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The electromagnetic mass of a non-Lorentz-contractible uniformly charged spherical shell in rectilinear motion

G. Vaman
Institute of Atomic Physics, P. O. Box MG-6, Bucharest, Romania
(vaman@ifin.nipne.ro)

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We calculate the electromagnetic mass of a non-Lorentz-contractible rigid and uniformly charged spherical shell of radius a whose center is moving on a rectilinear trajectory, starting from the expansion of the electromagnetic self-force given in Jackson's *Classical Electrodynamics*. We are summing all the terms of order $1/a$, linear and non-linear, and we find that the divergent term that appears in the expansion of the self-force around $a = 0$ is proportional to Abraham's longitudinal electromagnetic mass.

1. Introduction

The electromagnetic mass of a charged particle is a measure of the interaction of the particle with its own field. When the particle's characteristic length a tends to zero, i.e. in the point-limit, the electromagnetic self-force can be written as a series expansion around $a = 0$, and has a dominant divergent term of order $1/a$

$$\mathbf{F} = -m_e \dot{\mathbf{v}} + \mathcal{O}(a^0), \quad (1)$$

where e is the total charge, c is the velocity of light and \mathbf{v} is the velocity of the particle. The coefficient m_e is called the electromagnetic mass of the particle and it is divergent when $a \rightarrow 0$ as $1/a$.

For a non-Lorentz-contractible charged spherical shell of radius a , with a uniform charge density $e/(4\pi a^2)$, moving on a rectilinear trajectory $w(t)$ along the x -axis, the electromagnetic mass was found to be ([1] - [3])

$$m_{em} = \frac{e^2}{ac^2} \frac{1}{\beta^3} \left(\frac{\beta}{1-\beta^2} - \frac{1}{2} \ln \frac{1+\beta}{1-\beta} \right), \quad (2)$$

where $\beta = \dot{w}(t)/c$. The above result is known as Abraham's longitudinal electromagnetic mass [3]. The same result for the electromagnetic mass was found in [4] for a charged point particle moving arbitrarily on a rectilinear trajectory, after its self-force was calculated from its spherically averaged self-field.

There exist many methods of calculating the electromagnetic self-force. One of them is known in the literature as Jackson's method [5]. This method, based on the Taylor expansion of the retarded charge and current densities around the present time, was used in [6] to calculate the self-force of a spherically symmetric Lorentz-contractible charge density in its instantaneous rest frame. It is this method that we are studying in detail in this paper.

Let us summarize Jackson's method here, as it is presented in Ch.16 of [6], having made small changes of the notations. The electromagnetic self-force acting on a charged particle is given in terms of the Lorentz force density

$$\mathbf{F} = \int d\mathbf{r} \left(\rho(\mathbf{r}, t) \mathbf{E}_s(\mathbf{r}, t) + \frac{1}{c} \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}_s(\mathbf{r}, t) \right), \quad (3)$$

where the self-fields can be written in terms of the electromagnetic potentials as $\mathbf{E}_s(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$, $\mathbf{B}_s(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$. After writing the electromagnetic potentials in terms of retarded charge and current densities $\phi(\mathbf{r}, t) = \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t')}{R}$, $\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}', t')}{R}$, $R = |\mathbf{r} - \mathbf{r}'|$, $t' = t - R/c$, and expanding the retarded charge and current densities in Taylor series around $t' = t$, one obtains

$$\mathbf{F}(t) = \mathbf{F}^1(t) + \mathbf{F}^2(t), \quad (4)$$

$$\mathbf{F}^1(t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{n+1}{c^{n+2}} \frac{1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \mathbf{v}(t)), \quad (5)$$

$$\mathbf{F}^2(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{n-1}{c^{n+2}} \frac{1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-3} \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \mathbf{v}(t) \cdot (\mathbf{r} - \mathbf{r}') (\mathbf{r} - \mathbf{r}')). \quad (6)$$

Our Eqs. (4) - (6) are Eq. (16.25) combined with the second unnumbered equation after (16.26) in [6], with changed sign. After writing the self-force in this manner, the author of [6] makes a remark about a possible simplification of the integrand when the charge density is spherically symmetric (Eq. (16.27) in [6]). After neglecting the nonlinear terms in time derivatives of velocity, he writes the self-force in the form of the infinite series (16.28).

Jackson uses his formalism while making two important assumptions: the charged particle is studied in its instantaneously rest frame and, in this frame, it is rigid and spherically symmetric. The fact that the particle is studied in its instantaneously rest frame involves that the magnetic term in his Eq. (16.20) is zero, because the current

density is zero. The fact that the charge distribution is spherically symmetric is invoked when the author of [6] infers his Eq. (16.27).

