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# A zero-one law for invariant measures and a local limit theorem for coefficients of random walks on the general linear group

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**Abstract.** We prove a zero-one law for the stationary measure for algebraic sets generalizing the results of Furstenberg [13] and Guivarc'h and Le Page [21]. As an application, we establish a local limit theorem for the coefficients of random walks on the general linear group.

**Résumé.** Nous prouvons une loi zéro-un pour la mesure stationnaire pour des ensembles algébriques en généralisant les résultats de Furstenberg [13] et Guivarc'h et Le Page [21]. Comme application, nous établissons un théorème local limite pour les coefficients de marches aléatoires sur le groupe linéaire général.

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## 1. Introduction

### 1.1. Motivation and objectives

Let  $d \geq 2$  be an integer. We denote by  $V$  a  $d$ -dimensional vector space over  $\mathbb{R}$  and by  $V^*$  the dual space of  $V$ . We choose norms on  $V$  and  $V^*$  which will be both denoted by  $|\cdot|$ . For any  $v \in V$  and  $f \in V^*$ , the corresponding duality bracket is denoted by  $\langle f, v \rangle = f(v)$ ; sometimes instead of  $f(v)$  we shall use the reverse notation  $v(f) = \langle v, f \rangle$ . Denote by  $\mathbb{G} = \text{GL}(V)$  the group of linear automorphisms of  $V$ . The projective space  $\mathbb{P}(V)$  of  $V$  is the set of elements  $x = \mathbb{R}v$ , where  $v \in V \setminus \{0\}$ . The projective space of  $V^*$  is denoted by  $\mathbb{P}(V^*)$ . For any  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$  we define  $\delta(y, x) = \frac{|\langle f, v \rangle|}{|f||v|}$ . For any  $g \in \mathbb{G}$  and  $x = \mathbb{R}v \in \mathbb{P}(V)$  with  $v \in V \setminus \{0\}$ , let  $gx = \mathbb{R}gv \in \mathbb{P}(V)$ , where  $gv \in V$  is the image of the automorphism  $v \mapsto gv$  on  $V$ . Set  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .

Let  $\mu$  be a probability measure on  $\mathbb{G}$ . Consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega = \mathbb{G}^{\mathbb{N}^*}$ ,  $\mathcal{F}$  is the Borel  $\sigma$ -algebra on  $\Omega$  and  $\mathbb{P} = \mu^{\otimes \mathbb{N}^*}$ . If we denote by  $g_i$  the coordinate mapping on  $\Omega$ , then  $g_1, g_2, \dots$  is a sequence of independent and identically distributed random elements in  $\mathbb{G}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  with the same law  $\mu$ . Set  $G_n = g_n \dots g_1$ , for  $n \geq 1$ .

A measure  $\nu$  is called  $\mu$ -invariant (or equivalently  $\mu$ -stationary) if  $\mu * \nu = \nu$ , where  $*$  stands for the convolution of probability measures. Furstenberg [13] showed that under some mild conditions there exists a unique  $\mu$ -invariant probability measure  $\nu$  on  $\mathbb{P}(V)$  which is not supported by any proper projective hyperplane: for any projective hyperplane  $Y \subsetneq \mathbb{P}(V)$ ,

$$\nu(Y) = 0. \tag{1}$$

The Furstenberg zero-law (1) turns out to be one of the key properties in the study of random walks on the group  $\mathbb{G}$ . It is used in the proof of many limit theorems for the norm cocycle  $\log |G_n v|$ , where  $v \in V \setminus \{0\}$  is a

starting vector. We refer to Furstenberg and Kesten [14], Furstenberg [12], Le Page [25], Bougerol and Lacroix [9], Goldsheid and Guivarc'h [16], Guivarc'h [19], Benoist and Quint [3–5], Xiao, Grama and Liu [28], who established the law of large numbers, the central limit theorem, the local limit theorem and large deviation asymptotics. Some of these results have been extended to the coefficients  $\langle f, G_nv \rangle$ , where  $f \in V^*$  and  $v \in V$ , however much less is known in this respect. Guivarc'h and Raugi [22] have proved the law of large numbers and the central limit theorem for the coefficients. In the setting of reductive groups, Benoist and Quint [5] have established the law of iterated logarithm and the large deviation bounds. The approach developed in [22], [9] and [18] for the proof of the law of large numbers and of the central limit theorem for  $\log |\langle f, G_nv \rangle|$  is based on the use of the quantitative version of the property (1) called Hölder regularity of the stationary measure  $\nu$  which we state below: there exist positive constants  $\alpha$  and  $C$  such that for any  $y = \mathbb{R}f \in \mathbb{P}(V^*)$  and  $\varepsilon > 0$ ,

$$\nu(\{x \in \mathbb{P}(V) : \delta(y, x) \leq \varepsilon\}) \leq C\varepsilon^\alpha. \quad (2)$$

The next elementary identity relates the coefficient  $\langle f, G_nv \rangle$  to the norm  $|G_nv|$ : for any  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$  with  $|f| = 1$ ,

$$\log |\langle f, G_nv \rangle| = \log |G_nv| + \log \delta(y, G_nx), \quad (3)$$

where  $\delta(y, G_nx) = \frac{|\langle f, G_nv \rangle|}{|f||G_nv|}$ . From (2) one can deduce that for any  $\beta > 0$ ,

$$\lim_{n \rightarrow \infty} n^{-\beta} \log \delta(y, G_nx) = 0 \quad \mathbb{P}\text{-a.s.} \quad (4)$$

Now using (3) and (4) we can infer the limit behaviour of  $\log |\langle f, G_nv \rangle|$  from that of  $\log |G_nv|$ . This allows for instance to prove the law of large numbers and the central limit theorem: for any  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\langle f, G_nv \rangle| = \lambda \quad \mathbb{P}\text{-a.s.}$$

and for any  $t \in \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{\log |\langle f, G_nv \rangle| - n\lambda}{\sigma\sqrt{n}} \leq t \right) = \Phi(t),$$

where  $\lambda \in \mathbb{R}$  is a constant called the top Lyapunov exponent of  $\mu$ ,  $\sigma$  is a positive number given by  $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\log |G_nv| - n\lambda)^2]$ , and  $\Phi$  is the standard normal distribution function. For the law of iterated logarithm and large deviation bounds, Benoist and Quint [5, Lemma 14.11], following the approach of Bourgain, Furman, Lindenstrauss and Mozes [10], have developed another strategy based on the following inequality: for any  $a > 0$  there exist positive constants  $c, C$  and  $n_0$  such that for any  $n_0 \leq l \leq n$ ,  $x \in \mathbb{P}(V)$  and  $y \in \mathbb{P}(V^*)$ ,

$$\mathbb{P}(\delta(y, G_nx) \leq e^{-al}) \leq Ce^{-cl}. \quad (5)$$

The bound (5) implies (2), and therefore contains more information than (2). When studying the logarithm of the coefficients  $\log |\langle f, G_nv \rangle|$ , (5) gives an alternative way to prove the law of large numbers and the central limit theorem, but also allows to establish new results like the law of iterated logarithm and the large deviation bounds. However, many important properties such as the Berry-Esseen bounds, local limit theorems, large deviation principles and exact asymptotics of large deviations for the coefficients  $\langle f, G_nv \rangle$  cannot be obtained by this approach. For these latter statements the exact contribution of the term  $\delta(y, G_nx)$  should be accounted, which means that we need to establish more general theorems for the couple  $(G_nx, \log |G_nv|)$ .

We find out that in order to transfer many asymptotic properties from the couple  $(G_nx, \log |G_nv|)$  to that of the coefficients  $\log |\langle f, G_nv \rangle|$ , it is necessary to establish the identity (1) for subsets of  $\mathbb{P}(V)$  which are not projective hyperplanes, in particular, for the hypersurfaces  $\{x \in \mathbb{P}(V) : \delta(y, x) = t\}$ , where  $t \neq 0$ . The main goal of the paper is to extend the result of Furstenberg (1) from the special case of projective hyperplanes to arbitrary algebraic subsets  $Y$  of the projective space  $\mathbb{P}(V)$ . There is, however, an essential difference with the Furstenberg's result, which confers the mass 0 to a projective hyperplane. We show that for an arbitrary algebraic set  $Y$  of  $\mathbb{P}(V)$  it holds that  $\nu(Y)$  is 0 or 1. Contrary to the Furstenberg zero-law, it is possible, as we show in Example 2.3, that the invariant measure  $\nu$  is concentrated on an algebraic subset of dimension  $d - 2$  on the projective space  $\mathbb{P}(V)$ . It is also interesting to note that for projective hyperplanes the Furstenberg zero-law

can be strengthened to the regularity property (2), while for algebraic sets the quantitative analog of (2) has not yet been established.

Using the zero-one law of the stationary measure  $\nu$  we will prove the following local limit theorem for the coefficients  $\langle f, G_nv \rangle$ : under appropriate conditions, for any real numbers  $a_1 < a_2$ , any  $f \in V^* \setminus \{0\}$  and  $v \in V \setminus \{0\}$ , as  $n \rightarrow \infty$ ,

$$\mathbb{P}\left(\log |\langle f, G_nv \rangle| - n\lambda \in [a_1, a_2]\right) = \frac{a_2 - a_1}{\sigma\sqrt{2\pi n}}(1 + o(1)).$$

Based on a zero-one law for the invariant measure under the change of measure which is proved in Theorem 2.6, it is possible to establish a local limit theorem with large deviations for the coefficients of  $G_n$ , however this will be done elsewhere.

### 1.2. Idea of the proof of the local limit theorem

For illustration we show how to apply our zero-one law to establish the local limit theorem for the coefficients  $\langle f, G_nv \rangle$ , where  $f \in V^* \setminus \{0\}$  and  $v \in V \setminus \{0\}$ . Letting  $I = [a_1, a_2]$  be an interval of the real line, we have to handle the probability

$$\mathbb{P}(\log |\langle f, G_nv \rangle| - n\lambda \in I). \quad (6)$$

Using (3) and discretizing the values of  $\delta(y, G_nx)$ , the probability (6) is bounded from above by

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\log |G_nv| - n\lambda \in I_\eta + \eta k, G_nx \in Y_k^\eta\right), \quad (7)$$

where  $\eta > 0$  is sufficiently small and will be chosen later,  $I_\eta = [a_1 - \eta, a_2 + \eta]$  and  $Y_k^\eta = \{x \in \mathbb{P}(V) : -\log \delta(y, x) \in \eta[k - 1, k]\}$ . By the local limit theorem for the couple  $(G_nx, \log |G_nv|)$  (actually in the paper we circumvent it by using the spectral gap theory and some smoothing technique), each probability in the sum (7) is asymptotically bounded by  $\frac{\text{length}(I_\eta)}{\sigma\sqrt{2\pi n}} \nu(\bar{Y}_k^\eta)$ , with  $\bar{Y}_k^\eta = \{x \in \mathbb{P}(V) : -\log \delta(y, x) \in \eta[k - 1 - \varepsilon, k + \varepsilon]\}$ . Hence, the sum (7) asymptotically does not exceed

$$\frac{\text{length}(I_\eta)}{\sigma\sqrt{2\pi n}} \sum_{k=1}^{\infty} \nu(\bar{Y}_k^\eta). \quad (8)$$

An important issue is to show that the sum in (8) converges to 1 as  $\varepsilon \rightarrow 0$  and  $\eta \rightarrow 0$ . This turns out to be a difficult problem. By some easy calculations it reduces to showing that for any  $k \geq 0$ ,

$$\nu(\{x \in \mathbb{P}(V) : \log \delta(y, x) = -\eta k\}) = 0. \quad (9)$$

The set  $Y_0 = \{x \in \mathbb{P}(V) : \delta(y, x) = 0\}$  is a projective hyperplane in  $\mathbb{P}(V)$  of dimension  $d - 2$ . By (1), it holds that  $\nu(Y_0) = 0$ , under the strong irreducibility condition on the measure  $\mu$ . For  $k \geq 1$ , the equality (9) may not be true for an arbitrary  $\eta$ , as we show in the paper. In fact we establish a zero-one law for the invariant measure  $\nu$  (see Theorem 2.2), from which it follows that for any  $t < 0$ ,

$$\nu(\{x \in \mathbb{P}(V) : \log \delta(y, x) = t\}) = 0 \text{ or } 1.$$

This statement implies that (9) holds true for any  $k \geq 1$  if we choose an appropriate constant  $\eta$ . Indeed, if there exists  $t < 0$  such that  $\nu(\{x \in \mathbb{P}(V) : \log \delta(y, x) = t\}) = 1$ , then we can choose  $\eta$  such that  $-\eta k \neq t$  for any  $k$ , so that (9) holds true for any  $k$ . Otherwise  $\eta$  can be chosen arbitrarily. This proves that the sum in (8) converges to 1 as  $\varepsilon \rightarrow 0$  and  $\eta \rightarrow 0$  and so we obtain that  $\limsup_{n \rightarrow \infty}$  of the probability (6) is bounded from above by  $\frac{\text{length}(I_\eta)}{\sigma\sqrt{2\pi n}}$ . By some similar reasoning  $\liminf_{n \rightarrow \infty}$  of (6) is bounded from below by the same quantity.

## 2. Main results

The inverse of  $g \in \mathbb{G} = \text{GL}(V)$  is denoted by  $g^{-1}$ . The adjoint operator  $g^*$  of  $g \in \mathbb{G}$  is the automorphism  $g^*$  of  $V^*$  defined by  $(g^*f)(v) = f(gv)$ , where  $v \in V$  and  $f \in V^*$ . Let  $\mathcal{C}(\mathbb{P}(V))$  be the space of complex valued continuous functions on  $\mathbb{P}(V)$ .

All over the paper  $\mu$  is a probability measure on  $\mathbb{G}$ . Denote by  $\text{supp } \mu$  the support of  $\mu$  and by  $\Gamma_\mu := [\text{supp } \mu]$  the smallest closed subsemigroup of  $\mathbb{G}$  generated by  $\text{supp } \mu$ .

An element  $g \in \mathbb{G}$  is called *proximal* if it has an algebraically simple dominant eigenvalue, namely,  $g$  has an eigenvalue  $\lambda_g$  satisfying  $|\lambda_g| > |\lambda'_g|$  for all other eigenvalues  $\lambda'_g$  of  $g$ . It is easy to check that  $\lambda_g \in \mathbb{R}$ . We choose  $v_g^+$  an eigenvector with unit norm  $|v_g^+| = 1$ , corresponding to the eigenvalue  $\lambda_g$ , which will be called dominant eigenvector of  $g$ . The unique element  $x_g^+ = \mathbb{R}v_g^+ \in \mathbb{P}(V)$  will be called *attractor* of  $g$ . Note that it does not depend on the choice of  $v_g^+$ .

We need the following strong irreducibility and proximality conditions:

**A1** (Strong irreducibility). *No non-trivial finite union of proper subspaces of  $V$  is  $\Gamma_\mu$ -invariant, that is, no non-trivial finite union of proper subspaces  $V_1, \dots, V_k$  in  $V$  satisfies  $g(V_1 \cup \dots \cup V_k) \subset V_1 \cup \dots \cup V_k$  for any  $g \in \Gamma_\mu$ .*

**A2** (Proximality).  *$\Gamma_\mu$  contains at least one proximal element.*

Assume conditions **A1** and **A2**. By a well known result of Furstenberg [13], on the projective space  $\mathbb{P}(V)$  there exists a unique  $\mu$ -stationary probability measure  $\nu$  such that for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ ,

$$\int_{\mathbb{P}(V)} \int_{\mathbb{G}} \varphi(gx) \mu(dg) \nu(dx) = \int_{\mathbb{P}(V)} \varphi(x) \nu(dx). \quad (10)$$

Moreover, Furstenberg [13] (see also Bougerol and Lacroix [9, Chapter III, Proposition 2.3]) showed that under appropriate assumptions any proper projective subspace  $Y \subset \mathbb{P}(V)$  has  $\nu$ -measure 0.

**Theorem 2.1** (Furstenberg). *Assume condition **A1**. Then, for any  $\mu$ -stationary measure  $\nu$  and any proper projective subspace  $Y \subset \mathbb{P}(V)$ , it holds that  $\nu(Y) = 0$ .*

Our first result extends Theorem 2.1 to algebraic subsets of  $\mathbb{P}(V)$ . We recall that a subset  $X$  in  $V$  is algebraic if there exist polynomial functions  $p_1, \dots, p_k$  on  $V$  such that  $X = \{v \in V : p_1(v) = \dots = p_k(v) = 0\}$ . We say that  $X$  is homogeneous if for every  $t \in \mathbb{R}$  and  $v \in X$  it holds that  $tv \in X$ . A subset  $Y$  of  $\mathbb{P}(V)$  is algebraic if there exists an algebraic homogeneous subset  $X$  in  $V$  whose projective image on  $\mathbb{P}(V)$  is  $Y$ .

