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# On the semi-positive definition of the product of a positive diagonal matrix and a symmetric semi-positive definite matrix

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**Problem statement:** We have  $\Lambda$  a real symmetric semi-positive definite matrix and  $D$  a real positive diagonal matrix (thus also symmetric). The product  $\Lambda D$  is a square matrix but non-symmetric. We want to prove that  $\Lambda D$  quadratic form,  $x^T \Lambda D x$ , is semi-positive definite, i.e.  $x^T \Lambda D x \geq 0 \forall x \in \mathbb{R}^n \setminus \{0\}$ .

If the elements of the diagonal  $D$  are equal, i.e.  $D = aI$  with  $a \in \mathbb{R}^+ \setminus \{0\}$ , then the product  $\Lambda D = a\Lambda$  is a real symmetric semi-positive definite matrix. Thus, for a diagonal  $D$  with equal elements we have directly  $x^T \Lambda D x \geq 0 \forall x \in \mathbb{R}^n \setminus \{0\}$ .

For different elements on the diagonal, we have the following results:

- The eigenvalues of  $\Lambda D$  are real and non-negatives
- If  $\frac{\Lambda D + (\Lambda D)^T}{2}$  has non-negative eigenvalues, then the quadratic form of  $\Lambda D$  is semi-positive definite

*Proof.* Let us recall some useful matrix definitions and properties:

**Definition 1** (Real symmetric semi-positive definite matrix). *A is a real symmetric matrix (its eigenvalues are thus real): A is semi-positive definite  $\iff$  all its eigenvalues are non-negatives.*

**Definition 2** (Quadratic form semi-positive definition). *A matrix A is semi-positive definite  $\iff x^T A x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$*

**Property 1.** *If A is a real semi-positive definite matrix, then  $B^T A B$  is semi-positive definite for any matrix B.*

**Property 2** (Matrix congruent to a symmetric matrix). *Any matrix congruent to a symmetric matrix is again symmetric: If A is a symmetric matrix then so is  $B^T A B$  for any matrix B.*

## Proof that the eigenvalues of $\Lambda D$ are real and non-negatives

Because  $D$  is diagonal and positive we can write  $D = D^{\frac{1}{2}}D^{\frac{1}{2}}$ ,  $D^{\frac{1}{2}}$  is also real positive and symmetric thus invertible. Let us reformulate the matrix  $\Lambda D$ :

$$\begin{aligned}\Lambda D &= \Lambda D^{\frac{1}{2}}D^{\frac{1}{2}} \\ &= D^{-\frac{1}{2}}(D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}})D^{\frac{1}{2}}\end{aligned}\tag{1}$$

It corresponds to a change of basis of  $D^{\frac{1}{2}}$ . Because the eigenvalues (denoted  $\lambda$ ) are invariant to change of basis we have:

$$\lambda(\Lambda D) = \lambda(D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}})\tag{2}$$

Using the Properties. 1 and 2, because  $\Lambda$  is a real symmetric semi-positive definite matrix,  $(D^{\frac{1}{2}})^T\Lambda D^{\frac{1}{2}} = D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}}$  is a real symmetric semi-positive definite matrix. Thus its eigenvalues are real and non-negatives (Definition.1) and so are the ones of  $\Lambda D$  because of Eq.2. Then we have proven our first result: the eigenvalues of  $\Lambda D$  are real and non-negatives.

## Semi-positive definition of $\Lambda D$

In this part we look at a way to prove the semi-positive definition of  $\Lambda D$  by studying its symmetric part in the Toeplitz decomposition:

**Definition 3** (Toeplitz decomposition). *Every square matrix  $A$  can be decomposed uniquely as the sum of two matrices  $U$  and  $V$ , where  $U$  is symmetric and  $V$  is skew-symmetric.*

$$A = U + V = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\tag{3}$$

In our case  $A = \Lambda D$  and  $U = \frac{1}{2}(\Lambda D + (\Lambda D)^T)$  is symmetric.

We recall the fact that the quadratic form of a skew-symmetric matrix equals to zero. Indeed, by definition  $V^T = -V$  and thus  $x^T V x = (x^T V^T x)^T = -x^T V x$  which holds only if it equals to zero. Thus, the quadratic form of  $x^T \Lambda D x$  is the same one of  $x^T U x$ , i.e:

– If  $\frac{\Lambda D + (\Lambda D)^T}{2}$  has non-negative eigenvalues, then the quadratic form of  $\Lambda D$  is semi-positive definite

$$x^T \Lambda D x = x^T \left( \frac{\Lambda D + (\Lambda D)^T}{2} \right) x\tag{4}$$

One can thus prove the semi-positive definition of the symmetric matrix  $U$  to prove the semi-positive definition of  $\Lambda D$ . Indeed, if  $x^T U x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ , using Eq.4, we obtain  $x^T \Lambda D x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ : proving of the semi-positive definition of  $\Lambda D$ .

One way to prove the semi-positive definition of  $U$  is to look at its eigenvalues. Because  $U$  is symmetric, if its eigenvalues are non-negatives then  $U$  is semi-positive definite (see Definition 1). This gives our second result: if  $U$  has non-negative eigenvalues, then  $\Lambda D$  is semi-positive definite.

## On the eigenvalues of $U$

One may notice that we have further information on the eigenvalues of  $U$  with respect to the ones of  $\Lambda D$ . Using the following theorem of Fan on matrices [1] (Chapter 10, Theorem 10.28):

**Theorem 1.** *Let  $A$  be an  $n \times n$  matrix with eigenvalues  $\lambda_1(A), \dots, \lambda_n(A)$  and  $\mathcal{R}_e \lambda_i(A)$  their real parts. Then:*

$$\sum_{i=1}^n \mathcal{R}_e \lambda_i(A) \leq \sum_{i=1}^n \lambda_i\left(\frac{A + A^T}{2}\right) \quad (5)$$

Then, because the eigenvalues of  $\Lambda D$  are real and non-negatives (as proven in the first part), we have using the Theorem 1:

$$0 \leq \sum_{i=1}^n \lambda_i(\Lambda D) \leq \sum_{i=1}^n \lambda_i(U) \quad (6)$$

Thus, we know that the sum of the eigenvalues of  $U$  is non-negative, however to prove the semi-positive definition of  $U$  it is needed to prove that all its eigenvalues are non-negatives.  $\square$

## References

- [1] Fuzhen Zhang. *Matrix theory: basic results and techniques*. Springer, 2011.