But if we study a non-Lorentz-contractible spherical shell moving on a rectilinear trajectory in the laboratory frame, it is also spherically symmetric in the reference system in which we are studying it, and the magnetic part of the self-force is also zero because, for a current density $\mathbf{J} = (J_x, 0, 0)$, we have $(\mathbf{J} \times \mathbf{B})_x = 0$, and the other two components of $\mathbf{J} \times \mathbf{B}$ compensate each other because of symmetry. So, Jackson's formalism should be good to study a non-Lorentz-contractible spherical shell moving on a rectilinear trajectory in the laboratory frame.

In the next two sections, we are going to calculate the electromagnetic mass of a non-Lorentz-contractible uniformly charged spherical shell, in the laboratory frame, starting from Eqs. (4)-(6). In the third section we are discussing what happens if we use Jackson's trick (16.27).

2. Calculation of the force $\mathbf{F}^1(t)$

The charge density of a non-Lorentz-contractible spherical shell whose center is moving arbitrarily on a rectilinear trajectory $w(t)$ along the x -axis is

$$\rho(\mathbf{r}, t) = \frac{e}{4\pi a^2} \delta(|\mathbf{r} - \mathbf{i}w(t)| - a) = \frac{e}{4\pi a^2} \sum_{p=0}^{\infty} \frac{(-1)^p w^p(t)}{p!} \partial_x^p \delta(r - a), \quad (7)$$

where \mathbf{i} is the versor of the x -axis. We shall assume that, at the moment t , the center of the spherical shell is located at the origin of the coordinate system, which means $w(t) = 0$. We introduce (7) in (5) and, for the integration over \mathbf{r} , we use

$$\int d\mathbf{r} \delta(r - a) f(r, \theta, \phi) = a^2 \int_{S_a} dS_a f(a, \theta, \phi), \quad (8)$$

where

$$dS_a = \sin \theta d\theta d\phi, \quad (9)$$

$\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$ and $f(r, \theta, \phi)$ is an arbitrary function written in terms of spherical coordinates. In our notations, $\mathbf{v}(t) = \dot{w}(t)\mathbf{i}$. Using $\frac{d^{n+1}}{dt^{n+1}} \dot{w}(t)w^p(t) = \frac{1}{p+1} \frac{d^{n+2}}{dt^{n+2}} w^{p+1}(t)$, we have

$$\int d\mathbf{r} |\mathbf{r} - \mathbf{r}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} (v_x(t)\rho(\mathbf{r}, t)) = \frac{e}{4\pi} \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int_{S_a} dS_a \partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-1}. \quad (10)$$

From (10), (5), after writing $\rho(\mathbf{r}', t)$ in the form (7), we obtain

$$F_x^1(t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!c^{n+2}} \frac{n+1}{n+2} \int d\mathbf{r}' \frac{e}{4\pi a^2} \sum_{q=0}^{\infty} \frac{(-1)^q w^q(t)}{q!} \partial_{x'}^q \delta(r' - a) \cdot \frac{e}{4\pi} \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int_{S_a} dS_a \partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-1}. \quad (11)$$

As $w(t) = 0$, only the term $q = 0$ survives, so

$$F_x^1(t) = \frac{e^2}{16\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{n+1}}{n!c^{n+2}} \frac{n+1}{n+2} \frac{1}{(p+1)!} \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int_{S'_a} dS'_a \int_{S_a} dS_a \partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-1}. \quad (12)$$

The derivatives $\partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-1}$ can be calculated with the method used in [4] for the calculation of $\partial_x^p r^n$. From Eq.(12) of [4], after making the substitutions $n+1 \rightarrow p$, $i-1 \rightarrow n-1$, $r \rightarrow |\mathbf{r} - \mathbf{r}'|$, $x \rightarrow x - x'$, one obtains for p even

$$\frac{\partial^p}{\partial x^p} |\mathbf{r} - \mathbf{r}'|^{n-1} = \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{b+\frac{p}{2}} 2^{2b} p!}{(2b)! (\frac{p}{2} - b)!} \left(\frac{1-n}{2} \right)_{b+\frac{p}{2}} |\mathbf{r} - \mathbf{r}'|^{n-p-2b-1} (x - x')^{2b}, \quad (13)$$

and so on. After using in (12) the parametrization

$$\begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta \cos \phi \\ z &= a \sin \theta \sin \phi, \end{aligned} \quad (14)$$

we have to calculate integrals of the form

$$J = \int_{S_a} dS_a \int_{S'_a} dS'_a |\mathbf{r} - \mathbf{r}'|^n (x - x')^k. \quad (15)$$

Noting that, when we change $\mathbf{r} \rightarrow \mathbf{r}'$, J becomes $(-1)^k J$, it follows that these integrals are different from zero only for k even. From (12), (13), using the above remark, we can write

$$F_x^1(t) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!c^{n+2}} \frac{n+1}{n+2} \frac{1}{(p+1)!} \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^\pi d\theta' \sin \theta' \int_0^{2\pi} d\phi' \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{b+\frac{p}{2}} 2^{2b} p!}{(2b)! (\frac{p}{2} - b)!} \left(\frac{1-n}{2} \right)_{b+\frac{p}{2}} |\mathbf{r} - \mathbf{r}'|^{n-p-2b-1} (x - x')^{2b}. \quad (16)$$

Note that $|\mathbf{r} - \mathbf{r}'|^{n-p-2b-1}(x - x')^{2b}$ is of order $1/a^{p-n+1}$. We are interested in this paper only in the singular part of the self-force, denoted by $\mathbf{F}(\frac{1}{a})$, so we study only the term in (16) obtained for $n = p$.

