

Fuzziness, uncertainty and vagueness : toward a less blurry picture.

(Is a fuzzy set vague ?)

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Introduction

The notion of fuzzy set and its extensions have been understood in various ways in the literature: there are several notions that are appealed to in connection with fuzzy sets

- **Gradualness** : the idea that many categories (in natural language) are a matter of degree, including truth. Against the Boolean tradition
- **Epistemic Uncertainty** : the idea of representing partial or incomplete information by sets. Possibility theory, modal logic, against the probabilistic tradition.
- **Vagueness** : the idea that the extension of natural language predicates lack clear truth conditions. A nightmare for classical logic.

among others

Introduction

Gradualness, Epistemic Uncertainty, Vagueness seem to interact closely with one another within fuzzy set theory:

- A fuzzy set may account for epistemic uncertainty since it extends the notion of a set.
- Epistemic uncertainty is gradual since belief is often a matter of degree.
- Sometimes, membership functions may account for an ill-known crisp boundary and can be seen as modeling vagueness
- Higher order fuzzy sets try to capture ill-known membership functions of linguistic categories, and this seems to refer again to vagueness

A source of confusion in the field.

Aim of the talk

- recall basic issues so as to disentangle the notions of gradualness, and epistemic uncertainty,
- recall the information-oriented view of vagueness that enables some form of classification
- argue that the initial intuition behind fuzzy sets has little to do with some aspects of vagueness studied in the literature
- propose a tentative explanation on some controversies about vagueness and the role of fuzzy sets, based on the point of view of agents uttering non-Boolean statements or receiving them.

Conjunctive vs. disjunctive sets

A set S defined in extension, is often denoted by listing its elements, say, in the finite case $\{s_1, s_2, \dots, s_n\}$. What can such a notion represent ?

1. a collection of (sometimes physical) items that refer to some actual *well-known* complex object, formed by *conjunction* of its elements.
2. an epistemic construction representing incomplete information about an *ill-known* object. In this case, a set is used as a *disjunction* of its elements one of which is the right one.

Examples:

1. Conjunctive: A set-valued attribute, a region in an image, a time-interval spanning the course of a well-known activity.
2. Disjunctive : An event, a piece of incomplete information, an error or tolerance interval representing an imprecise measurement.

Conjunctive vs. Disjunctive fuzzy sets

A fuzzy set S is often denoted by its membership function $\mu_F : S \rightarrow L$, a complete lattice, the unit interval, a finite chain...

In the finite case $\{(s_1, \mu_F(s_1)), (s_2, \mu_F(s_1)), \dots, (s_n, \mu_F(s_1))\}$.

- *Conjunctive fuzzy sets* : precise entities (real things). A region in a grey-levelled image, a profile of ratings according to several criteria, the result of a fuzzy clustering procedure, the meaning of a linguistic term.
- *Disjunctive fuzzy sets* : higher order entities representing epistemic states. The membership function represents a possibility distribution $\pi = \mu_F$ on set S of possible worlds = a fuzzy set of mutually exclusive more or less possible values one of which is the right one.

The word *Tall* refers to a conjunctive fuzzy set μ_{Tall} of human sizes that can be qualified as tall.

If all I know is that *John is tall*, then *Tall* becomes a disjunctive fuzzy set describing knowledge, and we use $\pi = \mu_{Tall}$ as a possibility distribution.

Reasoning with disjunctive sets

Used to represent incomplete information, a disjunctive set is a higher-order construction: this set is an epistemic entity, only the ill-known object referred to is real.

- An element inside a disjunctive set D is a possible candidate for being the real value or state of the world, while elements outside D are considered impossible.
- A disjunctive set D representing an epistemic state underlies epistemic modalities of plausibility and certainty, which are special cases of modal possibility and necessity.
- an event A is considered
 - **possible(plausible)** if it is consistent with D ($A \cap D \neq \emptyset$), denoted $\models \diamond A$;
 - **certain (necessary)** if its negation A^c is inconsistent with D (hence it is entailed by D : $D \subseteq A$), denoted $\models \square A$.

