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# ASYMPTOTICS OF THE INERTIA MOMENTS AND THE VARIANCE CONJECTURE IN SCHATTEN BALLS

B. DADOUN, M. FRADELIZI, O. GUÉDON, AND P.-A. ZITT

ABSTRACT. We study the limit, as the dimension goes to infinity, of the moments of the Hilbert-Schmidt norm of a uniformly distributed matrix in the  $p$ -Schatten ball, with entries in the real, complex or quaternionic field. We also consider the restriction to the space of self-adjoint matrices. We build on the connection with spectral asymptotics of  $\beta$ -ensembles to adapt some fluctuation results due to Bekerman, Leblé and Serfaty [8]. When  $p > 3$ , this allows us to obtain the next asymptotic order for ratios of  $q$ -inertia moments of  $p$ -Schatten balls of self-adjoint matrices, and to establish a strong version of the variance conjecture for these families of convex bodies.

## 1. INTRODUCTION

Let  $\mathbb{F} := \mathbb{R}, \mathbb{C}$ , or the quaternionic field  $\mathbb{H}$ . For  $n \geq 1$ , we work on  $\mathbb{F}^n$  which is seen as a  $\beta n$ -dimensional vector space over  $\mathbb{R}$ , with  $\beta \in \{1, 2, 4\}$ . We generically denote by  $E$  either the space  $\mathcal{M}_n(\mathbb{F})$  of  $n \times n$  matrices with entries from the field  $\mathbb{F}$ , or the subspace of self-adjoint matrices in  $\mathcal{M}_n(\mathbb{F})$ . We view  $E$  as a vector space over  $n$ , whose dimension  $d_n$  depends on  $n$ ,  $\mathbb{F}$ , and whether or not we impose self-adjointness (see (5) below for the exact formula). In any case, we equip  $E$  with the Euclidean structure defined by  $\|T\|_{\text{HS}}^2 = \text{tr}(T^*T)$ , and with the Lebesgue measure denoted by  $dT$ . We write  $|A|$  for the Lebesgue measure of any Borel set  $A \subseteq E$ , when it is finite.

For every  $z \in \mathbb{F}^n$ , let

$$\|z\|_p := \left( \sum_{i=1}^n |z_i|^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty, \quad \text{and } \|z\|_\infty := \max_{1 \leq i \leq n} |z_i|.$$

For any matrix  $T \in \mathcal{M}_n(\mathbb{F})$ , let  $s(T) := (s_1(T), \dots, s_n(T)) \in \mathbb{R}^n$  be the set of singular values of  $T$ , i.e., the eigenvalues of  $\sqrt{T^*T}$ . For every  $1 \leq p \leq \infty$ , let  $\sigma_p(T) = \|s(T)\|_p$ : this defines a norm on  $\mathcal{M}_n(\mathbb{F})$ , called the  $p$ -Schatten norm (see [10, Chapter IV]), and the corresponding normed space is denoted by  $S_p^n$ , with unit ball

$$B(S_p^n) := \{T \in \mathcal{M}_n(\mathbb{F}) : \sigma_p(T) \leq 1\}.$$

In the special case  $p = 2$ ,  $\sigma_2(T) = \|T\|_{\text{HS}}$  and we recover the Euclidean structure.

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Let us note that, when  $E$  is a space of self-adjoint matrices, all matrices in  $E$  are diagonalizable, and  $\sigma_p(T)$  is then the  $p$ -norm of the vector  $(\lambda_1(T), \dots, \lambda_n(T))$  of eigenvalues of  $T$ . Taking random Gaussian entries for  $T$  then leads to the classical orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles in random matrix theory, see for instance [32, Chapter 2].

The purpose of this paper is to study the asymptotic behavior, as  $n$  goes to infinity, of the  $q$ -inertia moment

$$I_q(B_E(S_p^n)) := \frac{(\mathbb{E} \|T\|_{\text{HS}}^q)^{1/q}}{|B_E(S_p^n)|^{1/d_n}},$$

for  $T$  uniformly distributed in  $B_E(S_p^n) := B(S_p^n) \cap E$ . This quantity appears in various conjectures in asymptotic geometric analysis that we now present.

**1.1. Relevant conjectures of asymptotic geometric analysis.** The interplay between the classical conjectures of asymptotic geometric analysis that are the hyperplane conjecture, the Kannan-Lovász-Simonovits conjecture and the variance conjecture, is quite intricate; we state them here briefly to provide context and refer the interested reader to [1, 12, 31, 25] for an in-depth presentation of the field.

Let  $K$  be a symmetric convex body in  $\mathbb{R}^d$ . The isotropic constant  $L_K$  is defined by

$$(1) \quad L_K := \min_{\det(T)=1} \frac{1}{\sqrt{d}} \left( \frac{1}{|K|^{1+\frac{2}{d}}} \int_K \|T\mathbf{x}\|_2^2 \, d\mathbf{x} \right)^{\frac{1}{2}}$$

and its covariance matrix  $\Sigma$  is defined by

$$\Sigma_{i,j} := \frac{1}{|K|} \int_K x_i x_j \, d\mathbf{x}, \quad 1 \leq i, j \leq d.$$

We say that  $K$  is in isotropic position when  $\Sigma = \text{Id}$ , in which case  $L_K = \frac{1}{|K|^{1/d}}$  and the minimum in Equation (1) is attained for  $T = \text{Id}$ . The hyperplane conjecture asks for a uniform upper bound on  $L_K$  for all symmetric convex bodies and dimensions  $d$ , see [33, 25]. The Kannan-Lovász-Simonovits (KLS) conjecture [23] asks for the existence of a universal constant  $C$  such that, for every dimension  $d$ , every random vector  $X$  uniformly distributed on a symmetric convex body  $K$ , and every smooth function  $f$  on  $\mathbb{R}^d$ ,

$$(2) \quad \text{Var} f(X) \leq C \max_{\theta \in S^{d-1}} \mathbb{E} \langle X, \theta \rangle^2 \cdot \mathbb{E} \|\nabla f(X)\|_2^2.$$

This conjecture is satisfied for various families of convex bodies [1], like the balls  $B_p^n$  [39], the Orlicz balls [26, 6]. In a recent breakthrough, Yuansi Chen [13], using methods introduced by Eldan in [16] and developed in [30], proved that inequality (2) is valid for a value  $C$  still depending on the dimension but smaller than any power of  $d$ ,  $C = d^{o(d)}$ . Specifying the KLS conjecture to  $f(x) = \|x\|_2^2$  one gets the following weaker conjecture.

**Generalized variance conjecture.** *There is a constant  $C \geq 1$  such that for every dimension  $d$  and for every symmetric convex body  $K \subset \mathbb{R}^d$ , we have*

$$(3) \quad \text{Var} \|X\|_2^2 \leq C \max_{\theta \in S^{d-1}} \mathbb{E} \langle X, \theta \rangle^2 \cdot \mathbb{E} \|X\|_2^2 = C \|\Sigma\| \cdot \text{tr}(\Sigma)$$

where  $X$  is uniformly distributed on  $K$  and  $\Sigma$  is its covariance matrix.

Klartag [24] proved this conjecture for every unconditional convex bodies, that means invariant under coordinates hyperplane reflections, while Barthe and Cordero [5] studied these questions for convex bodies with more general symmetries. It was observed in [27], that the Schatten unit balls of  $\mathcal{M}_n(\mathbb{F})$  are in isotropic position. However, this is not the case for their restriction to self-adjoint matrices [4, 35].

These conjectures were studied in the particular case of the unit balls of the  $p$ -Schatten normed spaces. König, Meyer and Pajor [27] established the hyperplane conjecture for these families of particular bodies. Guédon and Paouris [17] proved a concentration of the volume through the study of the moments of the Hilbert-Schmidt norm of a random operator uniformly distributed in  $B_E(S_p^n)$ . This method was generalized by Radke and Vritsiou [35] and Vritsiou [40] to prove the generalized variance conjecture when the space of matrices is equipped with the operator norm, that is the case  $p = \infty$ .

**1.2. Main results.** Our first result is an exact computation of the limit of  $I_q(B_E(S_p^n))$ , correctly normalized by the square root of the dimension of  $E$  as a vector space over  $\mathbb{R}$ . This dimension will be denoted by  $d_n$ .

**Theorem 1.1** (Limit of the normalized inertia). *Let  $E$  be  $\mathcal{M}_n(\mathbb{F})$  or its restriction to self-adjoint matrices, equipped with the  $p$ -Schatten norm for some  $p \in [1, \infty)$ . Then for every  $q > 0$ ,*

$$(4) \quad \frac{I_q(B_E(S_p^n))}{\sqrt{d_n}} = \frac{(\mathbb{E} \|T\|_{\text{HS}}^q)^{1/q}}{\sqrt{d_n} |B_E(S_p^n)|^{1/d_n}} \xrightarrow{n \rightarrow \infty} e^{\frac{1}{2p} - \frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}},$$

where  $T$  is uniformly distributed in  $B_E(S_p^n)$ .

In particular, for  $q = 2$  and  $E = \mathcal{M}_n(\mathbb{F})$ , the left-hand side of (4) corresponds to the precise value of the limit as the dimension goes to infinity of the isotropic constant of the unit ball of  $S_p^n$ .

In the next theorems, we determine, in the self-adjoint case, the first two terms of the asymptotic expansion of the quotient  $I_q(B_E(S_p^n))/I_2(B_E(S_p^n))$ , which enables us to answer positively a strong form of the variance conjecture for these families of convex bodies.

**Theorem 1.2** (Asymptotic expansion of the  $q$ -inertia moment). *Let  $E$  be the space of real symmetric, complex Hermitian or Hermitian quaternionic matrices equipped with the  $p$ -Schatten norm for some  $p \in (3, \infty)$ . Then for every  $q > 0$ ,*

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = 1 - \frac{(q-2)(p-2)^2}{16p^2 d_n} + o\left(\frac{1}{d_n}\right)$$

as  $n$  goes to infinity, where  $T$  is uniformly distributed in  $B_E(S_p^n)$ .

Specializing to  $q = 4$  produces the following result.

**Theorem 1.3** (Asymptotic expansion of  $\text{Var} \|T\|_{\text{HS}}^2$ ). *Let  $E$  be the space of real symmetric, complex Hermitian or Hermitian quaternionic matrices equipped with the Schatten  $p$ -norm for some  $p \in (3, \infty)$ . Then*

$$\lim_{n \rightarrow \infty} d_n \frac{\text{Var}(\|T\|_{\text{HS}}^2)}{(\mathbb{E} \|T\|_{\text{HS}}^2)^2} = \frac{(p-2)^2}{2p^2},$$

where  $T$  is uniformly distributed in  $B_E(S_p^n)$ .

