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## Chapter 1

# Asymptotic comparison of the strain-gradient and micromorphic models when loading forces are widely spread.

Pierre Seppecher and Lukáš Jakabčín

**Abstract** In this paper we reconcile different homogenization results which describe the effective behavior of a heterogeneous material either by a strain-gradient model either by a micromorphic one. Indeed we prove that the solutions of both models are asymptotically very close when considering a loading with increasing wavelength. This result is obtained using the Fourier analysis on the tensor spaces and applies to a large class of micromorphic models. However, we provide an example of a micromorphic model that does not belong to this class and thus cannot be approximated by a strain-gradient model.

**Keywords:** Continuum mechanics · Strain-gradient · Micromorphic model

### 1.1 Introduction

The theory of elasticity is well founded since the work of Cauchy and generally gives satisfactory descriptions of the displacement field  $u$  of a solid submitted to an external load  $f$ . Let us recall that the deformation of the solid is, in the linear formulation, measured by the strain tensor  $e(u) := (\nabla u + {}^t\nabla u)/2$ . The behavior of the solid is described by a stiffness tensor  $C$ . The elastic energy of the solid contained in a domain  $\Omega$  is

$$\mathcal{E}^e(u) := \int_{\Omega} \frac{1}{2} e(u) : C : e(u) dx.$$

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At equilibrium, the displacement field  $u^e$  minimizes the total energy  $\mathcal{E}^e(u) - \int_{\Omega} f \cdot u \, dx$ . The existence of a unique solution of this minimization problem is ensured if suitable boundary conditions are imposed. Here, in order to compare our results with the literature (see for instance Smyshlyaev and Cherednichenko (2000)), we assume that  $f$  and  $u$  are  $[-L, L]^3$ -periodic function with vanishing mean value on any period  $\Omega$  ( $\int_{\Omega} u \, dx = 0$ ). Hence  $u^e$  is the unique  $[-L, L]^3$ -periodic function in  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$  with vanishing mean value and which satisfies, in the sense of distributions on  $\mathbb{R}^3$ , the Euler-Lagrange equation

$$\operatorname{div}(C : e(u^e)) + f = 0. \quad (1.1)$$

However Cauchy theory must sometimes be generalized. Indeed, when one focuses on small samples, scale effects are observed which cannot be explained by this theory (cf. Lam et al (2003)). Two main generalizations have been considered in order to encompass this difficulty.

- The first one, called “strain-gradient theory” consists in adding in the elastic energy a quadratic term depending on the gradient  $\nabla e(u)$  of the strain tensor (see among many others : Mindlin and Tiersten (1962); Mindlin (1965); Germain (2020); Casal (1972)).

$$\mathcal{E}^s(u) := \int_{\Omega} \left( \frac{1}{2} \nabla e(u) : \tilde{D} : \nabla e(u) + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

This model can alternatively be called “second-gradient model” as it is well known that any quadratic form  $\nabla e(u) : \tilde{D} : \nabla e(u)$  of  $\nabla e(u)$  is a quadratic form  $\nabla \nabla u : D : \nabla \nabla u$  of the second gradient of the displacement field and reciprocally. Hence

$$\mathcal{E}^s(u) := \int_{\Omega} \left( \frac{1}{2} \nabla \nabla u : D : \nabla \nabla u + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

The equilibrium  $u^s$  is the unique  $[-L, L]^3$ -periodic function in  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$  with zero mean value and which satisfies, in the sense of distributions on  $\mathbb{R}^3$ , the Euler-Lagrange equation

$$\operatorname{div} \left( -\operatorname{div}(D : \nabla \nabla u^s) + C : e(u^s) \right) + f = 0. \quad (1.2)$$

- The second generalization (see Forest and Sab (2020); Misra et al (2021); Forest (1999); Cosserat and Cosserat (1896)) consists in introducing a new kinematic descriptor  $\phi$  and assuming that the elastic energy couples  $\phi$  with  $\nabla u$  and also depends on the gradient of  $\phi$ , reading:

$$\mathcal{E}^c(u) := \inf_{\phi} \int_{\Omega} \left( \frac{1}{2} \nabla \phi : G : \nabla \phi + \frac{1}{2} (\nabla u - \phi) : H : (\nabla u - \phi) + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

The equilibrium displacement field  $u^c$  and the associated field  $\phi^c$  are the unique  $[-L, L]^3$ -periodic functions in  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$  and  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^{3 \times 3})$  which satisfy  $\int_{\Omega} u^c dx = 0$  and, in the sense of distributions on  $\mathbb{R}^3$ ,

$$\begin{cases} \operatorname{div} (H : (\nabla u^c - \phi^c) + C : e(u^c)) + f = 0, \\ \operatorname{div} (G : \nabla \phi^c) + H : (\nabla u^c - \phi^c) = 0. \end{cases}$$

Note that the new kinematic descriptor  $\phi$  is a tensor field of order two. In the sequel, we refer to this second generalization as the ‘‘micromorphic model’’. A particular case of this energy is the ‘‘Cosserat model’’ in which  $\phi$  is a skew-symmetric matrix coupled to the rotational of  $u$  (cf. Cosserat and Cosserat (1896)).

These two generalizations are strongly related (see Germain (1973)): some authors like considering  $\mathcal{E}^s$  as the limit of  $\mathcal{E}^c$  when  $G = D$  and  $H$  becomes very large while others consider  $\mathcal{E}^c$  as an approximation of  $\mathcal{E}^s$  more suitable for numerical simulations.

It is convenient to remark that all the aforementioned models are particular cases of a more general one which mix non-local and strain-gradient terms with an elastic energy  $\mathcal{E}^m(u)$  which is the infimum over  $\phi$ :

$$\mathcal{E}^m(u) := \inf_{\phi} \int_{\Omega} \left( \frac{1}{2} \nabla \nabla u : D : \nabla \nabla u + \frac{1}{2} \nabla \phi : G : \nabla \phi + \frac{1}{2} (\nabla u - \phi) : H : (\nabla u - \phi) + \frac{1}{2} e(u) : C : e(u) \right) dx. \quad (1.4)$$

The corresponding equilibrium solution  $(u^m, \phi^m)$  is the unique couple of  $[-L, L]^3$ -periodic functions in  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$  and  $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^{3 \times 3})$  which satisfy  $\int_{\Omega} u^m dx = 0$  and, in the sense of distributions on  $\mathbb{R}^3$ ,

$$\begin{cases} \operatorname{div} (-\operatorname{div} (D : \nabla \nabla u^m) + H : (\nabla u^m - \phi^m) + C : e(u^m)) + f = 0, \\ \operatorname{div} (G : \nabla \phi^m) + H : (\nabla u^m - \phi^m) = 0. \end{cases} \quad (1.5)$$

Our aim is to compare these two generalizations. In this direction, the choice we have made of a periodic framework is specially suitable. It avoids discussing about the effects of boundary conditions which cannot be identical for both models and about the presence of boundary layers.

