then  $m_{D(n)}$  would be the control maximizing the probability for player  $i$ 's state to be greater than  $D(n)$  at time  $T$ : Note that we can assume, without loss of generality, that  $\Omega$  is a complete, separable metric space and  $\mathcal{F}$  is its Borel- $\sigma$ -algebra. Then Theorem 5.3.18 in [20] implies that there exists a regular conditional probability  $Q : \Omega \times \mathcal{F} \rightarrow [0, 1]$  for  $\mathcal{F}$  given  $\sigma(D(n))$  and  $Q$  satisfies

$$Q\left(\omega, \left\{X_T^{i,(a-i,c)} > D(n)\right\}\right) = Q\left(\omega, \left\{X_T^{i,(a-i,c)} > b\right\}\right) \Big|_{b=D(\omega,n)}, \quad (18)$$

for a.e.  $\omega \in \Omega$ . Since  $W^i$  is also a Brownian motion under  $Q(\omega, \cdot)$ , because  $W^i$  and  $D(n)$  are independent, Lemma 3.4 implies that for any  $b \in \mathbb{R}$  we have

$$Q\left(\omega, \left\{X_T^{i,(a-i,c)} > b\right\}\right) \leq Q\left(\omega, \left\{X_T^{i,(a-i,m_b)} > b\right\}\right), \text{ for a.e. } \omega \in \Omega. \quad (19)$$

Hence, with (18) and (19)

$$\begin{aligned} P\left(X_T^{i,(a-i,c)} > D(n) | D(n)\right)(\omega) &= Q\left(\omega, \left\{X_T^{i,(a-i,c)} > D(n)\right\}\right) \\ &\leq Q\left(\omega, \left\{X_T^{i,(a-i,m_b)} > b\right\}\right) \Big|_{b=D(\omega,n)}, \end{aligned} \quad (20)$$

for a.e.  $\omega \in \Omega$ . The random variable  $X_T^{i,(a-i,m_b)}$  has the same distribution as the OBM with threshold level  $b$  at time  $T$ , and therefore with (20) we have

$$P\left(X_T^{i,(a-i,c)} > D(n) | D(n)\right) \leq 1 - F^b(b) \Big|_{b=D(n)}, \text{ a.s.}$$

This implies using (17)

$$P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) \leq E[1 - F^b(b) \Big|_{b=D(n)}]. \quad (21)$$

3. We can now combine the estimates in (15) and (21) above leading to

$$\begin{aligned} &P\left(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)\right) - P\left(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)\right) \\ &\leq E[F^{b^*}(A(n)) - F^{D(n)}(D(n))] \\ &\leq E\left|F^{b^*}(A(n)) - F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right)\right| \end{aligned} \quad (22)$$

$$+ \left|F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right) - F^{b^*}\left(q\left(\mu, 1 - \frac{n}{n-1}\alpha\right)\right)\right| \quad (23)$$

$$+ E\left|F^{b^*}\left(q\left(\mu, 1 - \frac{n}{n-1}\alpha\right)\right) - F^{b^*}(D(n))\right| \quad (24)$$

$$+ E\left|F^{b^*}(D(n)) - F^{D(n)}(D(n))\right|. \quad (25)$$

Using the Lipschitz continuity of  $F^{b^*}$  (see Lemma 2.5) we can estimate the term (22) as follows

$$\begin{aligned} &E\left|F^{b^*}(A(n)) - F^{b^*}\left(q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right)\right| \\ &\leq C_1 E\left[\left|A(n) - q\left(\mu, \frac{n}{n-1}(1 - \alpha)\right)\right| \wedge 2\right] \end{aligned}$$

$$\begin{aligned}
&= C_1 \int_0^\infty P(|A(n) - q(\mu, \frac{n}{n-1}(1-\alpha))| \wedge 2 > \varepsilon) d\varepsilon \\
&= C_1 \int_0^2 P(|A(n) - q(\mu, \frac{n}{n-1}(1-\alpha))| > \varepsilon) d\varepsilon
\end{aligned}$$

for some constant  $C_1 > 0$ . Moreover, Corollary 4.6 yields that for some  $C_2 > 0$  we have

$$P(|A(n) - q(\mu, \frac{n}{n-1}(1-\alpha))| > \varepsilon) \leq 2e^{-2nC_2\varepsilon^2}, \quad 0 < \varepsilon \leq 2,$$

and thus we obtain

$$\begin{aligned}
&E|F^{b^*}(A(n)) - F^{b^*}(q(\mu, \frac{n}{n-1}(1-\alpha)))| \\
&\leq C_1 \int_0^2 2e^{-2nC_2\varepsilon^2} d\varepsilon \leq \frac{\sqrt{2\pi}C_1}{\sqrt{C_2}} \frac{1}{\sqrt{n}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned} \tag{26}$$

