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PONTRYAGIN DUALITY FOR VARIETIES OVER $p$-ADIC FIELDS

THOMAS H. GEISSER AND BAPTISTE MORIN

Abstract. We define cohomological complexes of locally compact abelian groups associated with varieties over $p$-adic fields and prove a duality theorem under some assumption. Our duality takes the form of Pontryagin duality between locally compact motivic cohomology groups.

1. Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the ring of integers in $K$ and let $X$ be a regular, proper and flat scheme over $\mathcal{O}_K$ of dimension $d$. We denote by $X_K$ its generic fiber and by $i : X_s \to X$ its special fiber. It is a classical result that for any integer $m > 0$ we have a perfect duality of motivic cohomology with finite coefficients

$$H^i_{et}(X_K, \mathbb{Z}/m(n)) \times H^{2d-i}_{et}(X_K, \mathbb{Z}/m(d-n)) \to H^{2d}_{et}(X_K, \mathbb{Z}/m(d)) \to \mathbb{Z}/m.$$ 

However, this does not lift to a duality of integral groups \cite{?}. For example, even for a curve $X_K$, the dual of $H^1_{et}(X_K, \mathbb{Q}/\mathbb{Z}) \cong H^2_{et}(X_K, \mathbb{Z})$ has both contributions from $H^3_{et}(X_K, \mathbb{Z}(2))$ as well as from $H^4_{et}(X_K, \mathbb{Z}(2))$. The examples $H^1_{et}(X_K, \mathbb{Z}(1)) \cong K^\times$, or $H^2_{et}(X_K, \mathbb{Z}(1)) \cong \text{Pic}(X_K)$, an extension of a finitely generated group by a finitely generated $\mathbb{Z}_p$-module, also suggest that the cohomology groups are topological groups. Thus our goal is to construct topological cohomology groups which agree with étale cohomology groups with finite coefficients, but satisfy a Pontryagin duality. More generally, we conjecture the existence of a cohomology theory on the category of separated schemes of finite type over $\text{Spec}(\mathcal{O}_K)$, whose main expected properties are outlined in the last section of this paper.

Its existence was suggested by the "Weil-Arakelov cohomology" of arithmetic schemes, which is conditionally defined in \cite{?} for proper regular schemes over $\text{Spec}(\mathbb{Z})$. The aim of this paper is to give a possible construction of such groups.

Let LCA be the quasi-abelian category of locally compact abelian groups, and FLCA $\subseteq$ LCA the full subcategory consisting of locally compact abelian groups of finite ranks in the sense of \cite{?}. We consider the bounded derived category $\mathcal{D}^b(\text{LCA})$ and $\mathcal{D}^b(\text{FLCA})$, respectively, \cite{?}. The category $\mathcal{D}^b(\text{FLCA})$ is a closed symmetric monoidal category with internal homomorphisms $\mathcal{R}\text{Hom}(-, -)$ and

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Assuming certain expected properties of Bloch’s cycle complex $\mathbb{Z}(n)$, we construct for any $n \in \mathbb{Z}$ complexes in $D^b(LCA)$ fitting in an exact triangle

\[(1) \quad R\Gamma_{ar}(\mathcal{X}_s, Ri^\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)),\]

and we define

\[R\Gamma_{ar}(-, \mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(-, \mathbb{Z}(n)) \otimes \mathbb{L} \mathbb{R}/\mathbb{Z}.\]

We expect that this theory satisfies duality and many other properties, see Section 6.

To obtain unconditional results, we give an alternative construction, which conjecturally agrees with the above construction of the triangle (??) for $n = 0, d$, and show that this triangle belongs to $D^b(FLCA)$. Then we prove Theorem ?? below under the following hypothesis.

**Hypothesis 1.1.** The reduced scheme $(\mathcal{X}_s)^{\text{red}}$ is a simple normal crossing scheme, and the complex $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups, where $\mathbb{Z}^c(0)$ denotes the Bloch cycle complex in its homological notation [?] and $R\Gamma_W(\mathcal{X}_s, -)$ denotes Weil-étale cohomology.

The homology groups of the complex $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))[1]$ are and called arithmetic homology with compact support and denoted by $H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$ and an equivalence $\to R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d - n)), \mathbb{R}/\mathbb{Z}[-2d])$ in $D^b(FLCA)$, for $n = 0, d$.

**Theorem 1.2.** Suppose that either $d \leq 2$ or that $\mathcal{X}_s$ satisfies Hypothesis ??.

Then there is a trace map $H^2d_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$ and an equivalence

\[R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) \sim R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d - n)), \mathbb{R}/\mathbb{Z}[-2d])\]

in $D^b(FLCA)$, for $n = 0, d$.

Combining Theorem ?? with a consequence of Sato’s work [?], we obtain

**Corollary 1.3.** Suppose that $\mathcal{X}/\mathcal{O}_K$ has good or strictly semi-stable reduction and suppose that $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there is a perfect pairing of locally compact abelian groups

\[H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) \times H^{2d-i}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d - n)) \to H^{2d}_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}\]

for $n = 0, d$ and any $i \in \mathbb{Z}$.

In a forthcoming paper we will give applications of this result to class field theory of schemes over local fields.

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## 2. Locally Compact Abelian Groups

In this section we define and study the derived $\infty$-categories $D^b(LCA)$ and $D^b(FLCA)$. We also introduce a certain profinite completion functor.
2.1. Derived $\infty$-categories. Let $A$ be an additive category. Let $C^b(A)$ be the differential graded category of bounded complexes of objects in $A$ and let $\mathcal{N} \subset C^b(A)$ be a full subcategory which is closed under the formation of shifts and under the formation of mapping cones. If $N_{dg}(-)$ denotes the differential graded nerve [?, Construction 1.3.1.6], then $N_{dg}(C^b(A))$ is a stable $\infty$-category and $N_{dg}(\mathcal{N})$ is a stable $\infty$-subcategory of $N_{dg}(C^b(A))$ [?, Prop. 1.3.2.10].

The Verdier quotient is defined [?, Thm. I.3.3] as the Dyer-Kan localization

$$N_{dg}(C^b(A))/N_{dg}(\mathcal{N}) := N_{dg}(C^b(A))[W^{-1}],$$

where $W$ is the set of arrows in $N_{dg}(C^b(A))$ whose cone lies in $N_{dg}(\mathcal{N})$. The $\infty$-category $N_{dg}(C^b(A))[W^{-1}]$ is stable. Moreover, the functor $N_{dg}(C^b(A)) \to N_{dg}(C^b(A))/N_{dg}(\mathcal{N})$ is exact and induces an equivalence from the category of exact functors $N_{dg}(C^b(A))/N_{dg}(\mathcal{N}) \to \mathcal{E}$ to the category of exact functors $N_{dg}(C^b(A)) \to \mathcal{E}$ which send all objects of $N_{dg}(\mathcal{N})$ to zero objects in $\mathcal{E}$, for any (small) stable $\infty$-category $\mathcal{E}$. Finally, we have an equivalence of categories

$$h(N_{dg}(C^b(A))/N_{dg}(\mathcal{N})) \simeq h(N_{dg}(C^b(A)))/h(N_{dg}(\mathcal{N}))$$

where $h(-)$ denotes the homotopy category, and the right hand side is the classical Verdier quotient. Note that the homotopy category of a stable $\infty$-category is triangulated [?, Thm. 1.1.2.14].

If $A$ is a quasi-abelian category in the sense of [?], we define its bounded derived $\infty$-category

$$D^b(A) := N_{dg}(C^b(A))/N_{dg}(\mathcal{N}) \simeq N_{dg}(C^b(A))[S^{-1}]$$

where $\mathcal{N} \subset C^b(A)$ is the full subcategory of strictly acyclic complexes, and $S$ is the set of strict quasi-isomorphisms. The homotopy category

$$D^b(A) := h(D^b(A))$$

is equivalent to the bounded derived category of the quasi-abelian category $A$ in the sense of [?].

2.2. The category $D^b(LCA)$. We denote by LCA the quasi-abelian category of locally compact abelian groups. A morphism of locally compact abelian groups $f : A \to B$ has a kernel $\text{Ker}(f) = f^{-1}(0)$ and a cokernel $\text{Coker}(f) = B/f(A)$ where $f(A)$ is the closure of $f(A)$ in $B$. The morphism $f$ is said to be strict if the map $\text{Coker}(\text{Ker}(f)) \to \text{Ker}(\text{Coker}(f))$ is an isomorphism in LCA. Then $f$ is strict if and only if the induced monomorphism $\overline{f} : A/\text{Ker}(f) \to B$ is a closed embedding. Let FLCA $\subset$ LCA be the quasi-abelian category [?, Cor. 2.11] of locally compact abelian groups of finite ranks in the sense of [?, Def. 2.6]. Recall that $A \in$ LCA has finite ranks if the $\mathbb{R}$-vector spaces of continuous morphisms $\text{Hom}(\mathbb{R}, A)$ and $\text{Hom}(A, \mathbb{R})$ are finite dimensional and $p : A \to A$ is strict with finite kernel and cokernel for any prime number $p$.

Let $D^b(\text{LCA})$ and $D^b(\text{FLCA})$ be the bounded derived $\infty$-category of LCA and FLCA, respectively. Then $D^b(\text{LCA})$ and $D^b(\text{FLCA})$ are stable $\infty$-categories in the sense of [?] whose homotopy categories are the bounded derived categories $D^b(\text{LCA})$ and $D^b(\text{FLCA})$ as defined in [?], respectively. It is more convenient to work with the derived $\infty$-category $D^b(\text{LCA})$ rather than with its homotopy
category. For example, let $\text{Fun}(\Delta^1, D^b(\text{LCA}))$ be the $\infty$-category of arrows in $D^b(\text{LCA})$. Taking the mapping fiber (or cofiber) of a morphism defines a functor (see [?, Rem. 1.1.1.7])

$$\text{Fib} : \text{Fun}(\Delta^1, D^b(\text{LCA})) \to D^b(\text{LCA})$$

$$C \to C' \mapsto C \times_{C''} 0.$$ 

Let $\text{TA}$ be the quasi-abelian category of topological abelian groups, and define $D^b(\text{TA})$ and $D^b(\text{TA})$ as above. The inclusions $\text{FLCA} \subset \text{LCA} \subset \text{TA}$ send strict quasi-isomorphisms to strict quasi-isomorphisms, hence induce functors

$$D^b(\text{FLCA}) \to D^b(\text{LCA}) \to D^b(\text{TA}).$$ 

The functor $\text{disc} : \text{TA} \to \text{Ab}$, sending a topological abelian group to its underlying discrete abelian group, sends strict quasi-isomorphisms to usual quasi-isomorphisms. This yields a functor

$$\text{disc} : D^b(\text{TA}) \to D^b(\text{Ab}).$$ 

Recall that the Pontryagin dual $X^D := \text{Hom}(X, \mathbb{R}/\mathbb{Z})$ of the locally compact abelian group $X$ is the group of continuous homomorphisms $X \to \mathbb{R}/\mathbb{Z}$ endowed with the compact-open topology, and that Pontryagin duality gives an isomorphism of locally compact groups

$$X \simeq X^{DD}.$$ 

The functor $(-)^D$ sends strict quasi-isomorphisms to strict quasi-isomorphisms and locally compact compact abelian groups of finite ranks to locally compact groups of finite ranks. We obtain equivalences of $\infty$-categories

$$D^b(\text{LCA})^{\text{op}} \to D^b(\text{LCA})$$

$$X \mapsto X^D$$

and

$$D^b(\text{FLCA})^{\text{op}} \to D^b(\text{FLCA})$$

$$X \mapsto X^D.$$ 

In [?], the authors define functors

$$R\text{Hom}_{\text{LCA}}(-, -) : D^b(\text{LCA})^{\text{op}} \times D^b(\text{LCA}) \to D^b(\text{TA})$$

and

$$R\text{Hom}_{\text{FLCA}}(-, -) : D^b(\text{FLCA})^{\text{op}} \times D^b(\text{FLCA}) \to D^b(\text{FLCA}).$$ 

The construction of the functor $R\text{Hom}_{\text{FLCA}}(-, -)$ actually gives a functor of stable $\infty$-categories

$$R\text{Hom}(-, -) : D^b(\text{FLCA})^{\text{op}} \times D^b(\text{FLCA}) \to D^b(\text{FLCA}).$$ 

Indeed, let $I$ (resp. $P$) be the additive category of divisible (resp. codivisible) locally compact abelian groups $I$ (resp. $P$) of finite ranks such that $I_\mathbb{Z} = 0$ (such that $P_\mathbb{Z} = 0$), see [?, Def. 3.2]. Define

$$D^b(I) := N_{\text{dg}}(C^b(I))/N_{\text{dg}}(\mathcal{N}_I)$$

where $\mathcal{N}_I \subset C^b(I)$ is the $dg$-subcategory of strictly acyclic bounded complexes. We define similarly

$$D^b(P) := N_{\text{dg}}(C^b(P))/N_{\text{dg}}(\mathcal{N}_P).$$
The exact functor
\[ \text{N}_{\text{dg}}(C^{b}(I)) \to \text{N}_{\text{dg}}(C^{b}(\text{FLCA})) \to D^{b}(\text{FLCA}) \]
induces an exact functor
\[(2) \quad D^{b}(I) \to D^{b}(\text{FLCA})\]
of stable \(\infty\)-categories which induces an equivalences between the corresponding homotopy categories by [?, Cor. 3.10]. It follows that \((2)\) is an equivalence of stable \(\infty\)-categories. Similarly \(D^{b}(P) \to D^{b}(\text{FLCA})\) is an equivalence. We may therefore define
\[ R\text{Hom}(-,-) : D^{b}(\text{FLCA})^{\text{op}} \times D^{b}(\text{FLCA}) \to D^{b}(\text{FLCA}) \]
since the functor
\[ C^{b}(P)^{\text{op}} \times C^{b}(I) \to C^{b}(\text{FLCA}) \]
\[ (P,I) \mapsto \text{Hom}^{*}(P,I) := \text{Tot}(\text{Hom}(P,I)) \]
sends a pair of strict quasi-isomorphisms to a strict quasi-isomorphism [?, Cor. 3.7]. Here \(\text{Hom}(P,I)\) is the double complex of continuous maps endowed with the compact-open topology, and \(\text{Tot}\) denotes the total complex. Note that the Pontryagin dual \(X^{D}\) is given by the functor
\[ R\text{Hom}(-,\mathbb{R}/\mathbb{Z}) : D^{b}(\text{FLCA})^{\text{op}} \to D^{b}(\text{FLCA}) \]
\[ X \mapsto X^{D} \] Following [?], we define the derived topological tensor product
\[(3) \quad D^{b}(\text{FLCA}) \times D^{b}(\text{FLCA}) \to D^{b}(\text{FLCA}) \]
\[ (X,Y) \mapsto X \otimes^{L} Y := R\text{Hom}(X,Y^{D})^{D} \].

