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A LOCAL CLT FOR LINEAR STATISTICS OF 2D COULOMB GASES

THOMAS LEBLÉ AND OFER ZEITOUNI

ABSTRACT. We prove a local central limit theorem for fluctuations of linear statistics of smooth enough test functions under the canonical Gibbs measure of two-dimensional Coulomb gases at any positive temperature. The proof relies on the *global* central limit theorem of [BBNY19, LS18] and a new decay estimate for the characteristic function of such fluctuations.

1. INTRODUCTION

1.1. **Goal of the paper.** Let \mathbb{P}_N^β be the canonical Gibbs measure (defined below in (1.2)) of the two-dimensional Coulomb gas with N particles $\mathbf{X}_N = (x_1, \dots, x_N)$ at inverse temperature $\beta > 0$, let μ_0 be the associated *equilibrium measure* and let $\mathbf{X}_N := \sum_{i=1}^N \delta_{x_i}$ be the (random) point configuration associated to the N (random) positions of the particles. We introduce the random signed measure

$$(1.1) \quad \mathbf{M}_N := \mathbf{X}_N - N\mu_0$$

and, when φ is a continuous real-valued test function on \mathbb{R}^2 , we define the *fluctuations of φ* as:

$$(1.2) \quad \langle \varphi, \mathbf{M}_N \rangle := \int_{\mathbb{R}^2} \varphi(x) d\mathbf{M}_N(x) = \sum_{i=1}^N \varphi(x_i) - N \int_{\mathbb{R}^2} \varphi(x) d\mu_0(x).$$

The papers [LS18, BBNY19] prove a central limit theorem (CLT) for *fluctuations of linear statistics*, namely quantities of the type $\langle \varphi, \mathbf{M}_N \rangle$ where φ is smooth enough, possibly living at some mesoscopic scale, and supported within the support of μ_0 . The random variable $\langle \varphi, \mathbf{M}_N \rangle$ converges in distribution as $N \rightarrow \infty$ to a certain Gaussian random variable Z (with explicit mean and variance). The goal of the present paper is to prove a *local* central limit theorem, namely that:

$$\text{for any } \mathbf{a} \in (-\infty, +\infty), \mathbb{P}_N^\beta [\langle \varphi, \mathbf{M}_N \rangle \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)] \sim_{N \rightarrow \infty} \mathbb{P}[Z \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)],$$

where the characteristic size ε_N of the interval is allowed to go to 0 at certain speeds that we specify below, depending on the scale at which the test function lives.

The proof relies on the existing CLT and a crucial new estimate of the decay at infinity of the fluctuations' characteristic function. We use the transportation technique of [LS18, BLS18], which is closely related to the loop equations toolbox as in e.g. [BG13, BBNY19], but instead of using it to study the Laplace transform of a given fluctuation as in the papers cited, here we turn to study its Fourier transform. We plan to use the new decay estimate we obtained in further investigations of 2D Coulomb gases.

1.2. Setting.

- The *equilibrium measure* μ_0 will be taken as a probability measure on \mathbb{R}^2 that satisfies: μ_0 has a compact support Σ and possesses a density \mathbf{m}_0 with respect to the Lebesgue measure that is bounded below by a positive constant c_{\min} on Σ . We assume that Σ is the closure of its interior, that the boundary $\partial\Sigma$ is a piece-wise C^1 curve, and that the density \mathbf{m}_0 is of class C^3 on Σ .

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- The *confining potential* ζ will simply denote a non-negative continuous function that vanishes on the support Σ of μ_0 , is positive outside Σ , and grows at infinity in a strongly confining way, i.e. $\liminf_{|x| \rightarrow \infty} \frac{\zeta(x)}{\log|x|} > 2$. We will denote by $\vec{\zeta}(X_N)$ the quantity $\vec{\zeta}(X_N) := 2N \sum_{i=1}^N \zeta(x_i)$.

Definition 1.1 (The two-dimensional Coulomb energy). To any state of the system $X_N = (x_1, \dots, x_N)$ we associate its *logarithmic energy* $F(X_N, \mu_0)$ given by

$$(1.3) \quad F(X_N, \mu_0) := \frac{1}{2} \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} -\log|x-y| (d\mathbf{X}_N - d\mu_0)(x)(d\mathbf{X}_N - d\mu_0)(y),$$

where Δ denotes the diagonal in $\mathbb{R}^2 \times \mathbb{R}^2$. Recalling that $-\log$ is (up to a multiplicative constant) the interaction potential for two-dimensional electrostatics, we can think of $F(X_N, \mu_0)$ as being the electrostatic energy of a system consisting of N negative point charges placed at (x_1, \dots, x_N) and a charged background of opposite sign whose density is given by μ_0 .

Definition 1.2 (Canonical Gibbs measure of the two-dimensional Coulomb gas). For all $\beta > 0$ we consider the probability measure \mathbb{P}_N^β on $(\mathbb{R}^2)^N$, with density

$$(1.4) \quad d\mathbb{P}_N^\beta(X_N) := \frac{1}{Z_N^\beta} \exp\left(-\beta\left(F(X_N, \mu_0) + \vec{\zeta}(X_N)\right)\right) dX_N,$$

where Z_N^β is the normalizing constant, or partition function, given by the integral

$$Z_N^\beta := \int_{(\mathbb{R}^2)^N} \exp\left(-\beta\left(F(X_N, \mu_0) + \vec{\zeta}(X_N)\right)\right) dX_N.$$

The measure \mathbb{P}_N^β is called the *canonical Gibbs measure*, and β the *inverse temperature parameter*.

Henceforth, we fix $N \geq 2$ and an arbitrary value of $\beta > 0$. We let X_N be a random variable in $(\mathbb{R}^2)^N$ distributed according to the Gibbs measure \mathbb{P}_N^β , and \mathbf{X}_N be the associated random atomic measure of mass N on \mathbb{R}^2 .

Remark 1.3. Often, one defines a Coulomb Gibbs measure by choosing an “external potential” V that has a certain regularity and growth at infinity. That is, given such function V , one defines the measure

$$(1.5) \quad d\mathbb{Q}_N^\beta(X_N) := \frac{1}{\widehat{Z}_N^\beta} \exp(-\beta(F(X_N, 0) + V(X_N))) dX_N,$$

where \widehat{Z}_N^β is a normalization constant and $V(X_N) = N \sum_{i=1}^N V(x_i)$. Classical potential theory then yields, under the above mentioned mild assumptions on V , the existence of an equilibrium measure μ_0 so that, on its support, $\int \log|x-y| d\mu_0(y) = V(x) + C_{\mu_0}$, for an appropriate constant C_{μ_0} . One then obtains (1.4) with the confining potential ζ given, off the support of μ_0 , by the difference of V and the logarithmic potential of μ_0 . Here we prefer to forget V in order to work directly with the measure itself. Since we are restricting ourselves to test functions supported inside the support Σ of μ_0 , the confining potential ζ will play almost no role. We note that μ_0 is still the “equilibrium measure” of the system in the sense that as $N \rightarrow \infty$, the empirical measure L_N converges weakly to μ_0 almost surely.

1.3. Statements. We fix a compactly supported function $\overline{\varphi}$ of class C^5 on \mathbb{R}^2 and we take either:

- (Macroscopic case) $\varphi_N = \overline{\varphi}$, assuming $\overline{\varphi}$ supported in the interior of Σ . In this case we set $\ell_N := 1$.
- (Mesoscopic case) $\varphi_N := \overline{\varphi}\left(\frac{\cdot - \bar{x}_N}{\ell_N}\right)$, where ℓ_N is such that $N^{-1/2} \ll \ell_N \ll 1$ and $\{\bar{x}_N\}_N$ is a sequence of points in Σ that stays at a certain positive distance from $\partial\Sigma$.

We will usually omit the subscript N and simply write φ , with an implicit dependence on N in the mesoscopic case.

Remark 1.4. The test functions that we consider always live in the interior of the support of the equilibrium measure, in fact in both the macroscopic and mesoscopic cases their supports stay at some positive distance from the boundary. Allowing the test functions to have a support intersecting (or even close to) the boundary would raise several technical problems:

- There is the well-known fact, see e.g. [Pas06] in the one dimensional case, that a CLT does not hold in general if the support of μ_0 has more than one connected component (one needs to assume some compatibility conditions on the test function).
- The effect of such a “boundary” test function on the equilibrium measure and its support is very subtle, see [SS18] for an analysis
- The local laws that control the energy at small scales and that are needed to treat mesoscopic cases have not been proven close to the boundary.

The main result of the present paper is the following theorem. In it and the rest of the paper, we use the standard notation $a_N \ll b_N$ to mean that $a_N/b_N \rightarrow_{N \rightarrow \infty} 0$.

