Compressive learning for patch-based image denoising

Hui Shi, Yann Traonmilin, Jean-François Aujol

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Abstract. The Expected Patch Log-Likelihood algorithm (EPLL) and its extensions have shown good performances for image denoising. The prior model used by EPLL is usually a Gaussian Mixture Model (GMM) estimated from a database of image patches. Classical mixture model estimation methods face computational issues as the high dimensionality of the problem requires training on large datasets. In this work, we adapt a compressive statistical learning framework to carry out the GMM estimation. With this method, called *sketching*, we estimate models from a compressive representation (the *sketch*) of the training patches. The cost of estimating the prior from the sketch no longer depends on the number of items in the original large database. To accelerate further the estimation, we add another dimension reduction technique (low-rank modeling of the covariance matrices) to the compressing learning framework. To demonstrate the advantages of our method, we test it on real large-scale data. We show that we can produce denoising performances similar to performances obtained with models estimated from the original training database using GMM priors learned from the sketch with improved execution times.

Key words. Image denoising, Compressive learning, Sketching, Optimization,

AMS subject classifications. 68U10, 94A08, 49N30

1. Introduction.

We consider the classical noisy observation model of a clean natural image $u \in \mathbb{R}^N$ (composed of N pixels):

$$v = u + w \tag{1.1}$$

where v is the observed degraded version of u. The acquisition noise w is usually assumed to be an additive white Gaussian noise of variance σ, i.e. $w \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2 I_N)$. In the last two decades, non local patch-based methods have been proven successfull for denoising. Methods such as Piecewise Linear Estimators [52, 1], BM3D [9, 26] or NL-Bayes [28, 27, 50] are examples of non-local methods [2]. In patch-based image denoising, the noisy image v is divided into small patches $\{v_i\}_{i=1}^M$. Each patch $v_i \in \mathbb{R}^P$ (P is the patch size) can be seen as a vector in a high dimensional space. The denoising problem is considered on each patch:

$$v_i = u_i + w_i, \tag{1.2}$$

and a corresponding denoised version u_i^* of the true values u_i are estimated. To overcome the ill-posedness of this inverse problem, various denoising methods [29, 28, 27, 20] consider patch models within a Bayesian framework. According to the Bayes' theorem, the objective is to find u_i^* which maximizes the posterior probability distribution $f(u_i|v_i)$ under the prior

*Submitted to the editors DATE.

Funding: This work was funded by the French National Research Agency (ANR) under reference ANR-20-CE40-0001 (EFFIREG project).

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$p(u_i)$. The Maximum A Posteriori (MAP) problem is formulated as

$$u_i^* = \arg\max_{u_i \in \mathbb{R}^P} f(u_i|v_i) = \arg\max_{u_i \in \mathbb{R}^P} f(v_i|u_i)p(u_i) \propto \arg\max_{u_i \in \mathbb{R}^P} e^{-\frac{||u_i - v_i||^2}{2\sigma^2}} p(u_i) \tag{1.3}$$

where $\|\cdot\|$ denotes the ℓ^2-norm. This yields

$$u_i^* = \arg\min_{u_i \in \mathbb{R}^P} \frac{||u_i - v_i||^2}{2\sigma^2} - log(p(u_i)). \tag{1.4}$$

Ideally, the choice of the prior distribution should be determined by the nature of the image to be estimated. In practice, Gaussian Mixture Models (GMM) [52, 49, 20] have shown their effectiveness. With the GMM prior, the solution of problem (1.4) can be approximated by a Wiener filter solution.

Among these various non-local denoising methods, the Expected Patch Log-Likelihood algorithm (EPLL) [53] occupies a central position due to its efficient denoising performance. A large number of works build on the original EPLL formulation to deal with more general prior or go beyond the denoising problem [11, 33, 5, 30, 37, 47, 10, 39]. EPLL uses a GMM prior learned from a very large set of patches extracted from clean images. The key to the success of EPLL is to find a good prior distribution. Since in practice patch sizes are typically greater than 5×5, estimating prior distributions in such a high-dimensional space is a difficult task. Moreover, to estimate the best possible model, we need to maximize the redundancy of structural information and use training databases as large as possible. As the traditional empirical minimization approaches require access to the whole training dataset, when the collection size is large, the learning process can be extremely costly. For instance, in the case of the classical learning method Expectation Maximization (EM), the memory consumption and computation time depend on the size of the database (see section 3).

Leveraging ideas from compressive sensing [14] and streaming algorithms [8], R. Gribonval et al. propose a *sketching* method [23, 18, 19, 17, 16] to compress the training database. This scalable technique compress the whole training collection into a fixed-size representation (a vector): a *sketch* of the training dataset before learning. The sketch captures the necessary information for the considered learning task. For certain mixture model estimation, it is then possible to learn their parameters directly from the sketch, without access to the original dataset. Hence the space and time complexity of the learning algorithm no longer depends on the original database size, but only on the size of the sketch which is linked to the dimensionality of the model. Sketching has been already used successfully in machine learning [40, 16, 25, 6, 4, 36], generative networks [41], source localization [12, 13], independent component analysis [43] and depth imaging [44]. In [23], the sketching is implemented and evaluated on synthetic data to estimate a GMM with diagonal covariances. It is shown that on large synthetic data, for the estimation of GMM, the sketching produces precise results while requiring fewer memory space and computations. In this work, we explore the sketching method in the image patches context where GMM with full covariance must be estimated from the compressed database.

Due to the curse of dimensionality, it is computationally expensive to manipulate the GMMs' covariance matrices. [38] shows that most natural images and videos can be represented by a GMM with low-rank covariance matrices. The experiments have also shown the

efficiency of low rank covariance matrices applied to image denoising [34], image inpainting, high-speed video and hyperspectral imaging [51]. This motivates us to use such low rank covariances in the GMM modeling of patches and extend the sketching framework accordingly to gain computational speedup and to manage the modeling of the image patches in the most possible flexible way.

1.1. Contributions. A preliminary and short version of this work has appeared in [46]. In this paper, we provide a more detailed version of this work with a final consolidated version of the proposed learning algorithm, validated by extended numerical experiments.

Figure 1 summarizes the principle of our approach. We first construct a sketch by averaging random Fourier features computed over the whole image patch database. Then the model parameters are learned directly from the sketch by our Low-rank Continuous Orthogonal Matching Pursuit (LR-COMP) algorithm without access to the original database. Finally, the learned model is used with a Bayesian method (EPLL) for the denoising task.

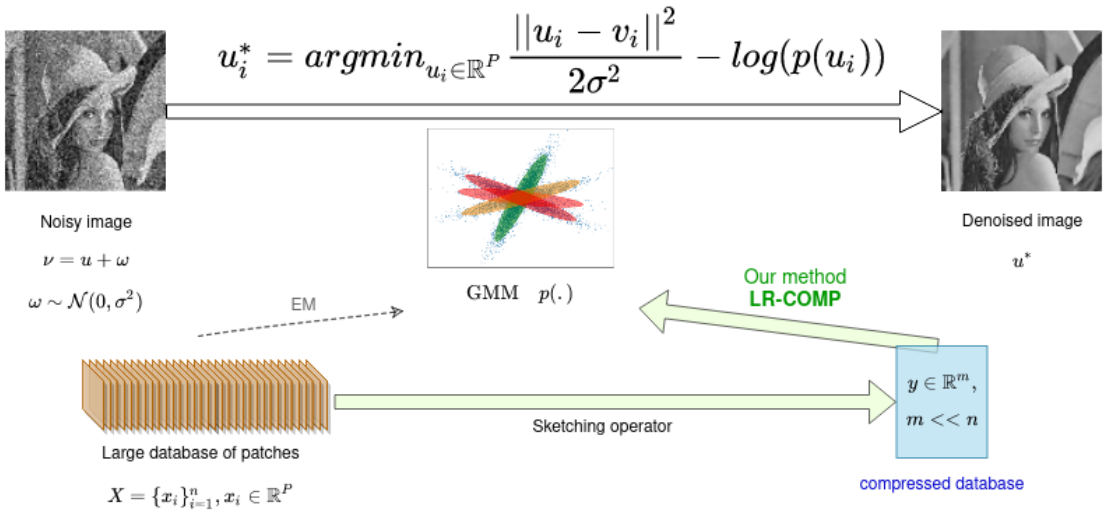

Figure 1. *A summary of our method*

Our contributions of this piece of work are the following:

- In this work, we propose an algorithm LR-COMP to estimate a GMM with non-diagonal and low-rank covariance matrices. Compared to previous work in [23], our extension to non-diagonal covariance matrices allows us to learn a GMM prior from a compressed database of patches in the context of image denoising. Moreover, with the low-rank approximation of the covariance matrices, we lighten the computation burden in the denoising process while keeping good denoising performances.
- We demonstrate the performance of our approach on real large-scale data (over 4 millions training samples of patch size of 7×7) for the task of patch-based image denoising. We show that using models trained with the compressed database, we

can obtain similar denoising performances compared to the models obtained with the classical EM algorithm. To the best of our knowledge, this is also the first time that the sketching framework has been applied with such high dimensional GMMs.
- Computationally, we estimate the model from a compressed database which is about 1000 times smaller than the original patch database. It leads to running time approximately two times faster compared to the EM method.