Supervaluation by Van Fraassen, as a mending of partial logic is like Boolean possibility theory: super-true = certainly true (D is then a partial interpretation).

Reasoning with disjunctive fuzzy sets

- Gradual notions of certainty and plausibility extend the modal notions to when the epistemic state is a fuzzy set .
 - The degree of possibility of an event A is $\Pi(A) = \max_{s \in A} \pi(s)$,
 - its degree of necessity is $N(A) = \min_{s \notin A} \nu(\pi(s)) = \nu(\Pi(A^c))$, where ν is the order-reversing map on L .

This is possibility theory, a theory of epistemic uncertainty devoted to the handling of incomplete information (initiated by Zadeh in 1978 to represent natural language statements with fuzzy predicates),

- Variants of it were proposed, sometimes earlier
 - by G.L.S Shackle (the theory of potential surprise, 1949 on)
 - David Lewis (comparative possibility, 1973)
 - Peter Gärdenfors (epistemic entrenchment, mid-80's) and A. Grove.
 - and Spohn (ordinal conditional functions, 1988).

Consequences of the distinction between conjunctive and disjunctive fuzzy sets

Scalar distance between fuzzy sets : $d : L^S \times L^S \rightarrow \mathbb{R}^+$

- Scalar distances between fuzzy sets $d(F, G)$ make sense for conjunctive fuzzy sets (measure how similar F and G are).
- The scalar distance $d(\pi_1, \pi_2)$ does not properly inform about the distance between the two ill-known values x_1 and x_2 referred to by π_1, π_2 .
- This distance $\delta(x_1, x_2) : S \times S \rightarrow \mathbb{R}^+$ is itself ill-known : a disjunctive fuzzy set of possible distances computed from π_1, π_2 via the extension principle.

Truth-values, uncertainty, and ill-known sets

The confusion between gradualness and uncertainty pervading fuzzy set theory : a variant of a confusion between truth values and belief degrees in logic.

- truth is in some sense objective (a representation convention); belief is a higher order construct (subjective, epistemic).
- From the inception of many-valued logics, it has been a temptation to attach an epistemic flavor to truth degrees.
 - Lukasiewicz 's 3d truth-value interpreted as “possible”
 - Partial logic based on Kleene's 3-valued logic, capturing ill-known models
 - Belnap's four-valued logic capturing inconsistency.
- A source of confusion : Elkan criticising the usual fuzzy connectives \max , \min , $1-$, as leading to an inconsistent approach.
- **Claim** We cannot consistently reason under incomplete or conflicting information about propositions by augmenting the set of "ontological" truth-values *true* and *false* with epistemic notions

Degrees of truth and degrees of belief.

Presenting classical (propositional) logic PL as the logic of the true and the false neglects the epistemic accounts of this logic

If a set B of well-formed Boolean formulae is understood as a *belief (or knowledge) base*:

1. p is believed (or known), if B implies p ;
2. its negation is believed (or known), if B implies $\neg p$;
3. neither p nor $\neg p$ is believed, if B implies neither $\neg p$ nor p .

Belief refers to the notion of validity of p in the face of B and is a matter of consequencehood, not truth-values. It is a ternary notion.

Belief (or knowledge if information is correct) corresponds to epistemic values, is induced by information states.

Belief/information states as *subsets of possible truth-values*

One can provide a semantic account of belief in PL by means of *disjunctive subsets of possible truth-values* left possible by a belief base B .

- Full belief in p corresponds to the singleton $\{1\}$ (only 'true' is left)
- full disbelief in p corresponds to the singleton $\{0\}$
- total ignorance about p for the agent corresponds to the set $\mathbf{2} = \{0, 1\}$ (both 0 and 1 are possible)

The characteristic function of $\{0, 1\}$ is viewed as a possibility distribution π on truth-values.