Theorem 1 (Local CLT for fluctuations). *Let Z be a Gaussian random variable with mean b_Z equal to 0 (in the mesoscopic case) or $\frac{1}{2\pi} \left(\frac{1}{\beta} - \frac{1}{4}\right) \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx$ (in the macroscopic case), and variance $\sigma_Z^2 = \frac{1}{2\pi\beta} \int_{\mathbb{R}^2} |\nabla\varphi|^2$. Let $\{\varepsilon_N\}_N$ be a sequence that satisfies¹:*

$$(1.6) \quad \frac{\ell_N^{-2}}{N} \log N \ll \varepsilon_N \ll 1.$$

Then the following local CLT holds²: for all real number \mathbf{a} , we have:

$$(1.7) \quad \mathbb{P}_N^\beta [\langle \varphi, \mathbf{M}_N \rangle \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)] = \mathbb{P} [Z \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)] (1 + o_N(1)),$$

and the convergence is uniform for \mathbf{a} in compacts.

The new technical ingredient in the proof of Theorem 1 is a decay estimate on the Fourier transform of fluctuations. The (sequence of) test functions φ having been chosen as above, we introduce the characteristic function \mathcal{F}_N :

$$(1.8) \quad \mathcal{F}_N : \omega \mapsto \mathcal{F}_N(\omega) := \mathbb{E} \left[e^{i\omega \langle \varphi, \mathbf{M}_N \rangle} \right].$$

We write $C(\overline{\varphi}, \mu_0, \beta)$ to denote a positive constant that depends only on $\overline{\varphi}, \mu_0$ and the value of β .

Proposition 1.5 (Decay estimate on the Fourier transform of fluctuations). *We have that*

$$(1.9) \quad |\mathcal{F}_N(\omega)| \leq C(\overline{\varphi}, \mu_0, \beta) \left(\frac{1}{\omega^2} + \frac{\ell_N^{-2}}{N} \right), \quad \text{for } |\omega| > 1.$$

Remark 1.6. In the related, one-dimensional context of linear statistics for unitary random matrices, a CLT with fast convergence in total variation norm (that in particular implies a local CLT) was proven in [Joh97] (see also the recent [JL20] for multidimensional extensions). In the Hermitian random matrix context, [BEYY16] study the Fourier transform of fluctuations in order to get better estimates on the speed of convergence of linear statistics to the limiting Gaussian random variable.

In Section 2, we present a proof of Theorem 1, assuming the decay estimate (1.9). We prove the latter in Section 3. In the Appendix, we gather existing results around the logarithmic interaction energy, its electric formulation, its behavior under a change of variables, and some useful *a priori* bounds.

¹The optimal lower bound probably does not have the $\log N$ factor.

²We observe that since Z has a Gaussian density the right hand-side of (1.7) is bounded above and below by positive constants (depending on \mathbf{a}) times ε_N .

1.4. Notation. Throughout the paper, constants K_i are universal, i.e. independent of ω or N (however, they may depend on μ_0 and β .) We use throughout the standard $o(\cdot)$ and $O(\cdot)$ notation. Thus, $b = o_n(a)$ means that $b/a \rightarrow_{n \rightarrow \infty} 0$ and $b = O_n(a)$ means that b/a remains bounded by a universal constant as n varies. When the parameter n is clear from the context, we write $a_n = o(b_n)$, etc. The notation $a_N \ll b_N$ is thus equivalent to the implicit notation $a = o_N(b)$. Finally, the notation $O_*(t)$ means a quantity whose absolute value is bounded above by Ct where C may depend on other parameters, e.g. N, ω , but not on t .

When no confusion can occur, we use the symbols \mathbb{P}, \mathbb{E} to denote probabilities and expectations with respect to the measure determined by the random variables involved. Thus, the left hand side of (1.7) equals $\mathbb{P}[\langle \varphi, \mathbf{M}_N \rangle \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)]$.

2. PROOF OF THEOREM 1 USING PROPOSITION 1.5

In this section we fix an arbitrary sequence of positive real numbers δ_N such that

$$\frac{\ell_N^{-2}}{N} \ll \delta_N \ll \varepsilon_N.$$

We let R be a C^∞ non-negative function of mass 1, supported on $(-1, 1)$ and we let $R_\delta := \frac{1}{\delta_N} R\left(\frac{\cdot}{\delta_N}\right)$. We use the function R_δ to regularize the random variable $\langle \varphi, \mathbf{M}_N \rangle$ at scale δ_N , that is we let \tilde{X}_δ be the sum of $\langle \varphi, \mathbf{M}_N \rangle$ and of an independent random variable with density R_δ . Since the random variable \tilde{X}_δ almost surely differs from the original variable by at most $\delta_N \ll \varepsilon_N$, it is enough to prove that for every \mathbf{a} real, we have:

$$\mathbb{P}[\tilde{X}_\delta \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)] = \mathbb{P}[Z \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)](1 + o_N(1)).$$

For simplicity of notation we take in the sequel $\mathbf{a} = 0$. In fact, as can be easily checked, the estimates are uniform for \mathbf{a} in compacts.

By regularization, \tilde{X}_δ has a density with respect to the Lebesgue measure, which is bounded by some function of N . We write the “local” probability $\mathbb{P}[\tilde{X}_\delta \in (-\varepsilon_N, \varepsilon_N)]$ as the integral of the indicator function $1_{[-\varepsilon_N, \varepsilon_N]}(t)$ against this density, and we use Parseval’s identity:

$$(2.1) \quad \mathbb{P}[\tilde{X}_\delta \in (-\varepsilon_N, \varepsilon_N)] = \int_{\mathbb{R}} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}_\delta(\omega) \mathcal{F}_N(\omega) d\omega = \int_{\mathbb{R}} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega.$$

We decompose the integral over \mathbb{R} in three parts: low frequencies where $|\omega| \leq K$ (where K is an arbitrarily large constant), intermediate frequencies where $K \leq |\omega| \leq \frac{1}{\delta_N}$, and high frequencies where $|\omega| \geq \frac{1}{\delta_N}$.

1. Low frequencies. We let

$$(2.2) \quad \mathcal{F}_\infty = \mathbb{E}(e^{i\omega Z}) = e^{-\omega^2 \frac{\sigma_Z^2}{2} - i\omega b_Z}$$

denote the characteristic function of Z . The central limit theorem of [LS18, BBNY19] implies that \mathcal{F}_N converges pointwise to \mathcal{F}_∞ . For any fixed K , we thus have by dominated convergence:

$$\int_{|\omega| \leq K} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega = (1 + o_N(1)) \int_{|\omega| \leq K} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}(\delta_N \omega) \mathcal{F}_\infty(\omega) d\omega.$$

Moreover, since \mathcal{F}_∞ is integrable, we have:

$$\left| \int_{|\omega| > K} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}(\delta_N \omega) \mathcal{F}_\infty(\omega) d\omega \right| = \varepsilon_N o_K(1),$$

and Parseval’s identity (2.1) holds for Z, \mathcal{F}_∞ as well, so we may write:

$$\int_{|\omega| \leq K} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{R}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega = \mathbb{P}[Z \in (-\varepsilon_N, \varepsilon_N)](1 + o_N(1)) + \varepsilon_N o_K(1).$$

2. Intermediate frequencies. We have, using (1.9), the bound $\frac{\sin(\varepsilon_N \omega)}{\omega} \leq \min(\varepsilon_N, \frac{1}{\omega})$, and a direct integration:

$$\left| \int_{K \leq |\omega| \leq \frac{1}{\delta_N}} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{\mathbf{R}}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega \right| \leq C(\overline{\varphi}, \mu_0, \beta) \left(\frac{1}{K} \varepsilon_N + \frac{\ell_N^{-2}}{N} \log \frac{1}{\delta_N} \right).$$

Since we have assumed that $\ell_N \gg N^{-1/2}$, that $\frac{\ell_N^{-2}}{N} \ll \delta_N \ll \varepsilon_N$ and that $\frac{\ell_N^{-2} \log N}{N} \ll \varepsilon_N \ll 1$, we obtain

$$\left| \int_{K \leq |\omega| \leq \frac{1}{\delta_N}} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{\mathbf{R}}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega \right| \leq \varepsilon_N (o_K(1) + o_N(1)).$$

3. High frequencies. In steps 1. and 2. we have simply bounded $\widehat{\mathbf{R}}$ by 1. For $|\omega| \geq \frac{1}{\delta_N}$ we use the smoothness of \mathbf{R} and write that:

$$\widehat{\mathbf{R}}(\delta_N \omega) \leq \frac{C(\mathbf{R})}{\delta_N^2 \omega^2}.$$

