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To cite this version:

Raphaël V Alexandre. Closed ray nil-affine manifolds and parabolic geometries. 2021. hal-03433549

HAL Id: hal-03433549

https://hal.science/hal-03433549v1

Preprint submitted on 17 Nov 2021

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Closed ray nil-affine manifolds and parabolic geometries

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November 18, 2021

Abstract

Ray nil-affine geometries are defined on nilpotent spaces. They occur in every parabolic geometry and in those cases, the nilpotent space is an open dense subset of the corresponding flag manifold.

We are interested in closed manifolds having a ray nil-affine structure. We show that under a rank one condition on the isotropy, closed manifolds are either complete or their developing map is a cover onto the complement of a nil-affine subspace. We prove that if additionally there is a parallel volume or if the automorphism group acts non properly then closed manifolds are always complete.

This paper is a sequel to a previous work on ray manifolds in affine geometry.

1 Introduction

Nilpotent spaces and nil-affine geometries are the natural generalization of Euclidean spaces and affine geometries. Interestingly, the study of nil-affine geometries has an importance in parabolic geometries.

If one has a non-compact symmetric space G/K, its visual boundary is generally not homogeneous but has several flag manifolds G/P as boundary components. The choice of a parabolic geometry $(G, G/P)$ corresponds to the choice of a parabolic subgroup P. A parabolic subgroup is a closed subgroup of G containing a minimal parabolic subgroup (a conjugate of the Borel subgroup $B \subset G$).

For instance, the conformal geometry is the (only) parabolic geometry of the hyperbolic space. More generally, the Furstenberg boundary is the parabolic manifold for the choice of the Borel subgroup as isotropy.

Once a geometry $(G, G/P)$ is chosen, one can try to classify manifolds with a geometric structure modeled on $(G, G/P)$ (it is the choice of an atlas with values in G/P and transitional maps in G). One of the most general questions is:

What are the closed manifolds that can be modeled on the geometry $(G, G/P)$ *?* This question is very open. In conformal geometry it is not known what are the closed manifolds modeled on it.

To investigate this question there are several possibilities. One can reduce G/P to $\Omega \subset G/P$ in order to obtain an invariant metric on Ω (invariant for the subgroup

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preserving Ω). For example the hyperbolic disc can be identified to a disc $\Omega \subset S^2$ in the conformal geometry of the sphere. In this direction, Zimmer [Zim18] studies closed manifolds with developing maps for which the image is *bounded in an affine chart* $\Omega \subset G/P$.

Another direction that we choose is to reduce G/P to an open dense subset. The open dense subset we choose is known as an open Schubert stratum in [KLP18] (see also [GJT98]). This open dense subset is a nilpotent space N that can be seen as a subgroup of P. So (P, N) is a subgeometry of $(G, G/P)$. The isotropy L of (P, N) is the *Levi* subgroup $L \subset P$ and we call $(P, N) = (P, P/L)$ a *Levi geometry*.

In rank one parabolic geometry (the parabolic geometries of the real rank one non-compact symmetric spaces), the study of this geometry $(P, P/L)$ has major consequences on the geometry $(G, G/P)$. For instance: if a closed manifold modeled on G/P avoids a point, then its developing map is a cover onto its image. We expect the study of $(P, P/L)$ to be again fruitful in higher rank parabolic geometries.

In rank one parabolic geometry, $P = N \rtimes L$ with $L = MA$ the combination of an abelian group A (that has dimension one) and a rotational centralizing group $M \subset K$. The study of P/L for rank one parabolic geometries has been achieved through the work of Fried [Fri80], Miner [Min90] and more recently by the author [Ale21b].

Those are the first examples of what we will call *ray nil-affine* geometries. Those geometries appear in very different contexts, and in particular in parabolic geometries of higher rank. They also have an interest in affine geometry and were studied in a previous work of the author [Ale21a].

A *ray geometry* $(N \rtimes KA, N)$ on a nilpotent space N is the data of a linear group KA such that A is an abelian subgroup and K centralizes A and is a compact subgroup. The *rank* of a ray geometry is the dimension of A. In a higher rank parabolic geometry, one can find different ray nil-affine geometries. They are more constrained than a Levi geometry $(P, P/L)$. However, we intend to show that they are rich examples of parabolic subgeometries that can also appear in other contexts.

We will give a generic construction (see proposition 3.3) to obtain ray geometries $(N \rtimes KA, N) \subset (P, N)$ in any Levi geometry.

In this paper, we address the question of the geometry of closed manifolds having a ray nil-affine structure. We extend the work in affine geometry [Ale21a] to a setting where the space can be nilpotent of order two. We believe those results should be true for any nilpotency order.

We define nil-affine subspaces to be the left translations of linear subspaces of the nilpotent Lie algebra. For instance, geodesic curves are (in this sense) left translations of linear line segments. See section 2.

Theorem (4.1)**.** *Let $\mathcal{N}$ be a commutative or two-step nilpotent space. Let $(G_1, \mathcal{N})$ be a ray geometry of rank one. Let M be a closed $(G_1, \mathcal{N})$-manifold. Then M is either complete or there exists a nil-affine subspace $I \subset \mathcal{N}$ such that $D\colon \widetilde{M} \to \mathcal{N} - I$ is a cover onto its image.*

Theorem (5.6)**.** *Let $\mathcal{N}$ be a commutative or two-step nilpotent space. Let $(G_1, \mathcal{N})$ be a ray geometry of rank one with parallel volume. Every closed $(G_1, \mathcal{N})$-manifold is complete.*

Theorem (5.8)**.** *Let $\mathcal{N}$ be a commutative or two-step nilpotent space. Let $(G_1, \mathcal{N})$ be a ray geometry of rank one. Let M be a closed $(G_1, \mathcal{N})$-manifold. If $\mathrm{Aut}(M)$ does not act properly on M then M is complete.*

Organization of the paper In section 2 we present the general setting of ray geometries. In section 3 we construct explicitly such geometries as subgeometries living in a flag manifold. This construction is general to any parabolic geometry. In section 4 we explain how to derive the proof of theorem 4.1 from [Ale21a] and additional arguments related to the nilpotency. In section 5 we study the reducibility of ray manifolds and prove theorems 5.6 and 5.8.

Acknowledgment This work is part of the author's doctoral thesis, under the supervision of Elisha Falbel. The author also enjoyed many conversations with Martin Mion-Mouton and Charles Frances on parabolic geometries.

2 Nilpotent spaces with ray geometries

Let N be a simply connected nilpotent Lie group. Denote $\mathfrak{n}$ its Lie algebra. The correspondance $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism and we have the Baker-Campbell-Hausdorff formula:

$$\exp(x)\exp(y) = \exp\left(x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] - [y,[x,y]]) + \dots\right). \tag{1}$$

Definition 2.1. *A* nil-affine subspace *of N is a left translation $L_x \exp(V)$ of the exponential in N of a linear subspace $V \subset \mathfrak{n}$.*

In particular when V is a line of $\mathfrak{n}$ we can deduce a notion of geodesic curve.

Definition 2.2. *A* geodesic segment *in N is the left-translation of a curve $\exp(tv)$ for $v \in \mathfrak{n}$ (the Lie algebra of N) and $t \in [0,1]$.*

Lemma 2.3. *When N is at most two-step nilpotent, the geodesic convex subsets are the affine convex subsets.*

Proof. The Baker-Campbell-Hausdorff formula shows that if $\exp(a+tv)$ is an affine segment then any left-translation $\exp(x)\exp(a+tv) = \exp(x+a+tv+1/2[x,a+tv]) = \exp(x+a+1/2[x,a]+t(v+1/2[x,v]))$ is again an affine segment. □

Definition 2.4. *Let $D\colon \widetilde{M} \to N$ be a local diffeomorphism. A curve $\gamma\colon [0,1] \to \widetilde{M}$ is* geodesic *if $D(\gamma)$ is a geodesic segment in N.*

For any $p \in \widetilde{M}$, define $V_p \subset \mathrm{T}_p\widetilde{M}$ the set of the vectors such that there exists a geodesic segment $\gamma\colon [0,1] \to \widetilde{M}$ with $\gamma(0) = p$, $\gamma'(0) \in V_p$. We say that V_p is the visibility set *from (or of) $p \in \widetilde{M}$.*

Definition 2.5. *Let $D\colon \widetilde{M} \to N$ be a local diffeomorphism. A subset $C \subset \widetilde{M}$ is* convex *if D is injective on C and $D(C)$ is convex. If $x \in \widetilde{M}$ and $y \in N$, we say that y* is visible from *x if there exists a geodesic segment from x to $z \in \widetilde{M}$ such that $D(z) = y$.*

Lemma 2.6. *The visible set $V_p \subset \mathrm{T}_p\widetilde{M}$ is open.*

Proof. It is necessarily a neighborhood of 0 since D is a local diffeomorphism and N is locally convex.

Consider $v \in V_p$ and its associated geodesic segment γ. Cover γ by finitely many convex opens U_i. Then $D(\gamma)$ can be continuously deformed into geodesic segments in $\bigcup D(U_i)$. They correspond to geodesic segments of $\widetilde{M}$ if they are always entirely contained in $\bigcup D(U_i)$. Since the U_i are in finite number, it can be done for an open set around $v \in \mathrm{T}_p\widetilde{M}$. □

Following the convexity arguments of Carrière [Car89] (see also [Kos65; Ben60]) we obtain:

Proposition 2.7 ([Ale21b])**.** *Let $D\colon \widetilde{M} \to N$ be a local diffeomorphism.*

- *If C_1, C_2 are convex subsets in $\widetilde{M}$ and $C_1 \cap C_2 \neq \emptyset$, then D is injective on $C_1 \cup C_2$.*
- *Let C be convex and containing $p \in \widetilde{M}$. Then $C \subset \exp_p(V_p)$.*
- *D is injective on $\exp_p(V_p)$ for any $p \in \widetilde{M}$.*
- *If for $p \in \widetilde{M}$, $\exp_p(V_p)$ is convex then $\exp_p(V_p) = \widetilde{M}$.*
- *D is a diffeomorphism if and only if for any $p \in \widetilde{M}$, $V_p = \mathrm{T}_p\widetilde{M}$.*

Definition 2.8. *A manifold M is a (G, X)-manifold for G a group acting on a space X if there exists (D, ρ) a pair of a local diffeomorphism $D\colon \widetilde{M} \to X$ (called the developing map) and a morphism $\rho\colon \pi_1(M) \to G$ (called the holonomy morphism) such that*

$$\forall \gamma \in \pi_1(M) \forall x \in \widetilde{M},\ D(\gamma \cdot x) = \rho(\gamma) D(x). \tag{2}$$

Let $X = \mathcal{N}$ be a nilpotent Lie group and $\mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$ be its largest linear automorphism group. Let $\mathrm{Aff}(N) = N \rtimes \mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$. We say that $(\mathrm{Aff}(N), \mathcal{N})$ is the nil-affine geometry *of $\mathcal{N}$ and if M has a $(G, \mathcal{N})$-structure then it is a* nil-affine manifold.

Definition 2.9. *Let M be a nil-affine manifold. It is* complete *if $D\colon \widetilde{M} \to \mathcal{N}$ is a diffeomorphism.*

They are several ways to describe ray geometries on a nilpotent space. We give here a simple axiomatic description.

