

MUSCL discretization for the fluid flow convection operator on staggered meshes

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Abstract We propose in this paper a second order discretization of the momentum convection operator for fluid flow simulation on staggered quadrangular or hexahedral meshes. The velocity is approximated by the Rannacher-Turek finite element. The implemented MUSCL-like approach is of algebraic type, in the sense that the limitation procedure does not invoke any slope reconstruction, and is independent from the geometry of the cells. The derived discrete convection operator applies both to constant or variable density flows; we perform here numerical tests for the barotropic and incompressible Navier-Stokes equations.

Key words: Convection operator, fluid flow, MUSCL, staggered grid

MSC (2010): 65M08, 76M12

1 Introduction

Several works combining a finite element approximation of diffusion terms with a finite volume discretization for the convection operator may be found in the literature. The implementation of such a technique to obtain monotone schemes for convection-diffusion equations may be found for instance in [1, 2, 3]. Since finite-volume convection operators (with suitable upwinding) also enjoy desirable L^2 -stability properties, they have been used for the discretization of the Navier-Stokes equations, preferably for compatible accuracy, *i.e.* low order, approximations. An application of this strategy for the discretization of stationary incompressible Navier-Stokes equations by Crouzeix-Raviart finite elements may be found in [4]; extension to quasi-incompressible unsteady flows, both with the Crouzeix-

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Raviart and Rannacher-Turek finite elements, is performed in [5]. These two works only consider a first order upwinding technique, and our aim in this paper is to develop a second order convection operator, based on an algebraic MUSCL-like technique [6]. The obtained operator is quite general, in the sense that it may be applied to incompressible constant or variable density flows as well as to compressible flows, either for Navier-Stokes or Euler equations. We show here some numerical applications for the barotropic and the incompressible Navier-Stokes equations.

The continuous momentum convection operator that we consider here takes the following generic form $\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u})$ where ρ is the fluid density and \mathbf{u} the velocity. It may be recast under the form of a transport operator (which is central for its stability) provided that a mass balance equation holds, that is

$$\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \quad (1)$$

which we suppose here. The problem in which the convection operator is involved is supposed to be posed over $\Omega \times [0, T)$ where $\Omega \subset \mathbb{R}^d$ (with $d = 2, 3$) is an open bounded domain of boundary $\partial\Omega$ and $[0, T)$ is a finite time interval.

2 Space and time discretizations

We first define a primal mesh \mathcal{M} by splitting Ω into a finite family of disjoint quadrangles (if $d = 2$) or hexahedra (if $d = 3$) denoted by K and called control volumes or cells. We then denote by \mathcal{E} the set of faces of the mesh \mathcal{M} ; for $K \in \mathcal{M}$, $\mathcal{E}(K)$ stands for the set of faces of K and we thus have $\partial K = \cup_{\sigma \in \mathcal{E}(K)} \sigma$. Any face $\sigma \in \mathcal{E}$ is either a part of the boundary of Ω , *i.e.* $\sigma \subset \partial\Omega$, in which case σ is said to be an external face, or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \sigma$: we denote in this case $\sigma = K|L$ and σ is said to be an internal face. We denote by \mathcal{E}_{ext} and \mathcal{E}_{int} the set of external and internal faces. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $|K|$ the measure of K and $|\sigma|$ the $(d - 1)$ -measure of the face σ .

The discretization is staggered in the sense that the scalar and vector unknowns are not collocated. Indeed, the pressure and density unknowns are associated with the cells of the primal mesh \mathcal{M} and denoted by p_K, ρ_K while the degrees of freedom for the velocity are defined on a dual mesh using the Rannacher-Turek non-conforming low-order finite element approximation [7] and are denoted $\mathbf{u}_\sigma = (u_{\sigma,1}, \dots, u_{\sigma,d})$. The dual mesh is constructed as follows: if $K \in \mathcal{M}$ is a rectangle or a rectangular cuboid, we denote by x_K the mass center of K and we construct $D_{K,\sigma}$ as the cone with basis σ and with vertex x_K ; this definition is extended to a general cell K , by supposing that K is split in the same number of sub-cells (the geometry of which does not need to be specified) and with the same connectivity. We now define D_σ , the dual cell of basis σ , as $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{\text{int}}$ and $D_\sigma = D_{K,\sigma}$ if $\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\text{ext}}$; its measure is denoted by $|D_\sigma|$. We then denote by $\tilde{\mathcal{E}}(D_\sigma)$ the set of dual faces of D_σ , and by $\mathcal{E} = D_\sigma|D_{\sigma'}$ the face separating two dual cells D_σ and $D_{\sigma'}$. All the components of the velocity are then approximated on each face of the mesh, and

their degrees of freedom are identified to the mean value of the velocity component over the face. An example of the discretization with a few control volumes is given on Figure 1.