Confusing truth-values and epistemic states comes down to confusing elements and subsets of a set.

NOTE: Subsets of truth-values may also be interpreted conjunctively: $\{0, 1\}$ is understood as the *simultaneous* attachment of "true" and "false" to p expressing a contradiction (Dunn). This is another convention based on necessity degrees.

Ill-known sets

Introducing uncertainty in crisp set theory : attaching to elements $s \in S$ one of the three non-empty subsets of $\mathbf{2}$: $L = \mathbf{2}^2 \setminus \{\emptyset\}$ (= intervals on $\mathbf{2}$)

An *ill-known set* denoted \hat{A} which is also an interval-valued set :

$$\mu_{\hat{A}}(s) = \begin{array}{ll} \{1\} & \text{if } s \text{ belongs for sure to the set} \\ \{0\} & \text{if } s \text{ for sure does not belong to the set} \\ \{0, 1\} & \text{if it is unknown whether } s \text{ belongs or not to the set} \end{array}$$

- It encodes a pair of nested sets (A_*, A^*) , where A_* is the set of sure elements, A^* the set of sure and unsure elements. $A^* \setminus A_*$ contains the elements with unknown membership.
- **Compositionality assumption** : Extending the standard Boolean connectives to such three-valued sets by means of interval computation yields Kleene three-valued logic (encoding $\{0, 1\}$ as $1/2$).
- $\hat{A} \cup \hat{B} = (A_* \cup B_*, A^* \cup B^*)$; $\hat{A} \cap \hat{B} = (A_* \cap B_*, A^* \cap B^*)$; $\hat{A}^c = ((A^*)^c, (A_*)^c)$.

Ill-known sets

- \hat{A} is not an object in itself, it is a representation of the incomplete knowledge of an agent about a set A of which all that is known is that $A_* \subset A \subset A^*$.
- The Kleene algebra structure does not address (but in a very approximate way) the issue of reasoning about the ill-known set A .
- **Example** of $\hat{A}^c = ((A^*)^c, (A_*)^c)$, obtained by switching $\{0\}$ and $\{1\}$.
 - Note $\hat{A} \cap \hat{A}^c = (A_* \cap (A^*)^c, A^* \cap (A_*)^c)$, where $A_* \cap (A^*)^c = \emptyset$ while $A^* \cap (A_*)^c \neq \emptyset$ are the uncertain elements : Boolean tautologies are lost!
 - Despite the fact that A is ill-known, $A \cap A^c = \emptyset$ since A is known to be Boolean (no reason to lose tautologies: supervaluation approach again)

Conclusion : Information is lost by considering subsets of truth-values as truth-values and acting compositionally. Uncertainty never goes along with compositionality.

Traditional views of vagueness

Following Keefe & Smith, there are three features of vagueness in natural language

- **The existence of borderline cases** : violation of the law of excluded middle, difficulty of making a decision as to the truth or the falsity of a vague sentence, even if a precise description of the case is available.
- **Unsharp boundaries**: possibly a specialisation of the above (for predicates referring to continuous scales). Against Boolean tradition.
- **Susceptibility to Sorites paradoxes**. Inappropriate for long inference chains.

But is vagueness

- a defect of natural language (it is not appropriate for devising formal proofs),
- or is it a quality (tolerance to error, flexibility in communication purpose)?

The main dispute seems to oppose those who claim a vague predicate to have unknown standard extension, vs. those who deny the existence of a decision threshold.

