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Jean-Jacques Marigo, Kim Pham, Agnès Maurel, Sébastien Guenneau. Effective model for elastic waves in a substrate supporting a array of plates/beams with flexural and longitudinal resonances. Journal of Elasticity, 2021, 146 (1), pp.143-177. 10.1007/s10659-021-09854-4 . hal-03431828

**HAL Id: hal-03431828**

**<https://hal.science/hal-03431828>**

Submitted on 16 Nov 2021

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# Effective model for elastic waves in a substrate supporting a array of plates/beams with flexural and longitudinal resonances

Jean-Jacques Marigo

*Laboratoire de Mécanique des solides, Ecole Polytechnique,  
Route de Saclay, 91120 Palaiseau, France*

Kim Pham

*IMSIA, ENSTA ParisTech - CNRS - EDF - CEA, Université Paris-Saclay,  
828 Bd des Maréchaux, 91732 Palaiseau, France*

Agnès Maurel

*Institut Langevin, ESPCI Paris, Université PSL, CNRS,  
1 rue Jussieu, Paris 75005, France*

Sébastien Guenneau

*Aix Marseille Univ, CNRS, Centrale Marseille, Institut Fresnel, Marseille, France*

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## Abstract

In a previous study [1] we have studied the effect of an array of plates or beams over an elastic ground on the propagation of waves hitting the interface. The study was restricted to the low frequency regime where only flexural resonances take place. The present study presents a generalization to higher frequencies which allows us to account for both flexural and longitudinal resonances and to evaluate their interplay. An effective model is obtained using asymptotic analysis and homogenization techniques, which can be expressed in terms of the ground alone with an effective dynamic (frequency-dependent) boundary conditions of the Robin's type. For an in-plane wave at oblique incidence, the scattered displacement fields and the reflection coefficients are obtained in closed forms and their effectiveness to reproduce the actual scattering is inspected by comparison with direct numerics in a two-dimensional setting.

**Keywords:** asymptotic analysis; elastic waves; metamaterials; metasurfaces; multimodal methods.

**2010 MSC:** 00-01, 99-00

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## 1. Introduction

The ability of arrays of resonant beams to modify the propagation of waves in a substrate which support them is now well established. Their study, primarily focused on the case of a plate substrate, have been then extended to the case of a half-plane substrate. In both cases, mass-spring oscillator models have been used as the simplest models, see *e.g.* [2, 3, 4]. Assuming point contacts between the resonators and the substrate, the displacement  $X$  of a single oscillator

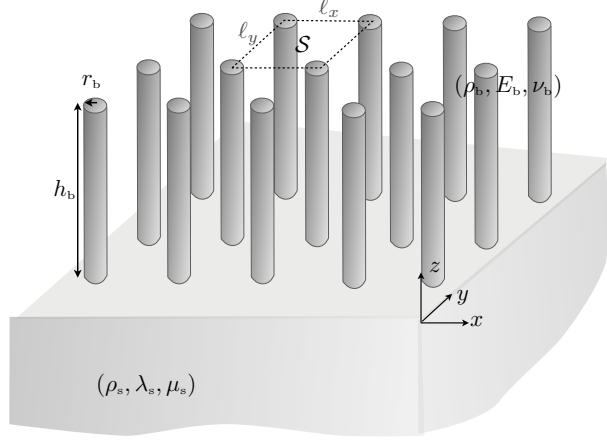


Figure 1: Array of cylindrical beams atop an isotropic substrate submitted to bending and longitudinal resonances; with  $k_T$  the wavenumber in the substrate,  $k_T h_b = O(1)$  and  $k_T \sqrt{S} = O(\eta^2)$ .

(with a mass  $M$  and a stiffness  $K$ ) satisfies  $M\ddot{X} = K(u_x - X)$  where  $u_x$  is the horizontal surface displacement. In the harmonic regime it follows that  $X = \frac{\omega_0^2}{\omega_0^2 - \omega^2} u_x$  ( $\omega_0^2 = \frac{K}{M}$ ). In the effective medium approach, for a wavelength much greater than the array spacing the average force per unit surface is identified to the normal stress along  $x$  hence  $\sigma_{xz} = \frac{K}{S}(X - u_x)$  ( $S$  is the surface area of the unit cell), resulting in

$$\sigma_{\alpha z} = Z_\alpha u_\alpha, \quad Z_\alpha = \frac{K}{S} \frac{\omega^2}{\omega_0^2 - \omega^2}. \quad (1)$$

2 If the physics of the substrate is accounted for, the dynamics of the resonators is considerably  
3 simplified. The complexity of actual beam resonators has been accounted for in more recent  
4 studies [5, 6, 7, 8, 9, 10, 11, 12]. The case of a plate substrate is simpler than that of a half-plane  
5 substrate, both experimentally and theoretically. From a theoretical point of view, this is because  
6 the dynamics of the resonator and of the substrate can be reduced to a one-dimensional problem.  
7 Besides a rigid plate substrate does not couple easily to the flexural motions of the resonators,  
8 which further simplifies the analysis to the coupling between the flexural modes in the plate and  
9 the longitudinal modes in the resonators [5, 6]. The influence of flexural modes in the resonators  
10 for less rigid plates has been studied theoretically in [7] and recently confirmed experimentally  
11 [10, 11].

12 The first experiments on the interaction of beams with a half-plane substrate have been re-  
13 ported in [13], and largely developed in the framework of the metaforest project [? ]. From a  
14 theoretical point of view, this situation is more involved; the reduction of model to one-dimension  
15 is not possible in the substrate and it is not straightforward in the resonator. Besides, the sub-  
16 strate can now couples efficiently to the beams both in flexion and compression; and as long as  
17 the full wave problem has not be specified, it is not possible to infer that displacements and *a*  
18 *fortiori* resonances in one direction can be neglected. In [7] the problem is solved considering  
19 the longitudinal resonances only. It is specified that the flexural motion could be accounted for  
20 but the analysis seems sufficient for the specific application considered by the authors. However,

21 there is a greyness in the term *rods* or *rod-like resonator* introduced by the authors and which  
 22 has been further used *e.g.* in [8, 12]: *beams* would support both flexural and longitudinal waves  
 23 while *rods* would only support compressional waves. This denomination suggests that the ability  
 24 of the resonators to couple in flexion or in compression with the substrate is an intrinsic property  
 25 of the resonator. However, the weak coupling in flexion observed for a plate substrate cannot  
 26 be attributable to the resonators on their own and the term "rod-like resonators" is *a minima*  
 27 confusing. The first study on the effect of flexural resonances has been proposed recently in  
 28 [9]; the interest is in the derivation of approximate dispersion relations of Rayleigh waves using  
 29 exotic junction conditions between the substrate and the array of beams. The authors introduce  
 30 the term "*beam-like resonator*" which refer to situations where the longitudinal motions could  
 31 be neglected; as for its "rod-like" version, this term is confusing as long as the wave context has  
 32 not been specified. Recently, we considered longitudinal and flexural motions of beams and we  
 33 restricted our analysis to the low frequency regime [1]. Low frequency meant that only flexural  
 34 resonances can take place namely frequencies below the first longitudinal resonance frequency  
 35 (see forthcoming figures 2 and 3).

36 The present study aims to generalize the results of [1] to higher frequencies. As in [1], both  
 37 flexural and longitudinal motions are accounted for and higher frequencies means that longitudi-  
 38 nal resonances, in addition to the flexural ones, can take place. We consider beams with circular  
 39 cross- section in three dimensions. (The case of plates in two-dimensions, much simpler, is given  
 40 in Appendix B.) The adaptation of the methodology proposed in [1] is demanding. The asymp-  
 41 totic analysis has to account for a separation of the length scales between the periodicity of the  
 42 array and the beam height which was already the case in [1]; however we have now to account for  
 43 an additional separation of the scales between the wavelength associated to flexural resonances  
 44 and that associated to the longitudinal resonances. The resulting full model which is the main  
 45 result of our study is given in §2 and discussed in view of existing models. The derivation of  
 46 the model based on asymptotic analysis and homogenization is detailed in §3. Eventually we  
 47 provide in §4 a validation of the model for a problem of scattering by comparison with direct  
 48 numerical simulations in a two-dimensional settings.

## 49 2. The actual problem and the effective problem

50 We consider an array of beams atop an isotropic elastic substrate in three-dimensions, with  
 51  $\mathbf{x} = (x, y, z)$ , see figure 1. The unit cell of the array has a section  $\mathcal{S} = \ell_x \times \ell_y$ ; the beams have  
 52 a circular cross-section with radius  $r_b$  and their height is  $h_b$  with  $h_b \gg r_b$ ; we define  $\varphi = \pi r_b^2 / \mathcal{S}$ .  
 53 We denote  $(\lambda_b, \mu_b, \rho_b)$  the Lamé coefficients and the mass density of the beams,  $(\lambda_s, \mu_s, \rho_s)$  those  
 54 of the substrate. We also introduce the Young's modulus  $E_b = \frac{\mu_b(3\lambda_b + 2\mu_b)}{\lambda_b + \mu_b}$  and Poisson ratio  
 55  $\nu_b = \frac{\lambda_b}{2(\lambda_b + \mu_b)}$  of the beam. We consider a low frequency regime for which the typical wavelength  
 56  $k$  in the substrate being of the order of magnitude of  $k_T = \frac{\omega}{c_T}$  or  $k_L = \frac{\omega}{c_L}$  satisfies  $k\sqrt{\mathcal{S}} \ll 1$ , with  
 57  $c_T = \sqrt{\frac{\mu_s}{\rho_s}}$ ,  $c_L = \sqrt{\frac{\lambda_s + 2\mu_s}{\rho_s}}$ , with  $\omega$  the angular frequency.

## 58 2.1. The physical problem

In the actual problem, the equations of elastodynamics apply

$$\begin{cases} \text{in the substrate, } z \in (-\infty, 0) : & \operatorname{div} \boldsymbol{\sigma} + \rho_s \omega^2 \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\sigma} = 2\mu_s \boldsymbol{\varepsilon} + \lambda_s \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + {}^t \nabla \mathbf{u}), \\ \text{and in the beams, } z \in (0, h_b) : & \operatorname{div} \boldsymbol{\sigma} + \rho_b \omega^2 \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\sigma} = 2\mu_b \boldsymbol{\varepsilon} + \lambda_b \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}, \end{cases} \quad (2)$$

59 where  $\mathbf{u}$  is the displacement vector,  $\boldsymbol{\sigma}$  the stress tensor and  $\boldsymbol{\varepsilon}$  the strain tensor;  $\mathbf{I}$  stands for the  
60 identity matrix. At each interface between an elastic medium (the beams or the substrate) and air,  
61 the stress free condition  $\boldsymbol{\sigma} \mathbf{n} = \mathbf{0}$  applies (with  $\mathbf{n}$  the normal to the interface). At each interface  
62 between the beams and the substrate, the displacement and the normal stress are continuous.  
63 This problem can be solved once the source  $\mathbf{u}^{\text{inc}}$  has been defined and accounting for the radiation  
64 condition for  $z \rightarrow -\infty$  which applies to the scattered field  $(\mathbf{u} - \mathbf{u}^{\text{inc}})$ .

## 65 2.2. The effective problem

Below we provide the main results of the analysis developed in the forthcoming §3. It will  
be shown that the problem reduces to a much simpler one set on the substrate on its own and  
where the effect of the array is reduced to impedance conditions. Specifically, we obtain that the  
effective problem reads

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho_s \omega^2 \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\sigma} = 2\mu_s \boldsymbol{\varepsilon} + \lambda_s \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}, \quad \text{for } z \in (-\infty, 0), \\ \sigma_{az}(\mathbf{x}', 0) = \mu_s k_T f_F(\omega, h_b) u_a(\mathbf{x}', 0), \quad a = x, y, \quad \sigma_{zz}(\mathbf{x}', 0) = \mu_s k_T f_L(\omega, h_b) u_z(\mathbf{x}', 0), \end{cases} \quad (3)$$

where  $\mathbf{x}' = (x, y)$  and

$$\begin{cases} f_F(\omega, h_b) = \varphi \frac{\rho_b k_T}{\rho_s \kappa} \frac{\operatorname{sh} \kappa h_b \cos \kappa h_b + \operatorname{ch} \kappa h_b \sin \kappa h_b}{1 + \cos \kappa h_b \operatorname{ch} \kappa h_b}, \quad \kappa = \left( \frac{4\rho_b \omega^2}{E_b r_b^2} \right)^{1/4}, \\ f_L(\omega, h_b) = \varphi \frac{\rho_b k_T}{\rho_s K} \tan K h_b, \quad K = \sqrt{\frac{\rho_b}{E_b}} \omega. \end{cases} \quad (4)$$

66 The boundary conditions on  $\sigma_{az}$ ,  $a = x, y$ , are the same as in [1], the impedance function  
67  $f_F(\omega, h_b)$  encapsulating the flexural resonances at roughly  $\kappa h_b \sim \frac{\pi}{2} + n\pi$ ,  $n$  integer<sup>1</sup>. Next, the  
68 impedance function  $f_L(\omega, h_b)$  encapsulates the longitudinal resonances at  $K h_b = \frac{\pi}{2} + n\pi$ ,  $n$  integer,  
69 and the first flexural resonance frequency  $\omega_F$  is much smaller than the first longitudinal resonance  
70  $\omega_L$  with  $\omega_L \sim \frac{h_b}{r_b} \omega_F \gg \omega_F$ . This is illustrated in figure 2 where we report typical variations of the  
71 impedance functions against  $h_b$ . Note that in [1], the analysis holds for small height  $h_b$  resulting  
72 in a linearized version of  $f_L$  whose validity has been inspected up to  $h_b = 30$  m, see figure 7 in [1]  
73 as (3) simplifies to  $\sigma_{zz}(\mathbf{x}', 0) = \varphi \rho_b h_b \omega^2 u_z(\mathbf{x}', 0)$ . For  $h_b \rightarrow 0$  which corresponds to roughnesses,  
74 we get  $\sigma_{iz}(\mathbf{x}', 0) = \varphi \rho_b h_b \omega^2 u_i(\mathbf{x}', 0)$ ,  $i = x, y, z$ , and in this limit the substrate couples to the array  
75 equally in flexion and compression by a term of added mass.

<sup>1</sup>The dispersion relation of the flexural resonances reads  $D(\kappa) = (1 + \cos \kappa h_b \operatorname{ch} \kappa h_b) = 0$  resulting in  $\kappa h_b = 1.875, 4.694, 7.855, 10.995, \dots$ ; the approximation  $\kappa h_b \sim \frac{\pi}{2} + n\pi$  provides  $\kappa h_b = 1.571, 4.712, 7.854, 10.995 \dots$  and the agreement is better as  $n$  increases (with  $D(\kappa) = 1$ ).

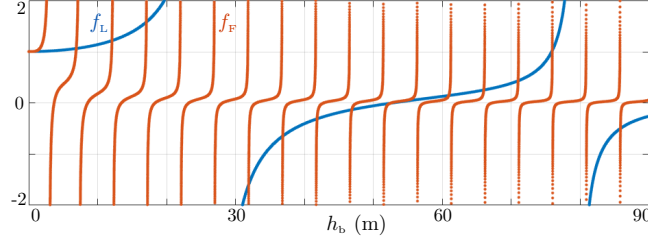


Figure 2: Variations of the impedance functions  $f_F$  and  $f_L$  in (4) against the beam height  $h_b$ . The flexural resonances correspond to  $|f_F| \rightarrow \infty$  and the longitudinal resonance to  $|f_L| \rightarrow \infty$ .

Once the effective problem (3) in the substrate has been resolved, the solution in the region of the beams can be post-processed. Indeed, the asymptotic analysis which is detailed in section 3 shows that the vertical displacement  $U_z$  in the region of the beams is solution to

$$\begin{cases} \frac{\partial^2 U_z}{\partial z^2} + K^2 U_z = 0, & \text{with } K = \sqrt{\frac{\rho_b}{E_b}} \omega, \\ U_z(\mathbf{x}', 0) = u_z(\mathbf{x}', 0), \quad \frac{\partial U_z}{\partial z}(\mathbf{x}', h_b) = 0, \end{cases} \quad (5)$$

and the horizontal displacements  $U_a$ ,  $a = x, y$ , are solutions to Euler-Bernoulli problems

$$\begin{cases} \frac{\partial^4 U_a}{\partial z^4} - \kappa^4 U_a = 0, & \text{with } \kappa = \left( \frac{4\rho_b \omega^2}{E_b I_b^2} \right)^{1/4}, \\ U_a(\mathbf{x}', 0) = u_a(\mathbf{x}', 0), \quad \frac{\partial U_a}{\partial z}(\mathbf{x}', 0) = 0, \quad \frac{\partial^2 U_a}{\partial z^2}(\mathbf{x}', h_b) = \frac{\partial^3 U_a}{\partial z^3}(\mathbf{x}', h_b) = 0. \end{cases} \quad (6)$$

In (5)-(6), boundary conditions at the top of the beams  $z = h_b$  correspond to free-end boundary conditions while at  $z = 0$ , we have a clamped boundary condition, meaning prescribed displacements at the junction with the substrate and zero rotation. Solving these two problems for which the inputs are the prescribed displacements  $\mathbf{u}(\mathbf{x}', 0)$  at the junction with the substrate gives

$$\begin{cases} U_a(\mathbf{x}) = u_a(\mathbf{x}', 0) V_F(z), \quad a = x, y, \quad U_z(\mathbf{x}) = u_z(\mathbf{x}', 0) V_L(z), \\ \text{with } V_L(z) = \cos(Kz) + \tan(Kh_b) \sin(Kz), \\ V_F(z) = a(\text{ch}\kappa(z - h_b) + \cos \kappa(z - h_b)) + b(\text{sh}\kappa(z - h_b) + \sin \kappa(z - h_b)), \end{cases} \quad (7)$$

76 with  $a = \frac{1}{d}(\text{ch}\kappa h_b + \cos \kappa h_b)$ ,  $b = \frac{1}{d}(\text{sh}\kappa h_b - \sin \kappa h_b)$ ,  $d = 2(1 + \text{ch}\kappa h_b \cos \kappa h_b)$ . It is worth noting  
 77 that at the dominant order, the effective surface is isotropic even for different spacings  $\ell_x$  along  $x$   
 78 and  $\ell_y$  along  $y$ .