**Theorem 2.2.** *Assume conditions **A1** and **A2**. Then, for any algebraic subset  $Y$  of  $\mathbb{P}(V)$ , it holds that either  $\nu(Y) = 0$  or  $\nu(Y) = 1$ .*

The statement of Theorem 2.2 can be interpreted as a statement about the behaviour of the trajectory  $(G_n x)_{n \geq 0}$ : either for some  $x \in \mathbb{P}(V)$  the trajectory  $(G_n x)_{n \geq 0}$  stays in  $Y$  with probability 1, or for every  $x \in \mathbb{P}(V)$  this trajectory mostly avoids  $Y$ . The case where the Zariski closure of  $\Gamma_\mu$  acts transitively on  $\mathbb{P}(V)$  has been considered in Proposition 10.1 (b) of Benoist and Quint [5], however, Theorem 2.2 covers also the case where this action is not transitive. For instance, in the example below this Zariski closure is contained in the group of isometries  $O(q)$  which does not act transitively on  $\mathbb{P}(V)$ . Another point of view of the way the random walk  $(G_n x)_{n \geq 0}$  avoids an algebraic subset is developed in the paper by Aoun [1].

The following example shows that there exist proper algebraic subsets of  $\mathbb{P}(V)$  of  $\nu$ -invariant measure 0 or 1.

**Example 2.3.** Let  $d \geq 3$ . Fix an integer  $p$  such that  $1 \leq p \leq d-1$  and equip  $V$  with the quadratic form  $q(v) = v_1^2 + v_2^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_d^2$ . Let  $O(q)$  be the group of isometries of  $q$ , that is the group of elements  $g \in \text{GL}(V)$  such that  $q(gv) = q(v)$  for all  $v \in V$ . We choose  $\mu$  to be any probability measure on  $O(q)$  such that  $\Gamma_\mu$  is proximal and strongly irreducible. For instance we can take any probability measure  $\mu$  with the full support  $O(q)$ ; then it will be proximal and strongly irreducible, since the group  $O(q)$  is proximal and strongly irreducible on  $V$  for  $d \geq 3$ . Denote by  $\nu$  the unique  $\mu$ -stationary probability measure on  $\mathbb{P}(V)$ . Let  $Y$  be the subset of  $\mathbb{P}(V)$  defined as the set of straight lines in  $V$  which are spanned by vectors  $v \in V$  with  $q(v) = 0$ . Then the  $\mu$ -invariant measure  $\nu$  is such that  $\nu(Y) = 1$ . In particular this implies that the support of the measure  $\nu$  is contained in  $Y$ .

Consider the case when  $p = 1$ . Let  $f$  be the linear functional  $f : u \in V \mapsto u_1 \in \mathbb{R}$  so that  $|f| = 1$  and let  $y = \mathbb{R}f \in \mathbb{P}(V^*)$ . Define the algebraic subset  $Y = \{x \in \mathbb{P}(V) : \delta(x, y) = 1/\sqrt{2}\}$ . We will show that  $Y$  contains the support of the measure  $\nu$ . Indeed, if  $v$  is such that  $q(v) = 0$ , then  $v_1^2 = v_2^2 + \dots + v_d^2$  and hence, with  $x = \mathbb{R}v \in \text{supp } \nu$ ,

$$\delta(x, y) = \frac{|\langle f, v \rangle|}{|f||v|} = \frac{1}{\sqrt{2}}.$$

Therefore  $\nu(Y) = 1$ . Moreover, if we define  $Y' = \{x \in \mathbb{P}(V) : \delta(x, y) = t\}$  with  $t \neq 1/\sqrt{2}$ , then  $\nu(Y') = 0$ .

**Corollary 2.4.** *Assume conditions [A1](#) and [A2](#). For any  $t \in (-\infty, 0)$ , define the hypersurface  $Y_0 = \{x \in \mathbb{P}(V) : \delta(x, y) = t\}$ . Then  $\nu(Y_0)$  is 0 or 1.*

**Proof.** Let  $f \in V^*$  be such that  $|f| = 1$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$ . Define the homogenous set  $X_0$  as the collection of all vectors  $v \in V$  satisfying  $\langle f, v \rangle^2 = t^2|v|^2$ . Then  $Y_0$  is the projective image of  $X_0 \setminus \{0\}$  on  $\mathbb{P}(V)$ . As the function  $\phi : v \in V \mapsto f(v)^2 - t^2|v|^2 = (f_1v_1 + \dots + f_dv_d)^2 - t^2(v_1^2 + \dots + v_d^2)$  is a polynomial on  $V$ , the set  $X_0$  is algebraic. Since it is also homogeneous, by definition the set  $Y_0$  is algebraic in  $\mathbb{P}(V)$ , and the conclusion follows from [Theorem 2.2](#).  $\square$

In order to state our second result we need to introduce the transfer operators and related notions. These operators play an important role in many problems related to random walks on the group  $\mathbb{G}$ , see for instance [Le Page \[25\]](#) or [Guivarc'h and Le Page \[21\]](#). We are going to show that the stationary measures related to these operators do not charge the algebraic subsets.

For any  $g \in \mathbb{G}$ ,  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$ , set

$$\sigma(g, x) = \log \frac{|gv|}{|v|}, \quad \sigma(g^*, y) = \log \frac{|g^*f|}{|f|}. \quad (11)$$

Denote  $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ . Consider the sets

$$I_\mu^+ = \left\{ s \geq 0 : \int_{\mathbb{G}} \|g\|^s \mu(dg) < +\infty \right\}, \quad I_\mu^- = \left\{ s \leq 0 : \int_{\mathbb{G}} N(g)^{-s} \mu(dg) < +\infty \right\}$$

and note that both  $I_\mu^+$  and  $I_\mu^-$  contain at least the element 0. Let  $s \in I_\mu^+ \cup I_\mu^-$ . Define the transfer operators  $P_s$  and  $P_s^*$  as follows: for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$  and  $x \in \mathbb{P}(V)$ ,

$$P_s \varphi(x) = \int_{\mathbb{G}} e^{s\sigma(g, x)} \varphi(gx) \mu(dg) \quad (12)$$

and for any  $\varphi \in \mathcal{C}(\mathbb{P}(V^*))$  and  $y \in \mathbb{P}(V^*)$ ,

$$P_s^* \varphi(y) = \int_{\mathbb{G}} e^{s\sigma(g^*, y)} \varphi(g^*y) \mu(dg). \quad (13)$$

The next assumption will be necessary to state the results for  $s < 0$ .

**A3** (Two-sided exponential moment). *There exists  $\alpha \in (0, 1)$  such that*

$$\int_{\mathbb{G}} N(g)^\alpha \mu(dg) < +\infty.$$

Let  $s \in I_\mu^+$ . By [\[21, Theorem 2.6\]](#), under conditions [A1](#) and [A2](#), there exist a unique  $\kappa(s) > 0$  and a unique probability measure  $\nu_s$  on  $\mathbb{P}(V)$  such that  $\nu_s$  is an eigenmeasure for the transfer operator  $P_s$  corresponding to the eigenvalue  $\kappa(s)$ :

$$P_s \nu_s = \kappa(s) \nu_s. \quad (14)$$

Similarly, the conjugate transfer operator  $P_s^*$  has a unique probability eigenmeasure  $\nu_s^*$  on  $\mathbb{P}(V^*)$  corresponding to the same eigenvalue  $\kappa(s)$ :

$$\nu_s^* P_s^* = \kappa(s) \nu_s^*.$$

For detailed account of the mentioned properties for  $s > 0$  we refer the reader to [\[20, 21\]](#), where it is proved that the mapping  $s \mapsto \kappa(s)$  is analytic on a complex neighborhood of the interval  $I_\mu^+$ . [Guivarc'h and Le Page \[21\]](#) have also proved that the measure  $\nu_s$  does not charge any proper projective subspace  $Y$  in  $\mathbb{P}(V)$ .

**Theorem 2.5** ([Guivarc'h and Le Page](#)). *Assume conditions [A1](#) and [A2](#). Then, for any  $s \in I_\mu^+$  and any proper projective subspace  $Y$  of  $\mathbb{P}(V)$ , it holds that  $\nu_s(Y) = 0$ .*

Our second result extends Theorem 2.5 to proper algebraic subsets of  $\mathbb{P}(V)$ .

**Theorem 2.6.** *Assume conditions A1 and A2. Then, for any  $s \in I_\mu^+$  and any proper algebraic subset  $Y$  of  $\mathbb{P}(V)$ , it holds that either  $\nu_s(Y) = 0$  or  $\nu_s(Y) = 1$ .*

We are able to prove a similar assertion for small negative  $s$ . First we show in Section 3 the existence and the uniqueness of the eigenmeasure  $\nu_s$  for small negative values  $s$ .

**Proposition 2.7.** *Assume conditions A1, A2 and A3. Then, there exists  $s_0 > 0$  with the following property: for any  $s \in [-s_0, 0)$ , there exist a unique  $\kappa(s) > 0$  and a unique probability measure  $\nu_s$  on  $\mathbb{P}(V)$  such that  $\nu_s$  is an eigenmeasure for the transfer operator  $P_s$  corresponding to the eigenvalue  $\kappa(s)$ . Similarly, the transfer operator  $P_s^*$  has a unique probability eigenmeasure  $\nu_s^*$  on  $\mathbb{P}(V^*)$  corresponding to the same eigenvalue  $\kappa(s)$ .*

For negative values of  $s$  we can prove the following analogue of Theorem 2.6.

**Theorem 2.8.** *Assume conditions A1, A2 and A3. Then, there exists  $s_0 > 0$  such that for any  $s \in [-s_0, 0)$  and any proper algebraic subset  $Y$  of  $\mathbb{P}(V)$ , it holds that either  $\nu_s(Y) = 0$  or  $\nu_s(Y) = 1$ .*

It is easy to see that all the conclusions of the Example 2.3 apply also to the measure  $\nu_s$ .

As an application of the stated results we use Theorem 2.2 to establish a local limit theorem for the coefficients of random walks on the general linear group  $\mathrm{GL}(V)$ .

**Theorem 2.9.** *Assume conditions A1, A2 and A3. Let  $-\infty < a_1 < a_2 < \infty$  be real numbers. Then, as  $n \rightarrow \infty$ , uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,*

$$\mathbb{P}\left(\log |\langle f, G_n v \rangle| - n\lambda \in [a_1, a_2]\right) = \frac{a_2 - a_1}{\sigma\sqrt{2\pi n}}(1 + o(1)).$$

To the best of our knowledge, a local limit theorem for the coefficients of random walks on the general linear group  $\mathrm{GL}(V)$  has not been established in the literature so far. Local limit theorem for sums of independent random variables have been studied by many authors: we refer the reader to Gnedenko [15], Stone [27], Borovkov and Borovkov [8], Breuillard [11]. For the norm cocycle of random walks on  $\mathrm{GL}(V)$ , local limit theorems have been proved by Le Page [25], Guivarc'h [19], Benoist and Quint [5]. The local limit theorems established in [5] play an important role for studying stationary measures on finite volume homogeneous spaces, see [2] for details.

Other potential applications of Theorem 2.2 are the Berry-Esseen bound, the Edgeworth expansion and the Cramér type moderate deviation. A paper in progress is [31], where a local limit theorem with moderate deviations has been established. In its turn Theorems 2.6 and 2.8 can be used to establish various limit theorems like the large deviation principle and local limit theorem with large deviations for the coefficients [30].

### 3. Properties of the stationary measure

#### 3.1. The existence and the uniqueness of the probability eigenmeasure for negative $s$

In this section we prove the existence and the uniqueness of the probability eigenmeasure of the transfer operator  $P_s$  for  $s < 0$ . Actually we shall establish it for  $s$  in a sufficiently small neighborhood of 0, i.e. for  $|s| < s_0$ , for some small  $s_0 > 0$ . The approach is inspired by that of Guivarc'h and Le Page [20, 21]. The existence of such a measure is deduced from a very general fixed point theorem due to Brouwer-Schauder-Tychonoff. To establish the uniqueness one can make use of the general results on the perturbation theory of linear operators, which simplifies slightly the proofs of Guivarc'h and Le Page [20, 21].

We proceed to state a fixed point theorem for measures in an abstract context. Let  $X$  be a compact topological space and  $\mathcal{C}(X)$  be the space of complex valued continuous functions on  $X$  equipped with the uniform norm. Denote by  $\mathcal{C}(X)'$  the topological dual space of  $\mathcal{C}(X)$  equipped with the weak-\* topology. Recall that the weak-\* topology is the weakest topology on  $\mathcal{C}(X)'$  for which the mapping  $\nu \in \mathcal{C}(X)' \mapsto \nu(\varphi) \in \mathbb{C}$  is continuous for any  $\varphi \in \mathcal{C}(X)$  and that by Riesz representation theorem, the space  $\mathcal{C}(X)'$  coincides with the space of complex valued Borel measures on  $X$ .

**Lemma 3.1.** *Let  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  be a bounded linear operator such that  $T(f) > 0$  for any  $f > 0$ . Then, there exist a constant  $\alpha > 0$  and a Borel probability measure  $\nu_0$  on  $X$  such that  $T'\nu_0 = \alpha\nu_0$ , where  $T' : \mathcal{C}(X)' \rightarrow \mathcal{C}(X)'$  is the adjoint operator of  $T$ .*

**Proof.** Recall the Brouwer-Schauder-Tychonoff theorem (see [6, Appendix]): Let  $\mathcal{P}$  be a convex and compact subset inside a topological vector space  $V$  and let  $A : \mathcal{P} \rightarrow \mathcal{P}$  be a continuous mapping. Then there exists  $\nu_0 \in \mathcal{P}$  such that  $A\nu_0 = \nu_0$ .

We shall apply Brouwer-Schauder-Tychonoff theorem with  $V = \mathcal{C}(X)'$ . By Riesz representation theorem, the space  $\mathcal{C}(X)'$  coincides with the space of complex valued Borel measures on  $X$ . Let  $\mathcal{P}$  be the subspace of  $\mathcal{C}(X)'$  formed by probability measures. Then the set  $\mathcal{P}$  is convex and by the Banach-Alaoglu theorem it is compact in the weak-\* topology. Since  $T1 > 0$  everywhere, the mapping  $\nu \in \mathcal{P} \mapsto \nu(T1) \in \mathbb{R}$  does not vanish on  $\mathcal{P}$ . Note that for any  $\varphi \in \mathcal{C}(X)$  with  $\varphi \geq 0$ , it holds  $T'\nu(\varphi) = \nu(T\varphi) \geq 0$  since  $T\varphi \geq 0$ . This allows to define the mapping  $A : \mathcal{P} \rightarrow \mathcal{P}$  by setting, for any  $\nu \in \mathcal{P}$ ,

$$A\nu = \frac{T'\nu}{\nu(T1)}.$$

As  $T$  is a bounded linear operator on  $\mathcal{C}(X)$ , the adjoint operator  $T'$  is continuous for the weak-\* topology. Since  $T1$  is a continuous function on  $X$ , the mapping  $\nu \mapsto \nu(T1)$  is also continuous for the weak-\* topology. Therefore, the mapping  $A$  is continuous for the weak-\* topology. By the Brouwer-Schauder-Tychonoff theorem, the mapping  $A$  has a fixed point  $\nu_0 : A\nu_0 = \nu_0$ . The assertion follows with  $\alpha = \nu_0(T1)$ .  $\square$

Let  $\gamma \in (0, 1)$ . Consider the Banach space  $\mathcal{B}_\gamma$  of  $\gamma$ -Hölder continuous functions on  $\mathbb{P}(V)$  endowed with the norm

$$\|\varphi\|_{\mathcal{B}_\gamma} = \sup_{x \in \mathbb{P}(V)} |\varphi(x)| + \sup_{x, x' \in \mathbb{P}(V): x \neq x'} \frac{|\varphi(x) - \varphi(x')|}{\mathbf{d}(x, x')^\gamma},$$

where  $\mathbf{d}(x, x')$  is the sin of the angle between the vector lines  $x = \mathbb{R}v$  and  $x' = \mathbb{R}v'$  in  $\mathbb{P}(V)$ :  $\mathbf{d}(x, x') = \sqrt{1 - \left(\frac{\langle v, v' \rangle}{|v||v'|}\right)^2}$ . The topological dual of  $\mathcal{B}_\gamma$  endowed with the induced norm is denoted by  $\mathcal{B}_\gamma^*$ . Denote by  $\mathcal{B}_\gamma^*$  the Banach space of  $\gamma$ -Hölder continuous functions on  $\mathbb{P}(V^*)$  endowed with the norm

$$\|\varphi\|_{\mathcal{B}_\gamma^*} = \sup_{y \in \mathbb{P}(V^*)} |\varphi(y)| + \sup_{y, y' \in \mathbb{P}(V^*): y \neq y'} \frac{|\varphi(y) - \varphi(y')|}{\mathbf{d}(y, y')^\gamma},$$

where  $\mathbf{d}(y, y')$  is the sin of the angle between the vector lines  $y = \mathbb{R}f$  and  $y' = \mathbb{R}f'$  in  $\mathbb{P}(V^*)$ :  $\mathbf{d}(y, y') = \sqrt{1 - \left(\frac{\langle f, f' \rangle}{|f||f'|}\right)^2}$ .