**Lemma 2.1.** The functor \(D^{b}(\text{FLCA}) \to D^{b}(\text{LCA})\) is an exact and fully faithful functor of stable \(\infty\)-categories.

**Proof.** The functor
\[ \text{N}_{\text{dg}}(C^{b}(\text{FLCA})) \to \text{N}_{\text{dg}}(C^{b}(\text{LCA})) \to D^{b}(\text{LCA}) \]
induces an exact functor \(D^{b}(\text{FLCA}) \to D^{b}(\text{LCA})\) by [?, Thm. I.3.3(i)]. It remains to check that this functor is fully faithful. The functors \(R\text{Hom}_{\text{LCA}}(-,-)\) and \(R\text{Hom}_{\text{FLCA}}(-,-)\) induce the same functor
\[ D^{b}(\text{FLCA})^{\text{op}} \times D^{b}(\text{FLCA}) \to D^{b}(\text{TA}) \]
by [?, Rem. 4.9]. Moreover, for any \(X,Y \in D^{b}(\text{FLCA})\) we have [?, Prop. 4.12(i)]
\[ \text{disc}(H^{0}(R\text{Hom}_{\text{LCA}}(X,Y))) \simeq \text{Hom}_{D^{b}(\text{LCA})}(X,Y) \]
and similarly\(^1\)
\[ \text{disc}(H^{0}(R\text{Hom}_{\text{FLCA}}(X,Y))) \simeq \text{Hom}_{D^{b}(\text{FLCA})}(X,Y) \].

Therefore, the map
\[ \text{Hom}_{D^{b}(\text{FLCA})}(X,Y) \to \text{Hom}_{D^{b}(\text{LCA})}(X,Y) \]
\[^{1}\text{One may adapt the proof of [?}, \text{Prop. 4.12(i)] to this case, using [?}, \text{Cor. 3.10(iii)].} \]
is an isomorphism of abelian groups, i.e. \( \mathcal{D}^b(FLCA) \rightarrow \mathcal{D}^b(LCA) \) is fully faithful. Hence

\[
(4) \quad \mathcal{D}^b(FLCA) \rightarrow \mathcal{D}^b(LCA)
\]

is an exact functor of stable \( \infty \)-categories which induces a fully faithful functor between the corresponding homotopy categories. It follows that (4) is fully faithful.

Therefore we may identify \( \mathcal{D}^b(FLCA) \) with its essential image in \( \mathcal{D}^b(LCA) \).

The stable \( \infty \)-category \( \mathcal{D}^b(LCA) \) is endowed with a \( t \)-structure by [?, Section 1.2.2], since a \( t \)-structure on a stable \( \infty \)-category is defined as a \( t \)-structure on its homotopy category [?, Def. 1.2.1.4]. We denote its heart by \( \mathcal{L}H(LCA) \). It is an abelian category containing \( LCA \) as a full subcategory. This also applies to \( \mathcal{D}^b(FLCA) \), and we denote \( \mathcal{L}H(FLCA) \) the heart of the corresponding \( t \)-structure.

**Remark 2.2.** By [?, Cor. 1.2.21], an object in \( \mathcal{L}H(LCA) \) can be represented by a monomorphism \( E_1 \rightarrow E_0 \) in \( LCA \), where \( E_0 \) is in degree zero. A common example appearing below is a monomorphism \( \mathbb{Z}^a \rightarrow \mathbb{Z}_p \). Its cokernel in \( LCA \) is trivial, but for the underlying discrete abelian groups the cokernel is

\[
(\oplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l) \oplus (\mathbb{Q}/\mathbb{Z})^{a-1} \oplus D
\]

with \( D \) uniquely divisible.

**Remark 2.3.** It follows from [?, Cor. 2.11] and [?, Prop. 1.2.19] that the fully faithful functor \( \mathcal{D}^b(FLCA) \rightarrow \mathcal{D}^b(LCA) \) is \( t \)-exact. Therefore, the induced functor \( \mathcal{L}H(FLCA) \rightarrow \mathcal{L}H(LCA) \) is exact and fully faithful.

**Notation 2.4.** For any \( X \in \mathcal{D}^b(LCA) \) and any \( i \in \mathbb{Z} \), we consider

\[
H^i(X) := \tau_{\geq 0} \tau_{\leq 0}(X[i]) \in \mathcal{L}H(LCA).
\]

In view of Remark 2.2, we identify \( \mathcal{L}H(FLCA) \) with a full subcategory of \( \mathcal{L}H(LCA) \).

**Lemma 2.5.** Let \( X \in \mathcal{D}^b(LCA) \). Then \( X \in \mathcal{D}^b(FLCA) \) if and only if \( H^i(X) \in \mathcal{L}H(FLCA) \) for any \( i \in \mathbb{Z} \).

**Proof.** If \( X \rightarrow Y \rightarrow Z \) is a fiber sequence in \( \mathcal{D}^b(LCA) \) such that \( X,Z \in \mathcal{D}^b(FLCA) \), then \( Y \in \mathcal{D}^b(FLCA) \). Indeed, a stable subcategory is closed under extensions. Let \( X \in \mathcal{D}^b(LCA) \) such that \( H^i(X) \in \mathcal{L}H(FLCA) \) for any \( i \). Note that \( H^i(X) = 0 \) for all but finitely many \( i \in \mathbb{Z} \). Therefore, \( X \) has a finite exhaustive filtration with \( i \)-graded piece \( H^i(X)[-i] \in \mathcal{D}^b(FLCA) \), so that \( X \) belongs to \( \mathcal{D}^b(FLCA) \) by induction.

The converse follows from the fact that the inclusion functor \( \mathcal{D}^b(FLCA) \rightarrow \mathcal{D}^b(LCA) \) is \( t \)-exact by Remark 2.2.

Recall that Pontryagin duality gives an equivalence

\[
\mathcal{D}^b(LCA)^{\text{op}} \rightarrow \mathcal{D}^b(LCA) \\
X \quad \mapsto \quad X^D.
\]
Lemma 2.6. Let $X \in D^b(LCA)$ such that $H^i(X) \in \mathcal{LH}(LCA)$ belongs to LCA for any $i \in \mathbb{Z}$. Then for any $i \in \mathbb{Z}$ we have a canonical isomorphism in LCA

$$H^i(X_D) \simeq (H^{-i}(X))^D.$$  

Proof. Let

$$X = [\cdots \to X^{i-1} \xrightarrow{d^i_{X^{-1}}} X^i \xrightarrow{d^i_X} X^{i+1} \to \cdots]$$

be an object of $D^b(LCA)$ such that $H^i(X) \in \mathcal{LH}(LCA)$ belongs to LCA for any $i \in \mathbb{Z}$. We first observe that the differentials $d^i_X$ all are strict morphisms. By [?, Prop. 1.2.19], the object $H^i(X)$ of $\mathcal{LH}(LCA)$ is given by the complex $[0 \to \text{Coim}(d^{i-1}_X) \xrightarrow{\delta} \text{Ker}(d^i_X) \to 0]$, where $\text{Ker}(d^i_X)$ sits in degree 0 and $\delta$ is a monomorphism. Since $H^i(X) \in \text{LCA}$, the map $\delta$ is strict by [?, Prop. 1.2.29], i.e. $\delta$ is a closed embedding. Since $\text{Ker}(d^i_X) \hookrightarrow X^i$ is a closed embedding as well, so is the map $\text{Coim}(d^{i-1}_X) = X^{i-1}/\text{Ker}(d^{i-1}_X) \to X^i$. Hence $d^{i-1}_X$ is strict.

We set $Y := X_D$ so that $Y^{-i} = (X^i)^D$ and $d^{i-1}_Y : Y^{-i} \to Y^{-i+1}$ is the map $d^{i-1}_Y = (d^{i-1}_X)^D$. The differentials $d^i_X$ are all strict morphisms, hence so are their duals $d^i_Y$.

We have the following isomorphisms of locally compact abelian groups:

$$\begin{align*}
(5) \quad H^i(X)^D & \simeq \left( \text{Coker}(\text{Coim}(d^{i-1}_X) \xrightarrow{\delta} \text{Ker}(d^i_X)) \right)^D \\
(6) \quad & \simeq \text{Ker}(\text{Coker}(d^i_X)^D \to \text{Im}(d^{i-1}_X)^D)) \\
(7) \quad & \simeq \text{Ker}(\text{Coker}(d^{i-1}_Y) \to \text{Im}(d^i_Y)) \\
(8) \quad & \simeq \text{Ker}(d^{i-1}_Y) \to Y^{-i+1}) \\
(9) \quad & \simeq \text{Ker}(Y^{-i}/d^{i-1}_Y(Y^{-i-1}) \to Y^{-i+1}) \\
(10) \quad & \simeq \text{Ker}(d^{i-1}_Y)/d^{i-1}_Y(Y^{-i-1}) \\
(11) \quad & \simeq \text{Ker}(d^{i-1}_Y)/\text{Im}(d^{i-1}_Y) \\
(12) \quad & \simeq H^{-i}(Y)
\end{align*}$$

where the kernels, cokernels, images, and coinages are all computed in LCA.

The isomorphism (5) is valid by [?, Prop. 1.2.29] since the map $\delta$ is strict, and (6) holds since Pontryagin duality $(-)^D : \text{LCA}^{\mathbb{Z}} \to \text{LCA}$ is an equivalence of categories with kernels and cokernels. The identification (7) is given by definition of the maps $d^i_Y$ and (8) holds since $\text{Im}(d^{i-1}_Y) \to Y^{-i+1}$ is a monomorphism. We have (9) in view of $\text{Coker}(d^{i-1}_Y) = Y^{-i}/d^{i-1}_Y(Y^{-i-1})$, which is valid since $d^{i-1}_Y(Y^{-i-1})$ is closed in $Y^{-i}$, as $d^{i-1}_Y$ is strict. The isomorphism of locally compact abelian groups (9) is clear; (10) holds since $\text{Coim}(d^{i-1}_Y) \to d^{i-1}_Y(Y^{-i-1}) = \text{Im}(d^{i-1}_Y)$ is an isomorphism in LCA since $d^{i-1}_Y$ is strict. Finally, (11) holds by [?, Props. 1.2.19 and 1.2.29] since $\text{Coim}(d^{i-1}_Y) \to \text{Ker}(d^{i-1}_Y)$ is strict; indeed $Y^{-i-1}/\text{Ker}(d^{i-1}_Y) \to Y^{-i}$ is a closed embedding hence so is $Y^{-i-1}/\text{Ker}(d^{i-1}_Y) \to \text{Ker}(d^{i-1}_Y)$. $\square$

The inclusion $\text{Ab} \subset \text{LCA}$ as discrete objects induces an exact functor

$$i : D^b(\text{Ab}) \to D^b(\text{LCA}).$$
Proposition 2.7. The exact functor $i : D^b(Ab) \to D^b(LCA)$ is fully faithful and left adjoint to $\text{disc} : D^b(LCA) \to D^b(Ab)$.

Proof. The functor $C^b(Ab) \xrightarrow{i} C^b(LCA) \xrightarrow{\text{disc}} C^b(Ab)$ is isomorphic to the identity functor of $C^b(Ab)$. We obtain a natural transformation
\[ \text{Id}_{D^b(Ab)} \cong \text{disc} \circ i. \]
Similarly, there is a natural transformation $i \circ \text{disc} \to \text{Id}_{D^b(LCA)}$.

Let $X \in D^b(Ab)$ and let $Y \in D^b(LCA)$. Let $F \xrightarrow{\sim} X$ be a bounded flat resolution, and let $Y \xrightarrow{\sim} D$ be a strict quasi-isomorphism where $D$ is a bounded complex of divisible locally compact abelian groups. Then $F$ is a bounded complex of codivisible\(^2\) discrete groups $F^i$ (in particular, such that $F^2_{S^1} = 0$). Therefore, we have
\[ R\text{Hom}_{LCA}(i(X), Y) \cong \text{Hom}^\bullet(F, D) := \text{Tot}(\text{Hom}(F, D)) \]
by [? , Cor. 4.7], where $\text{Hom}(F, D)$ is the double complex of continuous maps endowed with the compact-open topology, and $\text{Tot}$ denotes the total complex. We obtain
\[ \text{disc}(R\text{Hom}_{LCA}(i(X), Y)) \cong \text{disc}(\text{Hom}^\bullet(F, D)) \cong \text{Hom}^\bullet(F, \text{disc}(D)) \cong R\text{Hom}(X, \text{disc}(Y)). \]

In view of [? , Prop. 4.12] we have
\[ H^0(\text{disc}(R\text{Hom}_{LCA}(i(X), Y[-n]))) \cong \text{disc}(H^0(R\text{Hom}_{LCA}(i(X), Y[-n]))) \cong \text{Hom}_{D^b(LCA)}(i(X), Y[-n]) \cong \pi_0(\text{Map}_{D^b(LCA)}(i(X), \Omega^n Y)) \cong \pi_n(\text{Map}_{D^b(LCA)}(i(X), Y)) \]
where $\Omega(-) := 0 \times (-) 0$ is the loop space functor. Similarly we have
\[ H^0(R\text{Hom}(X, \text{disc}(Y[-n]))) \cong \pi_n(\text{Map}_{D^b(Ab)}(X, \text{disc}(Y))). \]
Hence the map
\[ \text{Map}_{D^b(LCA)}(i(X), Y) \to \text{Map}_{D^b(Ab)}(X, \text{disc}(Y)) \]
is an equivalence of $\infty$-groupoids. The result then follows from [? , Prop. 5.2.2.8] and from the fact that the unit transformation (??) is an equivalence. □

Definition 2.8. An object $X \in D^b(LCA)$ lies in the essential image of the functor $i : D^b(Ab) \to D^b(LCA)$ if and only if the co-unit map $i \circ \text{disc}(X) \to X$ is an equivalence. Such an object $X \in D^b(LCA)$ is called discrete.