### 79 2.3. Comparison with previous models

If the flexural motions are neglected, our impedance conditions (3) on  $\sigma_{zz}$  and the longitudinal displacements  $U_z$  in the region of the beams (7) are consistent with the findings of [5, 6]. As previously said, in these references, the flexural motions are disregarded since the beams are connected to a rigid plate substrate. In the present case, for a half-space substrate, we shall see

in §4 that neglecting the flexural resonances leads to incorrect predictions. Next, two recent studies have accounted for flexural resonances only [1, 9]. In [1], this is justified by the low frequency regime below the first longitudinal resonance and our present model includes this case, as previously stated. The comparison with the study [9] is made difficult by the fact that the authors consider different junction conditions among which some are exotic, but the case of vertical beams clamped to a substrate ( $U_a(\mathbf{x}', 0) = u_a(\mathbf{x}', 0)$ ,  $\frac{\partial U_a}{\partial z}(\mathbf{x}', 0) = 0$  from (6)) have not been considered. To be specific, 3 junction conditions have been considered in [9], called *simply supported*, *beams on a rail* and *fully matched*. They read, for  $a = x, y$ ,

$$\left\{ \begin{array}{ll} \text{simply supported} & U_a(\mathbf{x}', 0) = u_a(\mathbf{x}', 0), \quad \frac{\partial^2 U_a}{\partial z^2}(\mathbf{x}', 0) = 0, \\ \text{beams on the rails,} & \frac{\partial^2 U_a}{\partial z^2}(\mathbf{x}', 0) = 0, \quad \frac{\partial U_a}{\partial z}(\mathbf{x}', 0) = -\frac{\partial u_z}{\partial a}(\mathbf{x}', 0), \\ \text{fully matched,} & U_a(\mathbf{x}', 0) = u_a(\mathbf{x}', 0), \quad \frac{\partial U_a}{\partial z}(\mathbf{x}', 0) = -\frac{\partial u_z}{\partial a}(\mathbf{x}', 0). \end{array} \right.$$

80 The *simply supported* configuration appears to be dangerously unstable, as the condition of  
 81 vanishing moment  $\frac{\partial^2 U_a}{\partial z^2}(\mathbf{x}', 0) = 0$  allows for free rotation. While this condition is often used for  
 82 a horizontal beam maintained by point contact, a vertical beam free to rotate at its junction will  
 83 simply fall. The two other conditions assume that at  $z = 0$ , we have  $\frac{\partial U_a}{\partial z}(\mathbf{x}', 0) = -\frac{\partial u_z}{\partial a}(\mathbf{x}', 0)$  that  
 84 is a rotation prescribed in the beams by the gradient of the vertical displacement. This condition  
 85 is exotic but as explained by the authors, although counterintuitive, it can be understood in terms  
 86 of the so-called gyroscopic hinges as studied in [14].

### 87 3. Asymptotic analysis

The analysis is conducted assuming low frequencies which means that the typical array spacing  $\ell$ , with  $S = \ell_x \ell_y = \ell^2$ , is small compared to the typical incident wavelength  $1/k$ , hence  $k\ell \ll 1$  (with for instance  $k = k_\tau$ ). This ensures that incoming waves perceive the interface at  $z = 0$  essentially as an effective homogeneous interface. Next, in our analysis, we do not impose particular scalings on the material properties. We do not assume neither a sparse array of beams which means that  $r_b = O(\ell)$  hence  $\varphi = O(1)$ . However we cannot dispense with a careful length scalings in the vertical direction. Indeed, and as previously said, flexural and longitudinal resonances of beams are associated with different length scales

$$h_F = \kappa^{-1} \quad \text{and} \quad h_L = K^{-1}, \quad (8)$$

(with  $\kappa$  and  $K$  given in (4)) as illustrated with the typical deformation shown in figure 3. From (4), we have  $kh_L = O(1)$  while  $kh_F = O(\sqrt{k\ell}) \ll 1$ . It results a hierarchy of 3 scales  $\ell \ll h_F \ll h_L$  which is accounted for owing to a small parameter  $\eta$  such that

$$k\ell = \eta^2, \quad kh_F = O(\eta), \quad \text{and} \quad kh_L = O(1). \quad (9)$$

Accordingly, with  $\mathbf{x} = (x, y, z)$ , the analysis will be conducted owing to the rescaled spatial coordinates being

$$\tilde{z} = \frac{z}{\eta}, \quad (\hat{x}, \hat{y}, \hat{z}) = \frac{(x, y, z)}{\eta^2}.$$

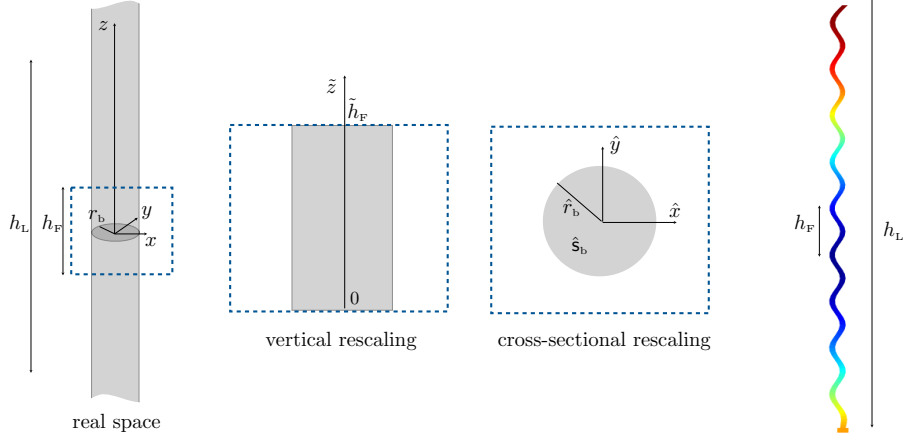


Figure 3: The two length scales in the vertical direction, with  $h_L \sim h_b$  the length scale associated with longitudinal resonances and  $h_F \ll h_b$  the length scale associated with flexural resonances; the center panels show the vertical rescaling with the  $\tilde{z}$ -coordinate,  $\tilde{z} \in (0, \tilde{h}_F)$  and the horizontal rescaling with  $\hat{\mathbf{x}}' = (\hat{x}, \hat{y}) \in \hat{\mathbf{S}}_b$ . The right panel show a typical deformation associated to the displacement  $U_x$  with variations at the scale  $h_F$  and colors associated to  $U_z$  with variations at the scale  $h_L$ .

(We define  $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z})$  and  $\hat{\mathbf{x}}' = (\hat{x}, \hat{y})$ .) The coordinate  $\tilde{z}$  aims to capture the rapid (in  $\frac{1}{\eta}$ ) spatial variations relative to flexural motions, while  $\hat{\mathbf{x}}$  aim to capture the very rapid (in  $\frac{1}{\eta^2}$ ) spatial variations relative to evanescent fields. Eventually the coordinate  $\mathbf{x}$  describe the slow variations (in  $O(1)$ ) associated with the wave propagation. Associated to these coordinates, rescaled lengths are defined

$$(\tilde{h}_F, \hat{r}_b, \hat{\ell}_x, \hat{\ell}_y) = \left( \frac{h_F}{\eta}, \frac{r_b}{\eta^2}, \frac{\ell_x}{\eta^2}, \frac{\ell_y}{\eta^2} \right).$$

### 3.1. Effective wave equation in the region of the beams

#### 3.1.1. Notations

In the region of the array of beams, the displacements and the stresses vary in the horizontal direction over small distances dictated by  $\ell$  ( $\ell = \sqrt{\ell_x \ell_y}$ ) and over large distances dictated by the incoming waves being of the same order of magnitude than  $h_b$ . These two scales are accounted for by the coordinates  $\mathbf{x}' = (x, y)$  and  $\hat{\mathbf{x}}' = (\hat{x}, \hat{y})$  with  $\hat{\mathbf{x}}' = \frac{\mathbf{x}'}{\eta^2}$ ; we define  $\hat{\mathbf{S}}_b$  the cross-section of a beam and

$$\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b = \{|\hat{\mathbf{x}}'| \in (0, \hat{r}_b)\}.$$

In the vertical direction, the displacements and the stresses vary over the intermediate distance  $h_F$  that we expect to play a role for the flexural motions only and over  $h_b$ ; this is accounted for by the rescaled coordinates  $z$  and  $\tilde{z}$ , with

$$\tilde{z} \in (0, \tilde{h}_F).$$

It follows that the fields  $(\mathbf{u}, \boldsymbol{\sigma})$  are written of the form

$$\mathbf{u} = \sum_{n \geq 0} \eta^n \mathbf{V}^n(\mathbf{x}, \hat{\mathbf{x}}', \tilde{z}), \quad \boldsymbol{\sigma} = \sum_{n \geq 0} \eta^n \boldsymbol{\Sigma}^n(\mathbf{x}, \hat{\mathbf{x}}', \tilde{z}), \quad (10)$$

90 with all the terms in the expansions being periodic with respect to  $\tilde{z} \in (0, \tilde{h}_r)$  (figure ??).

In the asymptotic analysis, the small scale  $\hat{\mathbf{x}}'$  aims to disappear since we are interested in macroscopic fields. Hence, we define at each order  $n$  the macroscopic displacement  $U_z^n$  and axial stress

$$U_z^n = \frac{1}{\hat{S}_b} \int_{\hat{S}_b} V_z^n d\hat{\mathbf{x}}', \quad N_{zz}^n = \int_{\hat{S}_b} \Sigma_{zz}^n d\hat{\mathbf{x}}', \quad (11)$$

where  $\hat{S}_b = |\hat{S}_b| = \pi \hat{r}_b^2$  is the rescaled surface area of beam section. We also define the macroscopic displacements, shear stresses and bending moments

$$U_a^n = \frac{1}{\hat{S}_b} \int_{\hat{S}_b} V_a^n d\hat{\mathbf{x}}', \quad T_a^n = \int_{\hat{S}_b} \Sigma_{az}^n d\hat{\mathbf{x}}', \quad a = x, y, \quad M_x^n = - \int_{\hat{S}_b} \hat{x} \Sigma_{zz}^n d\hat{\mathbf{x}}', \quad M_y^n = - \int_{\hat{S}_b} \hat{y} \Sigma_{zz}^n d\hat{\mathbf{x}}'. \quad (12)$$

### 91 3.1.2. The problem in the beam

According to the expansions in (10) the differential operator reads

$$\nabla \rightarrow \frac{1}{\eta^2} \nabla_{\hat{\mathbf{x}}'} + \frac{\mathbf{e}_z}{\eta} \frac{\partial}{\partial \tilde{z}} + \nabla_{\mathbf{x}},$$

where  $\mathbf{e}_z = (0, 0, 1)$ . Hence, the system in the region of the beams reads, from (2),

$$\begin{cases} (E_a)^n & \left( \frac{\partial \Sigma_{ax}^{n+2}}{\partial \hat{x}} + \frac{\partial \Sigma_{ay}^{n+2}}{\partial \hat{y}} \right) + \frac{\partial \Sigma_{az}^{n+1}}{\partial \tilde{z}} + \frac{\partial \Sigma_{ax}^n}{\partial x} + \frac{\partial \Sigma_{ya}^n}{\partial y} + \frac{\partial \Sigma_{az}^n}{\partial z} + \rho_b \omega^2 V_a^n = 0, \quad a = x, y, \\ (E_z)^n & \left( \frac{\partial \Sigma_{xz}^{n+2}}{\partial \hat{x}} + \frac{\partial \Sigma_{yz}^{n+2}}{\partial \hat{y}} \right) + \frac{\partial \Sigma_{zz}^{n+1}}{\partial \tilde{z}} + \frac{\partial \Sigma_{xz}^n}{\partial x} + \frac{\partial \Sigma_{yz}^n}{\partial y} + \frac{\partial \Sigma_{zz}^n}{\partial z} + \rho_b \omega^2 V_z^n = 0, \\ (C)^n & \Sigma^n = \mathbf{A} \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}'}(\mathbf{V}^{n+2}) + \mathbf{A} \boldsymbol{\varepsilon}^{\tilde{z}}(\mathbf{V}^{n+1}) + \mathbf{A} \boldsymbol{\varepsilon}^{\mathbf{x}}(\mathbf{V}^n), \end{cases} \quad (13)$$

where we have implicitly that  $\Sigma^m = \mathbf{V}^m = 0$  for  $m < 0$  and where  $\mathbf{A} \boldsymbol{\varepsilon}(V) = 2\mu_b \boldsymbol{\varepsilon}(V) + \lambda_b \text{tr}(\boldsymbol{\varepsilon}(V)) I$  which holds for  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}'}$ ,  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\tilde{z}}$  and  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathbf{x}}$ . We shall also use the stress-strain relation written in the form

$$(C)^n \quad \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}'}(\mathbf{V}^{n+2}) + \boldsymbol{\varepsilon}^{\tilde{z}}(\mathbf{V}^{n+1}) + \boldsymbol{\varepsilon}^{\mathbf{x}}(\mathbf{V}^n) = \frac{1 + \nu_b}{E_b} \Sigma^n - \frac{\nu_b}{E_b} \text{tr}(\Sigma^n) I. \quad (14)$$

The symmetric strain tensors with respect to  $\hat{\mathbf{x}}'$ ,  $\tilde{z}$  and  $\mathbf{x}$  have the following forms

$$\begin{aligned} \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}'}(\mathbf{V}) &= \frac{1}{2} \begin{pmatrix} 2 \frac{\partial V_x}{\partial \hat{x}} & \frac{\partial V_x}{\partial \hat{y}} + \frac{\partial V_y}{\partial \hat{x}} & \frac{\partial V_z}{\partial \hat{x}} \\ & 2 \frac{\partial V_y}{\partial \hat{y}} & \frac{\partial V_z}{\partial \hat{y}} \\ & & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}^{\tilde{z}}(\mathbf{V}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & \frac{\partial V_x}{\partial \tilde{z}} \\ & 0 & \frac{\partial V_y}{\partial \tilde{z}} \\ & & 2 \frac{\partial V_z}{\partial \tilde{z}} \end{pmatrix}, \\ \boldsymbol{\varepsilon}^{\mathbf{x}}(\mathbf{V}) &= \frac{1}{2} \begin{pmatrix} 2 \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} & \frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x} \\ & 2 \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \\ & & 2 \frac{\partial V_z}{\partial z} \end{pmatrix}. \end{aligned}$$

92 From now on, we shall use the hierarchy of equations in (13) and (14).

### 3.1.3. Useful forms of the displacements and stresses

We start by providing the explicit dependence with respect to the microscopic coordinate  $\hat{\mathbf{x}}'$  of the displacements ( $\mathbf{V}^0, \mathbf{V}^1$ ) and the of stress  $\Sigma^0$ .

– *Displacements at orders 0 and 1* – The dependance of  $\mathbf{V}^0$  and  $\mathbf{V}^1$  on  $\hat{\mathbf{x}}'$  read

$$\begin{cases} V_x^0 = U_x^0(\mathbf{x}', \tilde{z}), & V_y^0 = U_y^0(\mathbf{x}', \tilde{z}), & V_z^0 = U_z^0(\mathbf{x}), \\ V_x^1 = U_x^1(\mathbf{x}, \tilde{z}) + \Omega^1(\mathbf{x}) \hat{y}, & V_y^1 = U_y^1(\mathbf{x}, \tilde{z}) - \Omega^1(\mathbf{x}) \hat{x}, & V_z^1 = U_z^1(\mathbf{x}) - \frac{\partial}{\partial \tilde{z}} (U_x^0 \hat{x} + U_y^0 \hat{y}). \end{cases} \quad (15)$$

To get (15), we have used  $(C')^{-2}$  and  $(C')^{-1}$  in (14) from which  $\varepsilon^{\hat{\mathbf{x}}'}(\mathbf{V}^0) = \mathbf{0}$  and  $\varepsilon^{\hat{\mathbf{x}}'}(\mathbf{V}^1) + \varepsilon^{\tilde{z}}(\mathbf{V}^0) = \mathbf{0}$ . It follows that  $(V_x^m, V_y^m)$ ,  $m = 0, 1$ , are associated with rigid body motions (translation and rotation), namely  $V_x^m = U_x^m(\mathbf{x}, \tilde{z}) + \Omega^m(\mathbf{x}, \tilde{z}) \hat{y}$ ,  $V_y^m = U_y^m(\mathbf{x}, \tilde{z}) - \Omega^m(\mathbf{x}, \tilde{z}) \hat{x}$ . We also get that  $V_z^0$  depends only on  $\mathbf{x}$ , and that  $\frac{\partial}{\partial \tilde{z}} V_z^1 = -\frac{\partial}{\partial \tilde{z}} V_a^0$  for  $a = x, y$ . Doing so, we obtain the forms reported in (A.2)-(A.3). To obtain the forms given in (15), we have anticipated the following properties that we shall proof in the Appendix A:

$$\text{To be shown : } P_1 : \frac{\partial U_z^1}{\partial \tilde{z}} = 0 \quad P_2 : \frac{\partial \Omega^1}{\partial \tilde{z}} = 0, \quad P_3 : \Omega^0 = 0, \quad P_4 : \frac{\partial U_a^0}{\partial \tilde{z}} = 0, \quad a = x, y. \quad (16)$$

– *Stress at order 0* – The dependence of the stress  $\Sigma^0$  on  $\hat{\mathbf{x}}'$  can be obtained explicitly too. Indeed, from  $(E_a)^{-2}$ ,  $(E_z)^{-2}$  and from (C)<sup>0</sup> in (13), we have  $\text{div}_{\hat{\mathbf{x}}'} \Sigma^0 = \mathbf{0}$ ,  $\Sigma^0 = \mathbf{A} \varepsilon^{\hat{\mathbf{x}}'}(\mathbf{V}^2) + \mathbf{A} \varepsilon^{\tilde{z}}(\mathbf{V}^1) + \mathbf{A} \varepsilon^{\mathbf{x}}(\mathbf{V}^0)$ . As  $(\mathbf{V}^0, \mathbf{V}^1)$  are known from (15), the problem is set on  $(\Sigma^0, \mathbf{V}^2)$  for  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$ . Besides it can be decomposed into 2 decoupled boundary value problems. The first is set on the horizontal displacements  $(V_x^2, V_y^2)$  and the horizontal stress tensor  $\Sigma_{ab}^0$ ,  $a, b = x, y$ . The second is set on the vertical displacement  $V_z^2$  and the antiplane stress vector  $(\Sigma_{xz}^0, \Sigma_{yz}^0)$ . Both problems can be solved explicitly resulting in

$$\Sigma_{xx}^0 = \Sigma_{xy}^0 = \Sigma_{yy}^0 = \Sigma_{xz}^0 = \Sigma_{yz}^0 = 0, \quad \Sigma_{zz}^0 = E_b \left( \frac{\partial U_z^0}{\partial \tilde{z}} - \frac{\partial^2}{\partial \tilde{z}^2} (U_x^0 \hat{x} + U_y^0 \hat{y}) \right). \quad (17)$$

The associated displacement  $\mathbf{V}^2$  that we dont need in the following is given in (A.4).