If  $\Sigma$  is the covariance matrix of  $B_E(S_p^n)$ , then  $\mathbb{E} \|T\|_{\text{HS}}^2 = \text{tr}(\Sigma) \leq d_n \|\Sigma\|$ . Therefore, it is easy to deduce from Theorem 1.3 that there exists a universal constant  $C > 0$  such that for these spaces  $E$  of matrices and every  $p > 3$ , we have

$$\text{Var}(\|T\|_{\text{HS}}^2) \leq C \|\Sigma\| \cdot \text{tr}(\Sigma),$$

as conjectured in (3).

**1.3. Methods of proof and related works.** We now describe the strategy of the proof and the organization of the paper.

Recall that  $E$  is either  $\mathcal{M}_n(\mathbb{F})$  or the subspace of self-adjoint matrices in  $\mathcal{M}_n(\mathbb{F})$ . Using a strategy already developed by Saint-Raymond [37], König, Meyer and Pajor [27] and Guédon and Paouris [17], we first reduce the computations of  $I_q(B_E(S_p^n))$  to integrals over  $\mathbb{R}^n$  with respect to a Gibbs probability measure  $\mathbb{P}_{n,p}$ . This distribution governs a system of  $n$  unit charges interacting through a logarithmic Coulomb potential and confined with an external potential. More precisely, defining  $L_{n,p} := n^{-1} \sum_{i=1}^n \delta_{x_i}$  to be the random empirical measure under  $\mathbb{P}_{n,p}$ , we prove in Lemma 2.2 that the computation of  $I_q(B_E(S_p^n))$  can be expressed in terms of the normalization constant  $Z_{n,p}$  appearing in the definition of  $\mathbb{P}_{n,p}$ , and the moment  $\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2}$  of the linear statistics associated with the function  $h_2(x) := x^2$ . In the same Lemma 2.2, we also establish that the quotient appearing in Theorem 1.2 satisfies

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = \left(1 + O\left(\frac{1}{n^2}\right)\right) \frac{(\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2})^{1/q}}{(\mathbb{E} \langle L_{n,p}, h_2 \rangle)^{1/2}}.$$

Our task thus essentially amounts to estimating asymptotics for the normalizing constant  $Z_{n,p}$  and for these linear statistics associated with the function  $h_2$ . Some of these asymptotics are now well understood in the literature of random matrices [19, 18, 34]. This link between the study of the asymptotic volume of the  $S_p^n$  unit balls and random

matrix theory recently appeared in the work of Kabluchko, Prochno and Thäle [20, 22], where large deviations principles are used. We will clarify the connection with the seminal paper of Johansson [19] concerning the fluctuations of eigenvalues of random matrices to prove our results.

We gather in Section 3 some uniform moment bounds which are needed to accommodate the fact that we study a linear statistics for the unbounded function  $h_2$ . The estimates obtained in Theorem 3.1 are stronger than the previous bounds obtained in [27, 17].

The proof of our main results is done in Section 4. We first establish all our results for the self-adjoint cases, where the parameter  $a$  coming from Weyl's integration formula is equal to 1 (see Table 1). This case is studied in more detail in the literature, and therefore easier to handle. We first prove Theorem 1.1, combining our strengthened bounds from Section 3 with known results on the weak convergence of empirical spectral distributions initiated in [9, 19] and further generalized in [18, 34].

To prove Theorems 1.2 and 1.3, we need to understand more precisely the fluctuations of the linear statistics  $\langle L_{n,p}, h_2 \rangle$ . We rely here on recent results of Bekerman, Leblé, Serfaty [8] (see also Lambert, Ledoux, Web [29]) concerning the CLT for fluctuations of  $\beta$ -ensembles with general potential. Let us note that in these results, the regularity of the external potential  $V$  in the Gibbs measure  $\mathbb{P}_{n,p}$  plays a prominent role. It was supposed to be a polynomial of even degree with positive leading coefficient in [19]. This assumption has then been relaxed over the last two decades to include real-analytic potentials [28, 38, 11] and, more recently, potentials of class  $\mathcal{C}^r$  with  $r$  reasonably large [7, 8, 29]. However we are working in a very specific case where  $V(x)$  is proportional to  $|x|^p$ . Proposition 4.3 states that the result of [8] applies when  $p > 3$ . It is proved by tracing back in the proof of [8] various places where the regularity of  $V$  is needed, in particular to prove regularity of the solution to a key master equation. The details of this study are given in Section 5.2.

Finally, the proof of Theorem 1.1 in the non-symmetric case, where  $a = 2$ , is proved in Section 4.2.

## 2. REDUCTION TO INTEGRALS OVER $\mathbb{R}^n$

A function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be symmetric if for every  $x \in \mathbb{R}^n$  and every permutation  $\pi$  on  $\{1, \dots, n\}$  we have  $F(x_1, \dots, x_n) = F(x_{\pi(1)}, \dots, x_{\pi(n)})$ . Let  $E$  be  $\mathcal{M}_n(\mathbb{F})$  or its restriction to self-adjoint matrices. The following change of variable formula [2, Propositions 4.1.1 & 4.1.3], makes a connection between the Schatten unit ball  $B_E(S_p^n)$  and the classical unit ball  $B_p^n \subset \mathbb{R}^n$ : for every symmetric, continuous function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\int_{B_E(S_p^n)} F(s(T)) \, dT = c_n \int_{B_p^n} F(\mathbf{x}) f_{a,b,c}(\mathbf{x}) \, d\mathbf{x},$$

where

Matrix Ensemble	$\mathbb{F}$	$a$	$b$	$c$
real	$\mathbb{R}$	2	1	0
complex	$\mathbb{C}$	2	2	1
quaternion	$\mathbb{H}$	2	4	3
real symmetric	$\mathbb{R}$	1	1	0
complex Hermitian	$\mathbb{C}$	1	2	0
Hermitian quaternionic	$\mathbb{H}$	1	4	0

TABLE 1. Possible choices for  $E$ , and corresponding parameters.

In all cases,  $b = \beta = \dim_{\mathbb{R}}(\mathbb{F})$ . In the self-adjoint cases,  $a = 1$  and  $c = 0$ . In the cases where  $E = \mathcal{M}_n(\mathbb{F})$ , the computations depend on the singular values,  $a = 2$  and  $c = \beta - 1$ . See [2, Chapter 4] for details.

$$f_{a,b,c}(\mathbf{x}) := \prod_{1 \leq i < j \leq n} |x_i^a - x_j^a|^b \cdot \prod_{i=1}^n |x_i|^c, \quad \mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n,$$

defines a positively homogeneous function of degree  $d_n - n$ , with

$$(5) \quad d_n := \dim_{\mathbb{R}} E = ab \frac{n(n-1)}{2} + (c+1)n$$

being the dimension of the subspace  $E$  over  $\mathbb{R}$ , and where  $c_n$  is an explicit constant related to the volume of the unitary group  $U_n(\mathbb{F})$  and depending only on  $a$ ,  $b$  and  $n$  whose exact and asymptotic values are known and given in Lemma 4.7:

$$(6) \quad \sqrt{n} \cdot c_n^{1/d_n} \sim e^{\frac{3}{4}} \sqrt{\frac{4\pi}{ab}} \quad \text{as } n \rightarrow \infty.$$

Combining this change of variable with a classical trick in convexity leads to the following expression.

**Lemma 2.1** (Change of variables, [17]). *For every symmetric, continuous, positively homogeneous function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  of degree  $k \geq 0$ , one has*

$$\int_{B_E(S_p^n)} F(s(T)) \, dT = \frac{c_n}{\Gamma\left(1 + \frac{d_n+k}{p}\right)} \int_{\mathbb{R}^n} F(\mathbf{x}) e^{-\|\mathbf{x}\|_p^p} f_{a,b,c}(\mathbf{x}) \, d\mathbf{x},$$

where  $\Gamma$  is Euler's Gamma function.

We refer to [17] for the details of the computations. Let

$$(7) \quad \gamma_p := \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}\right)}.$$

Applying Lemma 2.1 with  $F \equiv 1$  (homogeneous of degree 0) yields, after a straightforward change of variables,

$$(8) \quad |B_E(S_p^n)| = (abn\gamma_p)^{\frac{d_n}{p}} \cdot \frac{c_n}{\Gamma\left(1 + \frac{d_n}{p}\right)} \cdot Z_{n,p},$$

where

$$(9) \quad Z_{n,p} := \int_{\mathbb{R}^n} e^{-abn\gamma_p \|\mathbf{x}\|_p^p} f_{a,b,c}(\mathbf{x}) \, d\mathbf{x}.$$

Using now Lemma 2.1 with  $F(\mathbf{x}) := (\sum_{i=1}^n x_i^2)^{q/2}$  (homogeneous of degree  $q$ ), we obtain, for  $T$  uniformly distributed in  $B_E(S_p^n)$ ,

$$(10) \quad \mathbb{E} \|T\|_{\text{HS}}^q = (abn\gamma_p)^{\frac{q}{p}} \cdot \frac{\Gamma\left(1 + \frac{dn}{p}\right)}{\Gamma\left(1 + \frac{dn+q}{p}\right)} \cdot n^{\frac{q}{2}} \int_{\mathbb{R}^n} \left(\frac{1}{n} \sum_{i=1}^n x_i^2\right)^{\frac{q}{2}} \mathbb{P}_{n,p}(d\mathbf{x}),$$

where  $\mathbb{P}_{n,p}(d\mathbf{x})$  is the probability

$$(11) \quad \mathbb{P}_{n,p}(d\mathbf{x}) := \frac{1}{Z_{n,p}} \cdot f_{a,b,c}(\mathbf{x}) e^{-abn\gamma_p \|\mathbf{x}\|_p^p} \, d\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n.$$

We conclude this section by expressing the quantities appearing in Theorems 1.1 and 1.2 in terms of the integral of  $h_2(x) := x^2$  with respect to the random empirical measure  $L_{n,p} := n^{-1} \sum_{i=1}^n \delta_{x_i}$ .