It is important to notice that all generalized models contain intrinsic lengths. Indeed any ratio of an entry of  $D$  or  $G$  to an entry of  $C$  is the square of such a length. Hence, deciding whether the supplementary terms in the energy are small perturbations of the Cauchy model is not a question about the constitutive laws of the material only but on the scale at which the effects of such supplementary terms are

observed. At a very large scale all models must be close to the Cauchy model. “Large scale” means here that the characteristic size of the domain and the characteristic wavelength of the applied load are large compared to the intrinsic length.

On the other hand, the use of generalized models is justified in the literature by several homogenization results. It is known that, when the tensor  $C$  oscillates periodically, with a very short period, the solution of associated Cauchy elasticity problems converges to the solution of a new problem in which the displacement minimizes the so-called “effective or homogenized energy”. The study of this asymptotic problem is now well understood from the mathematical point of view when  $C$  oscillates between fixed bounds: the effective energy is still of Cauchy type and the new tensor  $C^{hom}$  can be computed through the solution of an auxiliary problem set on the rescaled periodic cell (see Allaire (1992); Bakhvalov and Panasenko (2012); Bensoussan et al (1978); Sanchez-Palencia (1980)). When it oscillates between bounds whose ratio tends to infinity while the period length tends to zero, things are less clear. In this so-called “high-contrast” case, different results have been obtained (cf. Camar-Eddine and Seppecher (2003); Abdoul-Anziz and Seppecher (2018a,b)): some still give a Cauchy model, others lead to a strain-gradient model (cf. Pideri and Seppecher (1997); Briane and Camar-Eddine (2007); Alibert and Della Corte (2015); Turco et al (2016); dell’Isola et al (2016); Rahali et al (2015); Alibert et al (2003); Durand et al (2021)), still others lead to non-local models like micromorphic models (see Abdoul-Anziz and Seppecher (2018b); Jakabčín and Seppecher (2020)). In Jakabčín and Seppecher (2020) a mixed model of type (1.5) has even been obtained. Again we must emphasize that speaking of “high-contrast” is not a purely material property: indeed the ratio between the stiffness of the stiffest part of the material to the weakest one has to be compared to the ratio of the wavelength of these variations to the size of the domain or to the wavelength of the applied load.

To sum up, in a periodic homogenization framework, there exist at least three characteristic lengths. The first one is the period  $\varepsilon$  of the oscillations of the stiffness tensor : at such a scale homogenization is irrelevant. The second one is the intrinsic length  $\ell$  of an effective energy of strain-gradient or micromorphic type : at this scale strain-gradient or micromorphic effects are important. If such a scale is of the same order of magnitude as  $\varepsilon$ , strain-gradient or micromorphic effects are never important. The third one,  $L$ , is large compared to  $\ell$  : at this scale the material behaves essentially like a classical Cauchy material and strain-gradient or micromorphic terms are small corrections to the Cauchy energy.

In recent studies (Smyshlyaev and Cherednichenko (2000); Allaire et al (2016)), it has been proved that the first approximation of the solution of an elasticity problem, when the tensor  $C$  oscillates periodically with a very short period, minimizes at the leading order, the usual effective energy but that the first correction to this leading order solution corresponds to the solution of a strain-gradient model. This result could appear in contradiction with the results of Jakabčín and Seppecher (2020); Abdoul-Anziz and Seppecher (2018a,b); Abdoul-Anziz et al (2021); Camar-Eddine and Seppecher (2003) where non-local limits are obtained . It is not, because the

assumptions made by Smyshlyaev and Cherednichenko (2000) or Allaire et al (2016) prevent the appearance of a macroscopic intrinsic length  $\ell$  in the limit energy. In other words, all the intrinsic lengths  $\ell$  contained in the models obtained in these works are of the same order of magnitude as  $\varepsilon$  and tend to zero when  $\varepsilon$  tends to zero.

Though the results of Smyshlyaev and Cherednichenko (2000) or Allaire et al (2016) and those of Camar-Eddine and Seppecher (2003); Abdoul-Anziz and Seppecher (2018a) or Abdoul-Anziz et al (2021) seem to apply to different situations, a numerical study (Jakabčín and Seppecher (2020)) has suggested a strong correlation between them. In the present paper we show that this correlation is not fortuitous.

Our study results from the following observation: when the applied load is widely spread, then it often becomes very difficult to distinguish strain-gradient and Cosserat-type solutions (see Jakabčín and Seppecher (2020)). Considering a widely spread load is equivalent to considering the material at a large scale. As aforementioned, at such a scale, all extra energy terms become small corrections to standard elastic energy and all solutions converge toward the Cauchy solution. What we prove here is more surprising. We show that the corrections brought by strain-gradient terms or by the extra kinematic descriptor can be identical, up to a higher order correction.

The paper is organized as follows. After a short section where notation is fixed, we set the asymptotic problem when the applied load becomes wider spread in a large domain. A small parameter  $\eta$  characterizes this large wavelength. A change of variable brings back to a fix domain and the parameter  $\eta$  makes the strain-gradient and micromorphic terms small corrections of the classical Cauchy elasticity problem. We then write the equilibrium problem in the Fourier framework where it reduces to the inversion of a tensor which linearly relates the Fourier components of the applied force to the Fourier components of the solution. Our result comes from careful estimations of this inverse tensor.

These estimations need a fundamental assumption. Roughly speaking, if the micromorphic energy couples only a part of the extra kinematic variable with the gradient of the displacement field and if it also couples the gradient of this part with the gradient of the remaining part, then the micromorphic model is not asymptotically close to any strain gradient model. In section 1.6 we provide an explicit example of this rather rare situation.