### 3.1.4. Macroscopic relations for the longitudinal motions

Once the dependance on  $\hat{\mathbf{x}}'$  of the displacements and the stress at the order 0 are known, we can determine the macroscopic relations. We start with those associated to the longitudinal motions. The axial stress  $N_{zz}^0(\mathbf{x})$  in (11) is obtained by integration of  $\Sigma_{zz}^0$  in (17) over  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$  and owing to  $\int_{\hat{\mathbb{S}}_b} \hat{x} d\hat{\mathbf{x}}' = \int_{\hat{\mathbb{S}}_b} \hat{y} d\hat{\mathbf{x}}' = 0$  for a circular cross-section. The equilibrium is then determined using  $(E_z)^0$  in (13) which reads  $\frac{\partial}{\partial \tilde{z}} \Sigma_{xz}^2 + \frac{\partial}{\partial \tilde{z}} \Sigma_{yz}^2 + \frac{\partial}{\partial \tilde{z}} \Sigma_{zz}^1 + \frac{\partial}{\partial \tilde{z}} \Sigma_{zz}^0 + \rho_b \omega^2 U_z^0 = 0$  (we have used that  $\Sigma_{xz}^0 = \Sigma_{yz}^0 = 0$  from (17) and  $V_z^0 = U_z^0(\mathbf{x})$  from (15)). Integrating the equilibrium over  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$  and accounting for the boundary condition  $\Sigma_{xz}^2 n_x + \Sigma_{yz}^2 n_y = 0$  on  $\partial \hat{\mathbb{S}}_b$ , we eventually get

$$N_{zz}^0(\mathbf{x}) = E_b \hat{\mathbb{S}}_b \frac{\partial U_z^0}{\partial \tilde{z}}(\mathbf{x}), \quad \frac{\partial N_{zz}^0}{\partial \tilde{z}}(\mathbf{x}) + \rho_b \omega^2 \hat{\mathbb{S}}_b U_z^0(\mathbf{x}) = 0, \quad (18)$$

To get the equilibrium, we have used that  $\frac{\partial}{\partial \tilde{z}} \int_{\hat{\mathbb{S}}_b} \Sigma_{zz}^1 d\hat{\mathbf{x}}' = \frac{\partial}{\partial \tilde{z}} N_{zz}^1 = 0$ . Indeed, we have  $\frac{\partial}{\partial \tilde{z}} N_{zz}^1 = -(\frac{\partial}{\partial \tilde{z}} N_{zz}^0(\mathbf{x}) + \rho_b \omega^2 \varphi U_z^0(\mathbf{x}))$ , hence  $N_{zz}^1$  could be a linear function of  $\tilde{z}$ . However, such a linear

dependence is prevented by the periodicity of the field with respect to  $\tilde{z}$ , which allows us to conclude. It is worth noting that  $N_{zz}^0$  and  $U_z^0$  do not depend on  $\tilde{z}$  associated to flexural motions, as expected. Also, from (18), we recover the wave equation for longitudinal motion

$$\frac{\partial^2 U_z^0}{\partial z^2} + K^2 U_z^0 = 0, \quad \text{with } K = \sqrt{\frac{\rho_b}{E_b}} \omega. \quad (19)$$

### 3.1.5. Macroscopic relations for the flexural motions

In this section, we use  $a = x, y$ . Constitutive relations link the bending moments  $M_a^0$  in (12) to the horizontal displacements  $V_a^0$ . They are obtained by integrating  $\Sigma_{zz}^0$  in (17) over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$  after multiplication by  $\hat{a}$ . Owing to  $\int \hat{x} d\hat{\mathbf{x}}' = \int \hat{y} d\hat{\mathbf{x}}' = \int \hat{x}\hat{y} d\hat{\mathbf{x}}' = 0$  and introducing the area moment of inertia  $\hat{I}_b = \int \hat{a}^2 d\hat{\mathbf{x}}' = \frac{\pi}{4} \hat{r}_b^4$ , we get

$$M_a^0(\mathbf{x}', \tilde{z}) = E_b \hat{I}_b \frac{\partial^2 U_a^0}{\partial \tilde{z}^2}(\mathbf{x}', \tilde{z}), \quad a = x, y. \quad (20)$$

We now move on the equations of equilibrium. Since  $\Sigma_{az}^0 = 0$  from (17), we also have  $T_a^0 = 0$  in (12). Next, from  $(E_a)^0$  in (13) and using (17), we have  $\frac{\partial}{\partial \tilde{x}} \Sigma_{xa}^2 + \frac{\partial}{\partial \tilde{y}} \Sigma_{ya}^2 + \frac{\partial}{\partial \tilde{z}} \Sigma_{za}^1 + \rho_b \omega^2 V_a^0 = 0$  that we integrate over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$ . Accounting for the boundary condition  $\Sigma_{xa}^2 n_x + \Sigma_{ya}^2 n_y = 0$  on  $\partial \hat{\mathbf{S}}_b$  and for the forms of  $V_a^0$  in (15), we get

$$T_a^0 = 0, \quad \frac{\partial T_a^1}{\partial \tilde{z}}(\mathbf{x}', \tilde{z}) + \rho_b \omega^2 \hat{\mathbf{S}}_b U_a^0(\mathbf{x}', \tilde{z}) = 0, \quad a = x, y, \quad (21)$$

(which tells us that  $T_a^1$  does not depend on  $z$ ). We still have to determine the relations between bending moments and shear forces. To do so, we use  $(E_z)^{-1}$  in (13),  $\frac{\partial}{\partial \tilde{x}} \Sigma_{xz}^1 + \frac{\partial}{\partial \tilde{y}} \Sigma_{yz}^1 + \frac{\partial}{\partial \tilde{z}} \Sigma_{zz}^0 = 0$  that we integrate over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$  after multiplication by  $\hat{a}$ . Using that  $\int_{\hat{\mathbf{S}}_b} \hat{a} \left( \frac{\partial}{\partial \tilde{x}} \Sigma_{xz}^1 + \frac{\partial}{\partial \tilde{y}} \Sigma_{yz}^1 \right) d\hat{\mathbf{x}}' = -T_a^1$  (integrating by part and accounting for  $\Sigma_{xz}^1 n_x + \Sigma_{yz}^1 n_y = 0$  on  $\partial \hat{\mathbf{S}}_b$ ), we eventually get

$$\frac{\partial M_a^0}{\partial \tilde{z}}(\mathbf{x}', \tilde{z}) + T_a^1(\mathbf{x}', \tilde{z}) = 0, \quad a = x, y. \quad (22)$$

Gathering (20), (21) and (22), we recover the wave equation for the flexural motions

$$\frac{\partial^4 U_a^0}{\partial \tilde{z}^4} - \tilde{\kappa}^4 U_a^0 = 0, \quad a = x, y, \quad \text{with } \tilde{\kappa} = \left( \frac{4\rho_b \omega^2}{E_b \hat{r}_b^2} \right)^{1/4}, \quad (23)$$

which is the rescaled version of (6).

### 3.2. Matching conditions and associated boundary/transmission conditions

The set of beam equations found in the previous section are valid far from the extremities, bottom and top, of the beams. It has to be complemented with boundary conditions. These conditions are obtained by making a suitable asymptotic analysis near the extremities of the beams. Two regimes of boundary layers associated to two different length scales have to be distinguished (see figure 4).

The first boundary layer appears when, from the inner region of the beams that we have considered in the previous section, we approach  $z = 0$  (resp.  $z = h_b$ ) within a distance of order  $\eta$ . By

rescaling  $z$  as  $\eta\tilde{z}$  near  $z = 0$  (resp.  $z = \eta\tilde{z} + h_b$  near  $z = h_b$ ) with  $\tilde{z}$  lying in  $(0, +\infty)$  (resp.  $(-\infty, 0)$  near  $z = h_b$ ), we are entering an intermediate region where the fields loose their periodicity in  $\tilde{z}$ . These mesoscopic regions encapsulate the first boundary layers at the scale  $h_f = \kappa^{-1}$ .

The second boundary layer appears when, from the mesoscopic regions, we approach  $z = 0$  (resp.  $z = h_b$ ) within a distance of order  $\eta^2$ . Doing so, we entering a microscopic region where evanescent fields enable (i) that the fields in the beam connect those in the soil at  $z = 0$ , (ii) that the fields in the beam feel the traction-free termination at  $z = h_b$ . These microscopic regions encapsulate the second boundary layers at the scale  $\ell$ . A consistent dialogue between the microscopic, the mesoscopic and the macroscopic regions is ensured through suitable matching conditions on the displacements and stresses.

### 3.2.1. The mesoscopic region near the extremities : beams with boundary layers

In these intermediate regions where boundary layers for the beams are triggered, new expansions of the fields have to be thought. In particular, we have to drop the periodic assumption on  $\tilde{z}$  to encapsulate the boundary layers (periodic terms are typically  $\cos \kappa z$  and the loss of periodicity correspond to the appearance of terms in  $\cosh \kappa z$ ). Specifically we assume that near  $z = z^*$  with  $z^* = 0$  or  $z^* = h_b$ ,  $\mathbf{u}$  and  $\boldsymbol{\sigma}$  can be expanded as

$$\mathbf{u} = \sum_{n \geq 0} \eta^n \tilde{\mathbf{V}}^n|_{z^*}(\mathbf{x}', \hat{\mathbf{x}}', \tilde{z}), \quad \boldsymbol{\sigma} = \sum_{n \geq 0} \eta^n \tilde{\boldsymbol{\Sigma}}^n|_{z^*}(\mathbf{x}', \hat{\mathbf{x}}', \tilde{z}), \quad z^* = 0 \text{ or } h_b, \quad (24)$$

where the fast variables  $(\hat{\mathbf{x}}', \tilde{z})$  lie in  $\tilde{\mathbf{X}}_0$  if  $z^* = 0$  and in  $\tilde{\mathbf{X}}_{h_b}$  if  $z^* = h_b$  with

$$\tilde{\mathbf{X}}_0 = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \tilde{z} \in (0, +\infty)\}, \quad \tilde{\mathbf{X}}_{h_b} = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \tilde{z} \in (-\infty, 0)\}. \quad (25)$$

The expansions (24) in the two intermediate regions at the extremities have to match the  $\tilde{z}$ -periodic expansions (10) inside the beams when  $|\tilde{z}|$  goes to infinity. Making asymptotics on this mesoscopic problem is straightforward from the previous section (with  $\frac{\partial}{\partial z} = 0$ , see also (A.2) and (A.5)) and we get the following results. First, the hierarchies of equations as in (13) and (14) hold. Next, we find that the axial displacement  $\tilde{V}_z^0$  and the normal force  $\tilde{N}_{zz}^0$  do not depend on  $(\hat{\mathbf{x}}', \tilde{z})$  and  $\tilde{z}$  respectively. Using the matching conditions with the inner region of the beam, we get at the bottom of the beams and at the top of the beams

$$\tilde{V}_z^0|_{z^*} = U_z^0(\mathbf{x}', z^*), \quad \tilde{N}_{zz}^0|_{z^*} = N_{zz}^0(\mathbf{x}', z^*), \quad a = x, y, \quad z^* = 0, h_b. \quad (26)$$

Secondly, the tangential displacement still satisfies the Euler-Bernoulli equations as in the inner region of the beams, the difference being that the equations are now set on a semi-infinite domain as opposed to a periodic domain. Specifically the mesoscopic Euler-Bernoulli equations read at bottom and the top of the beams

$$\begin{cases} \tilde{V}_a^0|_{z^*} = \tilde{U}_a^0|_{z^*}(\mathbf{x}', \tilde{z}), & \tilde{T}_a^0|_{z^*} = 0, & \tilde{M}_a^0|_{z^*} = E_b \hat{I}_b \frac{\partial^2 \tilde{U}_a^0|_{z^*}}{\partial \tilde{z}^2} \\ \frac{\partial \tilde{M}_a^0|_{z^*}}{\partial \tilde{z}} + \tilde{T}_a^1|_{z^*} = 0, & \frac{\partial \tilde{T}_a^1|_{z^*}}{\partial \tilde{z}} + \rho_b \omega^2 \hat{S}_b \tilde{U}_a^0|_{z^*} = 0, & a = x, y, \quad z^* = 0, h_b, \end{cases} \quad (27)$$

hence

$$\frac{\partial^4 \tilde{U}_a^0|_{z^*}}{\partial \tilde{z}^4} - \tilde{\kappa}^4 \tilde{U}_a^0|_{z^*} = 0, \quad \text{with } \tilde{\kappa} = \left( \frac{4\rho_b \omega^2}{E_b \hat{I}_b^2} \right)^{1/4}. \quad (28)$$

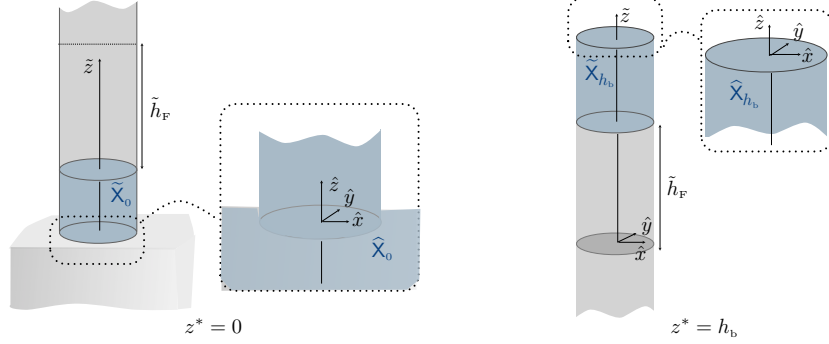


Figure 4: The intermediate regions and the microscopic regions near  $z = z^*$ ,  $z^* = 0$  and  $h_b$ . The intermediate regions  $\tilde{X}_0$  and  $\tilde{X}_{h_b}$ , correspond to the occurrence of boundary layers as the flexural displacements loose their  $\tilde{h}_F$ -periodicity (valid far from  $z = z^*$ ). The microscopic regions,  $\hat{X}_0$  and  $\hat{X}_{h_b}$ , correspond to the classical boundary layers at the scale of  $\ell$ .

The periodic behavior is retrieved when  $|\tilde{z}|$  goes to infinity, the solution in the intermediate region matching the  $\tilde{z}$ -periodic solution in the inner region of the beams

$$(\tilde{V}_a^0, \tilde{M}_a^0, \tilde{T}_a^1) \sim (U_a^0, M_a^0, T_a^1), \quad |\tilde{z}| \rightarrow \infty, \quad a = x, y. \quad (29)$$

Therefore intermediate scale can be fully solved upon the determination of the boundary conditions at  $\tilde{z} = 0$ . These latter are obtained from the matching conditions with the microscopic regions.

### 3.2.2. The microscopic regions near the extremities

The second step of the matching involves the microscopic region of extension  $\eta^2$  near  $z = z^*$ . In these regions, only the fast variable  $\hat{\mathbf{x}}$  acts in the three directions, while the slow variable  $\mathbf{x}'$  is kept to allow for displacements at the scale of the wavelength. Accordingly, the expansions are now thought of the form

$$\mathbf{u} = \sum_{n \geq 0} \eta^n \hat{\mathbf{V}}^n|_{z^*}(\mathbf{x}', \hat{\mathbf{x}}), \quad \boldsymbol{\sigma} = \sum_{n \geq 0} \eta^n \hat{\boldsymbol{\Sigma}}^n|_{z^*}(\mathbf{x}', \hat{\mathbf{x}}), \quad z^* = 0, h_b, \quad (30)$$

where the fast variable  $\hat{\mathbf{x}} = (\hat{\mathbf{x}}', \hat{z})$  lies in  $\hat{X}_0$  if  $z^* = 0$  and in  $\hat{X}_{h_b}$  if  $z^* = h_b$  with

$$\begin{cases} \hat{X}_0 = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \hat{z} \in (0, +\infty)\} \cup \{\hat{\mathbf{x}}' \in \mathbf{S}_s, \hat{z} \in (-\infty, 0)\}, & \mathbf{S}_s = \{\hat{\mathbf{x}}' \in (0, \hat{\ell}_x) \times (0, \hat{\ell}_y)\}, \\ \hat{X}_{h_b} = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \hat{z} \in (-\infty, 0)\}. \end{cases} \quad (31)$$

Eventually, according to (30), the differential operator reads  $\nabla \rightarrow \frac{1}{\eta^2} \nabla_{\hat{\mathbf{x}}} + \nabla_{\mathbf{x}'}$ .