Definition 2.10. *Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. We assume that $\mathfrak{n}$ is graded:*

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k \tag{3}$$

with $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$.

A ray geometry *on the space $\mathcal{N} = N$ is a pair $(G, \mathcal{N})$ with $G = N \rtimes KA$ and $KA \subset \mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$ such that the following conditions are verified.*

1. *The subgroup A is isomorphic to a multiplicative group $(\mathbf{R}_+^*)^r$ with $r = \dim A$ called the* rank of the ray geometry. *The subgroup K is compact and centralizes A.*
2. *There exists a basis $(e_1, \ldots, e_n)$ such that each e_i belongs to a single $\mathfrak{n}_j$. And in respect to the basis $(e_1, \ldots, e_n)$, there exists a linear isomorphism $(\alpha_1, \ldots, \alpha_r)\colon \mathfrak{a} \to \mathbf{R}^r$ and real numbers $d_{i,j}$ such that for any $\exp(a) \in A \subset \mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$:*

$$\exp(a) = \begin{pmatrix} \exp(\alpha_1(a))^{d_{1,1}} \cdots \exp(\alpha_r(a))^{d_{r,1}} & & \\ & \ddots & \\ & & \exp(\alpha_1(a))^{d_{1,n}} \cdots \exp(\alpha_r(a))^{d_{r,n}} \end{pmatrix}. \tag{4}$$

The second hypothesis on the basis $(e_1, \dots, e_n)$ allows to apply the following result. In general, one could ask under which condition a linear decomposition $\mathfrak{n} = L_1 \oplus L_2$ of any nilpotent Lie algebra gives a decomposition $N = \exp(L_1)\exp(L_2)$. In Bourbaki [Bou72, Prop. 17, p. 237] it is shown that:

Lemma 2.11 ([Bou72])**.** *Let $\mathfrak{n}$ be any nilpotent Lie algebra. Consider $\mathfrak{n} = \mathfrak{n}^0 \supset \mathfrak{n}^1 \supset \cdots \supset \mathfrak{n}^m = \{0\}$ any sequence of ideals of $\mathfrak{n}$ such that $[\mathfrak{n}, \mathfrak{n}^i] \subset \mathfrak{n}^{i+1}$.*

Consider a linear decomposition $\mathfrak{n} = L_1 \oplus L_2$. If for every $\mathfrak{n}^i$ we have $\mathfrak{n}^i = (\mathfrak{n}^i \cap L_1) \oplus (\mathfrak{n}^i \cap L_2)$ then for any $x \in N$ there exists a unique decomposition

$$x = \exp(x_1)\exp(x_2) \tag{5}$$

with $x_1 \in L_1$ and $x_2 \in L_2$.

The idea of the proof is to proceed by induction on the dimension, by taking the quotient by the centralizer. (On a commutative space we always have the property.)

In particular, in our context, one can consider

$$\mathfrak{n}^i = \mathfrak{n}_i \oplus \mathfrak{n}_{i+1} \oplus \cdots \oplus \mathfrak{n}_k \tag{6}$$

as sequence of decreasing ideals of $\mathfrak{n}$. The hypothesis $\mathfrak{n}^i = (\mathfrak{n}^i \cap L_1) \oplus (\mathfrak{n}^i \cap L_2)$ is necessarily verified if

$$\mathfrak{n}_i = (\mathfrak{n}_i \cap L_1) \oplus (\mathfrak{n}_i \cap L_2). \tag{7}$$

So if we were to consider L_1 and L_2 to be subspaces generated by subfamilies of $(e_1, \dots, e_n)$, the property would be verified.

A related consideration that is important in $\mathfrak{n}$ is:

$$\mathrm{Ad}_{\exp(x)} = \exp(\mathrm{ad}_x). \tag{8}$$

In consequence, we have in N:

$$\exp(x)\exp(y) = \exp\left(\sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}_x^n(y)\right)\exp(x) \tag{9}$$

$$= \exp\left(y + [x,y] + \frac{1}{2}[x,[x,y]] + \frac{1}{3!}[x,[x,[x,y]]] + \dots\right)\exp(x). \tag{10}$$

Limit sets Later we will need to study sequences of subsets that have a limit.

Definition 2.12. *Let B_n be a sequence of subsets. Its* limit set, *denoted $\lim B_n$, is the set of the points $\lim x_n$ for sequences $x_n \in B_n$ (with x_n chosen for each B_n).*

Proposition 2.13. *Let $g_i \in \pi_1(M)$ be a sequence of transformations and $S \subset \widetilde{M}$ be closed and convex. Assume that $g_i S$ has a limit point $y \in \widetilde{M}$. Consider $B_\infty = \lim D(g_i S) = \lim \rho(g_i) D(S)$ the limit set in the developing map. Then B_∞ is closed, convex and there exists a closed and convex subset S_∞ containing y such that $D(S_\infty) = B_\infty$.*

Proof. Observe that by convexity, each $D(g_i S)$ is closed and convex. Therefore B_∞ is again closed. It is also convex. If two points x_1, x_2 belong to the limit set B_∞ then we consider their associated converging sequences. We can form geodesics in each $g_i S$ that joins the points from each sequence. The limit of those geodesics exists, is geodesic and connects x_1 and x_2.

Similarly, $S_\infty = \lim g_i S$ is closed and convex. We show its developing image covers B_∞. Let $y_i \in g_i S$ be a sequence tending to y. Let $z \in B_\infty$ be the limit of $D(z_i)$ for $z_i \in g_i S$. The geodesics γ_i from y_i to z_i are developed into $D(\gamma_i)$ and those tend to the geodesic from $D(y)$ to $D(z)$. This geodesic is compact (defined on $[0,1]$) and each γ_i is completely visible. Therefore $\lim \gamma_i$ corresponds to a visible geodesic issued from y. It necessarily ends at z by injectivity. □

Example 1 Let Heis be the three-dimensional Heisenberg group, with the Lie bracket $[X,Y] = Z$. Let e_1 be the direction of X, e_2 of Y and e_3 of Z. Then the maximal group A acting on Heis is

$$\begin{pmatrix} \beta_1 & & \\ & \beta_2 & \\ & & \beta_1\beta_2 \end{pmatrix} \tag{11}$$

since we necessarily have $[\beta_1 X, \beta_2 Y] = \beta_1\beta_2 Z$. This ray geometry has many ray subgeometries of rank one by deciding any relation between β_1 and β_2. For example, the similarity geometry (as in [Ale21b]) is given by deciding $\beta_2 = \beta_1$. The volume-preserving geometry is given by deciding $\beta_2 = \beta_1^{-1}$.

Example 2 Any semisimple Lie groupe G has an Iwasawa decomposition $G = KAN$. Let M be the subgroup of K centralizing A, then the Borel subgroup $B = MAN$ is a ray geometry on N. Indeed, with a root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} g_\alpha \tag{12}$$

the Lie algebra $\mathfrak{n} = \mathfrak{n}_+$ consisting of the subspaces corresponding to positive roots $\Phi_+ \subset \Phi$ is a graded nilpotent Lie algebra on which $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$ acts and preserves each $\mathfrak{g}_\alpha$. We can find a basis of $\mathfrak{n}$ such that each e_i belongs to a single $\mathfrak{g}_\alpha$. The fact that A acts diagonally on $\mathfrak{n}$ corresponds to the diagonal action of the (positive simple) roots: $[a, e_i] = \sum \alpha_j(a) d_{i,j} e_i$.

Example 3 The similarity geometries defined in [Ale21b] are the ray geometries of rank one with the property that for every direction e_i, $d_i > 0$. It is the case for any Carnot group.

Notations The action of G on N is given by

$$(c, f)(x) = L_c(\exp(f(\ln(x)))). \tag{13}$$

In order to simplify the notations, if $x \in N$ and $f \in KA$, we define

$$f(x) := \exp(f(\ln(x))). \tag{14}$$

We denote by $+$ the group law of N. We should be careful that $+$ is not commutative in general, in opposition with the addition of $\mathfrak{n}$. If f is an automorphism of $\mathfrak{n}$, then

$$\forall x, y \in N,\ f(x+y) = f(x) + f(y). \tag{15}$$

(Note that this is an example of the use of $+$ for the group law of N.)

For any $x \in N$ and any $t \in \mathbf{R}$, we define

$$tx = \exp(t\ln(x)). \tag{16}$$

Note that if f is an automorphism of $\mathfrak{n}$, then by linearity of f on $\mathfrak{n}$,

$$\forall x, y \in N, \forall t \in \mathbf{R},\ f(x+ty) = f(x) + tf(y). \tag{17}$$

Therefore, for any nil-affine transformation $T \in N \rtimes \mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$, we can express T as

$$T(x) = c + f(x) \tag{18}$$

where $c \in N$ and $f \in \mathrm{GL}_{\mathrm{Aut}}(\mathfrak{n})$. We can see that geodesics are preserved:

$$T(x+tv) = c + f(x+tv) = c + f(x) + f(tv) = (c + f(x)) + tf(v). \tag{19}$$

Note also that if we change the base point from $0 = e \in N$ to any $y \in N$ then

$$T(x) = T(y - y + x) = c + f(y) + f(-y + x), \tag{20}$$

so f is constant but c is changed to $c + f(y)$. It characterizes the semi-direct product.

3 Ray nil-affine geometries in flag manifolds

Consider G a non-compact semi-simple Lie group with finite center. Say that P is a parabolic subgroup. Up to conjugation, it corresponds to a choice $\Sigma \subset \Delta$ of simple roots and to a grading (see [CS09, p. 309])

$$\mathfrak{g} = g_{-k,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{-1,\Sigma} \oplus \mathfrak{g}_{0,\Sigma} \oplus \mathfrak{g}_{1,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{k,\Sigma}. \tag{21}$$

The parabolic subgroup P corresponds to the Lie algebra of the null positive grades:

$$\mathfrak{p} = \mathfrak{g}_{0,\Sigma} \oplus \mathfrak{g}_{1,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{k,\Sigma} \tag{22}$$

and an opposite parabolic subgroup Q is given by the null and negative grades:

$$\mathfrak{q} = \mathfrak{g}_{0,\Sigma} \oplus \mathfrak{g}_{-1,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{-k,\Sigma} \tag{23}$$

The positive and negative grades give two isomorphic nilpotent subalgebras

$$\mathfrak{n}_{-\Sigma} = \mathfrak{g}_{-1,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{-k,\Sigma}, \tag{24}$$

$$\mathfrak{n}_{\Sigma} = \mathfrak{g}_{1,\Sigma} \oplus \cdots \oplus \mathfrak{g}_{k,\Sigma}. \tag{25}$$

The subalgebra

$$\mathfrak{l} = \mathfrak{g}_{0,\Sigma} = \mathfrak{p} \cap \mathfrak{q} \tag{26}$$

is the Levi factor and corresponds to the subgroup $L = P \cap Q$.

One can take the (opposite) nilpotent subgroups $N_+ \subset P$ and $N_- \subset Q$ corresponding to the subalgebras $\mathfrak{n}_\Sigma$ and $\mathfrak{n}_{-\Sigma}$ respectively. There is a natural embedding $N_+ \to G/Q$ (see [GJT98, p. 20-21]) and the image is known as an *open Schubert stratum* [KLP18, p. 175]. This open subset is dense inside G/Q.