\(^2\)A $\in LCA$ is said to be codivisible if $A^D$ is divisible.
Lemma 2.9. Let $X, Y \in \mathbf{D}^b(\text{Ab})$. If $iX$ and $iY$ belong to $\mathbf{D}^b(\text{FLCA})$, then there is a canonical map

$$i(\mathbf{R}\text{Hom}(X,Y)) \to \mathbf{R}\text{Hom}(iX,iY).$$

Moreover, if $X, Y$ are perfect complexes of abelian groups, then this map is an equivalence.

Proof. Let $P \sim \rightarrow iX$ and $iY \sim \rightarrow I$ be strict quasi-isomorphisms where $P \in \mathbf{C}^b(\mathbb{P})$ (resp. $I \in \mathbf{C}^b(\mathbb{I})$). We denote by $P^\delta := \text{disc}(P)$ and $I^\delta := \text{disc}(I)$ the underlying complexes of discrete abelian groups. Then the maps $P^\delta \sim \rightarrow X$ and $Y \sim \rightarrow I^\delta$ are quasi-isomorphisms in the usual sense. Hence we have

$$\text{Hom}^\cdot(X,I^\delta) \simeq \mathbf{R}\text{Hom}(X,Y),$$

where $\text{Hom}^\cdot$ denotes the total complex of the double complex of morphisms of abelian groups. We denote by $\text{Hom}^\cdot(P,I)$ the total complex of the double complex of continuous morphisms endowed with the compact-open topology. Then we have morphisms

$$\mathbf{R}\text{Hom}(X,Y) \simeq \text{Hom}^\cdot(X,I^\delta) \to \text{Hom}^\cdot(iX,I) \to \text{Hom}^\cdot(P,I) \simeq \mathbf{R}\text{Hom}(iX,iY).$$

Suppose now that $X$ and $Y$ are perfect complexes of abelian groups. We may suppose that $X_n$ is a finitely generated free abelian group for all $n \in \mathbb{Z}$, zero for almost all $n$, and similarly for $Y$. We have a strict quasi-isomorphism

$$iY \sim \rightarrow I := \text{Tot}[Y \otimes \mathbb{R} \rightarrow Y \otimes \mathbb{R}/\mathbb{Z}]$$

where $[Y \otimes \mathbb{R} \rightarrow Y \otimes \mathbb{R}/\mathbb{Z}]$ is seen as a double complex of locally compact abelian groups and Tot is the total complex. Then $iX \in \mathbf{C}^b(\mathbb{P})$ and $I \in \mathbf{C}^b(\mathbb{I})$ and we have a strict quasi-isomorphism

$$i\text{Hom}^\cdot(X,Y) \sim \rightarrow \text{Hom}^\cdot(iX,I).$$

We obtain

$$i\mathbf{R}\text{Hom}(X,Y) \simeq i\text{Hom}^\cdot(X,Y) \sim \rightarrow \text{Hom}^\cdot(iX,I) \simeq \mathbf{R}\text{Hom}(iX,iY).$$

\[ \square \]

2.3. Profinite completion.

Definition 2.10. We define a functor

$$(-)_{\otimes \mathbb{Z}} : \mathbf{D}^b(\text{Ab}) \rightarrow \mathbf{D}^b(\text{LCA}),$$

where we compute $\mathbf{R}\text{Hom}(X,Z/m)$ and the colimit $\text{colim} \mathbf{R}\text{Hom}(X,Z/m)$ over $m$ in the $\infty$-category $\mathbf{D}^b(\text{Ab})$. We define similarly

$$(-)_{\otimes \mathbb{Z}_p} : \mathbf{D}^b(\text{Ab}) \rightarrow \mathbf{D}^b(\text{LCA}),$$

where we compute $\mathbf{R}\text{Hom}(X,Z/p^\bullet)$ and the colimit $\text{colim} \mathbf{R}\text{Hom}(X,Z/p^\bullet)$ over $m$ in the $\infty$-category $\mathbf{D}^b(\text{Ab})$. We define similarly

For any $X \in \mathbf{D}^b(\text{LCA})$ we define\(^\text{3}\)

$$\mathbf{R}\text{Hom}(X,Z/m) := \text{Fib}(X^D \xrightarrow{m} X^D)$$

and

$$X_{\otimes^b \mathbb{Z}/m} := \text{Cofib}(X \xrightarrow{m} X).$$

\(^3\)This is compatible with the definition given in Section ??, which is only valid if $X \in \mathbf{D}^b(\text{FLCA})$.\]
Proposition 2.11. Let $X \in D^b(\text{Ab})$. Suppose that $R\text{Hom}(i(X), Z/m) \in D^b(\text{LCA})$ is discrete for any $m$. Then we have an equivalence

$$X \hat{\otimes} \hat{Z} \simeq \varprojlim (i(X) \otimes^L Z/m)$$

where the limit is computed in the $\infty$-category $D^b(\text{LCA})$ and an equivalence

$$\text{disc}(X \hat{\otimes} \hat{Z}) \simeq X \hat{\otimes} \hat{Z} := \varprojlim (X \otimes^L Z/m) \in D^b(\text{Ab}).$$

Proof. The co-unit map

$$i \circ \text{disc} R\text{Hom}(i(X), Z/m) \to R\text{Hom}(i(X), Z/m)$$

is an equivalence by assumption. Moreover we have

$$R\text{Hom}(X, Z/m) \simeq \text{disc} R\text{Hom}(i(X), Z/m)$$

hence

$$i R\text{Hom}(X, Z/m) \sim \to R\text{Hom}(i(X), Z/m).$$

We obtain

$$X \hat{\otimes} \hat{Z} := (i(\text{colim } R\text{Hom}(X, Z/m)))^D$$

$$\simeq (\text{colim } (i R\text{Hom}(X, Z/m)))^D$$

$$\simeq \text{lim } ((i R\text{Hom}(X, Z/m))^D)$$

$$\simeq \text{lim } (R\text{Hom}(i(X), Z/m)^D)$$

$$\simeq \text{lim } (i(X) \otimes^L Z/m)$$

since the left adjoint functor $i$ commutes with arbitrary colimits, and since $(-)^D$ transforms colimits into limits. Hence we have

$$\text{disc}(X \hat{\otimes} \hat{Z}) \simeq \text{disc} \left( \text{lim}(i(X) \otimes^L Z/m) \right)$$

$$\simeq \text{lim } \left( \text{disc}(i(X) \otimes^L Z/m) \right)$$

$$\simeq \text{lim } \left( \text{Cofib}(\text{disc} \circ i(X) \overset{m}{\longrightarrow} \text{disc} \circ i(X)) \right)$$

$$\simeq \text{lim } (X \otimes^L Z/m)$$

since the right adjoint functor $\text{disc}$ commutes with arbitrary limits. \qed

Remark 2.12. Suppose that $X \in D^b(\text{Ab})$ is such that the cohomology groups of $X \otimes^L Z/m$ are all finite. Then $R\text{Hom}(i(X), Z/m)$ is discrete.

Remark 2.13. We have

$$X \hat{\otimes} \hat{Z} \simeq R\text{Hom} \left( i \text{ colim } R\text{Hom}(X \otimes^L Z/m, Q/Z), \mathbb{R}/\mathbb{Z} \right).$$

Lemma 2.14. We have a canonical map $iX \to X \hat{\otimes} \hat{Z}$ in $D^b(\text{LCA})$.

Proof. The composite map

$$i(R\text{Hom}(X, Z/m)) \sim \to i \circ \text{disc}(R\text{Hom}(iX, Z/m)) \to R\text{Hom}(iX, Z/m)$$

$$\to R\text{Hom}(iX, \mathbb{R}/\mathbb{Z}) \simeq (iX)^D$$
induces
\[ i(\colim R\text{Hom}(X, \mathbb{Z}/m)) \simeq \colim i(R\text{Hom}(X, \mathbb{Z}/m)) \to (iX)^D. \]
We obtain
\[ iX \xrightarrow{\sim} (iX)^D \to (i(\colim R\text{Hom}(X, \mathbb{Z}/m)))^D =: X\hat{\otimes}\mathbb{Z}. \]

**Remark 2.15.** Let \( X \) be an object of \( D^b(\text{Ab}) \) whose image \( iX \in D^b(\text{LCA}) \)
belongs to \( D^b(\text{FLCA}) \). Then one may consider \( iX \hat{\otimes} \mathbb{Z} \) and \( iX \hat{\otimes} \mathbb{Z}_p \) where \( \hat{\otimes} \)
is the tensor product (??) in \( D^b(\text{FLCA}) \). There are canonical maps \( iX \hat{\otimes} \mathbb{Z} \to X \hat{\otimes} \mathbb{Z} \) and \( iX \hat{\otimes} \mathbb{Z}_p \to X \hat{\otimes} \mathbb{Z}_p \) but those maps are not equivalences in general.
For example, we have
\[ Q_p/\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_p \simeq Q_p/\mathbb{Z}_p \quad \text{and} \quad Q_p/\mathbb{Z}_p \simeq Q_p \]
while
\[ Q_p/\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_p \simeq \mathbb{Z}_p[1] \quad \text{and} \quad Q_p/\mathbb{Z}_p \simeq 0. \]

**Notation 2.16.** In the next sections, given \( X \in D^b(\text{Ab}) \) we often simply denote by \( X \) its image \( iX \) in \( D^b(\text{LCA}) \). In particular, for \( X, Y \in D^b(\text{Ab}) \), we denote by \( R\text{Hom}(X, Y) \in D^b(\text{Ab}) \subseteq D^b(\text{LCA}) \) the usual \( R\text{Hom} \) seen as an object of \( D^b(\text{LCA}) \).

### 3. Duality for schemes over finite fields

Let \( Y \) be a proper scheme over a finite field. If \( Y \) is smooth, then the Weil-étale cohomology \( R\Gamma_W(Y, \mathbb{Z}) \) of [?] is a perfect complex of abelian groups. In general, the Weil-h cohomology \( R\Gamma_{Wh}(Y, \mathbb{Z}) \) of [?] is a perfect complex of abelian groups provided resolution of singularities [?, Definition 2.4] holds (see Proposition ?? below). We show that if \( Y \) is a simple normal crossing scheme, then the Weil-étale cohomology \( R\Gamma_W(Y, \mathbb{Z}) \) is a perfect complex of abelian groups, and that the canonical map \( R\Gamma_W(Y, \mathbb{Z}) \to R\Gamma_{Wh}(Y, \mathbb{Z}) \) is an equivalence under resolution of singularities. In Section ??, we show that \( R\Gamma_W(Y, \mathbb{Z}) \) is dual to \( R\Gamma_W(Y, \mathbb{Z}_c(0)) \) under the assumption that \( R\Gamma_W(Y, \mathbb{Z}_c(0)) \) is perfect, where \( \mathbb{Z}_c(0) \) is the cycle complex.

#### 3.1. Finite generation of cohomology.

**Definition 3.1.** Let \( k \) be a finite field with algebraic closure \( \bar{k} \) and \( W_k \) be its Weil-group. For a scheme \( Y \) over \( k \) we let \( Y = Y \times_k \bar{k} \). For a scheme \( Y \) of finite type \( Y \) over \( k \) we define the Wh-cohomology of the constant sheaf \( \mathbb{Z} \) to be
\[ R\Gamma_{Wh}(Y, \mathbb{Z}) := R\Gamma(W_k, R\Gamma_{ch}(\bar{Y}, \mathbb{Z})). \]

**Proposition 3.2.** Let \( Y \) be a proper scheme over a finite field \( k \). Assume resolution of singularities for schemes over \( k \) of dimension \( \leq \dim(Y) \) [?, Def. 2.4]. Then \( R\Gamma_{Wh}(Y, \mathbb{Z}) \) is a perfect complex of abelian groups.

**Proof.** To prove perfectness of \( R\Gamma_{Wh}(Y, \mathbb{Z}) \), one first reduces to the smooth and projective case by [?, Prop. 3.2], in which case one can conclude with loc. cit. Theorem 4.3 and [?].

□
Definition 3.3. Let $k$ be a field and let $Y$ be a pure dimensional proper scheme over $k$ with irreducible components $Y_i$, $i = 1, \ldots, c$. Then $Y$ is said to be a simple normal crossing scheme if for all $I \subseteq \{1, \ldots, c\}$, $Y_I = \bigcap_{i \in I} Y_i$ is regular of codimension $|I| - 1$ in $Y$.

In fact for all the results in this paper we only need that $(Y_I)^{\text{red}}$ is regular.