*Effective boundary conditions at  $z = h_b$ .* **OJO, j'en suis LA** Near the top of the beams, each term of the expansion (30) satisfies the traction-free condition  $\hat{\boldsymbol{\Sigma}}^n|_{h_b} \mathbf{n} = \mathbf{0}$  on  $\partial \hat{\mathbf{S}}_b$ . Next, the whole expansions match those of the intermediate problem, (24) (set in  $\tilde{X}_{h_b}$ ). We shall need the

matching conditions for the stress only. Using  $\tilde{z} = \eta\hat{z}$  in (24), re-expanding and identifying with (30), we get

$$\widehat{\Sigma}^0|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\tilde{z} \rightarrow -\infty} \widehat{\Sigma}^0|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}), \quad \widehat{\Sigma}^1|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\tilde{z} \rightarrow -\infty} \left( \widehat{\Sigma}^1|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}) - \hat{z} \frac{\partial \widehat{\Sigma}^0|_{h_b}}{\partial \tilde{z}}(\mathbf{x}', \hat{\mathbf{x}}', 0) \right). \quad (32)$$

To derive the boundary conditions we only need the equations of equilibrium which read

$$(E)^n \quad \text{div}_{\hat{\mathbf{x}}} \widehat{\Sigma}^{n+2}|_{h_b} + \text{div}_{\mathbf{x}'} \widehat{\Sigma}^n|_{h_b} + \rho_b \omega^2 \widehat{\mathbf{V}}^n|_{h_b} = 0. \quad (33)$$

(We implicitly set  $\widehat{\Sigma}^m = \mathbf{0}$ ,  $\widehat{\mathbf{V}}^m = \mathbf{0}$  for  $m < 0$ .)

At the dominant order,  $n = 0$  from (33),  $\text{div}_{\hat{\mathbf{x}}} \widehat{\Sigma}^0|_{h_b} = 0$  that we integrate over  $\widehat{\mathbf{X}}_{h_b}$  to get  $\mathbf{0} = \int_{\widehat{\mathbf{X}}_{h_b}} \text{div}_{\hat{\mathbf{x}}} \widehat{\Sigma}^0|_{h_b} d\hat{\mathbf{x}} = - \lim_{\tilde{z} \rightarrow -\infty} \int_{\hat{\mathbf{S}}_b} \widehat{\Sigma}^0|_{h_b} \mathbf{e}_z d\hat{\mathbf{x}}'$ . From the matching condition (32) at dominant order, we thus have  $\widetilde{T}_a^0|_{h_b}(\mathbf{x}', 0) = \int_{\hat{\mathbf{S}}_b} \widehat{\Sigma}_{az}^0|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}', 0) d\hat{\mathbf{x}}' = 0$ ,  $a = x, y$ , which is simply consistent with (27). For the same reasons, we also get  $\widetilde{N}_{zz}^0|_{h_b}(\mathbf{x}', 0) = \int_{\hat{\mathbf{S}}_b} \widehat{\Sigma}_{zz}^0|_{h_b}(\mathbf{x}', \hat{\mathbf{x}}', 0) d\hat{\mathbf{x}}' = 0$  from (27) and as  $\widetilde{N}_{zz}^0$  does not depend on  $\tilde{z}$  from (26) we eventually have that  $\widetilde{N}_{zz}^0|_{h_b} = N_{zz}^0(\mathbf{x}', h_b) = 0$ . At the next order, we do the same (but we don't need  $N_{zz}^1$ ); we use that  $\frac{\partial \widetilde{T}_a^0|_{h_b}}{\partial \tilde{z}} = 0$  holds after derivation with respect to  $\tilde{z}$  in (27) hence  $\widetilde{T}_a^1|_{h_b}(\mathbf{x}', 0) = \lim_{\tilde{z} \rightarrow -\infty} \left( \int_{\hat{\mathbf{S}}_b} \widehat{\Sigma}_{az}^1|_{h_b} d\hat{\mathbf{x}}' - \hat{z} \int_{\hat{\mathbf{S}}_b} \frac{\partial \widetilde{T}_a^0|_{h_b}}{\partial \tilde{z}}(\mathbf{x}', 0) d\hat{\mathbf{x}}' \right) = 0$  from the matching conditions (32). At this stage we have obtained in the intermediate region at the top of the beams,

$$\widetilde{T}_a^n|_{h_b}(\mathbf{x}', 0) = 0, \quad a = x, y, \quad n = 0, 1, \quad \widetilde{N}_{zz}^0|_{h_b}(\mathbf{x}', \tilde{z}) = N_{zz}^0(\mathbf{x}', h_b) = 0. \quad (34)$$

To derive the boundary conditions on the bending moments, we come back to  $\text{div}_{\hat{\mathbf{x}}} \widehat{\Sigma}^0|_{h_b} = 0$  that we multiply by  $\mathbf{a} = -\hat{z}\mathbf{e}_x + \hat{x}\mathbf{e}_z$  and further integrate over  $\widehat{\mathbf{X}}_{h_b}$ . By construction  $\nabla \mathbf{a}$  is antisymmetric, hence  $\nabla \mathbf{a} \widehat{\Sigma}^0 = 0$ . It follows that  $0 = \int_{\partial \widehat{\mathbf{X}}_{h_b}} \mathbf{a} \cdot \widehat{\Sigma}^0|_{h_b} \mathbf{n} d\hat{\mathbf{x}} = - \lim_{\tilde{z} \rightarrow -\infty} \int_{\hat{\mathbf{S}}_b} \left( \hat{x} \widehat{\Sigma}_{zz}^0|_{h_b} - \hat{z} \widehat{\Sigma}_{xz}^0|_{h_b} \right) d\hat{\mathbf{x}}' = \int_{\hat{\mathbf{S}}_b} \hat{x} \widehat{\Sigma}_{xz}^0|_{h_b} d\hat{\mathbf{x}}'$  where we have accounted for  $\widetilde{T}_x^0|_{h_b}(\mathbf{x}', 0) = \lim_{\tilde{z} \rightarrow -\infty} \int_{\hat{\mathbf{S}}_b} \widehat{\Sigma}_{xz}^0|_{h_b} d\hat{\mathbf{x}}' = 0$  from (27) and (32), hence the boundary condition  $\widetilde{M}_x^0|_{h_b}(\mathbf{x}', 0) = 0$ . The procedure can be repeated with  $\mathbf{a} = -\hat{z}\mathbf{e}_y + \hat{y}\mathbf{e}_z$  resulting in  $\widetilde{M}_y^0|_{h_b}(\mathbf{x}', 0) = 0$ . We have thus obtained

$$\widetilde{M}_a^0|_{h_b}(\mathbf{x}', 0) = 0, \quad a = x, y. \quad (35)$$

*Effective transmission conditions at  $z = 0$ .* Near the interface between the beam and the substrate, we shall need the solution valid in the substrate far from  $z = 0$ ; this latter has variations associated to the typical wavelength only, hence the expansions are sought of the form

$$\mathbf{u} = \sum_{n \geq 0} \eta^n \mathbf{u}^n(\mathbf{x}), \quad \boldsymbol{\sigma} = \sum_{n \geq 0} \eta^n \boldsymbol{\sigma}^n(\mathbf{x}). \quad (36)$$

We first determine the transmission conditions for the displacements. Given the mesoscopic and microscopic expansions (24) and (30), the matching conditions for the displacements between

the soil, the microscopic region and the intermediate region read at the first orders

$$\begin{cases} \widehat{\mathbf{V}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\hat{z} \rightarrow +\infty} \widehat{\mathbf{V}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}), & \mathbf{u}^0(\mathbf{x}', 0) = \lim_{\hat{z} \rightarrow -\infty} \widehat{\mathbf{V}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}), \\ \widehat{\mathbf{V}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\hat{z} \rightarrow +\infty} \left( \widehat{\mathbf{V}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}) - \hat{z} \frac{\partial \widehat{\mathbf{V}}^0|_0}{\partial \hat{z}}(\mathbf{x}', \hat{\mathbf{x}}', 0) \right), & \mathbf{u}^1(\mathbf{x}', 0) = \lim_{\hat{z} \rightarrow -\infty} \widehat{\mathbf{V}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}). \end{cases} \quad (37)$$

In the microscopic region, the hierarchy of constitutive and equilibrium equations is given by

$$\begin{cases} (\mathbf{C})^n \quad \widehat{\boldsymbol{\Sigma}}^n|_0 = \mathbf{A} \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}}(\widehat{\mathbf{V}}^{n+2}|_0) + \mathbf{A} \boldsymbol{\varepsilon}^{\mathbf{x}'}(\widehat{\mathbf{V}}^n|_0), \\ (\mathbf{C}')^n \quad \boldsymbol{\varepsilon}^{\hat{\mathbf{x}}}(\widehat{\mathbf{V}}^{n+2}|_0) + \boldsymbol{\varepsilon}^{\mathbf{x}'}(\widehat{\mathbf{V}}^n|_0) = \frac{1 + \nu_b}{E_b} \widehat{\boldsymbol{\Sigma}}^n|_0 - \frac{\nu_b}{E_b} \text{tr}(\widehat{\boldsymbol{\Sigma}}^n|_0) \mathbf{I}. \end{cases} \quad (38)$$

We have from  $(\mathbf{C}')^{-2}$  and  $(\mathbf{C}')^{-1}$  that  $\widehat{\mathbf{V}}^n|_0 = \widehat{\boldsymbol{\Omega}}^n|_0(\mathbf{x}') \times \hat{\mathbf{x}} + \widehat{\mathbf{U}}^n|_0(\mathbf{x}')$ ,  $n = 0, 1$ , are piecewise rigid body motions for  $\hat{z} < 0$  and  $\hat{z} > 0$ . Invoking the periodic of  $\widehat{\mathbf{V}}^n|_0$  with respect to  $\hat{\mathbf{x}}'$  for  $\hat{z} < 0$  and the continuity of  $\widehat{\mathbf{V}}^n$  at  $\hat{z} = 0$ , these rigid body motions reduce to single translations  $\widehat{\mathbf{V}}^n|_0 = \widehat{\mathbf{U}}^n|_0(\mathbf{x}')$  independent of  $\hat{\mathbf{x}}$ . It follows from (37) that the beams are clamped to the substrate at the dominant order

$$u_a^0(\mathbf{x}', 0) = \widetilde{U}_a^0|_0(\mathbf{x}', 0), \quad \frac{\partial \widetilde{U}_a^0|_0}{\partial \hat{z}}(\mathbf{x}', 0) = 0. \quad (39)$$

125 In particular, we have used that since  $\widehat{\mathbf{V}}^1|_0 = \widehat{\mathbf{U}}^1|_0(\mathbf{x}')$  does not contain any contribution linear  
 126 in  $\hat{z}$ , then  $\frac{\partial \widetilde{U}_a^0}{\partial \hat{z}}$  has to cancel so that the matching condition (37) between  $\widehat{\mathbf{V}}^1|_0$  and  $\widetilde{\mathbf{V}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}', 0)$   
 127 can be fulfilled.

We now establish transmission conditions on the stresses. The matching conditions for the stresses between the soil, the microscopic region and the intermediate region read

$$\begin{cases} \widehat{\boldsymbol{\Sigma}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\hat{z} \rightarrow +\infty} \widehat{\boldsymbol{\Sigma}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}), & \boldsymbol{\sigma}^0(\mathbf{x}', 0) = \lim_{\hat{z} \rightarrow -\infty} \widehat{\boldsymbol{\Sigma}}^0|_0(\mathbf{x}', \hat{\mathbf{x}}), \\ \widehat{\boldsymbol{\Sigma}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}', 0) = \lim_{\hat{z} \rightarrow +\infty} \left( \widehat{\boldsymbol{\Sigma}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}) - \hat{z} \frac{\partial \widehat{\boldsymbol{\Sigma}}^0|_0}{\partial \hat{z}}(\mathbf{x}', \hat{\mathbf{x}}', 0) \right), & \boldsymbol{\sigma}^1(\mathbf{x}', 0) = \lim_{\hat{z} \rightarrow -\infty} \widehat{\boldsymbol{\Sigma}}^1|_0(\mathbf{x}', \hat{\mathbf{x}}). \end{cases} \quad (40)$$

By integration of  $\text{div}_{\hat{\mathbf{x}}} \widehat{\boldsymbol{\Sigma}}^0|_0 = \text{div}_{\hat{\mathbf{x}}} \widehat{\boldsymbol{\Sigma}}^1|_0 = 0$  over  $\widehat{\mathbf{X}}_0 = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \hat{z} \in (0, +\infty)\}$  and using (40) we get the following transmission conditions on the normal stress vector

$$\sigma_{a\hat{z}}^0(\mathbf{x}', 0) = \hat{\mathbf{S}}^{-1} \widetilde{T}_a^0|_0(\mathbf{x}', 0) = 0, \quad \sigma_{a\hat{z}}^1(\mathbf{x}', 0) = \hat{\mathbf{S}}^{-1} \widetilde{T}_a^1|_0(\mathbf{x}', 0), \quad a = x, y, \quad (41)$$

128 with  $\hat{\mathbf{S}} = \hat{\ell}_x \times \hat{\ell}_y$  being the rescaled periodic section area and where we have accounted for  
 129  $\frac{\partial \widetilde{T}_a^0|_0}{\partial \hat{z}}(\mathbf{x}', 0) = 0$ ,  $a = x, y$  from (27).

### 130 3.3. Construction of a unique problem

We construct the problem at the dominant order for the displacement  $U_z(\mathbf{x}) = U_z^0(\mathbf{x})$  and  $U_a(\mathbf{x}) = (U_a(\mathbf{x}', \tilde{z}), \widetilde{U}_a^0|_0(\mathbf{x}', \tilde{z}), \widetilde{U}_a^0|_{h_b}(\mathbf{x}', \tilde{z}))$  accounting for  $\tilde{z} = z/\eta$  and  $r_b = \eta^2 \hat{r}_b$ . Gathering the longitudinal and flexural equations (19), (23), (26), (28) in both the inner region of the beams

and the intermediate region near the extremities, we get the final set of homogenized equations for  $z \in (0, h_b)$  on the

$$\begin{cases} \frac{\partial^4 U_a}{\partial z^4} - \kappa^4 U_a = 0, & \kappa^4 = \frac{4\rho_b \omega^2}{E_b r_b^2}, \\ \frac{\partial^2 U_z}{\partial z^2} + K^2 U_z = 0, & K^2 = \frac{\rho_b}{E_b} \omega^2. \end{cases} \quad (42)$$

Gathering the transmission conditions (39) and (41) at  $z = 0$  and free end boundary conditions (34) and (35) at  $z = h_b$  leads to the boundary conditions

$$\begin{cases} \text{at } z = 0, & U_a = u_a(\mathbf{x}', 0), \quad \frac{\partial U_a}{\partial z} = 0, & \sigma_{az}(\mathbf{x}', 0) = -\varphi E_b \frac{r_b^2}{4} \frac{\partial^3 U_a}{\partial z^3}, \quad a = x, y, \\ & U_z = u_z(\mathbf{x}', 0), & \sigma_{zz}(\mathbf{x}', 0) = \varphi E_b \frac{\partial U_z}{\partial z}, \\ \text{at } z = h_b, & \frac{\partial^2 U_a}{\partial z^2} = \frac{\partial^3 U_a}{\partial z^3} = 0, \quad \frac{\partial U_z}{\partial z} = 0. \end{cases} \quad (43)$$

#### 131 4. Validation of the effective model

In this section, we shall inspect the ability of our effective model to reproduce the actual scattering properties for an incident plane wave in a two-dimensional setting, with plates infinite along  $y$  and incident waves in the  $(x, z)$  plane (figure 5). In this case, we have

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho_s \omega^2 \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\sigma} = 2\mu_s \boldsymbol{\varepsilon} + \lambda_s \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}, \quad \text{for } z \in (-\infty, 0), \\ \sigma_{xz}(x, 0) = \mu_s k_T f_F(\omega, h_b) u_x(\mathbf{x}', 0), \quad \sigma_{zz}(x, 0) = \mu_s k_T f_L(\omega, h_b) u_z(x, 0), \end{cases} \quad (44)$$

and

$$\begin{cases} f_F(\omega, h_b) = \varphi \frac{\rho_b k_T}{\rho_s \kappa} \frac{\operatorname{sh} \kappa h_b \cos \kappa h_b + \operatorname{ch} \kappa h_b \sin \kappa h_b}{1 + \cos \kappa h_b \operatorname{ch} \kappa h_b}, & \kappa = \left( \frac{3\rho_b \omega^2}{E_b^* r_b^2} \right)^{1/4}, \\ f_L(\omega, h_b) = \varphi \frac{\rho_b k_T}{\rho_s K} \tan K h_b, & K = \sqrt{\frac{\rho_b}{E_b^*}} \omega. \end{cases} \quad (45)$$

with

$$\varphi = \frac{2r_b}{\ell}, \quad E_b^* = \frac{4\mu_b(\lambda_b + \mu_b)}{\lambda_b + 2\mu_b} = \frac{E_b}{1 - \nu_b^2}. \quad (46)$$

We use as reference case:

$$\begin{aligned} & \text{At the angular frequency } \omega = 120 \text{ rad.s}^{-1}: \quad \ell = 1 \text{ m}, 2r_b = 0.5 \text{ m and } h_b \in (0, 90) \text{ m}, \\ & \text{Material properties:} \quad \rho_b = 500 \text{ kg.m}^{-3}, E_b = 2 \text{ GPa}, \nu_b = 0.3 \\ & \quad \rho_s = 1000 \text{ kg.m}^{-3}, E_s = 2 \text{ GPa}, \nu_s = 0.2, \end{aligned} \quad (47)$$

132 resulting in  $C = 0.574$  and  $\mathcal{K} = 0.091$ .