Using (1.9) again, we obtain (keeping in mind that $\frac{\ell_N^{-2}}{N} \ll \delta_N \ll \varepsilon_N$):

$$\begin{aligned} \left| \int_{|\omega| \geq \frac{1}{\delta_N}} \frac{\sin(\varepsilon_N \omega)}{\omega} \widehat{\mathbf{R}}(\delta_N \omega) \mathcal{F}_N(\omega) d\omega \right| &\leq C(\mathbf{R}) C(\overline{\varphi}, \mu_0, \beta) \int_{\frac{1}{\delta_N}}^{+\infty} \frac{1}{\omega} \frac{1}{\delta_N^2 \omega^2} \left(\frac{1}{\omega} + \frac{\ell_N^{-2}}{N} \right) d\omega \\ &\leq C(\mathbf{R}) C(\overline{\varphi}, \mu_0, \beta) \delta_N = \varepsilon_N o_N(1). \end{aligned}$$

In conclusion, we see that:

$$\mathbb{P} \left[\widetilde{X}_\delta \in (-\varepsilon_N, \varepsilon_N) \right] = \mathbb{P} [Z \in (-\varepsilon_N, \varepsilon_N)] (1 + o_N(1)) + \varepsilon_N (o_K(1) + o_N(1)).$$

We recall that $\mathbb{P} [Z \in (\mathbf{a} - \varepsilon_N, \mathbf{a} + \varepsilon_N)]$ is bounded above and below by positive constants times ε_N , uniformly for \mathbf{a} in every given line segment. Taking $N \rightarrow \infty$ followed by $K \rightarrow \infty$ concludes the proof of Theorem 1.

3. PROOF OF PROPOSITION 1.5

This section is devoted to the proof of Proposition 1.5, which goes in two steps. First, we establish a differential equation, see (3.16), for the characteristic function \mathcal{F}_N . The homogeneous part of the differential equation yields Gaussian decay. The non-homogeneous part contains a Fourier-like quantity associated to the so-called anisotropy term (see (3.17) and Section A.3), which we study next.

For the first step, we need to evaluate derivatives of various quantities, with respect to a parameter ε that is independent of N . The dependence in N will be irrelevant because we will take $\varepsilon \rightarrow 0$ with N fixed. For this reason, we recall that the notation $O_\star(\varepsilon^2)$ denotes a quantity that is bounded by ε^2 times a constant *possibly depending on N, ω* (and, of course, on $\overline{\varphi}, \beta, \mu_0$).

In order to quantify the decay of a Fourier transform, one could try to emulate the proof of Riemann-Lebesgue's lemma. The classical integration by parts cannot be easily performed, and we replace it here by an infinitesimal change of variables. The introduction of a change of variables (infinitesimal or not) and the analysis of its effect on the energy is not new in the present context, indeed there is a vast literature around loop equations, Ward identities, and other transportation techniques used to study fluctuations of Coulomb gases and related systems. See e.g. [Shc14, AHM⁺15, ZW06, BFG15] and the papers cited above.

3.1. Preliminaries. Let ε be a real number, we define the map $\mathbb{T}_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows:

$$(3.1) \quad \mathbb{T}_\varepsilon(x) := x + \frac{\varepsilon}{\omega N} \frac{\nabla \varphi(x)}{\mathbf{m}_0(x)}.$$

We recall that by assumption the support of φ is contained inside the support Σ of μ_0 , and that the density \mathbf{m}_0 is bounded below by a positive constant c_{\min} on Σ . Thus $\mathbb{T}_\varepsilon(x)$ is well-defined for all x and when $|\varepsilon|$

is smaller than $(C(\bar{\varphi}, \mu_0))^{-1} \omega N$, the map T_ε is a small (in C^1 norm) perturbation of the identity, hence a C^1 -diffeomorphism of \mathbb{R}^2 . We turn it into a change of variables on $(\mathbb{R}^2)^N$ by setting:

$$\vec{T}_\varepsilon(x_1, \dots, x_N) := (T_\varepsilon(x_1), \dots, T_\varepsilon(x_N)).$$

In this proof, we let μ_ε be the push-forward of μ_0 by the map T_ε . By the change of variable formula we can compute the density \mathbf{m}_ε of μ_ε as:

$$(3.2) \quad \mathbf{m}_\varepsilon = \mathbf{m}_0 - \operatorname{div} \left(\mathbf{m}_0 \frac{\varepsilon}{\omega N} \frac{\nabla \varphi}{\mathbf{m}_0} \right) + O_\star(\varepsilon^2) = \mathbf{m}_0 - \frac{\varepsilon}{\omega N} \Delta \varphi + O_\star(\varepsilon^2).$$

Let us write $\mathcal{F}_N(\omega)$ explicitly as³:

$$\mathcal{F}_N(\omega) = \frac{1}{Z_N^\beta} \int_{(\mathbb{R}^2)^N} \exp \left(-\beta F(Y_N, \mu_0) + \vec{\zeta}(Y_N) + i\omega \langle \varphi, \mathbf{Y}_N - N\mu_0 \rangle \right) dY_N,$$

and perform the change of variables $Y_N = \vec{T}_\varepsilon(X_N)$. We obtain:

$$(3.3) \quad \mathcal{F}_N(\omega) = \frac{1}{Z_N^\beta} \int_{(\mathbb{R}^2)^N} \exp \left(-\beta F \left(\vec{T}_\varepsilon(X_N), \mu_0 \right) + \vec{\zeta}(X_N) + i\omega \langle \varphi, \vec{T}_\varepsilon(X_N) - N\mu_0 \rangle \right) \left(\prod_{i=1}^N \det DT_\varepsilon(x_i) \right) dX_N.$$

The confining potential $\vec{\zeta}$ has not felt the effect of the change of variables because by assumption it vanishes on Σ and \vec{T}_ε is the identity outside Σ .

We now study the three components of the integrand in (3.3):

$$F \left(\vec{T}_\varepsilon(X_N), \mu_0 \right), \quad i\omega \langle \varphi, \vec{T}_\varepsilon(X_N) - N\mu_0 \rangle, \quad \prod_{i=1}^N \det DT_\varepsilon(x_i).$$

3.2. Effect of transport on the energy. We study the term $F \left(\vec{T}_\varepsilon(X_N), \mu_0 \right)$ in (3.3) and prove the following expansion in ε .

Claim 3.1. *With A_1 as in (A.5), we have:*

$$(3.4) \quad F \left(\vec{T}_\varepsilon(X_N), \mu_0 \right) = F(X_N, \mu_0) + \frac{\varepsilon}{N\omega} A_1 \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathbf{m}_0} \right) + \frac{2\pi\varepsilon}{\omega} \langle \varphi, \mathbf{M}_N \rangle + O_\star(\varepsilon^2).$$

The term A_1 in (3.4), whose definition is recalled in Section A.3, is the ‘‘anisotropy’’ term introduced in [LS18] and studied deeply in [Ser20]. We will rely on existing results (quoted in the Appendix) about A_1 and its behavior under a change of variables. This import of results motivated our assumptions in subsection 1.2.

Proof of Claim 3.1. Writing $\mu_0 = \mu_\varepsilon + (\mu_0 - \mu_\varepsilon)$ in the definition (1.3) of the energy yields:

$$(3.5) \quad F(\vec{T}_\varepsilon(X_N), \mu_0) = \text{I} + \text{II} + \text{III},$$

where the terms I, II, III are obtained by expanding the quadratic form associated to the energy:

$$\begin{aligned} \text{I} &:= F \left(\vec{T}_\varepsilon(X_N), T_\varepsilon \# \mu_0 \right), \quad \text{II} := \frac{1}{2} \iint -\log|x-y| (Nd\mu_\varepsilon - Nd\mu_0)^{\otimes 2}(x, y), \\ \text{III} &:= \iint -\log|x-y| \left(\vec{T}_\varepsilon(X_N) - Nd\mu_\varepsilon \right)(x) (Nd\mu_\varepsilon - Nd\mu_0)(y). \end{aligned}$$

- For I, using (A.6) of Lemma A.1 with $\mu = \mu_0$, $\psi = \frac{\nabla \varphi}{\mathbf{m}_0}$ and $t = \frac{\varepsilon}{N\omega}$, we obtain:

$$(3.6) \quad \text{I} := F \left(\vec{T}_\varepsilon(X_N), T_\varepsilon \# \mu_0 \right) = F(X_N, \mu_0) + \frac{\varepsilon}{N\omega} A_1 \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathbf{m}_0} \right) + O_\star(\varepsilon^2).$$

³Here we first choose Y_N to denote our ‘‘state of the system’’ variable, we will then perform a change of variables.