Lemma 3.4. Consider a cartesian square

\[
\begin{array}{ccc}
Y' & \overset{i'}{\longrightarrow} & T' \\
\downarrow & & \downarrow \pi \\
Y & \overset{i}{\longrightarrow} & T
\end{array}
\]

of schemes of finite type over a field $k$ with $i$ a closed embedding and $\pi$ finite such that $\pi|T' - Y'$ is an isomorphism to $T - Y$. Then there is a distinguished triangle

\[
R\Gamma_{\text{et}}(T, \mathbb{Z}) \rightarrow R\Gamma_{\text{et}}(T', \mathbb{Z}) \oplus R\Gamma_{\text{et}}(Y, \mathbb{Z}) \rightarrow R\Gamma_{\text{et}}(Y', \mathbb{Z}).
\]

In particular, if $k$ is a finite field, we obtain a triangle

\[
R\Gamma_W(T, \mathbb{Z}) \rightarrow R\Gamma_W(T', \mathbb{Z}) \oplus R\Gamma_W(Y, \mathbb{Z}) \rightarrow R\Gamma_W(Y', \mathbb{Z}).
\]

Proof. To get the first triangle, noting that $i^*$ and $\pi^*$ are exact, it suffices to show that the exact sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow \pi_*\mathbb{Z} \oplus i^*\mathbb{Z} \rightarrow (\pi \circ i')_*\mathbb{Z} \rightarrow 0
\]

of étale sheaves on $T$ is exact. But this follows by considering stalks at points $t \in T$. If $t \notin Y$, then the sequence reduces to the isomorphism $\mathbb{Z} \cong \pi_*\mathbb{Z}$, and if $t \in Y$, then $\mathbb{Z} \cong i_*\mathbb{Z}$ and $\pi_*\mathbb{Z} \cong (\pi \circ i')_*\mathbb{Z}$.

The second triangle can be obtain by applying $R\Gamma(W, -)$ to the first triangle after base extension to the algebraic closure. □

Proposition 3.5. If $T^{\text{red}}$ is a strict normal crossing scheme, then $R\Gamma_W(T, \mathbb{Z})$ is a perfect complex of abelian groups. Under resolution of singularities, we have a quasi-isomorphic $R\Gamma_W(T, \mathbb{Z}) \simeq R\Gamma_{\text{Wh}}(T, \mathbb{Z})$.

Proof. Since étale cohomology with coefficients in $\mathbb{Z}$ does not change if we replace $T$ by $T^{\text{red}}$, we can assume that $T$ is reduced. We proceed by induction on dimension of $T$ and the number of irreducible components of $T$. If the number of components is one, then $T^{\text{red}}$ is smooth and proper and the result follows from [?, Thm. 4.3]. In general, let $T = \bigcup_{i \in I} S_i$ and set $Y = S_1$ and $T' = \bigcup_{i \neq 1} S_i$. Then the hypothesis of Lemma ?? are satisfied, $Y$ is smooth, $T'$ is a normal crossing scheme with fewer irreducible components, and $Y'$ a normal crossing scheme of smaller dimension. Hence we obtain the first statement on perfectness, and the second statement by comparing with the corresponding triangle for Wh-cohomology. □

Note that we can have $H^2_W(T, \mathbb{Z}) \neq H^2_{\text{Wh}}(T, \mathbb{Z})$ for normal proper surfaces [?, Prop. 8.2].
3.2. Finite generation of homology. For later use, we record the two following conditional results on finite generation.

To get statements for homology, recall the following conjecture from [?].

**Conjecture** $P_n(X)$: For the smooth and proper scheme $X$ over a finite field, the group $CH_n(X, i)$ is torsion for all $i > 0$.

Conjecture $P_n(X)$ is known for all $n$ if $X$ is a curve. In general, it is a particular case of Parshin’s conjecture, which is equivalent to the statement $P_n(X)$ for all $n$. Parshin’s conjecture in turn is implied by the Beilinson-Tate conjecture [?, Thm. 1.2]. By the projective bundle formula, conjecture $P_n(X)$ for all $X$ of dimension at most $d$ implies conjecture $P_{n-1}(X)$ for all $X$ of dimension $d-1$. The following Proposition is [?, Prop. 4.2].

**Proposition 3.6.** If conjecture $P_0(X)$ holds for all smooth and proper schemes of dimension at most $\dim Y$, then the cohomology groups of $R\Gamma_W(Y, \mathbb{Z}^c(0))$ are finitely generated and vanish for almost all indices.

If $Y$ is a simple normal crossing scheme, then it suffices to assume that $P_0(Y_t)$ holds for all multiple intersections $Y_t$.

**Proposition 3.7.** If resolution of singularities and conjecture $P_{-1}(X)$ holds for all schemes of dimension at most $d-1$, then the cohomology groups of $R\Gamma_W(X, \mathbb{Z}(d))$ are finite and vanish for almost all indices.

**Proof.** Using blow-up squares and induction on the dimension it suffices to prove the statement for smooth and proper schemes $T$ of dimension at most $d-1$. By [?, Cor. 5.5] the Weil-eh cohomology groups agree with Weil-étale cohomology groups. By conjecture $P_{-1}(T)$ they are torsion, hence finite by comparison with étale cohomology groups.

3.3. Duality.

**Theorem 3.8.** Let $Y$ be a simple normal crossing scheme over a finite field $k$ such that $R\Gamma_W(Y, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there is a perfect pairing

$$R\Gamma_W(Y, \mathbb{Z}) \otimes^L R\Gamma_W(Y, \mathbb{Z}^c(0)) \to \mathbb{Z}[-1]$$

of perfect complexes of abelian groups.

**Proof.** Let $f : Y \to \text{Spec}(k) = s$ be the structure morphism. The push-forward map [?, Cor. 3.2]

$$Rf_*\mathbb{Z}^c(0)^Y \to \mathbb{Z}^c(0)^s \simeq \mathbb{Z}[0]$$

induces a trace map

$$R\Gamma_W(Y, \mathbb{Z}^c(0)) \to R\Gamma_W(s, \mathbb{Z}) \to \mathbb{Z}[-1].$$

We consider the map

$$(14) \quad R\Gamma_W(Y, \mathbb{Z}) \to R\text{Hom}(R\Gamma_W(Y, \mathbb{Z}^c(0)), \mathbb{Z}[-1])$$

induced by the pairing

$$R\Gamma_W(Y, \mathbb{Z}) \otimes^L R\Gamma_W(Y, \mathbb{Z}^c(0)) \to R\Gamma_W(Y, \mathbb{Z}^c(0)) \to R\Gamma_W(s, \mathbb{Z}^c(0)) \to \mathbb{Z}[-1].$$
which in turn is induced by the obvious pairing $\mathbb{Z} \otimes^L \mathbb{Z}_c(0) \to \mathbb{Z}_c(0)$. In order to show that the morphism of perfect complexes (??) is an equivalence, it is enough to show that (??) $\otimes^L \mathbb{Z}/m\mathbb{Z}$ is an equivalence for any integer $m$. But (??) $\otimes^L \mathbb{Z}/m\mathbb{Z}$ may be identified with the canonical map

$$R\Gamma_{et}(Y, \mathbb{Z}/m\mathbb{Z}) \longrightarrow R\text{Hom}(R\Gamma_{et}(Y, \mathbb{Z}_c(0)/m), \mathbb{Q}/\mathbb{Z}[-1])$$

(15)

since we have an equivalence of lax symmetric monoidal functors

$$R\Gamma_W(Y, (-) \otimes^L \mathbb{Z}/m\mathbb{Z}) \simeq R\Gamma_{et}(Y, (-) \otimes^L \mathbb{Z}/m\mathbb{Z}).$$

But (??) is an equivalence by [?, Thm. 5.1]. Hence (??) is an equivalence as well. \qed

4. **The complexes** $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ **in $D^b(LCA)$**.

Under the assumption that the pair $(\mathcal{X}, n)$ satisfies Hypothesis ?? below, we give in Section ?? a construction of complexes in a fiber sequence in $D^b(LCA)$

$$R\Gamma_{ar}(\mathcal{X}_s, R^i\mathbb{Z}(n)) \longrightarrow R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \longrightarrow R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$$

(16)

where

$$R\Gamma_{ar}(\mathcal{X}_s, R^i\mathbb{Z}(n)) := R\Gamma_{W}(\mathcal{X}_s, R^i\mathbb{Z}(n))$$

is defined below. Hypothesis ?? is known for $n = 0, 1$ and arbitrary $\mathcal{X}$, hence this construction is unconditional in those cases.

In Sections ?? and ?? we give an alternative definition of the triangle (??) for $n = 0$ and $n = d := \text{dim}(\mathcal{X})$, respectively, which are expected to coincide with the conditional definition of Section ??.

In Section ?? we show that these complexes in fact belong to $D^b(FLCA)$ under some conditions. In Section ??, we show that the cohomology of these complexes consists of locally compact abelian groups for $n = 0, d$.

4.1. **Notation.** Let $p$ be a prime number, let $K/\mathbb{Q}_p$ be a finite extension, let $\mathcal{O}_K$ be its ring of integers and let $\bar{K}/K$ be an algebraic closure. We denote by $K^{un}$ the maximal unramified extension of $K$ inside $\bar{K}$. Let $\mathcal{X}/\mathcal{O}_K$ be a regular, proper and flat scheme over $\text{Spec}(\mathcal{O}_K)$. Suppose that $\mathcal{X}$ is connected of Krull dimension $d$. Let $\mathcal{X}_s$ be its special fiber, where $s \in \text{Spec}(\mathcal{O}_K)$ is the closed point. We consider the following diagram.

$$\begin{align*}
\mathcal{X}_{K^{un}} & \xrightarrow{j} \mathcal{X}_{\mathcal{O}_{K^{un}}} \\
\mathcal{X}_s & \xrightarrow{i} \mathcal{X}.
\end{align*}$$

For any $n \geq 0$, we denote by $\mathbb{Z}(n)$ Bloch’s cycle complex considered as a complex of étale sheaves and $\mathbb{Z}/m(n) := \mathbb{Z}(n) \otimes^L \mathbb{Z}/m$ We denote by $G_{\kappa(s)} \simeq \hat{\mathbb{Z}}$ and by $W_{\kappa(s)} \simeq \mathbb{Z}$ the Galois group and the Weil group of the finite field $\kappa(s)$, respectively. We define Weil-étale cohomology groups

$$
R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_{K^{un}}, \mathbb{Z}(n))),
$$

$$
R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(n))),
$$

$$
R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(n))).
$$
Applying $R\Gamma(W_{\kappa(s)}, -)$ to the fiber sequence
\[ R\Gamma_{et}(X_s, R^i\mathbb{Z}(n)) \to R\Gamma_{et}(X_{\mathbb{K}^w}, \mathbb{Z}(n)) \to R\Gamma_{et}(X_{\mathbb{K}^u}, \mathbb{Z}(n)), \]
we obtain the fiber sequence
\[ R\Gamma_W(X_s, R^i\mathbb{Z}(n)) \to R\Gamma_W(X, \mathbb{Z}(n)) \to R\Gamma_W(X_K, \mathbb{Z}(n)). \]

4.2. Uniform conditional definition. Recall from [?] the $eh$-motivic cohomology $R\Gamma_{eh}(-, \mathbb{Z}(n))$ and $Wh$-motivic cohomology $R\Gamma_{Wh}(X_s, \mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{eh}(X_{\kappa}, \mathbb{Z}(n)))$.

**Hypothesis 4.1.** We have a reduction map
\[ \overline{i}^*: R\Gamma_{et}(X_{\mathbb{K}^w}, \mathbb{Z}(n)) \to R\Gamma_{eh}(X_s, \mathbb{Z}(n)) \]
and the complexes $R\Gamma_{et}(X, \mathbb{Z}(n))$, $R\Gamma_{eh}(X_s, \mathbb{Z}(n))$ and $R\Gamma_{et}(X_s, R^i\mathbb{Z}(n))$ are cohomologically bounded.

**Definition 4.2.** Under hypothesis [?], we apply the functor $R\Gamma(W_{\kappa(s)}, -)$ to the reduction map $\overline{i}^*$ and we obtain a map
\[ (17) \quad R\Gamma_W(X, \mathbb{Z}(n)) \to R\Gamma_{Wh}(X_s, \mathbb{Z}(n)). \]
We denote the cofiber of (17) by $C_W(X, n)$, so that we have a cofiber sequence
\[ (18) \quad R\Gamma_W(X, \mathbb{Z}(n)) \to R\Gamma_{Wh}(X_s, \mathbb{Z}(n)) \to C_W(X, n) \]
in $D^b(Ab)$.

**Proposition 4.3.** Assume Hypothesis [?]. Then there exist $R\Gamma_{ar}(X, \mathbb{Z}(n)) \in D^b(LCA)$ and $R\Gamma_{ar}(X_K, \mathbb{Z}(n)) \in D^b(LCA)$ endowed with fiber sequences
\[ (19) \quad R\Gamma_{ar}(X, \mathbb{Z}(n)) \to R\Gamma_{Wh}(X_s, \mathbb{Z}(n)) \to C_W(X, n) \widehat{\otimes} \mathbb{Z} \]
and
\[ R\Gamma_W(X_s, R^i\mathbb{Z}(n)) \to R\Gamma_{ar}(X, \mathbb{Z}(n)) \to R\Gamma_{ar}(X_K, \mathbb{Z}(n)) \]
in $D^b(LCA)$.

**Proof.** Composing the morphism in $D^b(Ab)$
\[ R\Gamma_{Wh}(X_s, \mathbb{Z}(n)) \to C_W(X, n) \]
and the morphism in $D^b(LCA)$
\[ C_W(X, n) \to C_W(X, n) \widehat{\otimes} \mathbb{Z} \]
given by Lemma [?], we obtain a morphism in $D^b(LCA)$
\[ (20) \quad R\Gamma_{Wh}(X_s, \mathbb{Z}(n)) \to C_W(X, n) \widehat{\otimes} \mathbb{Z}. \]
We define $R\Gamma_{ar}(X, \mathbb{Z}(n))$ as the fiber of (20) and we obtain the fiber sequence (20) in $D^b(LCA)$. Lemma [?] gives a map from (20) to (17) hence a map
\[ R\Gamma_W(X, \mathbb{Z}(n)) \to R\Gamma_{ar}(X, \mathbb{Z}(n)). \]
Then we define $R\Gamma_{ar}(X_K, \mathbb{Z}(n)) \in D^b(LCA)$ as the cofiber of the composite map
\[ R\Gamma_W(X_s, R^i\mathbb{Z}(n)) \to R\Gamma_W(X, \mathbb{Z}(n)) \to R\Gamma_{ar}(X, \mathbb{Z}(n)). \]
Remark 4.4. Since \( \mathbb{Z}(0) \cong \mathbb{Z} \) and \( \mathbb{Z}(1) \cong \mathbb{G}_m[-1] \), Hypothesis \(?\) holds for \( n = 0 \) and \( n = 1 \), so that \( R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \) and \( R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \) are unconditionally defined in these cases.