## 1.2 Notation

The different elastic energy densities that we have introduced contain quadratic forms. These forms are represented by tensors. We use the standard notation  $\otimes$  for the tensor product. We simply shorten some notation by writing  $T^{\otimes 2}$  for  $T \otimes T$ . Different conventions may be adopted for defining contraction products of tensors. Here we adopt the following ones: when  $Q$  is a fourth-order tensor and  $M$  and  $N$  are matrices,  $Q : M$  and  $N : Q : M$  stand for the matrix and the real defined respectively

by

$$(Q : M)_{ij} = \sum_{k,l} Q_{ijkl} M_{kl} \quad \text{and} \quad N : Q : M = \sum_{i,j,k,l} N_{ij} Q_{ijkl} M_{kl}.$$

Similarly when  $Q$  is a sixth-order tensor and  $M$  and  $N$  are third-order tensors,  $Q \dot{:} M$  and  $N \dot{:} Q \dot{:} M$  stand for the third-order tensor and the real defined by

$$(Q \dot{:} M)_{ijk} = \sum_{l,m,n} Q_{ijklmn} M_{lmn}, \quad N \dot{:} Q \dot{:} M = \sum_{i,j,k,l,m,n} N_{ijk} Q_{ijklmn} M_{lmn}.$$

Let  $\xi \in \mathbb{R}^3$ . To any quadratic form  $Q$  over matrices, we can associate the symmetric matrix  $\xi : Q : \xi$  defined<sup>1</sup> by setting, for any  $u \in \mathbb{R}^3$ ,

$$u \cdot (\xi \cdot Q \cdot \xi) \cdot u = (u \otimes \xi) : Q : (u \otimes \xi).$$

Similarly, to any quadratic form  $Q$  over third-order tensors, we can associate the quadratic form over matrices  $\xi \cdot Q \cdot \xi$  and, if  $N$  is a given matrix, the symmetric matrix  $N \cdot Q \cdot N$  defined<sup>2</sup> by setting, for any matrix  $M$  or any  $u \in \mathbb{R}^3$ ,

$$M : (\xi \cdot Q \cdot \xi) : M = (M \otimes \xi) : Q : (M \otimes \xi), \quad u \cdot (N : Q : N) \cdot u = (u \otimes N) \dot{:} Q \dot{:} (u \otimes N).$$

The tensors which represent quadratic forms are naturally symmetric: in (1.5)  $C$  and  $H$  are fourth order tensors and  $D$  and  $G$  are sixth order tensors satisfying, for any  $i, j, k, l, m, n$  in  $\{1, 2, 3\}$ ,

$$C_{ijkl} = C_{klij}, \quad H_{ijkl} = H_{klij}, \quad D_{ijklmn} = D_{lmnijk}, \quad G_{ijklmn} = G_{lmnijk}.$$

Moreover, due to the symmetric nature of the tensor  $e(u)$  on which  $C$  operates and to the natural right-symmetry of  $\nabla \nabla u$  (defined by  $(\nabla \nabla u)_{ijk} = \partial_j \partial_k u_i$ ), the tensors  $C$  and  $D$  which appear in (1.5) are assumed, without loss of generality, to satisfy the symmetries

$$C_{ijkl} = C_{ijlk} = C_{jikl}, \quad D_{ijklmn} = D_{ijklnm} = D_{ikjlmn}.$$

In the sequel we will have to compare several quadratic forms. If  $Q$  and  $\tilde{Q}$  are quadratic forms over the same space, writing  $Q \leq \tilde{Q}$  will simply mean that  $\tilde{Q} - Q$  is a non-negative quadratic form or equivalently<sup>3</sup> that, for any  $M$ ,  $M : Q : M \leq M : \tilde{Q} : M$ .

Any fourth-order tensor  $Q$  can also be considered as a linear operator on the space of matrices. Composition of  $Q$  and  $\tilde{Q}$  corresponds to the product  $Q \dot{:} \tilde{Q}$  defined by

<sup>1</sup> In terms of indices  $(\xi \cdot Q \cdot \xi)_{ij} = \sum_{k,l} Q_{ijkl} \xi_k \xi_l$ .

<sup>2</sup> In terms of indices  $(\xi \cdot Q \cdot \xi)_{ijkl} = \sum_{m,n} Q_{ijmklm} \xi_m \xi_n$  and  $((\xi \otimes \xi) : Q : (\xi \otimes \xi))_{ij} = \sum_{k,l,m,n} Q_{ijklmn} \xi_k \xi_l \xi_m \xi_n$ .

<sup>3</sup> The product used in this formula must be adapted to the space on which  $Q$  and  $\tilde{Q}$  apply.

$$(Q : \tilde{Q})_{ijkl} = \sum Q_{ijmn} \tilde{Q}_{mnkl}.$$

It is also in that sense that we will use its image or kernel  $Im(Q)$ ,  $Ker(Q)$ , its pseudo-inverse (Moore–Penrose inverse)  $Q^+$  and, when invertible, its inverse  $Q^{-1}$ .

The different minimization problems that we have introduced are all well-posed because we assume that  $C$  is a positive definite quadratic form over the space  $\mathcal{M}^{sym}$  of symmetric matrices and that  $D$ ,  $G$ ,  $H$  are non-negative forms. Specifically, we assume that there exists  $0 < \alpha < \beta$  such that, for any  $M$  in  $\mathcal{M}^{sym}$ , any matrix  $N$  and any third-order tensor  $T$ ,

$$\alpha \|M\|^2 \leq M : C : M \leq \beta \|M\|^2, \quad (1.6)$$

$$0 \leq T : D : T \leq \beta \|T\|^2, \quad 0 \leq T : G : T \leq \beta \|T\|^2 \quad (1.7)$$

$$0 \leq N : H : N \leq \beta \|N\|^2, \quad N : H^+ : N \leq \beta \|N\|^2. \quad (1.8)$$

Note that, applying assumption (1.6) to matrices  $M = (u \otimes \xi + \xi \otimes u)$ , gives

$$u \cdot (\xi \cdot C \cdot \xi) \cdot u = \frac{1}{4} M : C : M \geq \frac{\alpha}{4} \|M\|^2 \geq \frac{\alpha}{2} \|\xi\|^2 \|u\|^2. \quad (1.9)$$

Thus, for any  $\xi \neq 0$ , the matrix  $\xi \cdot C \cdot \xi$  is definite positive.

Note that these assumptions ensure that the equilibrium problem (1.5) is well-posed. Indeed, any minimizing sequence  $(u_n)$  of  $[-L, L]^3$ -periodic functions with zero mean value has bounded energy  $\mathcal{E}^m(u) \leq M$  and thus is bounded in  $H^1([-L, L]^3, \mathbb{R}^3)$ . As the energy functional is lower semi-continuous, the sequence converges, up to a sub-sequence, to a solution of (1.5).