So one can consider the subgeometry corresponding to $N_+ \subset G/Q$. The isotropy is L. The subgroup L acts on N_+ by conjugation since $Q = LN_-$ and N_+ acts on itself by left-translation.

This can be summarized by the following proposition.

Proposition 3.1. *Let G be a non-compact semi-simple Lie group with finite center and let $P \subset G$ be a parabolic subgroup. Let $Q \subset P$ be an opposite parabolic subgroup. Decompose $P = LN_+$ and $Q = LN_-$. Then (P, N_+) is a subgeometry of $(G, G/Q) \cong (G, G/P)$ and N_+ is an open and dense subset of G/Q.*

Definition 3.2. *We call (P, N_+) a* Levi subgeometry.

In general, a Levi subgeometry (P, N_+) is not a ray nil-affine geometry, except in the case where $P = B$ is the Borel subgroup. (Note that $P = B$ corresponds to $\Sigma = \emptyset$.) With Langland's decomposition

$$B = M_B A_B N_+, \ \mathfrak{b} = \mathfrak{m}_B \oplus \mathfrak{a}_B \oplus \mathfrak{n}_B \tag{27}$$

it is clear that $M_B A_B$ verifies the conditions to be a ray nil-affine isotropy group. (Note that we always have $\mathfrak{n}_\Sigma \subset \mathfrak{n}_B$.) In general, $\mathfrak{p}$ is decomposed as (see [GJT98; Kna02]):

$$P = M(\Sigma) A(\Sigma) N_+, \ \mathfrak{p} = \mathfrak{m}_\Sigma \oplus \mathfrak{a}_\Sigma \oplus \mathfrak{n}_\Sigma \tag{28}$$

and $M(\Sigma)A(\Sigma)$ always contains $M_B A_B$. The subgroup $A(\Sigma)$ still acts by diagonal transformations but $M(\Sigma)$ (that centralizes $A(\Sigma)$ in L) is no longer compact. For instance, $\mathfrak{m}_\Sigma \cap \mathfrak{a}_B$ has positive dimension if $P \neq B$. More specifically:

$$\mathfrak{a}_\Sigma = \{H \in \mathfrak{a} \mid \forall \phi \in \Sigma, \ \phi(H) = 0\}. \tag{29}$$

We can denote $K \subset G$ a maximal compact such that $M_B \subset K$. If we consider $K \cap M_\Sigma$ then it still centralizes A_Σ and becomes compact.

The construction we apply gives two different ray nil-affine geometries. It is summarized by the following.

Proposition 3.3. *Let (P, N_+) be a Levi subgeometry. We have two ray nil-affine geometries (that are different if $P \neq B$):*

$$(N_+ \rtimes M_B A_B, N_+), \tag{30}$$

$$(N_+ \rtimes (K \cap M_\Sigma) A_\Sigma, N_+). \tag{31}$$

The rank of the first ray geometry is the real rank of the underlying symmetric space. The rank of the second is the rank of the first minus the cardinal of Σ.

Example: real non-compact forms of $\mathrm{SL}(3, \mathbf{C})$ and $\mathrm{SL}(4, \mathbf{C})$

As an example, we give a table of the relevant characteristics of the parabolic geometries of the non-compact real forms of $\mathrm{SL}(3, \mathbf{C})$ and $\mathrm{SL}(4, \mathbf{C})$. It shows that the Levi subgeometries of dimension less than six have a nilpotency order of two, and therefore our results can be applied. Note however that in higher nilpotency order we still have a similar classification result on the Carnot group [Ale21b].

The case of $\mathrm{SU}(2,2)$

We workout a full example: the group $\mathrm{SU}(2,2)$. We choose the Hermitian form with signature $(2,2)$ determined by

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

G and real rank	$\Sigma \subset \Delta$	$\dim(\mathfrak{n}_\Sigma)$	Nil-order
$\mathrm{SL}(3,\mathbf{R})$, rank 2	$\emptyset$	3	2
	$\{\phi_i\}$	2	1
$\mathrm{SU}(2,1)$, rank 1	$\emptyset$	3	2
$\mathrm{SL}(4,\mathbf{R})$, rank 3	$\emptyset$	6	3
	$\{\phi_i\}$	5	2
	$\{\phi_i,\phi_{i+1}\}$	4	2
$\mathrm{SU}(3,1)$, rank 1	$\emptyset$	5	2
$\mathrm{SU}^*(4)$, rank 1	$\emptyset$	4	1
$\mathrm{SU}(2,2)$, rank 2	$\emptyset$	6	3
	$\{\phi_2\}$	5	2
	$\{\phi_1\}$	4	1

Table 1: The parabolic geometries of the real forms of $\mathrm{SL}_3(\mathbf{C})$ and $\mathrm{SL}_4(\mathbf{C})$.

We let, $x, y, z, w \in \mathbf{C}$ and $\alpha_1, \alpha_2, \beta, \gamma, \Delta, s, t \in \mathbf{R}$. The Lie algebra of $\mathrm{SU}(2,2) = \mathrm{SU}(J)$ is described as follows.

$$\mathfrak{su}(2,2) = \begin{pmatrix} \alpha_1 - \boldsymbol{i}\beta & -\overline{x} & -\overline{y} & \boldsymbol{i}t \\ -\overline{z} & \alpha_2 + \boldsymbol{i}\beta & \boldsymbol{i}\gamma & y \\ -\overline{w} & \boldsymbol{i}\delta & -\alpha_2 + \boldsymbol{i}\beta & x \\ \boldsymbol{i}s & w & z & -\alpha_1 - \boldsymbol{i}\beta \end{pmatrix} \tag{33}$$

By choosing the roots

$$\phi_1 = \alpha_1 - \alpha_2,\ \phi_2 = 2\alpha_2 \tag{34}$$

we can describe the structure by replacing variables with their corresponding root space.

$$\begin{pmatrix} \mathfrak{h} & \phi_1 & \phi_1+\phi_2 & 2\phi_1+\phi_2 \\ -\phi_1 & \mathfrak{h} & \phi_2 & \phi_1+\phi_2 \\ -(\phi_1+\phi_2) & -\phi_2 & \mathfrak{h} & \phi_1 \\ -(2\phi_1+\phi_2) & -(\phi_1+\phi_2) & -\phi_1 & \mathfrak{h} \end{pmatrix} \tag{35}$$

So, for example, the coordinate $\boldsymbol{i}t$ in the first matrix corresponds to the root space associated to the root $2\phi_1 + \phi_2$ as the second matrix indicates.

Now, they are several choices of parabolic geometries. Each gives a different dimension for $\mathfrak{n}_\Sigma$.

The Borel parabolic subgroup The case $\Sigma = \emptyset$ corresponds to the choice of the Borel subgroup.

$$\mathfrak{b} = \underbrace{\begin{pmatrix} -\boldsymbol{i}\beta & & & \\ & +\boldsymbol{i}\beta & & \\ & & +\boldsymbol{i}\beta & \\ & & & -\boldsymbol{i}\beta \end{pmatrix}}_{\mathfrak{m}_B} \oplus \underbrace{\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & -\alpha_2 & \\ & & & -\alpha_1 \end{pmatrix}}_{\mathfrak{a}_B} \oplus \underbrace{\begin{pmatrix} 0 & -\overline{x} & -\overline{y} & \boldsymbol{i}t \\ & 0 & \boldsymbol{i}\gamma & y \\ & & 0 & x \\ & & & 0 \end{pmatrix}}_{\mathfrak{n}_B} \tag{36}$$

In the coordinates (x,γ,y,t) of $\mathfrak{n}_B$ we can express the adjoint action of $\mathfrak{h}_B = \mathfrak{m}_B \oplus \mathfrak{a}_B$ by:

$$\mathfrak{h}_B|_{\mathfrak{n}_B} : \begin{pmatrix} (\alpha_1-\alpha_2)+2\boldsymbol{i}\beta & & & \\ & 2\alpha_2 & & \\ & & (\alpha_1+\alpha_2)+2\boldsymbol{i}\beta & \\ & & & 2\alpha_1 \end{pmatrix} \tag{37}$$

and it shows of course that it is indeed a ray geometry.

The 5-dimensional parabolic geometry We obtain the 5-dimension parabolic geometry by setting $\Sigma = \{\phi_2\}$. It corresponds to a CR geometry (that is in fact contact). This parabolic geometry is also the boundary geometry related to an homogeneous space in $\mathbf{CP}^3$. Indeed, consider the equation

$$\mathrm{Re}(x_1\overline{x_4}) + \mathrm{Re}(x_2\overline{x_3}) < 0. \tag{38}$$

It describes an homogeneous space with semi-simple group $\mathrm{SU}(2,2)$. The boundary

$$\mathrm{Re}(x_1\overline{x_4}) + \mathrm{Re}(x_2\overline{x_3}) = 0 \tag{39}$$

is a real submanifold of $\mathbf{CP}^3$ having dimension 5. The parabolic subgroup that we choose corresponds to the stabilizer of $(1:0:0:0)$. (Compare this geometry with [DGK18].)

The Lie algebra is given by:

$$\mathfrak{p}_{\phi_2} = \underbrace{\begin{pmatrix} \alpha_1-\boldsymbol{i}\beta & & & \\ & \alpha_2+\boldsymbol{i}\beta & \boldsymbol{i}\gamma & \\ & \boldsymbol{i}\delta & -\alpha_2+\boldsymbol{i}\beta & \\ & & & -\alpha_1-\boldsymbol{i}\beta \end{pmatrix}}_{\mathfrak{l}_{\phi_2}} \oplus \underbrace{\begin{pmatrix} 0 & -\overline{x} & -\overline{y} & \boldsymbol{i}t \\ & 0 & 0 & y \\ & & 0 & x \\ & & & 0 \end{pmatrix}}_{\mathfrak{n}_{\phi_2}} \tag{40}$$

Now, to obtain ray nil-affine geometries, we need to reduce the Levi factor $\mathfrak{l}_{\phi_2}$ into appropriated combinations of $\mathfrak{m} \oplus \mathfrak{a}$.

The first choice consists in taking the same diagonal part as the Borel subgroup, namely $\mathfrak{h}_B \subset \mathfrak{l}_{\phi_2}$. The adjoint action of $\mathfrak{h}_B$ is described by equation (37).