We shall write the powers $|\mathbf{r} - \mathbf{r}'|^n$ using Eqs. (5), (27b) of [7] and the addition theorem for spherical harmonics [6]. One obtains, for $r = r' = a$,

$$|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^n = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\left(\frac{-n}{2}\right)_l}{\left(\frac{1}{2}\right)_l} \frac{a^n}{2^{2l-n}} {}_2F_1\left(l - \frac{n}{2}, l + 1; 2l + 2; 1\right) \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta) P_l^m(\cos \theta') e^{im(\phi - \phi')} \quad (17)$$

or, after using [8]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (18)$$

$$|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^n = 2^n a^n \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\left(\frac{-n}{2}\right)_l}{\left(\frac{1}{2}\right)_l} \frac{(2l+1)! \left(\frac{n}{2}\right)! (l-m)!}{2^{2l} \left(l + \frac{n}{2} + 1\right)! l! (l+m)!} P_l^m(\cos \theta) P_l^m(\cos \theta') e^{im(\phi - \phi')}, \quad (19)$$

where we use the notation $\Gamma(z+1) = z!$ both for integer and non-integer z . Using (19) in (16), the integral over ϕ can be performed easily.

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' e^{im(\phi - \phi')} = 4\pi^2 \delta_{m,0}. \quad (20)$$

For the integral over θ , we use the result from [9]

$$\int_0^\pi d\theta \sin \theta \int_0^\pi d\theta' \sin \theta' (\cos \theta - \cos \theta')^{2b} P_l(\cos \theta) P_l(\cos \theta') = \begin{cases} \frac{(-1)^l 2^{2b+2} (b!)^2}{(2b+1)(b-l)!(b+l+1)!}, & b \geq l \\ 0, & b < l \end{cases}. \quad (21)$$

We obtain

$$F_x^1\left(\frac{1}{a}\right) = -\frac{e^2 \ddot{w}(t) \sqrt{\pi}}{8ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{p/2} \frac{\beta^p (p+1)^2 (-1)^{b+\frac{p}{2}} (-b - \frac{1}{2})! (2b+1)!}{2^{2b} (2b)! \left(\frac{p}{2} - b\right)! \left(b + \frac{1}{2}\right)! \left(b + \frac{1}{2}\right)!} \left(\frac{1-p}{2}\right)_{b+\frac{p}{2}} \cdot S_l, \quad (22)$$

where [10]

$$S_l = \sum_{l=0}^b \frac{(b + \frac{1}{2})_l (2l + 1)! (-2b)_l (2b - l)! (l + \frac{1}{2})!}{(\frac{1}{2})_l (l - b + \frac{1}{2})! l! (l + 1)_{l+1} (b - l)! (l + b + 1)!} = -\frac{\sqrt{\pi} (2b)! (b + 1) (2b - 1)}{\Gamma(\frac{3}{2} - b) \Gamma(b + 1) \Gamma(b + 2)}. \quad (23)$$

From the definition of the Pochhammer symbol [11] we have

$$\left(\frac{1-p}{2}\right)_{b+\frac{p}{2}} = \frac{\Gamma(\frac{1}{2} + b)}{\Gamma(\frac{1-p}{2})}, \quad (24)$$

and the sum over b in (22) can be performed [10] as

$$\begin{aligned} S_b &= \sum_{b=0}^{p/2} \frac{(-1)^b \left(\frac{1-p}{2}\right)_{b+\frac{p}{2}} (-b - \frac{1}{2})! (2b + 1)! (b + 1) (2b - 1)}{2^{2b} (\frac{p}{2} - b)! ((b + \frac{1}{2})!)^2 \Gamma(\frac{3}{2} - b) \Gamma(b + 1) \Gamma(b + 2)} \\ &= -\frac{4}{\Gamma(\frac{1-p}{2}) \Gamma(\frac{p+3}{2})}. \end{aligned} \quad (25)$$

From (22), (23), (25), we obtain

$$F_x^1\left(\frac{1}{a}\right) = -\frac{\pi e^2 \ddot{w}(t)}{4ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \frac{\beta^p (p+1)^2 (-1)^{\frac{p}{2}}}{\frac{(1+p)}{2} \Gamma(\frac{1}{2} - \frac{p}{2}) \Gamma(\frac{1}{2} + \frac{p}{2})}. \quad (26)$$