4.3. Working definition for the Tate twist \( n = 0 \). We assume that \( \mathcal{X}_s^{\text{red}} \) is a simple normal crossing scheme. To obtain unconditional definitions for \( n = 0 \), we replace \( R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}) \) by \( R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}) \) in the construction of Section \(?\). In view of Corollary \(?\), this will agree with the definition of Section \(?\) provided that resolution of singularities for schemes of dimension at most \( \dim(\mathcal{X}_s) \) exist.

There is a canonical map

\[
R\Gamma_{W}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z})
\]

whose cofiber we again denote by \( C_{W}(\mathcal{X}, 0) \). Following the construction of Section \(?\), we define \( R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \in D^b(\text{LCA}) \) and \( R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \in D^b(\text{LCA}) \) endowed with fiber sequences

\[
R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}) \to C_{W}(\mathcal{X}, 0)\hat{\boxtimes} \mathbb{Z}
\]

and

\[
R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})
\]

in \( D^b(\text{LCA}) \). We used bold letters for the complexes defined in Section \(?\) in order to distinguish them with the complexes defined in this Section.

Proposition 4.5. If \( \mathcal{X}_s^{\text{red}} \) is a simple normal crossing scheme, then the map

\[
R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z})
\]

is an equivalence. For arbitrary \( \mathcal{X} \), the map

\[
R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z})
\]

is an equivalence.

Proof. By proper base change, the map

\[
R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m)
\]

is an equivalence, hence \( C_{W}(\mathcal{X}, 0) \otimes^L \mathbb{Z}/m \simeq 0 \). We obtain \( C_{W}(\mathcal{X}, 0)\hat{\boxtimes} \mathbb{Z} \simeq 0 \). The first equivalence of the proposition follows. The second equivalence is obtained the same way, in view of the fact that

\[
R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m)
\]

is an equivalence, again by proper base change. \( \square \)

Proposition 4.6. If \( \mathcal{X}_s^{\text{red}} \) is a simple normal crossing scheme, then there is a canonical map of fiber sequences

\[
\begin{array}{cccc}
R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}) & \to & R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) & \to & R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \\
\| & & \| & & \\
R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}) & \to & R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) & \to & R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})
\end{array}
\]

If resolution of singularities for schemes over \( \kappa(s) \) of dimension at most \( d - 1 \) \([?\text{, Def. 2.4}]\) exist, then this morphism of fiber sequences is an equivalence.
Then there is a canonical map of fiber sequences

\[ \text{is an equivalence if } d \leq 3. \]

**Notation 4.7.** If \( \mathcal{X}_s \) is a simple normal crossing scheme, we denote by \( R\Gamma_{\text{ar}}(\mathcal{X}, \mathbb{Z}) \) and \( R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \) the complexes defined above. In view of Proposition 4.7, we set \( R\Gamma_{\text{ar}}(\mathcal{X}, \mathbb{Z}) := R\Gamma_{\text{ar}}(\mathcal{X}, \mathbb{Z}) \) and \( R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) := R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \) for arbitrary regular \( \mathcal{X} \) of dimension at most 3 or when we are assuming resolution of singularities.

### 4.4. Working definition for the Tate twist \( n = d \)

The complex \( R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \) is not known to be bounded below. However the complex

\[ R\Gamma_W(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d)) \]

induces an equivalence

\[ \tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \rightarrow \tau^{>b} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)). \]

**Definition 4.8.** Let \( a \ll 0 \). We define

\[ R\Gamma_{\text{ar}}(\mathcal{X}, \mathbb{Z}(d)) := (\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))). \]

If \( R\Gamma_W(\mathcal{X}_s, \mathbb{Z}(0)) \) is cohomologically bounded, we define \( R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \) as the cofiber of the composite map

\[ R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(d)) \rightarrow \tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \rightarrow R\Gamma_{\text{ar}}(\mathcal{X}, \mathbb{Z}(d)) \]

in \( D^b(\text{LCA}) \).

**Remark 4.9.** On the connected, \( d \)-dimensional and regular scheme \( \mathcal{X} \), we have \( \mathbb{Z}(d)^X = \mathbb{Z}(0)^X[-2d] \) by definition. By [?], Cor. 7.2, we have \( R^i\mathbb{Z}(0)^X = \mathbb{Z}(0)^X \), hence \( R^i\mathbb{Z}(d)^X = \mathbb{Z}(0)^X[-2d] \).

**Proposition 4.10.** Suppose that \( \mathcal{X} \) satisfies Hypothesis 4.7 for \( n = d \) and suppose that \( R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(d)) \) and \( R\Gamma_W(\mathcal{X}_s, \mathbb{Z}(0)) \) are cohomologically bounded. Then there is a canonical map of fiber sequences

\[ \text{If } R\Gamma_{\text{Wh}}(\mathcal{X}_s, \mathbb{Z}(d)) \text{ has finite cohomology groups, then this morphism of fiber sequences is an equivalence.} \]
Cohomological boundedness of $R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(d))$ in negative degrees is a special case of the Beilinson-Soulé conjecture stating that there is no negative motivic cohomology, and in positive degrees it follows for finite cohomological dimension reasons. See Proposition ?? and Proposition ?? for the other boundedness conditions.

Proof. If $R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(d))$ and $R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}_s, \mathbb{Z}(0))[-2d]$ are cohomologically bounded, then the same holds for $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$ by the localization triangle

$$\cdots \to R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(d)) \to \cdots .$$

In this case, $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \sim \tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$ is an equivalence for $a << 0$, hence

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))\hat{\otimes}\mathbb{Z}.$$ 

In view of (??) and (??), we obtain a map of fiber sequences

$$\xymatrix{ R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) \ar[r] \ar[d] & R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(d)) \ar[r] \ar[d] & C_W(\mathcal{X}, d)\hat{\otimes}\mathbb{Z} \ar[d] \ar[d] \\
R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) \ar[r] & R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(d))\hat{\otimes}\mathbb{Z} \ar[r] & C_W(\mathcal{X}, d)\hat{\otimes}\mathbb{Z} }$$

since $(-)\hat{\otimes}\mathbb{Z}$ is an exact functor. If the cohomology of $R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(d))$ consists of finite groups, then the middle vertical map

$$R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(d)) \to R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(d))\hat{\otimes}\mathbb{Z}$$

in the above diagram is an equivalence by Proposition ??, and the two fiber sequences are equivalent. \hfill \Box

Remark 4.11. It follows from Proposition ?? and Remark ?? that we have

$$\text{disc}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))) \simeq R\text{lim} (\tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m)$$

$$\simeq R\text{lim} R\Gamma_W(\mathcal{X}, \mathbb{Z}/m(d))$$

$$\simeq R\text{lim} R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d))$$

$$=: R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d))$$

where we denote $\mathbb{Z}/m(d) := \mathbb{Z}(d) \otimes^L \mathbb{Z}/m$.

4.5. Finite ranks.

Lemma 4.12. If $\mathcal{X}$ is normal, then we have an isomorphism

$$H^j_{et}(\mathcal{X}_s, R^i\mathbb{Q}/\mathbb{Z}) \cong H^{j+1}_{et}(\mathcal{X}_s, R^i\mathbb{Z})$$

of abelian groups of finite ranks for all $j \in \mathbb{Z}$.

Proof. The isomorphism follows because $\mathcal{X}$ normal implies that $\mathbb{Q} \cong Rj_*j^*\mathbb{Q}$, hence $R^i\mathbb{Q} \cong 0$. Since $H^j_{et}(\mathcal{X}_s, R^i\mathbb{Q}/\mathbb{Z})$ is torsion and discrete, it is both of finite $\mathbb{Z}$-rank and of finite $\mathbb{Z}^1$-rank. It remains to see that it is of finite $p$-rank for any prime number $p$. But $H^j_{et}(\mathcal{X}_s, R^i\mathbb{Z}/p\mathbb{Z})$ is a finite group for any $j \in \mathbb{Z}$, because of the fiber sequence

$$R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}/p\mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/p\mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}/p\mathbb{Z})$$
and classical finiteness results in étale and Galois cohomology.

\[\square\]

**Proposition 4.13.** a) Assume that \(X^\text{red}_s\) is a simple normal crossing scheme or assume resolution of singularities for schemes over \(\kappa(s)\) of dimension at most \(d-1\) \([?\text{, Def. 2.4}]. Then \(R\Gamma_{\text{et}}(X, \mathbb{Z})\) and \(R\Gamma_{\text{et}}(X_K, \mathbb{Z})\) belong to \(D^b(\text{FLCA})\).

b) Assume that \(R\Gamma_W(X_s, \mathbb{Z})\) is a perfect complex of abelian groups. Then \(R\Gamma_{\text{et}}(X, \mathbb{Z}(d))\) and \(R\Gamma_{\text{et}}(X_K, \mathbb{Z}(d))\) belong to \(D^b(\text{FLCA})\).

**Proof.** a) Under the hypothesis, the complexes \(R\Gamma_W(X_s, \mathbb{Z})\) and \(R\Gamma_{W,h}(X_s, \mathbb{Z})\) are perfect complexes of abelian groups by Proposition ?? and Proposition ??, respectively, hence they belong to \(D^b(\text{FLCA})\) by Lemma ???. The result for \(R\Gamma_{\text{et}}(X, \mathbb{Z})\) then follows from Proposition ?? (using Notation ??), and the result for \(R\Gamma_{\text{et}}(X_K, \mathbb{Z})\) follows from Proposition ?? and Lemma ???.

b) By the proof of Proposition ??, \(R\Gamma_{\text{et}}(X, \mathbb{Z}(d))\) is (up to a shift) dual to \(R\Gamma_{\text{et}}(X_s, R^i\mathbb{Q}/\mathbb{Z})\). Hence the result follows from Lemmas ?? and ???. The statement for \(R\Gamma_{\text{et}}(X_K, \mathbb{Z}(d))\) follows from the statement \(R\Gamma_{\text{et}}(X, \mathbb{Z}(d))\) together with the perfectness of \(R\Gamma_W(X_s, R^i\mathbb{Z}(d)) \simeq R\Gamma_W(X_s, \mathbb{Z}(0))[−2d]\) by hypothesis. \(\square\)

4.6. **The topology on cohomology groups.** Recall from Section ?? that \(X\) denotes a regular connected scheme which is proper and flat over \(O_K\). We refer to ??, Section 2.16 for the following definition.

**Definition 4.14.** Let \(X_{s,i}, i \in I\) be the irreducible components of \(X_s\). We set \(X_{s,J} = \cap_{i \in J} X_{s,i}\) for any non-empty subset \(J \subseteq I\). We say that \(X/O_K\) has strictly semi-stable reduction if \(X_s\) is reduced, \(X_{s,J}\) is a divisor on \(X\), and for each non-empty \(J \subseteq I\), the scheme \(X_{s,J}\) is smooth over \(\kappa(s)\) and has codimension \(|J|\) in \(X\).

If \(X/O_K\) has strictly semi-stable reduction then \(X_s\) is a simple normal crossing scheme over \(\kappa(s)\), in the sense of Definition ??.

**Theorem 4.15.** Suppose that \(X/O_K\) has strictly semi-stable reduction. Then for any \(i \in \mathbb{Z}\), the map

\[H^i_{\text{et}}(X, \mathbb{Q}_p(d)) \to H^i_{\text{et}}(X_K, \mathbb{Q}_p(d))\]

is injective.

**Proof.** Since \(X/O_K\) has strictly semi-stable reduction, the morphism \(X \to \text{Spec}(O_K)\) is log smooth with respect to the log structures associated with \(X_s\) and \(s\) respectively, where \(s\) is the closed point of \(\text{Spec}(O_K)\), and \(X_s\) is a normal crossing divisor on \(X\). Therefore, the results of ?? apply. We have isomorphisms

\[H^i_{\text{et}}(X, \mathbb{Q}_p(d)) \simeq H^i_{\text{et}}(X, \mathbb{Q}_p^S(d))\]

compatible with the map (??), where

\[R\Gamma_{\text{et}}(X, \mathbb{Q}_p^S(d)) := R\lim R\Gamma_{\text{et}}(X, \mathbb{Q}_p(d)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\]

is the complex studied in ?? and ???. Indeed, this follows from the equivalences

\[R\Gamma(X_{\text{et}}, \mathbb{Q}_p^S(d)) \simeq R\text{Hom}(R\Gamma_{\text{et}}(X_{\text{et}}, \mathbb{Z}/p^r), (\mathbb{Z}/p^r)[-2d-1]) \simeq R\Gamma(X_{\text{et}}, \mathbb{Z}/p^r(d)),\]
given by [?, Thm. 10.1.1] and [?, Proof of Thm. 7.5], and from the fact that (??) is induced by the dual of the map
\[
R\Gamma(\mathcal{X}_{K,\text{et}}, \mathbb{Z}/p^r)[-1] \to R\Gamma_{\text{et}}(\mathcal{X}, \mathbb{Z}/p^r).
\]
Hence we are reduced to show that the map
\[
H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}_p^S(d)) \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d))
\]
is injective. By [?, Prop. 3.4(1)], [?, Section 4.1] and [?, Thm. 5.3], there is a morphism of spectral sequences from
\[
H^i_j(G_K, H^j_{\text{et}}(\mathcal{X}_{\bar{K}}, \mathbb{Q}_p(d))) \Rightarrow H^{i+j}_{\text{et}}(\mathcal{X}, \mathbb{Q}_p^S(d))
\]
to
\[
H^i_j(G_K, H^j_{\text{et}}(\mathcal{X}_{\bar{K}}, \mathbb{Q}_p(d))) \Rightarrow H^{i+j}_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d))
\]
where the first spectral sequence degenerates into isomorphisms
\[
H^j_{\text{et}}(\mathcal{X}, \mathbb{Q}_p^S(d)) \cong H^j_{\text{et}}(\mathcal{X}, H^{j-1}_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d))).
\]
Since we have [?, Prop. 5.10(1)]
\[
H^0_j(G_K, H^j_{\text{et}}(\mathcal{X}_{\bar{K}}, \mathbb{Q}_p(d))) = H^0(G_K, H^j_{\text{et}}(\mathcal{X}_{\bar{K}}, \mathbb{Q}_p(d))) = 0
\]
for any \( j \in \mathbb{Z} \), we obtain a commutative square
\[
\begin{array}{ccc}
H^j_{\text{et}}(\mathcal{X}, \mathbb{Q}_p^S(d)) & \cong & H^j_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d)) \\
\downarrow & & \downarrow \\
H^j_{\text{et}}(\mathcal{X}, H^{j-1}_{\text{et}}(\mathcal{X}, \mathbb{Q}(d))) & \cong & H^1(G_K, H^{j-1}_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d)))
\end{array}
\]
where the vertical maps are edge morphisms of the corresponding spectral sequences. Here the left vertical map is an isomorphism and the lower horizontal map is injective. It follows that the upper horizontal map is injective as well. \( \square \)

**Lemma 4.16.** Suppose that \( \mathcal{X}_s \) is a simple normal crossing scheme. Then the group
\[
H^i_{\text{et}}(\mathcal{X}, \hat{\mathbb{Z}}'(d)) := H^i(R\text{lim}_{p^I} R\Gamma_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m)
\]
is finite for any \( i \in \mathbb{Z} \).