The second choice consists in setting $\phi_2 = 2\alpha_2 = 0$ and in choosing a compact factor in the middle square. We obtain the following Lie algebra.

$$\mathfrak{h}_{\phi_2} = \underbrace{\begin{pmatrix} -\boldsymbol{i}\beta & & & \\ & \boldsymbol{i}\beta & \boldsymbol{i}\gamma & \\ & \boldsymbol{i}\gamma & \boldsymbol{i}\beta & \\ & & & -\boldsymbol{i}\beta \end{pmatrix}}_{\mathfrak{m}_{\phi_2}} \oplus \underbrace{\begin{pmatrix} \alpha_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -\alpha_1 \end{pmatrix}}_{\mathfrak{a}_{\phi_2}} \tag{41}$$

And the adjoint action, with (x,y,t) as basis of $\mathfrak{n}_{\phi_2}$, is described by:

$$\mathfrak{h}_{\phi_2}|_{\mathfrak{n}_{\phi_2}} : \begin{pmatrix} \alpha_1+2\boldsymbol{i}\beta & \boldsymbol{i}\gamma & \\ \boldsymbol{i}\gamma & \alpha_1+2\boldsymbol{i}\beta & \\ & & 2\alpha_1 \end{pmatrix} \tag{42}$$

The 4-dimensional parabolic geometry The last parabolic geometry is given by $\Sigma = \{\phi_1\}$. We obtain the following Lie algebra.

$$\mathfrak{p}_{\phi_1} = \underbrace{\begin{pmatrix} \alpha_1 - \boldsymbol{i}\beta & -\overline{x} & & \\ -\overline{z} & \alpha_2 + \boldsymbol{i}\beta & & \\ & & -\alpha_2 + \boldsymbol{i}\beta & x \\ & & z & -\alpha_1 - \boldsymbol{i}\beta \end{pmatrix}}_{\mathfrak{l}_{\phi_1}} \oplus \underbrace{\begin{pmatrix} 0 & 0 & -\overline{y} & \boldsymbol{i}t \\ & 0 & \boldsymbol{i}\gamma & y \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}}_{\mathfrak{n}_{\phi_1}} \tag{43}$$

The first ray geometry is again obtained by taking the Borel Levi factor. The adjoint action of $\mathfrak{h}_B \subset \mathfrak{l}_{\phi_1}$ on $\mathfrak{n}_{\phi_1}$ is described by equation (37).

The second choice consists in reducing $\mathfrak{a}$ to the subspace $\mathfrak{a}_\Sigma$ where $\phi_1 = 0$, that is to say, with $\alpha_1 = \alpha_2$ since $\phi_1 = \alpha_1 - \alpha_2$. It gives the following algebra

$$\mathfrak{h}_{\phi_1} = \underbrace{\begin{pmatrix} -\boldsymbol{i}\beta & -\overline{x} & & \\ x & \boldsymbol{i}\beta & & \\ & & \boldsymbol{i}\beta & x \\ & & -\overline{x} & -\boldsymbol{i}\beta \end{pmatrix}}_{\mathfrak{m}_{\phi_1}} \oplus \underbrace{\begin{pmatrix} \alpha_1 & & & \\ & \alpha_1 & & \\ & & -\alpha_1 & \\ & & & -\alpha_1 \end{pmatrix}}_{\mathfrak{a}_{\phi_1}}. \tag{44}$$

The adjoint action of $\mathfrak{h}_{\phi_1}$ is more difficult to express as a single matrix. On $\mathfrak{h}_{\phi_1} \cap \mathfrak{h}_B$, we already know the result by equation (37). So it lasts the $\mathfrak{su}(2)$ part:

$$\left[\begin{pmatrix} 0 & -\overline{x} & & \\ x & 0 & & \\ & & 0 & x \\ & & -\overline{x} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\overline{y} & \boldsymbol{i}t \\ & 0 & \boldsymbol{i}\gamma & y \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & \boldsymbol{i}(t-\gamma)\overline{x} & 2\boldsymbol{i}\mathrm{Im}(x\overline{y}) \\ & 0 & 2\boldsymbol{i}\mathrm{Im}(y\overline{x}) & \boldsymbol{i}(t-\gamma)x \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \tag{45}$$

4 Ray closed manifolds

In this section we prove the following.

Theorem 4.1. *Let $\mathcal{N}$ be a commutative or two-step nilpotent space. Let $(G_1, \mathcal{N})$ be a ray geometry of rank one. Let M be a closed $(G_1, \mathcal{N})$-manifold. Then M is either complete or there exists a nil-affine subspace $I \subset \mathcal{N}$ such that $D\colon \widetilde{M} \to \mathcal{N} - I$ is a cover onto its image.*

And in fact, the proof is very similar to the proof in [Ale21a]. We will not repeat the proofs in [Ale21a] that can directly be used in our setting. We will recall the construction and the notations.

4.1 Fried dynamics

Let M be an incomplete closed nil-affine manifold. Choose any metric on M compatible with its topology. Denote $\pi\colon \widetilde{M} \to M$ its universal cover. If $x \in \widetilde{M}$ and $\gamma \subset \widetilde{M}$ is a geodesic issued from x, incomplete at $t = 1$, then the projection of γ in M is an open curve without any continuous completion at $t = 1$. Since M is closed, the projection $\pi(\gamma)$ has a recurrent point $y \in M$.

Let $U \subset M$ be a compact neighborhood of y, convex and trivializing the universal cover. For the choice of a decreasing sequence $\epsilon_i \to 0$ we can define t_i the time such that $\pi(\gamma(t_i))$ belongs to U and is at distance at most ϵ_i to $y \in M$. We ask that $\pi(\gamma)$

exits U between the times t_i and t_{i+1}. We have that $t_i \to 1$ since the geodesic is incomplete at $t = 1$.

We use the trivialization of the universal cover by U. It provides $U_i \subset \widetilde{M}$ such that $\gamma(t_i) \in U_i$. Let $y_i \in U_i \subset \widetilde{M}$ be the lifts of $y \in U \subset M$. Every U_i is again a compact convex neighborhood of y_i. Through the developing map $D\colon \widetilde{M} \to \mathcal{N}$, each $D(U_i)$ is compact, convex and they accumulate along the compactification $\overline{D(\gamma)}$ of $D(\gamma)$ at $t = 1$. Since the intersections of γ with the U_i's are transverse and disjoint along γ, the $D(U_i)$'s intersect transversally and disjointly $D(\gamma)$.

Definition 4.2. *Let $\gamma \subset \widetilde{M}$ be an incomplete geodesic at $t = 1$. Let $y \in M$ be a recurrent point of its projection into M. Let U be a convex, compact, trivializing neighborhood of y. Let $\epsilon_i \to 0$ be decreasing and let $t_i \to 1$ be an associated sequence of times such that (for the lifts $y_i \in U_i$ of $y \in U$) we have $\gamma(t_i) \in U_i$ and the distance between $\pi(\gamma(t_i))$ and y is lesser than ϵ_i. Define $g_{ji} \in \pi_1(M, y)$ the transformations of $\widetilde{M}$ verifiying*

$$g_{ji}(U_i) = U_j. \tag{46}$$

Those data define a Fried dynamics.

Note that a subsequence of the times $\{t_i\}$ (or the distances $\{\epsilon_i\}$) corresponds univocally to a subsequence of the pairs $\{y_i \in U_i\}$.

The transformations g_{ji} verify a cocycle property:

$$g_{ki} = g_{kj} g_{ji}. \tag{47}$$

For the nil-affine geometry $(G, \mathcal{N})$ considered, denote by T_{ji} the corresponding transformations by the holonomy morphism $T_{ji} = \rho(g_{ji}) \in G$.

We assume that $(G, \mathcal{N})$ is a ray geometry. We have a basis $(e_1, \dots, e_n)$ of $\mathfrak{n}$ such that $A \subset KA \subset G$ acts diagonally. Once a base point of $\mathcal{N}$ is chosen, we can express any $T_{ji} \in G$ by

$$T_{ji}(x) = c_{ji} + f_{ji}(x), \tag{48}$$

with $c_{ji} \in \mathcal{N}$ and $f_{ji} \in KA$. Recall that a change of the base point is expressed by

$$T_{ji}(x) = T_{ji}(y - y + x) = (c_{ji} + f_{ji}(y)) + f_{ji}(-y + x). \tag{49}$$

It preserves the linear part f_{ji}. Decompose each f_{ji} into

$$f_{ji} = f_{ji,K} f_{ji,A} \tag{50}$$

with $f_{ji,K} \in K$ et $f_{ji,A} \in A$. Recall that K centralizes A, so both factors commute. The cocycle property on T_{ji} implies that each factor also verifies the cocycle relation:

$$f_{ki,K} = f_{kj,K} f_{ji,K},\ f_{ki,A} = f_{kj,A} f_{ji,A} \tag{51}$$

Lemma 4.3. *Up to a subsequence of $\{y_i \in U_i\}$, we have*

$$\lim_{j\to\infty} \lim_{i\to\infty} f_{ji,K} = \mathrm{id}. \tag{52}$$

Let e_q be a basis vector. Denote by $\beta_{ji,q}$ the diagonal element of $f_{ji,A} \in A$ for the direction e_q. Up to a subsequence of $\{y_i \in U_i\}$, we have

$$\lim_{j\to\infty} \lim_{i\to\infty} \beta_{ji,q} = \omega_q \in \{0, 1, \infty\}. \tag{53}$$

It should be noted that if $\beta_{kj,q} \to \omega_q$ when $k \gg j \to \infty$, then we can obtain an information on how $\beta_{ki,q}$ can evolve when i is fixed and $k \to \infty$. Indeed, we have

$$\lim_{k\to\infty} \beta_{ki,q} = \lim_{k\to\infty} \lim_{j\to\infty} \beta_{ki,q} = \lim_{k\to\infty} \lim_{j\to\infty} \beta_{kj,q}\beta_{ji,q} = \omega_q \lim_{j\to\infty} \beta_{ji,q}. \tag{54}$$

If $\beta_{ki,q} \to r$ when $k \to \infty$ (up to choose a subsequence) then $r = \omega_q r$. If $\omega_q \in \{0, \infty\}$ it implies $r = \omega_q$.

Lemma 4.4. *Let $i > 0$ fixed. If $\beta_{ki,q} \to r$ and $\omega_q = 1$ then r must be a (finite) real positive number.*

Proposition 4.5. *There exists a subsequence of $\{y_i \in U_i\}$ such that:*

- *for $i > 0$ fixed, $f_{ji,K}$ converges;*
- *for $i > 0$ fixed, the sequence $\{\beta_{ji,q}\}$ is monotonic or constant for $j > i$.*

Definition 4.6. *We define a linear decomposition $E \oplus P \oplus F$ of $\mathfrak{n}$ by deciding that $e_q \in E$ if $\omega_q = 0$, $e_q \in P$ if $\omega_q = 1$ and $e_q \in F$ if $\omega_q = \infty$.*

Since A acts diagonally and K centralizes A, we have that f_{ji} preserve the decomposition $E \oplus P \oplus F$. Note that $\dim E > 0$ since $D(U_i)$ accumulates disjointly on $\overline{D(\gamma)}$. The other subspaces might be reduced to $\{0\}$.

If we chose any base point p, it gives three nil-affine subspaces based at p:

$$\forall L \in \{E, F, P\} \forall p \in N,\ L|_p := L_p \exp(L). \tag{55}$$

Note also that for example $[F, F] \subset F$ and $[F \oplus P, F \oplus P] \subset F \oplus P$ since A acts by automorphisms. It shows for instance that $\exp(F)$ is a subgroup of N,

We come back to M and discuss the choice of $U \subset M$. Choose $V \subset U$ a smaller neighborhood of y. Since we chose distances $\epsilon_i \to 0$, for $i \geq i_0$ large enough, every $\pi(\gamma(t_i))$ will also belong to V. Note that we can lift $V \subset M$ into $V_i \subset U_i \subset \widetilde{M}$.