Zadeh insists that fuzziness is not vagueness

- **Fuzziness** : *the transition between membership and non-membership is gradual rather than abrupt* (Zadeh, 1965).
- Zadeh (1978) argues in “PRUF: A meaning representation language for natural languages, Int. J. of Man-Machine Studies, 10, 395-460”:

”Although the terms fuzzy and vague are frequently used interchangeably in the literature, there is, in fact, a significant difference between them. Specifically, a proposition, p , is fuzzy if it contains words which are labels of fuzzy sets; and p is vague if it is both fuzzy and insufficiently specific for a particular purpose. For example, ”Bob will be back in a few minutes” is fuzzy, while ”Bob will be back sometime” is vague if it is insufficiently informative as a basis for a decision. Thus, the vagueness of a proposition is a decision-dependent characteristic whereas its fuzziness is not. ”

For Zadeh, fuzziness is gradualness, and vagueness is fuzziness + lack of specificity

An information setting for non-dichotomous representations of sets

UNIFYING PRINCIPLE

- A clear-cut crisp concept induces a dichotomy of the universe of discourse
- A vague concept induces a trichotomy of the universe (borderline cases)

Notations

- A finite set of objects or entities O
- A finite set A of attributes $a : O \rightarrow D_a$ each with domain D_a
- A property P referring to attribute a
- For a clear-cut property there is a non-empty subset $Y_P \subset D_a$, so that $Ext(P) = \{o \in O | a(o) \in Y_P\}$.

Situations leading to trichotomic representations

If $\neg P$ denotes the opposite property let $Ext(\neg P) = \{o \in O | a(o) \in N_a\}$ for some subset N_a of A . Then a property is classical if

- excluded-middle law: $Ext(P) \cup Ext(\neg P) = O$ (EML)
- the non-contradiction law: $Ext(P) \cap Ext(\neg P) = \emptyset$ (NCL)

We consider non-classical properties where EML or NCL does not seem to apply.

1. Gradual properties
2. Tolerant properties based on similarity
3. Vagueness due to ignorance or truth-value gap
4. Multiagent or voting approach
5. Ill-known attribute values
6. Insufficient language

Gradual properties

Properties P like *tall*, *young*, etc., seem to define an implicit complete ordering on the attribute domain D_a :

- A gradual property P is defined by a pair (D_a^P, \geq_P) , where $D_a^P \subset D_a$ such that
 - $u = a(o) \in D_a^P$ means that o is somewhat P .
 - $u = a(o) \notin D_a^P$ means that P is clearly false for o , and $\forall u, v \notin D_a^P, u =_P v$
 - $\forall u \in D_a^P, v \notin D_a^P, u >_P v$
 - $\forall u, v, \in u \in D_a^P, u >_P v$ where $u = a(o)$ and $u' = a(o')$ means that o is more P than o' .
 - Moreover, for the opposite $\neg P$, $u \geq_{\neg P} v$ if and only if $v \geq_P u$.
- Then the role of the membership function μ_P of P is to provide a numerical representation of this ordering. (the consistency profiles of Black as well)?

Gradual properties

- What can be the extension of P ?
 - one may admit there is no Boolean extension but a gradual one $\tilde{Ext}(P)$ with membership function μ_P
 - or one may define $Ext(P) = C(P)\{o \in O, a(o) \text{ maximal according to } \geq_P\}$ (the prototypes of P)
 - or one may use D_a^P .
- It leads to a trichotomy of the set of objects:
 - the prototypes of P ($C(P)$ as above),
 - the prototypes of $\neg P$, i.e., $C(\neg P) = (D_a^P)^c$
 - and the borderline cases ($O \setminus (C(P) \cup C(\neg P))$)
- A precise boundary separating objects such that P holds from those where $\neg P$ holds *does not exist* under the gradual view. EML and CL do not hold.

Some reasons for gradualness in natural language

- Some predicates refer to an *underlying continuous measurement scale* D_a for the attribute (tall : height; young: age). So there is no threshold on the real line separating the P 's from the non P 's (too precise to be cognitively relevant).
 - A test for detecting this kind of predicates: *very P* makes sense.
 - The use of the unit interval as a truth set is a just a way of rescaling the attribute domain D_a .
 - Truth-functionality for such fuzzy predicates is mathematically consistent (algebraic structures different from a Boolean algebra) even if not compulsory
- Some concepts underlie a *typicality ordering* within examples: Bird, Chair.
 - A penguin is a less typical bird than a swallow.
 - No clear membership function, just an ordering.