- For II, identity (3.2) shows that it is of second order in ε :

$$(3.7) \quad \text{II} = O_\star(\varepsilon^2).$$

- For III, using (3.2) again we find

$$\text{III} = \iint -\log|x-y| \left(-\frac{\varepsilon}{\omega} \Delta\varphi(y)\right) dy \left(\sum_{i=1}^N \delta_{T_\varepsilon(x_i)} - N(T_\varepsilon\#\mu_0)\right) (dx) + O_\star(\varepsilon^2).$$

The integral over y can be eliminated using the following identity, valid for f in $C_c^2(\mathbb{R}^2)$,

$$(3.8) \quad \int_{\mathbb{R}^2} -\log|x-y| \Delta f(y) dy = -2\pi f(x).$$

Changing variables by T_ε in the integral over x , we conclude that

$$\text{III} = 2\pi \frac{\varepsilon}{\omega} \int \varphi \circ T_\varepsilon(x) \left(\sum_{i=1}^N \delta_{x_i} - N\mu_0\right) (dx) = \frac{2\pi\varepsilon}{\omega} \langle \varphi \circ T_\varepsilon, \mathbf{M}_N \rangle.$$

Since φ is differentiable and $T_\varepsilon = \text{Id} + O_\star(\varepsilon)$ we have $\varphi \circ T_\varepsilon = \varphi + O_\star(\varepsilon)$ and thus

$$(3.9) \quad \text{III} = \frac{2\pi\varepsilon}{\omega} \langle \varphi, \mathbf{M}_N \rangle + O_\star(\varepsilon^2).$$

Combining (3.6), (3.7), (3.9) proves the claim. \square

3.3. Effect of transport on the fluctuations. We study the term $i\omega \langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle$ in (3.3) and prove the following expansion in ε :

Claim 3.2. *We have:*

$$(3.10) \quad i\omega \langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle = i\omega \langle \varphi, \mathbf{M}_N \rangle + i\varepsilon \int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx + \frac{i\varepsilon}{N} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2)$$

Proof of Claim 3.2. The following identity holds:

$$\langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle = \langle \varphi, \mathbf{X}_N - N\mu_0 \rangle + \sum_{i=1}^N (\varphi \circ T_\varepsilon(x_i) - \varphi(x_i)).$$

From the definition (3.1) of the map T_ε and Taylor's expansion we obtain that

$$\varphi \circ T_\varepsilon(x) = \varphi \left(x + \frac{\varepsilon}{N\omega} \frac{\nabla\varphi(x)}{\mathbf{m}_0(x)} \right) = \varphi(x) + \frac{\varepsilon}{N\omega} \nabla\varphi(x) \cdot \frac{\nabla\varphi(x)}{\mathbf{m}_0(x)} + O_\star(\varepsilon^2).$$

We can thus write $\langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - \mu_0 \rangle$ as

$$\langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle = \langle \varphi, \mathbf{M}_N \rangle + \frac{\varepsilon}{N\omega} \sum_{i=1}^N \frac{|\nabla\varphi(x_i)|^2}{\mathbf{m}_0(x_i)} + O_\star(\varepsilon^2).$$

We may compare the discrete sum to an integral against μ_0 by introducing a new fluctuation term:

$$\frac{\varepsilon}{N\omega} \sum_{i=1}^N \frac{|\nabla\varphi(x_i)|^2}{\mathbf{m}_0(x_i)} = \frac{\varepsilon}{\omega} \int_{\mathbb{R}^2} \frac{|\nabla\varphi(x)|^2}{\mathbf{m}_0(x)} \mathbf{m}_0(x) dx + \frac{\varepsilon}{N\omega} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle,$$

and we finally obtain:

$$i\omega \langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle = i\omega \langle \varphi, \mathbf{M}_N \rangle + i\varepsilon \int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx + \frac{i\varepsilon}{N} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2),$$

which proves the claim. \square

3.4. Computation of the Jacobian. We study the term $\prod_{i=1}^N \det DT_\varepsilon(x_i)$ in (3.3) and prove the following expansion in ε :

Claim 3.3. *We have:*

$$(3.11) \quad \prod_{i=1}^N \det DT_\varepsilon(x_i) = 1 + \frac{\varepsilon}{\omega N} \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx + \frac{\varepsilon}{\omega N} \left\langle \operatorname{div} \left(\frac{\nabla\varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2).$$

Proof of Claim 3.3. The proof follows a computation from [LS18]. We rewrite the left-hand side of (3.11) as $\exp\left(\sum_{i=1}^N \log \det DT_\varepsilon(x_i)\right)$, and compare the summand with an integral against μ_0 by introducing a new fluctuation term:

$$(3.12) \quad \sum_{i=1}^N \log \det DT_\varepsilon(x_i) = N \int_{\mathbb{R}^2} \log \det DT_\varepsilon(x) d\mu_0(x) + \langle \log \det DT_\varepsilon, \mathbf{M}_N \rangle.$$

By the change of variables formula, we have that $\det DT_\varepsilon(x) = \frac{\mathbf{m}_0(x)}{\mathbf{m}_\varepsilon \circ T_\varepsilon(x)}$ and therefore

$$\int_{\mathbb{R}^2} \log \det DT_\varepsilon(x) d\mu_0(x) = \int_{\mathbb{R}^2} \log \mathbf{m}_0(x) d\mu_0(x) - \int_{\mathbb{R}^2} \log \mathbf{m}_\varepsilon(x) d\mu_\varepsilon(x).$$

Using (3.2) and a first order Taylor expansion in the last display, we obtain that

$$(3.13) \quad \int_{\mathbb{R}^2} \log \det DT_\varepsilon(x) d\mu_0(x) = \frac{\varepsilon}{N\omega} \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx + O_\star(\varepsilon^2).$$

On the other hand, using a Taylor's expansion for \det around the identity matrix we have:

$$\log \det DT_\varepsilon(x) = \frac{\varepsilon}{N\omega} \operatorname{div} \left(\frac{\nabla\varphi}{\mathbf{m}_0} \right) (x) + O_\star(\varepsilon^2).$$

Combining the last display with (3.12) and (3.13), we obtain (3.11) and complete the proof of the claim. \square

3.5. Combining the effects of the transport. Using Claims 3.1, 3.2 and 3.3 in (3.3), we obtain:

$$\begin{aligned} \mathcal{F}_N(\omega) &= \frac{1}{Z_N^\beta} \int \exp \left(-\beta F(X_N, \mu_0) + \vec{\zeta}(X_N) + i\omega \langle \varphi, \mathbf{M}_N \rangle \right) \\ &\quad \times \left(1 - \frac{\beta\varepsilon}{N\omega} \mathbf{A}_1 \left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) - \frac{2\pi\beta\varepsilon}{\omega} \langle \varphi, \mathbf{M}_N \rangle + O_\star(\varepsilon^2) \right) \\ &\quad \times \left(1 + i\varepsilon \int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx + \frac{i\varepsilon}{N} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2) \right) \\ &\quad \times \left(1 + \frac{\varepsilon}{\omega} \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx + \frac{\varepsilon}{\omega N} \left\langle \operatorname{div} \left(\frac{\nabla\varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2) \right) dX_N. \end{aligned}$$

Since the left side in the last display is independent of ε , identifying the terms of order 1 in ε gives that:

$$\begin{aligned} \frac{1}{Z_N^\beta} \int \exp \left(-\beta F(X_N, \mu_0) + \vec{\zeta}(X_N) + i\omega \langle \varphi, \mathbf{M}_N \rangle \right) &\left(-\frac{\beta}{N\omega} \mathbf{A}_1 \left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) - \frac{2\pi\beta}{\omega} \langle \varphi, \mathbf{M}_N \rangle \right. \\ &+ i \int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx + \frac{i}{N} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle \\ &\left. + \frac{1}{\omega} \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx + \frac{1}{\omega N} \left\langle \operatorname{div} \left(\frac{\nabla\varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle \right) dX_N = 0, \end{aligned}$$

which can be seen as the expectation (under \mathbb{P}_N^β) of the quantity $\exp(i\omega \langle \varphi, \mathbf{M}_N \rangle)$ multiplied by some terms. This identity is analogous to e.g. [BBNY17, Prop 7.3], but this time with a complex-valued function f (here $i\varphi$). We now make the following observations:

1. The expectation of $-\frac{2\pi\beta}{\omega}\langle\varphi, \mathbf{M}_N\rangle$ yields a multiple of the derivative of $\mathcal{F}_N(\omega)$, indeed:

$$-\frac{2\pi\beta}{\omega}\mathbb{E}[\exp(i\omega\langle\varphi, \mathbf{M}_N\rangle)\langle\varphi, \mathbf{M}_N\rangle] = -\frac{2\pi\beta}{\omega}\frac{1}{i}\mathcal{F}'_N(\omega) = \frac{2i\pi\beta}{\omega}\mathcal{F}'_N(\omega).$$

We note in passing that the choice of the transport T_ε , see (3.1), was geared toward obtaining the derivative $\mathcal{F}'_N(\omega)$ in the last display. The fact that we are working in dimension $d = 2$ played a crucial role in the design of the transport, via the identity (3.8).