In exactly the same way one can estimate the term (24) and hence we have for some constant  $C_3 > 0$

$$E|F^{b^*}(q(\mu, 1 - \frac{n}{n-1}\alpha)) - F^{b^*}(D(n))| \leq \frac{C_3}{\sqrt{n-1}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \tag{27}$$

For the term (25) we obtain, using Lemma 2.6 and Corollary 4.6,

$$\begin{aligned}
&E|F^{b^*}(D(n)) - F^{D(n)}(D(n))| \\
&\leq C_4 E[|b^* - D(n)| \wedge 2] \\
&= C_4 \int_0^2 P(|b^* - D(n)| > \varepsilon) d\varepsilon \\
&\leq C_4 \int_0^2 2e^{-2nC_5\varepsilon^2} d\varepsilon \leq \frac{\sqrt{2\pi}C_4}{\sqrt{C_5}} \frac{1}{\sqrt{n}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)
\end{aligned} \tag{28}$$

for some constants  $C_4, C_5 > 0$ .

Recall that  $F^{b^*}(x) = \mu((-\infty, x])$ ,  $x \in \mathbb{R}$ , and hence the term (23) can be rewritten as follows

$$\begin{aligned}
&|F^{b^*}(q(\mu, \frac{n}{n-1}(1-\alpha))) - F^{b^*}(q(\mu, 1 - \frac{n}{n-1}\alpha))| \\
&= |\frac{n}{n-1}(1-\alpha) - (1 - \frac{n}{n-1}\alpha)| = \frac{1}{n-1} \in \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned} \tag{29}$$

Finally, equations (26)-(29) imply that

$$P(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1-\alpha)) - P(X_T^{i,a} > q(\mu^{n,a}, 1-\alpha)) \leq \frac{C}{\sqrt{n}} \in \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

for an appropriate constant  $C > 0$  independent of  $c$ . Thus, the feedback strategy  $a = (m_{b^*}, \dots, m_{b^*})$  is an  $\mathcal{O}(n^{-\frac{1}{2}})$ -Nash equilibrium.  $\square$

## 5 Large ranking games with drift

In this section we consider an extension of our  $n$ -player game model described in Section 1. We make the same assumptions as for the  $n$ -player game, but we add a drift component to the dynamics of each players' state process. Namely, for  $i = 1, \dots, n$  and any admissible control  $a \in \mathcal{A}_n^n$  we assume that the state process of player  $i$  is given by

$$dX_t^{i,a} = f(t)dt + a_i(t, X_t^{i,a}, X_t^{-i,a})dW_t^i, \quad X_0^{i,a} = 0, \quad (30)$$

where  $f : [0, T] \rightarrow \mathbb{R}$  is a measurable function and  $W = (W^1, \dots, W^n)$  is a Brownian motion on  $\mathbb{R}^n$ . Note that we can write (30) as the  $n$ -dimensional SDE

$$dX_t^a = F(t)dt + D(t, X_t^a)dW_t, \quad X_0^a = 0 \in \mathbb{R}^n, \quad (31)$$

where  $D(t, x)$  is the diagonal matrix with the entries  $a_1(t, x), \dots, a_n(t, x)$  on the diagonal, and  $F(t) \in \mathbb{R}^n$  is the vector with each entry equal to  $f(t)$ .

As before we suppose that each player aims at maximizing the probability of her own state to be greater than the empirical  $(1 - \alpha)$ -quantile of all states at time  $T$ , i.e. player  $i$  wants to maximize the quantity

$$P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha))$$

over all admissible controls  $a$ . In the case without drift we have shown that an approximate Nash equilibrium is given when each player chooses the threshold control  $m_{b^*}$  (see Theorem 4.2). We aim at adapting this result to the case with drift.