The paper is organized as follows. Section 2 is a reminder of the EPLL framework. Then we review the EM algorithm in section 3. In section 4, we explain the compressive learning method. In section 5, we focus on explaining how to adapt the sketching framework to learn a GMM in the image patch context. We also interpret the extension to low rank covariances and the implementation details of the adapted learning algorithm LR-COMP. In section 6, we provide numerical experiments that demonstrate the performance of our approach. Some conclusions and tracks for further works follow in section 7.

2. Image denoising with EPLL.

We review in this section the Expected Patch Log-Likelihood (EPLL) framework for image denoising. EPLL is a patch-based image restoration algorithm introduced by Zoran and Weiss [53]. The EPLL framework restores an image u by performing the following maximum a posteriori (MAP) estimation over all N patches:

$$u^* = \arg\min_{u\in\mathbb{R}^N} \frac{P}{2\sigma^2}\|u-v\|^2 - \sum_{i=1}^{N} log(p(\mathcal{P}_i u)) \tag{2.1}$$

where $\mathcal{P}_i : \mathbb{R}^N \to \mathbb{R}^P$ is a linear operator that extracts a patch of P pixels centered at the position i, typically $P = 7 \times 7$. The function $p(\cdot)$ is the density of the prior probability distribution of the patches.

2.1. Optimization.

Due to the non-convexity of $p(\cdot)$, direct optimization of the problem may be difficult. The authors of EPLL proposeto perform the optimization with "half-quadratic splitting" [15]. By introducing N auxiliary unknown vectors $z_i \in \mathbb{R}^P$ and a denoising parameter $\beta > 0$, the problem then is considered as:

$$u^* = \arg\min_{\substack{u\in R^N \\ z_1,\dots,z_N\in R^P}} \frac{P}{2\sigma^2}\|u-v\|^2 + \frac{\beta}{2}\sum_{i=1}^{N}\|\mathcal{P}_i u - z_i\|^2 - \sum_{i=1}^{N} log(p(z_i)) \tag{2.2}$$

The optimization (2.2) is accomplished by alternating the minimization of u and z_i.
- **Solving u for fixed** z_i — Problem (2.2) turns into a linear inverse problem with the Tikhonov regularization. It has a closed form solution:

$$\begin{aligned}\hat{u} &= \arg\min_{u\in\mathbb{R}^N} \frac{P}{2\sigma^2}\|u-v\|^2 + \frac{\beta}{2}\sum_{i=1}^{N}\|\mathcal{P}_i u - z_i\|^2 \\ &= (I + \frac{\beta\sigma^2}{P}\sum_{i=1}^{N}\mathcal{P}_i^T\mathcal{P}_i)^{-1}(v + \frac{\beta\sigma^2}{P}\sum_{i=1}^{N}\mathcal{P}_i^T z_i)\end{aligned} \tag{2.3}$$

where $\sum_{i=1}^{N} \mathcal{P}_i^T \mathcal{P}_i$ is a diagonal matrix of size $N \times N$, its i-th diagonal element corresponds to the number of patches overlapping the pixel in position i. The number is equal to P, which allows to express the solution as:

$$\hat{u} = (I + \sigma^2 \beta I)^{-1}(v + \sigma^2 \beta \bar{z}_i) \tag{2.4}$$

where $\bar{z}_i = (\sum_{i=1}^{N} \mathcal{P}_i^T \mathcal{P}_i)^{-1} \mathcal{P}_i^T z_i$ is the average of all overlapping patches $\hat{z}_i$.

- **Solving z_i for fixed $\mathbf{u}$** — (2.2) leads to a MAP estimation:

$$\hat{z}_i = \underset{z_1,\dots,z_N \in \mathbb{R}^P}{\arg\min} \frac{\beta}{2} \sum_{i=1}^{N} \|\mathcal{P}_i \hat{u} - z_i\|^2 - \sum_{i=1}^{N} log(p(z_i)) \tag{2.5}$$

The solution of this problem depends on the choice of patch prior $p(\cdot)$.

2.2. Denoising with a GMM prior. EPLL assumes that the prior is a finite Gaussian mixture model (GMM) with zero-means, i.e. we consider that a patch $x \in \mathbb{R}^P$ is a random vector generated from a distribution with density $p(x)$ defined as

$$p(x) = \sum_{k=1}^{K} \alpha_k \mathcal{N}_P(x; 0, \Sigma_k) \tag{2.6}$$

where K is the number of Gaussian components and $\alpha_k \geq 0$ are weights of each component such that $\sum_{k=1}^{K} \alpha_k = 1$. $\mathcal{N}_P(x; 0, \Sigma_k)$ denotes the density of a Gaussian distribution with zero-mean with covariance $\Sigma_k \in \mathbb{R}^{P \times P}$. Recall that the zero-mean Gaussian distribution density is:

$$\mathcal{N}_P(x; 0, \Sigma_k) = \frac{1}{(2\pi)^{P/2} |\Sigma_k|^{1/2}} e^{-\frac{1}{2} x^T \Sigma_k^{-1} x} \tag{2.7}$$

Hence, under the GMM prior, the problem (2.5) turns to:

$$\hat{z}_i = \underset{z_1,\dots,z_N \in \mathbb{R}^P}{\arg\min} \frac{\beta}{2} \sum_{i=1}^{N} \|\mathcal{P}_i \hat{u} - z_i\|^2 - \sum_{i=1}^{N} log(\sum_{k=1}^{K} \alpha_k \mathcal{N}_P(z_i; 0, \Sigma_k)) \tag{2.8}$$

This problem cannot be solved in closed form as the second term is the logarithm of a sum of exponential. [53] proposed to solve this problem by keeping only one Gaussian component. For a given patch $\tilde{z}_i = \mathcal{P}_i \hat{u}$, we chose the component k_i^* that maximizes the posterior probability $p(k_i | \tilde{z}_i)$. This leads to computationally efficient implementations. [48] also justified that only one component is required for good reconstructions. k_i^* is chosen by

$$\begin{aligned} k_i^* &= \underset{1 \leq k_i \leq K}{\arg\max}\, p(k_i | \tilde{z}_i) = \underset{1 \leq k_i \leq K}{\arg\max}\, p(k_i) p(\tilde{z}_i | k_i) \\ &= \underset{1 \leq k_i \leq K}{\arg\min} -2 \log \alpha_{ki} + \log \left| \Sigma_{k_i} + \frac{1}{\beta} I_P \right| + \tilde{z}_i^T (\Sigma_{k_i} + \frac{1}{\beta} I_P)^{-1} \tilde{z}_i \end{aligned} \tag{2.9}$$

With k_i^* (instead of a sum of K components), the solution of (2.8) is then a Wiener filtering solution:

$$\hat{z}_i = (\Sigma_{k_i^*} + \frac{1}{\beta} I_P)^{-1} \Sigma_{k_i^*} \tilde{z}_i. \tag{2.10}$$

2.3. Eigenspace implementation of EPLL. The matrix inversions in (2.9) and (2.10) can be done efficiently by using the singular value decomposition over the covariance matrices. We denote $\Sigma_k = U_k \Lambda_k U_k^T$, with $U_k \in \mathbb{R}^{P\times P}$ an unitary matrix and $\Lambda_k = \mathrm{diag}(\lambda_1^{(k)}, ..., \lambda_P^{(k)})$ a diagonal matrix. The diagonal entries $\lambda_i^{(k)}$ of Λ_k are the singular values of Σ_k. Then we can compute (2.9) by:

$$k_i^* = \underset{1\leq k\leq K}{\arg\min} -2\log\alpha_k + \sum_{j=1}^{P}\left(\log(\lambda_j^{(k)} + \frac{1}{\beta}) + \frac{[\tilde{v}_i^{(k)}]_j^2}{\lambda_j^k + \frac{1}{\beta}}\right) \tag{2.11}$$

where

$$\tilde{v}_i^{(k)} = U_k^T \tilde{z}_i \tag{2.12}$$

Then (2.10) leads to

$$\hat{z}_i = U_{k_i^*} S_{k_i^*} U_{k_i^*}^T \tilde{z}_i = U_{k_i^*} S_{k_i^*} \tilde{v}_i^{(k_i^*)} \tag{2.13}$$

with

$$S_{k_i^*} = \mathrm{diag}\left(\frac{\lambda_j^{(k_i^*)}}{\lambda_j^{(k_i^*)} + \frac{1}{\beta}}\right)_{j=1,...,P} \tag{2.14}$$

3. Learning a GMM with EM. The Expectation-Maximization (EM) algorithm is a classical mixture estimation approach. This algorithm starts with some initial estimates of model parameters and then iteratively updates the estimate until the the estimates are not changing much. See Appendix B for the details of the EM algorithm. In each iteration, it carries out two steps: the E-Step (expectation step) and the M-Step (maximization step). In E-Step, using the current estimate of the parameters, we evaluate the posterior probabilities. In the M-Step we compute parameters that maximize the probabilities found on the E-Step. These estimated parameters are then used to determine the distribution of the latent variables in the next E-Step.

