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# Proof semi-positive definition of the product of a positive diagonal matrix and a symmetric semi-positive definite matrix

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**Problem statement:** We have  $\Lambda$  a real symmetric semi-positive definite matrix and  $D$  a real positive diagonal matrix (thus also symmetric). The product  $\Lambda D$  is a square matrix but non-symmetric. We want to prove that  $\Lambda D$  quadratic form,  $x^T \Lambda D x$ , is semi-positive definite, i.e.  $x^T \Lambda D x \geq 0 \forall x \in \mathbb{R}^n \setminus \{0\}$ .

The proof is the following:

*Proof.* Let us recall some useful matrix definitions and properties:

**Definition 1** (Real symmetric semi-positive definite matrix). *A is a real symmetric matrix (its eigenvalues are thus real): A is semi-positive definite  $\iff$  all its eigenvalues are non-negative.*

**Definition 2** (Quadratic form semi-positive definition). *A matrix A is semi-positive definite  $\iff x^T A x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$*

**Property 1.** *If A is a real semi-positive definite matrix, then  $B^T A B$  is semi-positive definite for any matrix B.*

**Property 2** (Matrix congruent to a symmetric matrix). *Any matrix congruent to a symmetric matrix is again symmetric: If A is a symmetric matrix then so is  $B^T A B$  for any matrix B.*

Let us prove that  $\Lambda D$  is semi-positive definite with respect to the Definition 2.

**First, we prove that the eigenvalues of  $\Lambda D$  are reals and non-negatives:**

Because  $D$  is diagonal and positive we can write  $D = D^{\frac{1}{2}} D^{\frac{1}{2}}$ ,  $D^{\frac{1}{2}}$  is also real positive and symmetric thus invertible. Let us reformulate the matrix  $\Lambda D$ :

$$\begin{aligned} \Lambda D &= \Lambda D^{\frac{1}{2}} D^{\frac{1}{2}} \\ &= D^{-\frac{1}{2}} (D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}) D^{\frac{1}{2}} \end{aligned} \tag{1}$$

It corresponds to a change of basis of  $D^{\frac{1}{2}}$ . Because the eigenvalues (denoted  $\lambda$ ) are invariant to change of basis we have:

$$\lambda(\Lambda D) = \lambda(D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}) \tag{2}$$

Using the Properties.1 and 2, because  $\Lambda$  is a real symmetric semi-positive definite matrix,  $(D^{\frac{1}{2}})^T \Lambda D^{\frac{1}{2}} = D^{\frac{1}{2}} \Lambda D^{\frac{1}{2}}$  is a real symmetric semi-positive definite matrix. Thus its eigenvalues

are real and non-negatives (Definition.1) and so are the ones of  $\Lambda D$  because of Eq.2. Then we have proven that the eigenvalues of  $\Lambda D$  are real and non-negative.

**Now, let us prove the semi-positive definition of  $\Lambda D$ :**

**Property 3** (Types of Quadratic Forms). *Quadratic forms can be classified according to the nature of the eigenvalues of the matrix of the quadratic form:*

- *If all  $\lambda_i$  are positive, the form is said to be positive definite.*
- *If all  $\lambda_i$  are negative, the form is said to be negative definite.*
- *If all  $\lambda_i$  are non-negative (positive or zero), the form is said to be positive semi-definite.*
- *If all  $\lambda_i$  are non-positive (zero or negative), the form is said to be negative semi-definite.*
- *If the  $\lambda_i$  represent a mixture of positive, zero, and negative values, the form is said to be indefinite*

Using the Property 3 of [1], because the the eigenvalues of  $\Lambda D$  are real and non-negative we have directly the semi-positive definition of the quadratic form.

This property can be proven as follows:

**Definition 3** (Toeplitz decomposition). *Every square matrix  $A$  can be decomposed uniquely as the sum of two matrices  $U$  and  $V$ , where  $U$  is symmetric and  $V$  is skew-symmetric.*

$$A = U + V = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \quad (3)$$

In our case  $A = \Lambda D$  and  $U = \frac{1}{2}(\Lambda D + (\Lambda D)^T)$  is symmetric.

We recall the fact that the quadratic form of a skew-symmetric matrix equals to zero. Indeed, by definition  $V^T = -V$  and thus  $x^T V x = (x^T V^T x)^T = -x^T V x$  which holds only if it equals to zero. Thus, the quadratic form of  $x^T \Lambda D x$  is the same one of  $x^T U x$ , i.e:

$$x^T \Lambda D x = x^T \left( \frac{\Lambda D + (\Lambda D)^T}{2} \right) x \quad (4)$$

One can thus prove the semi-positive definition of the symmetric matrix  $U$  to prove the semi-positive definition of  $\Lambda D$ .

We use the following theorem on matrices [2] (Chapter 10, Theorem 10.28):

**Theorem 1.** *Let  $A$  be an  $n \times n$  matrix with eigenvalues  $\lambda_1(A), \dots, \lambda_n(A)$  and  $\mathcal{R}_e \lambda_i(A)$  their real part. Let  $\lambda(A) = \sum_{i=1}^n \lambda_i(A)$  and  $\mathcal{R}_e \lambda(A) = \sum_{i=1}^n \mathcal{R}_e \lambda_i(A)$ . Then:*

$$\mathcal{R}_e \lambda(A) \leq \lambda \left( \frac{A + A^T}{2} \right) \quad (5)$$

Then, because the eigenvalues of  $\Lambda D$  are real and non-negative (as proven in the previous part), we have using the Theorem 1:

$$0 \leq \lambda(\Lambda D) \leq \lambda(U) \quad (6)$$

Thus, the eigenvalues of  $U$  are non-negative, delivering the proof of the semi-positive definition of  $U$ , because  $U$  is symmetric (see Definition 1). Then,  $x^T U x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$  and using Eq.4, we obtain  $x^T \Lambda D x \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ : proving of the semi-positive definition of  $\Lambda D$ .

□

## References

- [1] J. D. Carroll and P. E. Green. *Decomposition of Matrix Transformations: Eigen structures and Quadratic Forms. Mathematical Tools for Applied Multivariate Analysis.* Elsevier Inc., 1997.
- [2] Fuzhen Zhang. *Matrix theory: basic results and techniques.* Springer, 2011.