**Theorem 5.1.** *Let  $n \geq 2$  and  $a = (a_1, \dots, a_n)$  be the tuple of strategies in  $\mathcal{A}_n^n$ , defined by*

$$a_i(t, x) := \begin{cases} \sigma_1, & \text{if } x_i \geq b^* + \int_0^t f(s) ds, \\ \sigma_2, & \text{if } x_i < b^* + \int_0^t f(s) ds, \end{cases} \quad (32)$$

for  $(t, x) \in [0, T] \times \mathbb{R}^n$  and

$$b^* := \begin{cases} -\sigma_2 \sqrt{T} \Phi^{-1} \left( \frac{\alpha(\sigma_1 + \sigma_2)}{2\sigma_2} \right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}, \\ \sigma_1 \sqrt{T} \Phi^{-1} \left( \frac{(1-\alpha)(\sigma_1 + \sigma_2)}{2\sigma_1} \right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1 + \sigma_2}. \end{cases}$$

Then there exists a sequence  $\varepsilon_n \geq 0$  with  $\lim_n \varepsilon_n = 0$  such that  $a = (a_1, \dots, a_n)$  is an  $\varepsilon_n$ -Nash equilibrium of the  $n$ -player game. We can choose  $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$ .

Note that the drift coefficient in (30) is purely time-dependent. Thus, to prove Theorem 5.1 we apply an appropriate shift to the state processes. This shift effects the strategies and our reward functional, that is given by  $g(x, \mu) = \mathbb{1}_{(1-\alpha, \infty)}(\mu((-\infty, x]))$  for  $x \in \mathbb{R}$  and  $\mu \in \mathcal{P}(\mathbb{R})$ , where  $\mathcal{P}(\mathbb{R})$  denotes the set of all probability measures on  $\mathbb{R}$ . We emphasize that the function  $g$  is *translation invariant* (see e.g. Definition 2.1 in [27]), meaning that for any  $x, y \in \mathbb{R}$  and  $\mu \in \mathcal{P}(\mathbb{R})$  we have  $g(x + y, \mu) = g(x, \mu(\cdot + y))$ . This fact plays a key role in the proof of Theorem 5.1.

First we show that for the strategy  $a$  pathwise uniqueness of the state equation holds true.

**Proposition 5.2.** *Let  $a \in \mathcal{A}_n^n$  be defined as in Theorem 5.1. Then pathwise uniqueness for (31) holds true and there exists a strong solution.*

*Proof.* Note that for the strategy  $a$  the equations (31) are decoupled and identical. Hence, it is enough to consider the one-dimensional SDE

$$dX_t = f(t)dt + m_{b^*} \left( X_t - \int_0^t f(s) ds \right) dW_t, X_0 = 0. \quad (33)$$

(33) has a weak solution. We show that also pathwise uniqueness holds true. This immediately entails the existence of a unique strong solution (see e.g. Section 5.3 in [20]).

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, X^1)$  and  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, X^2)$  be two weak solutions to (33) with the same filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  and Brownian motion  $W$ . Define  $Y^1$  and  $Y^2$  by  $Y_t^i := X_t^i - \int_0^t f(s) ds, i = 1, 2$ . Note that  $Y^1$  and  $Y^2$  solve the SDE

$$dY_t = m_{b^*}(Y_t)dW_t, Y_0 = 0, \quad (34)$$

on the same probability setup. But for (34) pathwise uniqueness holds true (see Remark 3.1). Hence,

$$P(X_t^1 = X_t^2, t \geq 0) = P(Y_t^1 = Y_t^2, t \geq 0) = 1.$$

□

*Proof of Theorem 5.1.* Let  $c \in \mathcal{A}_n$  and fix  $i \in \{1, \dots, n\}$ . As in the proof of Theorem 4.2, we compare  $a$  with the tuple where player  $i$  deviates from  $a_i$  by choosing a strategy  $c$ . To this end, we write  $(a_{-i}, c) = (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$ . Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, X^{(a_{-i}, c)})$  be a weak solution to (31) for the strategy  $(a_{-i}, c)$ . Proposition 5.2 implies that there is a unique solution  $X^a$  to (31) for the control  $a$  on the same filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  and with the same Brownian motion  $W$ . Our goal is to apply Theorem 4.2. To this end, we subtract from each component of  $X^a$  and  $X^{(a_{-i}, c)}$  the common drift.