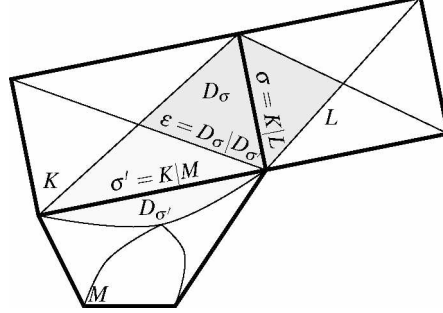


Fig. 1 Primal and dual meshes for the Rannacher-Turek elements.

Finally, a constant time step denoted by δt is used for the time discretization, and for $0 \leq n \leq N = T/\delta t$, we define $t^n = n \delta t$.

3 A second order discrete convection operator

Let us first address the discretization of the mass balance equation (1) over the primal mesh. Mimicking the divergence theorem, the discrete divergence reads (dropping the time exponents for short), for $K \in \mathcal{M}$:

$$\operatorname{div}(\rho \mathbf{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} \quad (2)$$

where $F_{K,\sigma}$ stands for the (primal) numerical mass flux across σ outward K and is defined by:

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad F_{K,\sigma} = |\sigma| \rho_{\sigma} \mathbf{u}_{\sigma} \cdot \mathbf{n}_{K,\sigma} \quad (3)$$

with $\mathbf{n}_{K,\sigma}$ the normal vector to the face σ outward K .

The dual mass fluxes $F_{\sigma,\varepsilon}$ for $\sigma \in \mathcal{E}$ and $\varepsilon \in \tilde{\mathcal{E}}(D_{\sigma})$, $\varepsilon \subset K$ are constructed from these primal fluxes so as to ensure that a discrete mass balance holds over the dual mesh D_{σ} ; this is obtained by computing the face densities $\rho_{D_{\sigma}}$ and the mass fluxes $F_{\sigma,\varepsilon}$ as a linear combination of the densities in the primal cells adjacent to σ and the mass fluxes through the primal faces of K , respectively; we refer to [8] for the exact expressions.

These dual fluxes are then used for the definition of the discrete momentum convection operator, *i.e.* the discretization of the continuous term $\text{div}(\rho u_i \mathbf{u})$. For $1 \leq i \leq d$ and $\sigma \in \mathcal{E}$, the term $\text{div}(\rho u_i \mathbf{u})_\sigma$ reads:

$$\text{div}(\rho u_i \mathbf{u})_\sigma = \frac{1}{|D_\sigma|} \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma, \varepsilon} u_{\varepsilon, i} \quad (4)$$

where $u_{\varepsilon, i}$ is an approximation of u_i over the face ε ; it is obtained by the algebraic MUSCL-like technique introduced in [6], which implements the following procedure. Let us consider the explicit part of the convection term

$$T_{\sigma, i} = \rho_{D_\sigma}^n u_{\sigma, i}^n - \delta t \text{div}(\rho^n u_i^n \mathbf{u}^n)_\sigma.$$

The discrete convection operator is said to be monotone if the term $T_{\sigma, i}$ can be written as a convex combination of degrees of freedom of u_i^n ; for instance, such a property would ensure a discrete maximum principle for the transport equation, or a convection-diffusion equation with a suitable (only available on specific meshes) discretization of the diffusion term. Such a formulation of the term $T_{\sigma, i}$ is possible by using the discrete mass balance over D_σ if the following condition holds for each $\varepsilon \in \tilde{\mathcal{E}}_{\text{int}}$ such as $\varepsilon = D_\sigma | D_{\sigma'}$:

$$\exists \alpha_\varepsilon^\sigma \in [0, 1], \exists \tilde{\sigma} \in \mathcal{E} \text{ such that } u_{\varepsilon, i} - u_{\sigma, i} = \begin{cases} \alpha_\varepsilon^\sigma (u_{\sigma, i} - u_{\tilde{\sigma}, i}) & \text{if } F_{\sigma, \varepsilon} \geq 0, \\ \alpha_\varepsilon^\sigma (u_{\tilde{\sigma}, i} - u_{\sigma, i}) & \text{otherwise,} \end{cases} \quad (5)$$

together with the following CFL condition:

$$\text{CFL} = \max_{\sigma \in \mathcal{E}} \left\{ \frac{\delta t}{|D_\sigma|} \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} |F_{\sigma, \varepsilon}| \right\} \leq 1. \quad (6)$$

We now deduce from the relation (5) a constructive process to compute the quantities $u_{\varepsilon, i}$. Let ε be an internal face separating an upwind dual cell D_{σ^-} from the downstream dual cell D_{σ^+} (*i.e.* $F_{\sigma^-, \varepsilon} \geq 0$). Let us now choose two sets $\mathcal{N}_\varepsilon(D_{\sigma^-})$ and $\mathcal{N}_\varepsilon(D_{\sigma^+})$ of neighbouring dual cells of D_{σ^-} and D_{σ^+} respectively. The following assumptions are then a transcription of Condition (5):

$$\exists D_{\tilde{\sigma}^+} \in \mathcal{N}_\varepsilon(D_{\sigma^+}) \text{ such that } u_{\varepsilon, i} \in I_1 = [u_{\tilde{\sigma}^+, i}, u_{\tilde{\sigma}^+, i} + \frac{\xi^+}{2} (u_{\sigma^+, i} - u_{\tilde{\sigma}^+, i})]; \quad (7a)$$

$$\exists D_{\tilde{\sigma}^-} \in \mathcal{N}_\varepsilon(D_{\sigma^-}) \text{ such that } u_{\varepsilon, i} \in I_2 = [u_{\tilde{\sigma}^-, i}, u_{\tilde{\sigma}^-, i} + \frac{\xi^-}{2} (u_{\sigma^-, i} - u_{\tilde{\sigma}^-, i})]; \quad (7b)$$

where ξ^+ and ξ^- are two numerical parameters lying in the interval $[0, 2]$.

Here we choose $\mathcal{N}_\varepsilon(D_{\sigma^+}) = \{D_{\sigma^-}\}$. Concerning $\mathcal{N}_\varepsilon(D_{\sigma^-})$, several choices are possible; we choose here the opposite cell to D_{σ^+} with regard to D_{σ^-} , which means that if D_{σ^-} and D_{σ^+} share the primal face ε , we choose the cell $D_{\sigma'}$ such that $D_{\sigma'}$ shares a face ε' with D_{σ^-} and ε and ε' share no vertex. With these choices, the

upwind value is always admissible, and in fact it is the only admissible cell if the two parameters are chosen equal to 0.

We are now in position to give the algorithm used to compute the quantities $u_{\varepsilon,i}$:

1. Compute a tentative value $\bar{u}_{\varepsilon,i}$ with a convex combination of the values (e.g. the centered choice) in the surrounding faces.
2. Evaluate $F_{\sigma,\varepsilon}$ to determine the upwind face D_{σ^-} and the downwind face D_{σ^+} , and choose accordingly the neighbouring sets $\mathcal{N}_{\varepsilon}(D_{\sigma^-})$ and $\mathcal{N}_{\varepsilon}(D_{\sigma^+})$.
3. Compute an admissible interval $I_1 \cap I_2$ for $u_{\varepsilon,i}$ by (7).
4. Compute $u_{\varepsilon,i}$ by projecting the tentative value $\bar{u}_{\varepsilon,i}$ into the interval obtained in the previous step.

Since this procedure is not linear, we cannot expect to derive an explicit formula to compute the values of the coefficients a_{ε}^{σ} . Their evaluation is however useless for an explicit scheme: indeed, the presented algorithm univocally defines the value $u_{\varepsilon,i}$. However, for this reason, we cannot easily define an implicit-in-time MUSCL scheme (this would require an iterative process for each time step).

4 Numerical tests

The computations presented here are performed with the open-source CALIF³S software developed at IRSN [9].