### 1.3 Spread loads

We consider equilibrium problems in a domain whose size is large compared with the lengths which are intrinsic to the micromorphic or second-gradient models. We also consider force fields whose characteristic wave lengths are comparable to the size of the domain. To make these assumptions precise we introduce a small parameter  $\eta > 0$  and we consider the domain  $Y_\eta := \frac{1}{\eta} Y$  where  $Y := [-\pi, \pi]^3$ . Let  $f$  be a  $Y$ -periodic vector-valued function with zero mean value. We assume that the material is submitted to the  $Y_\eta$ -periodic load  $f_\eta(x) := f(\eta x)$ .

We assume that the material is homogeneous : its elastic energy  $\mathcal{E}^m$  is given by (1.4) where the tensors  $C, D, G, H$  are constant. We look for a  $Y_\eta$ -periodic solution  $(u_\eta, \phi_\eta)$  with  $\int_{Y_\eta} u_\eta dx = 0$  of





As  $J_\eta^\xi \geq H$ , the kernels of  $H$  and  $J_\eta^\xi$  satisfy  $Ker(J_\eta^\xi) \subset Ker(H)$  and their images satisfy  $Im(H) = Ker(H)^\perp \subset Ker(J_\eta^\xi)^\perp = Im(J_\eta^\xi)$ . Introducing the pseudo-inverse  $(J_\eta^\xi)^+$  of  $J_\eta^\xi$ , the solutions of (1.13) read

$$\psi_\eta^\xi = (J_\eta^\xi)^+ : (H : (v_\eta^\xi \otimes \xi)) + \Delta$$

where  $\Delta$  is any element in  $Ker(J_\eta^\xi)$ . We have  $H : \psi_\eta^\xi = H : (J_\eta^\xi)^+ : (H : (v_\eta^\xi \otimes \xi))$  and we can eliminate  $\psi_\eta^\xi$  from Equation (1.12a). We get

$$K_\eta^\xi \cdot v_\eta^\xi = f^\xi \quad (1.14)$$

with

$$K_\eta^\xi = \xi \cdot (C + H - H : (J_\eta^\xi)^+ : H) \cdot \xi + \eta^2 (\xi \otimes \xi) : D : (\xi \otimes \xi). \quad (1.15)$$

The product  $J_\eta^{\xi+} : J_\eta^\xi$  is the orthogonal projection onto the image  $Im(J_\eta^\xi)$ . We have  $(J_\eta^\xi)^+ : J_\eta^\xi : (J_\eta^\xi)^+ = (J_\eta^\xi)^+$  and the fact that  $Im(H) \subset Im(J_\eta^\xi)$  implies  $H : J_\eta^{\xi+} : J_\eta^\xi = H$ . Using these remarks, one can check that  $H - H : (J_\eta^\xi)^+ : H$  is identical to

$$H \cdot (J_\eta^\xi)^+ : (J_\eta^\xi - H) : (J_\eta^\xi)^+ : H + (J_\eta^\xi - H) : (J_\eta^\xi)^+ : H : (J_\eta^\xi)^+ : (J_\eta^\xi - H).$$

Since  $J_\eta^\xi - H = \eta^2 \xi \cdot G \cdot \xi$  is non-negative, we deduce that  $H - H : (J_\eta^\xi)^+ : H$  is non-negative and that  $K_\eta^\xi \geq \alpha \|\xi\|^2 Id$  is definite positive. The solution of (1.14) is thus given by

$$v_\eta^\xi = (K_\eta^\xi)^{-1} \cdot f^\xi. \quad (1.16)$$

## 1.5 Comparison of different models at large scale

Our goal is to compare the solutions of problem (1.11) for different values of the material parameters  $C, D, G, H$  when  $\eta$  is small. In the Fourier setting, this will be obtained by estimating the tensor  $(K_\eta^\xi)^{-1}$ . To that aim we first focus on the term  $H : (J_\eta^\xi)^+ : H$ .

**Lemma 1.** *Let  $P = H^+ : H$  and  $Q = Id_4 - P$  be the projectors onto the image and the kernel of  $H$ . For any  $\xi \in \mathbb{R}^3$ , let us consider the following generalized Schur complement:*

$$G_{//}^\xi := P : (\xi \cdot G \cdot \xi) : P - (P : (\xi \cdot G \cdot \xi) : Q) : (Q : (\xi \cdot G \cdot \xi) : Q)^+ : (Q : (\xi \cdot G \cdot \xi) : P).$$

We have

$$H : (J_\eta^\xi)^+ : H = (H^+ + \eta^2 H^+ : G_{//}^\xi : H^+)^+.$$



$$\begin{aligned}
& \inf_{N \in \text{Ker}(H)} \{((P : M + N) \otimes \xi) : G : ((P : M + N) \otimes \xi)\} \\
&= \inf_{N \in \text{Ker}(H)} \{(P : M + N) : (\xi \cdot G \cdot \xi) : (P : M + N)\} \\
&= \inf_{N \in \text{Ker}(H)} \{M : (P : (\xi \cdot G \cdot \xi) : P) : M + 2N : (Q : (\xi \cdot G \cdot \xi) : P) : M \\
&\quad + N : (Q : (\xi \cdot G \cdot \xi) : Q) : N\}.
\end{aligned}$$

The infimum is reached when  $(Q : (\xi \cdot G \cdot \xi) : P) : M + (Q : (\xi \cdot G \cdot \xi) : Q) : N = 0$  that is when  $N = -(Q : (\xi \cdot G \cdot \xi) : Q)^+ : (Q : (\xi \cdot G \cdot \xi) : P) : M$ . At this minimum we have

$$\begin{aligned}
& N : (Q : (\xi \cdot G \cdot \xi) : Q) : N = -N : (Q : (\xi \cdot G \cdot \xi) : P) : M \\
&= M : (P : (\xi \cdot G \cdot \xi) : Q) : (Q : (\xi \cdot G \cdot \xi) : Q)^+ : (Q : (\xi \cdot G \cdot \xi) : P) : M
\end{aligned}$$

and we finally get

$$\inf_{N \in \text{Ker}(H)} \{((P : M + N) \otimes \xi) : G : ((P : M + N) \otimes \xi)\} = M : G_{//}^\xi : M.$$