Recall that  $\mathcal{C}(\mathbb{P}(V))$  is the space of the continuous complex valued functions on  $\mathbb{P}(V)$  and  $\mathcal{C}(\mathbb{P}(V))'$  is the space of the complex valued Borel measures on  $\mathbb{P}(V)$ .

**Lemma 3.2.** *Assume conditions **A1**, **A2** and **A3**. Then, there exists a positive constant  $s_0$  such that for any  $|s| < s_0$  the operator  $P_s$  has a unique probability eigenmeasure  $\nu_s$  associated with the unique eigenvalue  $\kappa(s)$ .*

**Proof.** Note that for  $s$  real and close to 0,  $P_s : \mathcal{C}(\mathbb{P}(V)) \rightarrow \mathcal{C}(\mathbb{P}(V))$  is a bounded linear operator. Moreover  $P_s\varphi > 0$  for any  $\varphi > 0$  on  $\mathbb{P}(V)$ . Denote by  $P_s' : \mathcal{C}(\mathbb{P}(V))' \rightarrow \mathcal{C}(\mathbb{P}(V))'$  the adjoint operator of  $P_s$ . Using Lemma 3.1 with  $X = \mathbb{P}(V)$ , we get that there exists a probability eigenmeasure  $\nu_s$  such that  $P_s'\nu_s = \alpha(s)\nu_s$ . By the duality, for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ , we have  $\nu_s(P_s\varphi) = P_s'\nu_s(\varphi)$ , so that

$$\nu_s(P_s\varphi) = \alpha(s)\nu_s(\varphi). \tag{15}$$

The measure  $\nu_s$  and the eigenvalue  $\alpha(s)$  might a priori not be unique. We shall prove the uniqueness using the perturbation theory of linear operators.

From the results of Le Page [25] it follows that the operator  $P_0$  has a spectral gap property on the Banach space  $\mathcal{B}_\gamma$ , for some  $\gamma > 0$ . Since the dependence of the operator  $P_s$  in  $s$  is analytic (cf. [9, Chapter V, Lemma 3.2] or [5, Lemma 11.17]), we can apply the perturbation theory of linear operators [24, Theorem III.8]. Thus there exist constants  $s_0 > 0$  and holomorphic mappings  $s \mapsto \theta_s \in \mathcal{B}_\gamma'$ ,  $s \mapsto r_s \in \mathcal{B}_\gamma$ ,  $s \mapsto \kappa_s \in \mathbb{C}$  on  $(-s_0, s_0)$  such that  $\theta_0 = \nu$ ,  $r_0 = 1$ ,  $\nu(r_s) = 1$ ,  $\theta_s(r_s) = 1$  and

$$\theta_s(P_s\varphi) = \kappa(s)\theta_s(\varphi), \quad P_s r_s = \kappa(s)r_s, \tag{16}$$



for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ . Moreover, one can choose  $s_0$  small enough so that there exist constants  $C > 0$  and  $\rho_0 \in (0, 1)$  such that for any complex  $|s| < s_0$ , we have  $|\kappa(s)| > \rho_0$  and

$$\|P_s^n \varphi - \kappa(s)^n \theta_s(\varphi) r_s\|_{\mathcal{B}_\gamma} \leq C \rho_0^n \|\varphi\|_{\mathcal{B}_\gamma}. \quad (17)$$

In particular, (17) implies that for any complex  $|s| < s_0$ , the complex number  $\kappa(s)$  is the unique eigenvalue of  $P_s$  in  $\mathcal{B}_\gamma$  with modulus strictly larger than  $\rho_0$  and the associated eigenspace is  $\mathbb{C}r_s$ . Indeed, let  $\varphi \in \mathcal{B}_\gamma$  with  $\varphi \neq 0$  and  $\lambda \in \mathbb{C}$  be such that  $P_s(\varphi) = \lambda\varphi$  and  $\lambda \neq \kappa(s)$ . Then

$$\lambda \theta_s(\varphi) = \theta_s(P_s \varphi) = \kappa(s) \theta_s(\varphi),$$

hence  $\theta_s(\varphi) = 0$ . Therefore, (17) gives that

$$|\lambda|^n \|\varphi\|_{\mathcal{B}_\gamma} = \|P_s^n \varphi\|_{\mathcal{B}_\gamma} \leq C \rho_0^n \|\varphi\|_{\mathcal{B}_\gamma}.$$

This implies that  $|\lambda| \leq \rho_0$ , which means that  $\kappa(s)$  is the unique eigenvalue with modulus strictly larger than  $\rho_0$ .

Let us now show that the eigenspace associated to  $\kappa(s)$  is spanned by the function  $r_s$ . Indeed, if  $\varphi$  is in this eigenspace then  $\varphi \in \mathcal{B}_\gamma$  and  $P_s \varphi = \kappa(s)\varphi$ . Again by (17), we have

$$|\kappa(s)|^n \|\varphi - \theta_s(\varphi) r_s\|_{\mathcal{B}_\gamma} = \|P_s^n \varphi - \kappa(s)^n \theta_s(\varphi) r_s\|_{\mathcal{B}_\gamma} \leq C \rho_0^n \|\varphi\|_{\mathcal{B}_\gamma}.$$

Since  $|\kappa(s)| > \rho_0$ , we get  $\varphi = \theta_s(\varphi) r_s$ .

We shall use the uniqueness property of  $\kappa(s)$  to show that  $\kappa(s)$  is real and that  $r_s$  takes real values for real  $s \in (-s_0, s_0)$ . Indeed, as  $s$  is real, for any  $\varphi \in \mathcal{B}_\gamma$  we have  $P_s \bar{\varphi} = \overline{P_s \varphi}$ , which gives  $P_s \bar{r}_s = \overline{P_s r_s} = \bar{\kappa}_s \bar{r}_s$ . Since  $\kappa(s)$  is the unique eigenvalue of  $P_s$  with modulus strictly larger than  $\rho_0$ , this proves  $\bar{\kappa}_s = \kappa_s$ . Besides, from the equation  $P_s \bar{r}_s = \kappa_s \bar{r}_s$  it follows that  $\bar{r}_s$  belongs to the eigenspace  $\mathbb{C}r_s$  associated to  $\kappa(s)$ , so there exists  $z \in \mathbb{C}$  such that  $\bar{r}_s = z r_s$ . Since  $\nu(r_s) = 1 = \nu(\bar{r}_s)$ , we get  $z = 1$  and hence  $\bar{r}_s = r_s$  as required.

Since  $r_0 = 1$ , we can assume that  $s_0$  is small enough such that  $r_s$  is strictly positive for real  $s \in (-s_0, s_0)$ . We now prove that  $\alpha(s) = \kappa(s)$  for real  $s \in (-s_0, s_0)$ . We put  $\varphi = r_s$  in (15) and use the second identity in (16) to obtain

$$\alpha(s) \nu_s(r_s) = \nu_s(P_s r_s) = \kappa(s) \nu_s(r_s). \quad (18)$$

Since  $r_s > 0$  we have  $\nu_s(r_s) > 0$ , which implies that  $\alpha(s) = \kappa(s)$  for real valued  $s$ .

Iterating (15) and using the fact that  $\alpha(s) = \kappa(s)$ , we have that for any  $\varphi \in \mathcal{B}_\gamma$ ,

$$\nu_s(P_s^n \varphi) = \kappa(s)^n \nu_s(\varphi). \quad (19)$$

From (19) and (17), taking the limit as  $n \rightarrow \infty$  we obtain that

$$\nu_s(\varphi) = \nu_s(\theta_s(\varphi) r_s) = \theta_s(\varphi) \nu_s(r_s),$$

from which it follows that for any  $\varphi \in \mathcal{B}_\gamma$ ,

$$\frac{\nu_s(\varphi)}{\nu_s(r_s)} = \theta_s(\varphi).$$

This proves that the linear functional  $\theta_s$  is indeed a non-negative Borel measure, and that any non-negative Borel measure which is an eigenmeasure of  $P_s$  is proportional to  $\theta_s$ .  $\square$

To show the uniqueness of the eigenfunction of the operator  $P_s$  we need more notation. For any real  $|s| < s_0$ , let  $r_s > 0$  be the function introduced in the proof of Lemma 3.2. Introduce the operator  $Q_s$  by setting, for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ ,

$$Q_s \varphi = \frac{P_s(r_s \varphi)}{\kappa(s) r_s}. \quad (20)$$

Then  $Q_s$  is a Markov operator, namely,  $Q_s \varphi \geq 0$  for any  $\varphi \geq 0$ , and  $Q_s(1) = 1$ .

**Lemma 3.3.** For any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ , one has

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{P}(V)} \left| Q_s^n(\varphi)(x) - \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)} \right| = 0 \quad (21)$$

and

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{P}(V)} \left| \frac{1}{\kappa(s)^n} P_s^n(\varphi)(x) - \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s(x) \right| = 0. \quad (22)$$

**Proof.** First note that from (17) we have that (22) holds for any  $\varphi \in \mathcal{B}_\gamma$ . This implies that (21) also holds for any  $\varphi \in \mathcal{B}_\gamma$ . As  $Q_s$  is a Markov operator, it has norm 1 in  $\mathcal{C}(\mathbb{P}(V))$ , so  $Q_s^n$  is uniformly bounded in the space of bounded operators of  $\mathcal{C}(\mathbb{P}(V))$ . Since  $\mathcal{B}_\gamma$  is dense in  $\mathcal{C}(\mathbb{P}(V))$ , the convergence (21) holds for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ . This in turn implies that (22) also holds for  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ .  $\square$

**Lemma 3.4.** Assume conditions **A1**, **A2** and **A3**. Then, for any  $|s| < s_0$ , the function  $r_s$  is the unique (up to a scaling constant) non-negative continuous eigenfunction of the operator  $P_s$ .

**Proof.** If we take  $\varphi$  to be a non-negative non-zero continuous eigenfunction of  $P_s$  associated to some eigenvalue  $\lambda$ , then by (22) we get

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{P}(V)} \left| \frac{1}{\kappa(s)^n} \lambda^n \varphi(x) - \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s(x) \right| = 0. \quad (23)$$

It follows that  $\lambda = \kappa(s)$  and  $\varphi = \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s$ , which means that the function  $\varphi$  coincides (up to a scaling constant) with the eigenfunction  $r_s$ .  $\square$

Applying the previous theory to the operator  $P_s^*$  and to the adjoint projective space  $\mathbb{P}(V^*)$  we obtain the following:

**Lemma 3.5.** Assume conditions **A1**, **A2** and **A3**. Then, there exists a positive constant  $s_0$  such that for any real  $|s| < s_0$  the operator  $P_s^*$  has a unique probability eigenmeasure  $\nu_s^*$  and a unique (up to a scaling constant) positive continuous eigenfunction  $r_s^*$  associated with the same unique eigenvalue  $\kappa^*(s)$ .

We will show in Lemma 3.10 that  $\kappa^*(s) = \kappa(s)$ .

### 3.2. The Hölder regularity of the stationary measure

In this section we establish the Hölder regularity of the stationary measure  $\nu_s$  defined in Lemma 3.2.

Note that  $\nu_0$  coincides with the stationary measure  $\nu$  defined by (10). The Hölder regularity of the stationary measure  $\nu$  has been established in [22] (see also [5, 9, 18]): under conditions **A1**, **A2** and **A3**, there exists a constant  $\alpha > 0$  such that

$$\sup_{y \in \mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(y, x)^\alpha} \nu(dx) < +\infty. \quad (24)$$

By the Frostman lemma (see [26]), the assertion (24) implies that the Hausdorff dimension of the stationary measure  $\nu$  is at least  $\alpha$ . As mentioned before, (24) plays a crucial role for establishing limit theorems such as the law of large numbers and the central limit theorem for the coefficients  $\langle f, G_n v \rangle$  (see [5, 9, 18, 22]). In the following we establish the Hölder regularity of the stationary measure  $\nu_s$  when  $s$  is in a small neighborhood of 0. The proof is based on (24) and the spectral gap properties of the transfer operator  $P_s$  established in subsection 3.1.

**Proposition 3.6.** Assume conditions **A1**, **A2** and **A3**. Then, there exist constants  $s_0 > 0$  and  $\alpha > 0$  such that

$$\sup_{s \in (-s_0, s_0)} \sup_{y \in \mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(y, x)^\alpha} \nu_s(dx) < +\infty. \quad (25)$$

In particular, there exist constants  $\alpha, C > 0$  such that for any  $0 < t < 1$ ,

$$\sup_{s \in (-s_0, s_0)} \sup_{y \in \mathbb{P}(V^*)} \nu_s(\{x \in \mathbb{P}(V) : \delta(y, x) \leq t\}) \leq Ct^\alpha. \quad (26)$$

One implication of the Proposition 3.6 is that the eigenmeasure  $\nu_s$  does not charge the projective hyperplanes. Precise formulation follows:

**Corollary 3.7.** *Assume conditions A1, A2 and A3. Then there exists a constant  $s_0 > 0$  such that for any  $s \in (-s_0, s_0)$  and any projective hyperplane  $Y$  of  $\mathbb{P}(V)$  it holds  $\nu_s(Y) = 0$ .*

Before proceeding to proving Proposition 3.6, let us first recall a change of measure formula which will be used in the proof of Proposition 3.6. For any  $s \in (-s_0, s_0)$ , the family of functions  $q_n^s(x, g) = \frac{e^{s\sigma(g, x)} r_s(gx)}{\kappa(s)^n r_s(x)}$ ,  $n \geq 1$ , satisfies the equation

$$q_{k+m}^s(x, g_2 g_1) = q_k^s(g_1 x, g_2) q_m^s(x, g_1)$$

for  $x \in \mathbb{P}(V)$  and  $g_1, g_2 \in \mathbb{G}$ . This, together with the fact that  $P_s r_s = \kappa(s) r_s$ , implies that the probability measures

$$\mathbb{Q}_{s,n}^x(dg_1, \dots, dg_n) = q_n^s(x, g_n \dots g_1) \mu(dg_1) \dots \mu(dg_n) \quad (27)$$

form a projective system on  $\mathbb{G}^{\mathbb{N}}$ . By the Kolmogorov extension theorem, there exists a unique probability measure  $\mathbb{Q}_s^x$  on  $\mathbb{G}^{\mathbb{N}}$  with marginals  $\mathbb{Q}_{s,n}^x$ . We denote by  $\mathbb{E}_{\mathbb{Q}_s^x}$  the corresponding expectation. For any measurable function  $\varphi$  on  $(\mathbb{P}(V) \times \mathbb{R})^n$ , it holds that

$$\begin{aligned} \frac{1}{\kappa(s)^n r_s(x)} \mathbb{E} \left[ r_s(G_n x) e^{s\sigma(G_n, x)} \varphi \left( G_1 x, \sigma(G_1, x), \dots, G_n x, \sigma(G_n, x) \right) \right] \\ = \mathbb{E}_{\mathbb{Q}_s^x} \left[ \varphi \left( G_1 x, \sigma(G_1, x), \dots, G_n x, \sigma(G_n, x) \right) \right]. \end{aligned} \quad (28)$$

Under the changed measure  $\mathbb{Q}_s^x$ , the Markov chain  $(G_n x)_{n \geq 0}$  has a unique stationary measure  $\pi_s$  defined by  $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$ , for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ .

We shall use the following result which has been established in [5, Lemma 14.11].