2. The term $i\int_{\mathbb{R}^2}|\nabla\varphi(x)|^2dx = i\|\varphi\|_{H^1}^2$ is a constant, and so is the term $\frac{1}{\omega}\int_{\mathbb{R}^2}\Delta\varphi(x)\log\mathbf{m}_0(x)dx$.

3. For the fluctuation terms, using the fact that ‘‘fluctuations are bounded in $L^1(\mathbb{P})$ ’’ as expressed in Lemma A.2, we can write:

$$(3.14) \quad \left|\mathbb{E}\left[\exp(i\omega\langle\varphi, \mathbf{M}_N\rangle)\frac{i}{N}\left\langle\frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N\right\rangle\right]\right| \leq \frac{1}{N}\mathbb{E}\left[\left|\left\langle\frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N\right\rangle\right|\right] \leq C(\overline{\varphi}, \beta, \mu_0)\frac{1}{N}\ell_N^{-2},$$

where $C(\overline{\varphi}, \beta, \mu_0)$ is some constant (depending on the test function $\overline{\varphi}$, on β and on μ_0). Indeed, since φ lives at scale ℓ_N , the gradient $\nabla\varphi$ is of order ℓ_N , the function $\frac{|\nabla\varphi|^2}{\mathbf{m}_0}$ is bounded by a constant (depending on \mathbf{m}_0) times ℓ_N^{-2} and every additional derivative loses a factor ℓ_N^{-1} . The *a priori* bound of Lemma A.2 states that the fluctuation of $\langle\frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N\rangle$ is of order ℓ_N^{-2} . Similarly, we have:

$$(3.15) \quad \left|\mathbb{E}\left[\exp(i\omega\langle\varphi, \mathbf{M}_N\rangle)\frac{1}{\omega N}\left\langle\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right), \mathbf{M}_N\right\rangle\right]\right| \leq \frac{1}{\omega N}\mathbb{E}\left[\left|\left\langle\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right), \mathbf{M}_N\right\rangle\right|\right] \leq C(\overline{\varphi}, \beta, \mu_0)\frac{1}{\omega N}\ell_N^{-2}.$$

Since we are interested in the *decay* of $\mathcal{F}_N(\omega)$ for $|\omega| > 1$, the bound (3.14) is the dominant one. We note in passing that in order to apply Lemma A.2, we need both $\frac{|\nabla\varphi|^2}{\mathbf{m}_0}$ and $\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right)$ to be of class C^3 , which is guaranteed when φ itself is of class C^5 and \mathbf{m}_0 of class C^3 , hence our regularity assumptions - which we have not tried to optimize.

4. Concerning the term $-\frac{\beta}{N\omega}\mathbf{A}_1\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right)$, in (A.5) of Section A.3 we decompose \mathbf{A}_1 into:

$$\mathbf{A}_1\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right) = \operatorname{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right) + \frac{1}{4}\sum_{i=1}^N\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right)(x_i).$$

The sum in the right-hand side can be compared to a continuous integral, and we obtain, after an integration by parts, that

$$\sum_{i=1}^N\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right)(x_i) = N\int_{\mathbb{R}^2}\Delta\varphi(x)\log\mathbf{m}_0(x)dx + \left\langle\operatorname{div}\left(\frac{\nabla\varphi}{\mathbf{m}_0}\right), \mathbf{M}_N\right\rangle,$$

with the same fluctuation term as the one obtained in Claim 3.3. (The first term in the right hand side of the last display contributed to the mean of Z in Theorem 1.) To summarize, we have:

$$-\frac{\beta}{N\omega}\mathbf{A}_1\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right) = -\frac{\beta}{N\omega}\operatorname{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right) - \frac{\beta}{4\omega}\int_{\mathbb{R}^2}\Delta\varphi(x)\log\mathbf{m}_0(x)dx + C(\overline{\varphi}, \beta, \mu_0)\frac{1}{\omega N}\ell_N^{-2}.$$

We may thus write, in conclusion:

$$(3.16) \quad \left(i\|\varphi\|_{H^1}^2 + \left(\frac{1}{\omega} - \frac{\beta}{4\omega}\right)\int_{\mathbb{R}^2}\Delta\varphi(x)\log\mathbf{m}_0(x)dx\right)\mathcal{F}_N(\omega) + \frac{2i\pi\beta}{\omega}\mathcal{F}'_N(\omega) \\ = \frac{\beta}{N\omega}\mathcal{G}(\omega) + C(\overline{\varphi}, \beta, \mu_0)\frac{1}{N}\ell_N^{-2},$$

where $\mathcal{G}(\omega)$ is defined as $\mathbb{E}\left[e^{i\omega\langle\varphi, \mathbf{M}_N\rangle}\operatorname{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right)\right]$, i.e.⁴:

$$(3.17) \quad \mathcal{G}(\omega) := \frac{1}{Z_N^\beta}\int_{(\mathbb{R}^2)^N}\exp\left(-\beta\mathbf{F}(Y_N, \mu_0) + \vec{\zeta}(Y_N) + i\omega\langle\varphi, \mathbf{Y}_N - N\mu_0\rangle\right)\operatorname{Ani}\left(Y_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0}\right)dY_N.$$

⁴Here we simply switch back to Y_N as the variable in the integral.

Remark 3.4. If the right-hand side of (3.16) were zero, we would obtain a simple ODE for $\mathcal{F}_N(\omega)$ of the form

$$\mathcal{F}'_N(\omega) = \left(-\frac{\omega \|\varphi\|_{H^1}^2}{2\pi\beta} + \frac{i}{2\pi} \left(\frac{1}{\beta} - \frac{1}{4} \right) \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx \right) \mathcal{F}_N(\omega),$$

whose solution (since $\mathcal{F}_N(0) = 1$) would indeed coincide with the Fourier transform of the limiting distribution for fluctuations of linear statistics as obtained in [LS18, BBNY19].

Using an *a priori* bound on Ani, we can see that $|\mathcal{G}(\omega)|$ is of order N . This is not enough for our purposes, which is why we need to go further and study the decay of $\mathcal{G}(\omega)$ itself. This is established in the next section.

3.6. Additional decay estimate.

Lemma 3.5. *There exists a constant $C(\bar{\varphi}, \mu_0, \beta)$ so that, for $|\omega| > 1$,*

$$(3.18) \quad |\mathcal{G}(\omega)| \leq C(\bar{\varphi}, \mu_0, \beta) \left(\frac{N}{\omega} + \ell_N^{-2} \right).$$

Proof of Lemma 3.5. We use the same strategy as above, and perform the same change of variables, namely $Y_N = \vec{T}_\varepsilon(X_N)$ in the right-hand side of (3.17). We get:

$$(3.19) \quad \mathcal{G}(\omega) = \frac{1}{Z_N^\beta} \int_{(\mathbb{R}^2)^N} \exp \left(-\beta F \left(\vec{T}_\varepsilon(X_N), \mu_0 \right) + \vec{\zeta}(X_N) + i\omega \langle \varphi, \vec{T}_\varepsilon(\mathbf{X}_N) - N\mu_0 \rangle \right) \\ \times \text{Ani} \left(\vec{T}_\varepsilon(X_N), \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) \left(\prod_{i=1}^N \det D\mathcal{T}_\varepsilon(x_i) \right) dX_N.$$

The effect of this change of variables has been previously computed for all the terms except for the anisotropy itself.