Using [11]

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z}, \quad (27)$$

we have

$$\Gamma\left(\frac{1}{2} - \frac{p}{2}\right) \Gamma\left(\frac{1}{2} + \frac{p}{2}\right) = (-1)^{\frac{p}{2}} \pi \quad (28)$$

and, after replacing $p \rightarrow 2p$, we obtain from (26)

$$F_x^1\left(\frac{1}{a}\right) = -\frac{e^2 \ddot{w}(t)}{2ac^2} \sum_{p=0}^{\infty} \beta^{2p} (2p + 1) = -\frac{e^2 \ddot{w}(t)}{2ac^2} \gamma^2 (2\gamma^2 - 1), \quad (29)$$

where $\gamma^2 = 1/(1 - \beta^2)$.

3. Calculation of the force $\mathbf{F}^2(t)$

As we did for $\mathbf{F}^1(t)$, we write the charge densities as given in (7), we use (13) and make the important observation that only the terms that contain even powers of $(x - x')$ give non-zero contribution. Then, the term of order $1/a$ in the expression of $F_x^2(t)$ is

$$\begin{aligned}
F_x^2\left(\frac{1}{a}\right) &= \frac{e^2 \ddot{w}(t)}{32\pi^2 c^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}-1} \frac{\beta^p (p^2 - 1) (-1)^{b+\frac{p}{2}-1} 2^{2b} \left(\frac{3-p}{2}\right)_{b+\frac{p}{2}-1}}{(2b)! \left(\frac{p}{2} - b - 1\right)!} \cdot J_1(b) \\
&+ \frac{e^2 \ddot{w}(t)}{32\pi^2 c^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}} \frac{\beta^p (p^2 - 1) (-1)^{b+\frac{p}{2}} 2^{2b} (2p - 2b + 1) \left(\frac{3-p}{2}\right)_{b+\frac{p}{2}}}{(2b+1)! \left(\frac{b}{2} - b\right)!} \cdot J_2(b),
\end{aligned} \tag{30}$$

where

$$J_1(b) = \int_{S'_a} dS'_a \int_{S_a} dS_a \frac{(x - x')^{2b}}{|\mathbf{r} - \mathbf{r}'|^{2b+1}}, \quad J_2(b) = J_1(b+1). \tag{31}$$

To calculate J_1 , we use again (19), (20), (21) (see also the Appendix). One obtains

$$J_1 = \frac{8\pi^2}{a} \frac{(-b - \frac{1}{2})! (b!)^2}{2b+1} S_{l1}, \tag{32}$$

where [10]

$$S_{l1} = \sum_{l=0}^b \frac{(-1)^l (b + \frac{1}{2})_l (2l+1)!}{2^{2l} (\frac{1}{2})_l l! (l - b + \frac{1}{2})! (b-l)! (b+l+1)!} = -\frac{2b-1}{(\frac{1}{2} - b)! (b!)^2}. \tag{33}$$

So, we have

$$J_1 = -\frac{8\pi^2}{a} \frac{(2b-1)}{(2b+1)} \frac{(-b - \frac{1}{2})!}{(\frac{1}{2} - b)!}, \tag{34}$$

$$J_2 = -\frac{8\pi^2}{a} \frac{(2b+1)}{(2b+3)} \frac{(-b - \frac{3}{2})!}{(-b - \frac{1}{2})!}. \tag{35}$$

Replacing (34), (35) in (30), we obtain

$$F_x^2\left(\frac{1}{a}\right) = -\frac{e^2\ddot{w}(t)}{4ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \frac{\beta^p(p^2-1)(-1)^{\frac{p}{2}}}{\Gamma\left(\frac{3}{2}-\frac{p}{2}\right)} \cdot (S_{b1} - S_{b2}), \quad (36)$$

where

$$S_{b1} = \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^b 2^{2b} (2p-2b+1) (b+\frac{1}{2})! (-b-\frac{3}{2})!}{(2b)! (\frac{p}{2}-b)! (2b+3) (-b-\frac{1}{2})!}, \quad (37)$$

$$S_{b2} = \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^b 2^{2b} (b-\frac{1}{2})! (2b-1) (-b-\frac{1}{2})!}{(2b+1)! (\frac{p}{2}-b-1)! (\frac{1}{2}-b)!}. \quad (38)$$