**Lemma 2.2** (From inertia to  $\beta$ -ensembles). *For every  $q > 0$ , we have*

$$(12) \quad \frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) = (1 + o(1)) \frac{e^{-\frac{1}{p} - \frac{3}{4}}}{\sqrt{2\pi}} Z_{n,p}^{-1/d_n} \left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}$$

and

$$(13) \quad \frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = \left( 1 - \frac{q-2}{abp^2 n^2} + o\left(\frac{1}{n^2}\right) \right) \frac{\left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2} \right)^{1/q}}{\left( \mathbb{E} \langle L_{n,p}, h_2 \rangle \right)^{1/2}}.$$

*Proof.* Observe that

$$\int_{\mathbb{R}^n} \left(\frac{1}{n} \sum_{i=1}^n x_i^2\right)^{\frac{q}{2}} \mathbb{P}_{n,p}(d\mathbf{x}) = \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}}.$$

By combining (10) and (8), the quantity appearing in Theorem 1.1 is

$$\begin{aligned} \frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) &= \frac{1}{\sqrt{d_n}} \frac{(\mathbb{E} \|T\|_{\text{HS}}^q)^{1/q}}{|B_E(S_p^n)|^{1/d_n}} \\ &= \frac{1}{\sqrt{d_n}} \left( \frac{\Gamma\left(1 + \frac{dn}{p}\right)}{\Gamma\left(1 + \frac{dn+q}{p}\right)} \right)^{\frac{1}{q}} \frac{\Gamma\left(1 + \frac{dn}{p}\right)^{\frac{1}{dn}}}{c_n^{1/d_n} Z_{n,p}^{1/d_n}} \sqrt{n} \left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}, \\ &\sim \sqrt{\frac{2}{abn}} e^{-\frac{1}{p}} c_n^{-1/d_n} Z_n^{-1/d_n} \left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}, \end{aligned}$$

using that  $d_n \sim abn^2/2$  as  $n \rightarrow \infty$ ,  $\Gamma(1+x)^{\frac{1}{x}} \sim x/e$  and  $\Gamma(1+x+\alpha)/\Gamma(1+x) \sim x^\alpha$  as  $x \rightarrow \infty$ , for any fixed number  $\alpha$ . Thus, by the asymptotics of  $c_n$  in (6),



$$\frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) \sim \frac{e^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}} Z_{n,p}^{-1/d_n} \left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Regarding the quantity involved in Theorem 1.2, two applications of (10) give

$$\begin{aligned} \frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} &= \frac{(\mathbb{E} \|T\|_{\text{HS}}^q)^{1/q}}{(\mathbb{E} \|T\|_{\text{HS}}^2)^{1/2}} \\ &= \left( \frac{\Gamma\left(1 + \frac{d_n+q}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{-\frac{1}{q}} \cdot \left( \frac{\Gamma\left(1 + \frac{d_n+2}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{\frac{1}{2}} \cdot \frac{(\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2})^{1/q}}{(\mathbb{E} \langle L_{n,p}, h_2 \rangle)^{1/2}}. \end{aligned}$$

We have  $d_n \sim abn^2/2$  and classical computations on the  $\Gamma$  function at infinity give that for any fixed  $r > 0$  and  $p \geq 1$ ,

$$\left( \frac{\Gamma\left(1 + \frac{d_n+r}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{-\frac{1}{r}} = \left( \frac{d_n}{p} \right)^{\frac{1}{p}} \cdot \left( 1 - \frac{p+r}{abp^2 n^2} + o\left(\frac{1}{n^2}\right) \right).$$

Therefore

$$\left( \frac{\Gamma\left(1 + \frac{d_n+q}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{-\frac{1}{q}} \cdot \left( \frac{\Gamma\left(1 + \frac{d_n+2}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{\frac{1}{2}} = 1 - \frac{q-2}{abp^2 n^2} + o\left(\frac{1}{n^2}\right)$$

which finishes the proof of (13). ■

### 3. MOMENT BOUNDS

Recall that  $\mathbb{P}_{n,p}(d\mathbf{x})$  is the probability whose density is defined on  $\mathbb{R}^n$  by

$$\mathbb{P}_{n,p}(d\mathbf{x}) := \frac{1}{Z_{n,p}} \cdot f_{a,b,c}(\mathbf{x}) e^{-abn\gamma_p \|\mathbf{x}\|_p^p} d\mathbf{x}, \quad \text{where } f_{a,b,c}(\mathbf{x}) := \prod_{1 \leq i < j \leq n} |x_i^a - x_j^a|^b \cdot \prod_{i=1}^n |x_i|^c.$$

In this section we consider a more general case when  $a$  is a positive integer,  $b > 0$  and  $c \geq 0$ . Prior to evaluating their asymptotics, we need to establish some uniform bounds on the quantities

$$(14) \quad \left( \mathbb{E} \langle L_{n,p}, h_r \rangle \right)^{\frac{1}{r}} = \left( \int_{\mathbb{R}^n} |x_1|^r \mathbb{P}_{n,p}(d\mathbf{x}) \right)^{\frac{1}{r}}$$

where  $h_r(x) := |x|^r$ . The evaluation of (14) in the case  $r = 2$  is the crucial part of the proof in [27] while it was also studied for larger values of  $r$  in [17]. We prove here a stronger result. Following the method of proof of Theorem 11.1.2 (i) in [34] we establish an upper bound of the tail of the first marginal density of  $\mathbb{P}_{n,p}(d\mathbf{x})$  denoted by  $p_n(x_1)$  and defined by

$$p_n(x_1) := \frac{1}{Z_{n,p}} \int_{\mathbb{R}^{n-1}} f_{a,b,c}(x_1, \dots, x_n) e^{-abn\gamma_p \sum_{i=1}^n |x_i|^p} dx_2 \cdots dx_n.$$

**Theorem 3.1** (Moment and tail bounds on the first marginal). *Let  $p \geq 1$  and  $R > 0$ . Then there exist constants  $C, C', C'', X_1, c_1, c' > 0$  depending only on  $a, b, c$  and on  $p, R$  such that, for every  $n \geq 1$ :*

(i) for every  $|x_1| \geq X_1$ ,

$$p_n(x_1) \leq e^{-c_1 n |x_1|^p};$$

(ii) for every  $r \geq 1$ ,

$$\left( \int_{\mathbb{R}^n} |x_1|^r \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \right)^{\frac{1}{r}} \leq X_1 + C \left( \frac{r}{n} \right)^{\frac{1}{p}},$$

and thus

$$\forall 1 \leq r \leq Rn, \quad \left( \int_{\mathbb{R}^n} |x_1|^r \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \right)^{\frac{1}{r}} \leq C'';$$

(iii) for every  $i = 1, \dots, n$ , and every  $B \geq X_1$ ,

$$\mathbb{P}_{n,p}(|x_i| \geq B) \leq C' e^{-c'n B^p}.$$

*Proof.* The proof of (i) goes into three steps.

1) We first prove that there exists a constant  $C_1 > 0$  independent of the dimension such that

$$\int_{\mathbb{R}^n} |x_1|^p \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}} |x_1|^p p_n(x_1) \mathrm{d}x_1 \leq C_1.$$

The proof of this inequality follows the lines of the proof of Corollary 7(a) in [17]. For  $t > 0$ , we define

$$g(t) := \int_{\mathbb{R}^n} f_{a,b,c}(\mathbf{x}) e^{-tabn\gamma_p \|\mathbf{x}\|_p^p} \mathrm{d}\mathbf{x}.$$

Then  $g(1) = Z_{n,p}$ . Changing variable by putting  $\mathbf{x} \leftarrow t^{-1/p} \mathbf{y}$  and using that  $f_{a,b,c}$  is positively homogeneous of degree  $d_n - n$ , where  $d_n := abn(n-1)/2 + (c+1)n$ , we get

$$g(t) = t^{-n/p} \int_{\mathbb{R}^n} f_{a,b,c}(t^{-1/p} \mathbf{y}) e^{-abn\gamma_p \|\mathbf{y}\|_p^p} \mathrm{d}\mathbf{y} = t^{-d_n/p} g(1).$$

It follows that  $g'(1) = -\frac{d_n}{p} g(1)$ . On the other hand, differentiating the integral formula which defines  $g(t)$ , we also get

$$\begin{aligned} g'(1) &= -abn\gamma_p \int_{\mathbb{R}^n} \|\mathbf{x}\|_p^p f_{a,b,c}(\mathbf{x}) e^{-abn\gamma_p \|\mathbf{x}\|_p^p} \mathrm{d}\mathbf{x} \\ &= -abn^2\gamma_p \int_{\mathbb{R}^n} |x_1|^p f_{a,b,c}(\mathbf{x}) e^{-abn\gamma_p \|\mathbf{x}\|_p^p} \mathrm{d}\mathbf{x}. \end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^n} |x_1|^p \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}} |x_1|^p p_n(x_1) \mathrm{d}x_1 = -\frac{g'(1)}{g(1)abn^2\gamma_p} = \frac{d_n}{abpn^2\gamma_p}.$$

It follows that  $\int_{\mathbb{R}^n} |x_1|^p \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \rightarrow 1/(2p\gamma_p)$  when  $n \rightarrow \infty$  and therefore is upper bounded by a constant  $C_1$ .

2) In the second step we prove that there exist  $c_2, X_2 > 0$  independent of the dimension such that, for all  $|x_1| \geq X_2$ ,

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}} e^{-c_2 n |x_1|^p}.$$

For  $\mathbf{x} \in \mathbb{R}^n$ , we denote

$$g_n(\mathbf{x}) := Z_{n,p} \frac{\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})}{\mathrm{d}\mathbf{x}} = \prod_{1 \leq i < j \leq n} |x_i^a - x_j^a|^b \cdot \prod_{i=1}^n |x_i|^c \cdot e^{-tabn\gamma_p \|\mathbf{x}\|_p^p}.$$

For  $\tilde{\mathbf{x}} := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$  and  $\mathbf{x} := (x_1, \tilde{\mathbf{x}})$ , using that  $n\|\mathbf{x}\|_p^p = n|x_1|^p + (n-1)\|\tilde{\mathbf{x}}\|_p^p + \|\tilde{\mathbf{x}}\|_p^p$  we get

$$g_n(\mathbf{x}) = g_n(x_1, \tilde{\mathbf{x}}) = g_{n-1}(\tilde{\mathbf{x}}) e^{-abn\gamma_p |x_1|^p} e^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} |x_1|^c \prod_{i=2}^n |x_i^a - x_1^a|^b.$$

Hence

$$\begin{aligned} (15) \quad & \frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^{n-1}} g_n(x_1, \tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \\ & = |x_1|^c e^{-abn\gamma_p |x_1|^p} \int_{\mathbb{R}^{n-1}} e^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \prod_{i=2}^n |x_i^a - x_1^a|^b \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}). \end{aligned}$$

Using that for any  $x, y \in \mathbb{R}$  one has  $|x - y| \leq |x| + |y| \leq (1 + |x|)(1 + |y|)$ , and letting  $D_2 := \max_{t \in \mathbb{R}} e^{-ab\gamma_p t^p} (1 + |t|^a)^b$ , we have

$$\frac{Z_{n,p}}{Z_{n-1,p}} p_n(x_1) = \frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^{n-1}} g_n(x_1, \tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \leq |x_1|^c e^{-abn\gamma_p |x_1|^p} (1 + |x_1|^a)^{(n-1)b} D_2^{n-1}.$$

We conclude that for  $c_2 = ab\gamma_p/2$  there exists  $X_2 > 0$  independent of the dimension such that, for  $|x_1| \geq X_2$ ,

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}} e^{-c_2 n |x_1|^p}.$$

3) In the third step, we establish that there exists  $c_3 > 0$  such that  $\frac{Z_{n-1,p}}{Z_{n,p}} \leq e^{-c_3 n}$ . Integrating equation (15) with respect to  $x_1 \in \mathbb{R}$  we get

$$\begin{aligned} (16) \quad & \frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^n} g_n(\mathbf{x}) \mathrm{d}\mathbf{x} \\ & = \int_{\mathbb{R}} |x_1|^c e^{-abn\gamma_p |x_1|^p} \int_{\mathbb{R}^{n-1}} e^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \prod_{i=2}^n |x_i^a - x_1^a|^b \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \mathrm{d}x_1. \end{aligned}$$

