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Proof semi-positive definition of the product of a positive diagonal matrix and a symmetric semi-positive definite matrix

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Problem statement: We have Λ a real symmetric semi-positive definite matrix and D a real positive diagonal matrix (thus also symmetric). The product ΛD is a square matrix but non-symmetric. We want to prove that ΛD quadratic form, $x^T \Lambda D x$, is semi-positive definite, i.e. $x^T \Lambda D x \geq 0 \forall x \in \mathbb{R}^n \setminus \{0\}$.

The proof is the following:

Proof. Let us recall some useful matrix definitions and properties:

Definition 1 (Real symmetric semi-positive definite matrix). *A is a real symmetric matrix (its eigenvalues are thus real): A is semi-positive definite \iff all its eigenvalues are non-negative.*

Definition 2 (Quadratic form semi-positive definition). *A matrix A is semi-positive definite $\iff x^T A x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$*

Property 1. *If A is a real semi-positive definite matrix, then $B^T A B$ is semi-positive definite for any matrix B.*

Property 2 (Matrix congruent to a symmetric matrix). *Any matrix congruent to a symmetric matrix is again symmetric: If A is a symmetric matrix then so is $B^T A B$ for any matrix B.*

Let us prove that ΛD is semi-positive definite with respect to the Definition 2.

First, we prove that the eigenvalues of ΛD are reals and non-negatives:

Because D is diagonal and positive we can write $D = D^{\frac{1}{2}} D^{\frac{1}{2}}$, $D^{\frac{1}{2}}$ is also real positive and symmetric thus invertible. Let us reformulate the matrix ΛD :

$$\begin{aligned} \Lambda D &= \Lambda D^{\frac{1}{2}} D^{\frac{1}{2}} \\ &= D^{-\frac{1}{2}} (D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}) D^{\frac{1}{2}} \end{aligned} \tag{1}$$

It corresponds to a change of basis of $D^{\frac{1}{2}}$. Because the eigenvalues (denoted λ) are invariant to change of basis we have:

$$\lambda(\Lambda D) = \lambda(D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}) \tag{2}$$

Using the Properties.1 and 2, because Λ is a real symmetric semi-positive definite matrix, $(D^{\frac{1}{2}})^T \Lambda D^{\frac{1}{2}} = D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}$ is a real symmetric semi-positive definite matrix. Thus its eigenvalues

are real and non-negatives (Definition.1) and so are the ones of ΛD because of Eq.2. Then we have proven that the eigenvalues of ΛD are real and non-negative.

Now, let us prove the semi-positive definition of ΛD :

Property 3 (Types of Quadratic Forms). *Quadratic forms can be classified according to the nature of the eigenvalues of the matrix of the quadratic form:*

- *If all λ_i are positive, the form is said to be positive definite.*
- *If all λ_i are negative, the form is said to be negative definite.*
- *If all λ_i are non-negative (positive or zero), the form is said to be positive semi-definite.*
- *If all λ_i are non-positive (zero or negative), the form is said to be negative semi-definite.*
- *If the λ_i represent a mixture of positive, zero, and negative values, the form is said to be indefinite*

Using the Property 3 of [1], because the the eigenvalues of ΛD are real and non-negative we have directly the semi-positive definition of the quadratic form.

This property can be proven as follows:

Definition 3 (Toeplitz decomposition). *Every square matrix A can be decomposed uniquely as the sum of two matrices U and V , where U is symmetric and V is skew-symmetric.*

$$A = U + V = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \quad (3)$$

In our case $A = \Lambda D$ and $U = \frac{1}{2}(\Lambda D + (\Lambda D)^T)$ is symmetric.

We recall the fact that the quadratic form of a skew-symmetric matrix equals to zero. Indeed, by definition $V^T = -V$ and thus $x^T V x = (x^T V^T x)^T = -x^T V x$ which holds only if it equals to zero. Thus, the quadratic form of $x^T \Lambda D x$ is the same one of $x^T U x$, i.e:

$$x^T \Lambda D x = x^T \left(\frac{\Lambda D + (\Lambda D)^T}{2} \right) x \quad (4)$$

One can thus prove the semi-positive definition of the symmetric matrix U to prove the semi-positive definition of ΛD .

We use the following theorem on matrices [2] (Chapter 10, Theorem 10.28):

Theorem 1. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$ and $\mathcal{R}_e \lambda_i(A)$ their real part. Let $\lambda(A) = \sum_{i=1}^n \lambda_i(A)$ and $\mathcal{R}_e \lambda(A) = \sum_{i=1}^n \mathcal{R}_e \lambda_i(A)$. Then:*

$$\mathcal{R}_e \lambda(A) \leq \lambda \left(\frac{A + A^T}{2} \right) \quad (5)$$

Then, because the eigenvalues of ΛD are real and non-negative (as proven in the previous part), we have using the Theorem 1:

$$0 \leq \lambda(\Lambda D) \leq \lambda(U) \quad (6)$$

Thus, the eigenvalues of U are non-negative, delivering the proof of the semi-positive definition of U , because U is symmetric (see Definition 1). Then, $x^T U x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and using Eq.4, we obtain $x^T \Lambda D x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$: proving of the semi-positive definition of ΛD .

□

References

- [1] J. D. Carroll and P. E. Green. *Decomposition of Matrix Transformations: Eigen structures and Quadratic Forms. Mathematical Tools for Applied Multivariate Analysis.* Elsevier Inc., 1997.
- [2] Fuzhen Zhang. *Matrix theory: basic results and techniques.* Springer, 2011.