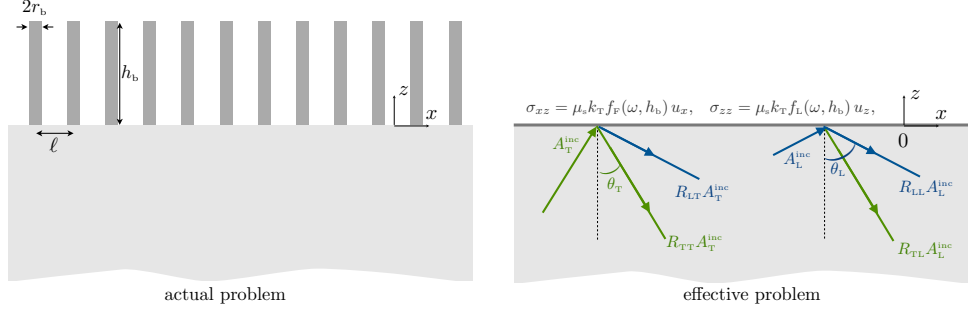


Figure 5: Scattering of an incident plane wave on an array of plates (two-dimensional settings); actual problem and effective problem with an interface ruled by (44)-(45). The incident wave is characterized by  $(A_L, A_T)$ , (50), and the solution reads (51) with four scattering coefficients (52).

#### 4.1. Explicit solution of the effective problem

To begin with, we stress that we don't have a one-to-one correspondance between frequency  $\omega$  and height  $h_b$  as the wavelength associated to the flexural motions scales as  $\sqrt{\omega}h_b$  while that of the longitudinal motions scales as  $\omega h_b$ . Next, we shall see that the effective conditions (44) involve two non-dimensional parameters and a non-dimensional frequency or a non-dimensional height.

##### 4.1.1. Result for given $h_b$ varying the frequency $\omega$

From (44)-(45), we have

$$\left\{ \begin{array}{l} \text{with } C = \varphi \sqrt{\frac{\rho_b E_b^*}{\rho_s \mu_s}}, \quad S = \frac{\sqrt{3}h_b}{r_b}, \quad \text{and} \quad \Omega = (\kappa h_b)^2 = \frac{\omega}{\omega_0}, \quad \omega_0 = \frac{1}{Sh_b} \sqrt{\frac{E_b^*}{\rho_b}}, \\ f_F(\Omega; C, S) = C \frac{\sqrt{\Omega}}{S} \frac{\text{sh } \sqrt{\Omega} \cos \sqrt{\Omega} + \text{ch } \sqrt{\Omega} \sin \sqrt{\Omega}}{1 + \cos \sqrt{\Omega} \text{ch } \sqrt{\Omega}}, \quad f_L(\Omega; C, S) = C \tan\left(\frac{\Omega}{S}\right). \end{array} \right. \quad (48)$$

The parameter  $C$  measures the effectiveness of the coupling between the plates and the substrate; higher coupling is obtained for heavier or stiffer plates. Next  $S$  is a measure of the slenderness of the plates; it does not affect the effectiveness of the longitudinal coupling between plates and substrate but large slendernesses weaken the flexural coupling. Eventually, the flexural resonances are given by constant  $\sqrt{\Omega}_{\text{flex. res.}} \simeq \frac{\pi}{2} + n\pi$ ,  $n$  integer, whatever the dimensions and material properties of the plates. In contrast, the positions of the longitudinal resonances with respect to the flexural ones depend on  $S$  through  $\Omega_{\text{long. res.}} = S\left(\frac{\pi}{2} + n\pi\right)$  (a larger slenderness produces a relative denser populations of flexural resonances, with  $N \sim \sqrt{S}$  flexural resonances between two longitudinal ones).

149 **4.1.2. Result for given  $\omega$  and varying the height  $h_b$**

We adapt the preceding expressions as  $\mathcal{S}$  and  $\omega_0$  depend on  $h_b$  which now varies (at prescribed frequency) and we obtain

$$\left\{ \begin{array}{l} \text{with } C = \varphi \sqrt{\frac{\rho_b E_b^*}{\rho_s \mu_s}}, \quad \mathcal{K} = \left( \frac{(Kr_b)^2}{3} \right)^{1/4}, \quad \text{and} \quad H = (\kappa h_b), \\ f_F(H; C, \mathcal{K}) = C \mathcal{K} \frac{\text{sh} H \cos H + \text{ch} H \sin H}{1 + \cos H \text{ch} H}, \quad f_L(H; C, \mathcal{K}) = C \tan(\mathcal{K} H). \end{array} \right. \quad (49)$$

150 The parameter  $C$  is still involved as it determines the coupling due to the contrast in the material  
 151 properties between the substrate and the plates. The new non-dimensional parameter  $\mathcal{K}$  is a mea-  
 152 sure of the frequency regime through wavenumber  $K = \omega \sqrt{\frac{\rho_b}{E_b^*}}$ . Higher frequency longitudinal  
 153 resonances take place for  $\mathcal{K}$  large enough. The flexural resonances still take place at  $H \sim \frac{\pi}{2} + n\pi$   
 154 and, at prescribed frequency, the density of flexural resonances between two longitudinal ones  
 155 goes as  $\frac{1}{\mathcal{K}^2}$ .

156 **4.1.3. Solution of the effective problem**

The scattering problem of an incident wave can be solved owing to the use of the Helmholtz decomposition for the elastic potentials  $(\phi, \psi)$ , with  $\mathbf{u} = \nabla \phi + \nabla \times (\psi \mathbf{e}_y)$ . The incident wave in the substrate is defined in terms of the incident potentials

$$\left\{ \begin{array}{l} \phi^{\text{inc}}(x, z) = A_L e^{i\alpha_L z} e^{i\beta x}, \quad \psi^{\text{inc}}(x, z) = A_T e^{i\alpha_T z} e^{i\beta x}, \\ \text{with } (\alpha_L, \beta) = k_L (\cos \theta_L, \sin \theta_L), \quad (\alpha_T, \beta) = k_T (\cos \theta_T, \sin \theta_T), \end{array} \right. \quad (50)$$

and  $k_L = \sqrt{\frac{\rho_s}{\lambda_s + 2\mu_s}} \omega$  and  $k_T = \sqrt{\frac{\rho_s}{\mu_s}} \omega$ . The solution in the substrate reads

$$\left\{ \begin{array}{l} \phi(x, z) = \phi^{\text{inc}}(x, z) + (R_{LL} A_L + R_{LT} A_T) e^{-i\alpha_L z} e^{i\beta x}, \\ \psi(x, z) = \psi^{\text{inc}}(x, z) + (R_{TL} A_L + R_{TT} A_T) e^{-i\alpha_T z} e^{i\beta x}. \end{array} \right. \quad (51)$$

157 The effective problem is a one dimensional problem and making use of (B.1) it can be solved  
 158 explicitly, specifically we get

$$\left\{ \begin{array}{l} R_{LL} = \frac{1}{D} \left[ \sin 2\theta_T \sin 2\theta_L - \xi^2 \cos^2 2\theta_T - i\xi (f_L \cos \theta_L - \xi f_F \cos \theta_T) - \xi f_L f_F \cos(\theta_L + \theta_T) \right], \\ R_{TT} = \frac{1}{D} \left[ \sin 2\theta_T \sin 2\theta_L - \xi^2 \cos^2 2\theta_T + i\xi (f_L \cos \theta_L - \xi f_F \cos \theta_T) - \xi f_L f_F \cos(\theta_L + \theta_T) \right], \\ R_{LT} = -\frac{\xi^2 \sin 2\theta_T}{D} (2 \cos 2\theta_T + f_L f_F), \quad R_{TL} = \frac{\sin 2\theta_L}{D} (2 \cos 2\theta_T + f_L f_F), \\ \text{where } D = \sin 2\theta_T \sin 2\theta_L + \xi^2 \cos^2 2\theta_T - i\xi (f_L \cos \theta_L + \xi f_F \cos \theta_T) - \xi f_L f_F \cos(\theta_L - \theta_T), \\ \text{with } (f_L, f_F) \text{ in (48) or (49), and } \xi = \sqrt{\frac{\lambda_s + 2\mu_s}{\mu_s}}. \end{array} \right. \quad (52)$$

<sup>2</sup>The parameter  $\mathcal{K}$  can be written  $\mathcal{K} = \frac{K}{\kappa} = \frac{h_F}{h_L}$ ; at prescribed frequency, it measures the relative wavelengths associated to flexural and longitudinal motions.

We get the displacement fields in the substrate  $z \in (-\infty, 0)$  with

$$\begin{cases} u_x(x, z) = ik_T \left[ \sin \theta_T (A_L e^{i\alpha_L z} + (R_{LL} A_L + R_{LT} A_T) e^{-i\alpha_L z}) - \cos \theta_T (A_T e^{i\alpha_T z} - (R_{TL} A_L + R_{TT} A_T) e^{-i\alpha_T z}) \right], \\ u_z(x, z) = ik_L \left[ \cos \theta_L (A_L e^{i\alpha_L z} - (R_{LL} A_L + R_{LT} A_T) e^{-i\alpha_L z}) + \sin \theta_L (A_T e^{i\alpha_T z} + (R_{TL} A_L + R_{TT} A_T) e^{-i\alpha_T z}) \right], \end{cases} \quad (53)$$

and in the region of the beams  $z \in (0, h_b)$ , with

$$\begin{cases} U_x(x, z) = u_x(x, 0) V_L(z), & U_z(x, z) = u_z(x, 0) V_F(z), \\ \begin{cases} u_x(x, 0) = \frac{ik_T}{D} [\sin 2\theta_L (2 \cos \theta_T - i f_L) A_L - 2\xi \cos \theta_T (\xi \cos 2\theta_T - i \cos \theta_L f_L) A_T] e^{i\beta x}, \\ u_z(x, 0) = \frac{i\xi^2 k_L}{D} [2 \cos \theta_L (\cos 2\theta_T - i \cos \theta_T f_F) A_L + \sin 2\theta_T (2 \cos \theta_L - i\xi f_F) A_T] e^{i\beta x}. \end{cases} \end{cases} \quad (54)$$

#### 4.2. Results

To begin with, we come back to the representation of the figure 12 in [1] where we stressed the limit of the model to the occurrence of the first longitudinal resonance and this limitation to low frequencies or small plates is now overcome<sup>3</sup>. We have computed numerically the actual displacements  $u_z(x, 0)$  and  $u_z(x, z_0)$  ( $z_0 = -5$  m) using the multimodal method presented in [15] in the reference case (47). We used ( $A_L = \frac{1}{2\beta}$ ,  $A_T = -\frac{1}{\alpha_T}$ ) resulting in an incident wave with unitary horizontal displacement at  $z = 0$ . Results are reported in figure 6 together with the displacements given by the effective model (53). For  $h_b \in (0, 90)$  m, striking variations associated to the 17 flexural resonances are visible, being superimposed to an underlying smooth curve dictated by the first two longitudinal resonances at  $h_b = 27.5$  m and 82 m. The overall agreement between

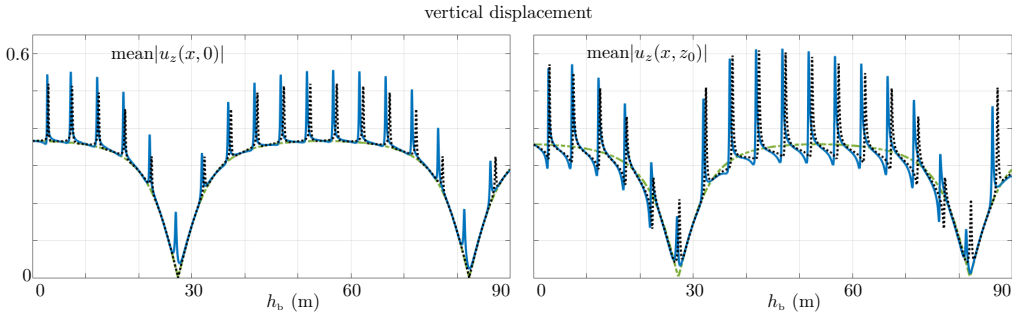


Figure 6: Variations of the mean vertical displacement  $u_z$  at  $z = 0$  and  $z = z_0 = -5$  m in the reference case, (47), for  $\theta_T = 45^\circ$ . Plain blue lines show  $u_z$  computed numerically, dotted black lines the close forms in (53) and green dashed-dotted lines the same when neglecting the flexural motions of the plates ( $f_F = 0$  in (52)).

the actual and the effective solutions is good although the actual resonances appear to be slightly shifted to frequencies lower than the predicted flexural resonances. We observe 11 resonances between the two first longitudinal resonances in agreement with a density  $\frac{1}{\mathcal{K}}$ , with  $\mathcal{K} \approx 0.092$

<sup>3</sup>At the occurrence of the first longitudinal resonance,  $f_L$  diverges in (45) resulting in  $u_z(x, 0) = 0$  in (44); this was not captured in the model of [1] as  $f_L = \varphi \frac{\rho_b}{\rho_s} k_T h_b$  was obtained as an approximation of  $f_L$  in (44) for  $K h_b \ll 1$ .

from (48). Next, because of the low contrast in the material properties the coupling  $C \simeq 0.574$  is weak, however the influence of both resonances is neat.

Eventually, a noticeable higher discrepancy is visible for  $u_z(x, 0)$  than for  $u_z(x, z_0 = -5 \text{ m})$ ; in particular the actual displacement  $u_z(x, 0)$  does not vanish at the longitudinal resonances for  $h_b \simeq 28 \text{ m}$  and  $82 \text{ m}$  as predicted by the effective model. We have checked that removing the evanescent field from the actual solution, hence keeping only the propagating waves in the substrate, the agreement is better and in particular the condition  $u_z(x, 0) = 0$  is recovered. At the deeper position  $z_0$ , the evanescent field is negligible and the agreement is better.

We now move on to the influence of the parameter  $(C, \mathcal{K})$  in (48). We have computed the actual reflection coefficients varying the wave incidence and the height of the plates for 3 different radii. The spectra of  $R_{\text{TT}}$  and  $R_{\text{LL}}$  are reported in figures 7 and 8<sup>4</sup>. The top panels show the actual

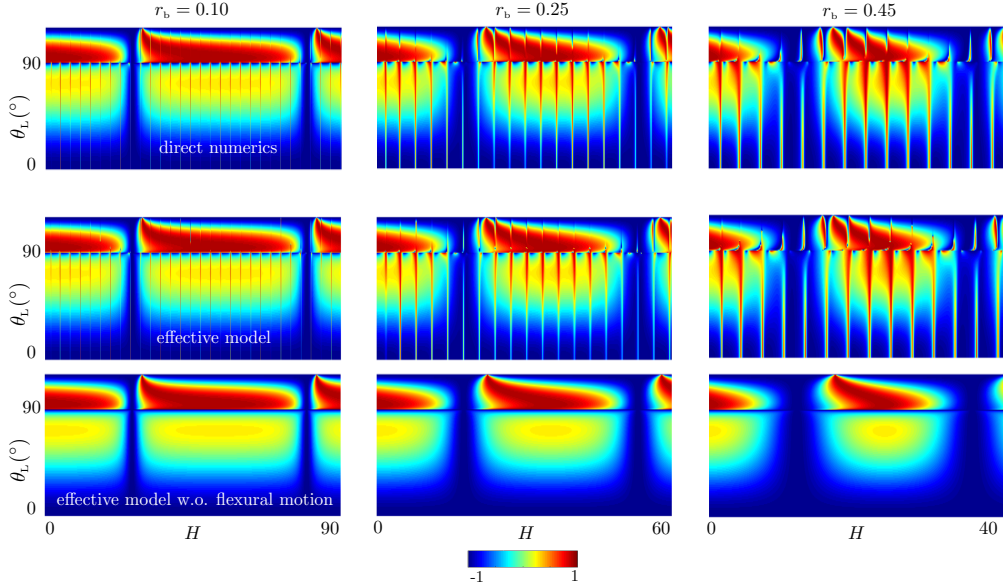


Figure 7: Real part of the reflection coefficient  $R_{\text{TT}}$  against  $H = \kappa h_b$  and the incidence angle  $\theta_L \in (0, 90^\circ)$ ; the upper part of the spectra above  $\theta_L = 90^\circ$  correspond to the range  $\theta_T \in (\theta_c, 90^\circ)$  with  $\theta_c \simeq 37.8^\circ$  where longitudinal waves are evanescent (hence  $|R_{\text{TT}}| = 1$ ).

spectra, together with those given by (52) (center panels). For comparison, the result when neglecting the flexural motions ( $f_L = 0$  in (52)) are shown in the bottom panels. With  $h_b \in (0, 90) \text{ m}$ , the two first longitudinal resonances are visible at  $\mathcal{K}H = \frac{\pi}{2}$  and  $\frac{3\pi}{2}$ , with  $\mathcal{K} = 0.057, 0.091$  and  $0.122$ ; accordingly, the density of flexural resonances between them decreases, with  $N \simeq 17, 11$  and  $8$ . The variations of  $R_{\text{TT}}$  and  $R_{\text{LL}}$  against  $\theta_L$  for the substrate on its own corresponds to  $H = 0$ ; for the thinnest plates,  $r_b = 0.10 \text{ m}$ , increasing  $H$  does not affect much this scattering

<sup>4</sup>The spectrum of  $R_{\text{TT}}$  is reported against  $\theta_L \in (0, 90^\circ)$  in its lower-part. As  $\theta_L = 90^\circ$ ,  $\theta_T = \theta_c$  with  $\theta_c = \arcsin \sqrt{\frac{\mu_s}{\lambda_s + 2\mu_s}} \simeq 37.8^\circ$ . Increasing further the incidence of the transverse wave above  $\theta_c$ , the longitudinal waves becomes evanescent and no mode conversion occurs (hence  $|R_{\text{TT}}| = 1$  and the figure 7 shows the real part). Note that the expressions of the reflection coefficients in (52) remain valid, with  $\sin \theta_L = \frac{\beta}{k_L} > 1$ , hence  $\alpha_L$  imaginary.

except at the 30 flexural resonances resulting in thin scars in the spectra and more neatly at the two longitudinal resonances  $H \simeq 27$  and 83. For thicker plates, the coupling with the substrate becomes significant ( $C = 0.230, 0.574, 1.033$ ) and this is particularly visible in the vicinities of the flexural resonances.