Applying Jensen's inequality to the integral on  $\mathbb{R}^{n-1}$  with the probability measure  $\mathbb{P}_{n-1,p}$  and the convex function being the exponential, and applying also the bound obtained in Step 1, we have

$$\begin{aligned}
& \int e^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \prod_{i=2}^n |x_i^a - x_1^a|^b \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \\
& \geq \exp \int \left( -ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p + b \sum_{i=2}^n \log |x_i^a - x_1^a| \right) \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \\
& \geq \exp \left( -(n-1)ab\gamma_p C_1 + b \int \sum_{i=2}^n \log |x_i^a - x_1^a| \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \right).
\end{aligned}$$

Plugging this inequality into Equation (16) and noticing that the left-hand side is equal to  $Z_{n,p}/Z_{n-1,p}$  we get

$$\frac{Z_{n,p}}{Z_{n-1,p}} \geq e^{-(n-1)ab\gamma_p C_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} |x_1|^c e^{-abn\gamma_p |x_1|^p} \exp \left( b \int \sum_{i=2}^n \log |x_i^a - x_1^a| \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \right) \mathrm{d}x_1.$$

Applying again Jensen's inequality but with the uniform probability measure on  $[-\frac{1}{2}, \frac{1}{2}]$  we deduce that

$$\begin{aligned}
\frac{Z_{n,p}}{Z_{n-1,p}} & \geq e^{-(n-1)ab\gamma_p C_1} \exp \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( c \log |x_1| - abn\gamma_p |x_1|^p \right. \\
& \quad \left. + b \int \sum_{i=2}^n \log |x_i^a - x_1^a| \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \right) \mathrm{d}x_1.
\end{aligned}$$

We then use Fubini's theorem and thus want to estimate from below the function  $g_a$  defined for  $a \geq 1$  being an integer and  $x \in \mathbb{R}$  by

$$g_a(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \log |x^a - y^a| \mathrm{d}y.$$

Namely we shall prove that  $g_a(x) \geq -a(2 \log 2 + 1)$ . First notice that for  $a$  even the function  $(x, y) \mapsto x^a - y^a$  is even in both variables, and for  $a$  odd, splitting the integral and changing variable one has

$$g_a(x) = \int_0^{\frac{1}{2}} (\log |x^a - y^a| + \log |x^a + y^a|) \mathrm{d}y = \int_0^{\frac{1}{2}} \log |x^{2a} - y^{2a}| \mathrm{d}y = \frac{1}{2} g_{2a}(x).$$

We are thus reduced to proving the lower bound of  $g_a$  in the case where  $a$  is even. Then the function  $g_a$  is even so we may assume that  $x \geq 0$  and we have

$$g_a(x) = 2 \int_0^{\frac{1}{2}} \log |x^a - y^a| \mathrm{d}y.$$

For  $x \geq \frac{1}{2}$  the function  $g_a$  is increasing hence  $g_a(x) \geq g_a(1/2)$ . Moreover for every  $x, y > 0$  one has  $|x^a - y^a| \geq y^{a-1}|x - y|$ , thus

$$g_a(x) \geq 2 \int_0^{\frac{1}{2}} ((a-1) \log y + \log |x - y|) \mathrm{d}y.$$

Let  $\varphi$  be the convex function defined by  $\varphi(x) = x \log x$  for  $x \geq 0$ . A simple calculation shows that, for every  $0 \leq x \leq \frac{1}{2}$ ,

$$g_a(x) \geq -(a-1)(\log 2 + 1) + 2\varphi(x) + 2\varphi\left(\frac{1}{2} - x\right) - 1.$$

From the convexity of  $\varphi$ , one has

$$\varphi(x) + \varphi\left(\frac{1}{2} - x\right) \geq 2\varphi\left(\frac{x + \frac{1}{2} - x}{2}\right) = 2\varphi\left(\frac{1}{4}\right) = -\log 2.$$

We conclude that  $g_a(x) \geq -a(2 \log 2 + 1)$ . Applying this inequality in the lower bound of  $Z_{n,p}/Z_{n-1,p}$  we deduce that

$$\begin{aligned} \frac{Z_{n,p}}{Z_{n-1,p}} &\geq \exp\left(- (n-1)ab\gamma_p C_1 - (2 \log 2 + 1)(c + (n-1)ab)\right) - \frac{nab\gamma_p}{(p+1)2^p} \\ &\geq e^{-c_3 n}, \end{aligned}$$

where  $c_3 := ab\gamma_p C_1 + (2 \log 2 + 1)(c + ab) + \frac{ab\gamma_p}{(p+1)2^p}$ , which finishes our third step. The final conclusion follows from combining the steps 2 and 3 which give that for every  $|x_1| \geq X_2$ , one has

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}} e^{-c_2 n |x_1|^p} \leq e^{-c_3 n - c_2 n |x_1|^p}.$$

Hence there exist  $c_1, X_1 > 0$  such that  $p_n(x_1) \leq e^{-c_1 n |x_1|^p}$  for all  $|x_1| \geq X_1$ .

(ii) The upper bound for the moment using the bound for the tail is standard and runs as follows. We first cut the integral into two parts and use the bound proved in (i):

$$\begin{aligned} \int_{\mathbb{R}^n} |x_1|^r \mathbb{P}_{n,p}(d\mathbf{x}) &= \int_{\mathbb{R}} |t|^r p_n(t) dt \leq X_1^r \int_{|t| \leq X_1} p_n(t) dt + \int_{|t| \geq X_1} |t|^r e^{-c_1 n |t|^p} dt \\ &\leq X_1^r + 2 \int_0^\infty t^r e^{-c_1 n t^p} dt. \end{aligned}$$

Changing variable, this last integral may be written as follows:

$$\int_0^\infty t^r e^{-c_1 n t^p} dt = \frac{1}{p(c_1 n)^{\frac{r+1}{p}}} \int_0^\infty s^{\frac{r+1}{p}-1} e^{-s} ds = \frac{1}{p(c_1 n)^{\frac{r+1}{p}}} \Gamma\left(\frac{r+1}{p}\right).$$

Using a standard bound on the Gamma function we conclude that there exists  $C > 0$  such that

$$\left( \int_{\mathbb{R}^n} |x_1|^r \mathbb{P}_{n,p}(d\mathbf{x}) \right)^{\frac{1}{r}} \leq X_1 + \left( \frac{2}{p(c_1 n)^{\frac{r+1}{p}}} \Gamma\left(\frac{r+1}{p}\right) \right)^{\frac{1}{r}} \leq X_1 + C \left(\frac{r}{n}\right)^{\frac{1}{p}}.$$

(iii) Using the bound (i), for  $B > X_1$  and choosing  $c' := c_1/2$ , we have

$$\mathbb{P}_{n,p}(|x_i| \geq B) \leq 2 \int_B^\infty e^{-c_1 n t^p} dt \leq 2 e^{-c' n B^p} \int_0^\infty e^{-c' n t^p} dt \leq C' e^{-c' n B^p}. \quad \blacksquare$$

Because of the above uniform bounds, the random measure  $L_{n,p}$  is mostly concentrated on a compact interval. As a result, testing  $L_{n,p}$  against a general function  $f$  is on average not very different from using a truncated version of  $f$  instead.

**Corollary 3.2** (Truncation argument). *Let  $c' > 0, X_1 \geq 1$  be given by Theorem 3.1. Let  $B \geq X_1$  and let  $\phi: \mathbb{R} \rightarrow [0, 1]$  be a smooth, compactly supported function such that  $\phi \equiv 1$  on  $[-B, B]$ . Then for all continuous, polynomially bounded functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ , one has*

$$\mathbb{E} \left| g(\langle L_{n,p}, f \rangle) - g(\langle L_{n,p}, f\phi \rangle) \right| = O(\alpha^n),$$

where  $\alpha := e^{-\frac{c'BP}{2}} < 1$ .

*Proof.* As  $f$  and  $g$  are polynomially bounded, there exist constants  $A_1, A_2, s, t \geq 1$  such that for every  $y \in \mathbb{R}$ ,  $|f(y)| \leq A_1 + |y|^s$  and  $|g(y)| \leq A_2 + |y|^t$ . Hence, we can find constants  $A, r \geq 1$  such that, for every  $\mathbf{x} \in \mathbb{R}^n$ ,

$$|g(\langle L_{n,p}(\mathbf{x}), f \rangle)| = \left| g\left(\frac{1}{n} \sum_{i=1}^n f(x_i)\right) \right| \leq A + \frac{1}{n} \sum_{i=1}^n |x_i|^r = A + \langle L_{n,p}(\mathbf{x}), h_r \rangle$$

(e.g.,  $A := 2^t A_1 + A_2$  and  $r := st$ , using Jensen's inequality). Since  $0 \leq \phi \leq 1$  on  $\mathbb{R}$ , we get the same bound for

$$|g(\langle L_{n,p}(\mathbf{x}), f\phi \rangle)| \leq A + \langle L_{n,p}(\mathbf{x}), h_r \rangle.$$

As  $\phi \equiv 1$  on  $[-B, B]$ ,

$$|g(\langle L_{n,p}(\mathbf{x}), f \rangle) - g(\langle L_{n,p}(\mathbf{x}), f\phi \rangle)| \leq 2\mathbf{1}_{\{\exists i: |x_i| > B\}} (A + \langle L_{n,p}(\mathbf{x}), h_r \rangle).$$

Since  $r$  is a constant, we have by Theorem 3.1.(ii),

$$\mathbb{E} \langle L_{n,p}(\mathbf{x}), h_r \rangle^2 = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^n |x_i|^r \right)^2 \leq \mathbb{E} \frac{1}{n} \sum_{i=1}^n |x_i|^{2r} \leq C''^{2r}.$$

Combining Theorem 3.1.(iii) with a union bound,

$$\mathbb{P}_{n,p}(\exists i: |x_i| > B) \leq nC' e^{-c'nBP}.$$

Therefore, by Cauchy-Schwarz inequality, we conclude that

$$\mathbb{E} \left| g(\langle L_{n,p}, f \rangle) - g(\langle L_{n,p}, f\phi \rangle) \right| \leq 2nC' e^{-c'nBP} (A + C''^r).$$

This is  $O(\alpha^n)$  with e.g.  $\alpha = e^{-\frac{c'BP}{2}}$ . ■

#### 4. ASYMPTOTICS

In this section, we prove Theorems 1.1 and 1.2 by completing the asymptotic expansions initiated in Lemma 2.2. As previously explained, we will appeal to the literature on random matrices, as  $L_{n,p}$  corresponds to the empirical distribution of the so called  $\beta$ -ensemble with potential  $V(x) \propto |x|^p$  and  $Z_{n,p}$  is the so called partition function. We split the discussion into two cases, according to Table 1. In the first case ( $a = 1$ ), we

prove both Theorems 1.1 and 1.2, where matrices in  $B_E(S_p^n)$  all have symmetries with respect to their diagonal and  $L_{n,p}$  thus corresponds to empirical distributions of real eigenvalues. The second case ( $a = 2$ ) pertains to Theorem 1.1 only and is treated more conveniently by working with  $\mathbb{R}^+$ -valued measures.

