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ASYMMETRIC ATTRACTIVE ZERO-RANGE PROCESSES WITH PARTICLE DESTRUCTION AT THE ORIGIN

CLÉMENT ERIGNOUX, MARIELLE SIMON, AND LINJIE ZHAO

ABSTRACT. We investigate the macroscopic behavior of asymmetric attractive zero-range processes on $\mathbb{Z}$ where particles are destroyed at the origin at a rate of order N^β, where $\beta \in \mathbb{R}$ and $N \in \mathbb{N}$ is the scaling parameter. We prove that the hydrodynamic limit of this particle system is described by the unique entropy solution of a hyperbolic conservation law, supplemented by a boundary condition depending on the range of β. Namely, if $\beta \geqslant 0$, then the boundary condition prescribes the particle current through the origin, whereas if $\beta < 0$, the destruction of particles at the origin has no macroscopic effect on the system and no boundary condition is imposed at the hydrodynamic limit.

1. INTRODUCTION

Asymmetric and conservative particle systems are among the most interesting microscopic dynamics currently under investigation in statistical physics. Their macroscopic behavior, due to the microscopic asymmetry of the particles' motion, is generally expected to be described by hyperbolic equations, whose solutions may be visualized as propagating waves. When the system is nonlinear, jump discontinuities, known as *shocks*, can arise in finite time, even if the initial profile is smooth. Even for linear systems, discontinuities can appear in presence of boundary effects. Therefore, when asymmetric particle systems are subject to strong boundary mechanisms, their macroscopic behavior is much less understood. In a general setting, imposing given boundary conditions at the macroscopic level starting from a given microscopic model is far from trivial. Because of discontinuities, hyperbolic equations do not in general have a unique solution; in order to fully describe the macroscopic limit of a given microscopic asymmetric system, one needs to rule out the non-physical solutions. The unique relevant one, called *entropy solution*, can be characterized in several different ways, and exhibits physically consistent behavior after the shocks (see the reference book [13], in particular Chapters 2.6 and 2.7 for the existence and uniqueness of solutions to hyperbolic conservation laws in bounded domains).

The characterization of the macroscopic behavior of asymmetric particle systems by hyperbolic equations has been initiated by Rezakhanlou in [15]. In the latter, the author derives the *hydrodynamic limit* –the law of large number characterizing a particle system's macroscopic behavior in terms of a given PDE– in cases where the microscopic dynamics are *attractive*. He proves in particular that the macroscopic density $\rho(t,u)$ of the Asymmetric Simple Exclusion Process (ASEP), in the hyperbolic time scale is given by the unique entropy solution to the hydrodynamic equation $\partial_t \rho = \partial_u(\rho(1-\rho))$ on the full line $\mathbb{R}$. Twenty years later Bahadoran [2] considered the *open* ASEP, where particles are created and absorbed at two boundaries with given rates, and obtained the convergence to the unique entropy solution on a bounded domain, satisfying the *Bardos-Leroux-Nédélec boundary condition* [3] in the sense of Otto [14]. Very recently, De Masi *et al.* [4] have carried on with the study of the open ASEP, and considered its *quasi-static* hydrodynamic limit with time dependent boundary rates changing on a slow time scale (w.r.t. the hyperbolic one). The quasi-static limit they obtain characterizes the slow transition between non-equilibrium stationary states due to the influence of the boundary. They prove in [4]

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that the density profile converges to the entropy solution of the quasi-static Burgers equation with the corresponding time dependent boundary condition.

In the case of asymmetric *zero-range* processes, the law of large numbers for the particle density on the full line has also been obtained in the attractive case by Rezankhanlou in [15]. Then, Landim in [12] focused on the *totally asymmetric* case and introduced a boundary effect at the origin by slowing down the jump rate at 0. He proves the convergence of the density profile towards the unique entropy solution of the hyperbolic conservation law with mass creation/annihilation at the origin.

In this work, we explore the effect of boundary conditions on asymmetric zero-range processes. We consider general asymmetric zero-range processes on $\mathbb{Z}$ and we introduce a microscopic mechanism which destroys particles at the origin: if there are $\omega_x \in \mathbb{N}$ particles at site $x \in \mathbb{Z}$, then one of these particles jumps to its right (resp. left) neighbour at rate $pg(\omega_x)$ (resp. $(1-p)g(\omega_x)$), and moreover, at the origin $x = 0$, one particle is destroyed at rate $\alpha N^\beta g(\omega_0)$. Here $g : \mathbb{N} \to \mathbb{R}_+$ is the rate function, $\alpha \geqslant 0$ and $\beta \in \mathbb{R}$ are two parameters that adjust the intensity of the destruction dynamics, $N \in \mathbb{N}$ being the hydrodynamic limit's scaling parameter. Our main result is the derivation of a family of hyperbolic equations on the one-dimensional space $\mathbb{R}$. The destruction mechanism enforces different boundary conditions at the origin, depending on the value of β: if $\beta < 0$, then the particle destruction has no macroscopic effect and the hydrodynamic equation is given by the hyperbolic conservation law on the full line $\mathbb{R}$ with no boundary condition; if $\beta \geqslant 0$, then the hyperbolic conservation law is now supplemented by a boundary condition which prescribes the particle current allowed through the origin. This current vanishes at the macroscopic scale if $\beta > 0$, and is an explicit function of the density left of the origin if $\beta = 0$ (see Theorem 2.3 for a precise statement).

All the results mentioned above [2, 4, 12, 15] strongly rely on the *attractiveness property* of both the ASEP and zero-range processes. Here, we keep attractiveness as a crucial assumption. Indeed, attractive zero-range processes have nice features which allow us to introduce the notion of *second-class particles* and to use coupling techniques. Similarly to [4] we need to control the microscopic boundary entropy flux and we use the notion of boundary entropy-entropy flux pairs introduced by Otto [14] and used in [12] in order to characterize the boundary conditions in the scalar hyperbolic equations. For non-attractive asymmetric processes, very little is known: in fact, Yau's classical *relative entropy* approach [16] is the unique method that derives the hydrodynamic behavior of such purely asymmetric processes in general, but it requires the solution to the hydrodynamic equation to be smooth. For this reason, it can usually only be applied to asymmetric systems up to the time of the first shock, and furthermore fails completely in the presence of boundary conditions such as those considered in this work.

Another option, which would allow to derive non-attractive hydrodynamic limits for asymmetric systems in the presence of discontinuities (past the first shock, in particular) would be to adapt Fritz's *compensated compactness arguments* [8, 9]. This technique, however, is not without its own shortcomings, and in particular requires to perturb the underlying microscopic dynamics by some symmetric stirring dynamics. This perturbation needs to be strong enough to ensure that one recovers at the hydrodynamic limit the entropy (or vanishing viscosity) solution to the hyperbolic equation, but weak enough so that the symmetric part of the dynamics does not appear in the limiting equation. This work is currently in progress in order to drop the attractiveness assumption in our model.

The article is organized as follows: in Section 2 we introduce the microscopic model and state the main result of the paper, namely the hydrodynamic limit for our attractive zero-range process with destruction of particles at the origin. In Section 3 we prove preliminary results on second-class particles and present the coupling argument, which will be used in Section 4 to prove the main theorem. In Appendix A we present a short alternative proof in the specific case where the jump rate is linear, using duality arguments, and in Appendix B we prove a technical result on the equivalence of two notions of entropy solutions.

2. Definitions, notations and results

2.1. **Zero-range process with source dynamics.** Let N be a positive integer and $\Omega = \mathbb{N}^{\mathbb{Z}}$ be the state space of the Markov process $(\omega(t))_{t\geqslant 0}$ which will be the focus of this article. its dynamics is characterized below by its infinitesimal generator $\mathscr{L}_N$. Given a configuration of particles $\omega \in \Omega$, $\omega_x \in \mathbb{N}$ represents the number of particles at site $x \in \mathbb{Z}$. Before defining the generator $\mathscr{L}_N$, let us fix:

- two parameters $\alpha \geqslant 0$ and $\beta \in \mathbb{R}$, which adjust the intensity of the particle destruction at the origin;
- the asymmetry parameter $p \neq \frac{1}{2}$. Without loss of generality, we assume $\frac{1}{2} < p \leqslant 1$.
- a function $g : \mathbb{N} \to \mathbb{R}_+$ which represents the rate at which one particle leaves a site. This rate function satisfies $g(0) = 0$, $g(k) > 0$ for any $k \geqslant 1$, and is assumed to be *Lipschitz*: there is a constant $a_0 > 0$ such that

$$\sup_{k\geqslant 0} |g(k+1) - g(k)| \leqslant a_0 \; ; \tag{2.1}$$

We consider in the present work the *attractive case*, which means that we further assume that

$$\text{the function } g \text{ is non-decreasing, } i.e.\ g(k) \leqslant g(k+1) \text{ for all } k \geqslant 0. \tag{H1}$$

We say that a function $f : \Omega \to \mathbb{R}$ is *local* if it depends on $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ only through a finite number of coordinates $(\omega_{x_1}, \dots, \omega_{x_k})$, and a local function $f : \Omega \to \mathbb{R}$ is *Lipschitz* if there is a constant $c > 0$ and a finite subset $\Lambda \subset \mathbb{Z}$ such that

$$|f(\eta) - f(\zeta)| \leqslant c \sum_{x\in\Lambda} |\eta(x) - \zeta(x)|, \qquad \text{for all } \eta,\, \zeta \in \Omega.$$

The generator $\mathscr{L}_N$ of the zero-range process considered here acts on local Lipschitz functions $f : \Omega \to \mathbb{R}$ by

$$\mathscr{L}_N f(\omega) = \mathscr{L}_N^{\mathrm{ZR}} f(\omega) + \mathscr{L}_N^{\alpha,\beta} f(\omega). \tag{2.2}$$

The generator $\mathscr{L}_N^{\mathrm{ZR}}$ of the *asymmetric zero-range dynamics* is given by

$$\mathscr{L}_N^{\mathrm{ZR}} f(\omega) = \sum_{x\in\mathbb{Z}} g(\omega_x) \Big\{ p\nabla_{x,x+1} f(\omega) + (1-p)\nabla_{x,x-1} f(\omega) \Big\} \tag{2.3}$$

where for $\omega \in \Omega$ and $x, y \in \mathbb{Z}$, we denote $\nabla_{x,y} f(\omega) = f(\omega^{x,y}) - f(\omega)$ and $\omega^{x,y} \in \Omega$ is the configuration obtained from ω after a particle jumps from site x to site y,

$$\omega_z^{x,y} = \begin{cases} \omega_x - 1 & \text{if } z = x, \\ \omega_y + 1 & \text{if } z = y, \\ \omega_z & \text{if } z \neq x, y. \end{cases}$$

The generator $\mathscr{L}_N^{\alpha,\beta}$ of the *source dynamics* is given by

$$\mathscr{L}_N^{\alpha,\beta} f(\omega) \;=\; \alpha N^{\beta} g(\omega_0) \left\{ f(\omega^{(0)}) - f(\omega) \right\},$$

where $\omega^{(0)} \in \Omega$ is the configuration obtained from Ω after the destruction of one particle at the origin

$$\omega_x^{(0)} = \omega_x - \mathbb{1}\{x = 0\},$$

$\beta \in \mathbb{R}$, and $\alpha > 0$ is a positive parameter, fixed throughout. We refer the readers to [1] for a rigorous construction of this infinite-volume dynamics, when $\alpha = 0$. The general case $\alpha \neq 0$ is a straightforward adaptation of [1].

2.2. **Invariant measures.** We now describe the invariant measures for the zero-range process *without* particle destruction. To that aim, following [11, Chapter 2], let us introduce the partition function $Z : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$Z(\zeta) = \sum_{k \geqslant 0} \frac{\zeta^k}{g(k)!}, \qquad \text{for } 0 \leqslant \zeta < \zeta^*,$$

where ζ^* is the radius of convergence of the above summation and $g(k)! := \prod_{\ell=1}^{k} g(\ell)$ with the convention $g(0)! = 1$. For each $0 \leqslant \zeta < \zeta^*$, let $\overline{\nu}_\zeta$ be the product measure on Ω with marginals given by

$$\overline{\nu}_\zeta(\omega_x = k) = \frac{1}{Z(\zeta)} \frac{\zeta^k}{g(k)!}, \quad x \in \mathbb{Z}, k \in \mathbb{N}.$$

It is easy to check that $\overline{\nu}_\zeta$, $0 \leqslant \zeta < \zeta^*$, are invariant measures for the generator $\mathscr{L}_N^{\mathrm{ZR}}$. Furthermore, the particle density $R(\zeta) = \int \omega_0 \overline{\nu}_\zeta(d\omega)$ is strictly increasing in ζ, hence has an inverse denoted by $\Phi(\cdot)$. To index by density the invariant measures of the generator $\mathscr{L}_N^{\mathrm{ZR}}$, we denote $\nu_\rho = \overline{\nu}_{\Phi(\rho)}$ and note that

$$\Phi(\rho) = \int g(\omega_0)\, \nu_\rho(d\omega). \tag{2.4}$$

Under our assumptions, $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and non-decreasing, where $\mathbb{R}_+ = [0, +\infty)$. Throughout, when 0 needs to be excluded from $\mathbb{R}_+$, we will write $(0, +\infty)$.