To perform the above two sums over b , we use [11]

$$\frac{(b+\frac{1}{2})!}{(2b)!} = \frac{\Gamma(b+\frac{3}{2})}{\Gamma(2b+1)} = \frac{(b+\frac{1}{2})\Gamma(b+\frac{1}{2})}{\Gamma(2b+1)} = \frac{(2b+1)\sqrt{\pi}}{2^{2b+1}b!}. \quad (39)$$

$$\frac{(-b-\frac{3}{2})!}{(-b-\frac{1}{2})!} = \frac{\Gamma(-b-\frac{1}{2})}{\Gamma(-b+\frac{1}{2})} = -\frac{2}{2b+1}, \quad (40)$$

and, in the expression of S_{b2} , we write $(2b+1)! = (2b+1)(2b)!$ and $(\frac{1}{2}-b)! = \Gamma(\frac{3}{2}-b) = (\frac{1}{2}-b)\Gamma(\frac{1}{2}-b) = (\frac{1}{2}-b)(-\frac{1}{2}-b)!$. Thus, we obtain [10]

$$S_{b1} = -\sqrt{\pi}(2p+1) \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^b}{b! (\frac{p}{2}-b)! (2b+3)} + 2\sqrt{\pi} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^b}{(b-1)! (\frac{p}{2}-b)! (2b+3)} = -\frac{\pi(p+2)}{2\Gamma\left(\frac{p+5}{2}\right)}, \quad (41)$$

$$S_{b2} = -2\sqrt{\pi} \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^b}{(2b+1)b! (\frac{p}{2}-b-1)!} = -\frac{\pi}{\Gamma\left(\frac{p+1}{2}\right)}. \quad (42)$$

Introducing (41), (42) in (36), after the substitution $p \rightarrow 2p$, we obtain

$$F_x^2 \left(\frac{1}{a} \right) = -\frac{\pi e^2 \ddot{w}(t)}{4ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (4p^2 - 1)}{\Gamma \left(\frac{3}{2} - p \right) \Gamma \left(\frac{1}{2} + p \right)} + \frac{\pi e^2 \ddot{w}(t)}{4ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (4p^2 - 1)(p+1)}{\Gamma \left(\frac{3}{2} - p \right) \Gamma \left(\frac{5}{2} + p \right)}. \quad (43)$$

To calculate the first term in (43), we write $\Gamma \left(\frac{3}{2} - p \right) \Gamma \left(\frac{1}{2} + p \right) = \left(\frac{1}{2} - p \right) \Gamma \left(\frac{1}{2} - p \right) \Gamma \left(\frac{1}{2} + p \right) = \left(\frac{1}{2} - p \right) \pi (-1)^p$, and we obtain that this first term is equal to $-F_x^1 \left(\frac{1}{a} \right)$. Then, it follows that the divergent term of the total self-force of a non-Lorentz-contractible uniformly charged spherical shell is

$$F_x \left(\frac{1}{a} \right) = F_x^1 \left(\frac{1}{a} \right) + F_x^2 \left(\frac{1}{a} \right) = \frac{\pi e^2 \ddot{w}(t)}{4ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (4p^2 - 1)(p+1)}{\Gamma \left(\frac{3}{2} - p \right) \Gamma \left(\frac{5}{2} + p \right)}. \quad (44)$$

Using $\Gamma \left(\frac{3}{2} - p \right) \Gamma \left(\frac{5}{2} + p \right) = \left(\frac{1}{2} - p \right) \left(\frac{3}{2} + p \right) \left(\frac{1}{2} + p \right) \Gamma \left(\frac{1}{2} - p \right) \Gamma \left(\frac{1}{2} + p \right) = (-1)^p \pi (1-2p)(3+2p)(1+2p)/8$, we have

$$F_x \left(\frac{1}{a} \right) = -\frac{2e^2 \ddot{w}(t)}{ac^2} \left(\sum_{p=0}^{\infty} \frac{\beta^{2p}}{3+2p} + \sum_{p=0}^{\infty} \frac{p\beta^{2p}}{3+2p} \right). \quad (45)$$

The first sum in the right hand side of (45) is $(\operatorname{arctanh} \beta - \beta)/\beta^3$ and the second sum can be obtained from the first by taking the derivative with respect to β as

$$\sum_{p=0}^{\infty} \frac{p\beta^{2p}}{3+2p} = \frac{\beta^2 \gamma^2 + 3}{2\beta^2} - \frac{3}{2\beta^3} \operatorname{arctanh} \beta. \quad (46)$$

So, we obtain

$$F_x \left(\frac{1}{a} \right) = \frac{e^2 \ddot{w}(t)}{ac^2 \beta^3} \left(\frac{1}{2} \ln \frac{1+\beta}{1-\beta} - \frac{\beta}{1-\beta^2} \right). \quad (47)$$

This is the same result as Eq. (30) in [4], where r is replaced with a . So, we proved that, starting from Eq.(16.25) of [6], *without* using Eq.(16.27), we obtained for the divergent term of the electromagnetic self-force of a non-Lorentz-contractible spherical shell a result that is proportional to the Abraham's longitudinal electromagnetic mass. This result coincides with the results from [1] - [3].