As for the time complexity of one iteration of this algorithm, it is linear in the number of model components K and the number of elements in the database n. However it is cubic with respect to the dimensions P due to the fact that we need to inverse the covariance matrix when calculating the density in E-Step. Thus, when estimating a K-components GMM on a database of n elements of dimension P, the computational complexity of one iteration of the EM algorithm is $\mathcal{O}(KP^3n)$. Learning parameters using EM technique face computational issues linked to the size of the dataset and the number of parameters to estimate, which would make the use of (very) large image patches databases impractical. In the next section we will see an alternive manner to learn parameters using compressive learning.

4. Sketching. Sketching is a dimensionality reduction method. The principle is to compress the whole dataset massively before learning. First, the dataset $\chi = \{x_i\}_{i=1}^n$ is summarized into a vector $y \in \mathbb{C}^m$ ($m \ll n$) called the *sketch*:

$$y := \mathrm{Sketch}(\chi). \tag{4.1}$$

Then we apply a learning procedure Υ that allows us to learn an estimate Ψ^* of some statistical parameters Ψ of the dataset directly from the sketch y, namely

$$\Psi^* = \Upsilon(y) = \Upsilon(\text{Sketch}(\chi)) \tag{4.2}$$

More specifically, learning from the sketch corresponds to a minimization problem

$$\Psi^* \in \arg\min_{\Psi} E(y, \Psi) \tag{4.3}$$

where the energy of the model $E(\cdot, \cdot)$ quantifies the fit between the sketch y and the parameter Ψ. In the context of statistical learning, the energy E can be seen as a proxy of the empirical risk. The principle of sketching is summarized in Fig. 2.

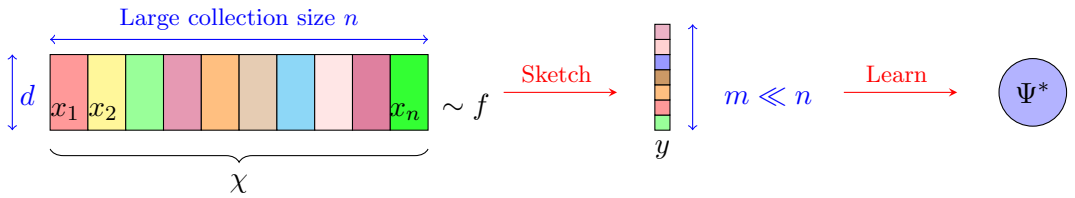

Figure 2. *Schema of sketching*

4.1. Compressive mixture estimation. In machine learning, the data $x_i \in \mathbb{R}^d$ are often modeled as i.i.d. random samples generated from a probability distribution parameterized by Θ with a density $f_\Theta \in \mathcal{D}$ ($\mathcal{D}$ is the set of probability measures over $\mathbb{R}^d$). The idea of sketching is to project the measure f_Θ on a low-dimensional vector space while keeping all the necessary information of the dataset. Mathematically, given a linear sketching operator $\mathcal{S}$:

$$\begin{aligned} \mathcal{S} : \mathcal{D} &\longrightarrow \mathbb{C}^m \\ z &= \mathcal{S} f \end{aligned} \tag{4.4}$$

and for some finite $K \in \mathbb{N}^*$, we define a K-sparse model $f_{\Theta,\alpha} \in \mathcal{D}$:

$$f_{\Theta,\alpha} = \sum_{k=1}^{K} \alpha_k f_{\theta_k} \tag{4.5}$$

where $f_{\theta_k} \in \mathcal{D}$ are elementary measures parametrized by θ_k, $\alpha_k \geq 0$ for all components and $\sum_{k=1}^{K} \alpha_k = 1$. We can express the vector z as

$$z = \mathcal{S} f_{\Theta,\alpha} = \sum_{k=1}^{K} \alpha_k \mathcal{S} f_{\theta_k}. \tag{4.6}$$

In practice we only have access to the empirical probability distribution $\tilde{y} = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$ where δ_{x_i} is a unit mass at x_i. So we can define the empirical sketch as $y = \frac{1}{n}\mathcal{S}\sum_{i=1}^{n} \delta_{x_i}$.

The goal of the sketching framework is to recover $f_{\Theta,\alpha}$ from y, hence we do the following minimization to estimate the parameters

$$(\Theta^*, \alpha^*) \in \underset{\substack{\Theta \in \mathbb{R}^K \\ \alpha \in \mathbb{R}^K, \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \|\mathcal{S} f_{\Theta,\alpha} - y\|_2^2. \tag{4.7}$$

The objective of sketched learning algorithms is to minimize a datafit functional between the compressed database and the sketch of the estimation. In other words, our aim is to find parameters α, Θ such that the sketch of the probability distribution parameterized by α, Θ is the closest to the empirical sketch y.

4.2. Recovery guarantees. It was shown in [17] that we can guarantee theoretically the success of this estimation with a condition on the sketch size. These guarantees necessitate a "Lower Restricted Isometry Property" (LRIP) of the sketching operator. This property, is verified with high probability, for GMM with sufficiently separated means and random Fourier sketching as long as the sketch size $m \geq O(K^2 d \text{polylog}(K, d))$, i.e. when the size of the sketch essentially depends on the parameters K (the number of components) and d (the model dimension). Empirical results seem to indicate that for d_{tot} the total number of parameters, a database size of the order of d_{tot} is sufficient. The excess risk of the GMM learning task is then controlled by the sum of an empirical error term and a modeling error term. This guarantees that the estimated GMM approximates well the distribution of the data [18].

Note that since the means of patches can be estimated from the noisy patches, the EPLL method uses a zero-means GMM as prior. The means of noisy patches are removed before the denoising process and added back in the end. Therefore, during the learning process, the patches are centered before sketching and we do not estimate the mean of Gaussians. In our case, the sketched GMM learning problem reduces to the estimation of the sum of k zero-mean Gaussians with covariances $\Theta = (\Sigma_k)_{k=1}^K$, i.e $f_{\Theta,\alpha} = \sum_{k=1}^K \alpha_k g_{\Sigma_k}$ where g_Σ is the zero mean Gaussian measure with covariance Σ. In this context, the notion of separation used to prove guarantees in [17] does not hold. We still show empirically that the sketching process is successful without this separation assumption.

4.3. Design of sketching operator: randomly sampling the characteristic function. In [23], the sketch is a sampling of the characteristic function (*i.e* the Fourier transform of the probability distribution f). Recall that the characteristic function ψ_f of a measure f is defined as:

$$\psi_f(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^T x} df(x) \quad \forall \omega \in \mathbb{R}^d \tag{4.8}$$

The sketching operator is therefore expressed as:

$$\mathcal{S} f = [\psi(\omega_1), ..., \psi(\omega_m)]^T \tag{4.9}$$

where $\{\omega_1, ..., \omega_m\}$ is a set of well chosen frequencies. In the spirit of Random Fourier Sampling, [23] proposes to draw the frequencies from a probability distribution, i.e. $(\omega_1, ..., \omega_m) \overset{i.i.d.}{\sim} \Delta$. The choice of frequencies is essential to the success of sketching, and we will discuss it in details in subsection 5.1.