**Lemma 3.8.** *Assume conditions A1, A2, A3. Then, for any  $\varepsilon > 0$ , there exist constants  $c_0 > 0$  and  $n_0 \geq 1$  such that for all  $n \geq k \geq n_0$ ,  $x \in \mathbb{P}(V)$  and  $y \in \mathbb{P}(V^*)$ ,*

$$\mathbb{P} \left( \delta(y, G_n x) \leq e^{-\varepsilon k} \right) \leq e^{-c_0 k}.$$

**Proof of Proposition 3.6.** *Step 1.* We choose a small enough constant  $s_0 > 0$  and show that for any  $\varepsilon > 0$ , there exist constants  $c_1 > 0$  and  $n_0 \geq 1$  such that for any  $n \geq n_0$ ,

$$\sup_{s \in (-s_0, s_0)} \sup_{y \in \mathbb{P}(V^*)} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x \left( \delta(y, G_n x) \leq e^{-\varepsilon n} \right) \leq e^{-c_1 n}. \quad (29)$$

To prove this, using (28) and the fact that the eigenfunction  $x \mapsto r_s(x)$  is strictly positive and bounded on  $\mathbb{P}(V)$  uniformly in  $s \in (-s_0, s_0)$ , we get

$$\begin{aligned} \mathbb{Q}_s^x \left( \delta(y, G_n x) \leq e^{-\varepsilon n} \right) &= \frac{1}{\kappa(s)^n r_s(x)} \mathbb{E} \left[ e^{s\sigma(G_n, x)} r_s(G_n x) \mathbb{1}_{\{\delta(y, G_n x) \leq e^{-\varepsilon n}\}} \right] \\ &\leq \frac{c}{\kappa(s)^n} \mathbb{E} \left[ e^{s\sigma(G_n, x)} \mathbb{1}_{\{\delta(y, G_n x) \leq e^{-\varepsilon n}\}} \right]. \end{aligned}$$

By Hölder's inequality, it follows that

$$\mathbb{Q}_s^x \left( \delta(y, G_n x) \leq e^{-\varepsilon n} \right) \leq \frac{c}{\kappa(s)^n} \left[ \mathbb{E} e^{2s\sigma(G_n, x)} \right]^{1/2} \left[ \mathbb{P}(\delta(y, G_n x) \leq e^{-\varepsilon n}) \right]^{1/2}. \quad (30)$$

It is easy to see that  $\mathbb{E} e^{2s\sigma(G_n, x)} \leq \left\{ \mathbb{E} [N(g_1)^{2|s|}] \right\}^n$ . Since  $\kappa(0) = 1$  and the function  $\kappa$  is continuous in a small neighborhood of 0, we can choose a sufficiently small constant  $s_0 > 0$  such that

$$\sup_{s \in (-s_0, s_0)} \sup_{x \in \mathbb{P}(V)} \frac{1}{\kappa(s)^n} \left[ \mathbb{E} e^{2s\sigma(G_n, x)} \right]^{1/2} \leq e^{c_2 n},$$

where  $c_2 > 0$  is a constant satisfying  $c_2 < c_0/4$  with  $c_0$  given in Lemma 3.8. This, together with (30) and Lemma 3.8, concludes the proof of (29) with  $c_1 = c_0/4$ .

*Step 2.* The invariance of  $\pi_s$  under the operator  $Q_s^n$  says that for any non-negative Borel measurable function  $\varphi$  on  $\mathbb{P}(V)$ , we have

$$\pi_s(\varphi) = \int_{\mathbb{P}(V)} \int_{\mathbb{G}^n} \varphi(g_n \dots g_1 x) \mathbb{Q}_{s,n}^x(dg_1, \dots, dg_n) \pi_s(dx),$$

where  $\mathbb{Q}_{s,n}^x$  is a probability measure on  $\mathbb{G}^n$  defined by (27). This, together with (29), gives that uniformly in  $s \in (-s_0, s_0)$  and  $y \in \mathbb{P}(V^*)$ ,

$$\pi_s(\{x \in \mathbb{P}(V) : \delta(y, x) \leq e^{-\varepsilon n}\}) = \int_{\mathbb{P}(V)} \mathbb{Q}_{s,n}^x(\delta(y, G_n x) \leq e^{-\varepsilon n}) \pi_s(dx) \leq e^{-c_1 n}. \quad (31)$$

We denote  $B_n := \{x \in \mathbb{P}(V) : e^{-\varepsilon(n+1)} \leq \delta(y, x) \leq e^{-\varepsilon n}\}$ . Choosing  $\alpha \in (0, c_1/\varepsilon)$ , we deduce from (31) that uniformly in  $s \in (-s_0, s_0)$  and  $y \in \mathbb{P}(V^*)$ ,

$$\begin{aligned} \int_{\mathbb{P}(V)} \frac{1}{\delta(y, x)^\alpha} \pi_s(dx) &= \int_{\{x \in \mathbb{P}(V) : \delta(y, x) > e^{-\varepsilon n_0}\}} \frac{1}{\delta(y, x)^\alpha} \pi_s(dx) + \sum_{n=n_0}^{\infty} \int_{B_n} \frac{1}{\delta(y, x)^\alpha} \pi_s(dx) \\ &\leq e^{\varepsilon n_0 \alpha} + \sum_{n=n_0}^{\infty} e^{\varepsilon \alpha(n+1)} e^{-c_1 n} < +\infty. \end{aligned}$$

This proves (25) by the relation  $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$  for any  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ . The inequality (26) is a direct consequence of (25) by the Markov inequality.  $\square$

Let  $s_0$  be small enough. For any real  $s$  such that  $|s| < s_0$ , any  $y \in \mathbb{P}(V^*)$  and bounded measurable function  $\varphi$  on  $\mathbb{P}(V)$  denote

$$\nu_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(x) \delta(x, y)^s \nu_s(dx).$$

**Corollary 3.9.** *There exists  $s_0 > 0$  such that for any  $s \in (-s_0, 0)$ , the mapping  $y \in \mathbb{P}(V^*) \mapsto \nu_s^y \in \mathcal{C}(\mathbb{P}(V))'$  is continuous for the total variation norm on  $\mathcal{C}(\mathbb{P}(V))'$ .*

**Proof.** Let  $s_0$  be small enough and  $s \in (-s_0, 0)$ . Let  $y, y' \in \mathbb{P}(V^*)$ . Since both  $\nu_s^y$  and  $\nu_s^{y'}$  are absolutely continuous with respect to  $\nu_s$ , we have

$$\begin{aligned} \|\nu_s^y - \nu_s^{y'}\|_{TV} &= \int_{\mathbb{P}(V)} |\delta(x, y)^s - \delta(x, y')^s| \nu_s(dx) \\ &= \int_{\mathbb{P}(V)} \frac{|\delta(x, y')^{-s} - \delta(x, y)^{-s}|}{\delta(x, y)^{-s} \delta(x, y')^{-s}} \nu_s(dx). \end{aligned}$$

Applying the Hölder inequality gives

$$\|\nu_s^y - \nu_s^{y'}\|_{TV}^3 \leq \left( \int_{\mathbb{P}(V)} |\delta(x, y')^{-s} - \delta(x, y)^{-s}|^3 \nu_s(dx) \right) \left( \int_{\mathbb{P}(V)} \delta(x, y)^{3s} \nu_s(dx) \right) \left( \int_{\mathbb{P}(V)} \delta(x, y')^{3s} \nu_s(dx) \right).$$

As  $y' \rightarrow y$ , the first term converges to 0 by the dominated convergence theorem, whereas the other two terms remain bounded by Proposition 3.6.  $\square$

### 3.3. The explicit form of the eigenfunction for negative $s$

We apply the results of the previous two sections to give explicit forms of the eigenfunctions of the operators  $P_s$  and  $P_s^*$  for  $s < 0$ . Let us recall the corresponding results for  $s > 0$  which have been established in [21]: under

conditions **A1** and **A2**, for any  $s \in I_\mu^+$ , the functions

$$r_s(x) = \int_{\mathbb{P}(V^*)} \delta(x, y)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}(V)} \delta(x, y)^s \nu_s(dx) \quad (32)$$

are the unique (up to a scaling constant) non-negative eigenfunctions of the operators  $P_s$  and  $P_s^*$ , respectively. The proof of these expressions for  $s < 0$  is quite different from that in the case  $s > 0$ ; it requires the Hölder regularity of the eigenmeasures  $\nu_s$  and  $\nu_s^*$ , which has been established in Proposition 3.6.

First we state the cohomological equation (see [5]) which will also be useful later on: for any  $g \in \mathbb{G}$ ,  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$ ,

$$\log \delta(y, gx) + \sigma(g, x) = \log \delta(x, g^*y) + \sigma(g^*, y). \quad (33)$$

For the ease of the reader we include a short proof of (33). By elementary transformations,

$$\log \frac{|\langle f, gv \rangle|}{|f||v|} = \log \frac{|\langle f, gv \rangle|}{|f||gv|} + \log \frac{|gv|}{|v|} = \log \delta(y, gx) + \sigma(g, x)$$

and, in the same way,

$$\log \frac{|\langle v, g^*f \rangle|}{|f||v|} = \log \frac{|\langle v, g^*f \rangle|}{|g^*f||v|} + \log \frac{|g^*f|}{|f|} = \log \delta(x, g^*y) + \sigma(g^*, y).$$

By the definition of the automorphism  $g^*$ , we have  $\langle f, gv \rangle = \langle v, g^*f \rangle$ , hence the identity (33) follows.

**Lemma 3.10.** *Assume conditions **A1**, **A2** and **A3**. Then, there exists a constant  $s_0 > 0$  such that for any  $s \in (-s_0, 0)$ , the eigenfunctions  $r_s$  and  $r_s^*$  are defined (up to a scaling constant) as follows: for  $x \in \mathbb{P}(V)$  and  $y \in \mathbb{P}(V^*)$ ,*

$$r_s(x) = \int_{\mathbb{P}(V^*)} \delta(x, y)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}(V)} \delta(x, y)^s \nu_s(dx). \quad (34)$$

Moreover,  $\kappa^*(s) = \kappa(s)$  for any  $s \in (-s_0, 0)$ .

**Proof.** Let  $x \in \mathbb{P}(V)$ . By Proposition 3.6, there exists  $s_0 > 0$  such that for any  $s \in (-s_0, 0)$ ,

$$\phi_s(x) = \int_{\mathbb{P}(V^*)} \delta(x, y)^s \nu_s^*(dy)$$

is well-defined and positive. By Corollary 3.9 (applied to the dual situation), the function  $\phi_s$  is continuous on  $\mathbb{P}(V)$  when  $s_0$  is small enough. We claim that  $P_s \phi_s = \kappa^*(s) \phi_s$  with  $\kappa^*$  from Lemma 3.5. Indeed, since  $\phi_s$  is uniformly bounded on  $\mathbb{P}(V)$ , using the cohomological identity (33) and Fubini's theorem we get

$$\begin{aligned} P_s \phi_s(x) &= \int_{\mathbb{G}} e^{s\sigma(g, x)} \left( \int_{\mathbb{P}(V^*)} \delta(gx, y)^s \nu_s^*(dy) \right) \mu(dg) \\ &= \int_{\mathbb{P}(V^*)} \int_{\mathbb{G}} e^{s\sigma(g^*, y) + s \log \delta(x, g^*y)} \mu(dg) \nu_s^*(dy). \end{aligned} \quad (35)$$

Note that the function  $y \mapsto \delta(x, y)^s$  belongs to the space  $L^1(\nu_s^*)$ . As the operator  $P_s^*$  is positive and  $\nu_s^*$  is a probability eigenmeasure of  $P_s^*$ , then  $P_s^*$  can be extended to be a bounded operator on  $L^1(\nu_s^*)$ , which we still denote by  $P_s^*$ . Therefore, by (35),

$$\begin{aligned} P_s \phi_s(x) &= \int_{\mathbb{P}(V^*)} P_s^*(\delta(x, \cdot)^s)(y) \nu_s^*(dy) \\ &= \kappa^*(s) \int_{\mathbb{P}(V^*)} \delta(x, y)^s \nu_s^*(dy) = \kappa^*(s) \phi_s(x). \end{aligned}$$

By Lemma 3.4 we get that  $\kappa^*(s) = \kappa(s)$  and  $r_s = c\phi_s$  for some constant  $c > 0$ .

The proof for  $r_s^*$  is similar and therefore will not be detailed here.  $\square$

To summarize, the same relations between  $P_s, P_s^*, \nu_s, \nu_s^*, r_s, r_s^*$  and  $\kappa(s)$  hold for both positive and small negative  $s$ : under appropriate conditions there exists  $s_0 > 0$  such that for any  $s \in (-s_0, 0) \cup I_\mu^+$ ,

$$P_s \nu_s = \kappa(s) \nu_s, \quad P_s^* \nu_s^* = \kappa(s) \nu_s^* \quad (36)$$

and

$$P_s r_s = \kappa(s) r_s, \quad P_s^* r_s^* = \kappa(s) r_s^*. \quad (37)$$

#### 3.4. Harmonicity of the invariant measure

Assume that  $s \in (-s_0, 0) \cup I_\mu^+$ , where  $s_0 > 0$  is small enough. For any  $y \in \mathbb{P}(V^*)$  and bounded measurable function  $\varphi$  on  $\mathbb{P}(V)$ , denote

$$v_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(x) \frac{\delta(x, y)^s}{r_s^*(y)} \nu_s(dx). \quad (38)$$

Due to the regularity of the eigenmeasure  $\nu_s$  we have  $0 < \delta(x, y) \leq 1$ ,  $\nu_s$ -a.s. on  $\mathbb{P}(V)$  (for large  $s > 0$  this follows from Theorem 2.5; for small  $s < 0$  it can be deduced from Proposition 3.6). Since the eigenfunction  $r_s^*$  is bounded and strictly positive on  $\mathbb{P}(V)$ , the measures  $\nu_s$  and  $v_s^y$  are equivalent.

**Lemma 3.11.** *The following two assertions hold:*

1. *Assume conditions **A1** and **A2**. Then, for any  $s \in I_\mu^+$ , the mapping  $y \in \mathbb{P}(V^*) \mapsto v_s^y \in \mathcal{C}(\mathbb{P}(V))'$  is continuous in the total variation norm  $\|\cdot\|_{\text{TV}}$ .*
2. *Assume conditions **A1**, **A2** and **A3**. Then, there exists a constant  $s_0 > 0$  such that for any  $s \in (-s_0, 0)$ , the mapping  $y \in \mathbb{P}(V^*) \mapsto v_s^y \in \mathcal{C}(\mathbb{P}(V))'$  is continuous in the total variation norm  $\|\cdot\|_{\text{TV}}$ .*

**Proof.** For positive  $s > 0$  the assertion of the lemma is easily proved due to the continuity of the mapping  $(x, y) \mapsto \delta(x, y)^s$  (see Lemma 3.5 of [21]). For  $s < 0$  the assertion of the lemma follows from Corollary 3.9.  $\square$

For any  $g \in \mathbb{G}$  and  $y \in \mathbb{P}(V^*)$ , set

$$q_s^*(g, y) = \frac{e^{s\sigma(g^*, y)} r_s^*(g^* y)}{\kappa(s) r_s^*(y)}. \quad (39)$$

By (37), for any  $y \in \mathbb{P}(V^*)$ , the function  $q_s^*(g, y)$  is a positive density on  $\mathbb{G}$ , i.e.

$$\int_{\mathbb{G}} q_s^*(g, y) \mu(dg) = 1, \quad (40)$$

and for any  $g \in \mathbb{G}$ ,

$$q_s^*(g, y) > 0. \quad (41)$$

For any  $g \in \mathbb{G}$ ,  $y \in \mathbb{P}(V^*)$  and bounded measurable function  $\varphi$  on  $\mathbb{P}(V)$ , define

$$g v_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(gx) v_s^y(dx), \quad (42)$$

where  $v_s^y$  is a probability measure given by (38). The following lemma can be viewed as a generalization of the stationary property  $\pi_s = \pi_s Q_s$ . For  $s > 0$  it has been obtained in [21, Lemma 3.6].

**Lemma 3.12.** *Assume either conditions **A1**, **A2** and  $s \in I_\mu^+$ , or conditions **A1**, **A2**, **A3** and  $s \in (-s_0, 0)$  with small enough  $s_0 > 0$ . Then, for any  $y \in \mathbb{P}(V^*)$  and bounded measurable function  $\varphi$  on  $\mathbb{P}(V)$ , we have*

$$v_s^y(\varphi) = \int_{\mathbb{G}} g v_s^{g^* y}(\varphi) q_s^*(g, y) \mu(dg).$$

**Proof.** For short we denote  $\psi_s^y(x) = \varphi(x)e^{s \log \delta(x,y)}/r_s^*(y)$ . By the definition (38) of the measure  $\nu_s^y$ , we have

$$\nu_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(x) \frac{e^{s \log \delta(x,y)}}{r_s^*(y)} \nu_s(dx) = \int_{\mathbb{P}(V)} \psi_s^y(x) \nu_s(dx).$$

From the identity  $P_s \nu_s(\psi_s^y) = \kappa(s) \nu_s(\psi_s^y)$ , it follows that

$$\begin{aligned} \nu_s^y(\varphi) &= \frac{1}{\kappa(s)} \int_{\mathbb{P}(V)} P_s \psi_s^y(x) \nu_s(dx) \\ &= \frac{1}{\kappa(s) r_s^*(y)} \int_{\mathbb{P}(V)} \left( \int_{\mathbb{G}} \varphi(gx) e^{s \log \delta(gx,y) + s \sigma(g,x)} \mu(dg) \right) \nu_s(dx). \end{aligned}$$

Using the cohomological identity (33), Fubini's theorem and (42), we get

$$\begin{aligned} \nu_s^y(\varphi) &= \frac{1}{\kappa(s) r_s^*(y)} \int_{\mathbb{G}} \left( \int_{\mathbb{P}(V)} \varphi(gx) e^{s \log \delta(x, g^*y) + s \sigma(g^*,y)} \nu_s(dx) \right) \mu(dg) \\ &= \int_{\mathbb{G}} \frac{e^{s \sigma(g^*,y)}}{\kappa(s) r_s^*(y)} \left( \int_{\mathbb{P}(V)} \varphi(gx) e^{s \log \delta(x, g^*y)} \nu_s(dx) \right) \mu(dg) \\ &= \int_{\mathbb{G}} \frac{e^{s \sigma(g^*,y)}}{\kappa(s)} \frac{r_s^*(g^*y)}{r_s^*(y)} g \nu_s^{g^*y}(\varphi) \mu(dg) \\ &= \int_{\mathbb{G}} q_s^*(g,y) g \nu_s^{g^*y}(\varphi) \mu(dg), \end{aligned}$$

as desired. □

## 4. Auxiliary statements

### 4.1. Stationary measures on finite extensions

Let  $G$  be a locally compact subgroup of  $\mathbb{G}$ . Assume that  $H < G$  is a closed subgroup of finite index, which means that the quotient  $G/H$  is a finite set. Let  $\mu$  be a probability measure on  $G$ . Denote by  $\Omega$  the set of infinite sequences  $(g_1, g_2, \dots)$  and equip it with the measure  $\mu^{\otimes \mathbb{N}^*}$ . For any  $\omega \in \Omega$ , set

$$\tau(\omega) = \min\{k \geq 1 : g_k \dots g_1 \in H\}.$$

The stopping time  $\tau$  is  $\mu^{\otimes \mathbb{N}^*}$ -a.s. finite, see Lemma 5.5 of [5]. Define the mapping  $\mathfrak{f} : \omega \in \Omega \mapsto g_{\tau(\omega)} \dots g_1 \in H$ . Let  $\mu_H$  be the image of the measure  $\mu^{\otimes \mathbb{N}^*}$  by the mapping  $\mathfrak{f}$ . We call  $\mu_H$  the probability measure induced by  $\mu$  on the subgroup  $H$ . From Lemma 5.7 of [5] we have the following assertion.