**4.1. The self-adjoint case ( $a = 1$ ).** In view of Table 1, we have  $a = 1$ ,  $b = \dim_{\mathbb{R}}(\mathbb{F})$  and  $c = 0$ . Hence, from (11), the probability  $\mathbb{P}_{n,p}$  can be written

$$(17) \quad \mathbb{P}_{n,p}(d\mathbf{x}) = \frac{1}{Z_{n,p}} \cdot e^{-\frac{b}{2}n^2 H_{n,p}(\mathbf{x})} d\mathbf{x},$$

with the Hamiltonian

$$H_{n,p}(\mathbf{x}) := \frac{2}{n} \sum_{i=1}^n \gamma_p |x_i|^p - \frac{1}{n^2} \sum_{i \neq j} \log |x_i - x_j|$$

where  $\gamma_p$  is defined in (7). Heuristically this Hamiltonian is approximated as  $n \rightarrow \infty$  by

$$\mathcal{I}_p(\mu) := 2 \int_{\mathbb{R}} \gamma_p |x|^p \mu(dx) - \iint_{\mathbb{R}^2_{\neq}} \log |x - y| \mu(dx) \mu(dy),$$

where  $\mathbb{R}^2_{\neq} := \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ , provided that the probability measure  $\mu$  is sufficiently close to the empirical measure  $L_{n,p}(\mathbf{x}) := n^{-1} \sum_{i=1}^n \delta_{x_i}$ . Thus, because of (17), we expect that  $L_{n,p}$  concentrates as  $n \rightarrow \infty$  around a probability measure minimizing the functional  $\mathcal{I}_p$ . This idea constitutes the cornerstone of large deviation principles for the empirical spectral distribution of large random matrices, as originally derived by Ben Arous and Guionnet [9]. The next two lemmas formalize what we need. We refer to the textbooks [14, Chapter 6] and [2, Section 2.6] or the recent article [15] for deeper results.

**Lemma 4.1** (Equilibrium measure). *There exists a unique element  $\mu_p$  minimizing the functional  $\mathcal{I}_p$  over the space  $\mathcal{P}(\mathbb{R})$  of real probability measures:*

$$\inf_{\mu \in \mathcal{P}(\mathbb{R})} \mathcal{I}_p(\mu) = \mathcal{I}_p(\mu_p).$$

Furthermore,  $\mu_p$  has the following properties:

- (i)  $\mu_p$  is compactly supported, with support  $[-1, 1]$ ;
- (ii)  $\mu_p$  is absolutely continuous with respect to the Lebesgue measure, with density

$$f_p(x) := \frac{p|x|^{p-1}}{\pi} \int_{|x|}^1 \frac{v^{-p}}{\sqrt{1-v^2}} dv, \quad |x| \leq 1;$$

- (iii)  $\mathcal{I}_p(\mu_p) = \log 2 + \frac{3}{2p}$ .

*Proof.* The existence and uniqueness of the minimizer  $\mu_p$  follows from [36, Theorem I.1.3] with the weight function  $w: x \mapsto \exp(-\gamma_p |x|^p)$  on  $\mathbb{R}$ . Properties (i)–(iii) are provided by [36, Theorem IV.5.1]. (We have rewritten the density with the change of variable  $v \leftarrow |x|/u$ .) ■

**Lemma 4.2** (Convergence towards the equilibrium measure). *Let  $L_{n,p}$  be the empirical measure of  $\mathbf{x}$  drawn from the probability  $\mathbb{P}_{n,p}$  defined in (17). Then:*

(i) *For every continuous, polynomially bounded functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ , it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E} g(\langle L_{n,p}, f \rangle) = g(\langle \mu_p, f \rangle).$$

(ii) *We have*

$$\lim_{n \rightarrow \infty} -\frac{1}{d_n} \log Z_{n,p} = \log 2 + \frac{3}{2p},$$

*where the normalizing constant  $Z_{n,p}$  is defined in (17).*

*Proof.* (i) By Corollary 3.2, we may reduce to  $f$  bounded. Thus the random variable  $g(\langle L_{n,p}, f \rangle)$  is uniformly bounded, and letting  $\Delta_{n,p}(f) := \langle L_{n,p}, f \rangle - \langle \mu_p, f \rangle$  we see from the Borel-Cantelli lemma and the dominated convergence theorem that it suffices to show that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\Delta_{n,p}(f)| > \varepsilon_n) < \infty,$$

for some null sequence  $(\varepsilon_n)$ . Thanks to [19, Theorem 2.1], we have that

$$\alpha_n := \frac{1}{n} \log \mathbb{E} \exp(n|\Delta_{n,p}(f)|) \xrightarrow{n \rightarrow \infty} 0.$$

Choosing now  $\varepsilon_n$  as, e.g.,  $\varepsilon_n := \alpha_n/2 + (\log n)/n$ , we see from Markov's inequality that

$$\mathbb{P}(|\Delta_{n,p}(f)| > 2\varepsilon_n) = \mathbb{P}(\exp(n|\Delta_{n,p}(f)|) > \exp(2n\varepsilon_n)) = O(e^{-(2\varepsilon_n - \alpha_n)n}) = O\left(\frac{1}{n^2}\right),$$

and the first point of the lemma is proved.

(ii) This point follows from [19, Corollary 4.3] and Lemma 4.1.(iii). ■

We now have all ingredients to prove Theorem 1.1 in the case  $a = 1$ .

*Proof of Theorem 1.1, case  $a = 1$ .* We start by recalling (12): as  $n \rightarrow \infty$ ,

$$\frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) \sim \frac{e^{-\frac{1}{p} - \frac{3}{4}}}{\sqrt{2\pi}} Z_{n,p}^{-1/d_n} \left( \mathbb{E} \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

By Lemma 4.6, we have  $Z_{n,p}^{-1/d_n} \rightarrow 2e^{\frac{3}{2p}}$  and

$$\lim_{n \rightarrow \infty} \mathbb{E} \langle L_{n,p}, h_2 \phi \rangle^{\frac{q}{2}} = \langle \mu_p, h_2 \phi \rangle^{\frac{q}{2}}.$$

Plugging in the value of  $\langle \mu_p, h_2 \rangle$  computed in Lemma 5.1, we conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) = e^{\frac{1}{2p} - \frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}}. \quad \blacksquare$$



The proof of Lemma 4.2 informs us rather poorly on the speed of convergence of the linear statistics  $\langle L_{n,p}, f \rangle$  towards  $\langle \mu_p, f \rangle$ . In fact, a necessary condition for Theorem 1.3 to hold is

$$\text{Var} \langle L_{n,p}, h_2 \rangle = O\left(\frac{1}{n^2}\right),$$

so we expect that  $\langle L_{n,p}, h_2 \rangle - \langle \mu_p, h_2 \rangle = O(n^{-1})$ . Besides, to identify the actual limit in Theorem 1.3 we need to understand precisely the asymptotic behavior of the fluctuations  $n(\langle L_{n,p}, h_2 \rangle - \langle \mu_p, h_2 \rangle)$ .

There has been an increasing wealth of literature on such fluctuations of linear statistics. Especially, a central limit theorem for general  $\beta$ -ensembles has been established since the pioneering work of Johansson [19], namely the convergence to a Gaussian distribution as  $n \rightarrow \infty$  of the random variable

$$(18) \quad F_n(\xi) := n\left(\langle L_n, \xi \rangle - \langle \mu_V, \xi \rangle\right),$$

where  $\xi$  is a given test function,  $\mu_V$  is the equilibrium distribution, and  $L_n := n^{-1} \sum_{i=1}^n \delta_{x_i}$  is the empirical distribution associated with an ensemble  $\mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$  of “eigenvalues” which are subject to some confining potential  $V$  and pairwise repulsive logarithmic interaction.

In our setting (specifically, of Theorem 1.2), the potential is  $V = V_p := 2\gamma_p h_p$ , the  $\beta$ -ensemble  $\mathbf{x}$  has the distribution  $\mathbb{P}_{n,p}$  with  $a = 1$ ,  $b = \beta$ , and  $c = 0$  (cf. (17)), and the equilibrium measure  $\mu_V$  is of course the probability distribution  $\mu_p$  of Lemmas 4.1 and 4.2. We only need to establish the central limit theorem for  $\xi = h_2$ , that is the convergence of  $F_n(h_2)$ . As  $V_p$  is not always a polynomial nor real-analytic, we choose to work with the currently most general version of Bekerman, Leblé and Serfaty [8]. Like in Johansson [19], the central limit theorem is obtained by establishing convergence of the moment generating function (a.k.a. Laplace transform) of (18). The result we can extract from their work that we need in order to prove Theorem 1.2 is summarized below. We mention that when  $p > 8$ , Proposition 4.3 is also a direct consequence of the CLT of Lambert, Ledoux and Webb [29, Theorem 1.2], which they derived alternatively using Stein’s method.

**Proposition 4.3** (Fluctuations of the linear statistics). *Let  $p > 3$  and let*

$$F_{n,p} := n\left(\langle L_{n,p}, h_2 \phi \rangle - \langle \mu_p, h_2 \rangle\right),$$

*with  $\phi$  being the truncature function of Corollary 3.2. Then  $(F_{n,p}^2)_{n \geq 1}$  is uniformly integrable, i.e.,*

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{F_{n,p}^2 > K\}} = 0.$$

*Furthermore,  $\lim_{n \rightarrow \infty} \text{Var} F_{n,p} = \frac{1}{4b}$ .*

*Proof.* We summarize the arguments of Bekerman, Leblé and Serfaty [8]. It suffices to show that the moment generating function of  $F_{n,p}$  converges to that of a Gaussian variable  $N$  with variance  $1/4b$ , i.e., that there exists  $m_p := m_p(h_2\phi) \in \mathbb{R}$  such that

$$(19) \quad \lim_{n \rightarrow \infty} \mathbb{E} e^{sF_{n,p}} = \exp\left(sm_p + \frac{s^2}{2} \left(\frac{1}{4b}\right)\right)$$

holds for all  $s \in \mathbb{R}$  with  $|s|$  sufficiently small. Indeed, this classically entails the convergence in distribution of  $F_{n,p}$  towards  $N \sim \mathcal{N}(m_p, \frac{1}{4b})$ , together with the convergence of all moments  $\mathbb{E} F_{n,p}^k \rightarrow \mathbb{E} N^k$ ,  $k \in \mathbb{N}$ . In particular,  $(F_{n,p}^2)_{n \geq 1}$  will be uniformly integrable.