5. Sketching image patches. In this section, we adapt the sketching framework to the context of image patches. Given a training set of n centered patches $\chi = \{x_1, ..., x_n\} \subset \mathbb{R}^P$, we define the empirical characteristic function with

$$\tilde{\psi}(w) = \frac{1}{n}\sum_{j=1}^{n} e^{-i\omega^T x_j} \quad \text{with} \quad \omega \in \mathbb{R}^P \tag{5.1}$$

Thus the empirical sketch y is expressed as

$$y = [\tilde{\psi}(\omega_1), ..., \tilde{\psi}(\omega_m)]^T = \frac{1}{n}\left[\sum_{j=1}^{n} e^{-i\omega_1^T x_j}, ..., \sum_{j=1}^{n} e^{-i\omega_m^T x_j}\right]^T \tag{5.2}$$

In other words, a sample of the sketched database is a P-dimensional frequency component calculated by averaging over patches (not to be mixed with usual 2D Fourier components of images). Thanks to the properties of the Fourier transform of Gaussians, the sketch of a single zero-mean Gaussian component g_{Σ_k} at frequency ω_l is

$$(\mathcal{S}(g_{\Sigma_k}))_l = \psi_{g_{\Sigma_k}}(\omega_l) = e^{-\frac{1}{2}\omega_l^T \Sigma_k \omega_l}. \tag{5.3}$$

Thus, given the weights $\alpha = (\alpha_k)_{k=1}^K$ and the covariance matrices $\Sigma = (\Sigma_k)_{k=1}^K$, the sketch of a zero-means GMM $f_{\Sigma,\alpha} = \sum_{k=1}^K \alpha_k g_{\Sigma_k}$ is

$$z = [\mathcal{S}(f_{\Sigma,\alpha})_l]_{l=1,...,m} = \left[\sum_{k=1}^{K} \alpha_k e^{-\frac{1}{2}\omega_l^T \Sigma_k \omega_l}\right]_{l=1,...,m}. \tag{5.4}$$

As a consequence, the problem (4.7) of estimating GMM parameters becomes

$$(\Sigma^*, \alpha^*) \in \underset{\substack{\Sigma \in \mathbb{R}^K \\ \alpha \in \mathbb{R}^K, \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \|y - \mathcal{S} f_{\Sigma,\alpha}\|_2^2 \tag{5.5}$$

i.e.

$$(\Sigma^*, \alpha^*) \in \underset{\substack{\Sigma_k \in \mathbb{R}^{P\times P}, \forall k \\ \alpha \in \mathbb{R}^K, \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \sum_{l=1}^{m} \left| \frac{1}{n}\sum_{j=1}^{n} e^{-i\omega_l^T x_j} - \sum_{k=1}^{K} \alpha_k e^{-\frac{1}{2}\omega_l^T \Sigma_k \omega_l} \right|^2. \tag{5.6}$$

5.1. Frequency sampling. The design of the probability distribution Δ for sampling the frequencies $\{\omega_1, ..., \omega_m\}$ is essential to the success of sketching. In our work, we draw frequencies from the *Adapted radius* frequency distribution proposed in [23]. The Adapted radius heuristic proposes to sample ω as

$$\omega = R\varphi \tag{5.7}$$

where $R \in \mathbb{R}_+$ is the norm of ω and $\varphi \in \mathbb{R}^P$ is the random direction. The radius R is chosen with a radius distribution $R \sim p_R(R;\eta) = ((\eta R)^2 + \frac{1}{4}(\eta R)^4)^{\frac{1}{2}} e^{-\frac{1}{2}(\eta R)^2}$ where η is a

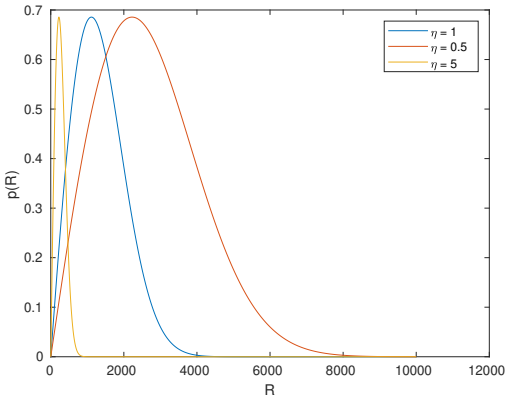

Figure 3. *Curve of the radius distribution density*

scale parameter which should be adjusted to the current dataset. By combining this radius distribution with the decomposition (5.7), we have a frequency distribution Δ_k referred as *Adapted radius* frequency distribution. See Appendix C for details. With this distribution, we avoid sampling very low frequencies. Figure 3 illustrates the curve of $p(R)$ with different values of η.

5.2. Extension to low rank covariances. Bayesian MAP theory permits to use a GMM with degenerate covariance matrices as a denoising prior. As we perform Wiener filtering, this is useful as we can reduce the number of parameters by just truncating the component of noisy patches supported on the lowest eigenvalues of the covariance. The experiments [34, 38] have shown that we can use low rank covariance matrices for denoising while keeping good performance. This motivates us to approximate the covariance matrices in the GMM prior by low-rank matrices.

Following classical Burer-Monteiro method [3, 7] in low-rank matrix estimation, we parameterize Σ_k by its factors X_k: $\Sigma_k = X_k X_k^T$. Supposing that $\|y - \mathcal{S} f_{X,\alpha}\|_2^2$ has a minimizer, we approximate the minimization (5.5) by

$$(\hat{X}, \hat{\alpha}) \in \underset{\substack{X \in \mathbb{R}^K \\ \alpha \in \mathbb{R}^K, \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \|y - \mathcal{S} f_{X,\alpha}\|_2^2 \tag{5.8}$$

i.e.

$$(\hat{X}, \hat{\alpha}) \in \underset{\substack{X_k \in \mathbb{R}^{P \times r}, \forall k \\ \alpha \in \mathbb{R}^K, \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \sum_{l=1}^{m} \left| y - \sum_{k=1}^{K} \alpha_k e^{-\frac{1}{2}\omega_l^T X_k X_k^T \omega_l} \right|^2 \tag{5.9}$$

where $\hat{X} = \{\hat{X}_1, ..., \hat{X}_K\}$ is the collection of factorized rank reduced covariances.

With the following proposition, we justify that the difference between the energy $E(y, \hat{\Phi})$ and the minimized energy in the full-rank case $E(y, \Phi^*)$ (where Φ^* is the result of minimization (5.5)) is associated with the smallest eigenvalues of the covariance matrices. We qualitatively validate this approximation since these eigenvalues are typically small.

Proposition 5.1. *Let* $\Phi^* = \{\Sigma_1^*, ..., \Sigma_K^*, \alpha_1^*, ..., \alpha_K^*\}$ *be a minimizer of* (5.5). *Suppose that there exists a minimizer* $\hat{\Phi} = \{\hat{X}_1, ..., \hat{X}_K, \hat{\alpha}_1, ..., \hat{\alpha}_K\}$ *for the problem* (5.8). *Let* $C = \frac{\sqrt{P}}{2}\sqrt{\sum_{l=1}^m \|\omega_l\|_2^2 \|\omega_l\|_\infty^2}$. *Then we have:*

$$\left\|\mathcal{S}f_{\hat{\Phi}} - y\right\|_2 - \|\mathcal{S}f_{\Phi^*} - y\|_2 \leq C \max_{1\leq k\leq K} \sqrt{\sum_{j\geq r+1} \sigma_j^2(\Sigma_k^*)}$$

where the $\sigma_j(\Sigma_k^*)$ *are the singular values of* Σ_k^* *sorted by decreasing order.*

The proof is detailed in Appendix D.

Ideally, we would like to obtain a similar bound for $\|\Sigma_k^* - \hat{X}_k\hat{X}_k^T\|_F$. We conjecture that a RIP (Restricted Isometry Property) would be needed for such a result. As the verification of RIP remains an open theoretical question in the zero-means GMM case, we leave this theoretical question for further work.

5.3. An algorithm for learning patch prior from a sketch : LR-COMP (Low Rank Continuous Orthogonal Matching Pursuit). Problem (4.7) can be solved approximately using the greedy Compressive Learning OMP called CL-OMP and a variation of CL-OMP called CL-OMP with Replacement (CL-OMPR) [23, 24]. These algorithms are based on the Matching Pursuit [31], Orthonormal Matching Pursuit [35] and Orthonormal Matching Pursuit with Replacement [21] for classical compressive sensing, which handle sparse approximation problems. It starts from an empty support and it expands the support by greedily adding new atoms to the current support Ω. Each new atom θ' is found by maximizing the correlation $\langle \mathcal{S}f_{\theta'}, r\rangle$ where r is the current residual. Then it updates the weights and reduces the cost function with a gradient descent initialized with the current parameters. For better practical recovery, the algorithms with Replacement extend the size of support more than the desired sparsity. Then it selects the K (the number of model components) largest weights and it deletes the extra atoms using a Hard Threshold.