4. Discussion and conclusions

Let us see now what happens if we use Eq. (16.27) of [6]. Looking again at the calculation from [6], we see that from (16.25) and the second equation after (16.26) it follows

$$\mathbf{F} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!c^{n+2}} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) R^{n-1} \cdot \frac{\partial^{n+1}}{\partial t^{n+1}} \left\{ \rho(\mathbf{r}, t) \mathbf{v}(t) \left[\frac{n+1}{n+2} - \frac{n-1}{n+2} \left(\frac{\mathbf{R} \cdot \mathbf{v}}{Rv} \right)^2 \right] \right\}, \quad (48)$$

while from (16.25) and (16.27) it follows

$$\mathbf{F} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!c^{n+2}} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) R^{n-1} \cdot \frac{\partial^{n+1}}{\partial t^{n+1}} \left\{ \rho(\mathbf{r}, t) \mathbf{v}(t) \left[\frac{n+1}{n+2} - \frac{n-1}{3(n+2)} \right] \right\}. \quad (49)$$

We define

$$\tilde{\mathbf{F}}^2 = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \frac{n-1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \mathbf{v}(t)). \quad (50)$$

Comparing (48), (49), if Eq. (16.27) from [6] is correct, we should have

$$\tilde{\mathbf{F}}^2 = \mathbf{F}_2. \quad (51)$$

Let us calculate the divergent part of \tilde{F}_x^2 , using the same method that we used in Secs. 2, 3 for the calculation of the divergent parts of F_x^1 and F_x^2 . The x -component of $\tilde{\mathbf{F}}^2$ is

$$\tilde{F}_x^2 = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \frac{n-1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \dot{w}(t)). \quad (52)$$

Using the charge density (7) and performing the angular integration, we obtain

$$\tilde{F}_x^2 \left(\frac{1}{a} \right) = \frac{e^2 \ddot{w}(t)}{48ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}} \sum_{l=0}^b \frac{(-1)^{b+\frac{p}{2}} \beta^p (p^2 - 1) \left(-\frac{1}{2} + b\right)!}{(2b)! \left(\frac{p}{2} - b\right)! \left(-\frac{1}{2} - \frac{p}{2}\right)!} \frac{\left(b + \frac{1}{2}\right)_l (2l+1)! \left(-b - \frac{1}{2}\right)! (-1)^l 2^{2b+2} (b!)^2}{l! \left(\frac{1}{2}\right)_l 2^{2l} \left(l - b + \frac{1}{2}\right)! (2b+1)(b-l)!(b+l+1)!}. \quad (53)$$

The sum over l is found to be [10]

$$\begin{aligned}\tilde{S}_l &= \sum_{l=0}^b \frac{(-1)^l (b + \frac{1}{2})_l (2l + 1)!}{l! (\frac{1}{2})_l 2^{2l} (l - b + \frac{1}{2})! (b + l + 1)! (b - l)!} \\ &= -\frac{(b + 1)(2b - 1)}{b! (\frac{1}{2} - b)! (b + 1)!}.\end{aligned}\quad (54)$$

From (53), (54) it follows that

$$\begin{aligned}\tilde{F}_x^2 \left(\frac{1}{a} \right) &= -\frac{e^2 \ddot{w}(t)}{12ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \frac{(-1)^{\frac{p}{2}} \beta^p (p^2 - 1)}{(-\frac{1}{2} - \frac{p}{2})!} \\ &\quad \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^b 2^{2b} (-\frac{1}{2} + b)! (-b - \frac{1}{2})! (b!)^2 (b + 1)(2b - 1)}{b! (2b)! (\frac{p}{2} - b)! (2b + 1) (\frac{1}{2} - b)! (b + 1)!}.\end{aligned}\quad (55)$$

The sum over b in the above equation can be performed after using

$$\frac{(-\frac{1}{2} + b)!}{(2b)!} = \frac{\Gamma(b + \frac{1}{2})}{\Gamma(2b + 1)} = \frac{\sqrt{\pi}}{2^{2b} b!}, \quad (56)$$

$$\frac{(-b - \frac{1}{2})!}{(-b + \frac{1}{2})!} = \frac{\Gamma(-b + \frac{1}{2})}{\Gamma(-b + \frac{3}{2})} = -\frac{2}{2b - 1}. \quad (57)$$

One obtains [10]

$$\tilde{S}_b = -2\sqrt{\pi} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^b}{b! (\frac{p}{2} - b)! (2b + 1)} = -\frac{\pi}{(\frac{p}{2} + \frac{1}{2})!}. \quad (58)$$

From (55), (58) it follows that

$$\tilde{F}_x^2 \left(\frac{1}{a} \right) = \frac{\pi e^2 \ddot{w}(t)}{12ac^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \frac{\beta^p (p^2 - 1) (-1)^{\frac{p}{2}}}{(-\frac{1}{2} - \frac{p}{2})! (\frac{p}{2} + \frac{1}{2})!}. \quad (59)$$