**Proof.** By definition we have \( \mathbb{Z}(d) \cong \mathbb{Z}^c(0)[-2d] \), and by [?, Prop. 7.10 a)] and Gabber’s purity theorem [?, §8] we have
\[
\mathbb{Z}^c/m(0)[-2d] \cong Rf^! \mathbb{Z}^c/m(0)[-2d] \cong Rf^! \mathbb{Z}/m[-2d] \cong \mu_m^\otimes d
\]
on \( \mathcal{X} \) for any \( m \) prime to \( p \). Moreover, by the proper base change theorem
\[
R\Gamma_{\text{et}}(\mathcal{X}, \mu_m^\otimes d) \cong R\Gamma_{\text{et}}(\mathcal{X}_s, \mu_m^\otimes d).
\]
Thus it suffices to show that the cohomology of the right hand side of
\[
R\Gamma_{\text{et}}(\mathcal{X}, \hat{\mathbb{Z}}'(d)) \cong \text{Rlim}_{p^I} R\Gamma_{\text{et}}(\mathcal{X}_s, \mu_m^\otimes d)
\]
if finite. By the analog of Proposition ?? and induction on the number of irreducible components of \( \mathcal{X}_s \) it suffices to show that the cohomology \( R\Gamma_{\text{et}}(Y, \mathbb{Z}_l(d)) \) of each connected component \( Y \) of each \( \mathcal{X}_s^{(i)} \) is finite for all \( l \neq p \) and zero for almost all \( l \). Since \( \hat{Y} \) is smooth and proper, this is known for the extension
\( \tilde{Y} \) to the algebraic closure by Gabber’s theorem [?], and this extends to \( Y \) by a weight argument because \( d > \dim Y \), hence the Frobenius does not have eigenvalue one on \( \mathcal{H}^i_{\text{et}}(Y, \mathbb{Z}_l(d)) \). \( \square \)

**Theorem 4.17.** a) Suppose that \( (\mathcal{X}_s)^{\text{red}} \) is a simple normal crossing scheme. Then for any \( i \in \mathbb{Z} \), the object \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\text{LCA}) \) is a discrete abelian group. More precisely, \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\text{LCA}) \) is an extension of a torsion abelian group by a finitely generated abelian group.

b) Suppose that \( \mathcal{X}/\mathcal{O}_K \) has strictly semi-stable reduction and suppose that \( R\mathcal{H}^i_{\text{et}}(\mathcal{X}, \mathbb{Z}_l(0)) \) is a perfect complex of abelian groups. Then for any \( i \in \mathbb{Z} \), the object \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \) is a locally compact abelian group. More precisely, \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \) is an extension of a finitely generated abelian group by a finitely generated \( \mathbb{Z}_p \)-module endowed with the \( p \)-adic topology.

**Proof.** a) We have a long exact sequence in the abelian category \( \mathcal{LH}(\text{LCA}) \)

\[ 
\cdots \to H^i_{\text{ar}}(\mathcal{X}, \mathbb{Z}) \to H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \to H^{i+1}_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Q}/\mathbb{Z}) \to \cdots
\]

where \( H^{i+1}_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Q}/\mathbb{Z}) \) is a discrete torsion abelian group (see the proof of Proposition ??) and \( H^i_{\text{ar}}(\mathcal{X}, \mathbb{Z}) \simeq H^i_{\text{W}}(\mathcal{X}_s, \mathbb{Z}) \) is a discrete finitely generated abelian group by assumption. Hence \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\text{LCA}) \) is an extension of a torsion abelian group by a finitely generated abelian group. It follows that \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \in \text{LCA} \) since \( \text{LCA} \subset \mathcal{LH}(\text{LCA}) \) is stable under extensions [?], Prop. 1.2.29(c)].

b) We have a long exact sequence in \( \mathcal{LH}(\text{LCA}) \)

\[ 
H^i_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}(d)) \to H^i_{\text{ar}}(\mathcal{X}, \mathbb{Z}(d)) \to H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \to H^{i+1}_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}(d))
\]

where \( H^i_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}(d)) \) is a discrete finitely generated abelian group by assumption. Moreover, \( H^i_{\text{ar}}(\mathcal{X}, \mathbb{Z}(d)) \in \mathcal{LH}(\text{LCA}) \) is the group (see Remark ??)

\[ 
H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \simeq \prod_l H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}_l(d)) := \prod_l H^i(R\mathcal{L}(R\mathcal{H}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/l^*)
\]

which by Lemma ?? is the product of a finite group and the finitely generated \( \mathbb{Z}_p \)-module \( H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}_p(d)) \) endowed with the \( p \)-adic topology. If we can show that the image of the map \( H^i_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}(d)) \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \) is finite, then it will follow that \( H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \) is an extension of a finitely generated abelian group by a profinite abelian group. Since we have an isomorphism of finitely generated \( \mathbb{Z}_p \)-modules

\[ 
H^i_{\text{W}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}(d)) \otimes \mathbb{Z} \simeq H^i_{\text{et}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}_p(d)),
\]

it is enough to show that the image of the map

\[ 
H^i_{\text{et}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Z}_p(d)) \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}_p(d))
\]

is finite, or equivalently that the map

\[ 
H^i_{\text{et}}(\mathcal{X}_s, \mathbb{R}^i\mathbb{Q}_p(d)) \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}_p(d))
\]

is the zero-map. This follows from Theorem ?? by the localization sequence. \( \square \)
5. Duality theorems

The goal of this section is to prove various duality theorems. In particular, we prove Theorem ?? and Corollary ?? of the introduction. Throughout this section, we use the notation and definitions introduced in Section ?? and Section ?? and we assume the following

**Hypothesis 5.1.** At least one of the following conditions holds:

- we have $d \leq 2$;
- the scheme $(X_s)_{\text{red}}$ is a simple normal crossing scheme and $R\Gamma_W(X_s, \mathcal{Z}(0))$ is a perfect complex of abelian groups.

In view of Proposition ??, $d \leq 2$ implies that $R\Gamma_W(X_s, \mathcal{Z}(0))$ is a perfect complex of abelian groups.

5.1. Duality with $\mathbb{Z}$-coefficients.

**Theorem 5.2.** Assume Hypothesis ?? Then there is a perfect pairing

$$R\Gamma_{ar}(X_K, \mathbb{Z}(d)) \otimes\mathbb{L} R\Gamma_{ar}(X_K, \mathbb{Z}) \rightarrow \mathbb{Z}[-2d]$$

in $D^b(FLCA)$.

The rest of Section ?? is devoted to the proof of Theorem ?? We assume Hypothesis ?? throughout.

**Proof.** Recall from Proposition ?? that $R\Gamma_{ar}(X_K, \mathbb{Z}(n))$ belongs to $D^b(FLCA)$ for $n = 0, d$, so that the tensor product

$$R\Gamma_{ar}(X_K, \mathbb{Z}(d)) \otimes\mathbb{L} R\Gamma_{ar}(X_K, \mathbb{Z})$$

defined in Section ??, makes sense. Moreover, the equivalence $Rf^!\mathcal{Z}(0)^X \simeq \mathcal{Z}(0)^X$ of Remark ?? and the push-forward map $Rf_s^!\mathcal{Z}(0)^X_s \rightarrow \mathcal{Z}(0)^s \simeq \mathbb{Z}[0]$ of [?, Cor. 3.2] induce trace maps

$$R\Gamma_W(X_s, Rf^!\mathcal{Z}(d)) \simeq R\Gamma_W(X_s, \mathcal{Z}(0)[−2d]) \rightarrow R\Gamma_W(s, \mathbb{Z}[−2d]) \rightarrow \mathbb{Z}[−2d − 1]$$

and

$$R\Gamma_{et}(X_s, Rf^!\mathcal{Z}/m(d)) \rightarrow R\Gamma_{et}(s, \mathbb{Z}/m[−2d]) \rightarrow \mathbb{Z}/m[−2d − 1]$$

We start with the following

**Proposition 5.3.** The canonical product map $\mathbb{Z} \otimes\mathbb{L} \mathbb{Z}(d) \rightarrow \mathbb{Z}(d)$ in the derived $\infty$-category of étale sheaves over $\mathcal{X}_{O_K^\text{un}}$ and $\mathcal{X}_{K^\text{un}}$ induce perfect pairings

$$R\Gamma_{et}(X_s, Rf^!\mathcal{Z}/m) \otimes\mathbb{L} R\Gamma_{et}(X, \mathbb{Z}/m(d)) \rightarrow R\Gamma_{et}(X_s, Rf^!\mathcal{Z}/m(d)) \rightarrow \mathbb{Z}/m[−2d − 1]$$

and

$$R\Gamma_{et}(X_s, Rf^!\mathcal{Z}/m(d)) \otimes\mathbb{L} R\Gamma_{et}(X, \mathbb{Z}/m) \rightarrow R\Gamma_{et}(X_s, Rf^!\mathcal{Z}/m(d)) \rightarrow \mathbb{Z}/m[−2d − 1]$$

for any $m$.

**Proof.** Consider the commutative square

$$
\begin{array}{ccc}
R\Gamma_{et}(X, \mathbb{Z}) \otimes\mathbb{L} R\Gamma_{et}(X, \mathbb{Z}(d)) & \rightarrow & R\Gamma_{et}(X, \mathbb{Z}(d)) \\
\downarrow & & \downarrow \\
R\Gamma_{et}(X_K, \mathbb{Z}) \otimes\mathbb{L} R\Gamma_{et}(X, \mathbb{Z}(d)) & \rightarrow & R\Gamma_{et}(X_K, \mathbb{Z}(d)).
\end{array}
$$
Taking the fibers of the vertical arrows induces the product map
\[(22) \quad R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}(d)),\]
and the product map
\[(23) \quad R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}(d)) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}(d))\]
is obtained similarly. By [? , Thm. 7.5] applied to \(\mathcal{F} = \mathbb{Z}/m\), the pairing
\[R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}/m) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1],\]
induced by (??), is perfect. The pairing induced by (??)
\[R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}/m(d)) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]\]
is perfect as well, since it reduces, by purity and proper base change, to
\[R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m[-2d]) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m[-2d]) \to \mathbb{Z}/m[-2d-1]\]
which is perfect by [? , Thm. 5.1] applied to \(\mathcal{F} = \mathbb{Z}/m\).
\[\square\]

For \(n = 0\) or \(n = d\), consider the product map
\[R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}(n)) \otimes^L R\Gamma_{et}(\mathcal{X}_\mathcal{O}_{\mathcal{X}_{un}}, \mathbb{Z}(d - n)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^1\mathbb{Z}(d)).\]
This product map is induced by the obvious product maps \(\mathbb{Z} \otimes^L \mathbb{Z}(d) \to \mathbb{Z}(d)\) in the derived \(\infty\)-category of étale sheaves over \(\mathcal{X}_\mathcal{O}_{\mathcal{X}_{un}}\) and \(\mathcal{X}_\mathcal{O}_{\mathcal{X}_{un}}\), as in the proof of Proposition ??. Applying \(R\Gamma(W_n(s), -)\) and composing with the trace map, we obtain
\[(24) \quad R\Gamma(W(\mathcal{X}_s, Ri^1\mathbb{Z}(n)) \otimes^L R\Gamma(\mathcal{X}, \mathbb{Z}(d - n)) \to R\Gamma(W(\mathcal{X}_s, Ri^1\mathbb{Z}(d)) \to \mathbb{Z}[-2d - 1].\]
This yields the morphisms
\[R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1])\]
which in turn induces
\[(25) \quad \tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1])\]
for \(a \ll 0\), since the right hand side is bounded. Composing (??) with the canonical map (see Lemma ??)
\[R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1]) \to R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1])\]
we obtain
\[(26) \quad \tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1]).\]

**Proposition 5.4.** The map (??) factors through an equivalence
\[(27) \quad R\text{Gamma}(\mathcal{X}, \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^1\mathbb{Z}), \mathbb{Z}[-2d - 1])\]
in \(D^b(\text{FLCA})\).
Proof. One has

\[ R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) := \tau^{>n} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \cong \bigoplus \bigoplus \mathbb{Z} \cong (\text{hocolim} R\text{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}/m(d)), \mathbb{Q}/\mathbb{Z}))^D \]

\[ \cong (\text{hocolim} R\text{Hom}(R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)), \mathbb{Q}/\mathbb{Z}))^D \]

\[ \cong R\text{Hom}(\text{hocolim} R\text{Hom}(R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)), \mathbb{Q}/\mathbb{Z}[{-2d - 1}]), \mathbb{R}/\mathbb{Z}[{-2d - 1}]) \]

\[ \cong R\text{Hom}(\text{hocolim} R\text{Hom}(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}/m), \mathbb{R}/\mathbb{Z}[{-2d - 1}]) \]

\[ \cong R\text{Hom}(R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]), \mathbb{R}/\mathbb{Z}[{-2d - 1}]) \]

\[ \cong R\text{Hom}(R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]), \mathbb{R}) \cong 0 \]

where we use Proposition ??, the vanishing

\[ R\text{Hom}(R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]), \mathbb{R}) \cong 0 \]

proven in Lemma ?? below and \( \mathbb{R}^1\mathbb{Q} \cong 0 \). \( \square \)

**Lemma 5.5.** We have

\[ R\text{Hom}(R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]), \mathbb{R}) \cong R\text{Hom}(\mathbb{R}, R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1])) \cong 0. \]

**Proof.** As observed above, we have \( R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]) \cong R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Q}/\mathbb{Z}) \).