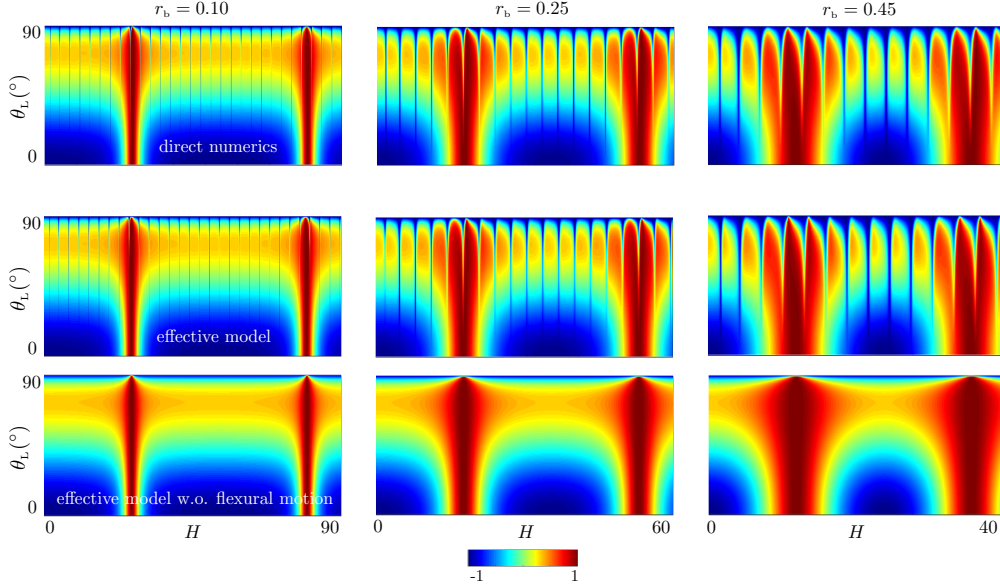


Figure 8: Real part of the reflection coefficient  $R_{LL}$ ; same representation as in figure 7.

Another way to modify the coupling between the plates and the substrate is to increase the contrast in the material properties. We have increased both  $\rho_b$  and  $E_b$  by a factor 5 (from  $C = 0.574$  to  $2.871$ ), hence we have left identical the resonance frequencies. We report the results in figure 8 by means of the profiles of  $(|R_{TT}|, |R_{LT}|)$  against  $H$  for  $\theta_T = 25.6^\circ$  ( $\theta_L = 45^\circ$ ). In the reference case with moderate contrast, the scars at the flexural resonances already produce a significant shift from the smooth underlying curve when accounting for  $f_L$  only (dashed-dotted green line). In particular, it is noticeable that the reflexion coefficients do not follow the simple prediction  $|R_{TT}| = 1$ ,  $|R_{LT}| = 0$  obtained when neglecting the flexural motions (dashed-dotted green lines). For  $|f_L| \rightarrow +\infty$ , (52) simplify to

$$R_{TT} = \frac{\cos \theta_L - i f_F \cos(\theta_L + \theta_T)}{\cos \theta_L - i f_F \cos(\theta_L - \theta_T)}, \quad R_{LT} = \frac{-2 i f_F \sin \theta_L \cos \theta_T}{\cos \theta_L - i f_F \cos(\theta_L - \theta_T)}, \quad (55)$$

hence their values depend on the relative position of the longitudinal resonance with the surrounding flexural ones. Eventually, for strong contrast, the variations due longitudinal motions only do not share much with the actual ones, as the influence of the flexural resonances are not reduced anymore to local scars. It is worth noting that the discrepancy between our model (dotted black lines) and that neglecting flexural motions (dashed-dotted green lines) is more important for the reported absolute values than for the real parts in figures 8 and 9. This is due to the fact that the imaginary parts of the reflexion coefficients have more pronounced variations around the resonances (they cancel for the substrate on its own, (52)).

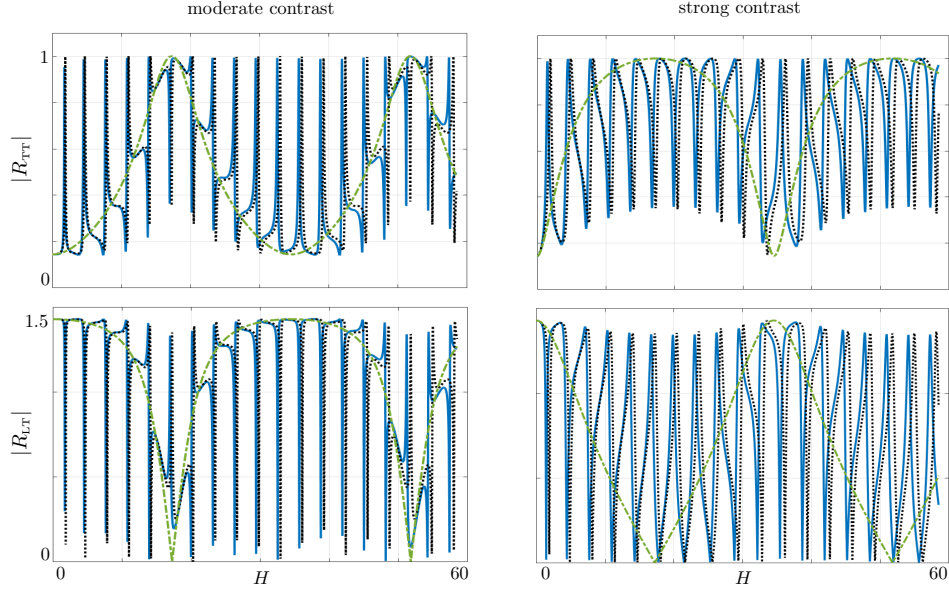


Figure 9: Variations of  $(|R_{TT}|, |R_{LT}|)$  against  $H$  for a moderate contrast, (47), and for a strong contrast (see main text). Plain blue lines shows the actual coefficients, dotted black lines those given by (52) and the dashed-dotted green lines the same when omitting the flexural motion.

We end this section with a slightly different representation by inspecting the conservation of the energy fluxes. Conservation of the fluxes is measured in terms of the conservation of the fluxes of the Poynting vector  $\Pi = \int_{z<0} \Im(\sigma_{xz}u_x + \sigma_{zz}u_z)dx$  where  $\Im$  means imaginary parts which, from (53), reads

$$\Re(\alpha_T |R_{TT}|^2 + \alpha_L |R_{LT}|^2) A_T^2 = \Re(\alpha_T A_T^2), \quad \Re(\alpha_L |R_{LL}|^2 + \alpha_T |R_{TL}|^2) A_L^2 = \Re(\alpha_L A_L^2),$$

(where  $\Re$  means real part) and the results holds by summing the two equations for an incident flux  $\Re(\alpha_T A_T^2 + \alpha_L A_L^2)$ . For an incident transverse wave, we have computed the normalized energy fluxes

$$\Phi_{TT} = |R_{TT}|^2, \quad \Phi_{LT} = \Re\left(\frac{\alpha_L}{\alpha_T}\right) |R_{LT}|^2, \quad (56)$$

and the conservation of the fluxes is ensured by  $\Phi_{TT} + \Phi_{LT} = 1$ . The figure 10 shows the variations of  $(\Phi_{TT}, \Phi_{LT})$  against  $H$  in the reference case for an incidence  $\theta_L = 30^\circ, 60^\circ$  and  $88^\circ$  (corresponding to  $\theta_T = 17.8^\circ, 32.0^\circ$  and  $37.7^\circ$ ). For the substrate on its own, the conversion from transverse to longitudinal wave is important,  $\Phi_{TT} \sim 0.1, \Phi_{LT} \sim 0.9$  depending on the incidence. The presence of the array drastically modifies the exchanges of energy between the two polarizations. In general, the sharpest variations take place in the vicinity of the flexural resonances where the conversion is forbidden. However, as  $H$  or  $\theta_T$  increases, less expected behaviors are observed, as the almost complete conversion  $\Phi_{LT} \simeq 1$  preceding or following the zero conversion at flexural resonances (see  $\theta_T = 32.0^\circ, H > 20$ ).

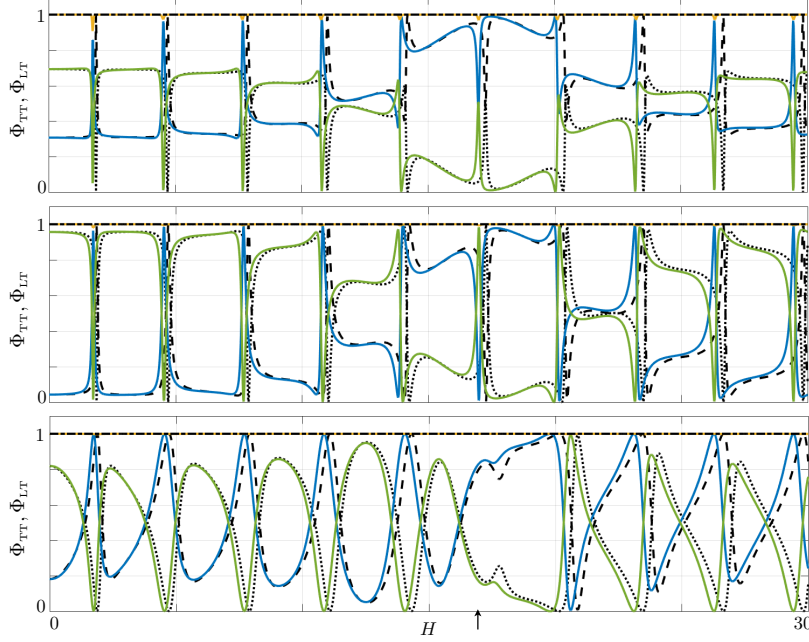


Figure 10: Variation of the energy fluxes  $\Phi_{TT}$  (blue lines) and  $\Phi_{LT}$  (green lines) against  $H$  in the reference case, (47), for  $\theta_T = 17.8^\circ$  (top panel),  $\theta_T = 32.0^\circ$  (center panel) and  $\theta_T = 37.7^\circ$  (bottom panel). The dashed and dotted black lines show  $\Phi_{TT}$  and  $\Phi_{LT}$  in the effective model with (52) in (56). The yellow and dashed-dotted black lines show  $(\Phi_{TT} + \Phi_{LT})$  in the numerics and in our model. The arrow indicates the position of the longitudinal resonance.

## 210 Appendix A. Proofs of the properties in(16)

Throughout the appendix, we shall use that for a symmetric tensor  $\tau_{ij}$  defined for  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$ , and for a vector  $a_i$  with antisymmetric tensor  $\partial_j a_i$ , we have

$$\int_{\hat{\mathbb{S}}_b} a_i \partial_j \tau_{ij} d\hat{\mathbf{x}}' = \int_{\hat{\mathbb{S}}_b} a_i \tau_{ij} n_j d\hat{\mathbf{x}}' - \int_{\hat{\mathbb{S}}_b} \partial_j a_i \tau_{ij} d\hat{\mathbf{x}}' = \int_{\hat{\mathbb{S}}_b} a_i \tau_{ij} n_j d\hat{\mathbf{x}}', \quad (\text{A.1})$$

211 since  $\partial_j a_i \tau_{ij} = 0$ .

### 212 Appendix A.1. Determination of $(\mathbf{V}^0, \mathbf{V}^1)$ and of $(\Sigma^0, \mathbf{V}^2)$

As previously said in §3.1.3, the dependance of  $\mathbf{V}^0$  and  $\mathbf{V}^1$  on  $\hat{\mathbf{x}}' = (\hat{x}, \hat{y})$  is obtained starting from  $(\mathbf{C}')^{-2}$  and  $(\mathbf{C}')^{-1}$  in (14) which tell us that there exists  $(\mathbf{U}^0(\mathbf{x}, \tilde{z}), \Omega^0(\mathbf{x}, \tilde{z}))$  and  $(\mathbf{U}^1(\mathbf{x}, \tilde{z}), \Omega^1(\mathbf{x}, \tilde{z}))$  such that

$$\begin{cases} V_x^0 = U_x^0(\mathbf{x}, \tilde{z}) + \Omega^0(\mathbf{x}, \tilde{z})\hat{y}, & V_y^0 = U_y^0(\mathbf{x}, \tilde{z}) - \Omega^0(\mathbf{x}, \tilde{z})\hat{x}, & V_z^0 = U_z^0(\mathbf{x}), \\ V_x^1 = U_x^1(\mathbf{x}, \tilde{z}) + \Omega^1(\mathbf{x}, \tilde{z})\hat{y}, & V_y^1 = U_y^1(\mathbf{x}, \tilde{z}) - \Omega^1(\mathbf{x}, \tilde{z})\hat{x}. \end{cases} \quad (\text{A.2})$$

Next, as  $\frac{\partial}{\partial \tilde{x}} V_z^1 = -\frac{\partial}{\partial \tilde{z}} V_x^0$ ,  $\frac{\partial}{\partial \tilde{y}} V_z^1 = -\frac{\partial}{\partial \tilde{z}} V_y^0$  from  $(\mathbf{C}')^{-1}$ , we have  $\frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} V_z^1 = \pm \frac{\partial}{\partial \tilde{z}} \Omega^0 = 0$ , hence

$$\frac{\partial \Omega^0}{\partial \tilde{z}} = 0, \quad V_z^1 = U_z^1(\mathbf{x}, \tilde{z}) - \frac{\partial}{\partial \tilde{z}} (U_x^0 \hat{x} + U_y^0 \hat{y}). \quad (\text{A.3})$$

The dependance of  $\mathbf{V}^2$  with respect to  $\hat{\mathbf{x}}'$  is more intricate as it requires to solve the following problem set in the section  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$  with  $(\Sigma^0, \mathbf{V}^2)$  as unknowns :

$$\begin{cases} \operatorname{div}_{\hat{\mathbf{x}}'} \Sigma^0 = \mathbf{0} & \text{in } \hat{\mathbb{S}}_b, & \Sigma^0 \mathbf{n} = \mathbf{0} & \text{on } \partial \hat{\mathbb{S}}_b, \\ \varepsilon^{\hat{\mathbf{x}}'}(\mathbf{V}^2) + \varepsilon^{\tilde{z}}(\mathbf{V}^1) + \varepsilon^{\mathbf{x}}(\mathbf{V}^0) = \frac{1 + \nu_b}{E_b} \Sigma^0 - \frac{\nu_b}{E_b} \operatorname{tr}(\Sigma^0) I, \end{cases}$$

with the input  $\mathbf{V}^0$  and  $\mathbf{V}^1$  given by (A.2). For a circular section, this problem can be solved explicitly. Specifically there exists  $(U^2(\mathbf{x}, \tilde{z}), \Omega^2(\mathbf{x}, \tilde{z}))$  such that the displacement  $\mathbf{V}^2$  is given by

$$\begin{cases} V_x^2 = U_x^2(\mathbf{x}, \tilde{z}) + \Omega^2(\mathbf{x}, \tilde{z})\hat{y} - \frac{\partial U_x^0}{\partial x}\hat{x} - \left(\frac{\partial U_y^0}{\partial x} + \frac{\partial U_x^0}{\partial y}\right)\frac{\hat{y}}{2} - \left(\frac{\partial \Omega^0}{\partial x}\hat{x} + \frac{\partial \Omega^0}{\partial y}\hat{y}\right)\hat{y} \\ \quad - \nu_b \left(\frac{\partial U_z^1}{\partial \tilde{z}} + \frac{\partial U_z^0}{\partial z}\right)\hat{x} + \nu_b \frac{\partial^2}{\partial \tilde{z}^2} \left(U_x^0 \frac{\hat{x}^2 - \hat{y}^2}{2} + U_y^0 \hat{x}\hat{y}\right), \\ V_y^2 = U_y^2(\mathbf{x}, \tilde{z}) - \Omega^2(\mathbf{x}, \tilde{z})\hat{x} - \frac{\partial U_y^0}{\partial y}\hat{y} - \left(\frac{\partial U_y^0}{\partial x} + \frac{\partial U_x^0}{\partial y}\right)\frac{\hat{x}}{2} + \left(\frac{\partial \Omega^0}{\partial x}\hat{x} + \frac{\partial \Omega^0}{\partial y}\hat{y}\right)\hat{x} \\ \quad - \nu_b \left(\frac{\partial U_z^1}{\partial \tilde{z}} + \frac{\partial U_z^0}{\partial z}\right)\hat{y} + \nu_b \frac{\partial^2}{\partial \tilde{z}^2} \left(U_x^0 \hat{x}\hat{y} - U_y^0 \frac{\hat{x}^2 - \hat{y}^2}{2}\right), \\ V_z^2 = U_z^2(\mathbf{x}, \tilde{z}) - \left(\frac{\partial U_x^1}{\partial \tilde{z}} + \frac{\partial U_z^0}{\partial x} + \frac{\partial U_x^0}{\partial z}\right)\hat{x} - \left(\frac{\partial U_y^1}{\partial \tilde{z}} + \frac{\partial U_z^0}{\partial y} + \frac{\partial U_y^0}{\partial z}\right)\hat{y}, \end{cases} \quad (\text{A.4})$$

and the corresponding stress tensor  $\Sigma^0$  by

$$\begin{cases} \Sigma_{xz}^0 = \mu_b \left(\frac{\partial \Omega^0}{\partial z} + \frac{\partial \Omega^1}{\partial \tilde{z}}\right)\hat{y}, & \Sigma_{yz}^0 = -\mu_b \left(\frac{\partial \Omega^0}{\partial z} + \frac{\partial \Omega^1}{\partial \tilde{z}}\right)\hat{x}, \\ \Sigma_{zz}^0 = E_b \left(\frac{\partial U_z^1}{\partial \tilde{z}} + \frac{\partial U_z^0}{\partial z} - \frac{\partial^2}{\partial \tilde{z}^2} (U_x^0 \hat{x} + U_y^0 \hat{y})\right), & \Sigma_{xx}^0 = \Sigma_{xy}^0 = \Sigma_{yy}^0 = 0. \end{cases} \quad (\text{A.5})$$