We adapt these algorithms in the GMMs context with our low-rank approximation. Several modifications are detailed below:

- **No Replacement.** Although the algorithms with Replacement show better results on synthetic data, our results tested on image patches show that the Replacement has a negligible effect. Therefore, we run our algorithm without this Hard Thresholding step to decrease the computation time.
- **Estimation the factors of covariance instead of the covariance matrices.** As we approximate the covariance matrices with their factors, in each step of the algorithm, we do operations directly on the factorized rank reduced covariance X instead of the covariance matrix Σ to lighten the computations.

The proposed algorithm is summarized in Algorithm 5.1. The main tool for the implementation of Algorithm 5.1 is to compute the necessary gradients for the optimization problems in Steps 1, 3 and 4. Our algorithm was implemented by extending the MATLAB toolbox [22]. The Matlab implementation of our approach is available at [45].

5.4. Expressions of the necessary gradients. For the following section, denote the vector $v(X) = \mathcal{S}f_X \in \mathbb{R}^m$.

Algorithm 5.1 LR-COMP: Compressive GMM estimation with low-rank covariances [45].

Input Empirical sketch y, sketching operator $\mathcal{S}$, sparsity K
$\hat{r} \leftarrow y$; $\Omega \leftarrow \emptyset$
for t $= 1$ **to** K **do**
Step 1: Find a X such that: $X \leftarrow \arg\max_X \operatorname{Re}\left\langle \frac{\mathcal{S}f_X}{\|\mathcal{S}f_X\|_2}, \hat{r} \right\rangle_2$, init = rand
Step 2: Extend the support $\Omega \leftarrow \Omega \cup \{X\}$
Step 3: Find weights: $\alpha \leftarrow \arg\min_\alpha \left\| y - \sum_{k=1}^{|\Omega|} \alpha_k \mathcal{S}f_{X_k} \right\|_2^2$
Step 4: Perform a gradient descent initialized with current parameters
$\Theta, \alpha \leftarrow \arg\min_{\Theta,\alpha} \left\| y - \sum_{k=1}^{|\Omega|} \alpha_k \mathcal{S}f_{X_k} \right\|_2^2$, init $= (\Omega, \alpha)$
Step 5: Update residual: $\hat{r} \leftarrow y - \sum_{k=1}^{|\Omega|} \alpha_k \mathcal{S}f_{X_k}$;
end for
Final adjustment: $\Theta, \alpha \leftarrow \arg\min_{\Theta,\alpha} \left\| y - \sum_{k=1}^{K} \alpha_k \mathcal{S}f_{X_k} \right\|_2^2$
Normalize the weights α_k such that $\sum_k \alpha_k = 1$
return Support Ω, weights α

5.4.1. The gradient for Step 1.

In step 1, we have the optimization problem

$$X \in \arg\max_{X \in \mathbb{R}^{P\times r}} \operatorname{Re}\left\langle \frac{\mathcal{S}f_X}{\|\mathcal{S}f_X\|_2}, \hat{r} \right\rangle_2 \quad \hat{r} \in \mathbb{C}^m \tag{5.10}$$

Let $F(X) = -\operatorname{Re}\left\langle \frac{\mathcal{S}f_X}{\|\mathcal{S}f_X\|_2}, \hat{r} \right\rangle_2 = -\frac{v(X)^T \operatorname{Re}(\hat{r})}{\|v(X)\|_2}$, then problem (5.10) turns to

$$X \in \arg\min_{X \in \mathbb{R}^{P\times r}} F(X) \tag{5.11}$$

With $W = [\omega_1, ..., \omega_m] \in \mathbb{R}^{P\times m}$ the frequency matrix and $\dot{*}$ the multiplication element by element, we express the gradient of $F(X)$ as :

$$\nabla_X F(X) = -\frac{1}{\|v(X)\|_2} W \left(W^T X \dot{*} \left(v\left(X\right) \dot{*} \left(\frac{F\left(X\right) v\left(X\right)}{\|v\left(X\right)\|_2} - \operatorname{Re}\left(\hat{r}\right) \right) \right) \right). \tag{5.12}$$

The detailed computation is in Appendix E.

5.4.2. Solution of Step 3.

The problem is

$$\alpha^* = \arg\min_{\alpha \in \mathbb{R}^{|\Omega|}} \left\| y - \sum_{k=1}^{|\Omega|} \alpha_k \mathcal{S}f_{X_k} \right\|_2^2 \quad y \in \mathbb{C}^m \tag{5.13}$$

Denote $V(X) = [v(X_1), ..., v(X_{|\Omega|})] \in \mathbb{R}^{m\times|\Omega|}, \alpha = [\alpha, ..., \alpha_{|\Omega|}]^T \in \mathbb{R}^{|\Omega|}$, then the problem can be expressed as a least-squares minimization

$$\alpha^* = \arg\min_{\alpha \in \mathbb{R}^{|\Omega|}} g(\alpha) = \arg\min_{\alpha \in \mathbb{R}^{|\Omega|}} \|y - V\alpha\|_2^2 \tag{5.14}$$

We thus have

$$\alpha^* = (V^T V)^{-1} V^T y. \tag{5.15}$$

5.4.3. The gradient for Step 4. The problem is

$$(\Theta, \alpha) \in \underset{\substack{\Theta \in \mathbb{R}^K, X_k \in \mathbb{R}^{P+P\times r} \\ \alpha \in \mathbb{R}^K}}{\arg\min} \left\| y - \sum_{k=1}^{K} \alpha_k \mathcal{S} f_{X_k} \right\|_2^2 \tag{5.16}$$

Denote $V = [v(X_1), ..., v(X_K)]$, $\alpha = [\alpha_1, ...\alpha_K]^T$, we express

$$h(\Theta, \alpha) = \|y - V\alpha\|_2^2 \tag{5.17}$$

so we have the gradients

$$\nabla_\alpha h(\Theta, \alpha) = 2V^T(V\alpha - y) \tag{5.18}$$

and

$$\nabla_{X_k} h(\Theta, \alpha) = 2\alpha_k \nabla_{X_k} v(X_k)^T (V\alpha - y) \tag{5.19}$$

In practice, as in Step 1, we compute the second gradient by

$$\nabla_{X_k} h(\Theta, \alpha) = -2\alpha_k W(W^T X_k \dot{*} v(X_k) \dot{*} (V\alpha - y)) \tag{5.20}$$

5.5. Complexity of LR-OMP. When estimating a K-components GMM, the proposed algorithm LR-OMP has a computational cost of the order of $O(mP^2rK^2)$. In each iteration, the computational cost is dominated by the matrix-vector product $W(W^TX)$ where W is a matrix of size $P \times m$ and W^TX is a matrix of size $m \times r$. As $m \ll n$, the computational cost of our algorithm is lower than that of the EM.

5.6. Denoising with low-rank covariance matrices. In this section, we describe some modifications required in EPLL to use our estimated model. The estimated parameters are $\hat{\Phi} = \{\hat{X}_1, ..., \hat{X}_K, \hat{\alpha}_1, ..., \hat{\alpha}_K\}$ with $\hat{X}_k \in \mathbb{R}^{P\times r}$ and $\alpha_k \in \mathbb{R}_+$. A singular value decomposition of $\hat{X}_k$ is given by $\hat{X}_k = \hat{U}_k \hat{S}_k \hat{U}_k^T$. $\hat{U}_k \in \mathbb{R}^{P\times P}$ is an orthogonal matrix and $\hat{S}_k = \text{diag}(\hat{s}_{k_1}, ..., \hat{s}_{k_r}) \in \mathbb{R}^{r\times r}$ is a diagonal matrix. The r-rank covariance matrix can be expressed with $\hat{\Sigma}_{kr} = \hat{X}_k \hat{X}_k^T = \hat{U}_k \hat{S}_k^2 \hat{U}_k^T$. We approximate the covariance matrix Σ_k with $\Sigma_k \simeq \hat{\Sigma}_k = \hat{U}_k \hat{\Lambda}_k \hat{U}_k^T$ where $\hat{\Lambda}_k$ is formed as:

$$\hat{\Lambda}_k = \begin{pmatrix} \hat{s}_{k_1}^2 & & & & \\ & \ddots & & 0 & \\ & & \hat{s}_{k_r}^2 & & \\ & & & \mu & \\ & 0 & & \ddots & \\ & & & & \mu \end{pmatrix} \tag{5.21}$$