**Lemma 4.1.** *Assume that the probability measure  $\mathfrak{m}$  is  $\mu$ -stationary on  $\mathbb{P}(V)$ . Then  $\mathfrak{m}$  is also a  $\mu_H$ -stationary probability measure.*

### 4.2. Determination of the support of the stationary measure

Since we will use facts from complex algebraic geometry, we now introduce some basic notions on complex algebraic sets. We write  $V_{\mathbb{C}}$  for the complexification of  $V$ . For example, one can define  $V_{\mathbb{C}}$  as the space  $V^2$  equipped with the complex vector space structure defined by  $(a+ib)(v, w) = (av - bw, aw + bv)$  for  $a, b \in \mathbb{R}$  and  $v, w \in V$ . One then identifies  $V$  with the set  $V \times \{0\} \subset V^2$ . We recall that a subset  $X$  in  $V_{\mathbb{C}}$  is algebraic if there exist complex polynomial functions  $p_1, \dots, p_k$  on  $V_{\mathbb{C}}$  such that  $X = \{v \in V_{\mathbb{C}} : p_1(v) = \dots = p_k(v) = 0\}$ . We say that  $X$  is homogeneous if for every  $t \in \mathbb{C}$  and  $v \in X$  it holds that  $tv \in X$ . The topological space  $X$  is called Noetherian if every non-increasing sequence of closed subsets is eventually constant.

We also need the projective space of  $V_{\mathbb{C}}$ , which is defined in a similar way:  $\mathbb{P}(V_{\mathbb{C}})$  is the set of elements  $x = \mathbb{C}v$ , where  $v \in V_{\mathbb{C}} \setminus \{0\}$ . A subset  $Y$  of  $\mathbb{P}(V_{\mathbb{C}})$  is algebraic if there exists an algebraic homogeneous subset  $X$  whose projective image is  $Y$ . With this notion, a set  $X$  in  $V$  is (real) algebraic if and only if there exists a

complex algebraic set  $X'$  such that  $X = X' \cap V$ . In the same way, a set  $Y$  in  $\mathbb{P}(V)$  is (real) algebraic if and only if there exists a complex algebraic set  $Y'$  such that  $Y = Y' \cap \mathbb{P}(V)$ .

From the Noetherian property of the ring of polynomial functions it follows that the sets defined as zeros of infinitely many polynomials are also algebraic. This implies that the algebraic sets are precisely the closed sets of the Zariski topology. The set  $\mathrm{GL}(V_{\mathbb{C}})$  is a Zariski open subset of the complex vector space  $\mathrm{End}(V_{\mathbb{C}})$  of complex endomorphisms of  $V_{\mathbb{C}}$ , so we can equip it with the Zariski topology of  $\mathrm{End}(V_{\mathbb{C}})$ . The closure of a set  $Y$  in the Zariski topology is denoted by  $\mathrm{Zc}(Y)$ .

The limit set of the semigroup  $\Gamma_{\mu}$  is a subset of  $\mathbb{P}(V_{\mathbb{C}})$  defined as follows:

$$\Lambda(\Gamma_{\mu}) = \overline{\{x_g^+ : g \in \Gamma_{\mu}, g \text{ is proximal}\}}.$$

It is well known that, under conditions **A1** and **A2**,  $\Lambda(\Gamma_{\mu})$  is the smallest nonempty closed  $\Gamma_{\mu}$ -invariant set in the projective space  $\mathbb{P}_{\mathbb{C}}^{d-1}$ .

**Lemma 4.2.** *Assume either conditions **A1**, **A2** and  $s \in I_{\mu}^+$ , or conditions **A1**, **A2**, **A3** and  $s \in (-s_0, 0)$  with small enough  $s_0 > 0$ . Then the support of  $\nu_s$  is  $\Lambda(\Gamma_{\mu})$ .*

**Proof.** If we apply Lemma 3.1 with  $T = P_s$  and  $X = \Lambda(\Gamma_{\mu})$ , then we obtain that  $P_s$  admits an eigenmeasure which is concentrated on  $\Lambda(\Gamma_{\mu})$ . By the uniqueness of the eigenmeasure  $\nu_s$  (see (14) and Lemma 3.5), we get  $\nu_s(\Lambda(\Gamma_{\mu})) = 1$ . This proves that the support of  $\nu_s$  is contained in  $\Lambda(\Gamma_{\mu})$ .

Conversely, let us prove that for any nonempty open subset  $U \subset \Lambda(\Gamma_{\mu})$  it holds that  $\nu_s(U) > 0$ . To show this we fix some point  $x \in \mathrm{supp}(\nu_s)$ . Since the orbit  $\Gamma_{\mu}x$  is dense in  $\Lambda(\Gamma_{\mu})$ , we can find an integer  $n \in \mathbb{N}$  and  $g \in \mathrm{supp}(\mu)^n$  such that  $gx \in U$ . As  $\mathrm{supp}(\mu)^n$  is the support of the measure  $\mu^{*n}$ , we get

$$\mathbb{P}(G_n x \in U) > 0.$$

This gives

$$P_s^n \mathbb{1}_U(x) = \mathbb{E} \left( e^{s\sigma(G_n, x)} \mathbb{1}_{\{G_n x \in U\}} \right) > 0.$$

Since the function  $P_s^n(\mathbb{1}_U)$  is upper semi-continuous, the set

$$V = \{x' \in \mathbb{P}(V) : P_s^n \mathbb{1}_U(x') > 0\}$$

is open. By assumption  $V \cap \mathrm{supp}(\nu_s)$  is nonempty, hence  $\nu_s(V) > 0$  and  $\nu_s(P_s^n \mathbb{1}_U) > 0$ . It follows that  $\nu_s(U) = \kappa(s)^{-n} \nu_s(P_s^n \mathbb{1}_U) > 0$ , as required.  $\square$

**Remark 4.3.** Let  $\Gamma_{\mu}^*$  be the adjoint semigroup of  $\Gamma_{\mu}$ , which is a subsemigroup of  $\mathrm{GL}(V^*)$ . Then assumptions **A1** and **A2** hold for  $\Gamma_{\mu}^*$  if and only if they hold for  $\Gamma_{\mu}$ . In particular, there exists a smallest nonempty closed  $\Gamma_{\mu}^*$ -invariant subset in the dual projective space  $\mathbb{P}(V^*)$ , which will be called limit set and denoted by  $\Lambda(\Gamma_{\mu}^*)$ .

Let  $H_{\mu} = \mathrm{Zc}(\Gamma_{\mu})$  be the Zariski closure of  $\Gamma_{\mu}$  in  $\mathrm{GL}(V_{\mathbb{C}})$ . Define  $X_{\mu} = \mathrm{Zc}(\Lambda(\Gamma_{\mu})) = \mathrm{Zc}(\mathrm{supp} \nu)$ . Note that  $X_{\mu}$  is an algebraic subset of  $\mathbb{P}(V_{\mathbb{C}})$ .

**Lemma 4.4.** *Assume conditions **A1** and **A2**. The algebraic set  $X_{\mu}$  is the unique closed  $H_{\mu}$ -orbit in  $\mathbb{P}(V_{\mathbb{C}})$ . In particular, every nonempty Zariski closed  $H_{\mu}$ -invariant subset contains  $X_{\mu}$ .*

**Proof.** It is a general fact in algebraic geometry (we refer to Borel [7, Corollary 1.8, p.53]) that there exists  $x_0 \in \mathbb{P}(V)$  such that  $Y_0 = H_{\mu}x_0$  is a Zariski closed orbit. We shall prove that  $X_{\mu} = Y_0$ .

The orbit  $Y_0$  is also analytically closed and, moreover, it is  $\Gamma_{\mu}$ -invariant (if  $g \in \Gamma_{\mu}$  then  $gY_0 = gH_{\mu}x_0 = \{gg'x_0 : g' \in H_{\mu}\} \subset Y_0$ , as  $gg' \in H_{\mu}$ ). Therefore, the minimal closed  $\Gamma_{\mu}$ -invariant set  $\Lambda(\Gamma_{\mu})$  is contained in  $Y_0$ :  $\Lambda(\Gamma_{\mu}) \subset Y_0$ . This implies that  $X_{\mu} = \mathrm{Zc}(\Lambda(\Gamma_{\mu})) \subset \mathrm{Zc}(Y_0) = Y_0$ , since  $Y_0$  is Zariski closed. So we have showed that  $X_{\mu} \subset Y_0$ .

Now we prove the converse inclusion  $Y_0 \subset X_{\mu}$ . Since  $X_{\mu}$  is a  $\Gamma_{\mu}$ -invariant Zariski closed subset, by the continuity of the map  $g \in \mathrm{GL}(V_{\mathbb{C}}) \mapsto gx \in \mathbb{P}(V_{\mathbb{C}})$  in the Zariski topology for any  $x \in \mathbb{P}(V_{\mathbb{C}})$ , the set  $X_{\mu}$  is also  $H_{\mu}$ -invariant. This means that  $H_{\mu}X_{\mu} \subset X_{\mu}$ . We know that for any  $x \in Y_0$  it holds that  $H_{\mu}x = Y_0$ . Since  $X_{\mu} \subset Y_0$ , we have that  $H_{\mu}x = Y_0$  for any  $x \in X_{\mu}$ . This implies that  $Y_0 = H_{\mu}x \subset X_{\mu}$ , which concludes the proof.  $\square$



To avoid any confusion with the notion of strong irreducibility of the set  $\Gamma_\mu$  introduced before, let us recall the notion of irreducibility of an algebraic subset. We say that an algebraic subset  $Y$  is irreducible if  $Y$  cannot be represented as the union of two proper closed algebraic subsets of  $Y$ .

**Lemma 4.5.** *Assume conditions [A1](#) and [A2](#). Then, the algebraic set  $X_\mu$  is irreducible.*

**Proof.** Let  $H_\mu^0$  be the Zariski connected component of  $H_\mu$  which contains the unit element of  $\mathbb{G}$ . Note that  $H_\mu^0$  is a finite index Zariski closed subgroup of  $H_\mu$ .

Let  $\mu_0$  be the measure on  $H_\mu^0$  induced by the measure  $\mu$  through the mapping  $\mathfrak{f}$  as explained in Section 4.1. By Lemma 4.1, the probability measure  $\nu$  defined by (10) is  $\mu_0$ -stationary. By the construction of  $\mu_0$  we have  $\Gamma_{\mu_0} = \Gamma_\mu \cap H_\mu^0$ . Since any finite index subgroup of a strongly irreducible group is still strongly irreducible, it follows that  $\Gamma_{\mu_0}$  is strongly irreducible (we note that the irreducibility is not necessarily preserved). It is also proximal since any positive power of an element of  $\Gamma_\mu$  is also proximal. By the result of Furstenberg [13] the measure  $\mu_0$  has a unique stationary probability measure  $\nu_0$  on  $\mathbb{P}(V)$  which (by uniqueness) coincides with  $\nu$ . This implies that  $X_\mu = X_{\mu_0} = \text{Zc}(\text{supp } \nu_0) = \text{Zc}(\Lambda(\Gamma_{\mu_0}))$ .

Applying Lemma 4.4 with  $\mu_0$  instead of  $\mu$ , we conclude that  $X_\mu = X_\mu^0$  is an  $H_\mu^0$ -orbit in  $\mathbb{P}(V)$ . Since  $H_\mu^0$  is connected (and therefore irreducible as an algebraic set), we conclude that  $X_\mu$  is irreducible as the image of  $H_\mu^0$  by the Zariski continuous mapping  $g \in H_\mu^0 \mapsto gx_0 \in X_\mu$ .  $\square$

### 4.3. The maximum principle

We shall use repeatedly the following simple fact which we call maximum principle.

**Lemma 4.6.** *Let  $\phi : (X, P) \mapsto \mathbb{R}$  be a measurable function on the measurable space  $X$  equipped with the probability measure  $P$  which satisfies  $\phi(x) \leq \beta$ ,  $P$ -a.s. and  $\int \phi(x)P(dx) = \beta$ , where  $\beta \in \mathbb{R}$  is a real number. Then  $\phi = \beta$ ,  $P$ -a.s.*

## 5. Proof of Theorem 2.2

We shall prove the following statement which implies Theorem 2.2. All over this section we assume conditions [A1](#) and [A2](#). First we recall that by the definition of  $X_\mu$  (see Section 4.2) we have  $X_\mu = \text{Zc}(\text{supp } \nu)$ , so that  $\nu(X_\mu) = 1$ .

**Proposition 5.1.** *For any proper algebraic subset  $Y$  of  $X_\mu$ , it holds  $\nu(Y) = 0$ .*

We need to recall some elementary notions from algebraic geometry (see, for instance Borel [7]). We say that a topological space  $X$  is irreducible if and only if  $X$  cannot be written as  $X = X_1 \cup X_2$ , where  $X_1$  and  $X_2$  are proper closed subsets of  $X$ . Recall that a topological space  $X$  is Noetherian if every non-increasing sequence of closed subsets is eventually constant. Any Noetherian topological space can be written as a finite union of closed irreducible subsets.

The (combinatorial) dimension of a Noetherian topological space  $X$  is the maximum length  $l$  of sequences  $X_0 \subsetneq \dots \subsetneq \dots \subsetneq X_l$  of distinct irreducible closed sets in  $X$ :

$$\dim(X) = \sup\{l : X_0 \subsetneq \dots \subsetneq \dots \subsetneq X_l \subsetneq X\} \in \mathbb{N} \cup \{\infty\}.$$

From this definition it is obvious that if  $X$  is irreducible and has finite dimension then any proper closed subset of  $X$  has strictly smaller dimension.

It is known that the projective space  $\mathbb{P}(V_{\mathbb{C}})$  is irreducible and Noetherian for the Zariski topology with  $\dim(\mathbb{P}(V_{\mathbb{C}})) = d - 1$ . In the following by the dimension of an algebraic subset of  $\mathbb{P}(V)$  we mean the combinatorial dimension.

Denote by  $\mathfrak{A}(X_\mu)$  the set of irreducible algebraic subsets  $Y$  of  $X_\mu$ . Set

$$d_0 = \min\{1 \leq r \leq d : Y \in \mathfrak{A}(X_\mu), \dim(Y) = r, \nu(Y) > 0\}.$$

It is easy to see that  $d_0 \leq \dim(X_\mu) \leq d - 1$ .

**Lemma 5.2.** *Let  $Y$  be an algebraic subset of  $\mathbb{P}(V_{\mathbb{C}})$  with  $\dim(Y) < d_0$ . Then  $\nu(Y) = 0$ .*

**Proof.** Write  $Y$  as a union of irreducible algebraic subsets  $Y_1, \dots, Y_r$  of  $\mathbb{P}(V_{\mathbb{C}})$ :  $Y = Y_1 \cup \dots \cup Y_r$  (see Proposition 1.5 of Hartshorne [23]). As each  $Y_k$  has dimension strictly less than  $d_0$ , we have  $\nu(Y_k) = 0$ , by the definition of  $d_0$ . So  $\nu(Y) = 0$ .  $\square$

We will establish the following assertion, which is the key point in the proof of Theorem 2.2.

**Proposition 5.3.** *The dimension of  $X_\mu$  is  $d_0$ :  $\dim(X_\mu) = d_0$ .*

The proof of this proposition requires some additional assertions. For any  $c > 0$ , set

$$\mathcal{W}_0(c) = \{Y \in \mathfrak{A}(X_\mu) : \dim(Y) = d_0, \nu(Y) \geq c\}.$$

We start by showing that this set is finite.