A straightforward consequence of this minimization formulation is that, for any matrix  $M$ ,

$$0 \leq M : G_{//}^\xi : M \leq \beta \|\xi\|^2 \|M\|^2. \quad (1.18)$$

**Lemma 2.** *We have the following estimations for  $K_\eta^\xi$  :*

$$K_\eta^\xi \geq \xi \cdot C \cdot \xi, \quad (1.19)$$

$$\begin{aligned}
K_\eta^\xi \geq \xi \cdot C \cdot \xi + \eta^2 (\xi \cdot G_{//}^\xi \cdot \xi + (\xi \otimes \xi) : D : (\xi \otimes \xi)) \\
- \eta^4 \xi \cdot (G_{//}^\xi : H^+ : G_{//}^\xi) \cdot \xi, \quad (1.20)
\end{aligned}$$

$$K_\eta^\xi \leq \xi \cdot C \cdot \xi + \eta^2 (\xi \cdot G_{//}^\xi \cdot \xi + (\xi \otimes \xi) : D : (\xi \otimes \xi)). \quad (1.21)$$

*Proof.* First estimation will be a consequence of

$$H : (J_\eta^\xi)^+ : H \leq H \quad (1.22)$$

while the two last ones are respectively equivalent to

$$H : (J_\eta^\xi)^+ : H \leq H - \eta^2 G_{//}^\xi + \eta^4 G_{//}^\xi : H^+ : G_{//}^\xi, \quad (1.23)$$

$$H : (J_\eta^\xi)^+ : H \geq H - \eta^2 G_{//}^\xi. \quad (1.24)$$

As all these quadratic forms vanish outside of  $\text{Im}(H)$ , it is enough to check the inequalities on this space. As, on this space,  $(H : (J_\eta^\xi)^+ : H)^+ = H^+ + \eta^2 H^+ : G_{//}^\xi : H^+$  is invertible, we can multiply previous inequalities by this tensor on the left and on the right. In order to shorten notation, let us temporarily introduce the non-negative

tensor  $X^\xi = H^+ : G_{//}^\xi : H^+$ . We are reduced to proving

$$\begin{aligned} H^+ + \eta^2 X^\xi &\leq (H^+ + \eta^2 X^\xi) : H : (H^+ + \eta^2 X^\xi), \\ H^+ + \eta^2 X^\xi &\leq (H^+ + \eta^2 X^\xi) : (H - \eta^2 G_{//}^\xi + \eta^4 G_{//}^\xi : H^+ : G_{//}^\xi) : (H^+ + \eta^2 X^\xi), \\ H^+ + \eta^2 X^\xi &\geq (H^+ + \eta^2 X^\xi) : (H - \eta^2 G_{//}^\xi) : (H^+ + \eta^2 X^\xi). \end{aligned}$$

This is obvious when developing the right-hand side of these three inequalities as they become respectively

$$\begin{aligned} 0 &\leq X^\xi + \eta^2 X^\xi : H : X^\xi, \\ 0 &\leq X^\xi : H : X^\xi : H : X^\xi + \eta^2 X^\xi : H : X^\xi : H : X^\xi : H : X^\xi, \\ 0 &\geq -X^\xi : H : X^\xi - \eta^2 X^\xi : H : X^\xi : H : X^\xi. \end{aligned}$$

□

*Remark 2.* From estimation (1.19), we deduce

$$\|f^\xi\| = \|K_\eta^\xi \cdot v_\eta^\xi\| \geq \alpha \|\xi\|^2 \|v_\eta^\xi\|. \quad (1.25)$$

We recover the fact that, when  $f$  belongs to  $L_{loc}^2$ , the equilibrium solution  $v_\eta$  belongs to  $H_{loc}^2$  :

$$\begin{aligned} \int_{\Omega} \|\nabla \nabla v_\eta\|^2 dx &= \pi \sum_{\xi \in \mathbb{Z}^2} \|v_\eta^\xi \otimes \xi \otimes \xi\|^2 \leq \pi \sum_{\xi \in \mathbb{Z}^2} \|v_\eta^\xi\|^2 \|\xi\|^4 \\ &\leq \pi \alpha^{-2} \sum_{\xi \in \mathbb{Z}^2} \|f^\xi\|^2 = \alpha^{-2} \int_{\Omega} \|f\|^2 dx. \end{aligned}$$

**Theorem 1.** *When  $\eta$  goes to zero, all solutions  $v_\eta$  converge to the solution  $v_0$  of the classical elasticity problem. More precisely, for any  $\xi \in \mathbb{Z}^2$ , we have the following estimation*

$$\|v_0^\xi - v_\eta^\xi\| \leq 2\eta^2 \frac{\beta}{\alpha^2} \|f^\xi\|$$

which directly implies  $\|v_\eta - v_0\|_{L^2} \leq 2\eta^2 \frac{\beta}{\alpha^2} \|f\|_{L^2}$ .

*Proof.* When  $\eta = 0$  (which is equivalent to assuming that  $D = G = 0$  and  $H = 0$ ), we have  $K_0^\xi = \xi \cdot C \cdot \xi$  and equation (1.14) reduces to  $(\xi \cdot C \cdot \xi) \cdot v_0^\xi = f^\xi$ . From  $K_\eta^\xi \cdot v_\eta^\xi = f^\xi = (\xi \cdot C \cdot \xi) \cdot v_0^\xi$ , we deduce

$$K_\eta^\xi \cdot (v_0^\xi - v_\eta^\xi) = (K_\eta^\xi - \xi \cdot C \cdot \xi) \cdot v_0^\xi.$$

Using Lemma 2, we get

$$\alpha \|\xi\|^2 \|v_0^\xi - v_\eta^\xi\| \leq \eta^2 \|(\xi \cdot G_{//}^\xi \cdot \xi + (\xi \otimes \xi) : D : (\xi \otimes \xi)) \cdot v_0^\xi\|.$$

Definition of  $G_{//}^\xi$  clearly shows that  $G_{//}^\xi \leq P : (\xi \cdot G \cdot \xi) : P$  and thus that  $\|(\xi \cdot G_{//}^\xi \cdot \xi) \cdot v_0^\xi\| \leq \beta \|\xi\|^4 \|v_0^\xi\|$ . As we have the same estimation for the term  $\|(\xi \otimes \xi) : D : (\xi \otimes \xi) \cdot v_0^\xi\|$ , we get

$$\|v_0^\xi - v_\eta^\xi\| \leq 2\eta^2 \frac{\beta}{\alpha} \|\xi\|^2 \|v_0^\xi\| \leq 2\eta^2 \frac{\beta}{\alpha^2} \|f^\xi\|.$$

Therefore

$$\|v_\eta - v_0\|_{L^2}^2 = \pi \sum_{\xi \in \mathbb{Z}^2} \|v_0^\xi - v_\eta^\xi\|^2 \leq 4\eta^4 \frac{\beta^2}{\alpha^4} \|f\|_{L^2}^2.$$

□

Previous theorem states that, in any case, the minimizer of a strain-gradient or a generalized continuum energy behaves, at the main order with respect to the characteristic length of the applied force, like a classical elastic model. Clearly different models sharing the same elasticity tensor  $C$  cannot be differentiated at this order.