Lemma 4.7. *For any $i > 0$, $g_{ji}^{-1}\gamma$ has y_i for limit point when $j \to \infty$.*

Definition 4.8. *In M, let $U_1 = U$. Define for a sequence of $\{0 < r < 1\}$ tending to 0 a sequence $\{U_r\}$ of compact convex neighborhoods of $y \in M$. Assume that U_r is decreasing for the inclusion and that $U_r \to \{y\}$ when $r \to 0$. In $\widetilde{M}$, define $U_{1,i} = U_i$ and $U_{r,i}$ the lift of U_r such that $U_{r,i} \subset U_{1,i}$. In $D(\widetilde{M})$, define*

$$C_{r,i} = D(U_{r,i}). \tag{56}$$

Recall that $U_1 = U$ is a convex, compact and trivializing neighborhood of $y \in M$. Every $C_{r,i}$ is convex and compact in $\mathcal{N}$ since U_r is a convex, compact and trivializing neighborhood of y in M.

Lemma 4.9. *For any i, j and $r \geq 0$, $g_{ji}U_{r,i} = U_{r,j}$ and by consequence $T_{ji}C_{r,i} = C_{r,j}$.*

As discussed before, since $U_r \subset U$ is a neighborhood of y, $D(\gamma)(1)$ is a limit point of $\{C_{r,i}\}$ since for $j > j_0$ large enough $D(\gamma(t_j))$ belongs to $C_{r,j}$.

Proposition 4.10. *Let $r > 0$. Any accumulation point of $\{C_{r,i}\}$ is a limit point.*

Definition 4.11. *For any fixed value $r > 0$, let $C_{r,\infty}$ be the limit set of the sequence $\{C_{r,i}\}$.*

Note that when $r = 0$, each $C_{r,i}$ is reduced to $D(y_i)$. Those have no reason to accumulate to $D(\gamma)(1)$. This is why we asked r to be different from 0.

Definition 4.12. *Define*

$$C_{0,\infty} = \bigcap_{r>0} C_{r,\infty}. \tag{57}$$

This definition makes sense because the $C_{r,\infty}$ are decreasing for the inclusion when $r \to 0$.

Lemma 4.13. *All the sets $C_{r,\infty}$ and $C_{0,\infty}$ are convex. The set $C_{0,\infty}$ is nonempty (it contains $D(\gamma)(1)$) and is a nil-affine subspace.*

Lemma 4.14. *Choose $D(\gamma)(1)$ as base point of $\mathbf{R}^n$.*

$$\forall r > 0,\ C_{r,\infty} \subset (P \oplus F)\,|_{D(\gamma)(1)} \tag{58}$$

$$C_{0,\infty} = F|_{D(\gamma)(1)}. \tag{59}$$

Fixed point Now we return to the general study. The next step is to find an asymptotic fixed point for T_{ji}.

Since $E \oplus P$ and F are generated by two subfamilies of $(e_1, \dots, e_n)$, we can apply lemma 2.11. We have that for any $x \in N$, there exists a unique decomposition $x = x_L + x_F$ with $x_L \in \exp(E \oplus P)$ and $x_F \in \exp(F)$.

Lemma 4.15. *Choose $D(\gamma)(1)$ as base point and decompose $T_{ji}(x) = c_{ji} + f_{ji}(x)$ with $f_{ji} \in KA$ and $c_{ji} \in N$. Let $c_{ji} = c_{ji,L} + c_{ji,F}$ be the decomposition given by the preceding lemma. Define $Q_{ji}(x) = c_{ji,F} + f_{ji}(x)$. Then*

$$\lim_{j\to\infty} T_{ji}(x) - Q_{ji}(x) = \lim_{j\to\infty} c_{ji,L} = 0 \tag{60}$$

and therefore this convergence is uniform for $x \in N$.

Proof. With $L = E \oplus P$ and the decomposition $L \oplus F = \mathfrak{n}$, we can show this result by showing that the coordinates of $c_{ji,L}$ in the Lie algebra tend to 0. By definition, $c_{ji,L}$ has vanishing coordinates on F.

For any $\epsilon > 0$, $T_{ji}(C_{\epsilon,i})$ has for limit $C_{\epsilon,\infty}$ when $j \to \infty$. When $\epsilon \to 0$ and $j \to \infty$, the set of the coordinates of $T_{ji}(C_{\epsilon,i})$ all tend to 0 in the linear subspace $L \subset \mathfrak{n}$. Indeed, $C_{0,\infty} = F|_{D(\gamma)(1)}$.

Note that $c_{ji,L}$ tends or not to 0 when $j \to \infty$ independently from the choice of ϵ.

Now decompose $T_{ji}(C_{\epsilon,i})$, and note that f_{ji} preserves the decomposition $L \oplus F$.

$$c_{ji} + f_{ji}(C_{\epsilon,i}) = c_{ji,L} + c_{ji,F} + f_{ji}(C_{\epsilon,i})_L + f_{ji}(C_\epsilon)_F \tag{61}$$

$$= \Big(c_{ji,L} + \exp(\mathrm{ad}_{c_{ji,F}})(f_{ji}(C_{\epsilon,i})_L)\Big) + \big(c_{ji,F} + f_{ji}(C_{\epsilon,i})_F\big) \tag{62}$$

The last term is exclusively in $\exp(F)$ since $[F,F] \subset F$ shows that F is a subgroup. So the coordinate in L is uniquely determined by the projection in L of

$$c_{ji,L} + \exp(\mathrm{ad}_{c_{ji,F}})(f_{ji}(C_{\epsilon,i})_L) = c_{ji,L} + \exp\big(f_{ji}(C_{\epsilon,i})_L + [c_{ji,F}, f_{ji}(C_{\epsilon,i})]_L\big) \tag{63}$$

and the projection in L must tend to 0 when $j \to \infty$ and $\epsilon \to 0$.

For any $j > i$, f_{ji} acts as a contraction on $(C_{\epsilon,i})_L$. Note that in two-step nilpotency, if the center does not intersect L then $[x,y]_L = 0$. By consequence, in either case, we can let $\epsilon v \in (C_{\epsilon,i})_L$ (in the center or not) so that $f_{ji}(\epsilon v) + [c_{ji,F}, f_{ji}(\epsilon v)]_L = f_{ji}(\epsilon v)$ and it can only tend to 0 when $j \to \infty$ and $\epsilon \to 0$. By consequence, when $j \to \infty$ and $\epsilon \to 0$, $c_{ji,L} \to 0$ since the sum tends to zero. It proves the lemma. □

Note that Q_{ji} has a fixed point on F, since F is preserved and f_{ji} acts by expansions on it.

Lemma 4.16. *Denote $q_{ji} \in F$ the fixed point of Q_{ji}. For $i > 0$, q_{ji} converges when $j \to \infty$ and we denote $q_i = \lim q_{ji}$. Also, $D(y_i) \in E|_{q_i}$.*

Asymptotic dynamic Now we can determine the limits of the orbits in positive time, and apply proposition 2.13.

Lemma 4.17. *Let $z \in E|_{q_i}$ and V a neighborhood of z. Then $\{T_{ji}(V)\}$ has $F|_{q_i}$ in its limit set.*

Proposition 4.18. *Let $S \subset \widetilde{M}$ be a convex containing the incomplete geodesic γ. Assume that $D(S)$ has a smooth boundary at $D(\gamma)(1)$. Let $i > 0$. We have the following properties.*

- *The orbit $T_{ji}^{-1}(D(S))$ tends to a product $E_{+,i} \times P_c$ of a half-space $E_{+,i} \subset E|_{q_i}$ and a neighborhood $P_c \subset P|_{q_i}$ of the origin.*
- *The product $E_{+,i} \times P_c$ is visible from y_i.*
- *The boundary of $E_{+,i}$ is described by the limit of $T_{ji}^{-1}(\mathrm{T}_{D(\gamma)(1)}D(S)) \cap E|_{q_i}$.*

To get a better description of $E_{+,i}$ inside $E|_{q_i}$, we must explain how to approximate T_{ji}^{-1} relatively to Q_{ji}^{-1}. Note that *a priori*

$$T_{ji}^{-1}(x) - Q_{ji}^{-1}(x) = -f_{ji}^{-1}(c_{ji,L}) \tag{64}$$

and might not be tending to zero, or even stay bounded.

The study of $-f_{ji}^{-1}(c_{ji,L})$ is technical. With the two following lemmas, we will prove that $-f_{ji}^{-1}(c_{ji,L}) \to 0$ when $j \gg i \to \infty$.

Lemma 4.19. *Assume that $E_{+,i}$ does not contain q_i. For any $i > 0$, there exists $M > 0$ such that for any $j > i$,*

$$c_{ji,L} = f_{ji}(b_{ji,E}) + b_{ji,P} \tag{65}$$

with $b_{ji,E} \in E$, $b_{ji,P} \in P$ verifying $b_{ji,P} \to 0$ and $\|b_{ji,E}\| < M$.

We will prove in the case of a rank one ray geometry that $E_{+,i}$ never contains q_i. So in fact $b_{ji,E}$ is bounded. The next lemma explains that in fact T_{ji}^{-1} is as well approximated by Q_{ji}^{-1} as $i > 0$ gets large.

Lemma 4.20. *Assume that for any i, $E_{+,i}$ never contains q_i. Then*

$$\lim_{k\to\infty} \lim_{j\to\infty} b_{kj,E} = 0. \tag{66}$$

Proof. The cocycle relation $T_{kj}T_{ji} = T_{ki}$ gives:

$$T_{ji} = f_{ji}(b_{ji,E}) + b_{ji,P} + c_{ji,F} + f_{ji} \tag{67}$$

$$T_{kj}T_{ji} = f_{kj}(b_{kj,E}) + b_{kj,P} + c_{kj,F} + f_{kj}(f_{ji}(b_{ji,E})) + f_{kj}(b_{ji,P}) + f_{kj}(c_{ji,F}) + f_{kj}f_{ji} \tag{68}$$

$$= f_{kj}(b_{kj,E}) + b_{kj,P} + c_{kj,F} + f_{ki}(b_{ji,E}) + f_{kj}(b_{ji,P}) + f_{kj}(c_{ji,F}) + f_{ki} \tag{69}$$

$$= T_{ki} = f_{ki}(b_{ki,E}) + b_{ki,P} + c_{ki,F} + f_{ki}. \tag{70}$$

and it implies by identification:

$$f_{ki}(b_{ki,E}) = f_{kj}(b_{kj,E}) + \left(\exp(\mathrm{ad}_{b_{kj,P}})\exp(\mathrm{ad}_{c_{kj,F}})f_{ki}(b_{ji,E})\right)_E. \tag{71}$$

Note that $b_{kj,P} \to 0$ when $k \to \infty$. So we omit this term. Also in two-step nilpotency,

$$\left(\exp(\mathrm{ad}_{c_{kj,F}})f_{ki}(b_{ji,E})\right)_E = \exp\left(f_{ki}(b_{ji,E}) + [c_{kj,F}, f_{ki}(b_{ji,E})]_E\right) \tag{72}$$

$$= f_{ki}\left(\exp\left(b_{ji,E} + [f_{ki}^{-1}(c_{kj,F}), b_{ji,E}]_E\right)\right). \tag{73}$$

Hence we obtain:

$$b_{ki,E} = f_{ji}^{-1}(b_{kj,E}) + \exp\left(b_{ji,E} + [f_{ki}^{-1}(c_{kj,F}), b_{ji,E}]_E\right) + o(1) \tag{74}$$

$$= \exp\left(f_{ji}^{-1}(b_{kj,E}) + b_{ji,E} + \frac{1}{2}[f_{ji}^{-1}(b_{kj,E}), b_{ji,E}] + [f_{ki}^{-1}(c_{kj,F}), b_{ji,E}]_E\right) + o(1). \tag{75}$$

We can decompose E into a grading $E_1 \oplus E_2$ so that the center intersect E exactly at E_2. Note that in two-nilpotency $[x, y]_E \in E_2$.