Claim 3.6 (Effect of the change of variables on the anisotropy). *When $Y_N = \vec{T}_\varepsilon(X_N)$, we have:*

$$(3.20) \quad \left| \text{Ani} \left(Y_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) - \text{Ani} \left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) \right| \leq C(\bar{\varphi}, \beta, \mu_0) \frac{\varepsilon}{\omega} \frac{\ell_N^{-3}}{N^{1/2}} (\text{EnerPts} + N\ell_N^2) + O_\star(\varepsilon^2).$$

The quantity EnerPts will be introduced in Section A.2 as the sum of the electric field Ele and the number of points Points on the support of φ , but its precise definition is irrelevant for the moment. It suffices to keep in mind that EnerPts is of order $N\ell_N^2$ in a sense made precise below. We postpone the proof of Claim 3.6 to Section A.5. We then argue as in Section 3.5. We obtain:

$$\mathcal{G}(\omega) = \frac{1}{Z_N^\beta} \int \exp \left(-\beta F(X_N, \mu_0) + \vec{\zeta}(X_N) + i\omega \langle \varphi, \mathbf{M}_N \rangle \right) \\ \times \left(1 - \frac{\beta\varepsilon}{N\omega} \mathbf{A}_1 \left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) - \frac{2\pi\beta\varepsilon}{\omega} \langle \varphi, \mathbf{M}_N \rangle + O_\star(\varepsilon^2) \right) \\ \times \left(1 + i\varepsilon \int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx + \frac{i\varepsilon}{N} \left\langle \frac{|\nabla\varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2) \right) \\ \times \left(1 + \frac{\varepsilon}{\omega} \int_{\mathbb{R}^2} \Delta\varphi(x) \log \mathbf{m}_0(x) dx + \frac{\varepsilon}{\omega N} \left\langle \text{div} \left(\frac{\nabla\varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle + O_\star(\varepsilon^2) \right) \\ \times \left(\text{Ani} \left(X_N, \mu_0, \frac{\nabla\varphi}{\mathbf{m}_0} \right) + C(\bar{\varphi}, \beta, \mu_0) \frac{\varepsilon}{\omega} \frac{\ell_N^{-3}}{N^{1/2}} (\text{EnerPts} + N\ell_N^2) + O_\star(\varepsilon^2) \right) dX_N.$$

Identifying again the terms of order 1 in ε , we find that

$$\begin{aligned} \frac{1}{Z_N^\beta} \int \exp \left(-\beta F(X_N, \mu_0) + \zeta(X_N) + i\omega \langle \varphi, \mathbf{M}_N \rangle \right) & \left[\text{Ani} \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathbf{m}_0} \right) \right. \\ & \times \left[-\frac{\beta}{N\omega} \mathbf{A}_1 \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathbf{m}_0} \right) - \frac{2\pi\beta}{\omega} \langle \varphi, \mathbf{M}_N \rangle \right. \\ & \quad \left. + i \int_{\mathbb{R}^2} |\nabla \varphi(x)|^2 dx + \frac{i}{N} \left\langle \frac{|\nabla \varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle \right. \\ & \left. + \frac{1}{\omega} \int_{\mathbb{R}^2} \Delta \varphi(x) \log \mathbf{m}_0(x) dx + \frac{1}{\omega N} \left\langle \text{div} \left(\frac{\nabla \varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle \right] \\ & \left. + C(\overline{\varphi}, \beta, \mu_0) \frac{1}{\omega} \frac{\ell_N^{-3}}{N^{1/2}} (\text{EnerPts} + N\ell_N^2) \right] dX_N = 0. \end{aligned}$$

We use the following a priori bounds, valid in $L^2(\mathbb{P}_N^\beta)$ (see Section A.6):

$$\begin{aligned} \left| \text{Ani} \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathbf{m}_0} \right) \right| & \leq C(\overline{\varphi}, \mu_0, \beta) N \quad (\text{Lemma A.3}) \\ |\langle \varphi, \mathbf{M}_N \rangle| & \leq C(\overline{\varphi}, \mu_0, \beta) \quad (\text{Lemma A.2}) \\ \left| \left\langle \frac{|\nabla \varphi|^2}{\mathbf{m}_0}, \mathbf{M}_N \right\rangle \right| & \leq C(\overline{\varphi}, \mu_0, \beta) \ell_N^{-2} \quad (\text{Lemma A.2}) \\ \left| \left\langle \text{div} \left(\frac{\nabla \varphi}{\mathbf{m}_0} \right), \mathbf{M}_N \right\rangle \right| & \leq C(\overline{\varphi}, \mu_0, \beta) \ell_N^{-2} \quad (\text{Lemma A.2}) \\ \text{EnerPts} & \leq C(\mu_0, \beta) N \ell_N^2 \quad (\text{Lemma A.3}) \end{aligned}$$

and, keeping only the dominant terms, we obtain that:

$$|\mathcal{G}(\omega)| \leq C(\overline{\varphi}, \mu_0, \beta) \left(\frac{N}{\omega} + \ell_N^{-2} \right),$$

which concludes the proof of Lemma 3.5. \square

Substituting the estimate from Lemma 3.5 into (3.16), we obtain a differential equation for \mathcal{F}_N of the form:

$$(iA + B)\mathcal{F}_N(\omega) + \frac{iC}{\omega} \mathcal{F}'_N(\omega) = \frac{D_1}{\omega^2} + \frac{D_2 \ell_N^{-2}}{N},$$

where A, C are positive constants (depending on $\beta, \overline{\varphi}$), D_1, D_2 are bounded (depending on $\beta, \overline{\varphi}, \mu_0$) and B is real (depending on $\beta, \overline{\varphi}, \mu_0$). Solving this equation yields:

$$|\mathcal{F}_N(\omega)| \leq C(\overline{\varphi}, \mu_0, \beta) \left(\frac{1}{\omega^2} + \frac{\ell_N^{-2}}{N} \right),$$

which concludes the proof of Proposition 1.5.

APPENDIX A. ELECTRIC FIELDS AND ANISOTROPY

Let μ be a probability measure with a bounded density and X_N a N -tuple of points in \mathbb{R}^2 .

In this section, we denote by $|f|_k$ the norm of the k -th derivative of f (with $|f|_0$ denoting the supremum norm of f). We recall that K denotes a universal constant, dependent only on μ_0, β and in particular independent of N, ω, ψ , whose value may change from line to line. We also recall the definition of $O_\star(t)$, see subsection 1.4.

A.1. The electric field formalism. We define the *electric field* ∇h^μ associated with μ and X_N as

$$\nabla h^\mu(z) = \left(\sum_{i=1}^N -\nabla \log |z - x_i| - N \int_{\mathbb{R}^2} -\nabla \log |z - x| d\mu(x) \right).$$

This vector field is in L^p_{loc} for every $p < 2$ but fails to be in L^2 because of its singularity around each point charge. Let $\vec{\eta} = (\eta_1, \dots, \eta_N)$ be a vector of positive numbers, we define the *truncated electric field* $\nabla h_{\vec{\eta}}^\mu$ as:

$$(A.1) \quad \nabla h_{\vec{\eta}}^\mu(z) := \nabla h^\mu(z) - \sum_{i=1}^N \nabla f_{\eta_i}(z - x_i),$$

where $f_{\eta_i}(z) := (\log \eta_i - \log |z|)_+$. We refer to e.g. [LS18, Section 2.2] and the references therein for a more detailed discussion of this procedure. It has the effect of canceling the contribution to the electric field coming from the particle x_i in the disk of radius η_i around x_i , thus effectively truncating the singularity around each charge.

A particularly convenient choice of the truncation parameter is the nearest-neighbor distances. For any $i = 1, \dots, N$ we let r_i be the following quantity:

$$(A.2) \quad r_i := \frac{1}{4} \min \left(\frac{1}{\sqrt{N}}, \min_{j \neq i} |x_i - x_j| \right),$$

and we let $\vec{r} := (r_1, \dots, r_N)$.

A.2. Electric energy (and number of points). For $\Omega \subset \mathbb{R}^2$,⁵ we introduce the notation $\text{Ele}(\Omega)$ to denote the “electric field energy” in Ω

$$(A.3) \quad \text{Ele}(\Omega) := \int_{\Omega} |\mathbf{E}_{\vec{r}}|^2.$$

We also let Points be the number of points of the configuration X_N in (an $N^{-1/2}$ -neighborhood of) Ω .

When working with a test function φ compactly supported in a domain of characteristic size ℓ_N , we write EnerPts for the sum of the electric energy and the number of points on the support of φ . This quantity is typically of order $N\ell_N^2$, see Lemma A.3.

A.3. The anisotropy. The “anisotropy” term was introduced in [LS18] as the first order effect of a transport map (applied both to the particles and the background measure) to the interaction energy. We will also quote results from [Ser20, Section 4]), where this quantity is studied extensively.

When μ has a bounded density and ψ is a C^1 vector field, we define $\text{Ani}(X_N, \mu, \psi)$ as:

$$(A.4) \quad \text{Ani}(X_N, \mu, \psi) := \frac{1}{2} \lim_{\eta \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left\langle \nabla h_{\vec{\eta}}^\mu, (2D\psi - (\text{div}\psi)\text{Id}) \nabla h_{\vec{\eta}}^\mu \right\rangle,$$

where $\nabla h_{\vec{\eta}}^\mu$ is the truncated electric field defined above, with truncation $\vec{\eta} := (\eta, \dots, \eta)$.