Let us now compare more precisely the solutions  $v_\eta$  and  $\tilde{v}_\eta$  of (1.11) associated respectively the sets of material tensors  $(C, D, H, G)$  and  $(C, D+L, 0, 0)$ . In Fourier setting, these equations read  $K_\eta^\xi \cdot v_\eta^\xi = f^\xi$  and  $\tilde{K}_\eta^\xi \cdot \tilde{v}_\eta^\xi = f^\xi$  with

$$\begin{aligned} K_\eta^\xi &= \xi \cdot (C + H - (H^+ + \eta^2 H^+ : G_{//}^\xi : H^+)^+) \cdot \xi + \eta^2 (\xi \otimes \xi) : D : (\xi \otimes \xi), \\ \tilde{K}_\eta^\xi &= \xi \cdot C \cdot \xi + \eta^2 (\xi \otimes \xi) : (D + L) : (\xi \otimes \xi). \end{aligned}$$

We have

$$\tilde{K}_\eta^\xi \cdot (v_\eta^\xi - \tilde{v}_\eta^\xi) = (\tilde{K}_\eta^\xi - K_\eta^\xi) \cdot v_\eta^\xi. \quad (1.26)$$

From Lemma 2 we deduce

$$\begin{cases} \tilde{K}_\eta^\xi - K_\eta^\xi \geq \eta^2 \xi \cdot (\xi \cdot L \cdot \xi - G_{//}^\xi) \cdot \xi, \\ \tilde{K}_\eta^\xi - K_\eta^\xi \leq \eta^2 \xi \cdot (\xi \cdot L \cdot \xi - G_{//}^\xi) \cdot \xi + \eta^4 \xi \cdot (G_{//}^\xi : H^+ : G_{//}^\xi) \cdot \xi. \end{cases} \quad (1.27)$$

In order to proceed further, we need an important assumption over the tensor  $G_{//}^\xi$  defined in Lemma 1 in terms of  $G, P$  and  $Q$ , that is in terms of  $H$  and  $G$ .

**Assumption 1** *The tensors  $H$  and  $G$  are such that there exists a six-order tensor  $G_{//}$  satisfying, for any vector  $\xi \in \mathbb{Z}^3$ ,*

$$\xi \cdot G_{//}^\xi \cdot \xi = (\xi \otimes \xi) : G_{//} : (\xi \otimes \xi). \quad (1.28)$$

We still call  $\beta$  a constant which, in addition to (1.6)-(1.7) satisfies, for any third-order tensor  $T$ ,

$$T : G_{//} : T \leq \beta \|T\|^2.$$

In many cases Assumption 1 is obtained as the consequence of stronger assumptions. Let us denote  $P : G : Q$  the sixth order tensor defined by  $(P : G : Q)_{ijklmn} := \sum_{p,q,u,v} P_{ijpq} G_{pqkuvn} Q_{uvlm}$ . The tensor  $P : G : P$  is defined in a similar way.

**Assumption 2**  $P : G : Q = 0$ .

Indeed, in that case, Assumption 1 is satisfied with  $G_{//} = P : G : P$ .

**Assumption 3**  $H$  is non degenerate.

Indeed, in that last case,  $P = Id_4$  and  $Q = 0$ , Assumption 2 is obviously satisfied and we have  $G_{//} = G$ .

**Theorem 2.** *Assume that  $f$  is smooth and that  $G$  and  $H$  satisfy Assumption 1. Then the solution  $v_\eta$  of (1.11) associated with the set of material tensors  $(C, D, H, G)$  shares the same asymptotic behavior at order  $\eta^2$  as the solution  $\tilde{v}_\eta$  of (1.11) associated with the set of material tensors  $(C, D + G_{//}, 0, 0)$ . More precisely, we have*

$$\|\tilde{v}_\eta^\xi - v_\eta^\xi\| \leq \eta^4 \frac{\beta^3}{\alpha^2} \|\xi\|^2 \|f^\xi\|$$

which directly implies  $\|v_\eta - \tilde{v}_\eta\|_{L^2(\Omega)} \leq \eta^4 \frac{\beta^3}{\alpha^2} \|f\|_{H^2(\Omega)}$ .

Reciprocally if, for any smooth force field  $f$ ,  $v_\eta$  shares the same asymptotic behavior at order  $\eta^2$  as the solution  $\tilde{v}_\eta$  of (1.11) associated with the set of material tensors  $(C, D + L, 0, 0)$ , then Assumption 1 is satisfied with  $G_{//} = L$ .

*Proof.* Indeed, replacing  $L$  by  $G_{//}$  and  $\xi \cdot G_{//}^\xi \cdot \xi$  by  $\xi \cdot (\xi \cdot G_{//} \cdot \xi) \cdot \xi$  in (1.27), we get

$$0 \leq \tilde{K}_\eta^\xi - K_\eta^\xi \leq \eta^4 \xi \cdot (G_{//}^\xi : H^+ : G_{//}^\xi) \cdot \xi.$$

From (1.26) we get the estimation

$$\alpha \|\xi\|^2 \|\tilde{v}_\eta^\xi - v_\eta^\xi\| \leq \eta^4 \|(\xi \cdot (G_{//}^\xi : H^+ : G_{//}^\xi) \cdot \xi) \cdot v_\eta^\xi\|.$$

Using (1.25) (1.18), we obtain the desired result

$$\|\tilde{v}_\eta^\xi - v_\eta^\xi\| \leq \eta^4 \frac{\beta^3}{\alpha^2} \|\xi\|^2 \|f^\xi\|.$$