After replacing $p \rightarrow 2p$ and using $(-\frac{1}{2} - p)! (p + \frac{1}{2})! = \Gamma(\frac{1}{2} - p) \Gamma(\frac{3}{2} + p) = (-1)^p \pi (1 + 2p)/2$, one obtains

$$\tilde{F}_x^2 \left(\frac{1}{a} \right) = \frac{e^2 \ddot{w}(t)}{6ac^2} \sum_{p=0}^{\infty} \beta^{2p} (2p - 1) = \frac{e^2 \ddot{w}(t)}{6ac^2} \gamma^2 (2\gamma^2 - 3) \quad (60)$$

Using (60) in (49) we obtain for the divergent term of the total self-force

$$F_x \left(\frac{1}{a} \right) = F_x^1 + \tilde{F}_x^2 = -\frac{2}{3} \frac{e^2 \ddot{w}(t)}{ac^2} \gamma^4. \quad (61)$$

Comparing (61) with (47), we see that these two results are equal only in the limit $\beta \rightarrow 0$. This happens because *the equality (51) is also valid only in the limit $\beta \rightarrow 0$*

$$\begin{aligned} \lim_{\beta \rightarrow 0} F_x^2 \left(\frac{1}{a} \right) &= \lim_{\beta \rightarrow 0} \left[\frac{e^2 \ddot{w}(t)}{2ac^2} \gamma^2 (2\gamma^2 - 1) + \frac{e^2 \ddot{w}(t)}{ac^2 \beta^3} \left(\frac{1}{2} \ln \frac{1+\beta}{1-\beta} - \frac{\beta}{1-\beta^2} \right) \right] = \\ &= \frac{e^2 \ddot{w}(t)}{6ac^2} = \lim_{\beta \rightarrow 0} \tilde{F}_x^2 \left(\frac{1}{a} \right). \end{aligned} \quad (62)$$

Why Eq. (51) is not valid for any value of β ? Because Eq. (16.27) from [6] is valid only when the integrand that appears in the calculation of the self-force depends on a spherically symmetric charge density. But, as we can see in our Eq.(48), the integrand does not depend only on the spherically symmetric charge density $\rho(\mathbf{r}, t) = \frac{e}{4\pi a^2} \delta(|\mathbf{r} - \mathbf{i}w(t)| - a)$, but also on its time derivatives that are not symmetric any more.

Using Jackson's formalism from Ch.16 of [6], we calculated the divergent term of the electromagnetic self-force of a non-Lorentz-contractible uniformly charged spherical shell in rectilinear motion. This divergent term is of order $1/a$, and it is proportional to the Abraham's longitudinal electromagnetic mass. By direct calculation it can be shown that the term of order $1/a^2$ is zero. Our result is in accordance with the results from [1] - [3]. We also showed that we must be careful when using Eq.(16.27) from [6], because the time derivatives of a spherically symmetric time-dependent charge density are not generally spherically symmetric.

Appendix

In this appendix, we calculate the double integral on a sphere of radius a

$$J_1(b) = \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r}-\mathbf{r}'|^{2b+1}}, \quad (63)$$

where \mathbf{r}, \mathbf{r}' are vectors of the same length a that span the surface of the sphere. We use the parametrization Eq.(14), for wich the surface element is given by Eq.(9). We calculate this integral by using two methods.

1. First method

Using the expansion Eq.(19) for $n \rightarrow -2b - 1$, after performing the integration over ϕ, ϕ' according to Eq.(20) and the integration over θ, θ' according to Eq.(21), one obtains

$$J_1(b) = \frac{8\pi^2 (b!)^2 (-b - \frac{1}{2})!}{a(2b+1)} \sum_{l=0}^{\infty} \frac{(-1)^l (b + \frac{1}{2})_l (2l+1)!}{l! (\frac{1}{2})_l 2^{2l} (l-b + \frac{1}{2})! (b-l)! (b+l+1)!}. \quad (64)$$

After performing the summation over l , one obtains

$$J_1(b) = -\frac{16\pi^2}{a(2b+1)}. \quad (65)$$

2. Second method

From

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'|_{r=r'=a} &= a\sqrt{2}\sqrt{1 - \cos\theta_{12}}, \\ \cos\theta_{12} &= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \end{aligned} \quad (66)$$

using the Taylor expansion, one obtains

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}} = \frac{1}{a^n 2^{\frac{n}{2}}} \sum_{k=0}^{\infty} (\cos\theta_{12})^k \frac{(k + \frac{n}{2} - 1)!}{k! (\frac{n}{2} - 1)!}. \quad (67)$$

Using [12]

$$(\cos\theta_{12})^k = \sum_{l=0 \text{ or } 1}^k \frac{(2l+1)k! P_l(\cos\theta_{12})}{2^{\frac{k-l}{2}} (\frac{k-l}{2})! (k+l+1)!}, \quad (68)$$

where k and l have the same parity, and the addition theorem of spherical harmonics [6], after changing the summation index $l \rightarrow q$, $k-l=2q$, we obtain