Now we look at the coordinates of each side inside E_1. On the left, we have the coordinates in E_1 of $b_{ki,E}$, which must be bounded. On the right, we get the coordinates in E_1 of $f_{ji}^{-1}(b_{kj,E}) + b_{ji,E}$. It can not be bounded unless the coordinates in E_1 of $b_{kj,E}$ tend to zero since f_{ji}^{-1} expands.

It implies also that $[f_{ji}^{-1}(b_{kj,E}), b_{ji,E}]$ is bounded since it only depends of the components in E_1.

In E_2, we get on the left the coordinates in E_2 of $b_{ki,E}$ that is again bounded. On the right, we get the coordinates in E_2 of $f_{ji}^{-1}(b_{kj,E}) + b_{ji,E}$ and the sum (completely in E_2) $\frac{1}{2}[f_{ji}^{-1}(b_{kj,E}), b_{ki,E}] + [f_{ki}^{-1}(c_{kj,F}), b_{ji,E}]_E$. The term $b_{ji,E}$ is bounded. The first bracket is bounded by what precedes. It lasts to prove that the second bracket is bounded.

Note that since Q_{mn} fixes $q_{mn} \in F|_{D(\gamma)(1)}$ we have

$$c_{mn,F} = q_{mn} - f_{mn}(q_{mn}). \tag{76}$$

Also, the expression of the cocycle $T_{kj}T_{ji} = T_{ki}$ and the two-step nilpotency show that

$$\left[c_{kj,F} + f_{kj}(c_{ji,F}), \cdot\right] = \left[c_{ki,F}, \cdot\right] \tag{77}$$

$$\left[f_{ki}^{-1}(c_{kj,F}), \cdot\right] = \left[f_{ki}^{-1}(c_{ki,F}) - f_{ji}^{-1}(c_{ji,F}), \cdot\right] \tag{78}$$

$$= \left[f_{ki}^{-1}(q_{ki}) - q_{ki} - f_{ji}^{-1}(q_{ji}) + q_{ji}, \cdot\right] \tag{79}$$

and when $j \to \infty$ and $k \to \infty$ we have

$$\lim f_{ki}^{-1}(q_{ki}) - q_{ki} - f_{ji}^{-1}(q_{ji}) + q_{ji} = 0 - q_i - 0 + q_i = 0. \tag{80}$$

By consequence, the second bracket tends to zero, and is therefore bounded.

Hence $f_{ji}^{-1}(b_{kj,E})$ is bounded and it implies that $b_{kj,E} \to 0$. □

Rank one ray manifolds

We examine closed manifolds with a rank one ray geometry. For those manifolds, the holonomy takes its values in $G_1 = N \rtimes KA_1$, where A_1 has dimension 1.

Lemma 4.21. *If $(G_1, \mathcal{N})$ is a rank one ray geometry, then there exists a decomposition $L_1 \oplus L_2 \oplus L_3$ such that for any Fried dynamics, $P = L_3$ and $\{E, F\} = \{L_1, L_2\}$.*

A consequence of this observation is that P is independent from the dynamics, since $d_i = 0$ is a condition on $A = A_1$. Therefore, from a Fried dynamics to another one E and F are either the same or exchanged. Also, every direction in P is completely visible by the following lemma.

Lemma 4.22. *The direction vector of an incomplete geodesic $D(\gamma)$ has a non-vanishing coordinate along E in the linear decomposition $\mathfrak{n} = E \oplus P \oplus F$ associated to its Fried dynamics.*

Proposition 4.23. *Let $S \subset \widetilde{M}$ be a convex containing γ. Assume that $D(S)$ has smooth boundary at $D(\gamma)(1)$. Let $i > 0$. The subspace $H_i = E_{+,i} \times (P \oplus F)$ is visible from y_i (this is a half-space of $\mathcal{N}$).*

Here we take the cartesian product in the sense of the group law. A point of $A \times B$ is a point $a + b$ with $a \in A$ and $b \in B$.

Proof. We use here the fact that a two-step nilpotent space has the property that any affine subset (in the classical sense) is a nil-affine subset and conversely.

By proposition 4.18, $E_{+,i}$ is a visible from y_i. Since the directions in P are always complete, the product $E_{+,i} \times P$ is fully visible from y_i. We show that we can extend $E_{+,i} \times P$ to a visible open H_i containing $E_{+,i} \times (P \oplus F)$.

Let $K \subset E_{+,i} \times P$ be convex and visible from y_i. Since the visible space from y_i is open, there exists an open convex $W \subset F$ such that $K \times W$ is visible and convex.

We show that we can extend K to $E_{+,i} \times P$. If that were not the case, there would exists $k + w \in \partial(K) \times W$ such that $z = k + w$ is invisible from $K \times W$. Now, at z, $E|_z$ is transverse to $\mathrm{T}_z(\partial K \times W)$. Therefore, we can consider $u \in E$ such that $z - tu$ is inside $K \times W$ for $t > 0$. We let $\eta(t) = z - u + tu$. It is an incomplete geodesic at $t = 1$ but lies inside $K \times W$ for $t < 1$.

Now, its limit set $C_{0,\infty}$ is contained in $\mathrm{T}_z(\partial K \times W)$ and therefore can only be $F|_z$. The limit set $F|_z$ must intersect $E_{+,i} \times P$. In fact it intersects it at k since, $z = k + w$ and $z + F = k + w + F = k + F$.

Therefore it intersects $E_{+,i} \times P$ and is simultaneously invisible since it is in $C'_{0,\infty} \cap (K \times W)$ and visible since $E_{+,i} \times P$ is completely visible, a contradiction.

Now, $(E_{+,i} \times P) \times W$ can be extended to H_i such that it contains $E_{+,i} \times (P \oplus F)$. Indeed, apply T_{ji} for $j \to \infty$ (note that $T_{ji}(x + y) = T_{ji}(x) + f_{ji}(y)$):

$$\begin{aligned} \lim_{j\to\infty} T_{ji}((E_{+,i} \times P) \times W) &= (E_{+,i} \times P) \times \lim_{j\to\infty} f_{ji}(W) \qquad (81) \\ &= (E_{+,i} \times P) \times F. \end{aligned}$$

□

Lemma 4.24. *The half-spaces H_i tend to a half-space denoted H_x when $i \to \infty$. For $i \to \infty$, q_i gets closer to $\overline{H_i}$.*

Lemma 4.25. *Let $x \in \widetilde{M}$. Let $D(S) \subset D(\widetilde{M})$ be the maximal Euclidean open ball such that $x \in S$ and S is visible from x. Then $H_x = \lim H_i$ contains x.*

Proof. We choose the Euclidean norm on $\mathfrak{n}$ associated to the basis $(e_1, \dots e_n)$. Those Euclidean balls are geodesically convex in two-step nilpotency.

We base $\mathcal{N}$ at $D(\gamma)(1)$. To prove $x \in \lim H_i$, we prove that $\langle x, \nu_{D(\gamma)(1)} H_x \rangle < 0$ with νH_x the choice of an exterior normal vector at $D(\gamma)(1) \in \partial H_x$. Note that for S the maximal Euclidean open ball we have $\langle x, \nu_{D(\gamma)(1)} S \rangle < 0$ since x is normal to the boundary of S at $D(\gamma)(1)$.

The normal vector at the boundary $\nu_{D(\gamma)(1)} H_x$ is described by $\lim f_{kj}^{-1}(\nu_{D(\gamma)(1)} S)$ when $j \to \infty$ and $k \to \infty$. Indeed, H_x is described by $\lim T_{kj}^{-1}(S)$. Let Δ be the tangent plane to S at $D(\gamma)(1)$. Then Δ contains $(P \oplus F)|_{D(\gamma)(1)}$. Since F is a subalgebra, and f_{mn} preserves F,

$$T_{kj}^{-1}(\Delta) = Q_{kj}^{-1}(\Delta) + o(1) \tag{82}$$

$$= c_{kj,F} + f_{kj}^{-1}(\Delta) + o(1) \tag{83}$$

$$= f_{kj}^{-1}(\Delta) + o(1) \tag{84}$$

Now, note that when $k \gg j \to \infty$, we get that f_{kj} is essentially its A-factor since the rotational part tends to the identity. The map $f_{kj,A}$ acts diagonally with positive factors. Denote $f_{kj,A}^{1/2}$ the transformation obtained by taking the square root of each diagonal factor.

Both x and $\nu_{D(\gamma)(1)}$ are normal to $\Delta \supset (F \oplus P)|_{D(\gamma)(1)}$. So both are contained in $E|_{D(\gamma)(1)}$. Note that $f_{kj}^{-1/2}$ acts by arbitrarily strong expansions on x and $\nu_{D(\gamma)(1)}$. By consequence

$$\langle x, \lim f_{kj}^{-1}(\nu_{D(\gamma)(1)} S) \rangle = \langle x, \lim f_{kj,A}^{-1}(\nu_{D(\gamma)(1)} S) \rangle + o(1) \tag{85}$$

$$= \lim \langle f_{kj}^{-1/2} x, f_{kj}^{-1/2}(\nu_{D(\gamma)(1)} S) \rangle + o(1) \tag{86}$$

$$\ll \langle x, \nu_{D(\gamma)(1)} S \rangle + o(1) < 0. \tag{87}$$

It proves that $x \in H_x$. □

For any $x \in \widetilde{M}$, we choose $S \subset \widetilde{M}$ the maximal convex open subset such that $D(S)$ is a Euclidean open ball. By applying the construction to the point x, the convex S and an incomplete geodesic $\gamma \subset S$ (which exists by maximality of S) we obtain an half-space $H_x = \lim H_i$ and $D(x) \in H_x$. This choice is now assumed.

Lemma 4.26. *Let $I \subset \partial H_x$ be the invisible set from the interior of H_x. (Note that $(F \oplus P)|_{D(\gamma)(1)} \subset I$.) Then I does not depend on $x \in \widetilde{M}$ and I is a nil-affine subspace.*

Proof of theorem 4.1. The nil-affine subspace I is constant and contains every $D(\gamma)(1)$ for any incomplete geodesic $\gamma \subset S$ in the maximal open ball of any $x \in \widetilde{M}$. We show it implies that $D\colon \widetilde{M} \to \mathcal{N} - I$ is a covering map.