Note that in [Ser20] however, the “anisotropy” is rather defined as:

$$(A.5) \quad \mathbf{A}_1(X_N, \mu, \psi) := \text{Ani}(X_N, \mu, \psi) + \frac{1}{4} \sum_{i=1}^N \text{div}\psi(x_i).$$

One advantage is that \mathbf{A}_1 admits an alternative non-asymptotic expression (no need to take a limit $\eta \rightarrow 0$ as in (A.4)), see [Ser20, (4.11)]. Generally, \mathbf{A}_1 is the natural quantity to consider in several questions. However, for our purposes, we sometimes prefer to consider the two terms in the right-hand side of (A.5) separately.

⁵We note that in the application in Section 3 we always take Ω to be the support of φ

A.4. Effect of a transport map. The following results are direct consequences of [Ser20, Proposition 4.3], upon noting that $\Xi(t)$ of the latter satisfies that $F((\text{Id} + t\psi)(X_N), (\text{Id} + t\psi)\#\mu) - \Xi(t)$ is independent of t , substituting [Ser20, (4.8)] into [Ser20, (4.9),(4.10)], and using [Ser20, (4.11)].

Lemma A.1 (Effect of a transport map). *Let ψ be in $C^2(\mathbb{R}^2, \mathbb{R}^2)$ with compact support. We have, as $t \rightarrow 0$:*

$$(A.6) \quad F((\text{Id} + t\psi)(X_N), (\text{Id} + t\psi)\#\mu) - F(X_N, \mu) = tA_1(X_N, \mu, \psi) + O_\star(t^2).$$

Further,

$$\begin{aligned} & |A_1((\text{Id} + t\psi)(X_N), (\text{Id} + t\psi)\#\mu, \psi) - A_1(X_N, \mu, \psi)| \\ & \leq Kt \left(|\psi|_1^2 + |\psi|_0|\psi|_2 + |\psi|_1|\psi|_2N^{-1/2} \right) \left| F(X_N, \mu) + \frac{1}{2}\#I_N \log N \right| + O_\star(t^2), \end{aligned}$$

where $\#I_N$ denotes the number of points of X_N in (a $N^{-1/2}$ -neighborhood of) the support of ψ .

We can easily translate both results in terms of Ani (instead of A_1) by writing

$$(A.7) \quad F((\text{Id} + t\psi)(X_N), (\text{Id} + t\psi)\#\mu) - F(X_N, \mu) = t\text{Ani}(X_N, \mu, \psi) + t \sum_{i=1}^N \text{div}\psi(x_i) + O_\star(t^2).$$

and

$$(A.8) \quad \begin{aligned} & |\text{Ani}((\text{Id} + t\psi)(X_N), \psi, (\text{Id} + t\psi)\#\mu) - \text{Ani}(X_N, \psi, \mu)| \\ & \leq Kt \left(\left(|\psi|_1^2 + |\psi|_0|\psi|_2 + |\psi|_1|\psi|_2N^{-1/2} \right) \left| F(X_N, \mu) + \frac{1}{2}\#I_N \log N \right| + \#I_N |\psi|_2 |\psi|_0 \right). \end{aligned}$$

(When applying [Ser20], one may wonder whether one needs to consider $I_N(X_N)$ or $I_N((\text{Id} + t\psi)(X_N))$. However, since we only care about the asymptotics as $t \rightarrow 0$, and since all points x_i are translated at most by $t|\psi(x)|$ and are only translated within the support of ψ , there is no difference which of the two we consider.) In our application, $|\psi|_0$ is of order ℓ_N^{-1} and each derivative loses a factor $\ell_N^{-1} \ll N^{1/2}$, so in the second comparison, the factor $|\psi|_1^2 + |\psi|_0|\psi|_2 + |\psi|_1|\psi|_2N^{-1/2}$ is of order ℓ_N^{-2} .

A.5. Proof of Claim 3.6. .

Proof of Claim 3.6. We again denote by μ_ε the push-forward of μ_0 by T_ε , and we let $Y_N = \vec{T}_\varepsilon(X_N)$. To prove the claim, we need to compare $\text{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$ and $\text{Ani}\left(Y_N, \mu_0, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$.

We first compare $\text{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$ with $\text{Ani}\left(Y_N, \mu_\varepsilon, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$. Applying (A.8) of the second part of Lemma A.1, with $\mu = \mu_0$, $\psi = \frac{\nabla\varphi}{\mathfrak{m}_0}$ and $t = \frac{\varepsilon}{N\omega}$, and using the scaling properties of φ and its derivatives we get:

$$(A.9) \quad \left| \text{Ani}\left(Y_N, \mu_\varepsilon, \frac{\nabla\varphi}{\mathfrak{m}_0}\right) - \text{Ani}\left(X_N, \mu_0, \frac{\nabla\varphi}{\mathfrak{m}_0}\right) \right| \leq C(\overline{\varphi}, \beta, \mu_0) \frac{\varepsilon \ell_N^{-4}}{N\omega} \text{EnerPts},$$

where we use the notation EnerPts as introduced in Section A.2. It remains to compare $\text{Ani}\left(Y_N, \mu_\varepsilon, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$ (that we will denote here by $\text{Ani}^{(\varepsilon)}$ for brevity) and $\text{Ani}\left(Y_N, \mu_0, \frac{\nabla\varphi}{\mathfrak{m}_0}\right)$ (denoted below by $\text{Ani}^{(0)}$). The points stay the same but the reference measure changes. It will be convenient to use some notation:

- Let $E^{(0,Y)}$ denote the electric field corresponding to the charges in Y_N and the background of density \mathfrak{m}_0 , namely

$$E^{(0,Y)}(z) := \int_{\mathbb{R}^2} -\nabla \log |z - x| (\mathbf{Y}_N - N\mu_0) (dx).$$

- Let $E^{(\varepsilon,Y)}$ denote the electric field corresponding to the charges in Y_N and the modified background μ_ε , namely:

$$E^{(\varepsilon,Y)}(z) := \int_{\mathbb{R}^2} -\nabla \log |z - x| (\mathbf{Y}_N - N\mu_\varepsilon) (dx),$$

- We let $E_{\bar{\eta}}^{(0,Y)}$, $E_{\bar{\eta}}^{(\varepsilon,Y)}$ be the corresponding truncated fields.

Using the expansion of μ_ε stated in (3.2), the difference between $E^{(0,Y)}$ and $E^{(\varepsilon,Y)}$ is fairly easy to compute. We get:

$$(A.10) \quad E_{\bar{\eta}}^{(0,Y)}(z) - E_{\bar{\eta}}^{(\varepsilon,Y)}(z) = E^{(0,Y)}(z) - E^{(\varepsilon,Y)}(z) = 2\pi \frac{\varepsilon}{\omega} \nabla \varphi(z) + O_\star(\varepsilon^2),$$

where we have used again the identity (3.8).

Going back to the definition (A.4) of Ani we see that (with $\psi = \frac{\nabla \varphi}{\mathfrak{m}_0}$ for brevity)

$$\begin{aligned} & 2 \left(\text{Ani}^{(\varepsilon)} - \text{Ani}^{(0)} \right) \\ &= \lim_{\eta \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left\langle E_{\bar{\eta}}^{(0,Y)} (2D\psi - (\text{div}\psi) \text{Id}) E_{\bar{\eta}}^{(\varepsilon,Y)} - E_{\bar{\eta}}^{(0,Y)} \right\rangle + \left\langle E_{\bar{\eta}}^{(\varepsilon,Y)} - E_{\bar{\eta}}^{(0,Y)}, (2D\psi - (\text{div}\psi) \text{Id}) E_{\bar{\eta}}^{(\varepsilon,Y)} \right\rangle. \end{aligned}$$

Both terms are bounded similarly. Hölder's inequality yields:

$$\begin{aligned} & \int_{\mathbb{R}^2} \left\langle E_{\bar{\eta}}^{(0,Y)} \left(2D \frac{\nabla \varphi}{\mathfrak{m}_0} - \left(\text{div} \frac{\nabla \varphi}{\mathfrak{m}_0} \right) \text{Id} \right) E_{\bar{\eta}}^{(\varepsilon,Y)} - E_{\bar{\eta}}^{(0,Y)} \right\rangle \\ & \leq C(\mu_0) \|E_{\bar{\eta}}^{(0,Y)}\|_{L^1(\text{supp}\varphi)} |\varphi|_2 \|E^{(\varepsilon,Y)} - E^{(0,Y)}\|_{L^\infty}. \end{aligned}$$