**Lemma 5.4.** *For any  $c > 0$ , the set  $\mathcal{W}_0(c)$  is finite. Moreover  $\text{card } \mathcal{W}_0(c) \leq c^{-1}$ .*

**Proof.** Let  $Y_1, \dots, Y_r$  be two by two distinct elements of  $\mathcal{W}_0(c)$ . For any  $1 \leq i < j \leq r$ , the intersection  $Y_i \cap Y_j$  is an algebraic subset of dimension strictly smaller than  $d_0$ . By Lemma 5.2 we have  $\nu(Y_i \cap Y_j) = 0$ . This implies that  $\nu(Y_1 \cup \dots \cup Y_r) \geq cr$ , so  $r \leq c^{-1}$ .  $\square$

Set

$$\beta = \sup\{\nu(Y) : Y \in \mathfrak{A}(X_\mu), \dim(Y) = d_0\}. \quad (43)$$

In the following it is important to show that the supremum in (43) is attained.

**Lemma 5.5.** *The following maximum is attained,*

$$\beta = \max\{\nu(Y) : Y \in \mathfrak{A}(X_\mu), \dim(Y) = d_0\}.$$

**Proof.** As  $\beta > 0$ , we have

$$\beta = \sup\{\nu(Y) : Y \in \mathcal{W}_0(\beta/2)\}.$$

By Lemma 5.4, the set  $\mathcal{W}_0(\beta/2)$  is finite. Therefore, the supremum is attained.  $\square$

All the algebraic sets  $Y \in \mathfrak{A}(X_\mu)$  for which the maximum  $\beta$  is realized are collected in the set

$$\mathcal{W} = \{Y \in \mathfrak{A}(X_\mu) : \dim(Y) = d_0, \nu(Y) = \beta\},$$

which is finite by Lemma 5.4 (as a subset of  $\mathcal{W}_0(\beta)$ ).

**Lemma 5.6.** *The set  $\mathcal{W}$  is  $\Gamma_\mu$ -invariant, which means that for any  $g \in \Gamma_\mu$  and any  $Y \in \mathcal{W}$  we have that  $gY \in \mathcal{W}$ .*

**Proof.** Let  $Y$  be an element of  $\mathcal{W}$ . Since the measure  $\nu$  is  $\mu$ -stationary, we have

$$\nu(Y) = \int_{\Gamma_\mu} \nu(g^{-1}Y) \mu(dg).$$

For any  $g \in \Gamma_\mu$ , we have that  $g^{-1}Y \in \mathfrak{A}(X_\mu)$  and  $g^{-1}Y$  has dimension  $d_0$ , hence  $\nu(g^{-1}Y) \leq \beta$ . Since

$$\beta = \nu(Y) = \int_{\Gamma_\mu} \nu(g^{-1}Y) \mu(dg),$$

by the maximum principle (see Lemma 4.6), we get that  $\nu(g^{-1}Y) = \beta$  for  $\mu$ -almost all  $g \in \Gamma_\mu$ . Since  $\mathcal{W}$  is finite (by Lemma 5.4), we get that  $g^{-1}\mathcal{W} = \mathcal{W}$  for  $\mu$ -almost all  $g \in \Gamma_\mu$ , and that the set  $\{g \in \Gamma_\mu : g^{-1}\mathcal{W} = \mathcal{W}\}$  is closed. We have shown that it has  $\mu$  measure 1, therefore it contains the  $\text{supp } \mu$  (by the definition of the latter). This means that for any  $g \in \text{supp } \mu$  we have  $g^{-1}\mathcal{W} \subset \mathcal{W}$ . Since the cardinality of  $\mathcal{W}$  is finite, we get that  $g^{-1}\mathcal{W} = \mathcal{W}$  for any  $g \in \text{supp } \mu$ . By the definition of the support of the measure  $\mu$ , the last statement implies that  $g^{-1}\mathcal{W} = \mathcal{W}$  for all  $g \in \Gamma_\mu$ . Since  $g^{-1}\mathcal{W} = \mathcal{W}$  is equivalent to  $g\mathcal{W} = \mathcal{W}$  for  $g \in \Gamma_\mu$ , the assertion follows.  $\square$

**Proof of Proposition 5.3.** By Lemma 5.6, the set  $\mathcal{W}$  is  $\Gamma_\mu$ -invariant ( $\Gamma_\mu \mathcal{W} \subset \mathcal{W}$ ), and by Lemma 5.4, the set  $\mathcal{W}$  is finite. Therefore, the set  $Z = \bigcup_{Y \in \mathcal{W}} Y$  is a Zariski closed  $\Gamma_\mu$ -invariant algebraic subset in  $X_\mu$ . As  $\Gamma_\mu$  is Zariski dense in  $H_\mu = \text{Zc}(\Gamma_\mu)$ , the algebraic set  $Z$  is  $H_\mu$ -invariant. By Lemma 4.4, we have that  $X_\mu \subset Z$ . On the other hand  $Z \subset X_\mu$ , so  $Z = X_\mu$ , which reads as  $X_\mu = \bigcup_{Y \in \mathcal{W}} Y$ . According to Lemma 4.5,  $X_\mu$  is irreducible, hence we get  $X_\mu = Y$  for some  $Y$  in  $\mathcal{W}$ . In particular,  $\dim(Y) = d_0 = \dim(X_\mu)$ , which concludes the proof of Proposition 5.3.  $\square$

Now we show that Proposition 5.3 implies Proposition 5.1.

**Proof of Proposition 5.1.** Let  $Y$  be an algebraic subset of  $X_\mu$  with  $\nu(Y) > 0$  and let us show that  $Y = X_\mu$ . By Proposition 1.5 of Hartshorne [23], we can decompose the set  $Y$  into a union of Zariski closed and Zariski irreducible subsets  $Z_1, \dots, Z_r$  of  $X_\mu$ :  $Y = Z_1 \cup \dots \cup Z_r$ . Then, there exists  $k$  such that  $\nu(Z_k) > 0$ , and hence  $\dim(Z_k) \geq d_0 = \dim(X_\mu)$ , by Proposition 5.3. It follows that  $Z_k = X_\mu$ , hence  $Y = X_\mu$  as claimed by our assertion.  $\square$

Now we prove that Proposition 5.1 implies Theorem 2.2.

**Proof of Theorem 2.2.** Let  $Y$  be an algebraic subset of the projective space  $\mathbb{P}(V_{\mathbb{C}})$ . If  $X_\mu \subset Y$ , we have  $\nu(Y) \geq \nu(X_\mu) = 1$ . Otherwise,  $Y \cap X_\mu$  is a proper algebraic subset of  $X_\mu$ . Since  $\nu(X_\mu) = 1$ , we have  $\nu(Y) = \nu(Y \cap X_\mu) = 0$ , by Proposition 5.1. The assertion of the theorem now follows from the fact that an algebraic subset of  $\mathbb{P}(V)$  is the intersection of an algebraic subset of the projective space  $\mathbb{P}(V_{\mathbb{C}})$  with  $\mathbb{P}(V)$ .  $\square$

## 6. Proof of Theorems 2.6 and 2.8

We shall first prove the following:

**Proposition 6.1.** *Assume conditions A1 and A2. Then, for any  $s \in I_\mu^+$  and for any proper algebraic subset  $Y$  of  $X_\mu$ , it holds  $\nu_s(Y) = 0$ .*

**Proposition 6.2.** *Assume conditions A1, A2 and A3. Then, there exists a constant  $s_0 > 0$  such that for any  $s \in [-s_0, 0)$  and for any proper algebraic subset  $Y$  of  $X_\mu$ , it holds  $\nu_s(Y) = 0$ .*

We will see at the end of this section that Propositions 6.1 and 6.2 imply Theorems 2.6 and 2.8, respectively.

To establish Propositions 6.1 and 6.2 we stick to the proof of Guivarc'h and Le Page [21]. The proofs that we give below will work in both cases  $s \in I_\mu^+$  and  $s \in [-s_0, 0)$ , where  $s_0 > 0$  is small enough.

We need a series of auxiliary statements. We use the notation  $\mathfrak{A}(X_\mu)$  from Section 5 and  $v_s^y$  from Section 3.4. For any  $y \in \mathbb{P}(V^*)$  define

$$d_0 = \min\{1 \leq r \leq d : Y \in \mathfrak{A}(X_\mu), \dim(Y) = r, v_s^y(Y) > 0\}.$$

Since the measures  $v_s^y$  and  $\nu_s$  are equivalent, there is no dependence on  $y$  in the above definition of  $d_0$ , so that

$$d_0 = \min\{1 \leq r \leq d : Y \in \mathfrak{A}(X_\mu), \dim(Y) = r, \nu_s(Y) > 0\}.$$

It is easy to see that  $d_0 \leq \dim(X_\mu) \leq d - 1$ .

**Lemma 6.3.** *Let  $Y$  be an algebraic subset of  $\mathbb{P}(V_{\mathbb{C}})$  with  $\dim(Y) < d_0$ . Then  $\nu_s(Y) = 0$ .*

**Proof.** The proof being similar to that of Lemma 5.2 is left to the reader.  $\square$

In the sequel we are going to prove the following assertion.

**Proposition 6.4.** *It holds that  $\dim(X_\mu) = d_0$ .*

We shall show below that Proposition 6.4 implies our Propositions 6.1 and 6.2.

The proof of Proposition 6.4 is based on several lemmas. For any  $c > 0$ , set

$$\mathcal{W}_0^y(c) = \{Y \in \mathfrak{A}(X_\mu) : \dim(Y) = d_0, v_s^y(Y) \geq c\}.$$

**Lemma 6.5.** *Let  $c > 0$  be a constant. Then, for any  $y \in (\mathbb{P}^{d-1})^*$ , the set  $\mathcal{W}_0^y(c)$  is finite. Moreover  $\text{card } \mathcal{W}_0^y(c) \leq c^{-1}$ .*

**Proof.** Let  $Y_1, \dots, Y_r$  be two by two distinct elements of  $W_0^y(c)$ . For any  $1 \leq i < j \leq r$ , the intersection  $Y_i \cap Y_j$  is an algebraic set of dimension strictly smaller than  $d_0$ , hence by Lemma 6.3,  $v_s^y(Y_i \cap Y_j) = 0$ . This implies that  $v_s^y(Y_1 \cup \dots \cup Y_r) \geq cr$ , so  $r \leq c^{-1}$ .  $\square$

Proceeding as in the proof of Lemma 5.5, from Lemma 6.5 it follows that for  $y \in \mathbb{P}(V^*)$ , the number

$$h(y) = \max \{v_s^y(Y) : Y \in \mathfrak{A}(X_\mu), \dim(Y) = d_0\} \quad (44)$$

is well-defined and the max is attained. By Lemma 3.11, the function  $h : \mathbb{P}(V^*) \rightarrow \mathbb{R}_+$  is continuous and thus attains its maximum on  $\mathbb{P}(V^*)$ . Therefore, we can define

$$\beta = \max_{y \in \mathbb{P}(V^*)} h(y). \quad (45)$$

**Lemma 6.6.** *If  $h(y) = \beta$  for some  $y \in \mathbb{P}(V^*)$ , then  $h(g^*y) = \beta$  for all  $g \in \Gamma_\mu$ . In particular  $h(y) = \beta$  for all  $y \in \Lambda(\Gamma_\mu^*)$ .*

**Proof.** Assume that  $y \in \mathbb{P}(V^*)$  and  $h(y) = \beta$ . Using (44) and (45), we see that there exists  $Y \in \mathfrak{A}(X_\mu)$  such that  $v_s^y(Y) = \beta$ . Note that  $\mathbb{1}_Y(gx) = \mathbb{1}_{g^{-1}Y}(x)$  for  $x \in \mathbb{P}(V)$ . By Lemma 3.12, it holds that

$$\beta = v_s^y(Y) = \int_{\mathbb{G}} v_s^{g^*y}(g^{-1}Y) q_s^*(g, y) \mu(dg). \quad (46)$$

Since  $g^{-1}Y$  is an irreducible algebraic subset of dimension  $d_0$ , we have that for any  $g^{-1} \in \Gamma_\mu$ ,

$$v_s^{g^*y}(g^{-1}Y) \leq \beta. \quad (47)$$

From (46), (47) and the fact that  $q_s^*(g, y) \mu(dg)$  is a probability measure, by the maximum principle (Lemma 4.6), it follows that  $v_s^{g^*y}(g^{-1}Y) = \beta$  for  $q_s^*(\cdot, y) d\mu$  almost all  $g \in \mathbb{G}$ . Using the fact that the measures  $q_s^*(\cdot, y) d\mu$  and  $\mu$  are equivalent, we obtain that  $v_s^{g^*y}(g^{-1}Y) = \beta$  for  $\mu$ -a.s.  $g \in \mathbb{G}$ . Therefore,  $h(g^*y) = \beta$  for  $\mu$ -a.s.  $g \in \mathbb{G}$ . Since the function  $h$  is continuous on  $\mathbb{P}(V^*)$ , we conclude that  $h(g^*y) = \beta$  for all  $g \in \text{supp } \mu$ , hence, by iteration, for all  $g \in \Gamma_\mu$ .

To prove the second assertion, define

$$Z = \{y \in \mathbb{P}(V^*) : h(y) = \beta\}. \quad (48)$$

The set  $Z$  is nonempty and, since the function  $h$  is continuous, it is closed. From the first assertion of the lemma we have  $\Gamma_\mu^* Z \subset Z$ . Since  $\Lambda(\Gamma_\mu^*)$  is the smallest nonempty closed  $\Gamma_\mu^*$ -invariant set (by Remark 4.3), we get  $\Lambda(\Gamma_\mu^*) \subset Z$ . Therefore, by the definition (48) of  $Z$ , we conclude that  $h(y) = \beta$  for all  $y \in \Lambda(\Gamma_\mu^*)$ .  $\square$

For any  $y \in \mathbb{P}(V^*)$ , we collect all the algebraic sets  $Y \in \mathfrak{A}(X_\mu)$  for which  $h(y)$  defined by (44) is realized, in the set

$$\mathcal{W}(y) = \left\{ Y \in \mathfrak{A}(X_\mu) : \dim(Y) = d_0, v_s^y(Y) = h(y) \right\}. \quad (49)$$

In the same way as in the proof of Lemma 5.5, using Lemma 6.5 one can show that  $\mathcal{W}(y)$  is a finite set:  $\text{card}(\mathcal{W}(y)) < \infty$ . For any  $y \in \mathbb{P}(V^*)$ , set

$$n(y) = \text{card}(\mathcal{W}(y)). \quad (50)$$

Note that for any  $Y \in \mathcal{W}(y)$  we have  $v_s^y(Y) = h(y) = \beta$ . So, by Lemma 6.5 we get that  $n(y) \leq \beta^{-1}$  for any  $y \in \mathbb{P}(V^*)$ . Set

$$r = \max \{n(y) : y \in \mathbb{P}(V^*), h(y) = \beta\}.$$

**Lemma 6.7.** *Suppose that  $y \in \mathbb{P}(V^*)$  is such that  $h(y) = \beta$  and  $n(y) = r$ . Then  $h(g^*y) = \beta$  and  $n(g^*y) = r$  for all  $g \in \Gamma_\mu$ . In particular  $h(y) = \beta$  and  $n(y) = r$  for all  $y \in \Lambda(\Gamma_\mu^*)$ .*

**Proof.** Define for any  $y \in \mathbb{P}(V^*)$ ,

$$h_r(y) = \max \left\{ v_s^y(Y) : Y = Y_1 \cup \cdots \cup Y_r, Y_k \in \mathfrak{A}(X_\mu), \dim(Y_k) = d_0, 1 \leq k \leq r \right\}.$$

By Lemma 3.11, the function  $h_r$  is continuous on  $\mathbb{P}(V^*)$ . We claim that for any  $y \in \mathbb{P}(V^*)$ , we have  $h_r(y) \leq \beta r$  and equality holds if and only if  $h(y) = \beta$  and  $n(y) = r$ . Indeed, for  $Y_1, \dots, Y_r$  from the definition of  $h_r$ , (44) and (45) imply that  $v_s^y(Y_k) \leq \beta$ , thus  $h_r(y) \leq \beta r$ . Moreover, if  $h_r(y) = \beta r$ , then there exist  $Y_1, \dots, Y_r \in \mathfrak{A}(X_\mu)$  with  $\dim(Y_k) = d_0$  for  $1 \leq k \leq r$  such that  $v_s^y(Y) = \beta r$ , where  $Y = Y_1 \cup \cdots \cup Y_r$ . Then, necessarily the sets  $Y_1, \dots, Y_r$  are two by two distinct and  $v_s^y(Y_i) = \beta$  for any  $1 \leq i \leq r$ . Thus  $h(y) = \beta$  and  $n(y) = r$ . The converse statement is obvious.