Similarity and tolerance

- A property P is clear-cut, but one is able to measure how close or similar is one element of the domain D_a to another.
- So it is possible to apply P to objects with some tolerance : even when P is false for o , it makes sense to say that P is not far from being true.
- Typically, when $a(o)$ is close to the boundary of P .
- One may model this situation as follows
 - Suppose d the distance function on D_a , and define a similarity relation on D_a as

$$S(u, v) = f(d(u, v))$$

with $f(0) = 1$, f decreasing and $f(x) = 0$ if $x \geq \theta$.

- The membership grade $\mu_P(o)$ is computed as the degree of similarity between o and those objects for which P is true:

$$\mu_P(o) = \sup\{S(a(o), a(o')), a(o') \in Y_P\}.$$

Similarity and tolerance

- $\max(\mu_P(o), \mu_{\neg P}(o)) = 1, \forall o \in Bo(p)$ (EML holds, not really NCL)
- But one may define $Bo(P) = \{o, \min(\mu_P(o), \mu_{\neg P}(o)) > 0\}$
- This is an example of truth-value glut.
- $C(P) = \{o, \mu_P(o) = 1\} \setminus Bo(P)$.
- This setting is a possible encoding of the idea of approximate truth by Weston (1987)
- There is no vagueness proper here, just the idea of gradual tolerance with respect to truth (or graceful degradation thereof).

Vagueness as ignorance or truth-value gaps

Vague properties P are such that the truth or falsity of P can be claimed for some objects, but not for all.

- Truth of P is not viewed as gradual.
- The borderline elements are those for which the truth or falsity of P is out of reach. For instance,
 - This truth-value does not exist (truth-value gaps: Kit Fine)
 - Or it exists but is unknown (epistemic view : Williamson)
- Truth of P can be asserted for an object o
 - Supervaluation : when it is true for all ways of making P sharp (super-truth for Fine)
 - Epistemic view : when it is known that P is true (Clearly true for Williamson)

In both approaches, the truth of P is established in the same conditions : the only difference is whether the “true” sharpening of P exists or not : the extension of P is either an ill-known set or a partially defined one

Vagueness as ignorance and truth-value gaps

- These views of vagueness also partition the set of objects into 3 subsets:
 $C(P), C(\neg P), Bo(P)$;
 - $C(P)$ = objects for which P is definitely true; $C(\neg P)$ = objects for which P is definitely false;
 - $Bo(P)$ = borderlines cases.
- Then P is clearly true or supertrue for o if $\forall S : C(P) \subseteq S \subseteq C(P) \cup Bo(P), o \in S$.
- The merit of these approaches is to preserve the laws of classical logic : it holds for each sharpening of P ; it holds for the “real” extension of P

Truth-value gap: The voting paradigm

- Different crisp representations of P provided by a set of n agents.
- each agent i perceives P as a classical property defining a partition (Y_P^i, N_P^i) of D_a .
- Then the trichotomy $C(P), C(\neg P), Bo(P)$ is retrieved letting
 - $C(P) = \{o \in O, a(o) \in \bigcap_{i=1, \dots, n} Y_P^i\}$
 - $C(\neg P) = \{o \in O, a(o) \in \bigcap_{i=1, \dots, n} N_P^i\}$
- “ o is P ” is super-true (false) if true (false) for all agents. Otherwise the truth-value is not defined.
- One may define the membership function $\mu_P(u) = \frac{|\{i, u \in Y_P^i\}|}{n}$. But it will not be truth-functional.
- Dually, one may ask for each agent i and each object o if “ o is P ” or not and compute $Prob(P|u)$.
- Then $\mu_P(u) = Prob(P|u)$ if each agent i declares “ o is P ” whenever $u \in Y_P^i$.