To prove the converse, let us assume that, for any smooth force field  $f$ ,

$$\frac{1}{\eta^2} \|v_\eta - \tilde{v}_\eta\|_{L^2(\Omega)} \rightarrow 0.$$

Dividing equation (1.26) by  $\eta^2$ , using (1.27) and passing to the limit  $\eta \rightarrow 0$ , we get

$$0 = (\xi \cdot (\xi \cdot L : \xi - G_{//}^\xi) \cdot \xi) \cdot v_0^\xi.$$

Let  $w$  be a  $C^\infty$   $Y$ -periodic function satisfying  $w^0 = 0$  and, for any  $\xi \neq 0$ ,  $w^\xi \neq 0$ . Let  $k \in \{1, 2, 3\}$ . Consider the load  $f := -\operatorname{div}(C : e(we^k))$ . Clearly, the solution

$v_0$  of the classical elasticity problem when this load is applied coincides with  $w e^k$ . For this load, previous equality reads

$$w^\xi ((\xi \otimes \xi) : L : (\xi \otimes \xi) - \xi \cdot G_{//}^\xi \cdot \xi) \cdot e^k = 0.$$

As, for  $\xi \neq 0$ ,  $w^\xi \neq 0$ , we have  $((\xi \otimes \xi) : L : (\xi \otimes \xi) - \xi \cdot G_{//}^\xi \cdot \xi) \cdot e^k = 0$  which remains true for  $\xi = 0$ . This being true for any  $k \in \{1, 2, 3\}$ , we get for any  $\xi \in \mathbb{Z}^3$ ,

$$(\xi \otimes \xi) : L : (\xi \otimes \xi) - \xi \cdot G_{//}^\xi \cdot \xi = 0. \quad (1.29)$$

□

## 1.6 Examples

We study in this section different energies which enter the general model introduced in (1.4): they correspond to the choice  $D = 0$  and to different choices of the tensors  $H$  and  $G$ . Let us first provide examples for which Theorem 2 applies and thus which can be approximated, up to order  $\eta^4$  by a strain-gradient model.

i) Micromorphic models of the type

$$\mathcal{E}(u) := \inf_{\phi} \int_{\Omega} \left( \frac{1}{2} \nabla \phi : G : \nabla \phi + \frac{1}{2} \|\nabla u - \phi\|^2 + \frac{1}{2} e(u) : C : e(u) \right) dx$$

correspond to the choice  $H = Id_4$ . They satisfy Assumption 3 and thus can be approximated, up to order  $\eta^4$  by strain-gradient models with energy

$$\mathcal{E}(u) := \inf_{\phi} \int_{\Omega} \left( \frac{1}{2} \nabla \nabla u : G : \nabla \nabla u + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

ii) Cosserat models of the type

$$\mathcal{E}(u) := \inf_w \int_{\Omega} \left( \frac{1}{2} \nabla w : M : \nabla w + \frac{1}{2} \|\nabla \times u - w\|^2 + \frac{1}{2} e(u) : C : e(u) \right) dx$$

correspond to the choice  $H_{ijkl} = \sum_p \epsilon_{ijp} \epsilon_{klp}$  and  $G_{ijklmn} = \sum_{p,q} \epsilon_{ijp} M_{pkqn} \epsilon_{lmq}$  where  $\epsilon_{ijm}$  stands for the usual Levi-Civita symbol. The expression in terms of  $w$  instead of  $\phi$  is simply obtained by substituting  $w_i = \epsilon_{ikj} \phi_{jk}$  in the original expression. We have  $P = \frac{1}{2} H$ ,  $G : P = G$  and thus  $G : Q = 0$ . Assumption 2 is satisfied : Cosserat models can be approximated, up to order  $\eta^4$  by strain-gradient models (actually couple-stress models) with energy

$$\mathcal{E}(u) := \inf_w \int_{\Omega} \left( \frac{1}{2} \nabla (\nabla \times u) : M : \nabla (\nabla \times u) + \frac{1}{2} e(u) : C : e(u) \right) dx.$$



iii) Consider now the following energy:

$$\int_{\Omega} \left( \frac{1}{2} \left( \left( \frac{\partial \phi_{11}}{\partial x_2} + \frac{\partial \phi_{22}}{\partial x_2} \right)^2 + \left( \frac{\partial \phi_{22}}{\partial x_2} \right)^2 \right) + \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} - \phi_{11} \right)^2 + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

This energy corresponds to the choice

$$\begin{cases} D = 0, & H = (e^1 \otimes e^1)^{\otimes 2}, & (1.30a) \\ G = (e^2 \otimes e^2 \otimes e^2)^{\otimes 2} + (e^1 \otimes e^1 \otimes e^2 + e^2 \otimes e^2 \otimes e^2)^{\otimes 2}. & (1.30b) \end{cases}$$

Neither assumption 3 nor Assumption 2 are satisfied. Computation of  $G_{//}^{\xi}$  needs some work. We check successively that  $P = H^+ = H$  and

$$\begin{aligned} \xi \cdot G \cdot \xi &= \xi_2^2 \left( (e^2 \otimes e^2)^{\otimes 2} + (e^1 \otimes e^1 + e^2 \otimes e^2)^{\otimes 2} \right), \\ P : (\xi \cdot G \cdot \xi) &: P = \xi_2^2 (e^1 \otimes e^1)^{\otimes 2}, \\ Q : (\xi \cdot G \cdot \xi) &: P = \xi_2^2 e^2 \otimes e^2 \otimes e^1 \otimes e^1, \\ P : (\xi \cdot G \cdot \xi) &: Q = \xi_2^2 e^1 \otimes e^1 \otimes e^2 \otimes e^2, \\ Q : (\xi \cdot G \cdot \xi) &: Q = 2\xi_2^2 (e^2 \otimes e^2)^{\otimes 2}, \\ (Q : (\xi \cdot G \cdot \xi) &: Q)^+ = (2\xi_2^2)^{-1} (e^2 \otimes e^2)^{\otimes 2}. \end{aligned}$$

We finally obtain

$$G_{//}^{\xi} = \frac{\xi_2^2}{2} (e^1 \otimes e^1)^{\otimes 2} = \xi \cdot \left( \frac{1}{2} (e^1 \otimes e^1 \otimes e^2)^{\otimes 2} \right) \cdot \xi.$$