Let $\delta\colon [0,1] \to \mathcal{N} - I$ be a path. Choose $x \in \widetilde{M}$ such that $D(x) = \delta(0)$. We need to prove that δ can be lifted to a path in $\widetilde{M}$ based at x. We can assume that δ can be lifted for $t < 1$. We show that is can be lifted at $t = 1$. Since $\delta(1) \notin I$, there exists a point $\delta(s)$ with $s < 1$ such that the maximal open ball based at $\delta(s)$ and avoiding I contains $\delta(t)$ for $s \leq t \leq 1$. But then for the lift at time s, the corresponding open ball S is convex and allows to lift δ for $s \leq t \leq 1$. □

To be more precise on the nature of I, we could use a discreteness argument very close to what Matsumoto [Mat92] proposed (see also [Ale21b, end of sec. 4] and [Ale21a]). It shows that in fact I is reduced to P if $F = \{0\}$. In particular, generalizations of Fried theorem [Fri80; Min90; Ale21b] are proven: if $F = P = \{0\}$ then the developing map is a cover onto the complement of a point.

5 Reducibility of incomplete manifolds

Before proving the theorems 5.6 and 5.8 on closed manifold with rank one ray structures, we examine the invisible subspace I. We will prove that $I \subset N$ is in fact a subgroup. This analysis does not depend on the fact that $\mathfrak{n}$ is two-step nilpotent.

Let $(G_1, \mathcal{N})$ be a rank one ray geometry. Let M be a closed incomplete manifold. Let $D\colon \widetilde{M} \to \mathcal{N} - I$ be its developing map, with I given by theorem 4.1.

We choose a base point in I that is also a asymptotic fixed point of a Fried dynamic. That is to say, $0 \in I$ is the asymptotic fixed point of T_{ji} for $i > 0$ fixed and $j \to \infty$.

We know that $I \supset (P \oplus F)$. We start by describing the rest of I in $\mathfrak{n}$, and notably in E.

Note that $[E, E] \subset E$. So E is a nilpotent subalgebra of $\mathfrak{n}$. Define a decomposition

$$E = E_1 \oplus \cdots \oplus E_q \tag{88}$$

where each E_i is the subspace of E on which A_1 (that has rank one) acts by homotheties with the degree d_i. Order the degrees such that $d_1 < d_2 < \cdots < d_q$. Observe that since A_1 acts by automorphisms, we have $[E_i, E_j] \subset E_{i+j}$.

It gives a graduation of the nilpotent Lie algebra E, and we can consider the sequence of ideals

$$E = E^1 \supset E^2 \supset \cdots \supset E^{q+1} = \{0\} \tag{89}$$

where

$$E^i = E_i \oplus E_{i+1} \oplus \cdots \oplus E_q \tag{90}$$

and it verifies $[E, E^i] \subset E^{i+1}$. With lemma 2.11 in sight we show the following.

Lemma 5.1. *The linear subspace I is decomposed by:*

$$I = P \oplus F \oplus \bigoplus E_i \cap I. \tag{91}$$

Proof. Since we know that $P \oplus F \subset I$, we only need to show that if $\Delta \subset I$ is a vectorial line in E that has a non vanishing coordinate on E_i for i minimal, then there exists a vectorial line Δ' of I fully contained in E_i.

Because we chose for base point the asymptotic fixed point of T_{ji}, we have that $T_{ji}(x) = c_{ji} + f_{ji}(x)$ with $c_{ji} \to 0$. So $T_{ji}(\Delta)$ tends to the same line as $f_{ji}(\Delta)$. Let Δ' be this limit. It is again in $I \cap E$ since f_{ji} preserves E. It is completely contained in E_i since it is the minimal degree of f_{ji} that acts on Δ.

Indeed, assume for example that $i = 1$. Then $f_{ji}(\Delta) = f_{ji}(\mathbf{R}x) = \mathbf{R}(\beta^{d_1} x_1, \ldots, \beta^d_q x_n)$. By normalizing by β^{d_1} we get $\mathbf{R}(x_1, \ldots, \beta^{d_j - d_1} x_l, \ldots, \beta^{d_q - d_1} x_n)$ and every coordinate that is after E_i tends to zero. □

Lemma 5.2. *The subspace I is a subalgebra of* $\mathfrak{n}$.

Proof. We already know that $P \oplus F$ is a subgroup since $[P \oplus F, P \oplus F] \subset P \oplus F$. To prove that I is a subgroup, we show that $[E \cap I, P \oplus F] \subset I$ and that $[E \cap I, E \cap I] \subset I$. Since $I = P \oplus F \oplus (E \cap I)$ (by the preceding lemma) it implies $[I, I] \subset I$ and therefore that I is a subalgebra.

The preceding lemma shows that $E \cap I$ is even the direct sum of every $E_m \cap I$. To prove $[E \cap I, P \oplus F] \subset I$ we let $x_m \in E_m \cap I$ and $e_k \in P \oplus F$ (an element of the basis $(e_1, \ldots, e_n)$) and we show that $[x_m, e_k] \in I$.

We let $\epsilon > 0$ and consider the point $\epsilon x_m \in I$. It gives a Fried dynamic T_{ji} centered asymptotically at a point in $F|_{\epsilon x_m}$. By consequence, without changing the base point of N, we use the decomposition $(E \oplus P) \oplus F$ at the origin and we have:

$$T_{ji}(x) = c_{E\oplus P} + c_F + f_{ji}(x) \tag{92}$$

where $c_{E\oplus P} \to \epsilon x_m$ by lemma 4.15. Now let $x = -f_{ji}^{-1}(c_F) + f_{ji}^{-1}(\epsilon e_k)$. Then $x \in I$ because f_{ji} preserves $P \oplus F$ and $P \oplus F$ is a subgroup. Therefore $T_{ji}(x) \in I$ and

$$T_{ji}(x) = c_{E\oplus P} + \epsilon e_k \to \epsilon x_m + \epsilon e_k \in I. \tag{93}$$

In the Lie algebra,

$$\ln(\epsilon x_m + \epsilon e_k) = \epsilon x_m + \epsilon e_k + \epsilon^2 \frac{1}{2}[x, e_k] + o(\epsilon^2) \tag{94}$$

and by consequence

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} 2\left(\ln(\epsilon x_m + \epsilon e_k) - \epsilon x_m - \epsilon e_k\right) = [x_m, e_k]. \tag{95}$$

But for any ϵ, $\ln(\epsilon x_m + \epsilon e_k) - \epsilon x_i - \epsilon e_k$ is in I. Hence $[x_m, e_k] \in I$.

Therefore we have proven that $[E \cap I, P \oplus F] \subset I$. It implies $[F, I] \subset I$.

To prove $[E \cap I, E \cap I] \subset I$ we repeat the same argument but with this time $x_k \in E_k \cap I$ instead of $e_k \in P \oplus F$. We only need to justify that the point $x = -f_{ji}^{-1}(c_F) + f_{ji}^{-1}(\epsilon x_k)$ is again in I. By invariance under f_{ji}, we prove that $F + E_k \cap I \subset I$.

For any point $\exp(p) \in I$ we have $\exp(p) + F \subset I$. Indeed, let $\exp(p)$ be a $D(\gamma)(1)$, then $F|_p \subset I$. Also for any $\exp(q) \in F$ and any $\exp(p) \in I$, since $\mathrm{Ad}(\exp) = \exp(\mathrm{ad})$ in $\mathfrak{n}$:

$$\exp(q) + \exp(p) = \exp\left(p + [q, p] + \frac{1}{2!}[q, [q, p]] + \dots\right) + \exp(q). \tag{96}$$

Every bracket belongs to I since $[F, I] \subset I$. So $\exp(p + [q, p] + \frac{1}{2!}[q, [q, p]] + \dots)$ belongs to I and therefore $\exp(q) + \exp(p)$ belongs to I since $I + \exp(q)$ does. By consequence, $F + E_k \cap I \subset I$. □

Lemma 5.3. *Under the same hypotheses, if $T \in \Gamma$ is any holonomy transformation with $T(x) = c + f(x)$. Then $f(I) = I$.*

Proof. The holonomy transformation must preserve I. Furthermore, since the base point is taken in I, we have $c + f(0) = c \in I$. So $f(I) = -c + I$ but $-c + I = I$ since I is a subgroup. □

Lemma 5.4. *Under the same hypotheses, there exists a decomposition $\mathfrak{n} = I \oplus V$ such that I and V are both invariant under the linear holonomy of M. Furthermore, lemma 2.11 applies and $N = \exp(I)\exp(V)$ by a diffeomorphism.*

Proof. On each E_i, the linear group KA_1 acts by similarity: A_1 acts by homotheties and since K centralizes A_1, it preserves E_i.

By the choice of the Fried dynamic T_{ji} with asymptotic fixed point $p \in I$, we know that $(P \oplus F)|_p \subset I$. So the linear subspaces L_j such that the degrees are $d_j \leq 0$ are all contained in I. To construct V, we only need to inspect the subspaces on which the degrees are $d_j > 0$ (it corresponds to the subspaces of $E|_p$).

Since A_1 acts by homotheties on E_i, we can consider a scalar product on E_i that is invariant under KA_1. Let $I_i = I \cap E_i$ and $V_i = I_i^{\perp}$. By lemma 5.3 the linear holonomy preserves I_i and therefore it preserves its orthogonal V_i. We let $V = V_1 \oplus \cdots \oplus V_k$. It is stable under the linear holonomy and gives $V \oplus I = \mathfrak{n}$.

It lasts to show that we have again a decomposition $N = \exp(I)\exp V$. First, in E we can decompose by $I \cap E \oplus V$. Indeed, each E_i is generated by $E_i \cap I$ and $E_i \cap V$. So lemma 2.11 applies and for any $x_E \in E$, there is a unique decomposition $x_E = x_{E\cap I} + x_V$.

Now decompose by $(P \oplus F) \oplus E$. We get that for any $x \in N$, $x = x_{P\oplus F} + x_E = x_{P\oplus F} + x_{E\cap I} + x_V$. Since I is a subgroup and $P \oplus F \subset I$, we have $x_{P\oplus F} + x_{E\cap I} = x_I \in I$.

It lasts to show that $x = x_I + x_V$ is unique. The point x_V is unique so $x_I' + x_V = x_I + x_V$ implies $x_I' = x_I$. □

5.1 Markus conjecture

We now address the link with Markus conjecture [Mar62]. This conjectures states that (in real affine geometry) *closed manifolds with parallel volume are complete.*

Definition 5.5. *A ray geometry* $(G, \mathcal{N})$ *has* parallel volume *if* G *preserves the volume form* $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$*, with* $(x_1, \ldots, x_n)$ *the dual coordinate functions of the basis* $(e_1, \ldots, e_n)$ *given by the geometry.*

If we were in real affine geometry, theorem 4.1 would show that closed manifolds with parallel volume are complete. Indeed, the fact that it preserves I implies that the holonomy is reducible. But by Goldman-Hirsch [GH84], the holonomy of such a closed manifold can never be reducible.

This argument that we used in [Ale21a] can not be applied here since Goldman-Hirsch construction relies on the commutative law of $\mathbf{R}^n$. Instead we prove theorem 5.6 by an argument similar to what Fried [Fri80] employed.