213 *Appendix A.2. Proof of  $P_1$  :  $\frac{\partial U_z^1}{\partial \tilde{z}} = 0$  in (16)*

We use  $(E_z)^{-1}$  in (13), specifically

$$\frac{\partial}{\partial \hat{x}} \Sigma_{xz}^1 + \frac{\partial}{\partial \hat{y}} \Sigma_{yz}^1 + \frac{\partial}{\partial \tilde{z}} \Sigma_{zz}^0 = 0,$$

that we integrate over  $\hat{\mathbf{x}}' \in \hat{\mathbb{S}}_b$ . Accounting for the boundary condition  $\Sigma_{xz}^1 n_x + \Sigma_{yz}^1 n_y = 0$  on  $\partial \hat{\mathbb{S}}_b$ , we obtain that  $\frac{\partial}{\partial \tilde{z}} \int_{\hat{\mathbb{S}}_b} \Sigma_{zz}^0 d\hat{\mathbf{x}}' = 0$ . Next, with the form of  $\Sigma_{zz}^0$  given in (A.5), and accounting for the facts that (i)  $U_z^0$  in (A.2) does not depend on  $\tilde{z}$  and (ii)  $\int_{\hat{\mathbb{S}}_b} \hat{x} d\hat{\mathbf{x}}' = \int_{\hat{\mathbb{S}}_b} \hat{y} d\hat{\mathbf{x}}' = 0$ , we obtain that  $\frac{\partial^2 U_z^1}{\partial \tilde{z}^2} = 0$ . The field  $U_z^1$  could be a linear function of  $\tilde{z}$  but this is prevented by its  $\tilde{z}$ -periodicity hence

$$\frac{\partial U_z^1}{\partial \tilde{z}} = 0. \quad (\text{A.6})$$

214 *Appendix A.3. Proof of  $P_2 : \frac{\partial \Omega^1}{\partial \tilde{z}} = 0$  in (16)*

We start by introducing the torsion force  $\mathcal{T}^0$

$$\mathcal{T}^0 = \int_{\hat{\mathbf{s}}_b} (\hat{y} \Sigma_{xz}^0 - \hat{x} \Sigma_{yz}^0) d\hat{\mathbf{x}}'. \quad (\text{A.7})$$

Using the forms of  $\Sigma_{xz}^0$  and  $\Sigma_{yz}^0$  in (A.5) we get that

$$\mathcal{T}^0(\mathbf{x}, \tilde{z}) = \mu_b J_b \left( \frac{\partial \Omega^0}{\partial \tilde{z}}(\mathbf{x}) + \frac{\partial \Omega^1}{\partial \tilde{z}}(\mathbf{x}, \tilde{z}) \right), \quad \text{with } J_b = \int_{\hat{\mathbf{s}}_b} (\hat{x}^2 + \hat{y}^2) d\hat{\mathbf{x}}' = \frac{\pi}{2} \hat{r}_b^4. \quad (\text{A.8})$$

( $J_b$  is the torsion constant for a circular cross-section.) Now, we use  $\text{div}_{\hat{\mathbf{x}}'} \Sigma^1 + \text{div}_{\tilde{z}} \Sigma^0 = \mathbf{0}$ , from  $(E_a)^{-1}$  and  $(E_z)^{-1}$  in (13), that we integrate over  $\hat{\mathbf{s}}_b$  after multiplication by  $\mathbf{a} = \hat{y} \mathbf{e}_x - \hat{x} \mathbf{e}_y$ ; we get

$$0 = \int_{\hat{\mathbf{s}}_b} \mathbf{a} \cdot \text{div}_{\hat{\mathbf{x}}'} \Sigma^1 d\hat{\mathbf{x}}' + \int_{\hat{\mathbf{s}}_b} \mathbf{a} \cdot \text{div}_{\tilde{z}} \Sigma^0 d\hat{\mathbf{x}}' = \int_{\partial \hat{\mathbf{s}}_b} \mathbf{a} \cdot (\Sigma^1 \mathbf{n}) d\hat{\mathbf{x}}' + \frac{\partial}{\partial \tilde{z}} \int_{\hat{\mathbf{s}}_b} \mathbf{a} \cdot (\Sigma^0 \mathbf{e}_z) d\hat{\mathbf{x}}'. \quad (\text{A.9})$$

In (A.9), we have applied (A.1) to the first integral involving  $\Sigma^1$ . Indeed, it suffices to notice that  $\text{div}_{\hat{\mathbf{x}}'} \Sigma^1 = \text{div}_{\hat{\mathbf{x}}'} \Sigma^1|_{\hat{\mathbf{x}}'}$  where  $\Sigma^1|_{\hat{\mathbf{x}}'}$  is the 2x2 symmetric restriction of  $\Sigma^1$  to the indices  $(x, y)$  and that  $\nabla_{\hat{\mathbf{x}}'} \mathbf{a}$  is antisymmetric. Since  $\Sigma^1|_{\hat{\mathbf{x}}'} \mathbf{n} = \Sigma^1 \mathbf{n} = \mathbf{0}$  over  $\partial \hat{\mathbf{s}}_b$ , we deduce that  $\int_{\partial \hat{\mathbf{s}}_b} \mathbf{a} \cdot (\Sigma^1 \mathbf{n}) d\hat{\mathbf{x}}' = 0$ .

Now, recalling the definition (A.7) of  $\mathcal{T}^0$  as well as (A.8), we find

$$0 = \frac{\partial}{\partial \tilde{z}} \int_{\hat{\mathbf{s}}_b} \mathbf{a} \cdot (\Sigma^0 \mathbf{e}_z) d\hat{\mathbf{x}}' = \frac{\partial \mathcal{T}^0}{\partial \tilde{z}} = \mu_b J_b \frac{\partial^2 \Omega^1}{\partial \tilde{z}^2}.$$

Thus,  $\Omega^1$  is independent of  $\tilde{z}$  (a linear dependance *w.r.t.*  $\tilde{z}$  is prevented by its  $\tilde{z}$ -periodicity), and  $\mathcal{T}^0$  reduces to

$$\mathcal{T}^0(\mathbf{x}) = \mu_b J_b \frac{\partial \Omega^0}{\partial \tilde{z}}(\mathbf{x}), \quad \frac{\partial \Omega^1}{\partial \tilde{z}} = 0. \quad (\text{A.10})$$

215 *Appendix A.4. Proof of  $P_3 : \Omega^0 = 0$  in (16)*

216 Here, we shall see that the equation for torsion (A.10) can be complemented to get the wave  
217 equation for  $\Omega^0$ ; in addition that the associated boundary conditions impose  $\Omega^0 = 0$ . (From  
218 (A.3) we already know that  $\Omega^0(\mathbf{x})$  does not depend of  $\tilde{z}$ .)

219 *Appendix A.4.1. The wave equation for torsion*

We start with the derivation of the equation satisfied by the macroscopic rotation  $\Omega^0$ . We use the relation  $\text{div}_{\hat{\mathbf{x}}'} \Sigma^2 + \text{div}_{\tilde{z}} \Sigma^1 + \text{div}_{\mathbf{x}} \Sigma^0 + \rho_b \omega^2 \mathbf{V}^0 = \mathbf{0}$  from  $(E_a)^0$  and  $(E_z)^0$  in (13), that we multiply by  $\mathbf{a} = \hat{y} \mathbf{e}_x - \hat{x} \mathbf{e}_y$  and integrate over  $(\hat{\mathbf{x}}', \tilde{z}) \in \hat{\mathbf{s}}_b \times (0, \tilde{h}_F)$ . We get

$$\int_{\hat{\mathbf{s}}_b \times (0, \tilde{h}_F)} \mathbf{a} \cdot \text{div}_{\mathbf{x}} \Sigma^0 d\hat{\mathbf{x}}' d\tilde{z} + \rho_b \omega^2 \int_{\hat{\mathbf{s}}_b \times (0, \tilde{h}_F)} \mathbf{a} \cdot \mathbf{V}^0 d\hat{\mathbf{x}}' d\tilde{z} = 0,$$

where we have accounted for  $\int_{\hat{\mathbf{s}}_b} \mathbf{a} \cdot \text{div}_{\hat{\mathbf{x}}'} \Sigma^2 d\hat{\mathbf{x}}' = 0$  as in (A.9) and for the  $\tilde{z}$ -periodicity of  $\Sigma^1$ . Next, owing to the form of  $\mathbf{V}^0$  in (A.2) (with  $\Omega^0 = \Omega^0(\mathbf{x})$  from (A.3)) and the form of  $\Sigma^0$  in (A.5) (with  $\frac{\partial}{\partial \tilde{z}} \Omega^1 = 0$  from (A.10)), we obtain

$$\frac{\partial \mathcal{T}^0}{\partial \tilde{z}}(\mathbf{x}) + \rho_b \omega^2 J_b \Omega^0(\mathbf{x}) = 0, \quad \rightarrow \quad \mu_b \frac{\partial^2 \Omega^0}{\partial \tilde{z}^2}(\mathbf{x}) + \rho_b \omega^2 \Omega^0(\mathbf{x}) = 0, \quad (\text{A.11})$$

220 with  $J_b$  defined in (A.8).

221 *Appendix A.4.2. The corresponding boundary conditions*

222 To conclude that  $\Omega^0 = 0$ , we have to specify the boundary conditions at  $z = 0$  and  $h_b$  by analyzing  
223 the torsional behavior near the extremities.

At  $z = h_b$ . In the intermediate region located near  $z = h_b$ , we introduce the torsion force  $\widetilde{\mathcal{T}}^0|_{h_b}(\mathbf{x}', \bar{z})$ . Repeating the same procedure than in Appendix A.3 in the region of the beam, we find that

$$\widetilde{\mathcal{T}}^0|_{h_b}(\mathbf{x}') = \mu_b J_b \frac{\partial \widetilde{\Omega}^0}{\partial \bar{z}}(\mathbf{x}'), \quad \frac{\partial \widetilde{\Omega}^1}{\partial \bar{z}} = 0. \quad (\text{A.12})$$

Next we go to the microscopic scale near  $z = h_b$  with the expansions (30). Multiplying  $\text{div}_{\hat{\mathbf{x}}} \widehat{\Sigma}^0|_{h_b} = \mathbf{0}$  by  $\mathbf{a} = \hat{\mathbf{y}}\mathbf{e}_x - \hat{\mathbf{x}}\mathbf{e}_y$ , integrating over  $\widehat{\mathbf{X}}_{h_b} = \{\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b, \hat{z} \in (-\infty, 0)\}$  and making use of the matching conditions and traction free boundary conditions, we deduce that

$$\widetilde{\mathcal{T}}^0|_{h_b}(\mathbf{x}') = \lim_{\bar{z} \rightarrow -\infty} \int_{\hat{\mathbf{S}}_b} (\hat{\mathbf{y}} \widehat{\Sigma}_{xz}^0|_{h_b} - \hat{\mathbf{x}} \widehat{\Sigma}_{yz}^0|_{h_b}) d\hat{\mathbf{x}}' = 0. \quad (\text{A.13})$$

Finally, using the matching condition between the intermediate region and the inner region of the beams we get

$$\mathcal{T}^0(\mathbf{x}', h_b) = \lim_{\bar{z} \rightarrow -\infty} \widetilde{\mathcal{T}}^0|_{h_b}(\mathbf{x}') = 0. \quad (\text{A.14})$$

At  $z = 0$ . In the intermediate region located near  $z = 0$ , in virtue of (14) which holds also for the intermediate expansions (24), we have  $\varepsilon^{\mathbf{x}'}(\widehat{\mathbf{V}}^0|_0) = \mathbf{0}$ , hence  $\widehat{\mathbf{V}}^0|_0 = \widehat{\mathbf{U}}^0|_0(\mathbf{x}', \bar{z}) + \widetilde{\Omega}^0|_0(\mathbf{x}')\hat{\mathbf{y}}$ . The goal is to show that  $\widetilde{\Omega}^0|_0(\mathbf{x}') = 0$ . For that, we go at the microscopic scale near  $z = 0$ , where we recall (see section (3.2.2)) that since  $\varepsilon^{\mathbf{x}'}(\widehat{\mathbf{V}}^0|_0) = \mathbf{0}$ ,  $\widehat{\mathbf{V}}^0|_0$  is a rigid body motion which can be reduced to a translation thanks to the periodic conditions acting at this scale. Therefore we get  $\widehat{\mathbf{V}}^0|_0 = \widehat{\mathbf{U}}^0|_0(\mathbf{x}')$ . It now suffices to apply the micro-meso matching conditions *i.e.*  $\widehat{\mathbf{V}}^0|_0(\mathbf{x}', 0) = \lim_{\bar{z} \rightarrow +\infty} \widehat{\mathbf{V}}^0|_0$ , to show that that  $\widetilde{\Omega}^0|_0(\mathbf{x}') = 0$ . Finally, using again matching conditions between the intermediate region and the inner region of the beams we deduce that

$$\Omega^0(\mathbf{x}', 0) = \lim_{\bar{z} \rightarrow +\infty} \widetilde{\Omega}^0|_0(\mathbf{x}') = 0. \quad (\text{A.15})$$

Given the wave equation (A.11) and the corresponding boundary conditions (A.14)-(A.15), we deduce that outside the set of torsional resonant frequencies which correspond to  $\omega_n = \sqrt{\frac{\mu_b}{\rho_b} \frac{n\pi}{2h_b}}$  with  $n \in \mathbb{Z}^*$ , we have

$$\Omega^0(\mathbf{x}) = 0. \quad (\text{A.16})$$

224 *Appendix A.5. Proof of  $P_4 : \frac{\partial U^0}{\partial z} = \frac{\partial U^0}{\partial z} = 0$  in (16)*

225 *Appendix A.5.1. Determination of  $\mathbf{V}^3$  and  $\Sigma^1$*

The first step is to solve the problem on  $(\Sigma^1, \mathbf{V}^3)$  which reads, from  $(E_a)^{-1}$  and  $(E_z)^{-1}$  in (13) and from  $(C')^1$  in (14),

$$\begin{cases} \text{div}_{\hat{\mathbf{x}}} \Sigma^1 + \text{div}_{\bar{z}} \Sigma^0 = \mathbf{0} & \text{in } \hat{\mathbf{S}}_b, \quad \Sigma^1 \mathbf{n} = \mathbf{0} & \text{on } \partial \hat{\mathbf{S}}_b, \\ \varepsilon^{\mathbf{x}'}(\mathbf{V}^3) + \varepsilon^{\bar{z}}(\mathbf{V}^2) + \varepsilon^{\mathbf{x}}(\mathbf{V}^1) = \frac{1 + \nu_b}{E_b} \Sigma^1 - \frac{\nu_b}{E_b} \text{tr}(\Sigma^1) I. \end{cases} \quad (\text{A.17})$$

We short-circuit the proof by assuming, or anticipating, that

$$\Sigma_{xx}^1 = \Sigma_{xy}^1 = \Sigma_{yy}^1 = 0, \quad (\text{A.18})$$

which can be verified *a posteriori*. As a result, from (A.17) and with  $\text{tr}(\Sigma^1) = \Sigma_{zz}^1$ , we have  $\Sigma_{zz}^1 = E_b \left( \frac{\partial V_z^2}{\partial \tilde{z}} + \frac{\partial V_z^1}{\partial \tilde{z}} \right)$ . Next, using  $V_z^2$  in (A.4),  $V_z^1$  in (A.3) and with  $V_z^0 = U_z^0(\mathbf{x})$  independent of  $\tilde{z}$  from (A.2), we obtain

$$\Sigma_{zz}^1 = E_b \left( \frac{\partial U_z^2}{\partial \tilde{z}} + \frac{\partial U_z^1}{\partial \tilde{z}} - \frac{\partial^2}{\partial \tilde{z}^2} (U_x^1 \hat{x} + U_y^1 \hat{y}) - 2 \frac{\partial^2}{\partial \tilde{z} \partial \tilde{z}} (U_x^0 \hat{x} + U_y^0 \hat{y}) \right). \quad (\text{A.19})$$

Let us consider now in (A.17) (second line) the strains associated to the 2D vanishing stress components  $(\Sigma_{xx}^1, \Sigma_{xy}^1, \Sigma_{yy}^1)$ . We obtain

$$E_b \left( \frac{\partial V_x^3}{\partial \hat{x}} + \frac{\partial V_x^1}{\partial x} \right) = -\nu_b \Sigma_{zz}^1, \quad E_b \left( \frac{\partial V_y^3}{\partial \hat{y}} + \frac{\partial V_y^1}{\partial y} \right) = -\nu_b \Sigma_{zz}^1, \quad \frac{\partial V_y^3}{\partial \hat{x}} + \frac{\partial V_x^3}{\partial \hat{y}} + \frac{\partial V_y^1}{\partial x} + \frac{\partial V_x^1}{\partial y} = 0.$$