Since \( R\text{Hom}(\mathbb{R}, -) \) and \( R\text{Hom}(-, \mathbb{R}) \) are exact functors, and using the \( t \)-structure on \( D^b(\text{FLCA}) \), we may suppose that \( R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Q}/\mathbb{Z}) \) is cohomologically concentrated in one degree. Hence one is reduced to show that

\[ R\text{Hom}(\mathbb{R}, A) \cong R\text{Hom}(-, \mathbb{R}) \cong 0 \]

for any torsion discrete abelian group of finite ranks \( A \). This follows from [??, Prop. 4.15 (i) and (vii)]. \( \square \)

**Corollary 5.6.** We have

\[ R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)), \mathbb{R}) \cong R\text{Hom}(\mathbb{R}, R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))) \cong 0. \]

**Proof.** In the proof of Proposition ??, we have shown that \( R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) \) is, up to a shift, Pontryagin dual to \( R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}[1]) \). Hence the corollary follows from Lemma ?? since \( R\text{Hom}(\mathcal{X}_s, \mathbb{R}^1\mathbb{Q}/\mathbb{Z}) = R\text{Hom}(\mathcal{X}_s, \mathbb{R}_s \mathcal{X}[1]) \) for any \( \mathcal{X}_s \in D^b(\text{FLCA}) \).

Similarly, we have the

**Proposition 5.7.** The map

\[ R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}(d)) \rightarrow R\text{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[{-2d - 1}]), \]

induced by \( (?) \), factors through an equivalence

\[ R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}(d)) \xrightarrow{\sim} R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[{-2d - 1}]). \]

**Proof.** Recall from Remark ?? that we have

\[ \mathbb{R}^1\mathbb{Z}(d) = \mathbb{R}^1\mathbb{Z}(0)[-2d] \cong \mathbb{Z}(0)[-2d]. \]

If \( \mathcal{X}_s \) is a simple normal crossing scheme, we may therefore identify the map

\[ R\Gamma_W(\mathcal{X}_s, \mathbb{R}^1\mathbb{Z}(d))[2d] \xrightarrow{\sim} R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[{-2d - 1}])[2d] \]
Proposition 5.8. There is an equivalence
\[ R\Gamma_W(\mathcal{X}_s, Z^0(0)) \cong R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Z), Z[-1]) \cong R\text{Hom}(R\Gamma_W(\mathcal{X}_s, Z), Z[-1]) \]
which is an equivalence of perfect complexes of abelian groups by Proposition ??, Theorem ??, and Lemma ??.

If \( d \leq 2 \), we may identify (?) with the morphism
\[ R\Gamma_W(\mathcal{X}_s, Z^0(0)) \cong R\text{Hom}(R\Gamma_{Wh}(\mathcal{X}_s, Z), Z[-1]) \cong R\text{Hom}(R\Gamma_{Wh}(\mathcal{X}_s, Z), Z[-1]) \]
which is an equivalence of perfect complexes of abelian groups by Proposition ?? and [?, Thm. 4.2] (using the fact that for a curve, étale and d-cohomology agree).

Note that, if \( d \leq 2 \), then the following diagram in \( D^b(\text{LCA}) \)
\[
\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(d)) & \xrightarrow{(??)} & R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, Z), Z[-2d - 1]) \\
\Downarrow[??] & & \Downarrow[\cong]
\end{array}
\]
\[
R\text{Hom}(R\Gamma_W(\mathcal{X}, Z), Z[-2d - 1]) \xrightarrow{\cong} R\text{Hom}(R\Gamma_{Wh}(\mathcal{X}_s, Z), Z[-2d - 1])
\]
commutes. If \( \mathcal{X}^{\text{red}}_s \) is a simple normal crossing scheme, then the same diagram with \( R\Gamma_{Wh}(\mathcal{X}_s, Z) \) replaced by \( R\Gamma_W(\mathcal{X}_s, Z) \) commutes as well.

We now combine Proposition ?? and Proposition ?? to prove our result for the generic fiber.

**Proposition 5.8.** There is an equivalence
\[ R\Gamma_{ar}(\mathcal{X}_K, Z(d)) \cong R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, Z), Z[-2d]) \]
such that, for any \( m \), there is a commutative square
\[
\begin{array}{ccc}
R\Gamma_{ar}(\mathcal{X}_K, Z(d)) & \xrightarrow{\cong} & R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, Z), Z[-2d]) \\
\Downarrow & & \Downarrow
\end{array}
\]
\[
R\Gamma_{et}(\mathcal{X}_K, Z/m(d)) \xrightarrow{\cong} R\text{Hom}(R\Gamma_{et}(\mathcal{X}_K, Z/m), \mathbb{Q}/\mathbb{Z}[-2d]),
\]
where the lower horizontal map is induced by duality for the usual étale cohomology of the variety \( \mathcal{X}_K \).

**Proof.** We start with the commutative diagram:
\[
\begin{array}{ccc}
R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_{O_{K^{an}}}, Z(d)) & \xrightarrow{} & R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d)) \\
\Downarrow & & \Downarrow
\end{array}
\]
\[
R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d)) \xrightarrow{} R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d))
\]
where the map
\[ R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d)) \rightarrow R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d)) \]
is induced by the map $\mathbb{Z} \otimes^L \mathbb{Z}(d) \to \mathbb{Z}(d)$ over $\mathcal{O}_{\mathbb{K}_{un}}$ as follows. Consider the morphism

$$\tilde{i}_* R_i^! \mathbb{Z} \to \mathbb{Z} \to R\text{Hom}_{\mathcal{O}_{\mathbb{K}_{un}}}(\mathbb{Z}(d), \mathbb{Z}(d))$$

and apply $\tilde{i}^*$, where $\text{Hom}$ denotes the internal Hom in the category of sheaves on the small étale site of the corresponding scheme. Applying $R\Gamma(W_{\mathbb{K}(s)}; -)$ to the diagram above, we obtain the following commutative diagram in $\textbf{D}(\text{Ab})$, where $\text{tr}$ is the trace map:

$$R\Gamma_W(\mathcal{X}, R_i^! \mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X}, R_i^! \mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to \mathbb{Z}[-2d - 1]$$

It gives the following commutative diagram in $\textbf{D}(\text{Ab})$

$$R\Gamma_W(\mathcal{X}, R_i^! \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d - 1])$$

We obtain the following commutative diagram

$$R\Gamma_W(\mathcal{X}, R_i^! \mathbb{Z}(d)) \to R\text{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d - 1])$$

in the derived $\infty$-category $\textbf{D}^b(\text{LCA})$, where the lower right map is given by Lemma ???. By construction of the maps (??) and (??), we obtain the following
commutative diagram

\[
\begin{array}{ccc}
\text{RHom}(\text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) & \xrightarrow{??} & \text{r} \gamma \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\end{array}
\]

hence the upper square in the commutative square

\[
\begin{array}{ccc}
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\end{array}
\]

It follows that the diagram is an equivalence of cofiber sequences in \( \mathbf{D}^b(\text{LCA}) \). Tensoring the upper commutative square with \( \mathbb{Z}/m \) gives a square equivalent to the commutative square

\[
\begin{array}{ccc}
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\text{R} \Gamma (X, \mathbb{Z}) & \xrightarrow{??} & \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\end{array}
\]

where the horizontal maps are induced by the perfect pairings of Proposition ???. This yields the commutative square of Proposition ???

It remains to prove that

\[
\text{R} \Gamma (X, \mathbb{Z}) \to \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z}[-2d])
\]
is an equivalence.

Lemma 5.9. The map

\[
\text{R} \Gamma (X, \mathbb{Z}) \to \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z})
\]
is an equivalence.

Proof. We have

\[
\text{R} \Gamma (X, \mathbb{Z}) \simeq \text{RHom} \, (X, \mathbb{Z}), \mathbb{Z})
\]
by Lemma ??, since $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z})$ is a perfect complex of abelian groups by Propositions ??, ??, and ??.

In view of the cofiber sequence

$$R\Gamma_W(\mathcal{X}, R^i\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$$

one is reduced to check that the map

$$R\Gamma_W(\mathcal{X}, R^i\mathbb{Z}) \to R\text{Hom}(R\text{Hom}(R\Gamma_W(\mathcal{X}, R^i\mathbb{Z}), \mathbb{Z}), \mathbb{Z})$$

is an equivalence. Recall from the proof of Proposition ?? that we have

$$R\text{Hom}(R\text{Hom}(R\Gamma_W(\mathcal{X}, R^i\mathbb{Z}), \mathbb{Z}), \mathbb{Z}) \
\simeq R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, Z(d))[2d + 1], \mathbb{Z}) \
\simeq R\text{Hom}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}/m)(2d + 1), \mathbb{R}/\mathbb{Z}[-1]) \
\simeq R\Gamma_{ar}(\mathcal{X}, Z(d))^D[-2d - 2] \
\simeq (\text{hocolim } R\text{Hom}(R\Gamma_W(\mathcal{X}, Z/m)(d), \mathbb{Q}/\mathbb{Z}))^{DD}[−2d − 2] \
\simeq \text{hocolim } R\text{Hom}(R\Gamma_W(\mathcal{X}, Z/m)(d), \mathbb{Q}/\mathbb{Z}[−2d − 1])[-1] \
\simeq R\Gamma_W(\mathcal{X}_K, R^i\mathbb{Q}/\mathbb{Z})[−1] \
\simeq R\Gamma_W(\mathcal{X}, R^i\mathbb{Z}).$$

where the second equivalence follows from Corollary ??.

Consider the pairing

\[ R\Gamma_{ar}(\mathcal{X}_K, Z(d)) \otimes \mathbb{L} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \longrightarrow \mathbb{Z}[-2d] \]

induced by the equivalence of Proposition ??.

The induced map

\[ R\Gamma_{ar}(\mathcal{X}_K, Z(d)) \xrightarrow{\sim} R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d]) \]  

is (tautologically) the equivalence of Proposition ??.

Moreover, the map

\[ R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{Z}[-2d]) \]

induced by (??) is an equivalence as well. Indeed, applying $R\text{Hom}(−, \mathbb{Z}[−2d])$ to (??) and using Lemma ??, we obtain the composite equivalence

\[ R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} R\text{Hom}(R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[−2d]), \mathbb{Z}[−2d]) \]

\[ \xrightarrow{\sim} R\text{Hom}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{Z}[−2d]) \]

which is, up to equivalence, the map (??).

5.2. Pontryagin duality. Recall that we denote by FLCA the category of locally compact abelian group of finite ranks in the sense of [?]. It follows from (??) and Proposition ?? that the following definition makes sense.

**Definition 5.10.** Assume Hypothesis ??.

For $n = 0, d$, we define

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes \mathbb{L} \mathbb{R}/\mathbb{Z};$$

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \otimes \mathbb{L} \mathbb{R}/\mathbb{Z}.$$
Corollary 5.11. Assume Hypothesis ???. Then one has equivalences

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \sim \to R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])
\]

and

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \sim \to R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])
\]
in \( D^b(\text{FLCA}) \).

Proof. By Theorem ?? and [?, Rem. 4.3(ii)], we have

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \sim \to R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d])
\]
\[
\sim \to R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}), R\text{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]))
\]
\[
\simeq R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \otimes^L \mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d])
\]
\[
:={R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}[-2d])}.
\]

Applying the functor \( R\text{Hom}(\mathbb{R}/\mathbb{Z}[-2d]) \) and using Pontryagin duality, we obtain the first equivalence of the Corollary.

Similarly, we have

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}) \sim \to R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{Z}[-2d])
\]
\[
\simeq R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d)) \otimes^L \mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d])
\]
\[
:={R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])}.
\]

\[\Box\]

Corollary 5.12. Suppose that \( \mathcal{X}/\mathcal{O}_K \) has strictly semi-stable reduction and suppose that \( R\Gamma_W(x, \mathcal{Z}^*(\mathcal{O})) \) is a perfect complex of abelian groups. Then for any \( i \in \mathbb{Z} \), we have an isomorphism of locally compact groups

\[
H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \sim \to H^{2d-i}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))^D
\]

and an isomorphism of discrete groups

\[
H^i(\mathcal{X}_K, \mathbb{Z}) \sim \to H^{2d-i}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)).
\]

Proof. In view of Theorem ?? and Lemma ??, the equivalence in \( D^b(\text{FLCA}) \)

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \sim \to R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))[-2d]
\]

induces isomorphisms

\[
H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \sim \to H^i(R\Gamma(\mathcal{X}_K, \mathbb{Z}(d))[-2d]) \simeq H^{2d-i}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))^D
\]
of locally compact abelian groups. Similarly, the equivalence in \( D^b(\text{FLCA}) \)

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \sim \to R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z})[-2d]
\]
induces isomorphisms

\[
H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \sim \to H^i(R\Gamma(\mathcal{X}_K, \mathbb{Z})[-2d]) \simeq H^{2d-i}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z})^D
\]
of compact abelian groups.

\[\Box\]
Remark 5.13. Corollary ?? as well as Corollary ?? can be extended to Tate twists \( n > d \), or equivalently \( n < 0 \). Let \( X \) be a regular, proper and flat scheme over \( \mathcal{O}_K \). Assume that \( X \) is connected of Krull dimension \( d \), and let \( n > d \). We moreover assume that \( R\Gamma_{et}(X_K, \mathbb{Z}(n)) \) is bounded\(^4\).