Therefore, to prove the first assertion of the lemma we need to show that the set  $\{y \in \mathbb{P}(V^*) : h_r(y) = \beta r\}$  is  $\Gamma_\mu^*$ -invariant. For any  $y$  in this set, let  $Y_1, \dots, Y_r \in \mathfrak{A}(X_\mu)$  be such that  $\dim(Y_k) = d_0$  for  $1 \leq k \leq r$  and  $v_s^y(Y) = \beta r$ , where  $Y = Y_1 \cup \cdots \cup Y_r$ . Using Lemma 3.12, we get

$$\beta r = v_s^y(Y_1 \cup \cdots \cup Y_r) = \int_{\mathbb{G}} v_s^{g^*y}(g^{-1}(Y_1 \cup \cdots \cup Y_r)) q_s^*(g, y) \mu(dg).$$

Since  $v_s^y(g^{-1}(Y_1 \cup \cdots \cup Y_r)) \leq \beta r$  for any  $g \in \mathbb{G}$  and  $y \in \mathbb{P}(V^*)$ , by the maximum principle (Lemma 4.6), it follows that  $q_s^*(\cdot, y) d\mu$ -a.s.

$$v_s^{g^*y}(g^{-1}(Y_1 \cup \cdots \cup Y_r)) = \beta r.$$

This means that  $h_r(g^*y) = \beta r$  for  $q_s^*(\cdot, y) d\mu$  almost all  $g \in \mathbb{G}$ . Since the measures  $q_s^*(\cdot, y) d\mu$  and  $\mu$  are equivalent, we deduce that  $h_r(g^*y) = \beta r$  holds for  $\mu$  almost all  $g \in \mathbb{G}$ . Thus we have proved that  $h_r(y) = \beta r$  implies  $h_r(g^*y) = \beta r$   $\mu$ -a.s. for  $\mu$  almost all  $g \in \mathbb{G}$ . Since the function  $h$  is continuous on  $\mathbb{P}(V^*)$ , it follows that  $h_r(g^*y) = \beta r$  for all  $g \in \text{supp } \mu$  and hence for  $g \in \Gamma_\mu$ . This means that  $\Gamma_\mu^* Z_r \subset Z_r$ , where

$$Z_r = \{y \in \mathbb{P}(V^*) : h_r(y) = \beta r\}.$$

By Lemma 6.6, we know that  $\Gamma_\mu^* Z \subset Z$ , where  $Z$  is defined by (48). Therefore  $\Gamma_\mu^*(Z_r \cap Z) \subset (Z_r \cap Z)$ . Noting that

$$Z_r \cap Z = \{y \in \mathbb{P}(V^*) : h(y) = \beta, n(y) = r\},$$

we get the first assertion of the lemma.

Now we prove the second assertion. Since the function  $h_r$  is continuous, the set  $Z_r$  is nonempty and closed. We have seen in the proof of Lemma 6.6 that the set  $Z$  is also nonempty and closed. Recalling that  $\Lambda(\Gamma_\mu^*)$  is the smallest closed  $\Gamma_\mu^*$ -invariant set (by Remark 4.3), we obtain  $\Lambda(\Gamma_\mu^*) \subset Z_r \cap Z$ , which is precisely the second assertion.  $\square$

**Lemma 6.8.** *The mapping  $y \in \Lambda(\Gamma_\mu^*) \mapsto \mathcal{W}(y)$  is locally constant. In particular, the number of distinct values of this mapping is finite, i.e.*

$$\text{card} \{ \mathcal{W}(y) : y \in \Lambda(\Gamma_\mu^*) \} < \infty.$$

**Proof.** For any  $y \in \mathbb{P}(V^*)$ , set

$$\varepsilon(y) = \max \{ v_s^y(Y) : Y \in \mathfrak{A}(X_\mu), \dim(Y) = d_0, v_s^y(Y) < \beta \},$$

where the maximum is attained (this can be established using Lemma 6.5 in the same way as in the proof of Lemma 5.5). Equip the space  $\mathcal{C}(\mathbb{P}(V))'$  of complex valued Borel measures on  $\mathbb{P}(V)$  with the total variation norm  $\| \cdot \|_{\text{TV}}$ . Since  $\varepsilon(y) < \beta$ , by the continuity of the mapping  $y \in \mathbb{P}(V^*) \mapsto v_s^y \in \mathcal{C}(\mathbb{P}(V))'$  (see Lemma 3.11), we get

$$\varepsilon_0 = \sup_{y \in \mathbb{P}(V^*)} \varepsilon(y) < \beta.$$

Note that any algebraic subset  $Y$  has the following property:

$$v_s^y(Y) = \beta \quad \text{or} \quad v_s^y(Y) \leq \varepsilon_0. \tag{51}$$

On the compact set  $\Lambda(\Gamma_\mu^*)$  there exists a neighborhood  $U_y \subset \Lambda(\Gamma_\mu)$  of  $y$  such that for any  $y' \in U_y$ ,

$$\|v_s^y - v_s^{y'}\|_{\text{TV}} < \beta - \varepsilon_0. \quad (52)$$

Let  $Y_1, \dots, Y_r$  be the elements of  $\mathcal{W}(y)$  that realize  $\beta$ , i.e.  $v_s^y(Y_k) = \beta$ ,  $k = 1, \dots, r$ . In particular, this implies that

$$v_s^y(Y_1 \cup \dots \cup Y_r) = \beta r. \quad (53)$$

Then, by (52) and (53),

$$\left| v_s^y(Y_1 \cup \dots \cup Y_r) - v_s^{y'}(Y_1 \cup \dots \cup Y_r) \right| = \left| \beta r - v_s^{y'}(Y_1 \cup \dots \cup Y_r) \right| < \beta - \varepsilon_0. \quad (54)$$

By (51) we have either  $v_s^{y'}(Y_k) = \beta$  or  $v_s^{y'}(Y_k) \leq \varepsilon_0$ , for  $k = 1, \dots, r$ . If there exists  $Y_k$  such that  $v_s^{y'}(Y_k) \leq \varepsilon_0$ , then this will lead to a contradiction. Hence we get that  $v_s^{y'}(Y_k) = \beta$  for  $k = 1, \dots, r$ , so that  $\mathcal{W}(y') \subset \mathcal{W}(y)$ . In turn, since  $\text{card } \mathcal{W}(y) = \text{card } \mathcal{W}(y') = r$ , we obtain that  $\mathcal{W}(y) = \mathcal{W}(y')$  for any  $y' \in U_y$ , which means that the mapping  $y \mapsto \mathcal{W}(y)$  is locally constant on  $\Lambda(\Gamma_\mu^*)$ .

Since  $\Lambda(\Gamma_\mu^*)$  is a closed set of the projective space  $\mathbb{P}(V^*)$ , it is also compact. We know that any locally constant mapping on a compact set has a finite range. Therefore, the mapping  $y \mapsto \mathcal{W}(y)$  on  $\Lambda(\Gamma_\mu^*)$  takes only finitely many values. This proves the second assertion.  $\square$

**Lemma 6.9.** *For any  $y \in \Lambda(\Gamma_\mu^*)$ , we have  $\mathcal{W}(g^*y) = g^{-1}\mathcal{W}(y)$  for  $\mu$ -a.s. all  $g \in \Gamma_\mu$ .*

**Proof.** Let  $y \in \Lambda(\Gamma_\mu^*)$  and  $Y_1, \dots, Y_r$  be two by two distinct elements of  $\mathcal{W}(y)$  that realize  $\beta$ , i.e. such that  $Y_k \in \mathfrak{A}(X_\mu)$ ,  $\dim(Y_k) = d_0$  and  $v_s^y(Y_k) = \beta$  for any  $k = 1, \dots, r$ . Since the sets  $Y_1, \dots, Y_r$  are two by two distinct, for any  $1 \leq i < j \leq r$ , the intersection  $Y_i \cap Y_j$  is an algebraic set of dimension strictly smaller than  $d_0$ , in view of Lemma 6.3 it holds that  $v_s^y(Y_i \cap Y_j) = 0$ . This implies  $\beta r = v_s^y(Y_1 \cup \dots \cup Y_r)$ . Taking into account Lemma 3.12, we have

$$\beta r = v_s^y(Y_1 \cup \dots \cup Y_r) = \int_{\mathbb{G}} v_s^{g^*y}(g^{-1}(Y_1 \cup \dots \cup Y_r)) q_s^*(g, y) \mu(dg).$$

Since  $v_s^y(g^{-1}(Y_1 \cup \dots \cup Y_r)) \leq \beta r$  for any  $y \in \mathbb{P}(V^*)$ , by the maximum principle (Lemma 4.6), it follows that  $\mu$ -a.s. on  $\mathbb{G}$ , and therefore following the same reasoning as in the proof of Lemma 6.7 for all  $g \in \text{supp } \mu$ ,

$$v_s^{g^*y}(g^{-1}(Y_1 \cup \dots \cup Y_r)) = \beta r.$$

Note that  $v_s^{g^*y}(g^{-1}Y_i) \leq \beta$ , so that for all  $g \in \text{supp } \mu$ ,

$$\beta r = v_s^{g^*y}(g^{-1}(Y_1 \cup \dots \cup Y_r)) \leq \sum_{i=1}^r v_s^{g^*y}(g^{-1}Y_i) \leq \beta r.$$

This implies that  $v_s^{g^*y}(g^{-1}Y_i) = \beta$  for all  $g \in \text{supp } \mu$  and  $i = 1, \dots, r$ . Thus  $g^{-1}\mathcal{W}(y) \subset \mathcal{W}(g^*y)$  for all  $g \in \text{supp } \mu$ . Noticing that both sets have the same cardinality  $r$ , we obtain that  $g^{-1}\mathcal{W}(y) = \mathcal{W}(g^*y)$  for all  $g \in \text{supp } \mu$ . The desired assertion follows.  $\square$

**Proof of Proposition 6.4.** The proof is similar to that of Proposition 5.3. Set

$$\mathcal{W} = \bigcup_{y \in \Lambda(\Gamma_\mu^*)} \mathcal{W}(y).$$

By Lemma 6.9, the set  $\mathcal{W}$  is  $\Gamma_\mu$ -invariant:  $\Gamma_\mu \mathcal{W} = \mathcal{W}$ . By Lemma 6.8, the set  $\mathcal{W}$  is finite, hence  $Z := \bigcup_{Y \in \mathcal{W}} Y$  is a nonempty  $\Gamma_\mu$ -invariant algebraic subset in  $X_\mu$ . Since  $\Gamma_\mu$  is Zariski dense in  $H_\mu = \text{Zc}(\Gamma_\mu)$ , the algebraic set  $Z$  is  $H_\mu$ -invariant. By Lemma 4.4  $X_\mu$  is the minimal  $H_\mu$  invariant algebraic subset, so  $X_\mu \subset Z$ . On the other hand, we have  $Z = \bigcup_{Y \in \mathcal{W}} Y \subset X_\mu$ , therefore, we get  $Z = X_\mu$ . This reads as  $X_\mu = \bigcup_{Y \in \mathcal{W}} Y$ . From Lemma 4.5, we know that  $X_\mu$  is irreducible, therefore, we obtain that  $X_\mu = Y$  for some  $Y$  in  $\mathcal{W}$ . In particular, it holds that  $\dim(Y) = d_0 = \dim(X_\mu)$ , which concludes the proof of Proposition 6.4.  $\square$

We are now prepared to prove Propositions 6.1 and 6.2. The proof is based on Proposition 6.4 and the arguments already used in Section 5.

**Proof of Propositions 6.1 and 6.2.** Let  $Y$  be an algebraic subset of the set  $X_\mu$  with  $\nu_s(Y) > 0$ . As the set  $Y$  can be reducible, we decompose it as a finite union of irreducible algebraic subsets. Specifically, according to Proposition 1.5 of [23], there exist irreducible algebraic subsets  $Z_1, \dots, Z_r$  such that  $Y = Z_1 \cup \dots \cup Z_r$ . From  $\nu_s(Y) > 0$ , it follows that  $\nu_s(Z_k) > 0$  for some  $1 \leq k \leq r$ . Recalling that  $d_0$  is the minimal dimension of irreducible algebraic subsets  $U$  satisfying  $\nu_s(U) > 0$ , we get  $\dim(Z_k) \geq d_0$ . By Proposition 6.4, we know that  $\dim(X_\mu) = d_0$ . Hence we have  $\dim(Z_k) \geq d_0 = \dim(X_\mu)$ . This implies that  $Z_k = X_\mu$ , hence  $Y = X_\mu$ . The latter obviously proves the claims of Propositions 6.1 and 6.2.  $\square$

We end the section by establishing Theorems 2.6 and 2.8.

**Proof of Theorems 2.6 and 2.8.** Let  $Y$  be an algebraic subset of the projective space  $\mathbb{P}(V_{\mathbb{C}})$ . If  $X_\mu \subset Y$ , we have  $\nu_s(Y) \geq \nu_s(X_\mu) = 1$ , by Lemma 4.2. Otherwise,  $Y \cap X_\mu$  is a proper algebraic subset of  $X_\mu$ . Since  $\nu_s(X_\mu) = 1$ , it follows that  $\nu_s(Y) = \nu_s(Y \cap X_\mu) = 0$ , according to Propositions 6.1 and 6.2 for  $s > 0$  and  $s < 0$  respectively. The assertion of both theorems now follows from the fact that an algebraic subset of  $\mathbb{P}(V)$  is the intersection of an algebraic subset of the projective space  $\mathbb{P}(V_{\mathbb{C}})$  with  $\mathbb{P}(V)$ .  $\square$

## 7. Proof of local limit theorems for coefficients

In this section we show how to apply the zero-one law for the stationary measure  $\nu$  to establish a local limit theorem for the coefficients  $\langle f, G_n v \rangle$  of the random walk  $G_n$ , which to the best of our knowledge cannot be found in the literature.

### 7.1. Auxiliary results

Let us fix a non-negative density function  $\rho$  on  $\mathbb{R}$  with  $\int_{\mathbb{R}} \rho(u) du = 1$ , whose Fourier transform  $\widehat{\rho}(t) = \int_{\mathbb{R}} e^{-itu} \rho(u) du$ ,  $t \in \mathbb{R}$ , is supported on  $[-1, 1]$ . For any  $0 < \varepsilon < 1$ , define the rescaled density function  $\rho_\varepsilon$  by  $\rho_\varepsilon(u) = \frac{1}{\varepsilon} \rho(\frac{u}{\varepsilon})$ ,  $u \in \mathbb{R}$ , whose Fourier transform has a compact support on  $[-\varepsilon^{-1}, \varepsilon^{-1}]$ . Set  $\mathbb{B}_\varepsilon(u) = \{u' \in \mathbb{R} : |u' - u| \leq \varepsilon\}$ . For any non-negative integrable function  $\psi$ , we define

$$\psi_\varepsilon^+(u) = \sup_{u' \in \mathbb{B}_\varepsilon(u)} \psi(u') \quad \text{and} \quad \psi_\varepsilon^-(u) = \inf_{u' \in \mathbb{B}_\varepsilon(u)} \psi(u'), \quad u \in \mathbb{R}. \quad (55)$$

We need the following smoothing inequality from [17], which gives two-sided bounds for the function  $\psi$ .

**Lemma 7.1.** *Suppose that  $\psi$  is a non-negative integrable function and that  $\psi_\varepsilon^+$  and  $\psi_\varepsilon^-$  are measurable for any  $\varepsilon > 0$ . Then, there exists a positive constant  $C_\rho(\varepsilon)$  with  $C_\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , such that*

$$\psi_\varepsilon^- * \rho_{\varepsilon^2}(u) - \int_{|w| \geq \varepsilon} \psi_\varepsilon^-(u-w) \rho_{\varepsilon^2}(w) dw \leq \psi(u) \leq (1 + C_\rho(\varepsilon)) \psi_\varepsilon^+ * \rho_{\varepsilon^2}(u), \quad u \in \mathbb{R}.$$

Define the perturbed operator  $P_{it}$  as follows: for any  $t \in \mathbb{R}$  and  $\varphi \in \mathcal{C}(\mathbb{P}(V))$ ,

$$P_{it}(\varphi)(x) = \int_{\mathbb{G}} e^{it(\sigma(g,x) - \lambda)} \varphi(gx) \mu(dg), \quad x \in \mathbb{P}(V). \quad (56)$$

The following proposition which is taken from [29] will be used in the proof of Theorem 2.9. Recall that  $\nu$  is the unique stationary measure of the Markov chain  $(G_n x)_{n \geq 0}$  on the projective space  $\mathbb{P}(V)$ . Let  $\varphi$  be a  $\gamma$ -Hölder continuous function on  $\mathbb{P}(V)$ . Assume that  $\psi : \mathbb{R} \mapsto \mathbb{C}$  is a continuous function with compact support in  $\mathbb{R}$ , and moreover,  $\psi$  is differentiable in a small neighborhood of 0 on the real line.