The invariant measures for the full process generated by $\mathscr{L}_N$ will be given in Section 3.1. In the following, for a probability measure ν on a state space Ω, we denote by $E_\nu[f]$ the expectation of a function f defined on Ω with respect to ν.

2.3. **Entropy solutions.** To state our main result, we first need to introduce the notion of *entropy solutions* to the hydrodynamic limit in the various cases for the parameter β. We start with the standard definition when no boundary condition is imposed, and then we define the entropy solution to *initial-boundary value problems*, as given for instance in [2, 12]. In the following we denote by $C_K^{k,\ell}(U \times V)$ the set of functions $H : U \times V \to \mathbb{R}$ which are compactly supported, and C^k-regular (resp. C^ℓ) in the first (resp. second) variable.

Definition 2.1 (Hydrodynamic equation on $\mathbb{R}$ for $\beta < 0$)**.** *Assume that $\rho \mapsto \Phi(\rho)$ is a smooth function. We say that $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution to*

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, & t > 0, u \in \mathbb{R}, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}, \end{cases} \tag{2.5}$$

if

- *for any non-negative test function $H \in C_K^{1,1}((0,+\infty) \times \mathbb{R})$, and any constant $c \geqslant 0$,*

$$\int_0^\infty \int_{\mathbb{R}} \left\{ \partial_t H(t,u) \big|\rho(t,u) - c\big| + (2p-1)\partial_u H(t,u) \big|\Phi(\rho(t,u)) - \Phi(c)\big| \right\} du\, dt \geqslant 0, \tag{2.6}$$

- *for every $A > 0$,*

$$\lim_{t \to 0} \int_{-A}^{A} \big|\rho(t,u) - \rho_0(u)\big|\, du = 0.$$

Remark 2.1. *It is known that entropy solutions are continuous in (t,u) for all but at most a countable number of shock lines (t,u) (see [13, Chapter 2] for example). Therefore there exists a version ρ solution to (2.5) which is continuous at $(t,0)$ for all but a countable number of t. Then it is easy to check*

$$\int_{\mathbb{R}_+} \left\{ \rho(t,u) - \rho_0(u) \right\} du = \int_0^t (2p-1)\Phi(\rho(s,0))\, ds$$

so that in particular

$$\partial_t \int_{\mathbb{R}_+} \big\{\rho(t,u)-\rho_0(u)\big\}\,du = (2p-1)\Phi(\rho(t,0))$$

and $t \mapsto \Phi(\rho(t,0))$ *is almost everywhere continuous on* $\mathbb{R}_+$.

In the cases where $\beta \geqslant 0$, we will need to separate both parts of the real line because of the boundary condition at the origin.

Definition 2.2 (Hydrodynamic equation on $\mathbb{R}^+$ for $\beta = 0$)**.** *Assume that* $\rho \mapsto \Phi(\rho)$ *is a smooth non-decreasing function. Let* $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ *be an almost-everywhere continuous function. We say that* $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ *is an entropy solution to*

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, & t>0, u>0, \\ \rho(t,0) = \varrho(t), & t>0, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}_+, \end{cases} \tag{2.7}$$

if

- *there exists* $M>0$ *such that for any non-negative test function* $H \in C_K^{1,1}((0,\infty)\times\mathbb{R})$ *and any constant* $c \geqslant 0$*,*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \big\{\partial_t H(t,u)(\rho(t,u)-c)^\pm + (2p-1)\partial_u H(t,u)(\Phi(\rho(t,u))-\Phi(c))^\pm\big\}\,du\,dt \\ &\qquad + M\int_0^\infty H(t,0)(\varrho(t)-c)^\pm\,dt \geqslant 0, \end{aligned} \tag{2.8}$$

  where for $r \in \mathbb{R}$, $r^+ = \max\{r,0\}$ *and* $r^- = \max\{-r,0\}$*;*
- *for every* $A>0$

$$\lim_{t\to 0}\int_0^A \big|\rho(t,u)-\rho_0(u)\big|\,du = 0.$$

Definition 2.3 (Hydrodynamic equation on $\mathbb{R}^+$ for $\beta > 0$)**.** *Assume that* $\rho \mapsto \Phi(\rho)$ *is a smooth non-decreasing function. Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be an almost-everywhere continuous function. We say that* $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ *is an entropy solution to*

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, & t>0, u>0, \\ \partial_t \int_{\mathbb{R}_+} \rho(t,u)du = f(t), & t>0, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}_+, \end{cases} \tag{2.9}$$

if

- *for any non-negative test function* $H \in C_K^{1,1}((0,\infty)\times(0,\infty))$ *and any constant* $c \geqslant 0$*,*

$$\int_0^\infty \int_0^\infty \big\{\partial_t H(t,u)\big|\rho(t,u)-c\big| + (2p-1)\partial_u H(t,u)\big|\Phi(\rho(t,u))-\Phi(c)\big|\big\}\,du\,dt \geqslant 0; \tag{2.10}$$

- *for every* $A>0$ *and every* $T>0$*,*

$$\lim_{t\to 0}\int_0^A \big|\rho(t,u)-\rho_0(u)\big|\,du = 0, \tag{2.11}$$

$$\lim_{u\to 0}\int_0^T \big|(2p-1)\Phi(\rho(t,u))-f(t)\big|\,dt = 0. \tag{2.12}$$

Remark 2.2 (Uniqueness of the entropy solution)**.** *By Kružkov's uniqueness Theorem* [10, Theorem A.2.5.3]*, the entropy solution to* (2.5) *is unique. We refer the readers to* [2, Theorem 2.1] *for the uniqueness of the entropy solution to* (2.7)*, and to* [12, Theorem A.1] *for* (2.9)*.*

We adress in Appendix B the equivalence of Definitions 2.2 and 2.3. We are now ready to state the main result of this paper.

2.4. **Statements of main results.** Fix an initial density profile ρ_0 on $\mathbb{R}$, we assume that ρ_0 is bounded and Riemann-integrable on any finite segment of $\mathbb{R}$, and we define

$$\mu_N(d\omega) = \bigotimes_{x\in\mathbb{Z}} \mathscr{P}_{\rho_0(x/N)}(d\omega_x)$$

the product measure with marginal at site x given by the Poisson distribution $\mathscr{P}_{\rho_0(x/N)}$ on $\mathbb{N}$, with parameter $\rho_0(x/N)$.

We consider the process $(\omega(t))_{t\geqslant 0}$ in the hyperbolic scaling, *i.e.* driven by the *accelerated* generator $N\mathscr{L}_N$, and with initial distribution given by μ_N on Ω. Denote by $\mathbb{P}_{\mu_N}$ the distribution of this process $(\omega(t))_{t\geqslant 0}$ on Ω, and $\mathbb{E}_{\mu_N}$ the corresponding expectation. We are now ready to state our main result.

Theorem 2.3 (Hydrodynamic limit)**.** *For any compactly supported and continuous function $H:\mathbb{R}\to\mathbb{R}$, for any $t\geqslant 0$ and for any $\delta>0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\left|\frac{1}{N}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)\omega_x(t) - \int_{\mathbb{R}} H(u)\rho(t,u)\,du\right| > \delta\right) = 0,$$

where

(i) if $\beta<0$, then $\rho(t,u)$ is the unique entropy solution to (2.5)*;*

(ii) if $\beta=0$, then $\rho(t,u)=\rho_L(t,u)\mathbb{1}\{u<0\}+\rho_R(t,u)\mathbb{1}\{u\geqslant 0\}$, where ρ_L is the unique entropy solution to (2.5) *and ρ_R is the unique entropy solution to* (2.7) *with*

$$\varrho(t) = R\left(\frac{2p-1}{2p-1+\alpha}\,\Phi\big(\rho_L(t,0)\big)\right), \tag{2.13}$$

where the function R is the inverse of Φ (recall (2.4)*);*

(iii) if $\beta>0$, then $\rho(t,u)=\rho_L(t,u)\mathbb{1}\{u<0\}+\rho_R(t,u)\mathbb{1}\{u\geqslant 0\}$, where ρ_L is the unique entropy solution to (2.5) *and ρ_R is the unique entropy solution to* (2.9) *with $f(t)=0$.*

Remark 2.4. *Note that from Remark 2.1, ϱ defined in* (2.13) *is almost everywhere continuous, so that ρ_R is well defined, as given in Definition 2.2.*

Remark 2.5 (Linear case)**.** *If we assume $g(k)=k$ for all $k\geqslant 0$, namely the* linear case*, then the solution can be explicitly computed. To that aim, let us introduce $\widetilde{\alpha}=\widetilde{\alpha}(\beta)$ as*

$$\widetilde{\alpha} = \begin{cases} 1 & \text{if } \beta>0,\\ \alpha/(\alpha+2p-1) & \text{if } \beta=0,\\ 0 & \text{if } \beta<0. \end{cases} \tag{2.14}$$

Then, $\rho(t,u)$ is a weak solution to the formal equation

$$\begin{cases} \partial_t\rho(t,u)+(2p-1)\partial_u\rho(t,u) = -\widetilde{\alpha}\rho(t,0)\delta_0(u), & t>0, u\in\mathbb{R},\\ \rho(0,\cdot)=\rho_0, & u\in\mathbb{R}, \end{cases}$$

and is given, for any $(t,u)\in\mathbb{R}_+\times\mathbb{R}$, by:

$$\rho(t,u) = \big(1-\widetilde{\alpha}\,\mathbb{1}\{0\leqslant u<(2p-1)t\}\big)\,\rho_0(u-(2p-1)t). \tag{2.15}$$

We provide in Appendix A an alternative (short) proof of Theorem 2.3 in the linear case, using duality tools.

The rest of the paper is devoted to the proof of Theorem 2.3.

3. Preliminary results

In this section we give some preliminary results which will be crucial to prove Theorem 2.3. We start in Section 3.1 by defining the invariant measures for the zero-range process with destruction at the origin. Then, in Section 3.2 we introduce the notion of *second-class particles*. In Section 3.3 we prove an entropy inequality at the microscopic level using coupling techniques.

3.1. **Invariant measures for the process with destruction.** For a function $m : \mathbb{Z} \to \mathbb{R}_+$, let $\overline{\nu}_{m(\cdot)}$ be the product probability measure on Ω with marginals given by

$$\overline{\nu}_{m(\cdot)}(\omega_x = k) = \overline{\nu}_{m(x)}(\omega_x = k), \quad x \in \mathbb{Z}, k \in \mathbb{N}. \tag{3.1}$$

In order for $\overline{\nu}_{m(\cdot)}$ to be invariant for the whole process generated by $\mathscr{L}_N$, the function m needs to solve some discrete system, as stated by the following lemma:

Lemma 3.1. *Assume that $m : \mathbb{Z} \to \mathbb{R}_+$ is solution to*

$$\begin{cases} pm_{x-1} + (1-p)m_{x+1} - m_x = 0 & \text{for } x \neq 0, \\ pm_{-1} + (1-p)m_1 - (1+\alpha N^\beta)m_0 = 0. \end{cases} \tag{3.2}$$

Then the product measure $\overline{\nu}_{m(\cdot)}$ is invariant for the dynamics generated by $\mathscr{L}_N$.