We define the processes  $Y^a$  and  $Y^{(a_{-i}, c)}$  by setting for  $j = 1, \dots, n$  and  $t \in [0, T]$

$$\begin{aligned} Y_t^{j,a} &= X_t^{j,a} - \int_0^t f(s) ds, \\ Y_t^{j,(a_{-i},c)} &= X_t^{j,(a_{-i},c)} - \int_0^t f(s) ds. \end{aligned}$$

Note that  $a_j(t, x_j + \int_0^t f(s) ds) = m_{b^*}(x_j)$  for any  $t \geq 0$  and  $x \in \mathbb{R}^n$ . Hence,  $Y^{j,a}, j = 1, \dots, n$ , and  $Y^{j,(a_{-i},c)}, j \neq i$ , solve (1) with the control  $m_{b^*}$ . Their trajectories agree a.s. due to pathwise uniqueness. Moreover, the process  $Y^{i,(a_{-i},c)}$  solves (1) with control  $\tilde{c}(t, x) = c \left( t, x_1 - \int_0^t f(s) ds, \dots, x_n - \int_0^t f(s) ds \right), t \in [0, T], x \in \mathbb{R}^n$ . The states  $Y^a$  and  $Y^{(a_{-i},c)}$  correspond to the  $n$ -player game without drift.  $Y^a$  are the states controlled with the approximate Nash equilibrium, given in Theorem 4.2, and  $Y^{(a_{-i},c)}$  represents the

states where one player deviates from the approximate Nash equilibrium strategy by choosing the control  $\tilde{c}$ . Therefore, Theorem 4.2 implies that

$$P(Y_T^{i,(a-i,d)} > q(\nu^{n,(a-i,d)}, 1 - \alpha)) - P(Y_T^{i,a} > q(\nu^{n,a}, 1 - \alpha)) \leq \varepsilon_n, \quad (35)$$

for some sequence  $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$  independent of  $i, c$  and the choice of the weak solution  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P), W, X^{(a-i,c)}$ .

Now, the translation invariance of our cost function implies that for  $d \in \{a, (a-i, c)\}$

$$P(X_T^{i,d} > q(\mu^{n,d}, 1 - \alpha)) = P(Y_T^{i,d} > q(\nu^{n,d}, 1 - \alpha)), \quad (36)$$

where  $\mu^{n,d} = \frac{1}{n} \sum_{j=1}^n \delta_{X_T^{j,d}}$  and  $\nu^{n,d} = \frac{1}{n} \sum_{j=1}^n \delta_{Y_T^{j,d}}$ . Finally, (35) and (36) imply that

$$\begin{aligned} & P(X_T^{i,(a-i,c)} > q(\mu^{n,(a-i,c)}, 1 - \alpha)) - P(X_T^{i,a} > q(\mu^{n,a}, 1 - \alpha)) \\ & \leq P(Y_T^{i,(a-i,c)} > q(\nu^{n,(a-i,c)}, 1 - \alpha)) - P(Y_T^{i,a} > q(\nu^{n,a}, 1 - \alpha)) \leq \varepsilon_n. \end{aligned}$$

□

## 6 Comparison with the two player case

In this section we consider our game with two players. We show that there exists a Nash equilibrium in an explicit form by using classical verification techniques for the Isaacs equations of the game. Our main result is as follows:

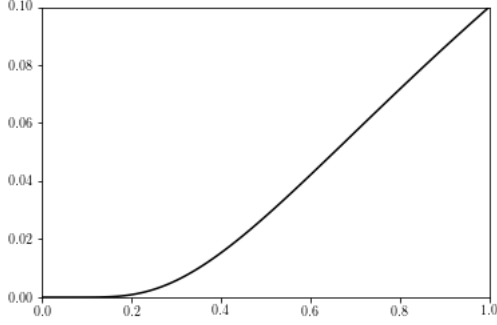
**Theorem 6.1.** *Let  $\alpha = \frac{1}{2}$  and  $n = 2$ . Then the strategy  $(a_1^*, a_2^*) \in \mathcal{A}_2^2$  is a Nash equilibrium of the two player game, where  $(a_1^*, a_2^*)$  is defined by*

$$a_1^*(x, y) = \begin{cases} \sigma_1, & \text{if } x \geq y, \\ \sigma_2, & \text{if } x < y, \end{cases} \quad (37)$$

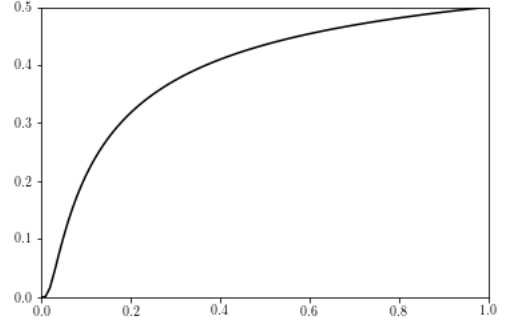
and  $a_2^*(x, y) = a_1^*(y, x)$  for  $(x, y) \in \mathbb{R}^2$ .