First, by Theorem 3.1.(iii) and union bound, we can choose  $B > 0$  sufficiently large such that  $\mathbb{P}_{n,p}(\exists i \leq n : |x_i| \geq B) \leq C'n e^{-c'nB^p}$  for some constants  $c', C' > 0$ . Observing that  $\|F_{n,p}(h_2\phi)\|_\infty = O(n)$  we deduce that as soon as  $|s|$  is sufficiently small,

$$\lim_{n \rightarrow \infty} \mathbb{E} e^{sF_{n,p}} \mathbf{1}_{\{\exists i \leq n : |x_i| \geq B\}} = 0.$$

Thus, we may restrict to  $\mathbf{x} \in U_0^n$  with  $U_0 := (-B, B)$ . Now, the strategy, adopted in [8] and developed already in [19], is to perturb the Hamiltonian  $H_{n,p}$  in (17) as follows:

$$H_{n,p}^t(\mathbf{x}) := H_{n,p}(\mathbf{x}) + \frac{t}{n} \sum_{i=1}^n x_i^2 = \frac{1}{n} \sum_{i=1}^n (V_p(x_i) + tx_i^2) - \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} \log |x_i - x_j|,$$

where we have set  $t := -\frac{2s}{bn}$ . Then, we write

$$\mathbb{E} e^{sF_{n,p}} = o(1) + e^{-sn\langle \mu_p, h_2 \rangle} \mathbb{E} e^{sn\langle L_{n,p}, h_2 \rangle} \mathbf{1}_{U_0^n} = o(1) + \frac{e^{-sn\langle \mu_p, h_2 \rangle}}{Z_{n,p}} \int_{U_0^n} e^{-\frac{b}{2}n^2 H_{n,p}^t(\mathbf{x})} d\mathbf{x}.$$

Next, one applies in the last integral a change of variables  $x_i \leftarrow \vartheta_t(y_i)$  for  $1 \leq i \leq n$  and for some well-chosen  $\mathcal{C}^1$ -diffeomorphism  $\vartheta_t: U_t \rightarrow U_0$  depending on  $t$ . We get

$$\begin{aligned} \mathbb{E} e^{sF_{n,p}} &= o(1) + \frac{e^{-sn\langle \mu_p, h_2 \rangle}}{Z_{n,p}} \int_{U_t^n} e^{-\frac{b}{2}n^2 H_{n,p}^t \circ \vartheta_t(\mathbf{y})} \prod_{i=1}^n |\vartheta_t'(y_i)| d\mathbf{y} \\ &= o(1) + e^{-sn\langle \mu_p, h_2 \rangle} \mathbb{E} e^{-\frac{b}{2}n^2 (H_{n,p}^t \circ \vartheta_t - H_{n,p}) + n\langle L_{n,p}, \log |\vartheta_t'| \rangle} \mathbf{1}_{U_t^n}. \end{aligned}$$

The idea is that, for a judicious choice of  $\vartheta_t$ , the exponent in the last expectation becomes simple enough to establish the convergence towards the moment generating function of a Gaussian distribution. The diffeomorphism is chosen as  $\vartheta_t(y) := y + t\psi_p(y)$  where, in our setting,  $\psi_p: \mathbb{R} \rightarrow \mathbb{R}$  is the solution to the equation

$$(20) \quad \Xi_p \psi = \frac{1}{2}h_2 + c_{h_2}$$

for some constant  $c_{h_2}$ , where  $\Xi_p$  is the so called ‘‘master operator’’ acting on  $\mathcal{C}^1$  functions and defined for  $x \in \mathbb{R}$  by

$$\Xi_p \psi(x) = -\frac{1}{2}V_p'(x)\psi(x) + \int \frac{\psi(x) - \psi(y)}{x - y} \mu_p(dy).$$

We show in Section 5.2 that  $\psi_p$  is an odd function, we give its explicit expression and check that it is of class  $\mathcal{C}^{[p]-1} \subset \mathcal{C}^1$ . Then, by the local inversion theorem,  $\vartheta_t = \text{Id} + t\psi_p$  becomes a  $\mathcal{C}^1$ -diffeomorphism  $U_t \rightarrow U_0$  provided  $|t| = 2|s|/bn \leq \tau$  is sufficiently small (equivalently,  $n$  is sufficiently large), and we have

$$(21) \quad \mathbb{E} e^{sF_{n,p}} = o(1) + e^{-sn\langle \mu_p, h_2 \rangle} \mathbb{E} e^{-\frac{b}{2} n^2 (H_{n,p}^t \circ \vartheta_t - H_{n,p}) + n\langle L_{n,p}, \log(1+t\psi_p') \rangle} \mathbb{1}_{U_t^n},$$

where  $U_t := (-A_t, A_t)$  and  $\vartheta_t(A_t) = A_t + t\psi_p(A_t) = B$ . Furthermore, we know from the implicit function theorem that  $A_t$ ,  $|t| \leq \tau$ , depends continuously on  $t$ , and thus choosing  $\tau$  small enough we can assume that  $1 < \inf_{|t| \leq \tau} A_t < \sup_{|t| \leq \tau} A_t < \infty$ .

The remaining of the proof is to Taylor-expand as  $t \rightarrow 0$  (i.e.,  $n \rightarrow \infty$ ) the terms  $H_{n,p}^t \circ \vartheta_t - H_{n,p}$  and  $\log(1+t\psi_p')$  appearing in the right-hand side of (21) up to the order  $O(t^3)$ . After a rather lengthy but not difficult calculation (see [8, Section 4]), and using that  $\psi_p$  solves (20), we obtain that

$$(22) \quad \mathbb{E} e^{sF_{n,p}} = o(1) + e^{sm_p + \frac{s^2}{2} \Sigma^2(h_2)} \mathbb{E} \exp \left\{ -\frac{s}{n} \mathbf{A}_n[\psi_p] + O(nt^2) + O(n^2t^3) \right\} \mathbb{1}_{U_t^n},$$

where  $O(nt^2)$  and  $O(n^2t^3)$  are random quantities converging uniformly to 0 as  $n \rightarrow \infty$  (recall that  $t := -\frac{2s}{bn}$ ,  $m_p := \langle \mu_p, \psi_p' \rangle$  is the limiting mean,

$$\Sigma^2(h_2) := \frac{1}{2b\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{(x+y)^2(1-xy)}{\sqrt{1-x^2}\sqrt{1-y^2}} dx dy$$

is the limiting variance<sup>1</sup>, and, lastly,

$$\mathbf{A}_n[\psi_p] := n^2 \iint \frac{\psi_p(x) - \psi_p(y)}{x-y} (L_{n,p} - \mu_p)(dx)(L_{n,p} - \mu_p)(dy)$$

is the so called *anisotropy term*. We can see from the proof of [8, Proposition 5.4] that, for  $|s|$  sufficiently small,

$$(23) \quad \lim_{n \rightarrow \infty} \log \mathbb{E} \exp \left\{ -\frac{s}{n} \mathbf{A}_n[\psi_p] \right\} = 0$$

holds provided  $\psi_p$  is of class  $\mathcal{C}^3$ . The general theorem of [8] states that this is the case if  $p > 6$ . However, since in our specific situation we can derive an explicit expression for  $\psi_p$  (see Section 5.2), the  $\mathcal{C}^3$ -regularity is true as soon as  $p > 3$  (Lemma 5.3). It then follows by the Cauchy-Schwarz inequality that the expectation in the right-hand side of (22) tends to 1. As the value of  $\Sigma^2(h_2)$  simplifies to  $1/4b$  thanks to Lemma 5.2, this establishes (19), and the proof is complete.  $\blacksquare$

**Remark 4.4.** Note that (23) is the only place where a restriction on  $p$  seems necessary.

It is now easy to establish Theorem 1.2.

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<sup>1</sup>It is a known ‘‘universality’’ fact of random matrix theory that this variance does not depend on the potential.

*Proof of Theorem 1.2.* Recall the earlier computation (13) (with  $a = 1$ ): as  $n \rightarrow \infty$ ,

$$(24) \quad \frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = \left(1 - \frac{q-2}{bpn^2} + o\left(\frac{1}{n^2}\right)\right) \frac{(\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2})^{1/q}}{(\mathbb{E} \langle L_{n,p}, h_2 \rangle)^{1/2}}.$$

We must expand both the numerator and denominator of that latter fraction. By Corollary 3.2, referring to the truncature function  $\phi$  there, we may replace  $h_2$  by  $h_2\phi$  as this only induces a  $o(n^{-2})$  error:

$$(25) \quad \frac{\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2}}{(\mathbb{E} \langle L_{n,p}, h_2 \rangle)^{q/2}} = \frac{\mathbb{E} \langle L_{n,p}, h_2\phi \rangle^{q/2} + o(\frac{1}{n^2})}{(\mathbb{E} \langle L_{n,p}, h_2\phi \rangle + o(\frac{1}{n^2}))^{q/2}}.$$

Applying Taylor-Lagrange's formula to the function  $x \mapsto x^{q/2}$  between  $x_0 := \langle \mu_p, h_2 \rangle > 0$  and  $x_0 + h$  yields

$$\left| (x_0 + h)^{\frac{q}{2}} - x_0^{\frac{q}{2}} - \frac{q}{2}h - \frac{q(q-2)}{8}h^2 \right| \mathbf{1}_{\{|h| \leq \frac{x_0}{2}\}} \leq C_{q,x_0} |h|^3,$$

for some constant  $C_{q,x_0}$  depending on  $q$  and  $x_0$ . Up to enlarging  $C_{q,x_0}$ , we also have

$$\left| x_0^{\frac{q}{2}} + \frac{q}{2}h + \frac{q(q-2)}{8}h^2 \right| \mathbf{1}_{\{|h| > \frac{x_0}{2}\}} \leq C_{q,x_0} h^2 \mathbf{1}_{\{|h| > \frac{x_0}{2}\}}.$$

We write  $\langle L_{n,p}, h_2\phi \rangle = x_0 + h$  with  $h := \frac{1}{n}F_{n,p} = \langle L_{n,p}, h_2\phi \rangle - \langle \mu_p, h_2 \rangle$  and note that  $|h| \leq \|h_2\phi\|_\infty + x_0$ . Then, on the one hand, using the above inequalities according to whether  $|h| \leq x_0/2$  or  $|h| > x_0/2$  gives

$$(26) \quad \begin{aligned} & n^2 \left| \mathbb{E} \langle L_{n,p}, h_2\phi \rangle^{\frac{q}{2}} \mathbf{1}_{\{|F_{n,p}| \leq \frac{nx_0}{2}\}} - \langle \mu_p, h_2 \rangle^{\frac{q}{2}} - \frac{q}{2n} \mathbb{E} F_{n,p} - \frac{q(q-2)}{8n^2} \mathbb{E} F_{n,p}^2 \right| \\ & \leq C_{q,x_0} \left( \frac{\mathbb{E} |F_{n,p}|^3}{n} + \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{|F_{n,p}| > \frac{nx_0}{2}\}} \right) \\ & \leq C_{q,x_0} \left( \frac{K^{3/2}}{n} + (\|h_2\phi\|_\infty + x_0) \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{F_{n,p}^2 > K\}} + \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{|F_{n,p}| > \frac{nx_0}{2}\}} \right) \\ & \leq C_{q,x_0} \left( \frac{K^{3/2}}{n} + (\|h_2\phi\|_\infty + x_0 + 1) \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{F_{n,p}^2 > K\}} \right), \end{aligned}$$

for any  $0 < K \leq n^2 x_0^2/4$ . On the other hand, for such  $n$  and  $K$ ,

$$(27) \quad n^2 \mathbb{E} \langle L_{n,p}, h_2\phi \rangle^{\frac{q}{2}} \mathbf{1}_{\{|F_{n,p}| > \frac{nx_0}{2}\}} \leq \frac{4}{x_0^2} \|h_2\phi\|_\infty^{q/2} \mathbb{E} F_{n,p}^2 \mathbf{1}_{\{F_{n,p}^2 > K\}}.$$