Remark 3.2. (i) *If $p = 1$, namely the* totally asymmetric case*, the solution to* (3.2) *is*

$$m_x = \begin{cases} m_-, & \text{if } x \leqslant -1, \\ m_+, & \text{if } x \geqslant 0, \end{cases}$$

where m_-, m_+ are non-negative and satisfy $m_- = (1+\alpha N^\beta)m_+$.
(ii) *If $\frac{1}{2} < p < 1$, then any solution to* (3.2) *has the following form:*

$$m_x = \begin{cases} c_1 \frac{p^x}{(1-p)^x} + c_2 & \text{if } x \leqslant 0, \\ c_3 \frac{p^x}{(1-p)^x} + c_4 & \text{if } x \geqslant 0, \end{cases}$$

where $\{c_i\}_{1 \leqslant i \leqslant 4}$ are real numbers such that $m_x \geqslant 0$ for all $x \in \mathbb{Z}$, and moreover they satisfy

$$\begin{cases} c_1 + c_2 = c_3 + c_4, \\ c_1(1-p) + c_2 p + c_3 p + c_4(1-p) = (1+\alpha N^\beta)(c_1 + c_2). \end{cases}$$

Solving the above equations, we have

$$\begin{cases} c_3 = c_1 + \frac{\alpha N^\beta}{(2p-1)}(c_1 + c_2), \\ c_4 = c_2 - \frac{\alpha N^\beta}{(2p-1)}(c_1 + c_2). \end{cases}$$

Proof of Lemma 3.1. We need to show that for any local function f such that $\int f \, d\overline{\nu}_{m(\cdot)} = 0$, we have

$$\int \mathscr{L}_N f(\omega) \overline{\nu}_{m(\cdot)}(d\omega) = 0.$$

Direct calculations show that the left hand side equals

$$\int \sum_{x \in \mathbb{Z}} \Big\{ pg(\omega_x)\big(f(\omega^{x,x+1}) - f(\omega)\big) + (1-p)g(\omega_{x+1})\big(f(\omega^{x+1,x}) - f(\omega)\big) \Big\} \overline{\nu}_{m(\cdot)}(d\omega) \\ + \int \alpha N^\beta g(\omega_0)\big(f(\omega^{(0)}) - f(\omega)\big) \overline{\nu}_{m(\cdot)}(d\omega). \tag{3.3}$$

Since, by change of variables,

$$m_{x+1}g(\omega_x)\,\overline{\nu}_{m(\cdot)}(d\omega) = m_x g(\omega_{x+1}+1)\,\overline{\nu}_{m(\cdot)}(d\omega^{x,x+1}),$$
$$m_x g(\omega_{x+1})\,\overline{\nu}_{m(\cdot)}(d\omega) = m_{x+1} g(\omega_x+1)\,\overline{\nu}_{m(\cdot)}(d\omega^{x+1,x}),$$

making the transformations $\omega^{x,x+1} \mapsto \omega$ and $\omega^{x+1,x} \mapsto \omega$ respectively, we rewrite the first line of (3.3) as

$$\begin{aligned}\int \sum_{x\in\mathbb{Z}} \Big\{ \frac{(1-p)m_{x+1}-pm_x}{m_x} g(\omega_x) - \frac{(1-p)m_{x+1}-pm_x}{m_{x+1}} g(\omega_{x+1}) \Big\} f(\omega)\,\overline{\nu}_{m(\cdot)}(d\omega)\\ = \int \sum_{x\in\mathbb{Z}} \frac{(1-p)m_{x+1}+pm_{x-1}-m_x}{m_x} g(\omega_x) f(\omega)\,\overline{\nu}_{m(\cdot)}(d\omega)\\ = \int \alpha N^{\beta} g(\omega_0) f(\omega)\,\overline{\nu}_{m(\cdot)}(d\omega).\end{aligned}$$

In the first identity above we have reindexed the second sum, whereas in the second identity, we have used that m is a solution to (3.2). Similarly, using the change of variable

$$g(\omega_0)\,\overline{\nu}_{m(\cdot)}(d\omega) = m_0\,\overline{\nu}_{m(\cdot)}(d\omega^{(0)}),$$

and using the fact that f has mean 0 w.r.t. $\overline{\nu}_{m(\cdot)}$, the second line of (3.3) equals

$$\int \alpha N^{\beta}\Big(m_0 - g(\omega_0)\Big) f(\omega)\,\overline{\nu}_{m(\cdot)}(d\omega) = -\int \alpha N^{\beta} g(\omega_0) f(\omega)\,\overline{\nu}_{m(\cdot)}(d\omega).$$

Therefore (3.3) vanishes. This proves the result. □

3.2. **Second-class particles.** We now consider an auxiliary process $(\zeta(t))_{t\geqslant 0}$ composed of *second-class particles*. Instead of killing a particle at the origin, we turn the killed particle into a *second-class particle*, as explained below. The process $(\omega(t), \zeta(t))_{t\geqslant 0}$ evolves as follows:

- at site $x \neq 0$, an ω-particle jumps at rate $Ng(\omega_x)$, and a ζ-particle jumps at rate
$$N(g(\omega_x+\zeta_x) - g(\omega_x))\,;$$
- when a particle jumps, it jumps to the right with probability p and to the left w.p. $1-p$;
- at site $x = 0$, an ω-particle is transformed into a ζ-particle at rate $\alpha N^{1+\beta} g(\omega_0)$.

It is easy to see that the process $(\omega(t)+\zeta(t))_{t\geqslant 0}$ evolves as the usual zero-range process according to the generator $N\mathscr{L}_N^{\mathrm{ZR}}$, and $(\omega(t))_{t\geqslant 0}$ according to the generator $N\mathscr{L}_N$. Denote by $\overline{\mathbb{P}}_{\mu_N}$ the law of the process $(\omega(t), \zeta(t))$, with no second-class particle at the initial time, *i.e.* $(\omega(0), \zeta(0)) \sim \mu_N \otimes \delta_\emptyset$ where $\delta_\emptyset$ is the Dirac measure on the empty configuration. Denote by $\overline{\mathbb{E}}_{\mu_N}$ the corresponding expectation.

We first state a lemma to bound the number of second-class particles up to time t. The following lemma is sufficient to prove the hydrodynamic limit in the case $\beta < 0$.

Lemma 3.3. *Assume that the initial distribution μ_N is stochastically dominated by $\nu_{\rho^\star}$ for some density $\rho^\star \in (0,+\infty)$. Let $K_t := \sum_{x\in\mathbb{Z}} \zeta_x(t)$ be the number of second-class particles up to time t. Then there exists a finite constant $C > 0$ independent of N such that*

$$\overline{\mathbb{E}}_{\mu_N}[K_t] \leqslant Ct \min\{\alpha N^{1+\beta}, N\}.$$

Proof. Since $K_t - \int_0^t \alpha N^{1+\beta} g(\omega_0(s))\,ds$ is a mean zero martingale and since the process $(\omega(t))_{t\geqslant 0}$ is attractive and stochastically dominated by $\nu_{\rho^\star}$,

$$\overline{\mathbb{E}}_{\mu_N}[K_t] \leqslant \alpha N^{1+\beta} t\, \mathrm{E}_{\nu_{\rho^\star}}[g(\omega_0)] \leqslant a_0 \rho^\star \alpha N^{1+\beta} t.$$

where a_0 is the Lipschitz constant of g, introduced in (2.1), which satisfies $g(k) \leqslant a_0 k$. We now show that $\overline{\mathbb{E}}_{\mu_N}[K_t] \leqslant CNt$ for some constant $C > 0$. Observe that K_t is bounded by the number of particles visiting the origin up to time t. Let Y_t be a homogeneous Poisson point process on $\mathbb{R}$ with intensity

a_0N. Then the displacement of a typical first-class particle up to time t is bounded stochastically by Y_t. Dividing $\mathbb{Z}$ into the unions of $[ja_0Nt,(j+1)a_0Nt]$, $j\in\mathbb{Z}$, we have

$$\overline{\mathbb{E}}_{\mu_N}[K_t]\leqslant\sum_{j=3}^{\infty}2a_0\rho^\star Nt\overline{\mathbb{P}}_{\mu_N}[Y_t>ja_0Nt]+6a_0\rho^\star Nt.$$

Above, the first term on the right hand side comes from the total number of particles which are initially at $\mathbb{Z}\backslash[-3a_0Nt,3a_0Nt]$ and visit the origin before time t, and the second term from the total initial number of particles in the interval $[-3a_0Nt,3a_0Nt]$. Since $\overline{\mathbb{P}}_{\mu_N}[Y_t>ja_0Nt]\leqslant\exp\{-a_0(j-2)Nt\}$, the first term on the right hand side vanishes as $N\to\infty$. This is enough to complete the proof. □

The following lemma will be used to identify the limiting density profile to the left of the origin in the case $\beta\geqslant 0$.

Lemma 3.4. *For any $t\geqslant 0$ and for any $\beta\in\mathbb{R}$,*

$$\lim_{N\to\infty}\overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{x\leqslant 0}\zeta_x(t)\right]=0. \tag{3.4}$$

Proof. We only need to prove

$$\lim_{N\to\infty}\overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{x\leqslant-\sqrt{N}}\zeta_x(t)\right]=0$$

since by attractiveness of the process, we have

$$\overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{-\sqrt{N}\leqslant x\leqslant 0}\zeta_x(t)\right]\leqslant\rho^\star N^{-1/2}.$$

Recall that K_t is the number of second-class particles at time t. Denote the positions of the second-class particles by $X_i(t)$, for $1\leqslant i\leqslant K_t$. Then

$$\frac{1}{N}\sum_{x\leqslant-\sqrt{N}}\zeta_x(t)=\frac{1}{N}\sum_{i=1}^{K_t}\mathbb{1}\{X_i(t)\leqslant-\sqrt{N}\}.$$

Let S_k be a discrete time simple random walk which jumps to the right with probability p and to the left with probability $1-p$. Then

$$\begin{aligned}\overline{\mathbb{P}}_{\mu_N}\big(X_1(t)\leqslant-\sqrt{N}\big)&=\sum_{j=0}^{\infty}\overline{\mathbb{P}}_{\mu_N}(X_1(t)\text{ jumps }j\text{ times up to time }t)\times\overline{\mathbb{P}}_{\mu_N}(S_j\leqslant-\sqrt{N})\\&\leqslant\sup_{j\geqslant\sqrt{N}}\overline{\mathbb{P}}_{\mu_N}(S_j\leqslant-\sqrt{N})\leqslant\sup_{j\geqslant\sqrt{N}}\frac{j}{((2p-1)j+\sqrt{N})^2}\leqslant CN^{-1/2}\end{aligned}$$

for some finite constant C independent of N. Together with Lemma 3.3, we have

$$\overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{i=1}^{K_t}\mathbb{1}\{X_i(t)\leqslant-\sqrt{N}\}\right]\leqslant CN^{-3/2}\overline{\mathbb{E}}_{\mu_N}[K_t]\leqslant CN^{-1/2}.$$

This completes the proof. □

Since the process $(\omega(t)+\zeta(t))_{t\geqslant 0}$ is the usual asymmetric attractive zero-range process, we conclude this paragraph with a well-known result concerning the hydrodynamic limit of $(\omega(t)+\zeta(t))_{t\geqslant 0}$, which has been proved in [15].

Theorem 3.5 ([15, Theorem 1.3])**.** *Suppose the initial density profile ρ_0 is bounded. For any compactly supported and continuous function $H:\mathbb{R}\to\mathbb{R}$, for any $t\geqslant 0$, $\delta>0$, $\beta\in\mathbb{R}$,*

$$\lim_{N\to\infty}\overline{\mathbb{P}}_{\mu_N}\left(\left|\frac{1}{N}\sum_{x\in\mathbb{Z}}H\left(\tfrac{x}{N}\right)(\omega_x(t)+\zeta_x(t))-\int_{\mathbb{R}}H(u)\rho(t,u)\,du\right|>\delta\right)=0,$$

where $\rho(t,u)$ is the unique entropy solution to (2.5).

3.3. **Microscopic entropy inequality.** In this subsection, we shall couple two copies of our Markov process, denoted by $\omega(t)$ and $\varpi(t)$. Recall that the process is supposed to be attractive (Assumption H1). If both processes have the same number of particles on a given site, a particle at this site will perform at the normal rate a particle jump/destruction, identical in both processes. If the processes do not have the same number of particles at a given site, at the minimal rate $g(\omega_x)\wedge g(\varpi_x)$, a particle will perform the same jump/destruction in both processes. To make up for the loss of jump rate, the process with the highest number of particles will make a particle jump/destruction while the other one will remain unchanged at a compensated rate $|g(\varpi_x)-g(\omega_x)|$. The generator of the coupled process is given by its action on local Lipschitz functions $f:\Omega\times\Omega\to\mathbb{R}$ by

$$\begin{aligned}\widetilde{\mathscr{L}_N}f(\omega,\varpi)=&\sum_{x\in\mathbb{Z}}\big(g(\omega_x)\wedge g(\varpi_x)\big)\left\{pf(\omega^{x,x+1},\varpi^{x,x+1})+(1-p)f(\omega^{x,x-1},\varpi^{x,x-1})-f(\omega,\varpi)\right\}\\&+\sum_{x\in\mathbb{Z}}\big(g(\omega_x)-g(\omega_x)\wedge g(\varpi_x)\big)\left\{pf(\omega^{x,x+1},\varpi)+(1-p)f(\omega^{x,x-1},\varpi)-f(\omega,\varpi)\right\}\\&+\sum_{x\in\mathbb{Z}}\big(g(\varpi_x)-g(\omega_x)\wedge g(\varpi_x)\big)\left\{pf(\omega,\varpi^{x,x+1})+(1-p)f(\omega,\varpi^{x,x-1})-f(\omega,\varpi)\right\}\\&+\alpha N^{\beta}\big(g(\omega_0)\wedge g(\varpi_0)\big)\left\{f(\omega^{(0)},\varpi^{(0)})-f(\omega,\varpi)\right\}\\&+\alpha N^{\beta}\big(g(\omega_0)-g(\omega_0)\wedge g(\varpi_0)\big)\left\{f(\omega^{(0)},\varpi)-f(\omega,\varpi)\right\}\\&+\alpha N^{\beta}\big(g(\varpi_0)-g(\omega_0)\wedge g(\varpi_0)\big)\left\{f(\omega,\varpi^{(0)})-f(\omega,\varpi)\right\}.\end{aligned}$$

One can easily check that under this coupling, if the process (ω,ϖ) evolves according to $\widetilde{\mathscr{L}_N}$, then both marginal processes $\omega(t)$ and $\varpi(t)$ evolve according to $\mathscr{L}_N$. For a probability measure $\widetilde{\mu}_N$ on $\Omega\times\Omega$, let $\widetilde{\mathbb{P}}_{\widetilde{\mu}_N}$ be the distribution on $D([0,T],\Omega\times\Omega)$ of the process $(\omega(t),\varpi(t))$ with generator $N\widetilde{\mathscr{L}_N}$ and with initial distribution $\widetilde{\mu}_N$.