**Proposition 7.2.** *Assume conditions A1, A2 and A3. Then, there exist constants  $\delta > 0$ ,  $c > 0$ ,  $C > 0$  such that for all  $x \in \mathbb{P}(V)$ ,  $|t| \leq \frac{1}{\sqrt{n}}$ ,  $\varphi \in \mathcal{B}_\gamma$  and  $n \geq 1$ ,*

$$\left| \sigma \sqrt{n} e^{\frac{nt^2}{2\sigma^2}} \int_{\mathbb{R}} e^{-itln} P_{it}^n(\varphi)(x) \psi(t) dt - \sqrt{2\pi} \nu(\varphi) \psi(0) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma + \frac{C}{n} \|\varphi\|_\gamma \sup_{|t| \leq \delta} (|\psi(t)| + |\psi'(t)|) + C e^{-cn} \|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)| dt.$$

The explicit dependence of the bound on the target function  $\varphi$  as well as the rate of convergence established in Proposition 7.2 will be used in the proof of Theorem 2.9.

### 7.2. Proof of Theorem 2.9

The goal of this subsection is to establish Theorem 2.9 using Theorems 2.1 and 2.2, Lemma 3.8 and Proposition 7.2.

**Proof of Theorem 2.9.** The basic idea is to decompose the logarithm of the coefficient  $\log |\langle f, G_n v \rangle|$  as a sum of the norm cocycle  $\sigma(G_n, x)$  and of  $\log \delta(y, G_n x)$ , where  $x = \mathbb{R}v \in \mathbb{P}(V)$  and  $y = \mathbb{R}f \in \mathbb{P}(V^*)$  with  $|v| = 1$  and  $|f| = 1$ , see (3). Specifically, we have the decomposition:

$$\begin{aligned} J &:= \sigma\sqrt{2\pi n} \mathbb{P}\left(\log |\langle f, G_n v \rangle| - n\lambda \in [a_1, a_2]\right) \\ &= \sigma\sqrt{2\pi n} \mathbb{P}\left(S_n^v + \log \delta(y, G_n x) \in [a_1, a_2]\right), \end{aligned} \quad (57)$$

where

$$S_n^v = \log |G_n v| - n\lambda.$$

For any fixed small constant  $0 < \eta < 1$ , we denote

$$I_k := (-\eta k, -\eta(k-1)], \quad k \in \mathbb{N}^*.$$

In the sequel, let  $[a]$  denote the integral part of  $a \in \mathbb{R}$ . To apply limit theorems established for the norm cocycle  $\log |G_n v|$ , we are led to discretize the function  $x \mapsto \log \delta(y, x)$  to obtain that: with a sufficiently large constant  $C_1 > 0$ ,

$$\begin{aligned} J &= \sigma\sqrt{2\pi n} \mathbb{P}\left(S_n^v + \log \delta(y, G_n x) \in [a_1, a_2], \log \delta(y, G_n x) \leq -\eta \lfloor C_1 \log n \rfloor\right) \\ &\quad + \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{P}\left(S_n^v + \log \delta(y, G_n x) \in [a_1, a_2], \log \delta(y, G_n x) \in I_k\right) \\ &=: J_1 + J_2. \end{aligned}$$

*Upper bound of  $J_1$ .* The term  $J_1$  is easily handled by using the fact that, with very small probability, the Markov chain  $(G_n x)_{n \geq 0}$  stays close to the hyperplane  $\ker f$ . Indeed, by Lemma 3.8, we get that there exists a constant  $c_\eta > 0$  such that

$$J_1 \leq \sigma\sqrt{2\pi n} \mathbb{P}\left(\log \delta(y, G_n x) \leq -\eta \lfloor C_1 \log n \rfloor\right) \leq \sigma\sqrt{2\pi n} e^{-c_\eta \lfloor C_1 \log n \rfloor} \rightarrow 0, \quad (58)$$

as  $n \rightarrow \infty$ , since the constant  $C_1 > 0$  is sufficiently large.

*Upper bound of  $J_2$ .* Note that on the set  $\{\log \delta(y, G_n x) \in I_k\}$ , we have  $0 < \log \delta(y, G_n x) + \eta k \leq \eta$  and hence

$$J_2 \leq \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\left(\mathbf{1}_{\{S_n^v - \eta k \in [a_1, a_2 + \eta]\}} \mathbf{1}_{\{\log \delta(y, G_n x) \in I_k\}}\right).$$

We denote  $\psi_1(u) = \mathbf{1}_{\{u \in [a_1, a_2 + \eta]\}}$ ,  $u \in \mathbb{R}$ , and  $\psi_\varepsilon^+(u) = \sup_{u' \in \mathbb{B}_\varepsilon(u)} \psi_1(u')$  as in (55), for  $0 < \varepsilon < 1$ . Using the upper bound in Lemma 7.1 gives

$$J_2 \leq (1 + C_\rho(\varepsilon))\sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\left[(\psi_\varepsilon^+ * \rho_{\varepsilon^2})(S_n^v - \eta k) \mathbf{1}_{\{\log \delta(y, G_n x) \in I_k\}}\right]. \quad (59)$$

For small enough constant  $\varepsilon_1 > 0$ , we define the density function  $\bar{\rho}_{\varepsilon_1}$  by setting  $\bar{\rho}_{\varepsilon_1}(u) := \frac{1}{\varepsilon_1} (1 - \frac{|u|}{\varepsilon_1})$  for  $u \in [-\varepsilon_1, \varepsilon_1]$ , and  $\bar{\rho}_{\varepsilon_1}(u) = 0$  otherwise. For any  $k \in \mathbb{N}^*$ , with the notation  $\chi_k(u) := \mathbf{1}_{\{u \in I_k\}}$  and  $\chi_{k, \varepsilon_1}^+(u) =$



$\sup_{u' \in \mathbb{B}_{\varepsilon_1}(u)} \chi_k(u')$ , one can check that

$$\chi_k(u) \leq (\chi_{k,\varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(u) \leq \chi_{k,2\varepsilon_1}^+(u), \quad u \in \mathbb{R}. \quad (60)$$

For  $y \in \mathbb{P}(V^*)$ , we denote for short  $\varphi_k^y(x) = (\chi_{k,\varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(\log \delta(y, x))$ ,  $x \in \mathbb{P}(V)$ , which is Hölder continuous on  $\mathbb{P}(V)$ . Using (60) leads to

$$J_2 \leq (1 + C_\rho(\varepsilon)) \sigma \sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E} [\varphi_k^y(G_n x) (\psi_\varepsilon^+ * \rho_{\varepsilon^2})(S_n^v - \eta k)]. \quad (61)$$

Denote by  $\widehat{\psi}_\varepsilon^+$  the Fourier transform of  $\psi_\varepsilon^+$ , then by the Fourier inversion formula,

$$\psi_\varepsilon^+ * \rho_{\varepsilon^2}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt, \quad u \in \mathbb{R}.$$

Substituting  $u = S_n^v - \eta k$  and using Fubini's theorem, we obtain

$$J_2 \leq (1 + C_\rho(\varepsilon)) \sigma \sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \int_{\mathbb{R}} e^{-it\eta k} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt, \quad (62)$$

where  $P_{it}$  is the perturbed operator defined by (56). We shall apply Proposition 7.2 to deal with each integral in (62) for fixed  $k \geq 1$ . Note that  $e^{\frac{Ck^2}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ , uniformly in  $1 \leq k \leq \lfloor C_1 \log n \rfloor$ . Since the function  $\widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$  is compactly supported in  $\mathbb{R}$ , using Proposition 7.2 with  $\varphi = \varphi_k^y$ ,  $\psi = \widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$  and  $l = \frac{\eta k}{n}$ , we obtain that for any fixed  $k \geq 1$ , as  $n \rightarrow \infty$ , uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,

$$\left| \sigma \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} e^{-it\eta k} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt - \widehat{\psi}_\varepsilon^+(0) \widehat{\rho}_{\varepsilon^2}(0) \nu(\varphi_k^y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi_k^y\|_\gamma.$$

Note that  $\widehat{\rho}_{\varepsilon^2}(0) = 1$  and

$$\widehat{\psi}_\varepsilon^+(0) = \int_{\mathbb{R}} \sup_{y' \in \mathbb{B}_\varepsilon(u)} \psi_\varepsilon^+(u') du = \int_{\mathbb{R}} \mathbf{1}_{\{u \in [a_1 - \varepsilon, a_2 + \eta + \varepsilon]\}} du = a_2 - a_1 + \eta + 2\varepsilon.$$

One can calculate that  $\gamma$ -Hölder norm  $\|\varphi_k^y\|_\gamma$  is dominated by  $C \frac{e^{\eta\gamma k}}{(1 - e^{-2\varepsilon_1})^\gamma}$ , uniformly in  $y \in \mathbb{P}(V^*)$ . Taking sufficiently small  $\gamma > 0$ , we obtain that the series  $\frac{C}{\sqrt{n}} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \frac{e^{\eta\gamma k}}{(1 - e^{-2\varepsilon_1})^\gamma}$  converges to 0 as  $n \rightarrow \infty$ . Consequently, we are allowed to interchange the limit as  $n \rightarrow \infty$  and the sum over  $k$  in (62) to obtain that, uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,

$$\limsup_{n \rightarrow \infty} J_2 \leq (1 + C_\rho(\varepsilon)) (a_2 - a_1 + \eta + 2\varepsilon) \sum_{k=1}^{\infty} \nu(\varphi_k^y). \quad (63)$$

Observe that for any  $x \in \mathbb{P}(V)$ ,

$$\varphi_k^y(x) \leq \mathbf{1}_{\{\log \delta(y, \cdot) \in I_k\}}(x) + \mathbf{1}_{\{\log \delta(y, \cdot) \in I_{k,\varepsilon_1}\}}(x), \quad (64)$$

where  $I_{k,\varepsilon_1} = (-\eta k - 2\varepsilon_1, -\eta k] \cup (-\eta(k-1), -\eta(k-1) + 2\varepsilon_1]$ . For the first part in (64), we have that for any  $y \in \mathbb{P}(V^*)$ ,

$$\sum_{k=1}^{\infty} \nu(x \in \mathbb{P}(V) : \log \delta(y, x) \in I_k) = 1. \quad (65)$$

For the second part in (64), we need to apply Theorem 2.1 and the zero-one law for the stationary measure  $\nu$  established in Theorem 2.2. By the Lebesgue dominated convergence theorem, we get

$$E := \lim_{\varepsilon_1 \rightarrow 0} \sum_{k=1}^{\infty} \nu(x \in \mathbb{P}(V) : \log \delta(y, x) \in I_{k,\varepsilon_1})$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \nu \left( x \in \mathbb{P}(V) : \log \delta(y, x) = -\eta k \right) + \sum_{k=1}^{\infty} \nu \left( x \in \mathbb{P}(V) : \log \delta(y, x) = -\eta(k-1) \right) \\
&= 2 \sum_{k=1}^{\infty} \nu \left( x \in \mathbb{P}(V) : \log \delta(y, x) = -\eta k \right),
\end{aligned}$$

where in the last equality we used Theorem 2.1. We are going to apply Theorem 2.2 to prove that  $E = 0$ . In fact, for any  $y \in \mathbb{P}(V^*)$  and any set  $Y_{y,t} = \{x \in \mathbb{P}(V) : \log \delta(y, x) = t\}$  with  $t \in (-\infty, 0)$ , by Theorem 2.2 it holds that either  $\nu(Y_{y,t}) = 0$  or  $\nu(Y_{y,t}) = 1$ . If  $\nu(Y_{y,t}) = 0$  for all  $y \in \mathbb{P}(V^*)$  and  $t \in (-\infty, 0)$ , then clearly we get that  $E = 0$ . If  $\nu(Y_{y_0, t_0}) = 1$  for some  $y_0 \in \mathbb{P}(V^*)$  and  $t_0 \in (-\infty, 0)$ , then we can always choose  $0 < \eta < 1$  in such a way that  $-\eta k \neq t_0$  for all  $k \geq 1$ , so that we also obtain  $E = 0$  for all  $y \in \mathbb{P}(V^*)$ . Therefore, combining this with (63), (64) and (65), letting first  $\varepsilon \rightarrow 0$  and then  $\eta \rightarrow 0$ , and noting that  $C_\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we obtain the upper bound: uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,

$$\limsup_{n \rightarrow \infty} J_2 \leq a_2 - a_1. \quad (66)$$

*Lower bound of  $J_2$ .* Since  $0 < \log \delta(y, G_n x) + \eta k \leq \eta$  on the set  $\{\log \delta(y, G_n x) \in I_k\}$ , we have

$$J_2 \geq \sigma \sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E} \left[ \mathbf{1}_{\{S_n^v - \eta k \in [a_1 + \eta, a_2 - \eta]\}} \mathbf{1}_{\{\log \delta(y, G_n x) \in I_k\}} \right].$$

We denote  $\psi_2(u) = \mathbf{1}_{\{u \in [a_1, a_2 + \eta]\}}$ ,  $u \in \mathbb{R}$ , and recall that  $\psi_\varepsilon^-(u) = \inf_{u' \in \mathbb{B}_\varepsilon(u)} \psi_2(u')$  is defined by (55), for  $0 < \varepsilon < 1$ . By Lemma 7.1, we get

$$\begin{aligned}
J_2 &\geq \sigma \sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E} \left[ (\psi_\varepsilon^- * \rho_{\varepsilon^2})(S_n^v - \eta k) \mathbf{1}_{\{\log \delta(y, G_n x) \in I_k\}} \right] \\
&\quad - \sigma \sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \int_{|w| \geq \varepsilon} \mathbb{E} \left[ \psi_\varepsilon^-(S_n^v - \eta k - w) \mathbf{1}_{\{\log \delta(y, G_n x) \in I_k\}} \right] \rho_{\varepsilon^2}(w) dw \\
&=: J_3 - J_4.
\end{aligned} \quad (67)$$

For any  $k \in \mathbb{N}$ , define  $\chi_k(u) := \mathbf{1}_{\{u \in I_k\}}$  and  $\chi_{k, \varepsilon_1}^-(u) = \inf_{u' \in \mathbb{B}_{\varepsilon_1}(u)} \chi_k(u')$ . It is easy to verify that

$$\chi_k(u) \geq (\chi_{k, \varepsilon_1}^- * \bar{\rho}_{\varepsilon_1})(u) \geq \chi_{k, 2\varepsilon_1}^-(u), \quad u \in \mathbb{R}, \quad (68)$$

where  $\bar{\rho}_{\varepsilon_1}$  is the density function introduced in (60). For short, we denote  $\tilde{\varphi}_k^y(x) = (\chi_{k, \varepsilon_1}^- * \bar{\rho}_{\varepsilon_1})(\log \delta(y, x))$ ,  $x \in \mathbb{P}(V)$ , which is Hölder continuous on  $\mathbb{P}(V)$ .

*Lower bound of  $J_3$ .* Using (68), we get

$$J_3 \geq \sigma \sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E} \left[ \tilde{\varphi}_k^y(G_n x) (\psi_\varepsilon^- * \rho_{\varepsilon^2})(S_n^v - \eta k) \right].$$

In an analogous way as in the proof of (63), we obtain

$$\liminf_{n \rightarrow \infty} J_3 \geq (a_2 - a_1 - 2\eta - 2\varepsilon) \sum_{k=1}^{\infty} \nu(\tilde{\varphi}_k^y). \quad (69)$$

Proceeding in a similar way as in the proof of the upper bound (66) for  $J_2$ , using Theorem 2.2, we can obtain the lower bound for  $J_3$ : uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,

$$\liminf_{\eta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} J_3 \geq a_2 - a_1. \quad (70)$$

*Upper bound of  $J_4$ .* Note that  $\psi_\varepsilon^- \leq \psi$ , then it follows from Lemma 7.1 that  $\psi_\varepsilon^- \leq (1 + C_\rho(\varepsilon))\widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$ . Moreover, using (60), we get  $\mathbb{1}_{\{\log \delta(y, G_n x) \in I_k\}} \leq (\chi_{k, \varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(G_n x)$ . Similarly to (62), we have that  $J_4$  defined in (67) is bounded from above by

$$(1 + C_\rho(\varepsilon))\sigma\sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \int_{|w| \geq \varepsilon} \left[ \int_{\mathbb{R}} e^{-it(\eta k + w)} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt \right] \rho_{\varepsilon^2}(w) dw.$$

Applying Proposition 7.2 with  $\varphi = \varphi_k^y$  and  $\psi = \widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$ , it follows from the Lebesgue dominated convergence theorem that

$$\liminf_{n \rightarrow \infty} J_4 \leq (1 + C_\rho(\varepsilon)) \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \nu(\varphi_k^y) \widehat{\psi}_\varepsilon^+(0) \widehat{\rho}_{\varepsilon^2}(0) \int_{|w| \geq \varepsilon} \rho_{\varepsilon^2}(w) dw \rightarrow 0,$$

as  $\varepsilon \rightarrow 0$ . Combining this with (67) and (70), we get the lower bound for  $J_2$ : uniformly in  $f \in V^*$  and  $v \in V$  with  $|f| = 1$  and  $|v| = 1$ ,

$$\liminf_{n \rightarrow \infty} J_2 \geq a_2 - a_1. \tag{71}$$

Putting together (58), (66) and (71), we conclude the proof of Theorem 2.9.  $\square$

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