μ is a user parameter. Denoting $\hat{U}_k^r \in \mathbb{R}^{P\times r}$ the matrix formed by the first r columns of $\hat{U}_k$ and $\hat{\Lambda}_k^r$ the matrix formed with the first r rows and r columns of $\hat{\Lambda}_k$, we have:

$$\left(\Sigma_k + \frac{1}{\beta} I_P\right)^{-1} = \hat{U}_k^r (\hat{\Lambda}_k^r + \frac{1}{\beta} I_r)^{-1} \hat{U}_k^{rT} + \frac{\beta}{\beta\mu + 1} (I_p - \hat{U}_k^r \hat{U}_k^{rT}) \tag{5.22}$$

and

$$\left(\Sigma_k + \frac{1}{\beta} I_P\right)^{-1} \Sigma_k = \hat{U}_k^r (\hat{\Lambda}_k^r + \frac{1}{\beta} I_r)^{-1} \hat{\Lambda}_k^r \hat{U}_k^{rT} + \frac{\beta\mu}{\beta\mu + 1} (I_p - \hat{U}_k^r \hat{U}_k^{rT}) \tag{5.23}$$

Then the Gaussian selection step of EPLL (2.11) becomes

$$k_i^* = \underset{1 \le k \le K}{\arg\min} -2 \log \alpha_k + \sum_{j=1}^{r} \left(\log(\hat{s}_{k_j}^2 + \frac{1}{\beta}) + \frac{[\hat{v}_i^{(k)}]_j^2}{\hat{s}_{k_j}^2 + \frac{1}{\beta}} - \frac{\beta}{\beta\mu + 1} [\hat{v}_i^{(k)}]_j^2 \right) \tag{5.24}$$

where

$$\hat{v}_i^{(k)} = \hat{U}_k^{rT} \tilde{z}_i \tag{5.25}$$

With the optimal component k_i^*, the estimated patch (2.10) becomes

$$\begin{aligned} \hat{z}_i &= (\Sigma_{k_i^*} + \frac{1}{\beta} I_P)^{-1} \Sigma_{k_i^*} \tilde{z}_i \\ &= \hat{U}_{k_{i^*}^r} (\hat{\Lambda}_{k_i^*}^r + \frac{1}{\beta} I_r)^{-1} \hat{\Lambda}_{k_i^*}^r \hat{U}_{k_i^*}^{rT} \tilde{z}_i + \frac{\beta\mu}{\beta\mu + 1} (I_p - \hat{U}_{k_i^*}^r \hat{U}_{k_i^*}^{rT}) \tilde{z}_i \\ &= \hat{U}_{k_i^*}^r \hat{\Lambda}'_{k_i^*} \hat{v}_i^{(k_i^*)} + \frac{\beta\mu}{\beta\mu + 1} (\tilde{z}_i - \hat{U}_{k_r^*} \hat{v}_i^{(k_i^*)}) \end{aligned} \tag{5.26}$$

with

$$\hat{\Lambda}'_{k_i^*} = (\hat{\Lambda}_{k^*}^r + \frac{1}{\beta} I_r)^{-1} \hat{\Lambda}_{k^*}^r = \operatorname{diag}\left(\frac{\hat{s}_{k_{ij}^*}^2}{\hat{s}_{k_{ij}^*}^2 + \frac{1}{\beta}} \right)_{j=1,\dots,r}. \tag{5.27}$$

6. Experimental Results. In this section we present several numerical experiments to illustrate the benefits of our approach. The noisy images are obtained by adding zero-mean Gaussian noise with standard deviations $\sigma^2 = 20$ to the test images. The denoising is performed with EPLL[1] . To evaluate the quality of denoised images, we use two measures: PSNR (Peak Signal to Noise Ratio) and SSIM (Structural Similarity).

The prior model used for EPLL is learned from a sketch that compresses $n = 4 \times 10^6$ patches of size $P = 7 \times 7$. The patches are randomly extracted from the training images of the Berkeley Segmentation Database (BSDS) [32]. Based on observations from numerical simulations, the scale parameter in C.5 needs to be adjusted with different tasks [42]. In [23], the authors propose to estimate this parameter with a small sketch on a small subset from the dataset. In our work, we choose the optimal parameter η by hand. We learn a mixture model of $K = 20$ Gaussian components, the rank of covariance matrices are reduced to $r = 20$. Our experiments showed that we cannot reduce the rank further to keep good denoising performance.

We compare the denoised results with the results obtained with a prior learned by EM. For the comparison, we train the prior from the same database using the EM algorithm. The experimental results are shown in Figure 4 and Figure 5. We observe that for most of images, we obtain similar or better values of PSNR and SSIM. To reproduce the results below, you can use the code at [45].

[1]Matlab implementation based on the code of [34].

Figure 4. *From left to right: Original images, noisy images with noise $\sigma^2 = 20$, results with EM model, results with LR-COMP model. The denoising results are evaluated with PSNR/SSIM. Similar denoising performances are obtained with LR-COMP with a 1000 times smaller compressed database. To estimate the prior model, our method is 2 times faster than the EM algorithm.*

Figure 5. *From left to right: Original images, noisy images with noise $\sigma^2 = 20$, results with EM model, results with LR-COMP model. The denoising results are evaluated with PSNR/SSIM. Similar denoising performances are obtained with LR-COMP with a 1000 times smaller compressed database. To estimate the prior model, our method is 2 times faster than the EM algorithm.*

6.1. The sketch size and compression rate. Theoretically, we can successfully estimate a GMM with sufficiently separated means and random Fourier sketching with high probability as long as the sketch size $m \geq O(K^2 P \text{polylog}(K, P))$. In our case, we learn zero-mean Gaussians. Empirical results indicate that it is sufficient when the sketch size is the order of the number of parameters. We set $m = 10K(P \times r + 1) \approx 2 \times 10^5$, i.e the compressed database is approximately 1000 times smaller than the original patch database. The gains in terms of memory is approximately $\frac{n}{m}$ times compared to the EM approach.

6.2. Learning time. In terms of time complexity, the running time depends on the number of components K and the complexity of the gradient descent algorithm. In our approach, we use the Limited-memory BFGS algorithm to handle the optimization problems in Step 1 and 4. The latter is the most time-consuming part of the algorithm. To get the model that achieves the denoising performance of our experiments, it takes less than 2 hours on a computer with 2 * 32 cores AMD EPYC 7452 @ 2,35 GHz. With the same environment, our learning algorithm is about 2 times faster than the EM algorithm[2].

7. Conclusions. In this work, we adapt the sketching framework in the context of image patches. We propose an algorithm LR-COMP to estimate a GMM with low-rank approximation and provide an implementation of the algorithm. Experiments illustrate that a high-dimensional GMM can be learned from a compressed database and then used for patch-based denoising. We achieve denoising performances close to state-of-the art model based methods while the learning procedure uses less memory and time than the classical EM algorithm.

In future works, we can generalize our approach to other models such as GGMM (Generalized Gaussian Mixture Model) for a better denoising performance [10]. We also aim to adapt the sketching to more inverse problems such as image super-resolution, image deblurring, etc. Another perspective is to extend our model to the study of video denoising method as the potential of the technique for video restoration remains unexplored. In our work, we estimate a GMM with zero-means. In this context, the notion of separation used to prove guarantees in [17] does not hold. We still show empirically that the sketching process is successful without this separation assumption. This opens interesting new theoretical questions for the study of the success of compressive learning in patch-based image processing.

Appendix A. Definitions and theorems.

Definition A.1. ***Singular values*** *For $A \in \mathbb{C}^{m \times n}$ and $i = 1, ..., \min(m, n)$, the singular values $\sigma_i(A)$ (that we suppose sorted by decreasing order) of the matrix A are the absolute values of the eigenvalues of the matrix AA^T:*

$$\sigma_i^2(A) = \lambda_i(AA^T) \tag{A.1}$$

Definition A.2. ***Frobenius norm.*** *For a matrix $A \in \mathbb{C}^{m \times n}$, the Frobenius norm of A is*

[2] Mo Chen (2021). EM Algorithm for Gaussian Mixture Model (EM GMM) (https://www.mathworks.com/matlabcentral/fileexchange/26184-em-algorithm-for-gaussian-mixture-model-em-gmm), MATLAB Central File Exchange. Retrieved October 11, 2021.

defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{i,j}|^2} = \sqrt{\text{trace}\,(A^T A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}}\sigma_i^2(A)} \tag{A.2}$$

where $\sigma_i(A)$ are the singular values of A.