For $x\in\mathbb{Z}$ and a positive integer ℓ, denote by $\omega_x^\ell=(2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}\omega_y$ the average number of particles in a box of size $2\ell+1$ around x. Given the coupling above, we now state a microscopic version of entropy inequality.

Lemma 3.6 (Microscopic entropy inequality)**.** *Fix $\beta<1$. Let $\widetilde{\mu}_N$ be a probability measure on $\Omega\times\Omega$ with both marginals stochastically dominated by $\nu_{\rho^\star}$ for some $\rho^\star>0$. Then for any $\varepsilon>0$ and for any non-negative smooth function H with compact support in $(0,\infty)\times\mathbb{R}$,*

$$\begin{aligned}\lim_{\ell\to\infty}\liminf_{N\to\infty}\widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Bigg[\int_0^\infty N^{-1}\sum_{x\in\mathbb{Z}}\Big\{&\partial_tH(t,\tfrac{x}{N})\left|\omega_x^\ell(t)-\varpi_x^\ell(t)\right|\\&+(2p-1)\partial_uH(t,\tfrac{x}{N})\left|\Phi(\omega_x^\ell(t))-\Phi(\varpi_x^\ell(t))\right|\Big\}dt\geqslant-\varepsilon\Bigg]=1.\end{aligned}\tag{3.5}$$

Remark 3.7. *Using the same argument, we could prove*

$$\begin{aligned}\lim_{\ell\to\infty}\liminf_{N\to\infty}\widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Bigg[\int_0^\infty N^{-1}\sum_{x\in\mathbb{Z}}\Big\{&\partial_tH(t,\tfrac{x}{N})\big(\omega_x^\ell(t)-\varpi_x^\ell(t)\big)^{\pm}\\&+(2p-1)\partial_uH(t,\tfrac{x}{N})\left(\Phi(\omega_x^\ell(t))-\Phi(\varpi_x^\ell(t))\right)^{\pm}\Big\}dt\geqslant-\varepsilon\Bigg]=1.\end{aligned}\tag{3.6}$$

We will use (3.5) *in the case* $\beta > 0$*, and* (3.6) *in the case* $\beta = 0$*. Note that the case* $\beta < 0$ *is a direct consequence of the hydrodynamic limit stated in Theorem 3.5, because then the particle destruction at the origin is too weak to affect the macroscopic density.*

Proof of Lemma 3.6. We follow the strategy presented in [10, Chapter 8] and [12, Proposition 5.1]. For this reason we only sketch the proof here.

Step 1. The first step is to prove that in the limit $N \uparrow \infty$ the configurations ω and ϖ are ultimately locally ordered. To be precise, for any $A > 0$, any $t > 0$, and any $y \in \mathbb{Z}$,

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}_N}\Big[\int_0^t N^{-1} \sum_{|x|\leqslant AN} G_{x,x+y}(\omega(s),\varpi(s))\,ds\Big] = 0, \tag{3.7}$$

where $G_{x,x'}(\omega,\varpi) = \mathbb{1}\{\omega_x < \varpi_x, \omega_{x'} > \varpi_{x'}\} + \mathbb{1}\{\omega_x > \varpi_x, \omega_{x'} < \varpi_{x'}\}$. The idea is to investigate the mean zero martingale defined as

$$\frac{1}{N}\sum_{|x|\leqslant AN} |\omega_x(t) - \varpi_x(t)| - \frac{1}{N}\sum_{|x|\leqslant AN} |\omega_x(0) - \varpi_x(0)| - \int_0^t \sum_{|x|\leqslant AN} \widetilde{\mathscr{L}_N}\big(|\omega_x(s) - \varpi_x(s)|\big)\,ds.$$

By calculating directly the term involving the generator $\widetilde{\mathscr{L}_N}$, we show that

$$\begin{aligned}\lim_{N\to\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}_N}\Big[\int_0^t N^{-1} \sum_{|x|\leqslant AN} \Big\{ & pG_{x,x-1}(\omega(s),\varpi(s))\big|g(\omega_{x-1}(s)) - g(\varpi_{x-1}(s))\big| \\ & + (1-p)G_{x,x+1}(\omega(s),\varpi(s))\big|g(\omega_{x+1}(s)) - g(\varpi_{x+1}(s))\big|\Big\} + \alpha N^{\beta-1}\big|g(\omega_0(s)) - g(\varpi_0(s))\big|\,ds\Big] = 0.\end{aligned}$$

By an induction argument as in the proof of [10, Lemma 8.2.2], one is able to prove (3.7) from the above formula.

Step 2. For any non-negative smooth function H with compact support in $(0,\infty)\times\mathbb{R}$, consider the mean zero martingale

$$\mathfrak{M}_t^N := \frac{1}{N}\sum_{x\in\mathbb{Z}} H(t,\tfrac{x}{N})|\omega_x(t) - \varpi_x(t)| - \int_0^t (\partial_s + N\widetilde{\mathscr{L}_N})\Big(\frac{1}{N}\sum_{x\in\mathbb{Z}} H(s,\tfrac{x}{N})|\omega_x(s) - \varpi_x(s)|\Big)\,ds. \tag{3.8}$$

Direct calculations show that if $\beta < 1$, then $\lim_{N\to\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}_N}[\sup_{0\leqslant s\leqslant t}(\mathfrak{M}_s^N)^2] = 0$. Since H has compact support in $(0,\infty)\times\mathbb{R}$, we can take t large enough for the first term on the right hand side to vanish. By direct calculations, we bound the term $\sum_{x\in\mathbb{Z}} H(s,\frac{x}{N})\widetilde{\mathscr{L}_N}|\omega_x - \varpi_x|$ from above by

$$\sum_{x\in\mathbb{Z}} H(s,\tfrac{x}{N})\big(p|g(\omega_{x-1}) - g(\varpi_{x-1})| + (1-p)|g(\omega_{x+1}) - g(\varpi_{x+1})| - |g(\omega_x) - g(\varpi_x)|\big).$$

An integration by parts allows us to write the first term above as

$$\frac{1}{N}\sum_{x\in\mathbb{Z}} (2p-1)\partial_u H(s,\tfrac{x}{N})|g(\omega_x) - g(\varpi_x)| + o_N(1).$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{N\to\infty} \widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big[\int_0^\infty N^{-1}\sum_{x\in\mathbb{Z}} \Big\{ & \partial_t H(t,\tfrac{x}{N})\big|\omega_x(t) - \varpi_x(t)\big| \\ & + (2p-1)\partial_u H(t,\tfrac{x}{N})\,\big|g(\omega_x(t)) - g(\varpi_x(t))\big|\Big\}dt \geqslant -\varepsilon\Big] = 1.\end{aligned} \tag{3.9}$$

Step 3. Due to the smoothness and compactness of H, in (3.9) we can replace $|\omega_x - \varpi_x|$ and $|g(\omega_x) - g(\varpi_x)|$ with their spatial averages over sites in a box of size $2\ell + 1$ around x. Moreover, by

(3.7) and repeating the proof of [10, Lemma 8.2.3], we can replace

$$(2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}|g(\omega_y(t))-g(\varpi_y(t))| \quad \text{resp.} \quad (2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}|\omega_y(t)-\varpi_y(t)|$$

i.e. the average of the absolute value, by

$$\left|(2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}g(\omega_y(t))-g(\varpi_y(t))\right| \quad \text{resp.} \quad |\omega_x^\ell(t)-\varpi_x^\ell(t)|$$

i.e. the absolute value of the average. Therefore, we have

$$\begin{aligned}\lim_{N\to\infty}\widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Bigg[\int_0^\infty \frac{1}{N}\sum_{x\in\mathbb{Z}}\Big\{\partial_t H(t,\tfrac{x}{N})\big|\omega_x^\ell(t)-\varpi_x^\ell(t)\big|+(2p-1)\partial_u H(t,\tfrac{x}{N})\\ \times\Big|(2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}g(\omega_y(t))-(2\ell+1)^{-1}\sum_{|y-x|\leqslant\ell}g(\varpi_y(t))\Big|\Big\}dt\geqslant-\varepsilon\Bigg]=1.\end{aligned} \tag{3.10}$$

We complete the proof by using the one-block estimate as stated in Lemma 3.8 below. □

Lemma 3.8 (One-block estimate)**.** *If $\beta<1$, then for any $\varepsilon>0$, any $A>0$ and any $t>0$,*

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big(\int_0^t\frac{1}{N}\sum_{|x|\leqslant AN}\tau_x V_\ell(\omega(s))\,ds\geqslant\varepsilon\Big)=0,$$

where

$$V_\ell(\eta)=\Big|(2\ell+1)^{-1}\sum_{|y|\leqslant\ell}g(\omega_y)-\Phi(\omega_0^\ell)\Big|.$$

The same result obviously holds with $\omega(t)$ replaced by $\varpi(t)$.

The proof of the above lemma is quite standard, we omit the proof here. We refer the readers to [12, Lemma 5.4] for its detailed implementation.

4. Proof of Theorem 2.3

In this section we give the proof of Theorem 2.3. We distinguish four cases, namely $\beta<0$, $\beta=0$, $0<\beta<1$ and $\beta\geqslant 1$.

4.1. **The case $\beta<0$.** By Lemma 3.3, if $\beta<0$, then for any compactly supported smooth function H,

$$\overline{\mathbb{E}}_N\Big[\frac{1}{N}\sum_{x\in\mathbb{Z}}H\big(\tfrac{x}{N}\big)\zeta_x(t)\Big]\leqslant\frac{||H||_\infty}{N}\overline{\mathbb{E}}_N[K_t]\leqslant C||H||_\infty\alpha t N^\beta\xrightarrow[N\to\infty]{}0,$$

where ζ is the process of second-class particles introduced in Section 3.2. In particular, both processes $\omega(t)$ and $\omega(t)+\zeta(t)$ have the same macroscopic limit. Theorem 2.3 therefore follows from Theorem 3.5.

4.2. **The case $\beta=0$.** We now prove Theorem 2.3 in the case $\beta=0$. Several results proved in this subsection will also be used in the case $\beta<1$, we will make it explicit in the statements. We first introduce some notation. Denote by $\mathscr{M}_+(\mathbb{R})$ the space of positive Radon measures on $\mathbb{R}$ endowed with the vague topology. For each configuration $\omega\in\Omega$, define the empirical measure $\alpha^N\in\mathscr{M}_+(\mathbb{R})$ as

$$\alpha^N(\omega)=\frac{1}{N}\sum_{x\in\mathbb{Z}}\omega_x\delta_{x/N},$$

where δ_u is the Dirac measure at $u\in\mathbb{R}$, and denote $\alpha_t^N=\alpha^N(\omega(t))$. Fix a time $T>0$. Let $\mathbb{Q}_N$ be the distribution on the path space $D\big([0,T],\mathscr{M}_+(\mathbb{R})\big)$ of the path (α_t^N), where α_0^N is distributed according to the pushforward distribution $\mu_N\circ(\alpha^N)^{-1}$.

Lemma 4.1 (Tightness)**.** *Suppose the initial measure μ_N is stochastically dominated by $\nu_{\rho^\star}$ for some $\rho^\star > 0$. If $\beta < 1$, then the sequence $(\mathbb{Q}_N)_{N\in\mathbb{N}}$ is tight. Moreover, all limit points are concentrated on weakly continuous paths which are absolutely continuous with respect to the Lebesgue measure and with density bounded by $\rho^\star$.*

Proof of Lemma 4.1. The proof of tightness is quite standard, see [15, Lemma 4.1] for example. To prove Lemma 4.1, it is enough to show that for any $H \in C_c(\mathbb{R})$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\sup_{|t-s|\leqslant\delta}\Big|N^{-1}\sum_{x\in\mathbb{Z}}H\big(\tfrac{x}{N}\big)\big(\omega_x(t)-\omega_x(s)\big)\Big|\Big]=0.$$

The only difference with [15] is that Dynkin's formula has an extra term coming from the destruction part of the dynamics, so that we only detail how the Glauber contribution is controlled, and refer to [15] for the other estimates. For any function f,

$$f(\omega(t)) = f(\omega(0)) + \int_0^t N\mathscr{L}_N f(\omega(s))ds + \mathfrak{M}_t, \tag{4.1}$$

where $\mathfrak{M}_t$ is a martingale whose quadratic variation is given by

$$\mathbb{E}_{\mu_N}\big[\mathfrak{M}_t^2\big] = \mathbb{E}_{\mu_N}\Big[\int_0^t N\Big(\mathscr{L}_N f^2 - 2f\mathscr{L}_N f\Big)(\omega(s))ds\Big]. \tag{4.2}$$

Choosing $f(\omega) = N^{-1}\sum H(\frac{x}{N})\omega_x$, in addition to the proof performed in [15] we have two extra terms to control:

(i) first, in (4.1), the contribution of the Glauber dynamics is given by

$$\begin{aligned}\int_0^t N\mathscr{L}_N^{\alpha,\beta} f(\omega(s))ds &= \int_0^t H(0)\mathscr{L}_N^{\alpha,\beta}\omega_0(s)\,ds\\ &= \frac{H(0)}{N}(\omega_0(t)-\omega_0(0)) - H(0)\int_0^t \mathscr{L}_N^{\mathrm{ZR}}\omega_0(s)\,ds + \mathfrak{N}_{0,t},\end{aligned}$$

where $\mathfrak{N}_{0,t}$ is a martingale whose quadratic variation has order $\max\{N^{\beta-1}, N^{-1}\}$. All terms but the third term on the right hand side vanish in $L^2(\mathbb{P}_N)$. To control the third term, since

$$\mathscr{L}_N^{\mathrm{ZR}}\omega_0 = pg(\omega_{-1}) + (1-p)g(\omega_1) - g(\omega_0),$$

and the function g is Lipschitz, it remains to prove

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\sup_{|t-s|\leqslant\delta}\Big|\int_s^t \omega_x(\tau)d\tau\Big|\Big]=0. \tag{4.3}$$

for any $x\in\mathbb{Z}$. Using the Cauchy-Schwarz inequality twice, the expectation above is bounded by

$$\delta^{1/2}\times\Big(\mathbb{E}_{\mu_N}\Big[\int_0^T \omega_x^2(\tau)d\tau\Big]\Big)^{1/2}.$$

Since the initial measure is bounded by $\nu_{\rho^\star}$ and the process is attractive, the quantity above is uniformly bounded in N. This concludes the proof of (4.3).