4.1 Compressible Navier-Stokes equations

We first show an application to the barotropic compressible case:

$$\partial_t(\rho u_i) + \operatorname{div}(\rho u_i \mathbf{u}) + \partial_i p - \operatorname{div}(\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t))_i = 0 \quad (i \in \llbracket 1, d \rrbracket) \quad (8a)$$

$$\partial_t(\rho) + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (8b)$$

$$p = a\rho^{\gamma}, \quad a > 0, \gamma \geq 1. \quad (8c)$$

where p stands for the pressure.

The scheme - A first-order forward Euler time-discretization of System (8) reads:

$$\forall K \in \mathcal{M}, \quad \frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \operatorname{div}(\rho^n \mathbf{u}^n)_K = 0, \quad (9a)$$

$$\text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E},$$

$$\begin{aligned} \frac{1}{\delta t}(\rho_{D_{\sigma}^{n+1}}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_{\sigma}^n}^n u_{\sigma,i}^n) + \operatorname{div}(\rho^n u_i^n \mathbf{u}^n)_{\sigma} \\ + (\nabla p)_{\sigma,i}^n - \operatorname{div}(\mu(\nabla \mathbf{u}^n + \nabla(\mathbf{u}^n)^t))_{\sigma,i} = 0, \end{aligned} \quad (9b)$$

$$\forall K \in \mathcal{M}, p_K^{n+1} = a(\rho_K^{n+1})^{\gamma}, \quad (9c)$$

with the previously described discrete convection terms and with the discrete pressure gradient and momentum diffusion term as given in [10]. Second order in time is obtained by using the Heun scheme, which reads

$$\mathbf{W}^{n+\frac{1}{3}} = S(\mathbf{W}^n), \quad \mathbf{W}^{n+\frac{2}{3}} = S(\mathbf{W}^{n+\frac{1}{3}}), \quad \mathbf{W}^{n+1} = \frac{1}{2}(\mathbf{W}^n + \mathbf{W}^{n+\frac{2}{3}}),$$

where $\mathbf{W}^n = (\rho^n, u^n, p^n)$ is the vector of unknowns at step n and $S(\mathbf{W}^n)$ is obtained by one step of the forward Euler (9). The Heun scheme is used in the following numerical test.

Translated standing vortex - Here we assess the convergence rate of the proposed scheme on a test case built to this purpose. We first derive an analytical solution of the steady barotropic Euler equations consisting in a standing vortex; then this solution is made unsteady by adding a constant velocity translation. A solution for the Navier-Stokes equations is finally derived by compensating the viscous forces (that appear on the left hand side of equation (8a)) with a source term. We refer to [11] for the expression of this solution.

The viscosity μ is chosen so that the Reynolds number is equal to 50, $a = 9.81/2$ and $\gamma = 2$. The domain is the square $\Omega = [-1.2, 2]^2$ and the computation is run on the time interval $[0, 0.8]$. Uniform $n \times n$ grids are used, starting from $n = 32$ and doubling the number of control volumes in each direction up to $n = 256$ mesh. The time step is set to $0.03125 \times h$, with $h = 3.2/n$, which yields a CFL number with respect to the celerity of the fastest wave close to 0.01 (the material velocity and the speed of sound are in the range of 1.45 and 0.76 respectively); this low value of the CFL number is imposed by the explicit discretization of the diffusion term (the constraint stems from the necessity to be stable up to the finest mesh). On Figure

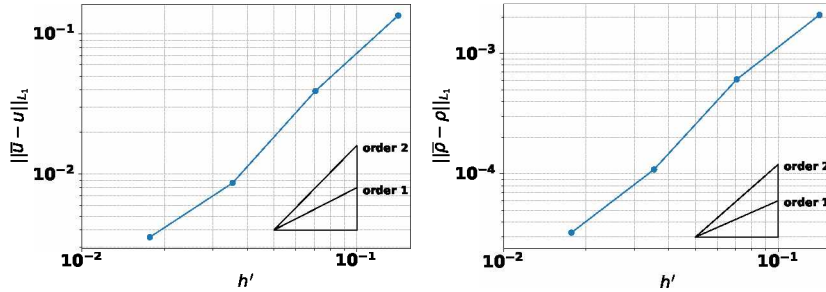


Fig. 2 L^1 norm error for the MUSCL scheme for the velocity and the density. Here $h' = \max_{K \in \mathcal{M}} \text{diam}(K) = \sqrt{2}h$.