Unknown threshold : limited perception

- One reason for vagueness maybe the difficulty to perceive the difference between close values in D_a : if $d(u, v) \leq \epsilon$ then u is perceived as being the same value as v (Parikh).
- If the real extension of P is $Ext(P)$, two objects o and o' such that $o \in Ext(P)$, $o' \in Ext(P')$ will be borderline for P whenever $d(a(o), a(o')) \leq \epsilon$.
- the boundary of the extension of P exists but it will be perceived as thick (of width 2ϵ).
- it corresponds to the epistemic view

Unknown extension : ill-known attribute values

- Suppose a clear-cut property P defining a partition on the corresponding attribute domain $D_a : (Y_P, N_P)$
- The attribute values $a(o)$ are ill-known: $a(o) \in E(o) \subset D_a$.
- Then the set of objects such that “ o is P ” is ill-known :
 - “ o is P ” is certainly true if $E(o) \subset Y_P$
 - “ o is P ” is certainly false if $E(o) \subset N_P$
 - The truth of “ o is P ” is unknown otherwise.

In this case one may say that the reason for vagueness is that the objects to which predicates are applied are themselves ill-known.

Unknown extension: insufficient language

- Suppose a subset of objects given in extension $S = \{o_1, \dots, o_n\}$ corresponding to a concept P : “ o is P ” if and only if $o \in S$.
- Suppose a set $A = \{a_1, \dots, a_k\}$ of attributes. It defines a partition of O into subsets O_1, \dots, O_N induced by the equivalence relation oRo' if and only if $a_j(o) = a_j(o') \forall j = 1, \dots, k$.
- The set S defining property P cannot be precisely described by means of the attributes in A supposedly known for all objects (a **Pawlak rough set**)
 - The set $P_* = \cup_{O_i \subseteq S} O_i$ is the set of objects that are certainly P ;
 - The set $P^* = \cup_{O_i \cap S \neq \emptyset} O_i \supseteq P_*$ is the set of objects that are possibly P .
 - $Bo(P) = P^* \setminus P_*$
- The set S remains out of reach, so that the truth of P for a given object remains unknown, due the lack of expressivity of the language in which P is expressed. Only P_*, P^* can be precisely expressed.

Connectives

- Connectives can be truth-functional under the gradual semantics, extending truth-tables to the unit interval, up to a loss of properties of the Boolean algebra.
- Connectives are not truth-functional under the epistemic and truth-value gap semantics, but the Boolean nature of extensions (or precisiations) is retained. This is not surprising because truth-functionality is lost under partial ignorance.
- Connectives are not truth-functional for rough sets (except for union), ill-known sets and the like.
- Connectives are not truth-functional under the similarity semantics of gradual properties (except for union).

Is there a threshold underlying a fuzzy concept?

How to reconcile the epistemic and the gradual views on fuzzy concepts ? we suggest that it all depends on the situation in which a vague statement is used.

Two different situations:

- **Asserting a gradual statement** : an agent declares “*Jon is tall*”. This claim is unambiguous so (s)he must use an implicit threshold.
- **Receiving a gradual statement** : an agent receives a piece of information of the form “*Jon is tall*”. There is no need for any threshold when interpreted this statement as being gradual.

Asserting a gradual statement

- Asserting a statement “ $P = o$ is F ” is a Boolean event (the agent asserts it or not) whether the statement is vague, gradual, or not.
- If F is a fuzzy set (gradual statement) then there must be a threshold α in $[0, 1]$ such that statement “ $P = o$ is F ” was asserted because $\mu_F(a(o)) \geq \alpha$.
- This threshold according to which F is a fuzzy label appropriate for object o may not be stable across several such asserted statements .
- In any case this threshold (that makes P temporarily crisp) is ill-known (epistemic view).

In practice, when asserting a gradual statement gradualness can be dispensed with, and membership degrees may then just reflect the probability that a label is appropriate to an object (like in the voting paradigm).