(ii) second, in the integral in (4.2) appears the Glauber contribution

$$\mathbb{E}_{\mu_N}\Big[\int_0^t\Big(N\mathscr{L}_N^{\alpha,\beta}f^2 - 2Nf\mathscr{L}_N^{\alpha,\beta}f\Big)(\omega(s))ds\Big] = \mathscr{O}(N^{\beta-1})$$

which vanishes in the limit as $N\to\infty$ since $\beta<1$.

□

We now characterize the macroscopic density profile to the left of the origin.

Lemma 4.2 (Hydrodynamic limit to the left of the origin)**.** *Assume $\beta < 1$. For any continuous function H with compact support included in $(-\infty, 0]$, and for any $t \geqslant 0$, $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\left|\frac{1}{N}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)\omega_x(t) - \int_{\mathbb{R}} H(u)\rho(t,u)\,du\right| > \delta\right) = 0,$$

where $\rho(t,u) = \rho_L(t,u)$ is defined in Theorem 2.3 as the unique entropy solution to (2.5).

Proof. Recall the second-class particle's process introduced in Section 3.2. We start by proving

$$\lim_{N\to\infty} \overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{x\in\mathbb{Z}} |H\left(\tfrac{x}{N}\right)|\zeta_x(t)\right] = 0. \tag{4.4}$$

Since H has its support included in $(-\infty, 0]$, we bound

$$\overline{\mathbb{E}}_{\mu_N}\left[\frac{1}{N}\sum_{x\in\mathbb{Z}} |H\left(\tfrac{x}{N}\right)|\zeta_x(t)\right] \leqslant \|H\|_\infty \frac{1}{N}\sum_{x\leqslant 0} \overline{\mathbb{E}}_{\mu_N}[\zeta_x(t)]$$

and the right hand side vanishes from Lemma 3.4. Note that the function H is not necessarily continuous on $\mathbb{R}$, however by maximum principle (applied to the vanishing viscosity limit, see e.g. [13, Lemma 4.46, p. 73]), and since ω is stochastically dominated by an equilibrium measure $\nu_{\rho^\star}$, H can be approximated in $L^1(\mathbb{R})$ by a sequence of smooth, compactly supported functions H^ε. We can now use straightforwardly Theorem 3.5 and (4.4) to complete the proof. □

We now characterize the macroscopic density profile to the right of the origin. Let $\mathscr{M}_+(\mathbb{R}_+^2)$ be the space of positive Radon measures on $\mathbb{R}_+^2$ endowed with the vague topology. For any integer ℓ and any configuration $\omega \in \Omega$, the *Young measure* $\pi^{N,\ell}(du, d\lambda) \in \mathscr{M}_+(\mathbb{R}_+^2)$ is defined by its action on continuous function $G : \mathbb{R}_+^2 \to \mathbb{R}$ with compact support as

$$\langle \pi^{N,\ell}, G\rangle := N^{-1}\sum_{x>\ell} G\left(\tfrac{x}{N}, \omega_x^\ell\right).$$

Denote $\pi_t^{N,\ell} = \pi^{N,\ell}(\omega_t)$. The mapping $\omega_t \mapsto (\alpha_t^N, \pi_t^{N,\ell})$ induces a probability measure $\mathbb{Q}_{N,\ell}$ on the space $D([0,T], \mathscr{M}_+(\mathbb{R}) \times \mathscr{M}_+(\mathbb{R}_+^2))$. It is not hard to show that if $\beta < 1$, then the sequence of measures $\mathbb{Q}_{N,\ell}$ is tight (cf. *e.g.* [10, Lemma 8.1.2]). Denote by $\mathbb{Q}^*$ any limit point of the sequence $\mathbb{Q}_{N,\ell}$ as $N \to \infty$ then $\ell \to \infty$. To prove the hydrodynamic limit in our setting, the main step is to prove the following microscopic version of (2.8).

Lemma 4.3. *Let $\beta = 0$. Assume the initial measure μ_N is stochastically bounded by $\nu_{\rho^\star}$ for some $\rho^\star > 0$. There exists $M > 0$ such that for any non-negative test function $H \in C_K^{1,1}((0,\infty)\times\mathbb{R})$ and any constant $c \geqslant 0$, $\mathbb{Q}^*$-almost surely,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \left\{\partial_t H(t,u)(\lambda - c)^\pm + (2p-1)\partial_u H(t,u)(\Phi(\lambda) - \Phi(c))^\pm\right\}\pi_t(du, d\lambda)\,dt$$
$$+ M\int_0^\infty H(t,0)\big(\varrho(t) - c\big)^\pm\,dt \geqslant 0.$$

Proof. Fix $c \geqslant 0$. We couple two processes $(\omega(t), \varpi(t))$ according to the generator $N\widetilde{\mathscr{L}_N}$ as described in Subsection 3.3. Denote by $\widetilde{\mu}_{N,i}$, $i = 1, 2$, the i-th marginal of $\widetilde{\mu}_N$. Take the first marginal $\widetilde{\mu}_{N,1}$ of $\widetilde{\mu}_N$ to be the initial measure μ_N defined in Subsection 2.4, and the second marginal $\widetilde{\mu}_{N,2}$ to be the invariant measure of the process with generator $\mathscr{L}_N$, with the four parameters $\{c_i\}_{1\leqslant i\leqslant 4}$ as stated in Remark 3.2 given by

$$c_1 = -\frac{\alpha N^\beta}{2p-1}\Phi(c), \quad c_2 = \frac{\alpha N^\beta + 2p - 1}{2p-1}\Phi(c), \quad c_3 = 0, \quad c_4 = \Phi(c).$$

It is easy to check

$$\lim_{N\to\infty} \mathbb{E}_{\widetilde{\mu}_{N,2}}[\omega_{[Nu]}] = \begin{cases} R\big(\frac{2p-1}{\alpha+2p-1}\Phi(c)\big) & \text{if } u<0, \\ c & \text{if } u>0, \end{cases}$$

where R is the inverse function of Φ. Since the process $\varpi(t)$ is time invariant, by law of large numbers, for any $\varepsilon>0$ and for any non-negative test function H with compact support in $(0,+\infty)\times\mathbb{R}$, we have

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big[\int_0^\infty \frac{1}{N}\sum_{x\leqslant 0} H(t,\tfrac{x}{N})\big|\varpi_x^\ell(t)-R\big(\tfrac{2p-1}{\alpha+2p-1}\Phi(c)\big)\big|dt \geqslant \varepsilon\Big]=0,$$

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big[\int_0^\infty \frac{1}{N}\sum_{x\geqslant 0} H(t,\tfrac{x}{N})|\varpi_x^\ell(t)-c|dt \geqslant \varepsilon\Big]=0.$$

For any $\delta,\ \gamma>0$, define $H_{\gamma,\delta}(t,u)=\psi_\gamma * \mathbb{1}\{\cdot>-\delta\}(u)H(t,u)$, where ψ_γ is an *approximation of the identity*, so that $H_{\gamma,\delta}(t,u)$ converges as $\gamma\to 0$ to $H(t,u)\mathbb{1}\{u>-\delta\}$. Taking $H_{\gamma,\delta}$ as test function in (3.6), and letting $\gamma\to 0$, we have $\mathbb{Q}^*$-almost surely,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \Big\{\partial_t H(t,u)\big(\lambda-c\big)^\pm + (2p-1)\partial_u H(t,u)\big(\Phi(\lambda)-\Phi(c)\big)^\pm\Big\}\pi_t(du,d\lambda)\,dt \\
&+\int_0^\infty\int_{-\delta}^0 \Big\{\partial_t H(t,u)\Big(\rho_L(t,u)-R\big(\tfrac{2p-1}{\alpha+2p-1}\Phi(c)\big)\Big)^\pm + (2p-1)\partial_u H(t,u)\Big(\Phi(\rho_L(t,u))-\tfrac{2p-1}{\alpha+2p-1}\Phi(c)\Big)^\pm\Big\}\,du\,dt \\
&+\int_0^\infty (2p-1)H(t,-\delta)\Big(\Phi(\rho_L(t,-\delta))-\tfrac{\alpha+2p-1}{2p-1}\Phi(c)\Big)^\pm dt \geqslant 0.
\end{aligned}$$

Since $\rho_L(t,u)$ is bounded, the second line above is bounded from above by $C\delta$ for some finite constant C. Furthermore, because Φ is increasing and Lipschitz continuous, $|\Phi(u)-\Phi(v)|\leqslant a_0|u-v|$, we have

$$\Big(\Phi(\rho_L(t,-\delta))-\tfrac{\alpha+2p-1}{2p-1}\Phi(c)\Big)^\pm \leqslant \frac{a_0(\alpha+2p-1)}{2p-1}\Big(R\big(\tfrac{2p-1}{\alpha+2p-1}\Phi(\rho_L(t,-\delta))\big)-c\Big)^\pm.$$

We conclude the proof by letting $\delta\to 0$. □

The rest of the proof is quite standard [15, Section 5], and we only sketch the proof here. We first introduce the concept of *measure-valued entropy solutions*. Let $\pi_{t,u}(d\lambda)$ be a measurable map from $\mathbb{R}_+^2$ into the space $\mathscr{M}_+(\mathbb{R}_+)$. We say $\pi_{t,u}(d\lambda)$ is *bounded* if there exists a finite constant C such that $\pi_{t,u}(d\lambda)$ is supported in $[0,C]$ for a.e. $(t,u)\in\mathbb{R}_+^2$. We say $\pi_{t,u}(d\lambda)$ is a *measure-valued entropy solution* to (2.7) if

- there exists $M>0$ such that for any non-negative test function $H\in C_K^{1,1}((0,\infty)\times\mathbb{R})$ and any constant $c\geqslant 0$,

$$\begin{aligned}
\int_0^\infty\int_0^\infty \Big\{\partial_t H(t,u)\int_0^\infty(\lambda-c)^\pm\pi_{t,u}(d\lambda) + (2p-1)\partial_u H(t,u)\int_0^\infty(\Phi(\lambda)-\Phi(c))^\pm\pi_{t,u}(d\lambda)\Big\}\,du\,dt \\
+M\int_0^\infty H(t,0)(\varrho(t)-c)^\pm\,dt \geqslant 0; \quad (4.5)
\end{aligned}$$

- for every $A>0$

$$\lim_{t\to 0}\int_0^A\int_0^\infty |\lambda-\rho_0(u)|\pi_{t,u}(d\lambda)\,du = 0. \quad (4.6)$$

It is obvious that if $\rho(t,u)$ is an entropy solution to (2.7), then $\delta_{\rho(t,u)}(d\lambda)$ is a bounded measure-valued entropy solution. By [2, Theorem 2.1], the converse is also true: there exists a unique bounded measure-valued entropy solution $\pi_{t,u}(d\lambda)$ to (2.7), which is of the form $\pi_{t,u}(d\lambda)=\delta_{\rho(t,u)}(d\lambda)$, $\rho(t,u)$ being the unique entropy solution.

We now state a lemma concerning the properties of the limiting measure $\mathbb{Q}^*$.

Lemma 4.4. *Assume the initial measure μ_N is stochastically bounded by ν_ρ. for some $\rho^\star$. For all $\beta \in \mathbb{R}$, we have $\mathbb{Q}^*$-almost surely,*

(i) $\alpha_t(du)$ and $\pi_t(du, d\lambda)$ are both absolutely continuous with respect to the Lebesgue measure du, whose densities are denoted by $\rho(t,u)$ and $\pi_{t,u}(d\lambda)$ respectively. Moreover,

$$\int_0^\infty \lambda \pi_{t,u}(d\lambda) = \rho(t,u);$$

(ii) $\pi_{t,u}(d\lambda)$ is bounded, with support on $[0,\rho^\star]$ for almost every $(t,u) \in \mathbb{R}_+^2$.