Note that this computation is valid only when  $\xi_2 \neq 0$  but one can easily check that the result remains true when  $\xi_2 = 0$ . This micromorphic model satisfies Assumption 1 with  $G_{//} = \frac{1}{2} (e^1 \otimes e^1 \otimes e^2)^{\otimes 2}$  and Theorem 2 again applies. The model can be approximated, up to order  $\eta^4$  by the strain-gradient model with energy

$$\int_{\Omega} \left( \frac{1}{4} \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} \right)^2 + \frac{1}{2} e(u) : C : e(u) \right) dx.$$

iv) Let us now show that there exist energies for which the conclusion of Theorem 2 does not apply. We modify a little bit the previous example by considering the following energy:

$$\inf_{\phi} \int_{\Omega} \left( \frac{1}{2} \left( \left( \frac{\partial \phi_{11}}{\partial x_2} + \frac{\partial \phi_{22}}{\partial x_1} \right)^2 + \left( \frac{\partial \phi_{22}}{\partial x_2} \right)^2 \right) + \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} - \phi_{11} \right)^2 + \frac{1}{2} e(u) : C : e(u) \right) dx,$$

which corresponds to the choice

$$\begin{cases} D = 0, & H = (e^1 \otimes e^1)^{\otimes 2}, \\ G = (e^2 \otimes e^2 \otimes e^2)^{\otimes 2} + (e^1 \otimes e^1 \otimes e^2 + e^2 \otimes e^2 \otimes e^1)^{\otimes 2}. \end{cases} \quad (1.31a)$$

Again we have  $P = H^+ = H$ . Computation of  $G_{//}^{\xi}$  is straightforward: for any  $\xi \neq 0$ , we have

$$\xi \cdot G \cdot \xi = \xi_2^2 (e^2 \otimes e^2)^{\otimes 2} + (\xi_2 e^1 \otimes e^1 + \xi_1 e^2 \otimes e^2)^{\otimes 2},$$

$$P : (\xi \cdot G \cdot \xi) : P = \xi_2^2 (e^1 \otimes e^1)^{\otimes 2},$$

$$Q : (\xi \cdot G \cdot \xi) : P = \xi_1 \xi_2 e^2 \otimes e^2 \otimes e^1 \otimes e^1,$$

$$P : (\xi \cdot G \cdot \xi) : Q = \xi_1 \xi_2 e^1 \otimes e^1 \otimes e^2 \otimes e^2,$$

$$Q : (\xi \cdot G \cdot \xi) : Q = (\xi_1^2 + \xi_2^2) (e^2 \otimes e^2)^{\otimes 2},$$

$$(Q : (\xi \cdot G \cdot \xi) : Q)^+ = (\xi_1^2 + \xi_2^2)^{-1} (e^2 \otimes e^2)^{\otimes 2}.$$

$$\begin{aligned} (P : (\xi \cdot G \cdot \xi) : Q) : (Q : (\xi \cdot G \cdot \xi) : Q)^+ : (Q : (\xi \cdot G \cdot \xi) : P) \\ = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} (e^1 \otimes e^1)^{\otimes 2} \end{aligned}$$

and finally

$$G_{//}^{\xi} = \frac{\xi_2^4}{\xi_1^2 + \xi_2^2} (e^1 \otimes e^1)^{\otimes 2} \quad \text{and} \quad \xi \cdot G_{//}^{\xi} \cdot \xi = \frac{\xi_2^4 \xi_1^2}{\xi_1^2 + \xi_2^2} e^1 \otimes e^1.$$

Assume, by contradiction that Assumption 1 is satisfied: for any  $\xi \neq 0$  in  $\mathbb{Z}^3$  we would have

$$\xi \cdot (\xi \cdot G_{//} \cdot \xi) \cdot \xi - \frac{\xi_2^4 \xi_1^2}{\xi_1^2 + \xi_2^2} e^1 \otimes e^1 = 0.$$

that is, for any  $\xi$  in  $\mathbb{Z}^3$ ,

$$(\xi_1^2 + \xi_2^2) \xi \cdot (\xi \cdot G_{//} \cdot \xi) \cdot \xi - \xi_2^4 \xi_1^2 e^1 \otimes e^1 = 0.$$

This polynomial identity extends to whole  $\mathbb{C}^3$ . We get a contradiction by considering for instance  $\xi = (1, i, 0)$ . By converse statement of Theorem 2 we know that there is no strain-gradient model which can approximate the considered model more accurately than what is done at order  $\eta^2$  by the standard model

$$\int_{\Omega} \frac{1}{2} e(u) : C : e(u) dx.$$

## 1.7 Conclusion

Our study has practical implications. Assume that you are analyzing the results of an experimental campaign, that is the equilibrium displacement fields of samples submitted to different force fields. Assume moreover that you have noticed scale effects and thus that you are suspecting your material to behave either like a strain-gradient one or a micromorphic one. Our study shows that it is rather difficult to differentiate these two possibilities. Indeed, in general, the characteristic wavelengths of the applied force fields are much larger than the intrinsic lengths of the suspected models. Let  $\eta \ll 1$  be their ratio. Then both strain-gradient and micromorphic models provide a correction to the Cauchy model of order  $\eta^2$  but, for well tuned material parameters, the difference between the two corrections is extremely small: of order  $\eta^4$ . In other words, in order to decide whether your material is better described by a micromorphic model or by a strain-gradient one, you must be extremely precise or use force fields which vary extremely rapidly in space.

Our study gives also a new insight on recent results about periodic homogenization of elastic materials : results of Smyshlyaev and Cherednichenko (2000); Allaire et al (2016) state that the first correction which must be applied to the classical homogenized strain model is always obtained by adding in the energy a small strain-gradient term. On the other hand, results from Abdoul-Anziz and Seppecher (2018a,b); Jakabčín and Seppecher (2020) provide micromorphic effective behaviors. These seemingly contradictory results may become coherent when spread enough forces are considered.

We must emphasize that our results apply only for micromorphic models which satisfy assumption 1. The question of the status of models which do not satisfy this assumption remains open: previous conclusion does not apply. We suspect that such models cannot be obtained through periodic homogenization. We recall that the example that we provide in Section 1.6 corresponds to a quadratic, objective and lower-semi-continuous functional and thus, as proved in Camar-Eddine and Seppecher (2003), that it can be obtained as the limit of heterogeneous classical elastic continua. However the results of Camar-Eddine and Seppecher (2003) are obtained using non-periodic heterogeneities. This leads to a new general question : what is the subclass of all functionals which can be obtained as the limit of *periodically* heterogeneous classical elastic continua ? And, in particular, is it equivalent, for a functional  $\mathcal{E}^m$  defined by (1.4), to belong to this class and to satisfy Assumption 1?

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