Proposition 4.3 tells us that the right-hand sides of (26) and (27) vanish (letting  $n \rightarrow \infty$  first then  $K \rightarrow \infty$ ), and that  $\text{Var} F_{n,p} \rightarrow \frac{1}{4b}$  as  $n \rightarrow \infty$ . Hence, by triangle inequality

$$\mathbb{E} \langle L_{n,p}, h_2\phi \rangle^{\frac{q}{2}} = \langle \mu_p, h_2 \rangle^{\frac{q}{2}} + \frac{q}{2n} \mathbb{E} F_{n,p} + \frac{q(q-2)}{8n^2} \mathbb{E} F_{n,p}^2 + o\left(\frac{1}{n^2}\right).$$

This holds in particular when  $q = 2$ , and going back to (25) we get

$$\begin{aligned}
\frac{\mathbb{E} \langle L_{n,p}, h_2 \rangle^{q/2}}{(\mathbb{E} \langle L_{n,p}, h_2 \rangle)^{q/2}} &= \frac{\mathbb{E} \langle L_{n,p}, h_2 \phi \rangle^{q/2} + o\left(\frac{1}{n^2}\right)}{\left(\mathbb{E} \langle L_{n,p}, h_2 \phi \rangle + o\left(\frac{1}{n^2}\right)\right)^{q/2}} \\
&= \frac{\langle \mu_p, h_2 \rangle^{\frac{q}{2}} + \frac{q}{2n} \mathbb{E} F_{n,p} + \frac{q(q-2)}{8n^2} \mathbb{E} F_{n,p}^2 + o\left(\frac{1}{n^2}\right)}{\langle \mu_p, h_2 \rangle^{\frac{q}{2}} + \frac{q}{2n} \mathbb{E} F_{n,p} + \frac{q(q-2)}{8n^2} (\mathbb{E} F_{n,p})^2 + o\left(\frac{1}{n^2}\right)} \\
&= 1 + \frac{q(q-2)}{8 \langle \mu_p, h_2 \rangle} \cdot \frac{\text{Var} F_{n,p}}{n^2} + o\left(\frac{1}{n^2}\right) \\
&= 1 + \frac{q(q-2)(p+2)^2}{8bp^2 n^2} + o\left(\frac{1}{n^2}\right),
\end{aligned}$$

where we also replaced the value of  $\langle \mu_p, h_2 \rangle$  computed in Lemma 5.1. Raising this expansion to the power  $1/q$  and returning to (24), we obtain

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = 1 - \frac{(q-2)(p-2)^2}{16p^2 d_n} + o\left(\frac{1}{d_n}\right),$$

because  $d_n \sim bn^2/2$ . ■

**4.2. The case  $E = \mathcal{M}_n(\mathbb{F})$  ( $a = 2$ ).** We suppose in this section that  $a = 2$ . We reduce to measures and integrals over  $\mathbb{R}^+$  by performing the change of variables  $y_i := |x_i|^2$  for all  $1 \leq i \leq n$ . Specifically, the pushforward of  $\mathbb{P}_{n,p}$  by the map  $\mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{y} \in (\mathbb{R}^+)^n$  is the probability<sup>2</sup>

$$\tilde{\mathbb{P}}_{n,p}(\mathrm{d}\mathbf{y}) := \frac{1}{Z_{n,p}} \cdot \prod_{1 \leq i < j \leq n} |y_i - y_j|^b \cdot \prod_{i=1}^n y_i^{\frac{c-1}{2}} \cdot e^{-2bn\gamma_p \|\mathbf{y}\|_{p/2}^{p/2}} \mathrm{d}\mathbf{y}, \quad \mathbf{y} \in (\mathbb{R}^+)^n.$$

In particular, for every measurable functions  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ , and  $g: \mathbb{R} \rightarrow \mathbb{R}$ ,

$$(28) \quad \int_{(\mathbb{R}^+)^n} g\left(\frac{1}{n} \sum_{i=1}^n f(y_i)\right) \tilde{\mathbb{P}}_{n,p}(\mathrm{d}\mathbf{y}) = \int_{\mathbb{R}^n} g\left(\frac{1}{n} \sum_{i=1}^n f(x_i^2)\right) \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}).$$

It is here convenient to work with the empirical probability measure  $\tilde{L}_{n,p} := n^{-1} \sum_{i=1}^n \delta_{y_i}$  where  $\mathbf{y} \in (\mathbb{R}^+)^n$  is sampled from  $\tilde{\mathbb{P}}_{n,p}$ , and we note that Corollary 3.2 obviously still applies with  $\tilde{L}_{n,p}$  in place of  $L_{n,p}$ . Similarly to the previous section, we introduce

$$\tilde{\mathcal{I}}_p(\mu) := 4 \int_{\mathbb{R}^+} \gamma_p |y|^{p/2} \mu(\mathrm{d}y) - \iint_{(\mathbb{R}^+)^2_{\neq}} \log |x - y| \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$

The counterparts of Lemmas 4.1 and 4.2 are as follows.

---

<sup>2</sup>The normalizing constant  $Z_{n,p}$  remains unchanged; it is just as in (9) but with  $a = 2$ .

**Lemma 4.5** (Equilibrium measure). *There exists a unique element  $\tilde{\mu}_p$  minimizing the functional  $\tilde{\mathcal{I}}_p$  over the space  $\mathcal{P}(\mathbb{R}^+)$  of probability measures on  $\mathbb{R}^+$ :*

$$\tilde{\mathcal{I}}_p(\tilde{\mu}_p) = \inf_{\mu \in \mathcal{P}(\mathbb{R}^+)} \tilde{\mathcal{I}}_p(\mu).$$

Furthermore,  $\tilde{\mu}_p$  coincides with the image measure of  $\mu_p$  by the map  $x \mapsto x^2$ , and it has the following properties:

- (i)  $\tilde{\mu}_p$  is compactly supported, with support  $[0, 1]$ ;
- (ii)  $\tilde{\mu}_p$  is absolutely continuous with respect to the Lebesgue measure, with density

$$\frac{d\tilde{\mu}_p}{dy} = \frac{py^{\frac{p}{2}-1}}{\pi} \int_{\sqrt{y}}^1 \frac{u^{-p}}{\sqrt{1-u^2}} du, \quad 0 \leq y \leq 1;$$

- (iii)  $\tilde{\mathcal{I}}_p(\tilde{\mu}_p) = 2 \log 2 + \frac{3}{p}$ .

*Proof.* In the previous section, the minimization over all *real* probability measures of the functional  $\mathcal{I}_p$  corresponding to the weight function  $w: x \mapsto \exp(-\gamma_p|x|^p)$  gave rise to the minimizer  $\mu_p$ . We are now facing the minimization problem for probability measures on  $\mathbb{R}^+ = \{x^2 : x \in \mathbb{R}\}$ , with the weight function being  $v: y \mapsto \exp(-2\gamma_p y^{p/2}) = w(\sqrt{y})^2$ . According to [36, Theorem IV.1.10.(f)], the solution of the latter is simply the image measure of  $\mu_p$  by the map  $x \mapsto x^2$ . From this, (i)–(iii) easily follow.  $\blacksquare$

**Lemma 4.6** (Convergence towards the equilibrium measure).

- (i) For every continuous, polynomially bounded functions  $f, g: \mathbb{R}^+ \rightarrow \mathbb{R}$ , it holds that

$$\lim_{n \rightarrow \infty} \mathbb{E} g(\langle \tilde{L}_{n,p}, f \rangle) = g(\langle \tilde{\mu}_p, f \rangle).$$

- (ii) We have

$$\lim_{n \rightarrow \infty} -\frac{1}{d_n} \log Z_{n,p} = \log 2 + \frac{3}{2p}.$$

*Proof.* We apply<sup>3</sup> [18, Theorem 5.5.1]. This theorem provides the limit of  $Z_{n,p}^{1/d_n}$  stated in (ii), as well as a large deviation principle with good rate function  $\frac{b}{2}(\tilde{\mathcal{I}}_p - \tilde{\mathcal{I}}_p(\tilde{\mu}_p))$  for the random probability measures  $\tilde{L}_{n,p}$ . By the Borel-Cantelli lemma (see e.g. [18, p. 212] for details), this entails that  $\tilde{L}_{n,p}$  converges almost surely towards  $\tilde{\mu}_p$  in the sense of weak convergence of probability measures, that is  $\langle \tilde{L}_{n,p}, f \rangle \rightarrow \langle \tilde{\mu}_p, f \rangle$   $\mathbb{P}$ -a.s. for every continuous, bounded function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ . As before, point (i) then follows from the dominated convergence theorem and Corollary 3.2.  $\blacksquare$

We conclude this section with the proof of Theorem 1.1 in the case  $a = 2$ .

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<sup>3</sup>At first sight, the application of Theorem 5.5.1 in [18] seemingly requires that  $\gamma(n) := \frac{c-1}{2}$  be nonnegative (which is not true if  $c = 0$ ). It appears this condition is stated merely for simplicity and, referring to the notation there, the proof of the theorem remains valid as long as  $2\alpha\beta + \gamma \geq 0$  and  $\gamma > -\alpha$ .

*Proof of Theorem 1.1, case  $a = 2$ .* Recalling (12) and taking into account the change of variable (28), we have

$$\frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) \sim \frac{e^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}} Z_{n,p}^{-1/d_n} \left( \mathbb{E} \langle \tilde{L}_{n,p}, h_1 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}},$$

where  $h_1(y) := y$ . Lemma 4.6.(ii) tells us that  $Z_{n,p}^{-1/d_n} \sim 2e^{\frac{3}{2p}}$ , while Lemma 4.6.(i) gives

$$\lim_{n \rightarrow \infty} \mathbb{E} \langle \tilde{L}_{n,p}, h_1 \rangle^{\frac{q}{2}} = \langle \tilde{\mu}_p, h_1 \rangle^{\frac{q}{2}}.$$

Plugging in the value of  $\langle \tilde{\mu}_p, h_1 \rangle$  given by Lemma 5.1, we find

$$\frac{1}{\sqrt{d_n}} \frac{(\mathbb{E} \|T\|_{\text{HS}}^q)^{1/q}}{|B_E(S_p^n)|^{1/d_n}} \sim e^{\frac{1}{2p}-\frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}}. \quad \blacksquare$$

**4.3. Asymptotics of  $c_n$ .** It remains to establish the asymptotics (6) for the coefficient  $c_n$  involved in Weyl's integration formulas.