The above lemma is an analogue of [15, Lemma 5.5]. We omit the proof here. The main ingredients are attractiveness of the process and the following observation: for any continuous function with compact support $H : \mathbb{R}_+ \to \mathbb{R}$,

$$\langle \alpha^N, H\rangle = \langle \pi^{N,\ell}, H\lambda\rangle$$

plus error terms that vanish uniformly as $N \to \infty$.

Now we are ready to prove Theorem 2.3 for $\beta = 0$.

Proof of Theorem 2.3 in the case $\beta = 0$. We lay out the main arguments, but omit some technical details, for which we refer the reader to [15, Lemma 5.2]. Fix a smooth function H with compact support, that we decompose as $H(u) = H(u)\mathbb{1}\{u \geqslant 0\} + H(u)\mathbb{1}\{u < 0\}$. The part corresponding to $H(u)\mathbb{1}\{u < 0\}$ is treated in Lemma 4.2, so that we only need to prove that for any function H which is continuous and compactly supported in $\mathbb{R}_+$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\left|\frac{1}{N}\sum_{x\geqslant 0} H\left(\tfrac{x}{N}\right)\omega_x(t) - \int_0^{+\infty} H(u)\rho(t,u)\,du\right| > \delta\right) = 0,$$

where $\rho = \rho_R$ is the unique entropy solution to (2.7) with boundary condition given by (2.13). the proof is divided into several steps.

Step 1. We first assume the initial density profile ρ_0 restricted to $\mathbb{R}_+$ is a Lipschitz continuous function with compact support, so that we can use the proof of [15, Lemma 5.6]. From the latter, it follows straightforwardly that the initial condition (4.6) holds $\mathbb{Q}^*$-almost surely. By Lemmas 4.3 and 4.4, $\pi_{t,u}(d\lambda)$ is bounded and the inequality (4.5) holds $\mathbb{Q}^*$-almost surely. This implies the limit $\mathbb{Q}^*$ is concentrated on trajectories such that $\pi_{t,u}(d\lambda)$ is a bounded measure-valued entropy solution to (2.7). Since there exists a unique bounded measure-valued entropy solution, which is given by a Dirac measure $\delta_{\rho(t,u)}(d\lambda)$, and by Lemma 4.4, $\mathbb{Q}^*$ is concentrated on trajectories such that $\rho(t,u)$ is entropy solution to (2.7). By the uniqueness of the entropy solution, we conclude the proof for Lipschitz continuous initial density profile ρ_0 with compact support.

Step 2. Now we assume the initial density profile ρ_0 is integrable on $\mathbb{R}_+$ and bounded. On $\mathbb{R}_+$, we approximate ρ_0 by $\rho_{0,\varepsilon} \in C_K^\infty(\mathbb{R}_+)$ such that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \left|(\rho_{0,\varepsilon} - \rho_0)(u)\right| du = 0,$$

and leave $\rho_{0,\varepsilon} = \rho_0$ unchanged on $(-\infty, 0)$. We shall use the coupling introduced in Subsection 3.3. We take the initial measure $\widetilde{\mu}_N^\varepsilon$ of the coupled process to be such that

$$\widetilde{\mu}_{N,1}^\varepsilon = \mu_N, \quad \widetilde{\mu}_{N,2}^\varepsilon(d\omega) = \bigotimes_{x\in\mathbb{Z}} \mathscr{P}_{\rho_{0,\varepsilon}(x/N)}(d\omega_x)$$

and that

$$\omega_x \leqslant \varpi_x \quad \text{if} \quad \rho_0(\tfrac{x}{N}) \leqslant \rho_{0,\varepsilon}(\tfrac{x}{N}), \quad \text{and} \quad \omega_x \geqslant \varpi_x \quad \text{if} \quad \rho_0(\tfrac{x}{N}) \geqslant \rho_{0,\varepsilon}(\tfrac{x}{N})$$

for all $x \in \mathbb{Z}$ and with probability one with respect to $\widetilde{\mu}_N^\varepsilon$. Since under the coupling, the number of discrepancies only decreases in time, for any $t > 0$,

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}_N^\varepsilon}\Big[\frac{1}{N}\sum_{x\geqslant 0}\big|\omega_t(x)-\varpi_t(x)\big|\Big] \leqslant \lim_{N\to\infty} \widetilde{\mathbb{E}}_{\widetilde{\mu}_N^\varepsilon}\Big[\frac{1}{N}\sum_{x\in\mathbb{Z}}\big|\omega_0(x)-\varpi_0(x)\big|\Big] \leqslant \int_0^{+\infty}\big|\rho_{0,\varepsilon}(u)-\rho_0(u)\big|du,$$

which vanishes as $\varepsilon \to 0$. Let $\widetilde{\mathbb{Q}}_{N,\varepsilon}$ be the law of the pair $(\alpha_t^N, \alpha_t^{N,\varepsilon})$ when $(\omega(t), \varpi(t))$ starts from $\widetilde{\mu}_N^\varepsilon$, where

$$\alpha_t^{N,\varepsilon} = \frac{1}{N}\sum_{x\in\mathbb{Z}} \varpi_x(t)\delta_{x/N}(du).$$

Let $\widetilde{\mathbb{Q}}_\varepsilon$ be any limit point of $\widetilde{\mathbb{Q}}_{N,\varepsilon}$ as $N \to \infty$, whose support is concentrated on absolutely continuous measures $(\alpha_t, \alpha_t^\varepsilon) = (\delta_{\rho(t,u)}du, \delta_{\rho_\varepsilon(t,u)}du)$. Passing to the limit in the last inequality, we have

$$\lim_{\varepsilon\to 0} \mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\Big[\int_0^\infty \big|\rho(t,u)-\rho_\varepsilon(t,u)\big|du\Big] \leqslant \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\big|\rho_{0,\varepsilon}(u)-\rho_0(u)\big|du = 0, \tag{4.7}$$

where, as a consequence of the first step $\rho_\varepsilon(t,u)$ is the unique entropy solution to (2.7) with initial datum $\rho_{0,\varepsilon}$ and with boundary condition (2.13) independent of ε. To justify the last statement, note that the boundary condition (2.13) only depends on the density left of the origin, which is initially independent from ε. By the L^1-contraction principle (see [13, Theorem 7.28] for instance) ρ_ε converges in L^1 to the solution of (2.7) with boundary condition given by (2.13) and initial condition ρ_0. We conclude the proof by using (4.7) which proves that $\rho(t,u)$ must be the solution of (2.7) with b.c. (2.13).

Step 3. To extend the result to bounded initial datum ρ_0, we use the approximated functions $\rho_{0,k}(u) = \rho_0(u)\mathbb{1}\{|u| \leqslant k\}$ and let $k \to \infty$. See [15, Theorem 5.1] for details. This concludes the proof for the case $\beta = 0$. □

4.3. **The case $0 < \beta < 1$.** In this subsection we prove Theorem 2.3 for the case $0 < \beta < 1$. Following the same steps presented in the last subsection, we only need to prove the following lemma.

Lemma 4.5. *Assume $0 < \beta < 1$. Every limit point $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{N,\ell}$ is concentrated on paths $\pi_{t,u}(d\lambda)$ such that*

$$\lim_{u\to 0}\int_0^T\int_0^\infty \Phi(\lambda)\,\pi_{t,u}(d\lambda)\,dt = 0.$$

Proof. To prove this result, we adapt the arguments laid out in [12, Lemma 6.4]. Fix $c \geqslant 0$. We couple two processes $(\omega(t), \varpi(t))$ according to the generator $N\widetilde{\mathscr{L}_N}$ as stated in Subsection 3.3. Let $\widetilde{\mu}_{N,1} = \mu_N$. We take the second marginal $\widetilde{\mu}_{N,2}$ of $\widetilde{\mu}_N$ to be the invariant measure as stated in Remark 3.2, with the four parameters given by

$$c_1 = -\frac{\alpha N^\beta}{2p-1+\alpha N^\beta}\Phi(c), \quad c_2 = \Phi(c), \quad c_3 = 0, \quad c_4 = \frac{2p-1}{2p-1+\alpha N^\beta}\Phi(c).$$

With the above choice,

$$\lim_{N\to\infty} \mathbb{E}_{\widetilde{\mu}_{N,2}}[\omega_{[Nu]}] = \begin{cases} c & \text{if } u < 0, \\ 0 & \text{if } u > 0. \end{cases}$$

Since the process $\varpi(t)$ is time invariant, by law of large numbers, for any $\varepsilon > 0$ and for any non-negative test function H with compact support in $(0,\infty)\times\mathbb{R}$, we have

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big[\int_0^\infty N^{-1}\sum_{x\leqslant 0} H(t,\tfrac{x}{N})\big|\varpi_x^\ell(t)-c\big|dt \geqslant \varepsilon\Big] = 0,$$

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} \widetilde{\mathbb{P}}_{\widetilde{\mu}_N}\Big[\int_0^\infty N^{-1}\sum_{x\geqslant 0} H(t,\tfrac{x}{N})\varpi_x^\ell(t)dt \geqslant \varepsilon\Big] = 0.$$

By Lemma 3.6,

$$\begin{aligned}&\int_0^\infty\int_0^\infty\Big\{\partial_t H(t,u)\int_0^\infty \lambda\pi_{t,u}(d\lambda)+(2p-1)\partial_u H(t,u)\int_0^\infty \Phi(\lambda)\pi_{t,u}(d\lambda)\Big\}\,du\,dt\\&+\int_0^\infty\int_{-\infty}^0\Big\{\partial_t H(t,u)|\rho_L(t,u)-c|+(2p-1)\partial_u H(t,u)|\Phi(\rho_L(t,u))-\Phi(c)|\Big\}\,du\,dt\geqslant 0\end{aligned}\tag{4.8}$$

with $\mathbb{Q}^*$-probability one. The proof of Lemma 4.5 is adapted from [5, Theorem 4.1]. In order not to burden with technical details, we now present a formal argument, and refer to the latter for additional technical details. From (4.8), we first obtain formally, for any c, t, u,

$$\begin{aligned}&\langle\partial_t\pi_{t,u}\mathbb{1}\{u\geqslant 0\},\lambda\rangle+(2p-1)\langle\partial_u\pi_{t,u}\mathbb{1}\{u\geqslant 0\},\Phi(\lambda)\rangle\\&\qquad+\langle\partial_t\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\lambda-c|\rangle+(2p-1)\langle\partial_u\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\Phi(\lambda)-\Phi(c)|\rangle\leqslant 0\end{aligned}\tag{4.9}$$

in the sense of distributions, where we define $\pi_{t,u}\mathbb{1}\{u\leqslant 0\}=\delta_{\rho_L(t,u)}\mathbb{1}\{u\leqslant 0\}$. Observe that

$$\begin{aligned}0&=\partial_t\langle\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\lambda-\rho_L(t,u)|\rangle+(2p-1)\partial_u\langle\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\Phi(\lambda)-\Phi(\rho_L(t,u))|\rangle\\&=\langle\partial_t\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\lambda-\rho_L(t,u)|\rangle+(2p-1)\langle\partial_u\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\Phi(\lambda)-\Phi(\rho_L(t,u))|\rangle\\&\quad+\langle\pi_{t,u}\mathbb{1}\{u\leqslant 0\},\partial_t|\lambda-\rho_L(t,u)|+(2p-1)\partial_u|\Phi(\lambda)-\Phi(\rho_L(t,u))|\rangle\\&\leqslant\langle\partial_t\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\lambda-\rho_L(t,u)|\rangle+(2p-1)\langle\partial_u\pi_{t,u}\mathbb{1}\{u\leqslant 0\},|\Phi(\lambda)-\Phi(\rho_L(t,u))|\rangle\end{aligned}$$

in the sense of distributions. In the last inequality, we use the fact that $\rho_L(t,u)$ satisfies the entropy inequality and thus the penultimate line is non-positive in the sense of distributions. Therefore, choosing $c=\rho_L(t,u)$ in (4.9) yields

$$\langle\partial_t\pi_{t,u}\mathbb{1}\{u\geqslant 0\},\lambda\rangle+(2p-1)\langle\partial_u\pi_{t,u}\mathbb{1}\{u\geqslant 0\},\Phi(\lambda)\rangle\leqslant 0$$

in the sense of distributions. More explicitly, this identity yields for any non-negative test function H that

$$\int_0^\infty\int_0^\infty\Big\{\partial_t H(t,u)\int_0^\infty \lambda\pi_{t,u}(d\lambda)+(2p-1)\partial_u H(t,u)\int_0^\infty \Phi(\lambda)\pi_{t,u}(d\lambda)\Big\}\,du\,dt\geqslant 0\tag{4.10}$$

with $\mathbb{Q}^*$-probability one. Fix $\delta>0$. Using the same strategies presented in the proof of Lemma 4.3, we approximate $H(t,u)=\mathbb{1}\{t\leqslant T\}\times\mathbb{1}\{u\leqslant\delta\}$ by smooth functions with compact support, and together with (4.10), we obtain that there exists $C>0$ such that

$$\int_0^T\int_0^\infty \Phi(\lambda)\pi_{t,\delta}(d\lambda)\,dt\leqslant C\delta\tag{4.11}$$

with $\mathbb{Q}^*$-probability one. This completes the proof by letting $\delta\to 0$. □