Definition A.3. ***Operator norm.*** *For a continuous linear operator $A : V \to W$, the operator norm of A is defined as*

$$\begin{aligned}\|A\|_{op} &= \inf\{c \geq 0 : \|Av\| \leq c\|v\| \quad \forall v \in V\} \\ &= \sup\left\{\frac{\|Av\|}{\|v\|} : v \neq 0 \quad \textit{and} \quad v \in V\right\}\end{aligned} \tag{A.3}$$

Theorem A.4. ***Eckart-Young-Mirsky theorem.*** *Let $D = U\Sigma V^\top \in \mathbb{R}^{m\times n}, \quad m \geq n$ be the singular value decomposition of D with $\Sigma = \text{diag}(\sigma_1, ..., \sigma_m)$. Let U_r (resp. V_r) be the matrix formed by the first r columns of U (resp. V) and $\Sigma_r = \text{diag}(\sigma_1, ..., \sigma_r)$. Then the r-rank matrix, obtained from the truncated singular value decomposition: $D^* = U_r\Sigma_r V_r^T$ is the minimizer of the low-rank approximation:*

$$\|D - D^*\|_F = \min_{rank(D')\leq r}\|D - D'\|_F = \sqrt{\sum_{j\geq r+1}\sigma_j^2(D)} \tag{A.4}$$

The minimizer D^* is unique if and only if $\sigma_{r+1} < \sigma_r$.

Appendix B. EM algorithm. Given a data set of n clean training patches $\chi = \{x_1, ..., x_n\} \subset \mathbb{R}^{P\times n}$, the EM algorithm for estimating a GMM can be summarized as follows:

1. Define the number of components K. For each component k, we initialize the parameters $\Theta_k = (\mu_k, \Sigma_k, \alpha_k)$ randomly, and we compute the log likelihood

$$log\mathcal{L}(\Theta_k; x_1, ..., x_n) = \sum_{i=1}^{n} log(\sum_{k=1}^{K}\alpha_k\mathcal{N}_P(x_i; \mu_k, \Sigma_k)) \tag{B.1}$$

2. E-Step
   Compute the posterior function $\Gamma_{i,k}$ with the current parameters Θ_k:

$$\Gamma_{i,k} = \frac{\alpha_k\mathcal{N}_P(x_i; \mu_k, \Sigma_k)}{\sum_{j=1}^{K}\alpha_j\mathcal{N}_P(x_i; \mu_j, \Sigma_j)} \tag{B.2}$$

3. M-Step
   Re-estimate the parameters Θ_k^{new} with the $\Gamma_{i,k}$ obtained in the E-Step:

$$\mu_k^{new} = \frac{1}{N_k}\sum_{i=1}^{n}\Gamma_{i,k}x_i \tag{B.3}$$

$$\Sigma_k^{new} = \frac{1}{N_k}\sum_{i=1}^{n}\Gamma_{i,k}(x_i-\mu_k)^T(x_i-\mu_k) \tag{B.4}$$

$$\alpha_k^{new} = \frac{N_k}{\sum_{k=1}^{K}N_k} \tag{B.5}$$

where $N_k = \sum_{i=1}^{n}\Gamma_{i,k}$.

4. Re-evaluate the log likelihood. Iterate E-Step and M-Step until the log likelihood or the parameters are not changing much.

Appendix C. Design of adapted radius distribution. The frequency distribution is chosen as with a heuristic called adapted radius [25]. Assuming that we want to estimate a P-dimensional Gaussian $g = \mathcal{N}(0, I_P)$, we can compute the characteristic function $\psi_g(\omega)$ associated with g:

$$\psi_g(\omega) = e^{-\frac{1}{2}\omega^T\omega} \tag{C.1}$$

The Adapted radius heuristic proposes not to sample ω directly but rather to sample the radius of the P-dimensional Gaussian $R = \sqrt{\omega^T\omega}$. Thus, we draw the frequency $\omega \in \mathbb{R}^P$ as

$$\omega = R\varphi \tag{C.2}$$

The radius $R \in \mathbb{R}_+$ is chosen with a radius distribution $R \sim p_R(R;\eta)$. The direction $\varphi \in \mathbb{R}^P$ is uniformly generated on the l_2 unit sphere S_{P-1}, i.e. $\varphi \sim \mathcal{U}(S_{P-1})$. Then, the characteristic function $\psi_g(\omega)$ reduces to

$$\psi_g(\omega) = \psi_g(R\varphi) = e^{-\frac{1}{2}R^2} = \psi(R) \tag{C.3}$$

We obtain a one-dimensional Gaussian distribution for R. To design the radius distribution, we consider the estimation of a Gaussian $g = \mathcal{N}(0,1)$. We aim at sampling the radius R leading to large variations of the characteristic function when the parameters are closed to the true parameters. In other words, when parameters (μ, σ^2) are closed to (0,1), we want have a large $\left|\psi_{(\mu,\sigma^2)}(R) - \psi_{(0,1)}(R)\right|$. This can be accomplished by promoting the radius R which makes the norm of the gradient $\left\|\nabla\psi_{(\mu,\sigma^2)}(R)\right\|_2$ large. Recall that $\psi_{(\mu,\sigma^2)}(R) = e^{-i\mu R}e^{-\frac{1}{2}\sigma^2R^2}$ and the norm of the gradient is:

$$\left\|\nabla\psi_{(\mu,\sigma^2)}(R)\right\|_2^2 = \left|-iR\psi_{(\mu,\sigma^2)}(R)\right|^2 + \left|-\frac{1}{2}R^2\psi_{(\mu,\sigma^2)}(R)\right|^2 = (R^2 + \frac{1}{4}R^4)e^{-\sigma^2R^2} \tag{C.4}$$

Therefore, $\|\nabla\psi_{(0,1)}(R)\|_2 = (R^2 + \frac{1}{4}R^4)^{\frac{1}{2}}e^{-\frac{1}{2}R^2}$. It yields the density of a radius distribution :

$$p_R(R;\eta) = ((\eta R)^2 + \frac{1}{4}(\eta R)^4)^{\frac{1}{2}}e^{-\frac{1}{2}(\eta R)^2}. \tag{C.5}$$

Appendix D. Proof of Proposition 5.1.

Proof. Let $\Phi_k^* = (\Sigma_k^*, \alpha_k^*)$ be the minimizer of the problem (5.5), i.e.

$$\Phi_k^* \in \underset{\substack{\Sigma_k \in \mathbb{R}^{P\times P} \\ \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \sum_{l=1}^m \left| \sum_{k=1}^K \alpha_k e^{-\frac{1}{2}\omega_l^T \Sigma_k \omega_l} - y_l \right|^2 \tag{D.1}$$

and suppose that there exists a minimizer $\hat{\Phi}_k = (\hat{X}_k, \hat{\alpha}_k)$ for the problem (5.8):

$$\hat{\Phi}_k \in \underset{\substack{X_k \in \mathbb{R}^{P\times r} \\ \alpha_k \geq 0, \sum_{k=1}^K \alpha_k = 1}}{\arg\min} \sum_{l=1}^m \left| \sum_{k=1}^K \alpha_k e^{-\frac{1}{2}\omega_l^T X_k X_k^T \omega_l} - y_l \right|^2 \tag{D.2}$$

Let $\tilde{\Sigma}_k$ be the best rank-r approximation of Σ_k^* with rank r i.e.

$$\tilde{\Sigma}_k \in \underset{\Sigma, rank(\Sigma)=r}{\arg\min} \|\Sigma_k^* - \Sigma\|_F^2 \tag{D.3}$$

Define $\tilde{\Phi} = (\tilde{\Sigma}_k, \alpha_k^*)$. According to the definition (D.2) and the triangle inequality, we have

$$\begin{aligned} \left\|\mathcal{S}f_{\hat{\Phi}} - y\right\|_2 &\leq \left\|\mathcal{S}f_{\tilde{\Phi}} - y\right\|_2 \\ &= \left\|\mathcal{S}f_{\tilde{\Phi}} - \mathcal{S}f_{\Phi^*} + \mathcal{S}f_{\Phi^*} - y\right\|_2 \\ &\leq \left\|\mathcal{S}f_{\tilde{\Phi}} - \mathcal{S}f_{\Phi^*}\right\|_2 + \left\|\mathcal{S}f_{\Phi^*} - y\right\|_2 \end{aligned} \tag{D.4}$$