We find that there exists  $(U_x^3(\mathbf{x}, \tilde{z}), U_y^3(\mathbf{x}, \tilde{z}), \Omega^3(\mathbf{x}, \tilde{z}))$  such that the displacements  $(V_x^3, V_y^3)$  read

$$\begin{cases} V_x^3 = U_x^3(\mathbf{x}, \tilde{z}) + \Omega^3(\mathbf{x}, \tilde{z}) \hat{y} - \frac{\partial \Omega^1}{\partial x} \hat{x} \hat{y} - \frac{\partial \Omega^1}{\partial y} \hat{y}^2 - \frac{\partial U_x^1}{\partial x} \hat{x} - \left( \frac{\partial U_y^1}{\partial x} + \frac{\partial U_x^1}{\partial y} \right) \frac{\hat{y}}{2} \\ \quad - \nu_b \left( \frac{\partial U_z^2}{\partial \tilde{z}} + \frac{\partial U_z^1}{\partial \tilde{z}} \right) \hat{x} + \nu_b \frac{\partial}{\partial \tilde{z}} \left( \left( \frac{\partial U_x^1}{\partial \tilde{z}} + 2 \frac{\partial U_x^0}{\partial \tilde{z}} \right) \frac{\hat{x}^2 - \hat{y}^2}{2} + \left( \frac{\partial U_y^1}{\partial \tilde{z}} + 2 \frac{\partial U_y^0}{\partial \tilde{z}} \right) \hat{x} \hat{y} \right), \\ V_y^3 = U_y^3(\mathbf{x}, \tilde{z}) - \Omega^3(\mathbf{x}, \tilde{z}) \hat{x} + \frac{\partial \Omega^1}{\partial y} \hat{x} \hat{y} + \frac{\partial \Omega^1}{\partial x} \hat{x}^2 - \frac{\partial U_y^1}{\partial y} \hat{y} - \left( \frac{\partial U_y^1}{\partial x} + \frac{\partial U_x^1}{\partial y} \right) \frac{\hat{x}}{2} \\ \quad - \nu_b \left( \frac{\partial U_z^2}{\partial \tilde{z}} + \frac{\partial U_z^1}{\partial \tilde{z}} \right) \hat{y} + \nu_b \frac{\partial}{\partial \tilde{z}} \left( \left( \frac{\partial U_x^1}{\partial \tilde{z}} + 2 \frac{\partial U_x^0}{\partial \tilde{z}} \right) \hat{x} \hat{y} + \left( \frac{\partial U_y^1}{\partial \tilde{z}} + 2 \frac{\partial U_y^0}{\partial \tilde{z}} \right) \frac{\hat{x}^2 - \hat{y}^2}{2} \right). \end{cases}$$

The remaining part of the solution,  $(\Sigma_{xz}^1, \Sigma_{yz}^1)$  and  $V_z^3$ , is more demanding. From (A.17) with  $\Sigma_{zz}^0$  in (A.5) and using that  $\frac{\partial}{\partial \tilde{z}} U_z^0 = \frac{\partial}{\partial \tilde{z}} U_z^1 = 0$  from (A.2) and (A.6), these fields satisfy

$$\begin{cases} \frac{\partial \Sigma_{xz}^1}{\partial \hat{x}} + \frac{\partial \Sigma_{yz}^1}{\partial \hat{y}} = -\frac{\partial \Sigma_{zz}^0}{\partial \tilde{z}} = E_b \frac{\partial^3}{\partial \tilde{z}^3} (U_x^0 \hat{x} + U_y^0 \hat{y}) \quad \text{in } \hat{\mathbf{S}}_b, \\ \Sigma_{xz}^1 = \mu_b \left( \frac{\partial V_z^3}{\partial \hat{x}} + \frac{\partial V_x^2}{\partial \tilde{z}} + \frac{\partial V_z^1}{\partial x} + \frac{\partial V_x^1}{\partial \tilde{z}} \right), \quad \Sigma_{yz}^1 = \mu_b \left( \frac{\partial V_z^3}{\partial \hat{y}} + \frac{\partial V_y^2}{\partial \tilde{z}} + \frac{\partial V_z^1}{\partial y} + \frac{\partial V_y^1}{\partial \tilde{z}} \right), \\ \Sigma_{xz}^1 n_x + \Sigma_{yz}^1 n_y = 0 \quad \text{on } \partial \hat{\mathbf{S}}_b. \end{cases} \quad (\text{A.20})$$

(We have used the constitutive equation (C')<sup>1</sup> instead of (C')<sup>1</sup> in the second line.) Now, we rearrange the constitutive behavior using (i)  $(V_x^2, V_y^2)$  in (A.4) (with  $\Omega^0 = 0$  from (A.16)), (ii)  $(V_x^1, V_y^1, V_z^1)$  in (A.2)-(A.3) and (iii)  $\frac{\partial}{\partial \tilde{z}} U_z^0 = 0$  from A.2, resulting in

$$\frac{1}{\mu_b} \underbrace{\left( \frac{\Sigma_{xz}^1}{\Sigma_{yz}^1} - \mu_b S \hat{y} \right)}_{\Sigma'} = \underbrace{\nabla_{\hat{\mathbf{x}}'} (V_z^3 + V_z^1)}_{V'} + \nu_b \frac{\partial^3 U_x^0}{\partial \tilde{z}^3} \left( \frac{\hat{x}^2 - \hat{y}^2}{2} \right) + \nu_b \frac{\partial^3 U_y^0}{\partial \tilde{z}^3} \left( \frac{\hat{x} \hat{y}}{-\hat{x}^2 + \hat{y}^2} \right), \quad (\text{A.21})$$

where

$$\left\{ \begin{array}{l} V'_z = \left( \frac{\partial U_x^2}{\partial \bar{z}} + \frac{\partial U_z^1}{\partial x} + \frac{\partial U_x^1}{\partial z} \right) \hat{x} + \left( \frac{\partial U_y^2}{\partial \bar{z}} + \frac{\partial U_z^1}{\partial y} + \frac{\partial U_y^1}{\partial z} \right) \hat{y} \\ \quad - \frac{\partial}{\partial \bar{z}} \left( \frac{\partial U_x^0}{\partial x} \hat{x}^2 + \frac{\partial U_y^0}{\partial y} \hat{y}^2 + \left( \frac{\partial U_y^0}{\partial x} + \frac{\partial U_x^0}{\partial y} \right) \hat{x} \hat{y} \right), \\ S = \frac{\partial \Omega^2}{\partial \bar{z}} + \frac{\partial \Omega^1}{\partial z} - \frac{1}{2} \frac{\partial}{\partial \bar{z}} \left( \frac{\partial U_y^0}{\partial x} - \frac{\partial U_x^0}{\partial y} \right). \end{array} \right. \quad (\text{A.22})$$

Basically, the idea is to work on the auxiliary displacement variable  $V'$  which has absorbed  $V'_z$  associated to a gradient. Thus, it is now sufficient to solve the problem on  $(\Sigma', V')$  which satisfy (A.21) along with  $\text{div}_{\hat{\mathbf{x}}} \Sigma' = E_b \frac{\partial^3}{\partial \bar{z}^3} (U_x^0 \hat{x} + U_y^0 \hat{y})$  and  $\Sigma' \mathbf{n} = 0$ ; by linearity, we can set

$$\left\{ \begin{array}{l} \Sigma'(\mathbf{x}, \bar{z}, \hat{\mathbf{x}}') = \frac{\partial^3}{\partial \bar{z}^3} U_x^0(\mathbf{x}, \bar{z}) \sigma^x(\hat{\mathbf{x}}') + \frac{\partial^3}{\partial \bar{z}^3} U_y^0(\mathbf{x}, \bar{z}) \sigma^y(\hat{\mathbf{x}}'), \\ V'(\mathbf{x}, \bar{z}, \hat{\mathbf{x}}') = \frac{\partial^3}{\partial \bar{z}^3} U_x^0(\mathbf{x}, \bar{z}) v^x(\hat{\mathbf{x}}') + \frac{\partial^3}{\partial \bar{z}^3} U_y^0(\mathbf{x}, \bar{z}) v^y(\hat{\mathbf{x}}'), \end{array} \right.$$

and solve

$$\left\{ \begin{array}{l} \text{div} \sigma^x = E_b \hat{x} \quad \text{in } \hat{\mathbf{S}}_b, \quad \sigma^x \mathbf{n} = 0 \quad \text{on } \partial \hat{\mathbf{S}}_b, \\ \sigma^x = \mu_b \nabla v^x + \mu_b \nu_b \left( \frac{\hat{x}^2 - \hat{y}^2}{2} \right), \end{array} \right. \quad \left\{ \begin{array}{l} \text{div} \sigma^y = E_b \hat{y} \quad \text{in } \hat{\mathbf{S}}_b, \quad \sigma^y \mathbf{n} = 0 \quad \text{on } \partial \hat{\mathbf{S}}_b, \\ \sigma^y = \mu_b \nabla v^y + \mu_b \nu_b \left( \frac{\hat{x} \hat{y}}{-\frac{\hat{x}^2 - \hat{y}^2}{2}} \right). \end{array} \right.$$

The above problems can be solved in polar coordinates with  $\hat{x} = \hat{r} \cos \theta$ ,  $\hat{y} = \hat{r} \sin \theta$  by looking for a solution of the form  $v^x = g(\hat{r}) \cos \theta$  and  $v^y = g(\hat{r}) \sin \theta$ , with  $g$  satisfying  $g'' + \frac{g'}{\hat{r}} - \frac{g}{\hat{r}^2} = 2\hat{r}$ , along with the boundary condition  $(2g'(\hat{r}_b) + \nu_b \hat{r}_b^2) = 0$ . It follows that

$$g(\hat{r}) = \frac{\hat{r}^3}{4} - \left( \frac{3}{4} + \frac{\nu_b}{2} \right) \hat{r}_b^2 \hat{r}. \quad (\text{A.23})$$

Eventually the solution reads

$$\left\{ \begin{array}{l} V_z^3 = -V'_z + \frac{\partial^3}{\partial \bar{z}^3} \left( (U_x^0 \hat{x} + U_y^0 \hat{y}) \frac{g}{\hat{r}} \right), \\ \frac{1}{\mu_b} \Sigma_{xz}^1 = S \hat{y} + \frac{\partial^3}{\partial \bar{z}^3} \left[ U_x^0 \left( \left( \frac{g'}{\hat{r}^2} + \frac{\nu_b}{2} \right) \hat{x}^2 + \left( \frac{g}{\hat{r}^3} - \frac{\nu_b}{2} \right) \hat{y}^2 \right) + U_y^0 \left( \frac{g'}{\hat{r}^2} - \frac{g}{\hat{r}^3} + \nu_b \right) \hat{x} \hat{y} \right], \\ \frac{1}{\mu_b} \Sigma_{yz}^1 = -S \hat{x} + \frac{\partial^3}{\partial \bar{z}^3} \left[ U_x^0 \left( \frac{g'}{\hat{r}^2} - \frac{g}{\hat{r}^3} + \nu_b \right) \hat{x} \hat{y} + U_y^0 \left( \left( \frac{g'}{\hat{r}^2} + \frac{\nu_b}{2} \right) \hat{y}^2 + \left( \frac{g}{\hat{r}^3} - \frac{\nu_b}{2} \right) \hat{x}^2 \right) \right], \end{array} \right. \quad (\text{A.24})$$

226 with  $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2}$  and with  $(V'_z, S)$  defined in (A.22).

#### 227 Appendix A.5.2. Macroscopic equation for the flexural motions $U_a^1$

Now, we want to derive the relations equivalent to (20), (21) and (22). To do so, we iterate the procedure conducted in §3.1.5. First, we obtain  $M_a^1$ ,  $a = x, y$ , by integrating  $\Sigma_{zz}^1$  (whose form is in (A.19)) over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$  after multiplication by  $\hat{x}$  then by  $\hat{y}$ . Next, the equilibrium is obtained starting with  $(E_a)^1$  in (13):  $\frac{\partial}{\partial \bar{x}} \Sigma_{xa}^3 + \frac{\partial}{\partial \bar{y}} \Sigma_{ya}^3 + \frac{\partial}{\partial \bar{z}} \Sigma_{az}^2 + \frac{\partial}{\partial z} \Sigma_{az}^1 + \rho_b \omega^2 V_a^1 = 0$ ,  $a = x, y$  (since

$\Sigma_{xa}^1 = \Sigma_{ya}^1 = 0$  from (A.18)), that we integrate over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$ ; then we use  $\Sigma_{az}^1$  in (A.24) and  $V_a^1$  in (A.2). Eventually, the relations between bending moments and shear forces are written from  $(E_z)^0$  in (13):  $\frac{\partial}{\partial \hat{x}} \Sigma_{xz}^2 + \frac{\partial}{\partial \hat{y}} \Sigma_{yz}^2 + \frac{\partial}{\partial \hat{z}} \Sigma_{zz}^1 + \frac{\partial}{\partial z} \Sigma_{zz}^0 + \rho_b \omega^2 V_z^0 = 0$  (since  $\Sigma_{xz}^0 = \Sigma_{yz}^0 = 0$  from (17)), that we integrate over  $\hat{\mathbf{x}}' \in \hat{\mathbf{S}}_b$  after multiplication by  $\hat{a} = \hat{x}, \hat{y}$ ; then, we use  $\Sigma_{zz}^0$  in (17) and  $V_z^0$  in (A.2). We obtain

$$\begin{cases} M_a^1 = E_b I_b \frac{\partial^2 U_a^1}{\partial \tilde{z}^2} + 2E_b I_b \frac{\partial^2 U_a^0}{\partial z \partial \tilde{z}}, & \frac{\partial T_a^2}{\partial \tilde{z}} + \rho_b \omega^2 \varphi U_a^1 + \mu_b A_b \frac{\partial^4 U_a^0}{\partial z \partial \tilde{z}^3} = 0, \\ T_a^2 + \frac{\partial M_a^1}{\partial \tilde{z}} + E_b I_b \frac{\partial^3 U_a^0}{\partial z \partial \tilde{z}^2} = 0, \end{cases}$$

with  $A_b = -\frac{1}{2}(1 + \nu_b)\pi r_b^4$ . The above system conduces to

$$\frac{\partial^4 U_a^1}{\partial \tilde{z}^4} - \kappa^4 U_a^1 = B_b \frac{\partial^4 U_a^0}{\partial z \partial \tilde{z}^3}, \quad a = x, y, \quad (\text{A.25})$$

with  $\kappa^4 = \frac{\rho_b \varphi}{E_b I_b} \omega^2$  and  $B_b = \frac{\mu_b A_b}{E_b I_b} - 3 = -4$ . If  $U_a^0$  depend on  $z$  (and we want to show that it is not the case), then  $U_a^0$  solution of (23) reads  $U_a^0(\mathbf{x}, \tilde{z}) = A_a(\mathbf{x}) \sin \kappa \tilde{z} + B_a(\mathbf{x}) \cos \kappa \tilde{z}$ . Integrating (A.25) over  $\tilde{z} \in (0, 2\pi/\kappa)$  after multiplication by  $\cos \kappa \tilde{z}$  (resp. by  $\sin \kappa \tilde{z}$ ) and integrating by part four times provides  $\frac{\partial}{\partial z} A_a = 0$  (resp.  $\frac{\partial}{\partial z} B_a = 0$ ), which allows us to conclude that

$$\frac{\partial U_x^0}{\partial z} = \frac{\partial U_y^0}{\partial z} = 0. \quad (\text{A.26})$$

## 228 Appendix B. The case of a two-dimensional array of plates

We provide in this section the effective model for the alternative situation of a periodic array of identical plates supported by a soil substrate. We transpose the notations of the beams case, assuming that the plates have a height  $h_b$ , a thickness  $2r_b$  and that the periodicity of the array is  $\ell$ . We denote  $(\lambda_b, \mu_b, \rho_b)$  the material parameters of the plates and  $(\lambda_s, \mu_s, \rho_s)$  those of the substrate. The same scaling as for the beams case is used, see (9). For in-plane incident wave, the resulting displacement fields remain in-plane with

$$U_y = 0, \quad \frac{\partial U_x}{\partial y} = \frac{\partial U_z}{\partial y} = 0, \quad \sigma_{xy} = \sigma_{yz} = 0.$$

Following the same asymptotic procedure as for the beams, we derive an effective model for the array of plates. In the substrate, the displacement field is governed by the classical balance of linear momentum equation and constitutive equation of the substrate. The array of the plates is replaced by frequency dependent boundary conditions which encapsulate both the longitudinal and flexural resonant motions of the plates

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho_s \omega^2 \mathbf{u} = \mathbf{0}, & \boldsymbol{\sigma} = 2\mu_s \boldsymbol{\varepsilon} + \lambda_s \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}, \quad \text{for } z \in (-\infty, 0), \\ \sigma_{xz}(x, 0) = \mu_s k_T f_F(\omega, h_b) u_x(x, 0), & \sigma_{zz}(x, 0) = \mu_s k_T f_L(\omega, h_b) u_z(x, 0), \end{cases} \quad (\text{B.1})$$

and

$$\begin{cases} f_F(\omega, h_b) = \varphi \frac{\rho_b}{\rho_s} \frac{k_T}{\kappa} \frac{\text{sh}\kappa h_b \cos \kappa h_b + \text{ch}\kappa h_b \sin \kappa h_b}{1 + \cos \kappa h_b \text{ch}\kappa h_b}, & \kappa = \left( \frac{3\rho_b \omega^2}{E_b^* r_b^2} \right)^{1/4}, \\ f_L(\omega, h_b) = \varphi \frac{\rho_b}{\rho_s} \frac{k_T}{K} \tan K h_b, & K = \sqrt{\frac{\rho_b}{E_b^*}} \omega. \end{cases} \quad (\text{B.2})$$

with

$$\varphi = \frac{2r_b}{\ell}, \quad E_b^* = \frac{4\mu_b(\lambda_b + \mu_b)}{\lambda_b + 2\mu_b} = \frac{E_b}{1 - \nu_b^2}. \quad (\text{B.3})$$

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