4.4. **The case $\beta\geqslant 1$.** To indicate the dependence of the process on the parameter β, let $\omega^{(\beta)}(t)$ be the process with generator $N(\mathscr{L}_N^{\mathrm{ZR}}+\mathscr{L}_N^{\alpha,\beta})$. Without confusions and to make notations simple, in this subsection we denote $\omega(t):=\omega^{(1/2)}(t)$ and $\eta(t)=\omega^{(\infty)}(t)$. In other words, in η any particle that reaches the origin is destroyed. We still denote by $\mathbb{P}_{\mu_N}$ the distribution of the coupled processes, and by $\mathbb{E}_{\mu_N}$ the corresponding expectation. We shall couple $\eta(t)$ and $\omega(t)$ together such that if $\omega(0)=\eta(0)$ at the initial time, then

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\Big[\big|\eta_x(t)-\omega_x(t)\big|\Big]=0.\tag{4.12}$$

This is enough to prove Theorem 2.3 for the case $\beta\geqslant 1$, since the case $\beta=1/2$ has already been proved. Indeed, since the process is attractive, for any compactly supported and continuous function

$H:\mathbb{R}\to\mathbb{R}$ and for any $\beta\geqslant 1$,

$$\mathbb{E}_{\mu_N}\Big[\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}H\big(\tfrac{x}{N}\big)(\omega_x^{(\beta)}(t)-\omega_x(t))\Big|\Big]\leqslant\frac{||H||_\infty}{N}\sum_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\Big[\big|\omega_x^{(\beta)}(t)-\omega_x(t)\big|\Big]$$
$$\leqslant\frac{||H||_\infty}{N}\sum_{x\in\mathbb{Z}}\mathbb{E}_{\mu_N}\Big[\big|\eta_x(t)-\omega_x(t)\big|\Big]\to 0\quad\text{as } N\to\infty.$$

We now describe the coupling. We label the η-particles by $Y^{k,i}(t)$, and the ω-particles by $Z^{k,i}(t)$, where $k\in\mathbb{Z}$, $i=1,2,\dots,\eta_k(0)$. Let

$$\eta_x(t)=\sum_{k\in\mathbb{Z}}\sum_{i=1}^{\eta_k(0)}\mathbb{1}\{Y^{k,i}(t)=x\},\qquad \omega_x(t)=\sum_{k\in\mathbb{Z}}\sum_{i=1}^{\eta_k(0)}\mathbb{1}\{Z^{k,i}(t)=x\}.$$

If $Y^{k,i}$ and $Z^{k,i}$ are at the same site, then we say these two particles are coupled; otherwise they are uncoupled. We shall see that $\eta(t)\leqslant\omega(t)$ for all $t\geqslant 0$ and that Y-particles are always coupled with Z-particles. The dynamics is as follows:

- at rate $Ng(\eta_x(t))$, a pair of coupled particles is chosen uniformly from the coupled particles at site x, and jumps to site $x+1$ with probability p, or to site $x-1$ with probability $1-p$;
- at rate $N(g(\omega_x(t))-g(\eta_x(t)))$, an uncoupled Z-particle is chosen uniformly from the uncoupled Z-particles at site x, and jumps to site $x+1$ with probability p, or to site $x-1$ with probability $1-p$;
- at the origin, Y-particles die instantaneously and at rate $(N+\alpha N^{3/2})g(\omega_0(t))$, a Z-particle is chosen uniformly from those particles at the origin, then it jumps to site $x+1$ with probability $p/(1+\alpha N^{1/2})$, or to site $x-1$ with probability $(1-p)/(1+\alpha N^{1/2})$, or dies with probability $\alpha N^{1/2}/(1+\alpha N^{1/2})$.

With the above coupling, we have

$$\sum_{x\in\mathbb{Z}}\big|\eta_x(t)-\omega_x(t)\big|=\sum_{k\in\mathbb{Z}}\sum_{i=1}^{\eta_k(0)}\mathbb{1}\{Z^{k,i}\text{ visits the origin before time } t \text{ and survives by time } t\}.$$

Using the same argument as in the proof of Lemma 3.3, we can restrict the above summation to a large box $\{k:k\in[-AN,AN]\}$ for some $A>0$. Depending on whether $Z^{k,i}$ is at the origin or not at time t, we can bound the expectation of the last expression by

$$\mathbb{E}_{\mu_N}[\omega_0(t)]+\sum_{k\in[-An,An]}\mathbb{E}_{\mu_N}[\eta_k(0)]\frac{1}{1+\alpha N^{1/2}},$$

which divided by N converges to zero as $N\to\infty$. This proves (4.12).

Appendix A. The linear case

In this section, we provide a simpler, alternative proof of the hydrodynamic limit (Theorem 2.3) in the linear case where the jump rate is given by $g(k)=k$. Note that in this case, (cf. Remark 2.5), the hydrodynamic limit follows a linear advection equation, and thus does not develop shock. Proving existence and uniqueness of weak solutions is then straightforward. The alternative proof of Theorem 2.3 we propose in the linear case follows the *duality approach* already used in [6, 7] for symmetric exclusion processes. However, it is significantly easier for the linear asymmetric zero-range process, because the estimation of the two-points correlations of the system is straightforward.

For any sites $x,y\in\mathbb{Z}$, any $t\geqslant 0$, we define the following discrete *density field*

$$\rho_x^N(t):=\mathbb{E}_{\mu_N}\big[\omega_x(t)\big],\tag{A.1}$$

as well as the *two-points correlation field*

$$\varphi_{x,y}^N(t):=\mathbb{E}_{\mu_N}\big[(\omega_x(t)-\rho_x^N(t))(\omega_y(t)-\rho_y^N(t))\big].\tag{A.2}$$

As detailed in [6], the hydrodynamic limit in the linear case is a straightforward consequence of Lemmas A.1 and A.2 below. The first lemma states that the discrete density field, once properly rescaled, converges to the hydrodynamic limit.

Lemma A.1. *The density field ρ^N defined in (A.1) converges as $N \to \infty$ towards the function ρ defined in (2.15), in the sense that: for any $t \geqslant 0$, any $u \in \mathbb{R}$,*

$$\lim_{N\to\infty} \rho^N_{\lfloor Nu \rfloor}(t) = \rho(t,u), \tag{A.3}$$

where the convergence is uniform in u over compact subsets of $\mathbb{R} \setminus \{0,(2p-1)t\}$. In the identity above, the function ρ is the linear hydrodynamic limit defined by (2.15). In particular, for any $H \in C_K(\mathbb{R})$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)\rho^N_x(t) = \int_{\mathbb{R}} H(u)\rho(t,u)\,du.$$

The second lemma states that the two-points correlation field vanishes away from the diagonal.

Lemma A.2. *For any $\varepsilon > 0$, and any $t \geqslant 0$,*

$$\lim_{N\to\infty} \sup_{\substack{x,y\in\mathbb{Z}\\ |x-y|\geqslant \varepsilon N}} |\varphi^N_{x,y}(t)| = 0. \tag{A.4}$$

We will not detail the proof that the hydrodynamic limit follows from these two estimates, we simply sketch the main steps and refer the interested reader to [6, Section 5] for a detailed implementation. The main argument comes from the straightforward bound

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(\frac{1}{N}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)\omega_x(t) - \int_{\mathbb{R}} H(u)\rho(t,u)\Bigg)^2\Bigg]&\\ \leqslant \frac{2}{N^2}\sum_{x,y\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)H\left(\tfrac{y}{N}\right)\varphi^N_{x,y}(t) + 2\Bigg(\frac{1}{N}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{N}\right)\rho^N_x(t) - \int_{\mathbb{R}} H(u)\rho(t,u)\,du\Bigg)^2,&\end{aligned}$$

which is obtained by adding and subtracting $N^{-1}\sum_x H(x/N)\rho^N_x(t)$ inside the parenthesis. The second term in the right hand side vanishes according to Lemma A.1. The first term can be split, for any $\varepsilon > 0$ into a term that vanishes as $N \to \infty$ according to Lemma A.2, and a second sum which contains $\mathcal{O}((\varepsilon N)^2)$ terms. Because the process is stochastically dominated by some equilibrium distribution $\nu_{\rho^\star}$, the $\varphi^N_{x,y}(t)$ are bounded, uniformly in x, y, t, by some constant $C(\rho^\star)$. In particular, each of the terms in the sum are of order $C(H,\rho^\star)/N^2$, and this last contribution vanishes as $N \to \infty$ and $\varepsilon \to 0$, which proves the hydrodynamic limit.

Proof of Lemma A.1. By Dynkin's formula, the values $\rho^N_x(t) := \rho^N(t, \frac{x}{N})$ are the solutions of the following system:

$$\begin{cases} \frac{d}{dt}\rho^N_x(t) = N\Big((1-p)\rho^N_{x+1}(t) + p\rho^N_{x-1}(t) - \rho^N_x(t)\Big), & x \neq 0\\ \frac{d}{dt}\rho^N_0(t) = N\Big((1-p)\rho^N_1(t) + p\rho^N_{-1}(t) - (1+\alpha N^\beta)\rho^N_0(t)\Big). \end{cases} \tag{A.5}$$

The main ingredient to prove Lemma A.1 is a duality relationship that expresses the density field $\rho^N(t)$ as a function of an asymmetric random walk X on $\mathbb{Z}$. Define a cemetery state $\mathfrak{d}$, and consider a continuous-time random walk X_t on $\mathbb{Z}$, killed at rate $\alpha N^{1+\beta}$ at the origin, at which point it goes to the cemetery state $\mathfrak{d}$. The random walk is asymmetric, and jump at rate Np to the left and $N(1-p)$ to the right, *i.e.*, its drift is inverted w.r.t. the particles in our zero-range process. Denote by $\mathscr{L}^\dagger_N$ the generator of this process, acting on local functions $f : \mathbb{Z} \cup \{\mathfrak{d}\} \to \mathbb{R}$ as

$$(\mathscr{L}^\dagger_N f)_x = Np\,\{f_{x-1} - f_x\} + N(1-p)\,\{f_{x+1} - f_x\} + \alpha N^{1+\beta}\,\{f_{\mathfrak{d}} - f_x\}\,\mathbb{1}\{x = 0\}. \tag{A.6}$$

We denote by $\tau \in [0,+\infty]$ the time at which the random walk is killed, namely $\tau = \inf\{t>0,\ X_t = \mathfrak{d}\}$, by $\mathbb{P}_x$ the distribution of X_t started from x, and by $\mathbb{E}_x$ the corresponding expectation. For convenience, we let the random walk run after it is killed by letting it jump at infinite rate from $\mathfrak{d}$ to 0, but keep track that it died.

With these notations, (A.5) yields

$$\begin{cases} \frac{d}{dt}\rho_x^N(t) = \mathscr{L}_N^\dagger \rho_x^N(t) & x\in\mathbb{Z},\ t>0,\\ \rho_{\mathfrak{d}}^N(t) = 0 & t>0,\\ \rho_x^N(0) = \rho_0(\frac{x}{N}) & x\in\mathbb{Z}, \end{cases} \tag{A.7}$$

so that by Feynman-Kac's formula, we have the dual identity

$$\rho_x^N(t) = \mathbb{E}_x\left[\rho_0\Big(\frac{X_t}{N}\Big)\,\mathbb{1}\{\tau>t\}\right]. \tag{A.8}$$

We now consider the behavior of the random walk X_t. Define p_N, q_N, r_N as

$$p_N = \frac{\alpha N^\beta}{1+\alpha N^\beta},\quad q_N = \frac{2p-1}{1+\alpha N^\beta}\quad \text{and}\quad r_N = \frac{2-2p}{1+\alpha N^\beta} = 1-p_N-q_N,$$

which are the probabilities that: the random walk X, initially at the origin, gets killed before jumping out (p_N), the random walk is not killed and escapes to infinity without ever coming back to the origin (q_N), and the random walk is not killed immediately but eventually comes back to the origin (r_N).

From this, we deduce that the probability that, starting at the origin, the probability that X_t gets killed before escaping to infinity is

$$\widetilde{\alpha}_N := \mathbb{P}_0(\tau<\infty) = p_N\sum_{k\geqslant 0} r_N^k = \frac{\alpha N^\beta}{\alpha N^\beta + 2p-1} \underset{N\to\infty}{\longrightarrow} \widetilde{\alpha}. \tag{A.9}$$

where $\widetilde{\alpha}$ was defined in (2.14).