One can interpret Theorem 6.1 and 3.5 as providing Nash equilibria for the extreme cases  $n = 2$  and  $n = \infty$ . Observe that the equilibrium strategies of the two extreme cases differ considerably. In the two player equilibrium the *relative* position of the own state with respect to the opponent's state determines the choice of volatility. In contrast, the strategy of the mean field equilibrium only depends on the *absolute* position of the own state. For implementing (37) a player needs to observe her own state process and the one of the opponent, whereas the mean field equilibrium only requires observation of the own state. The results thus indicate that any information about the other players' states becomes less useful as the number of players increases. This is plausible since in a symmetric equilibrium the empirical distribution of all players' states converges to the distribution of the own state as  $n \rightarrow \infty$ .

A comparison of Theorem 6.1 and 3.5 further reveals that a larger number of players incentivizes players to choose a high volatility more often. Indeed, in the two player



(a) Parameters:  $\sigma_1 = 1$ ,  $\sigma_2 = 2$ ,  $T = 1$ ,  $\alpha = 0.1$



(b) Parameters:  $\sigma_1 = 1$ ,  $\sigma_2 = 2$ ,  $T = 1$ ,  $\alpha = 0.5$

Figure 1: Part of the players choosing the small volatility  $\sigma_1$  in dependence of time.

case with  $\alpha = \frac{1}{2}$  the player with the higher state always chooses the small volatility. In the large game with  $\alpha = \frac{1}{2}$  the percentage of players choosing the smaller volatility increases over time and reaches 50% at  $T$ . One can show that all players choose  $\sigma_2$  at the beginning (see Figure 1b).

A related two player diffusion control game has been studied in McNamara [34]. In [34], however, the two players control the same single state, whereas in our game each player controls her own process. As in our two player game only the relative position matters, one can derive the equilibrium from Theorem 6.1 by studying the difference of the state processes, which has a similar dynamics as the state in the game of [34].

We first describe the two player game in more detail before proving Theorem 6.1. We denote by  $X$  the state of player 1 and by  $Y$  the state of player 2. If the players choose the strategy  $a \in \mathcal{A}_2^2$ , their dynamics are governed by the SDE

$$\begin{aligned} dX_s^{t,x,y,a} &= a_1(X_s^{t,x,y,a}, Y_s^{t,x,y,a})dW_s^1, \\ dY_s^{t,x,y,a} &= a_2(X_s^{t,x,y,a}, Y_s^{t,x,y,a})dW_s^2, \quad s \in [t, T], \end{aligned} \quad (38)$$

with initial data  $(X_t, Y_t) = (x, y) \in \mathbb{R}^2, t \in [0, T)$ . As mentioned already in Section 1, there exists a weak solution of (38) for any control  $a \in \mathcal{A}_2^2$ . Moreover, for any control  $a \in \mathcal{A}_2^2$  uniqueness in law of the SDE (38) holds true because Theorem 3 in [22] applies for two-dimensional equations. Although for any weak solution the filtered probability space and the Brownian motion depend on the chosen control, we omit this dependence in this section to simplify notation.

**Lemma 6.2.** *For all  $a \in \mathcal{A}_2^2$ ,  $t \in [0, T)$  and  $(x, y) \in \mathbb{R}^2$  we have*

$$P(X_T^{t,x,y,a} = Y_T^{t,x,y,a}) = 0.$$

*Proof.* Let  $Z_s = X_s^{t,x,y,a} - Y_s^{t,x,y,a}$  for all  $s \in [t, T]$ . Moreover, we use the short hand notation  $a_{i,s} = a_i(X_s^{t,x,y,a}, Y_s^{t,x,y,a})$ . Notice that  $\tilde{W}$ , given by

$$\tilde{W}_s = \int_t^s \frac{1}{\sqrt{a_{1,r}^2 + a_{2,r}^2}}(a_{1,r}dW_r^1 - a_{2,r}dW_r^2), \quad s \geq t,$$

is a Brownian motion, and

$$dZ_s = \sqrt{a_{1,s}^2 + a_{2,s}^2} d\tilde{W}_s, \quad s \in [t, T]. \quad (39)$$

Observe that  $\sqrt{a_{1,s}^2 + a_{2,s}^2}$  is bounded and bounded away from zero. Theorem 1 in [33] entails that  $P(Z_T = 0) = 0$ , and hence we obtain the result.  $\square$

Note that for  $\alpha = \frac{1}{2}$  and  $n = 2$  the game described in Section 1 is for each player equivalent to the task of maximizing the probability of being ahead at time  $T$ . Moreover, for each player maximizing the probability of being ahead is equivalent to minimizing the probability for the opponent to be ahead at time  $T$ . Hence, it is enough to consider the payoff of player 1, which is given by

$$J(t, x, y, a) := P(X_T^{t,x,y,a} \geq Y_T^{t,x,y,a}) = P(X_T^{t,x,y,a} > Y_T^{t,x,y,a}).$$

The last equality follows from Lemma 6.2.