Theorem 5.6. *Let* $\mathcal{N}$ *be a commutative or two-step nilpotent space. Let* $(G_1, \mathcal{N})$ *be a ray geometry of rank one with parallel volume. Every closed* $(G_1, \mathcal{N})$*-manifold is complete.*

Proof. Consider a $(G, \mathcal{N})$-manifold that has its developing map covering $\mathcal{N} - I$. We show a contradiction.

Denote $H \subset G$ the stabilizer of I. Each point of I provides a Fried dynamic.

Consider the decomposition $\mathfrak{n} = I \oplus V$ given by the preceding lemma. By lemma 2.11, for any $x \in N$ there exists a unique decomposition $x = x_I + x_V$ with $x_I \in \exp(I)$ and $x_V \in \exp(V)$. On $N - I$ we define the vector field X by:

$$X(x) = (L_x)_* \ln(x_V) \tag{97}$$

The vector field X is H-invariant. Indeed, if $T \in H$ then $T(x) = c_I + f(x)$ with $c_I \in \exp(I)$ since I is stable. So

$$X(T(x)) = X(c_I + f(x_I) + f(x_V)) = (L_{T(x)})_* f(\ln(x_V)) = T_* X(x) \tag{98}$$

since $c_I + f(x_I)$ must belong to $\exp(I)$ and V is stable under the linear holonomy.

Now, similarly to the last argument of Fried [Fri80], we show that X can't preserve the parallel volume, in contradiction with the fact that M is closed. Recall that we have the parallel volume in the coordinates $(x_1, \ldots, x_n)$ associated to the $(e_1, \ldots, e_n)$:

$$\mathrm{vol}_N = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n. \tag{99}$$

By hypothesis, vol_N is preserved by G, and so it is also by H. The pulled-back $D^*\mathrm{vol}_N$ defines a volume form on $\widetilde{M}$ that is invariant under $\pi_1(M)$. So it gives a volume form on the closed manifold M. Any flow of any vector field on M must preserve the volume of M since it is closed.

But X is H-invariant, so by the same procedure, provides a vector field Y on M. The flow of Y corresponds to the flow of X. Choose $\cup U_i$ a cover of M by open sets so that there exists corresponding lifts $\cup \widetilde{U}_i$ in $\widetilde{M}$ that are each diffeomorphically sent to V_i in N by $D\colon \widetilde{M} \to N$. Choose a partition of the unity $\{\rho_i\}$ in M associated to $\{U_i\}$. Denote R_t the flow of X in N and Φ_t the flow of Y in M. Then the conservation of the volume in M implies that for any $t \in \mathbf{R}$:

$$\int_M \Phi_t^* \mathrm{vol}_M = \int_M \mathrm{vol}_M \tag{100}$$

$$\iff \sum_j \int_{U_j} \rho_j \Phi_t^* \mathrm{vol}_M = \sum_j \int_{U_j} \rho_j \mathrm{vol}_M \tag{101}$$

$$\iff \sum_j \int_{V_j} \rho_j R_t^* \mathrm{vol}_N = \sum_j \int_{V_j} \rho_j \mathrm{vol}_N \tag{102}$$

$$\iff \sum_j \int_{V_j} \rho_j \left(R_t^* \mathrm{vol}_N - \mathrm{vol}_N\right) = 0. \tag{103}$$

But now, observe that

$$R_t(x_I + x_V) = x_I + e^t x_V. \tag{104}$$

Indeed, for any fixed x, $R_t(x)$ must be the solution to the differential equation $R_t(x)^*\omega_N = \omega_N(X(R_t(x)))$, where ω_N is the Maurer-Cartan form of N. By left-invariance and the notations,

$$R_t(x)^*\omega_N = \exp(e^t \ln(x_V))^*\omega_N. \tag{105}$$

Note that the exponential map $f(t) = \exp(tv)$ is precisely the solution to the equation $f^*\omega_N = v$ with $f(0) = e$. Let $t \in \mathbf{R}$ and $\epsilon \to 0$, by the Baker-Campbell-Hausdorff formula and $[x_V, x_V] = 0$,

$$\exp(e^{t+\epsilon}\ln(x_V)) = \exp(e^t(1+\epsilon+o(\epsilon))\ln(x_V)) \tag{106}$$

$$= \exp(e^t \ln(x_V))\exp((\epsilon + o(\epsilon))e^t \ln(x_V)) \tag{107}$$

and by left-invariance, we obtain

$$\exp(e^t \ln(x_V))^*\omega_N = e^t \ln(x_V) = \omega_N(X(R_t(x))). \tag{108}$$

Therefore

$$R_t^* \mathrm{vol}_N = e^{t \dim V} \mathrm{vol}_N \tag{109}$$

and it contradicts the preservation of the volume since for $t > 0$ large enough $(R_t^*\mathrm{vol}_N - \mathrm{vol}_N)$ is a positive function. □

5.2 The automorphism group

It is a vague conjecture [DG91] that geometric manifolds with large automorphism groups should be classifiable.

Definition 5.7. *Let M be a (G,X)-manifold. An* automorphism *$f\colon M \to M$ is a diffeomorphism such that if $\widetilde{f}\colon \widetilde{M} \to \widetilde{M}$ is any lift then there exists a unique $\chi(\widetilde{f}) \in N_G(\Gamma)$ in the normalizer of the holonomy group, such that $D(\widetilde{f}(x)) = \chi(\widetilde{f})D(x)$.*

The group $\mathrm{Aut}(M)$ *of its automorphism is sent by χ into a subgroup of $\Gamma \backslash N_G(\Gamma)$.*

By unicity of $\chi(\widetilde{f})$, if $\widetilde{f_1}$ and $\widetilde{f_2}$ are two lifts of f then $\widetilde{f_2} = g\widetilde{f_1}$ for $g \in \pi_1(M)$ and therefore $\chi(\widetilde{f_2}) = \rho(g)\chi(\widetilde{f_1})$.

The hypothesis that $\chi(\widetilde{f})$ normalizes Γ follows from the fact that if $\widetilde{f}$ lifts f then it must preserves the fibers $\pi_1(M)$ of $\widetilde{M}$ and therefore

$$\chi(\widetilde{f})\Gamma \cdot D(x) = D(\widetilde{f}(\pi_1(M) \cdot x)) = D(\pi_1(M) \cdot \widetilde{f}(x)) = \Gamma \cdot \chi(\widetilde{f})D(x). \tag{110}$$

If the rank one ray geometry $(G_1, \mathcal{N})$ is a subgeometry of a Levi geometry of a rank one parabolic geometry, then by Ferrand-Obata [Fer96; Oba71], Schoen-Webster [Sch95; Web77] and Frances [Fra07] we would get that if the automorphism group acts non properly then the manifold is complete.

We show that this phenomenon is only dependent on the fact that the ray geometry has rank one on an at most two-step nilpotent space. It is indeed the case if it comes from a rank one parabolic geometry.

Theorem 5.8. *Let $\mathcal{N}$ be a commutative or two-step nilpotent space. Let $(G_1, \mathcal{N})$ be a ray geometry of rank one. Let M be a closed $(G_1, \mathcal{N})$-manifold. If* $\mathrm{Aut}(M)$ *does not act properly on M then M is complete.*

Proof. Assume that M is not complete, then by theorem 4.1, we get that $D\colon \widetilde{M} \to \mathcal{N} - I$ is a cover. Therefore the holonomy Γ preserves I and so does the normalizer $N(\Gamma)$. Both must be subgroups of $I \rtimes KA_1$, the subgroup of G_1 stabilizing I. We show that $\mathrm{Aut}(M)$ must act properly.

Choose a base point $p \in I$ which is the asymptotic fixed point of a Fried dynamic and consider the decomposition $I \oplus V = \mathfrak{n}$ given by lemma 5.4. We identify I with its linear subspace in $\mathfrak{n}$. In rank one, I contains $(P \oplus F)|_p$ and therefore V is contained in $E|_p$. By lemma 2.11 we get that any $x \in N$ has a unique decomposition $x = x_I + x_V$ with $x_I \in \exp(I)$ and $x_V = \exp(V)$.

Assume that $\mathrm{Aut}(M)$ acts non properly. It is equivalent to the existence of $x_n \to x$ in $\widetilde{M}$, of $g_n \in N(\Gamma)$ with Γg_n escaping every compact of $\Gamma \backslash N(\Gamma)$, and of $y \in \widetilde{M}$ such that $\Gamma g_n D(x_n) \to \Gamma D(y)$.

Write $g_n(x) = c_n + f_n(x)$ with $c_n \in I$ and $f_n \in KA_1$. The Fried dynamic $T_{ji} \in \Gamma$ considered can be written $T_{ji}x = c_{ji} + f_{ji}(x)$. Note that

$$T_{ji}g_n(x) = c_{ji} + f_{ji}(c_n) + f_{ji}f_n(x). \tag{111}$$

It shows that we can assume $f_{ji}f_n$ bounded in KA_1 since A_1 has rank one (up to exchange T_{ji} with T_{ji}^{-1}).

Hence there exists $\gamma_n \in \Gamma$ such that $\gamma_n g_n = h_n$ has its KA_1-factor that converges. The convergence $\Gamma g_n D(x_n) \to \Gamma D(y)$ says that there exists $\eta_n \in \Gamma$ such that $\eta_n h_n D(x_n) \to D(y)$.

Write $\eta_n(x) = b_n + q_n(x)$ and $h_n(x) = c_n + r_n(x)$ we have by construction $r_n \to r$ and

$$(\eta_n h_n)(D(x_n)) = b_n + q_n(c_n) + q_n r_n(D(x_n)) \to D(y). \tag{112}$$

We again have a decomposition

$$(\eta_n h_n)(D(x_n)) = ((\eta_n h_n)(D(x_n)))_I + ((\eta_n h_n)(D(x_n)))_V. \tag{113}$$

The term $b_n + q_n(c_n)$ must belong to I since $c_n \in I$. So the term $((\eta_n h_n)(D(x_n)))_V$ tending to $D(y)_V$ is determined by $(q_n r_n(D(x_n)))_V$. But since $D(y)_V \neq 0$ and $V \subset E$, it shows that q_n itself must converge to $q \in KA_1$.

The term $((\eta_n h_n)(D(x_n)))_I$ tending to $D(y)_I$ has the same limit as the I-factor of $b_n + q_n(c_n) + qr(D(x_n))$. Since $D(y)_I$ and $D(x) = \lim D(x_n)$ are both finite, $b_n + q(c_n)$ must converge.

Therefore $\eta_n h_n$ converges in $I \rtimes KA_1$, contradicting the fact that Γg_n escapes every compact of $\Gamma \backslash N(\Gamma)$. □

In higher rank As in [Ale21a], we can give a relatively generic example that in rank $r > 1$ this phenomenon is no longer true. Let (x, y, z) be coordinates of the Heisenberg group in dimension three and consider the rank two diagonal group be given by $(\beta_1 x, \beta_2 y, \beta_1\beta_2 z)$. Then consider the radiant manifold given by $\mathcal{N} - \{0\}$ quotiented by the subgroup generated by $(2x, 2y, 4z)$. Then let $f(x, y, z) = (x/2, y, z/2)$ and $p = (x, 1, 0)$. Then $f^n(p) \to (0, 1, 0)$ and it corresponds in M to an automorphism acting non properly.

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