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^n} &= \frac{1}{a^n 2^{\frac{n}{2}}} \sum_{k=0}^{\infty} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=-k+2q}^{k-2q} \frac{(k + \frac{n}{2} - 1)!}{k! (\frac{n}{2} - 1)!} \cdot \frac{(2k-4q+1)k!}{2^q q! (2k-2q+1)!} \\ &\quad \cdot \frac{(k-2q-m)!}{(k-2q+m)!} P_{k-2q}^m(\cos\theta) P_{k-2q}^m(\cos\theta') e^{im(\phi-\phi')}. \end{aligned} \quad (69)$$

Using this expansion for $n=2b+1$ and introducing it in (63), after performing the integrals over ϕ, ϕ' and θ, θ' as in the first method, we obtain

$$\begin{aligned} J_1(b) &= \frac{4\pi^2}{a 2^{b+\frac{1}{2}}} \sum_{k=0}^{\infty} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k+b-\frac{1}{2})!}{(b-\frac{1}{2})!} \cdot \frac{(2k-4q+1)}{2^q q! (2k-2q+1)!} \\ &\quad \cdot \frac{(-1)^k 2^{2b+2} (b!)^2}{(2b+1)(b-k+2q)!(b+k-2q+1)!}. \end{aligned} \quad (70)$$

Changing the order of summation and using the fact that $k \leq 2q+b$ (as follows from Eq.(21) when $l \rightarrow k-2q$), one obtains

$$J_1(b) = \frac{16\pi^2 2^{b-\frac{1}{2}}}{a} \cdot \frac{(b!)^2}{(2b+1)(b-\frac{1}{2})!} \sum_{q=0}^{\infty} \frac{1}{2^q q!} \cdot S_k, \quad (71)$$

where

$$S_k = \sum_{k=2q}^{2q+b} \frac{(-1)^k (k + b - \frac{1}{2})! (2k - 4q + 1)}{(2k - 2q + 1)!! (b - k + 2q)! (b + k - 2q + 1)!}. \quad (72)$$

The above sum S_k can be easily expressed in terms of finite generalized hypergeometric functions ${}_3F_2$ as

$$S_k = S_k^1 + S_k^2, \quad (73)$$

where

$$\begin{aligned} S_k^1 &= \frac{\sqrt{\pi}}{2^{q+1}} \frac{\Gamma(b + 2q + \frac{1}{2})}{\Gamma(b + 1)\Gamma(b + 2)\Gamma(q + \frac{3}{2})} {}_3F_2 \left(1, -b, b + 2q + \frac{1}{2}; b + 2, q + \frac{3}{2}; \frac{1}{2} \right), \\ S_k^2 &= \frac{\sqrt{\pi}}{2^{q+1}} \frac{\Gamma(b + 2q + \frac{3}{2})}{\Gamma(b)\Gamma(b + 3)\Gamma(q + \frac{5}{2})} {}_3F_2 \left(2, 1 - b, b + 2q + \frac{3}{2}; b + 3, q + \frac{5}{2}; \frac{1}{2} \right). \end{aligned} \quad (74)$$

We introduce (73), (74) in (71), and perform the summation over q as follows:

$$\begin{aligned} \sum_{q=0}^{\infty} \frac{1}{2^q q!} S_k^1 &= \frac{\sqrt{2}\Gamma(b + \frac{1}{2})}{2^b(1 - 2b)\Gamma(b + 1)\Gamma(b + 2)} {}_3F_2 \left(-b, b + \frac{1}{2}, 1; b + 2, \frac{3}{2} - b; 1 \right), \\ \sum_{q=0}^{\infty} \frac{1}{2^q q!} S_k^2 &= \frac{4\sqrt{2}\Gamma(b + \frac{3}{2})}{2^b(1 - 2b)(3 - 2b)\Gamma(b)\Gamma(b + 3)} {}_3F_2 \left(1 - b, b + \frac{3}{2}, 2; b + 3, \frac{5}{2} - b; 1 \right). \end{aligned} \quad (75)$$

The hypergeometric functions in (75) can be written in terms of Gamma functions using ([11])

$$\begin{aligned} {}_3F_2(-n, a, b; c, 2 + a + b - c - n; 1) &= \frac{(c - a - 1)_n (c - b)_n}{(c)_n (c - a - b - 1)_n} \\ &\cdot \left[1 + \frac{nb}{(c - a - 1)(b - c - n + 1)} \right], \\ {}_3F_2(-n, a, b; c, 1 + a + b - c - n; 1) &= \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n} \end{aligned} \quad (76)$$

and, after doing some simplifications using the properties of the Gamma functions and of the Pochhammer symbols, one obtains again the result (65).

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