For  $(t, x, y) \in [0, T) \times \mathbb{R}^2$  the lower value of the game is then defined by

$$\underline{V}(t, x, y) = \sup_{a_1 \in \mathcal{A}} \inf_{a_2 \in \mathcal{A}} J(t, x, y, a_1, a_2),$$

and the upper value of the game is defined by

$$\overline{V}(t, x, y) = \inf_{a_2 \in \mathcal{A}} \sup_{a_1 \in \mathcal{A}} J(t, x, y, a_1, a_2).$$

Notice that the definitions imply  $\underline{V} \leq \overline{V}$ .

We now prove Theorem 6.1 by constructing a solution to the Isaacs equations of the game and then show, by classical verification techniques, that they agree with the upper and lower value function of the game. Moreover, we show that  $(a_1^*, a_2^*)$  is a saddle point of  $J$ , i.e.

$$\inf_{a_2 \in \mathcal{A}} J(t, x, y, a_1^*, a_2) = \sup_{a_1 \in \mathcal{A}} J(t, x, y, a_1, a_2^*) = V(t, x, y),$$

where the value function  $V$  is given by  $V = \overline{V} = \underline{V}$ . Thus we obtain that  $(a_1^*, a_2^*)$  is a Nash equilibrium.

The upper and lower Isaacs equation are given by

$$-w_t(t, x, y) - \frac{1}{2} \inf_{b \in [\sigma_1, \sigma_2]} \sup_{a \in [\sigma_1, \sigma_2]} (a^2 w_{xx} + b^2 w_{yy})(t, x, y) = 0, \quad (40)$$

$$-v_t(t, x, y) - \frac{1}{2} \sup_{a \in [\sigma_1, \sigma_2]} \inf_{b \in [\sigma_1, \sigma_2]} (a^2 v_{xx} + b^2 v_{yy})(t, x, y) = 0, \quad (41)$$

with  $w(T, x, y) = \mathbf{1}_{(0, \infty)}(x - y)$  and  $v(T, x, y) = \mathbf{1}_{(0, \infty)}(x - y)$ , respectively. Note that (40) and (41) coincide and hence it is enough to consider either one of those equations. Moreover, one can easily verify that the function  $G$ , defined by

$$G(t, x, y) := \Phi \left( \frac{x - y}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}} \right), \quad (t, x, y) \in [0, T) \times \mathbb{R}^2,$$



and  $G(T, x, y) := \mathbb{1}_{(0, \infty)}(x - y)$ , solves both (40) and (41). We show by a classical verification that  $G$  coincides with  $\underline{V}$  and  $\overline{V}$ .

**Lemma 6.3.**  $\underline{V}(t, x, y) = \overline{V}(t, x, y) = G(t, x, y)$  for  $(t, x, y) \in [0, T) \times \mathbb{R}^2$  and hence the value function of the problem is given by

$$V(t, x, y) = \Phi \left( \frac{x - y}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}} \right), \quad (t, x, y) \in [0, T) \times \mathbb{R}^2.$$

*Proof.* We show that  $G = \overline{V}$  and  $G = \underline{V}$  on  $[0, T) \times \mathbb{R}^2$ . Hence we obtain all the statements of Lemma 6.3. Note that for  $t \in [0, T)$  and  $(x, y) \in \mathbb{R}^2$  we have

$$\partial_{xx}G(t, x, y) = \partial_{yy}G(t, x, y) \begin{cases} \leq 0, & \text{if } x \geq y, \\ > 0, & \text{if } x < y, \end{cases} \quad (42)$$

and

$$\partial_t G(t, x, y) + \frac{1}{2} \partial_{xx}G(t, x, y) (\sigma_1^2 + \sigma_2^2) = 0 \quad (43)$$

since  $G$  satisfies (40) and (41).