The first term is

$$\begin{aligned} \left\|\mathcal{S}f_{\tilde{\Phi}} - \mathcal{S}f_{\Phi^*}\right\|_2^2 &= \left\| \sum_{k=1}^K \alpha_k^* \mathcal{S}(f_{\tilde{\Sigma}_k} - f_{\Sigma_k^*}) \right\|_2^2 = \sum_{l=1}^m \left| \sum_{k=1}^K \alpha_k^* \left(e^{-\frac{1}{2}\omega_l^T \tilde{\Sigma}_k \omega_l} - e^{-\frac{1}{2}\omega_l^T \Sigma_k^* \omega_l} \right) \right|^2 \\ &= \sum_{l=1}^m \left| \sum_{k=1}^K \alpha_k^* e^{-\frac{1}{2}\omega_l^T \tilde{\Sigma}_k \omega_l} \left(1 - e^{-\frac{1}{2}\omega_l^T (\Sigma_k^* - \tilde{\Sigma}_k)\omega_l} \right) \right|^2 . \end{aligned} \tag{D.5}$$

Using the convexity inequality $|1 - e^{-x}| \leq |x|$ and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left| e^{-\frac{1}{2}\omega_l^T \tilde{\Sigma}_k \omega_l} (1 - e^{-\frac{1}{2}\omega_l^T (\Sigma_k^* - \tilde{\Sigma}_k)\omega_l}) \right| &\leq \left| 1 - e^{-\frac{1}{2}\omega_l^T (\Sigma_k^* - \tilde{\Sigma}_k)\omega_l} \right| \\ &\leq \frac{1}{2} \left| \omega_l^T (\Sigma_k^* - \tilde{\Sigma}_k)\omega_l \right| = \frac{1}{2} \left| \langle \omega_l, (\Sigma_k^* - \tilde{\Sigma}_k)\omega_l \rangle \right| \\ &\leq \frac{1}{2} \|\omega_l\|_2 \|(\Sigma_k^* - \tilde{\Sigma}_k)\omega_l\|_2 \end{aligned} \tag{D.6}$$

We define the linear operator Ω_l:

$$\begin{aligned} \Omega_l : &\mathbb{R}^{P\times P} \longrightarrow \mathbb{R}^P \\ &\Sigma = (S_{i,j})_{1\leq i\leq P, 1\leq j\leq P} \longrightarrow \Omega_l(\Sigma) = \Sigma\omega_l \end{aligned} \tag{D.7}$$

We have

$$\begin{aligned}\|\Sigma\omega_l\|^2 &= \sum_{i=1}^{P}\left|\sum_{j=1}^{P} S_{i,j}(\omega_l)_j\right|^2 \\ &\leq \sum_{i=1}^{P}\left(\sum_{j=1}^{P}|S_{i,j}(\omega_l)_j|\right)^2 \leq \max_j |\omega_l|_j^2 P\sum_{i=1}^{P}\sum_{j=1}^{P}|S_{i,j}|^2 = P\|\omega_l\|_\infty^2\,\|\Sigma\|_F^2\end{aligned} \tag{D.8}$$

We deduce that $\|\omega_l\|_{op} \leq \sqrt{P}\|\omega_l\|_\infty$. Since $\alpha_k^* \geq 0$ and $\sum_{k=1}^K \alpha_k^* = 1$, we get:

$$\begin{aligned}\left|\sum_{k=1}^{K}\alpha_k^* e^{-\frac{1}{2}\omega_l^T\tilde{\Sigma}_k\omega_l}\left(1-e^{-\frac{1}{2}\omega_l^T(\Sigma_k^*-\tilde{\Sigma}_k)\omega_l}\right)\right| &= \sum_{k=1}^{K}\alpha_k^* e^{-\frac{1}{2}\omega_l^T\tilde{\Sigma}_k\omega_l}\left(1-e^{-\frac{1}{2}\omega_l^T(\Sigma_k^*-\tilde{\Sigma}_k)\omega_l}\right) \\ &\leq \frac{\sqrt{P}}{2}\|\omega_l\|_2\|\omega_l\|_\infty\sum_{k=1}^{K}\alpha_k^*\|\Sigma_k^*-\tilde{\Sigma}_k\|_F\end{aligned} \tag{D.9}$$

Therefore, we can bound the energy (D.5) by

$$\begin{aligned}\left\|\mathcal{S}f_{\tilde{\Phi}}-\mathcal{S}f_{\Phi^*}\right\|_2^2 &\leq \frac{P}{4}\sum_{l=1}^{m}\|\omega_l\|_2^2\|\omega_l\|_\infty^2\left(\sum_{k=1}^{K}\alpha_k^*\|\Sigma_k^*-\tilde{\Sigma}_k\|_F\right)^2 \\ &\leq \frac{P}{4}\sum_{l=1}^{m}\|\omega_l\|_2^2\|\omega_l\|_\infty^2\max_{1\leq k\leq K}\|\Sigma_k^*-\tilde{\Sigma}_k\|_F^2\end{aligned} \tag{D.10}$$

By the Eckart and Young theorem, $\|\Sigma_k^*-\tilde{\Sigma}_k\|_F^2 = \sum_{j\geq r+1}\sigma_j^2(\Sigma_k^*)$, where σ_j are the singular values. Thus

$$\left\|\mathcal{S}f_{\tilde{\Phi}}-\mathcal{S}f_{\Phi^*}\right\|_2 \leq \frac{\sqrt{P}}{2}\max_{1\leq k\leq K}\sqrt{\sum_{j\geq r+1}\sigma_j^2(\Sigma_k^*)}\sqrt{\sum_{l=1}^{m}\|\omega_l\|_2^2\|\omega_l\|_\infty^2} \tag{D.11}$$

Denoting $C = \frac{\sqrt{P}}{2}\sqrt{\sum_{l=1}^{m}\|\omega_l\|_2^2\|\omega_l\|_\infty^2}$, we have from (D.4) that:

$$\left\|\mathcal{S}f_{\hat{\Phi}}-y\right\|_2 \leq C\max_{1\leq k\leq K}\sqrt{\sum_{j\geq r+1}\sigma_j^2(\Sigma_k^*)} + \|\mathcal{S}f_{\Phi^*}-y\|_2 \tag{D.12}$$

■

Appendix E. Calculation of the gradient.

The expression of (5.12) is computed as follows: Denote $F(X) = -\frac{v(X)^T\hat{r}}{\|v(X)\|_2}$, where $r \in \mathbb{R}^m$ is the real part of $\hat{r}$. We compute the gradient of F as follows:

$$\begin{aligned}\nabla_X F(X) &= -\frac{1}{\|v(X)\|_2^2}\left(\left(\nabla_X v(X)\right)^T r\|v(X)\|_2 - \frac{v(X)^T r(\nabla_X v(X))^T v(X)}{\|v(X)\|_2}\right) \\ &= -\frac{(\nabla_X v(X))^T}{\|v(X)\|_2}\left(r+\frac{v(X)^T r v(X)}{\|v(X)\|_2^2}\right) \\ &= \frac{(\nabla_X v(X))^T}{\|v(X)\|_2}\left(\frac{F(X)v(X)}{\|v(X)\|_2}-r\right)\end{aligned} \tag{E.1}$$

For each component $v_l(X) = e^{-\frac{1}{2}\omega_l^T X X^T \omega_l}$, we have

$$\frac{\partial v_l(X)}{\partial X} = -v_l(X) X^T \omega_l \omega_l^T \tag{E.2}$$

Then for a given vector $\hat{y} \in \mathbb{R}^m$

$$\langle \nabla_X v(X), \hat{y} \rangle = -\sum_{l=1}^{m} y_l v_l(X) X^T \omega_l \omega_l^T \tag{E.3}$$

In practice, we compute the scalar product with

$$\langle \nabla_X v(X), \hat{y} \rangle = -W(W^T X \dot{*} (v(X) \dot{*} \hat{y})) \tag{E.4}$$

where $W = [\omega_1, ..., \omega_m] \in M_{P,m}(\mathbb{R})$ the frequency matrix and $\dot{*}$ the multiplication element by element. As a consequence,

$$\nabla_X F(X) = -\frac{1}{\|v(X)\|_2} W \left(W^T X \dot{*} \left(v(X) \dot{*} \left(\frac{F(X) v(X)}{\|v(X)\|_2} - r \right) \right) \right). \tag{E.5}$$

Acknowledgments. The authors acknowledge the support of the French National Research Agency (ANR) under reference ANR-20-CE40-0001 (EFFIREG project). Experiments presented in this paper were carried out using the PlaFRIM experimental testbed, supported by Inria, CNRS (LABRI and IMB), Université de Bordeaux, Bordeaux INP and Conseil Régional d'Aquitaine (see https://www.plafrim.fr/).

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