Fix now $u\in\mathbb{R}$, and assume that $X_0 = \lfloor Nu\rfloor$. Recall that we let the random walk X jump asymmetrically after it dies, we further claim that

$$\mathbb{P}_{\lfloor Nu\rfloor}\left(\sup_{s\leqslant t}\left|\frac{X_s}{N} - u + (2p-1)s\right| \geqslant 2N^{-1/4}\right) = \mathscr{O}(N^{-1/8}). \tag{A.10}$$

We define the martingale $M_s := X_s - \lfloor Nu\rfloor + N(2p-1)s$, which is a martingale w.r.t. the natural filtration of $X_\cdot$. Fix $t>0$, by Doob's inequality

$$\mathbb{P}_{\lfloor Nu\rfloor}\left(\sup_{s\leqslant t}|M_s|\geqslant N^{3/4}\right) \leqslant \mathbb{E}_{\lfloor Nu\rfloor}[|M_t|]N^{-3/4} \leqslant N^{-3/4}\left[1+\int_1^\infty \mathbb{P}_{\lfloor Nu\rfloor}(|M_t|\geqslant v)dv\right]. \tag{A.11}$$

Fix $K>0$, we define for $k\leqslant N$

$$Y_k = \big(M_{t(k+1)/N} - M_{tk/N}\big)\,\mathbb{1}\big\{|M_{t(k+1)/N} - M_{tk/N}|\leqslant K\big\}.$$

By Azuma's inequality,

$$\mathbb{P}_{\lfloor Nu\rfloor}\left(\left|\sum_{k=0}^{N-1} Y_k\right|\geqslant v\right) \leqslant 2e^{-v^2/2NK^2}.$$

In particular,

$$\mathbb{P}_{\lfloor Nu\rfloor}(|M_t|\geqslant v) \leqslant 2e^{-v^2/2NK^2} + N\max_{k\in\{0,\dots,N-1\}}\mathbb{P}_{\lfloor Nu\rfloor}\big(|M_{t(k+1)/N} - M_{tk/N}|\geqslant K\big).$$

Assume that $K>2t$, the probability in the right hand side is less than the probability that the random walk X jumps at least $K/2$ times in the time interval $[t(k+1)/N, tk/N]$, which is less than $Ce^{-K/t}$ since the number of jumps in this interval is distributed as a Poisson random variable of parameter t. This finally yields

$$\mathbb{P}_{\lfloor Nu\rfloor}(|M_t|\geqslant v) \leqslant 2e^{-v^2/2NK^2} + NCe^{-K/t},$$

so that, choosing $K = (\log N \log \nu)^2/t$, (A.11) yields

$$\mathbb{P}_{\lfloor Nu \rfloor}\left(\sup_{s\leqslant t}|M_s| \geqslant N^{3/4}\right) = \mathcal{O}(N^{-1/8}), \tag{A.12}$$

which proves (A.10).

We now get back to (A.8). For any $u \notin [0,(2p-1)t]$, for any N large enough, according to (A.10), $\mathbb{P}_{\lfloor Nu \rfloor}(\tau < t) \leqslant \mathbb{P}_{\lfloor Nu \rfloor}(\exists\, s \leqslant t, X_s = 0) = \mathcal{O}(N^{-1/8})$. In particular, recall from (2.15) the definition of ρ, and recall that we assumed ρ_0 to be uniformly c_0-Lipschitz, (A.8) yields

$$\left|\rho^N_{\lfloor Nu \rfloor}(t) - \rho(t,u)\right| \leqslant 2N^{-1/4}c_0 + \mathcal{O}(N^{-1/8}) \underset{N\to\infty}{\longrightarrow} 0.$$

We now consider $u \in (0,(2p-1)t)$, for which we can write, using the same bounds,

$$\left|\rho^N_{\lfloor Nu \rfloor}(t) - \rho(t,u)\mathbb{P}_{\lfloor Nu \rfloor}(\tau \geqslant t)\right| \leqslant 2N^{-1/4}c_0 + \mathcal{O}(N^{-1/8}) \underset{N\to\infty}{\longrightarrow} 0.$$

To conclude the proof, we only need to show that $\mathbb{P}_{\lfloor Nu \rfloor}(\tau \geqslant t)$ converges to $1-\widetilde{\alpha}$ as $N \to \infty$. To do so, we show that $\mathbb{P}_{\lfloor Nu \rfloor}(\tau \geqslant t) = \mathbb{P}_0(\tau = \infty) + o_N(1)$. We start by writing

$$\mathbb{P}_{\lfloor Nu \rfloor}(\tau \geqslant t) = \mathbb{P}_{\lfloor Nu \rfloor}(\tau = \infty) + \mathbb{P}_{\lfloor Nu \rfloor}(t \leqslant \tau < \infty).$$

By Markov property, the first term in the right hand side is $\mathbb{P}_0(\tau = \infty)$ because an asymmetric random walk X started from $x > 0$ hits the origin a.s. in finite time. Regarding the second term, we can write using (A.10) that for some *negative* u',

$$\mathbb{P}_{\lfloor Nu \rfloor}(t \leqslant \tau < \infty) \leqslant \mathbb{P}_{\lfloor nu' \rfloor}(\tau < \infty) + \mathcal{O}(N^{-1/8}).$$

Indeed, at time t the random walk has not died, and is w.h.p. of order $1-\mathcal{O}(N^{-1/8})$ to the right of $\lfloor nu' \rfloor$, where $u' = (u-(2p-1)t)/2 < 0$. However, the random walk being asymmetric, the probability $\mathbb{P}_{\lfloor Nu' \rfloor}(\tau < \infty)$ is $\mathcal{O}(e^{-cN})$ since it visits the origin with probability $\mathcal{O}(e^{-cN})$ for some positive constant $c = c(u',p)$. Finally (A.9) yields as wanted

$$\mathbb{P}_{\lfloor Nu \rfloor}(\tau \geqslant t) = \mathbb{P}_0(\tau = \infty) + \mathcal{O}(N^{-1/8}) = 1 - \widetilde{\alpha} + \mathcal{O}(N^{-\varepsilon}),$$

where $\varepsilon = |\beta| \wedge 1/8$.

All estimates above are uniform away from the boundary points $u = 0$, $u = (2p-1)t$, so that the second statement of the lemma is now clear. The last statement of the lemma is straightforward, we do not detail it. □

Proof of Lemma A.2. In the asymmetric case, the proof of Lemma A.2 is straightforward, we simply sketch it. In the same way that we obtained (A.7), one can write using Dynkin's formula

$$\begin{cases} \frac{d}{dt}\varphi^N_{x,y}(t) = \mathcal{L}^{\dagger,2}_N \varphi^N_{x,y}(t) & x,\ y \in \mathbb{Z}, |x-y| > 1,\ t > 0, \\ \varphi^N_{\mathfrak{d},y}(t) = \varphi^N_{x,\mathfrak{d}}(t) = 0 & t > 0,\ x,\ y \in \mathbb{Z}, \\ \varphi^N_{x,y}(0) = 0 & x,\ y \in \mathbb{Z}, \end{cases} \tag{A.13}$$

where $\mathcal{L}^{\dagger,2}_N$ is the generator of a two-dimensional asymmetric random walk $\mathbf{X} = (X,Y)$ on $\mathbb{Z}^2$, where both X and Y are driven by the generator $\mathcal{L}^{\dagger}_N$ defined in (A.6). In other words, under $\mathcal{L}^{\dagger,2}_N$, $\mathbf{X}$ jumps from (x,y) to either $(x+1,y)$ or $(x,y+1)$ at rate $1-p$, to $(x-1,y)$ or $(x,y-1)$ at rate p, and jumps from $(x,0)$ or $(0,y)$ to the cemetery state $(x,\mathfrak{d})$, $(\mathfrak{d},y)$ at rate $\alpha N^{1+\beta}$. Note that not to burden the proof, we say nothing about the behavior of the random walk $\mathbf{X} = (X,Y)$ close to the diagonal, *i.e.* when $|X-Y| \leqslant 1$. When the random walk gets close to the diagonal, we will simply use the crude bound yielded by attractiveness, namely

$$\sup_{\substack{x,y\in\mathbb{Z}\\ t\geqslant 0}} |\varphi^N_{x,y}(t)| \leqslant C(\rho^\star).$$

Once again, by Feynman-Kac's formula, one can write for any $|x-y|>1$, $t\geqslant 0$,

$$\varphi^N_{x,y}(t)=\mathbb{E}_{x,y}\left[\varphi_{\mathbf{X}_{\tau'}}(t-\tau')\,\mathbb{1}_{\{\tau'<t\wedge\tau\}}\right]\leqslant C(\rho^\star)\mathbb{P}_{x,y}(\tau'<t), \tag{A.14}$$

where $\mathbb{P}_{x,y}$, $\mathbb{E}_{x,y}$ denotes the distribution of the random walk $\mathbf{X}_t=(X_t,Y_t)$ started from $\mathbf{X}_0=(x,y)$, and the corresponding expectation. In the identity above, τ' is the first time the random walk $\mathbf{X}_t$ hits the set $|x-y|\leqslant 1$, whereas $\tau=\tau(X)\wedge\tau(Y)$ is the first time either X or Y hits the cemetery state. Equation (A.14) comes from the fact that we have three boundaries to the system (A.13) : if either X or Y hits the cemetery state, φ^N vanishes. If the random walk reaches time t, then we query the initial time correlations, which also vanish. The only non-zero contribution therefore comes from getting close to the diagonal. However, given $\varepsilon>0$, uniformly in $|x-y|\geqslant\varepsilon N$, $\mathbb{P}_{x,y}(\tau'<t)$ vanishes as $N\to\infty$, because X_t and Y_t will be respectively close (up to a fluctuation of order $\mathcal{O}(\sqrt{N})$) to $x+(2p-1)Nt$ and $y+(2p-1)Nt$ and x and y are far apart. This proves (A.4).

□

Appendix B. Equivalence of Definitions 2.2 and 2.3

In Definition 2.2 the boundary condition is given by the density $\varrho(t)$ at the origin, while in Definition 2.3 the boundary term is given by the flux $f(t)$ into the system. In this section, we show that the two definitions coincide if $f(t)=(2p-1)\Phi(\varrho(t))$.

Indeed, assuming ρ is an entropy solution to (2.7), we shall prove it is also an entropy solution to (2.9) with $f(t)=(2p-1)\Phi(\varrho(t))$. Then by uniqueness of the weak entropy solutions, these two definitions are the same. Taking $H\in C^{1,1}_K((0,\infty)\times(0,\infty))$ in (2.8) proves that ρ satisfies (2.9). It remains to prove ρ also satisfies (2.12). Fix $T>0$. We also need to assume furthermore that ϱ has a finite number of discontinuities on the time interval $[0,T]$. For any interval $[t_i,t_{i+1}]$ and $a>0$, taking $H(t,u)=\mathbb{1}_{[t_i,t_{i+1}]}(t)\times\mathbb{1}_{[-a,a]}$ and $c=\varrho(t_i)$ in (2.8), we have

$$\begin{aligned}
&\int_0^a\big\{|\rho(t_i,u)-\varrho(t_i)|-|\rho(t_{i+1},u)-\varrho(t_i)|\big\}du+M\int_{t_i}^{t_{i+1}}|\varrho(t)-\varrho(t_i)|dt\\
&\quad\geqslant\int_{t_i}^{t_{i+1}}(2p-1)\big|\Phi(\rho(t,a))-\Phi(\varrho(t_i))\big|dt\\
&\quad\geqslant\int_{t_i}^{t_{i+1}}(2p-1)\big|\Phi(\rho(t,a))-\Phi(\varrho(t))\big|dt-\int_{t_i}^{t_{i+1}}(2p-1)\big|\Phi(\varrho(t))-\Phi(\varrho(t_i))\big|dt.
\end{aligned}\tag{B.1}$$

Since Φ is smooth,

$$\int_{t_i}^{t_{i+1}}(2p-1)\big|\Phi(\varrho(t))-\Phi(\varrho(t_i))\big|dt\leqslant C(\Phi)\delta_\varrho(t_i,t_{i+1})(t_{i+1}-t_i),$$

where $\delta_\varrho(t_i,t_{i+1})=\sup_{t\in(t_i,t_{i+1})}|\varrho(t)-\varrho(t_i)|$. Similarly, the second term on the left hand side of (B.1) is bounded by $M\delta_\varrho(t_i,t_{i+1})(t_{i+1}-t_i)$. Taking a partition $\{t_i\}$ of the time interval $[0,T_0]$ and summing over the partition in (B.1), we have

$$\begin{aligned}
&\sum_{t_i}\int_0^a\big\{|\rho(t_i,u)-\varrho(t_i)|-|\rho(t_{i+1},u)-\varrho(t_i)|\big\}du+(C(\Phi)+M)T_0\sup_i\delta_\varrho(t_i,t_{i+1})\\
&\quad\geqslant\int_0^{T_0}(2p-1)\big|\Phi(\rho(t,a))-\Phi(\varrho(t))\big|dt.
\end{aligned}$$

Since $\inf_{\{t_i\}}\sup_i\,\delta_\varrho(t_i,t_{i+1})=0$ and $\rho(t,u)$ is bounded,

$$
\begin{aligned}
Ca &\geqslant \inf_{\{t_i\}} \sum_{t_i} \int_0^a \big\{ |\rho(t_i,u) - \varrho(t_i)| - |\rho(t_{i+1},u) - \varrho(t_i)| \big\} du \\
&\geqslant \int_0^T (2p-1) \big| \Phi(\rho(t,a)) - \Phi(\varrho(t)) \big| dt.
\end{aligned}
$$

We conclude the proof by letting $a \to 0$.

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