Assertional approaches to vagueness

Several approaches refer to assertional vagueness

- The likelihood view of membership functions (Hisdal, Scozzafava and Coletti) :
 $\mu_F(o) = Prob(\text{Asserting } o \text{ is } F|o)$.
- The voting paradigm (conflict between individual crisp representations on the attribute domain)
- The betting approach of Giles, the game-theoretic view of Fermueller: membership degrees = money involved in commitment to assert vague propositions.
- Jon Lawry 's label semantics : collecting from a group of individuals the sets of labels that are appropriate to be attached to objects.

Receiving a gradual statement

- The receiver, upon hearing “ $P = o$ is F ”, knows that F is gradual.
- Its acceptance does not require any decision pertaining to an underlying threshold separating F and $non - F$
- The only decision made is one of accepting the statement as true (its reliability is high enough, above a reliability threshold).
- The receiver assumes that his/her membership function of F is close enough to the one of the emitter (= mutual understanding of the context and vocabulary).
- The membership function μ_F is viewed as a possibility distribution over D_a , hence over possible objects: $\pi(o) = \mu_F(a(o))$.
- It expands or revises the prior epistemic state of the receiver.
- The issue of representing gradual information with a receiver’s point of view is the purpose of Zadeh ’s possibility theory.

Bivalence vs. gradualness

- There seems to be a strong tradition for Bivalence in logic that would need an investigation of its own
- The existence of a decision threshold in the epistemic view is a realistic point of view a la Plato ???.
- The status of truth in philosophy is so prominent that it is taken as an objective notion whose perfection cannot go along shades of truth.
- Yet, bivalence of propositions can be viewed as a convention (thanks, De Finetti), not at all a matter of actual fact. Hence, truth-values *True*, *False* are also a matter of convention.
- Gradual truth is another convention more in agreement with the use of some terms in natural language.

The epistemic view on gradual predicates

Is there a contradiction between gradualness and the epistemic view of vagueness ? NO.

- Even if a gradual predicate should rather be modelled by a fuzzy set than a crisp set, it is not clear at all that this membership function is well-known.
- A “*gradual epistemic view* of vagueness could postulate that the membership function of a gradual, but one is partially ignorant about it.
- As it looks much more difficult to define membership functions of gradual precisely than crisp extensions of clearly bivalent ones, it is natural that most gradual concepts sound more vague than crisp ones.
- The vagueness of a fuzzy concept can be modelled via intervals of truth-values, or even fuzzy sets thereof.
- Likewise, the super-valuationism of Kit fine could be accommodated in a gradual setting: a vague statement is super- α -true if it is at least α -true in all of its gradual precisiations (precise membership functions).

Vagueness is uncertainty about meaning of categories that maybe gradual.

The case of Interval-valued fuzzy sets

An interval-valued fuzzy set F is defined by an interval-valued membership function:

$$F(u) = [F_*(u), F^*(u)], \forall u \in U.$$

- Under the epistemic view: there exists a real membership function $f \in F$.
- Supervaluationism : all $f \in F$ are to be used as none is the true one.

The union, intersection and complementation of IVF's is obtained by canonically extending fuzzy set-theoretic operations to interval-valued operands in the sense of *interval arithmetics*.

$$F \cap G(u) = [\min(F_*(u), G_*(u)), \min(F^*(u), G^*(u))];$$

$$F \cup G(u) = [\max(F_*(u), G_*(u)), \max(F^*(u), G^*(u))];$$

$$F^c(u) = [1 - F^*(u), 1 - F_*(u)].$$

An IVF is also a special case of type 2 fuzzy set (also introduced by Zadeh).

Paradoxes of Interval-valued fuzzy sets

Paradoxes of IVFs are less blatant than those of ill-known sets because the lack of excluded-middle law for Boolean concepts is a striking anomalous feature.