Then we define
\[
R\Gamma_{ar}(X_K, \mathbb{Z}(n)) := R\Gamma_{et}(X_K, \mathbb{Z}(n)) \hat{\otimes} \mathbb{Z} \simeq \lim_{\leftarrow} \left( R\Gamma_{et}(X_K, \mathbb{Z}(n)) \hat{\otimes} \mathbb{Z}/m \right)
\]

where the limit is computed in the \( \infty \)-category \( D^b(LCA) \), see Proposition ?? and Remark ??.

Dually, we define
\[
R\Gamma_{ar}(X_K, \mathbb{Z}(d-n)) := R\Gamma_{et}(X_K, \mathbb{Q}/\mathbb{Z}(d-n))[-1] \simeq \lim_{\rightarrow} R\Gamma_{et}(X_K, \mathbb{Z}/m\mathbb{Z}(d-n))[-1] \simeq \lim_{\rightarrow} R\Gamma_{et}(X_K, \mu_m^d \otimes m)[-1]
\]

where the colimit is computed in the \( \infty \)-category \( D^b(LCA) \), see Proposition ??.

Here we follow the abuse of Notation ??.

Poincaré duality for étale cohomology of \( X_K \) together with Tate duality for Galois cohomology of the local field \( K \) gives an equivalence
\[
R\Gamma_{et}(X_K, \mu_m^d) \sim R\text{Hom}(R\Gamma_{et}(X_K, \mu_m^d), \mathbb{R}/\mathbb{Z}[-2d]).
\]

of discrete complexes in \( D^b(FLCA) \). We obtain an equivalence
\[
R\Gamma_{et}(X_K, \mathbb{Z}(n)) \sim R\text{Hom}(R\Gamma_{et}(X_K, \mu_m^d), \mathbb{R}/\mathbb{Z}[-2d])
\]
in \( D^b(FLCA) \) and an isomorphism of compact groups of finite ranks
\[
H^i_{ar}(X_K, \mathbb{Z}(n)) \sim H^{2d-i}_{ar}(X_K, \mathbb{R}/\mathbb{Z}(d-n))^D
\]
for any \( i \in \mathbb{Z} \).

Remark 5.14. It might also be possible, although probably not so trivial, to prove the analogue of Corollary ?? and Corollary ?? in the case \( n = 1, d = 2 \). We refer to the work of Karpuk [?, Thm. 4.2.2] for a first step in this direction.

It would be interesting to translate Karpuk’s result in the LCA-language used in this paper, in order to obtain a perfect Pontryagin duality between locally compact abelian groups of finite ranks.

\(^4\)This rather strong condition can be avoided using the trick of Definition ??.
6. The conjectural picture

We conjecture the existence of a cohomology theory on the category of separated schemes of finite type over \(\text{Spec}(\mathcal{O}_K)\), with values in \(\mathbf{D}^b(\text{FLCA})\), which we denote by

\[
R\Gamma_{\text{ar}}(-, A(n))
\]

for any \(A \in \text{FLCA}\) and any \(n \in \mathbb{Z}\). Furthermore we conjecture that the conclusion of Theorem ?? holds in full generality: For any smooth proper \(\mathcal{X}_K\) over \(K\) of pure dimension \(d - 1\), any Tate twist \(n \in \mathbb{Z}\), and any \(A \in \text{FLCA}\), there is an equivalence

\[
R\Gamma_{\text{ar}}(\mathcal{X}_K, A^D(n)) \xrightarrow{\sim} R\text{Hom}(R\Gamma_{\text{ar}}(\mathcal{X}_K, A(d - n)), \mathbb{R}/\mathbb{Z}[−2d])
\]

induced by a trace map \(H^d_{\text{ar}}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}\), where \(A^D\) denotes the Pontryagin dual of \(A\). However, we do not expect the analog of Corollary ?? to be true in general, since the groups \(H^i_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(n))\) cannot be expected to be locally compact for arbitrary Tate twist \(n\) (as one can see from [?] for \(n = 1\)). Instead, they could be seen as condensed abelian groups (or in the language used in this paper, as objects of the heart \(\mathcal{L}H(\text{LCA})\)) of the left \(t\)-structure on \(\mathbf{D}^b(\text{LCA})\) in the sense of [?] and [?]). In contrast, we do expect isomorphisms of compact abelian groups

\[
H^i_{\text{ar}}(\mathcal{X}, A(n)) \simeq H^{2d+1-i}_{\text{ar}}(\mathcal{X}, R^i A^D(d - n))_{\mathbb{D}}
\]

for any \(i, n \in \mathbb{Z}\) and any compact \(A \in \text{FLCA}\). Concerning the relationship between \(R\Gamma_{\text{ar}}(-, A(n))\) and known cohomology theories, we expect the following, for \(\mathcal{X}\) a regular proper flat scheme over \(\text{Spec}(\mathcal{O}_K)\) of pure Krull dimension \(d\).

- For any positive integer \(m\), we have
  \[
  R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}/m(n)) \simeq R\Gamma_{\text{et}}(\mathcal{X}_K, \mathbb{Z}/m(n)).
  \]
  In particular, for any prime \(l\), one has equivalences

  \[
  R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}_l(n)) \simeq R\Gamma_{\text{et}}(\mathcal{X}_K, \mathbb{Z}_l(n)) \hat{\otimes} \mathbb{Z}_l \simeq R\Gamma_{\text{et}}(\mathcal{X}_K, \mathbb{Z}_l(n))
  \]

  where \((-) \hat{\otimes} \mathbb{Z}_l := R\text{lim}((- \hat{\otimes} \mathbb{Z}/l^n))\) is the \(l\)-adic completion functor.

- The canonical map

  \[
  R\Gamma_{\text{ar}}(-, \mathbb{Z}(n)) \otimes^L A \xrightarrow{\sim} R\Gamma_{\text{ar}}(-, A(n))
  \]

  is an equivalence for \((-) = \mathcal{X}, \mathcal{X}_s\) and any ring object \(A\), and for \((-) = \mathcal{X}_K\) if \(A\) has no topological \(p\)-torsion\(^5\). For example the map

  \[
  R\Gamma_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes^L A \xrightarrow{\sim} R\Gamma_{\text{ar}}(\mathcal{X}_K, A(n))
  \]

  is an equivalence for \(A = \mathbb{R}\) and \(A = \mathbb{Q}_l\) if \(l \neq p\).

---

\(^5\)The locally compact group \(A\) has a unique filtration by closed subgroups with graded pieces \(A_{\geq 1}, A_1\) and \(A_2\) of type \(S^1\), \(\mathbb{A}\) and \(\mathbb{Z}\) respectively. Then \(A_1\) is the direct sum of a finite dimensional \(\mathbb{R}\)-vector space and topological torsion group \(A_{\text{top-tor}}\), which in turn has a topological \(p\)-torsion component \(A_p\) (see [?, Section 2]). We say that \(A\) has no topological \(p\)-torsion if \(A_p = 0\).
We have
\[ R\Gamma_{ar}(X_s, Z(n)) \simeq R\Gamma_{Wh}(X_s, Z(n)) \]
where \( R\Gamma_{Wh}(X_s, Z(n)) \) is motivic \( Wh \)-cohomology in the sense of [?], see Section ???. Moreover, the cofiber
\[ C_{ar}(X, n) := \text{Cofib}(R\Gamma_{ar}(X, Z(n)) \to R\Gamma_{ar}(X_s, Z(n))) \]
is a perfect complex of \( Z_p \)-modules such that
\[
C_{ar}(X, n) \otimes \mathbb{Q} \simeq R\Gamma(X^\ell, \Omega^{<n}_{X^\ell/K})
\]
where the right hand side denotes de Rham cohomology modulo the \( n \)-step of the Hodge filtration.

We have equivalences
\[
R\Gamma_{ar}(X, Z(n)) \otimes \mathbb{Z}_p \otimes \hat{\mathbb{Z}}_p \simeq R\Gamma_{ar}(X, Z_p(n)) \simeq R\Gamma_{et}(X, Z(n)) \hat{\otimes} \mathbb{Z}_p =: R\Gamma_{et}(X, Z_p(n))
\]
where \( R\Gamma_{et}(X, Z(n)) \) denotes étale motivic cohomology, i.e. étale hypercohomology of Bloch’s motivic complex, see Section ???. Note that \( R\Gamma_{et}(X, Z(n)) \hat{\otimes} \mathbb{Q}_p \) is equivalent to the syntomic cohomology of Fontaine-Messing [?], at least if \( X/O_K \) is smooth and \( n < p - 1 \), see [?, Thm. 1.3] and [?, Prop. 7.21, Rem. 7.23]. For general regular proper flat \( X \) and arbitrary \( n \), a conjectural syntomic description of \( R\Gamma_{et}(X, Z(n)) \hat{\otimes} \mathbb{Q}_p \) is given by [?, Cor. 7.17].

One has
\[ R\Gamma_{ar}(X_s, R^i\mathbb{Z}(n)) \simeq R\Gamma_W(X_s, R^i\mathbb{Z}(n)) \]
where the right hand side is defined as in Section ???. On the one hand, Bloch’s cycle complex \( Z(n) \), seen as a complex of étale sheaves, is expected\(^6\) to satisfy
\[ \tau^{\leq n+1} R^i\mathbb{Z}(n) \simeq Z^c(d - n)[-2d] \]
where \( Z^c(d - n) \) denotes Bloch’s cycle complex in its homological notation as in [?]. On the other hand, the cohomology groups of \( R\Gamma_W(X_s, \tau^{> n+1} R^i\mathbb{Z}(n)) \) are of the form \( (\mathbb{Q}_p/\mathbb{Z}_p)^a \) up to a finite group. In particular, we have
\[ R\Gamma_W(X_s, \tau^{> n+1} R^i\mathbb{Z}(n)) \hat{\otimes} \mathbb{Q}_p \simeq 0 \]
Thus we expect
\[
R\Gamma_{ar}(X_s, R^i\mathbb{Z}(n)) \hat{\otimes} \mathbb{Q}_p \simeq R\Gamma_W(X_s, Z^c(d - n)) \hat{\otimes} \mathbb{Q}_p[-2d] \\
\simeq R\Gamma_W(X_s, Z^c(d - n)) \hat{\otimes} \mathbb{Q}_p[-2d] \\
\simeq R\Gamma_{et}(X_s, Z^c(d - n)) \hat{\otimes} \mathbb{Q}_p[-2d] \\
=: R\Gamma_{et}(X_s, \mathbb{Q}_p(d - n))[-2d].
\]
\(^6\)This is known at least if \( X/O_K \) is smooth by [?, Thm. 1.2.1].
The equivalence (77) is justified by the fact that \( R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(d - n)) \) is expected to be a perfect complex of abelian groups. We also expect\(^7\) an equivalence

\[
R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(d - n)) \xrightarrow{\sim} \mathcal{R}\text{Hom}(R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(d - n)), \mathbb{Z}[-1])
\]

of perfect complexes of abelian groups.

By contrast, one has

\[
R\Gamma_{ar}(\mathcal{X}_s, R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p =: R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Q}_p(n)).
\]

Therefore, the map

\[
(41) \quad R\Gamma_{ar}(\mathcal{X}_s, R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \to R\Gamma_{ar}(\mathcal{X}_s, R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p
\]

is expected to be an equivalence if and only if one has

\[
(42) \quad R\Gamma_{et}(\mathcal{X}_s, \tau^{>n+1}R^i\mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq 0.
\]

Now we observe that (77) holds if and only if \( n \geq d \). This condition is indeed sufficient by [7, Cor. 7.2 (a)]. In view of the de Rham description (??) of \( C_{ar}(\mathcal{X}, d - n) \), the condition \( n \geq d \) is also necessary for (77) to hold, because we expect an isomorphism of discrete torsion groups

\[
(43) \quad H^i_{et}(\mathcal{X}_s, \tau^{>n+1}R^i\mathbb{Z}(n)) \xrightarrow{\sim} H^{2d+1-i}(C_{ar}(\mathcal{X}, d - n))^D.
\]

Hence the map (??) is an equivalence if and only if \( n \geq d \). Moreover, it follows from (??) and (??) that (??) is an equivalence in cohomological degrees \( \leq n + 1 \), for any Tate twist \( n \).

- For any \( n \geq 0 \), one has an equivalence

\[
(44) \quad R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq R\Gamma_{syn}(\mathcal{X}_K, n)
\]

where the right hand side is the Nekovar-Nizioł syntomic cohomology [7]. Indeed, by (??) and (??), one has a cofiber sequence

\[
R\Gamma_{et}(\mathcal{X}_s, \mathbb{Q}_p(d - n)[−2d]) \to R\Gamma_{et}(\mathcal{X}, \mathbb{Q}_p(n)) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p
\]

where the left map is induced by the adjunction maps \( \tau^{\leq n+1}R^i\mathbb{Z}(n) \to R^i\mathbb{Z}(n) \) and \( Ri_nR^i \to \text{Id} \). But \( R\Gamma_{syn}(\mathcal{X}_K, n) \) lies in the same cofiber sequence by [7, Cor. 7.13 and Corollary 7.17], hence (??) follows. The induced map

\[
(45) \quad R\Gamma_{syn}(\mathcal{X}_K, n) \simeq R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq R\Gamma_{et}(\mathcal{X}_K, \mathbb{Q}_p(n))
\]

is an equivalence if and only if \( n \geq d \), as (??) is an equivalence if and only if \( n \geq d \). For any Tate twist \( n \geq 0 \), the map (??) is an equivalence in cohomological degrees \( \leq n \), as (??) is an equivalence in cohomological degrees \( \leq n + 1 \).

\(^7\)See Theorem ?? for a special case.
• We have
\[
\dim_{\mathbb{Q}} H^i_{et}(\mathcal{X}_K, \mathbb{Q}_l(n)) = \dim_{\mathbb{R}} H^i_{et}(\mathcal{X}_K, \mathbb{R}(n))
\]
for any \(i, n \in \mathbb{Z}\) and any prime \(l \neq p\). In particular, the left hand side is independent on \(l \neq p\).

REFERENCES