Using (A.10) and the scaling properties of φ , which is supported on a disk of radius ℓ_N , we get:

$$\left| \text{Ani}^{(\varepsilon)} - \text{Ani}^{(0)} \right| \leq C(\bar{\varphi}, \mu_0) \frac{\varepsilon}{\omega} \ell_N^{-3} \|E^{(0,Y)}\|_{L^1(\text{supp}\varphi)} + O_\star(\varepsilon^2)$$

Note that the L^1 norm of $E^{(0,Y)}$ is indeed finite. We can compare it to the electric energy by writing

$$(A.11) \quad \|E^{(0,Y)}\|_{L^1(\text{supp}\varphi)} \leq \|E_{\bar{r}}^{(0,Y)}\|_{L^1(\text{supp}\varphi)} + \sum_{x_i \in \text{supp}\varphi} \|\nabla f_{r(x_i)}\|_{L^1} \\ \leq \|E_{\bar{r}}^{(0,Y)}\|_{L^2(\text{supp}\varphi)} (\ell_N^2)^{1/2} + \text{Points}(\text{supp}\varphi) N^{-1/2},$$

up to some universal multiplicative constant. In (A.11) we first have re-introduced a truncation (using the nearest-neighbor distance r defined in Section A.1) and then used Hölder's inequality. The contribution of the truncation terms $\nabla f_{r(x_i)}$ can be bounded explicitly, using the fact that $r(x_i)$ is, by definition, always smaller than $N^{-1/2}$, and the elementary bound (see [Ser20, Sec. 3.3]): $\int_{\mathbb{R}^2} |\nabla f_{(r)}| \leq C\eta$.

We can thus write:

$$\begin{aligned} & \left| \text{Ani} \left(Y_N, \mu_\varepsilon, \frac{\nabla \varphi}{\mathfrak{m}_0} \right) - \text{Ani} \left(Y_N, \mu_0, \frac{\nabla \varphi}{\mathfrak{m}_0} \right) \right| \\ & \leq C(\bar{\varphi}, \mu_0) \frac{\varepsilon}{\omega} \ell_N^{-3} \left(\|E_{\bar{r}}^{(0,Y)}\|_{L^2(\text{supp}\varphi)} (\ell_N^2)^{1/2} + \text{Points}(\text{supp}\varphi) N^{-1/2} \right) + O_\star(\varepsilon^2). \end{aligned}$$

The L^2 norm of the truncated electric field corresponds to the square root of the electric energy as defined in (A.3), so we write

$$\|E_{\bar{r}}^{(0,Y)}\|_{L^2(\text{supp}\varphi)} (\ell_N^2)^{1/2} = \frac{\sqrt{\text{Ele}(\text{supp}(\varphi))}}{N^{1/4}} (\ell_N N^{1/4}) \leq \frac{\text{Ele}(\text{supp}(\varphi))}{N^{1/2}} + \ell_N^2 N^{1/2},$$

where we expect both terms to be of the same size. Combining with (A.9) and keeping only the dominant terms, we obtain:

$$\left| \text{Ani} \left(Y_N, \mu_0, \frac{\nabla \varphi}{\mathfrak{m}_0} \right) - \text{Ani} \left(X_N, \mu_0, \frac{\nabla \varphi}{\mathfrak{m}_0} \right) \right| \leq C(\bar{\varphi}, \beta, \mu_0) \frac{\varepsilon}{\omega} \frac{\ell_N^{-3}}{N^{1/2}} (\text{EnerPts} + N \ell_N^2) + O_\star(\varepsilon^2)$$

which proves the claim. \square

A.6. A priori bounds.

Lemma A.2 (Fluctuations are bounded in L^1, L^2). *Assume $\{\xi_N\}_N$ is a sequence of test functions supported on a disk of radius ℓ_N , and such that $|\xi_N|_k \leq M\ell_N^{-k}$ for $k \leq 3$, with a uniform constant $M > 1$. Then the sequence of random variables $\{\langle \xi_N, \mathbf{M}_N \rangle\}_N$ is bounded in $L^1(\mathbb{P}_N^\beta), L^2(\mathbb{P}_N^\beta)$ by some constant (depending only on β, μ_0) times M .*

Proof. This is a consequence of [Ser20, Corollary 2.1], which proves in fact boundedness of the *exponential moments*, with an explicit bound in terms of the constant M . In particular, taking the parameter $\tau = \frac{1}{CM}$ in [Ser20, equation (2.21)] and applying Jensen's inequality yields a control by (some constant times) M in L^1 and L^2 norm. \square

Lemma A.3 (Bound on the electric energy, the number of points, the anisotropy). *Let Ω be a disk of radius r contained in the interior of Σ (the support of μ_0). Then*

- (1) *Ele(Ω) is bounded in $L^1(\mathbb{P}_N^\beta), L^2(\mathbb{P}_N^\beta)$ by some constant (depending only on β, μ_0) times Nr^2 .*
- (2) *Points(Ω) is bounded in $L^1(\mathbb{P}_N^\beta), L^2(\mathbb{P}_N^\beta)$ by some constant (depending only on β, μ_0) times Nr^2 .*
- (3) *If ψ is C^1 and supported on Ω , then $\text{Ani}(X_N, \mu_0, \psi)$ is bounded in $L^1(\mathbb{P}_N^\beta), L^2(\mathbb{P}_N^\beta)$ by some constant (depending only on β, μ_0) times $|\psi|_1 Nr^2$.*

Moreover the constants are uniform for disks at a given distance of the boundary $\partial\Sigma$.

Proof. The first two points follow from the *local laws* for two-dimensional Coulomb gases, obtained in [Leb17, BBNY17] and refined in [AS19], see also [Ser20, Proposition 3.7] for a summary. The bound on the anisotropy can be inferred from combining [Ser20, Proposition 4.3] and the aforementioned controls on both the electric energy and the number of points. \square

REFERENCES

- [AHM⁺15] Yacin Ameur, Haakan Hedenmalm, Nikolai Makarov, et al. Random normal matrices and Ward identities. *The Annals of Probability*, 43(3):1157–1201, 2015.
- [AS19] S. Armstrong and S. Serfaty. Local laws and rigidity for Coulomb gases at any temperature. *arXiv preprint arXiv:1906.09848*, 2019.
- [BBNY17] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. Local density for two-dimensional one-component plasma. *Communications in Mathematical Physics*, 356(1):189–230, 2017.
- [BBNY19] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. *Advances in Theoretical and Mathematical Physics*, 23(4), 2019.
- [BEYY16] Paul Bourgade, Laszlo Erdős, Horng-Tzer Yau, and Jun Yin. Fixed energy universality for generalized Wigner matrices. *Communications on Pure and Applied Mathematics*, 69(10):1815–1881, 2016.
- [BFG15] Florent Bekerman, Alessio Figalli, and Alice Guionnet. Transport maps for β -matrix models and universality. *Communications in mathematical physics*, 338(2):589–619, 2015.
- [BG13] G. Borot and A. Guionnet. Asymptotic expansion of β matrix models in the one-cut regime. *Communications in Mathematical Physics*, 317(2):447–483, 2013.
- [BLS18] F. Bekerman, T. Leblé, and S Serfaty. CLT for fluctuations of β -ensembles with general potential. *Electronic Journal of Probability*, 23, 2018.
- [JL20] Kurt Johansson and Gaultier Lambert. Multivariate normal approximation for traces of random unitary matrices. *arXiv preprint arXiv:2002.01879*, 2020.
- [Joh97] Kurt Johansson. On random matrices from the compact classical groups. *Annals of mathematics*, pages 519–545, 1997.
- [Leb17] T. Leblé. Local microscopic behavior for 2D Coulomb gases. *Probability Theory and Related Fields*, 169(3-4):931–976, 2017.
- [LS18] T. Leblé and S. Serfaty. Fluctuations of two dimensional Coulomb gases. *Geom. Funct. Anal.*, 28(2):443–508, 2018.
- [Pas06] L. Pastur. Limiting laws of linear eigenvalue statistics for Hermitian matrix models. *J. Math. Phys.*, 47(10):103303, 22, 2006.
- [Ser20] S. Serfaty. Gaussian fluctuations and free energy expansion for 2D and 3D Coulomb gases at any temperature. *arXiv preprint arXiv:2003.11704*, 2020.
- [Shc14] M. Shcherbina. Change of variables as a method to study general β -models: Bulk universality. *Journal of Mathematical Physics*, 55(4):043504, apr 2014.
- [SS18] S. Serfaty and J. Serra. Quantitative stability of the free boundary in the obstacle problem. *Analysis & PDE*, 11(7):1803–1839, 2018.

- [ZW06] A Zabrodin and P Wiegmann. Large- N expansion for the 2D Dyson gas. *Journal of Physics A: Mathematical and General*, 39(28):8933, 2006.

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