But interval-valued fuzzy sets have a weaker structure than the fuzzy set algebra they extend

For instance, the weak form of the contradiction law ($\min(F(u), F^c(u)) \leq 0.5$) does not hold for IVFs

- $F(u) = [0, 1]$ implies $\min(F(u), 1 - F(u)) = [0, 1]$ (loss of information).
- if $t(p) = F(u) = [a, b]$, the truth-functional calculus yields $t(p \wedge \neg p) = \min(F(u), 1 - F(u)) = [\min(a, 1 - b), \min(b, 1 - a)]$, sometimes not included in $[0, \frac{1}{2}]$.

Like the Kleene calculus applied to ill-known set, the paradox of IVF theory is that it consider sets of truth-values as (new) truth-values. Same flaw with type 2 fuzzy sets calculi.

Reasoning with ill-known truth-values

The interval-valued membership function F is generally not the *known* membership function of an IVF. It represents *what we know about* an ill-known *standard* membership function $\mu \in F$.

- The generic reasoning problem in interval-valued fuzzy logic viewed as a problem of uncertainty management is of the following form:

Given a set of weighted many-valued propositional formulas

$\{(p_i, [a_i, b_i]), i = 1, \dots, n\}$, find the most narrow interval $[a, b]$ such that $(p, [a, b])$ can be deduced. It corresponds to the following optimization problem: maximize (resp. minimize) $t(p)$ under the constraints $t(p_i) \in [a_i, b_i], i = 1, \dots, n$.

- For instance computing the truth-value range of a fuzzy logic formula Φ given the range of truth-values of its atomic constituents is a matter of constrained optimisation, not of applying IVF's truth tables.

Reasoning with ill-known truth-values

Example : finding the interval range of the membership function $F \cap F^c$ when F is an IVF representing an ill-known membership function μ comes down to solving for each element of the universe of discourse the following problem:

$$\text{maximize (resp. minimize) } f(x) = \min(x, 1 - x)$$

under the constraint $x \in [a, b]$.

Since the function f is not monotonic, the solution is obviously not (always) the interval $[\min(a, 1 - b), \min(b, 1 - a)]$ suggested by IVF connectives, it is as follows:

$$f(x) \in [a, b] \text{ if } b \leq 0.5;$$

$$f(x) \in [\min(a, 1 - b), 0.5] \text{ if } a \leq 0.5 \leq b;$$

$$f(x) \in [1 - b, 1 - a] \text{ if } a \geq 0.5.$$

IVF's: interval-truth values vs. ill-known truth-values

There are two ways of interpreting interval membership grades:

- Truth values ARE intervals: $t(p) = F(u)$. Then, apply the truth-functional calculus of IV-fuzzy set theory. (see triangle algebras , by Van Gasse, Kerre....). IVFs are then a special case of L-fuzzy sets in the sense of Goguen.
- Truth -values BELONG to $[0, 1]$ but they are *constrained* by intervals (because ill-known): $t(p) \in F(u)$.

Claim *Fuzzy sets equipped with fixed connectives have a given well-defined structure.*

This structure should be valid whether the membership grades are known or not.

So the operations and algebra of IVF's (L -fuzzy sets with $L = \{[a, b], 0 \leq a \leq b \leq 1\}$)

CANNOT faithfully account for the handling of ill-known membership functions.

They only calculate a conservative approximation of the range of uncertainty of truth values

Conclusion

- Fuzziness refers to gradual truth, not to vagueness regarded as uncertainty about meaning.
- Crisp sets whose boundaries are ill-known may account for some form of vagueness.
- Gradual predicates do not underlie an ill-known crisp sets (hence are not Boolean and may fail to satisfy EML NCL)
- Membership functions summarizing uncertainty about an ill-known threshold are not systematically compositional.
- **Perspectives for vagueness**
 - The study of vagueness might benefit from being cast in context : asserting vs. receiving vague information.
 - Neither the epistemic view of vagueness nor supervaluationism look incompatible with the idea of gradual predicates : they can be combined.