2, we draw the L^1 norm of the numerical error for the velocity and the density as a function of the mesh step. This error is obtained by taking the difference between the computed velocity or density at the final time and the piecewise constant function defined by taking on each dual cell the value of the continuous solution at the cell

center. The measured convergence orders are close to 1.8 and 2 for the velocity and the density respectively.

4.2 Incompressible Navier-Stokes equation

We now turn to the incompressible Navier-Stokes equations (with $\rho = 1$):

$$\partial_t u_i + \operatorname{div}(u_i \mathbf{u}) + \partial_i p - \operatorname{div}(\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t))_i = 0 \quad (i \in \llbracket 1, d \rrbracket), \quad (10a)$$

$$\operatorname{div}(\mathbf{u}) = 0. \quad (10b)$$

The scheme - This system is solved using a projection scheme, which consists in the two following steps:

Prediction step – Solve for $\tilde{\mathbf{u}}^{n+1}$:

$$\begin{aligned} \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}, \quad & \frac{1}{\delta t} (\tilde{\mathbf{u}}_{\sigma,i}^{n+1} - \mathbf{u}_{\sigma,i}^n) + \operatorname{div}(\tilde{\mathbf{u}}_i^n \mathbf{u}^n)_\sigma \\ & + (\nabla p)_\sigma^n - \operatorname{div}(\mu(\nabla \tilde{\mathbf{u}}^{n+1} + (\nabla \tilde{\mathbf{u}}^{n+1})^t))_{\sigma,i} = 0. \end{aligned} \quad (11a)$$

Correction step – Solve for p^{n+1} and \mathbf{u}^{n+1} :

$$\text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}, \quad \frac{1}{\delta t} (\mathbf{u}_{\sigma,i}^{n+1} - \tilde{\mathbf{u}}_{\sigma,i}^{n+1}) + (\nabla p^{n+1})_{\sigma,i} - (\nabla p^n)_{\sigma,i} = 0, \quad (11b)$$

$$\forall K \in \mathcal{M}, \quad \operatorname{div}(\mathbf{u}^{n+1})_K = 0. \quad (11c)$$

The convection terms are defined in the previous section, with the density set to 1 in the mass flux. We refer once again for short to [10] for the definition of the discrete pressure gradient and the momentum diffusion term.

Flow past a cylinder - We consider here a flow past a cylinder studied in a literature benchmark [12]. The geometry of the domain is given in [12, Figure 1]. The present test corresponds to the **Test Case 2D-2** of [12]. The viscosity μ is chosen so that the Reynolds number is equal to 5000, so the convection is strongly dominant. The computations are performed using a very coarse grid with 4033 cells, representative of what may be encountered in very complex 3D industrial simulations. We compare the results with the present convection scheme with the results obtained with (implicit-in-time) upwind and centered convection operators.

The main quantities of interest are the pressure difference ΔP between the front and end points of the cylinder, the Strouhal number, the maximum drag coefficient and the maximal and minimal lift coefficients. They are gathered in Table 1 and Table 2, together with reference values obtained with a converged-in-space computation. Even if the convergence is far from being reached with the (intentionally) very coarse mesh used in this study, the MUSCL scheme seems able to capture at least the order of magnitude of the reported quantities. This is supported by an examination of the computed flow structure: the vortex shedding phenomenon is qual-

itatively reproduced by the MUSCL scheme while the upwind and centered ones yield, respectively, unrealistic small and large recirculation zones.

Reference		
ΔP_{max}	S_t	
3.33080	0.3371	
Scheme	ΔP_{max}	S_t
Upwind	2.37020	0.2304
Centered	2.00400	0.2591
MUSCL	2.63470	0.2660

Table 1 ΔP and Strouhal number.

Reference			
$c_{d,max}$	$c_{l,max}$	$c_{l,min}$	
3.46088	2.8479	-2.78536	
Scheme	$c_{d,max}$	$c_{l,max}$	$c_{l,min}$
Upwind	3.23544	0.52157	-0.51295
Centered	3.18728	0.10793	-0.16129
MUSCL	3.42478	1.23162	-1.16792

Table 2 Lift and drag coefficients.

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