We start with showing that  $G \leq \overline{V}$ . To this end fix  $a_2 \in \mathcal{A}$ . By (42) and (43) we have

$$0 \leq \partial_t G(t, x, y) + \frac{1}{2} \partial_{xx}G(t, x, y) (a_1^*(x, y)^2 + a_2(x, y)^2) \quad (44)$$

for all  $(t, x, y) \in [0, T) \times \mathbb{R}^2$ . Now fix  $(t, x, y) \in [0, T) \times \mathbb{R}^2$ , set  $\hat{a} := (a_1^*, a_2)$  and let  $(X, Y) = (X^{t, x, y, \hat{a}}, Y^{t, x, y, \hat{a}})$  be a weak solution of the state SDE (38) on the time interval  $[t, T]$  with initial condition  $(x, y)$ . Recall that the solution  $(X, Y)$  is unique in law. By Itô's formula and (44), for all  $\delta \in (0, T - t)$  we have

$$\begin{aligned} & G(T - \delta, X_{T-\delta}, Y_{T-\delta}) \\ &= G(t, x, y) + M_{T-\delta} + \int_t^{T-\delta} \left( \partial_t G + \frac{1}{2} \partial_{xx}G ((a_1^*)^2 + a_2^2) \right) (s, X_s, Y_s) ds \\ &\geq G(t, x, y) + M_{T-\delta}, \end{aligned}$$

where  $M_s$ ,  $s \in [t, T)$ , is a martingale with  $M_t = 0$ . By first taking expectations, and then letting  $\delta \downarrow 0$ , we arrive by dominating convergence at

$$G(t, x, y) \leq \lim_{\delta \downarrow 0} E[G(T - \delta, X_{T-\delta}, Y_{T-\delta})] = P(X_T \geq Y_T) = J(t, x, y, a_1^*, a_2).$$

Note that we can apply dominated convergence since by Lemma 6.2 the discontinuity points of  $G(T, \cdot, \cdot)$  are attained with probability zero. Since  $a_2$  is arbitrary, we also have  $G(t, x, y) \leq \inf_{a_2 \in \mathcal{A}} J(t, x, y, a_1^*, a_2) \leq \overline{V}(t, x, y)$ .

We next show that  $G \geq \overline{V}$ . To this end fix  $a_1 \in \mathcal{A}$ . By (42) and (43) we have

$$0 \geq \partial_t G(t, x, y) + \frac{1}{2} \partial_{xx}G(t, x, y) (a_1(x, y)^2 + a_2^*(x, y)^2) \quad (45)$$

for all  $(t, x, y) \in [0, T) \times \mathbb{R}^2$ . Now fix  $(t, x, y) \in [0, T) \times \mathbb{R}^2$ , set  $\hat{a} := (a_1, a_2^*)$  and let  $(X, Y) = (X^{t,x,y,\hat{a}}, Y^{t,x,y,\hat{a}})$  be a weak solution of the state SDE (38) on the time interval  $[t, T]$  with initial condition  $(x, y)$ . With Itô's formula and (45) one can derive similar to the reasoning above that

$$E[G(T - \delta, X_{T-\delta}, Y_{T-\delta})] \leq G(t, x, y)$$

for any  $\delta \in (0, T - t)$ . Letting  $\delta \downarrow 0$  we obtain  $G(t, x, y) \geq J(t, x, y, a_1, a_2^*)$ , and hence also  $G(t, x, y) \geq \bar{V}(t, x, y)$ . Thus, we have shown  $G = \bar{V}$  and

$$\sup_{a_1 \in \mathcal{A}} J(t, x, y, a_1, a_2^*) = J(t, x, y, a_1^*, a_2^*) = \inf_{a_2 \in \mathcal{A}} J(t, x, y, a_1^*, a_2). \quad (46)$$

The proof of  $G = \underline{V}$  is similar and therefore omitted.  $\square$

*Proof of Theorem 6.1.* The statement follows immediately from the proof of Lemma 6.3: (46) implies that  $(a_1^*, a_2^*)$  is a saddle point of  $J$  and thus  $(a_1^*, a_2^*)$  is a Nash equilibrium of the two player game.  $\square$

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## References

- [1] B. Baldacci and D. Possamai. Governmental incentives for green bonds investment. *arXiv preprint arXiv:2101.00648*, 2021.
- [2] A. Barrasso and N. Touzi. Controlled diffusion Mean Field Games with common noise, and McKean-Vlasov second order backward SDEs. *arXiv preprint arXiv:2005.07542*, 2020.
- [3] E. Bayraktar, J. Cvitanic, and Y. Zhang. Large tournament games. *Ann. Appl. Probab.*, 29(6):3695–3744, 2019.
- [4] E. Bayraktar and Y. Zhang. A rank-based mean field game in the strong formulation. *Electron. Commun. Probab.*, 21:Paper No. 72, 12, 2016.
- [5] E. Bayraktar and Y. Zhang. Terminal ranking games. *Forthcoming, Mathematics of Operations Research*, 2019.
- [6] S. Callander. Searching for good policies. *American Political Science Review*, 105(4):643–662, 2011.

- [7] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games*. Princeton University Press, 2019.
- [8] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field FBSDEs, control, and games.
- [9] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field games with common noise and master equations.
- [10] C. Deng, X. Su, and C. Zhou. Relative wealth concerns with partial information and heterogeneous priors. *arXiv preprint arXiv:2007.11781*, 2020.
- [11] R. Élie, E. Hubert, T. Mastrolia, and D. Possamaï. Mean-field moral hazard for optimal energy demand response management. *Mathematical Finance*, 31(1):399–473, 2021.
- [12] R. Elie and D. Possamaï. Contracting theory with competitive interacting agents. *SIAM Journal on Control and Optimization*, 57(2):1157–1188, 2019.
- [13] G.-E. Espinosa and N. Touzi. Optimal investment under relative performance concerns. *Math. Finance*, 25(2):221–257, 2015.
- [14] G. Fu, X. Su, and C. Zhou. Mean field exponential utility game: A probabilistic approach. *arXiv preprint arXiv:2006.07684*, 2020.
- [15] D. Glick and C. D. Myers. Learning from others: an experimental test of brownian motion uncertainty models. *Journal of Theoretical Politics*, 27(4):588–612, 2015.
- [16] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58:13–30, 1963.
- [17] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized  $\epsilon$ -Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.
- [18] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.
- [19] E. Hubert. Continuous-time incentives in hierarchies. *arXiv preprint arXiv:2007.10758*, 2020.
- [20] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

- [21] J. Keilson and J. A. Wellner. Oscillating Brownian motion. *J. Appl. Probability*, 15(2):300–310, 1978.
- [22] N. V. Krylov. On Itô’s stochastic integral equations. *Theory of Probability & Its Applications*, 14(2):330–336, 1969.
- [23] N. V. Krylov. *Controlled Diffusion Processes*, volume 14. Springer Science & Business Media, 1980.
- [24] N. V. Krylov. On time inhomogeneous stochastic Itô equations with drift in  $L_{D+1}$ . *Ukrainian Mathematical Journal*, 72(9):1420–1444, 2021.
- [25] D. Lacker. A general characterization of the mean field limit for stochastic differential games. *Probability Theory and Related Fields*, 165(3):581–648, 2016.
- [26] D. Lacker. On the convergence of closed-loop nash equilibria to the mean field game limit. *The Annals of Applied Probability*, 30(4):1693–1761, 2020.
- [27] D. Lacker and K. Webster. Translation invariant mean field games with common noise. *Electronic Communications in Probability*, 20:no. 42, 13, 2015.
- [28] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
- [29] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
- [30] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
- [31] A. Lejay and P. Pigato. Statistical estimation of the oscillating Brownian motion. *Bernoulli*, 24(4B):3568–3602, 2018.
- [32] J. M. McNamara. Optimal control of the diffusion coefficient of a simple diffusion process. *Math. Oper. Res.*, 8(3):373–380, 1983.
- [33] J. M. McNamara. A regularity condition on the transition probability measure of a diffusion process. *Stochastics*, 15(3):161–182, 1985.
- [34] J. M. McNamara. A stochastic differential game with safe and risky choices. *Probability in the Engineering and Informational Sciences*, 2(1):31–39, 1988.
- [35] J. M. McNamara and O. Leimar. *Game theory in biology: concepts and frontiers*. Oxford University Press, USA, 2020.
- [36] S. Nakao. On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka Journal of Mathematics*, 9(3):513–518, 1972.
- [37] M. Nutz and Y. Zhang. A mean field competition. *Math. Oper. Res.*, 44(4):1245–1263, 2019.

- [38] C. Seel and P. Strack. Gambling in contests. *J. Econom. Theory*, 148(5):2033–2048, 2013.
- [39] C. Seel and P. Strack. Continuous time contests with private information. *Math. Oper. Res.*, 41(3):1093–1107, 2016.