**Lemma 4.7** (Asymptotics of  $c_n$ ). *We have*

$$\sqrt{n} \cdot c_n^{1/d_n} \sim e^{\frac{3}{4}} \sqrt{\frac{4\pi}{ab}} \quad \text{as } n \rightarrow \infty.$$

*Proof.* The coefficient  $c_n$  is related to the volume of the unitary group  $U_n(\mathbb{F})$ . Specifically, supposing first  $a = 1$ , we know after [2, Propositions 4.1.1] that

$$c_n = \frac{|U_n(\mathbb{F})|}{|U_1(\mathbb{F})|^n n!},$$

where, according to [2, Proposition 4.1.14],

$$|U_n(\mathbb{F})| = (2\pi)^{\frac{bn(n+1)}{4}} \cdot 2^{n(1-\frac{b}{2})} \left/ \prod_{k=1}^n \Gamma\left(\frac{b}{2}k\right) \right.$$

Since  $\log \Gamma(z) = z \log z - z + o(z)$  as  $z \rightarrow \infty$ , we have

$$\log \Gamma\left(\frac{b}{2}k\right) = \frac{b}{2}k \log k + \frac{b}{2} \left( \log \frac{b}{2} - 1 \right) k + o(k), \quad k \rightarrow \infty.$$

Therefore, using that  $\sum_{k=1}^n k \log k = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + o(n^2)$  and  $d_n \sim bn^2/2$ , we find

$$\frac{1}{d_n} \log \prod_{k=1}^n \Gamma\left(\frac{b}{2}k\right) = \frac{1}{2} \log n + \frac{1}{2} \log \frac{b}{2} - \frac{3}{4} + o(1).$$

Hence

$$\frac{1}{d_n} \log c_n = \frac{1}{d_n} \log |U_n(\mathbb{F})| + o(1) = -\frac{1}{2} \log n + \frac{1}{2} \log \frac{4\pi}{b} + \frac{3}{4} + o(1),$$

which leads to the statement in the case  $a = 1$ . When  $a = 2$ , we have instead  $d_n \sim bn^2$  and, after [2, Proposition 4.1.3],

$$c_n = \frac{|U_n(\mathbb{F})|^2}{|U_1(\mathbb{F})|^n n!} 2^{-\frac{bn(n-1)}{2}}.$$

So the difference with above is that, here,

$$\begin{aligned} \frac{1}{d_n} \log c_n &= \frac{2}{d_n} \log |U_n(\mathbb{F})| - \frac{1}{2} \log 2 + o(1) \\ &= -\frac{1}{2} \log n + \frac{1}{2} \log \frac{2\pi}{b} + \frac{3}{4} + o(1). \end{aligned}$$

Hence the statement in the case  $a = 2$ . ■

**Remark 4.8.** *We can recover the asymptotic volumes of the Schatten unit balls. These were recently derived in [21, Theorem 3.1] and [20, Theorem 1], completing the much earlier computations of Saint-Raymond [37]. Indeed, starting from the expression of the volume of  $B_E(S_p^n)$  in (8) and plugging in the asymptotics of  $Z_{n,p}$  (Lemmas 4.2 and 4.6) and of  $c_n$  (Lemma 4.7), we find*

$$|B_E(S_p^n)|^{\frac{1}{d_n}} = (abn\gamma_p)^{\frac{1}{p}} \cdot \frac{c_n^{1/d_n}}{\Gamma\left(1 + \frac{d_n}{p}\right)^{1/d_n}} \cdot Z_{n,p}^{1/d_n} \sim (abn\gamma_p)^{\frac{1}{p}} \cdot \frac{e^{3/4} \sqrt{\frac{4\pi}{nab}}}{\left(\frac{d_n}{ep}\right)^{1/p}} \cdot \frac{1}{2} e^{-\frac{3}{2p}}.$$

Hence, because  $d_n \sim abn^2/2$ ,

$$\lim_{n \rightarrow \infty} n^{\frac{1}{2} + \frac{1}{p}} |B_E(S_p^n)|^{\frac{1}{d_n}} = (2p\gamma_p)^{\frac{1}{p}} e^{\frac{3}{4} - \frac{1}{2p}} \sqrt{\frac{\pi}{ab}}. \quad \blacksquare$$

## 5. PROPERTIES OF THE EQUILIBRIUM MEASURES

In this section we gather some auxiliary results on the equilibrium measure  $\mu_p$  of Lemma 4.1. We first carry out some easy computations. Next, we explain why it is sufficient to assume  $p > 3$  for Proposition 4.3 to hold.

**5.1. Some integral computations.** We recall the notation  $h_2(x) := x^2$ ,  $h_1(y) := y$ , and the equilibrium measures  $\mu_p$  and  $\tilde{\mu}_p$ , see Lemmas 4.1 and 4.5. We further denote by  $\varrho(dx) := (\pi\sqrt{1-x^2})^{-1} \mathbf{1}_{\{|x|<1\}}$  the standard Arcsine distribution on  $[-1, 1]$ .

**Lemma 5.1.** *We have*

$$\langle \mu_p, h_2 \rangle = \langle \tilde{\mu}_p, h_1 \rangle = \frac{p}{2p+4}.$$

*Proof.* First,  $\langle \mu_p, h_2 \rangle = \langle \tilde{\mu}_p, h_1 \rangle$  because  $\tilde{\mu}_p$  is the image measure of  $\mu_p$  by  $h_2$ . Next we know [3, Lemma 4.1] that  $\mu_p$  is the distribution of  $AB$  where  $A$  and  $B$  are independent variables with  $A$  following  $\varrho$  and  $B$  following the Beta( $p, 1$ ) distribution. We easily conclude that

$$\langle \mu_p, h_2 \rangle = \mathbb{E} A^2 \cdot \mathbb{E} B^2 = \frac{1}{2} \cdot \frac{p}{p+2}. \quad \blacksquare$$



**Lemma 5.2.** *We have*

$$\frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{(x+y)^2(1-xy)}{\sqrt{1-x^2}\sqrt{1-y^2}} dx dy = \frac{1}{2}.$$

*Proof.* Let  $X, Y$  be independent variables with law  $\varrho$ . Then the left-hand side is

$$\begin{aligned} \mathbb{E}(X+Y)^2(1-XY) &= \mathbb{E}(X+Y)^2 - 2\mathbb{E}X^2\mathbb{E}Y^2 - \mathbb{E}X^3\mathbb{E}Y - \mathbb{E}X\mathbb{E}Y^3 \\ &= \frac{1}{2}, \end{aligned}$$

using that  $\mathbb{E}X = \mathbb{E}Y = 0$  and  $\mathbb{E}X^2 = \mathbb{E}Y^2 = \frac{1}{2}$ . ■

**5.2. Regularity of the solution to the master equation.** Proposition 4.3, on which Theorem 1.2 crucially relies, deals with the fluctuations

$$(29) \quad F_{n,p}(h_2) := n \left( \langle L_{n,p}, h_2 \rangle - \langle \mu_p, h_2 \rangle \right)$$

of the linear statistics  $\langle L_{n,p}, h_2 \rangle$  (equivalently, of the truncated version  $\langle L_{n,p}, h_2 \phi \rangle$ ) about its limiting value  $\langle \mu_p, h_2 \rangle$ . To understand them, a sufficient approach is to appeal to the general central limit theorem of Bekerman, Leblé, and Serfaty [8] specialized to our setting: namely, for a  $\beta$ -ensemble with Hamiltonian (cf. (17) with  $a = 1$ ,  $b = \beta$  and  $c = 0$ )

$$H_{n,p}(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^n 2\gamma_p |x_i|^p - \frac{1}{n^2} \sum_{i \neq j} \log |x_i - x_j|, \quad \mathbf{x} \in \mathbb{R}^n,$$

i.e., the ensemble is subject to the potential  $V_p := 2\gamma_p h_p$ . Indeed, their Theorem 1 states that, for general  $\beta$ -ensembles, the moment generating function of the fluctuations converges to that of a certain Gaussian random variable. A key requirement nonetheless is the existence and  $\mathcal{C}^3$ -regularity of the solution  $\psi$  to a certain functional “master” equation. For our matter, this is Equation (30) below, and it is the purpose of this section to check that its solution is of class  $\mathcal{C}^3$  as soon as  $p > 3$ .

We first observe that we may apply [8, Theorem 1] directly<sup>4</sup> if we assume  $p > \mathfrak{p} := 6$ : then, given that  $V := V_p$  is of class  $\mathcal{C}^{\mathfrak{p}}$  and  $\xi := h_2$  is of class  $\mathcal{C}^r$  with  $r := \infty$ , the theorem implies that there exists for some constant  $c_{h_2}$  a solution  $\psi_p: \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^{(\mathfrak{p}-3) \wedge (r-1)} = \mathcal{C}^3$  to the equation

$$(30) \quad \Xi_p \psi = \frac{1}{2} h_2 + c_{h_2},$$

where

$$\Xi_p \psi(x) := -\frac{1}{2} V_p'(x) \psi(x) + \int \frac{\psi(x) - \psi(y)}{x - y} \mu_p(dy), \quad x \in \mathbb{R},$$

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<sup>4</sup>We are in the so called “one-cut” regime, which corresponds to [8, Theorem 1] with  $m = n = k = 0$ .

and furthermore<sup>5</sup>, the moment generating function

$$\mathbb{E} e^{sF_{n,p}(h_2)}, \quad s \in \mathbb{R},$$

of the fluctuations (29) converges to that of a Gaussian random variable. However, as [8, Theorem 1] is very general and covers cases far beyond our very specific situation of the potential being  $V = V_p$ , it has the drawback of inducing a sub-optimal regularity threshold of  $\mathfrak{p} = 6$ . Indeed, we shall show that the solution  $\psi_p$  is still of class  $\mathcal{C}^3$  if we only assume  $p > 3$ .

A general expression for the solution to the master equation in terms of the test function and of the equilibrium measure is given in [8, Lemma 3.3], see also Section B.5 there. The inversion of the master operator was first achieved in [7, Lemma 3.2]. In our framework, the expression of  $\psi_p$  can be made quite explicit: first, we compute  $c_{h_2} = -1/4$ , and

$$(31) \quad \begin{aligned} \psi_p(x) &= -\frac{x\sqrt{1-x^2}}{2\pi f_p(x)}, & \text{if } |x| \leq 1, \\ \psi_p(x) &= -\frac{|x|\sqrt{x^2-1}}{2\zeta'_p(x)}, & \text{if } |x| > 1, \end{aligned}$$

where

$$(32) \quad f_p(x) := \frac{px^{p-1}}{\pi} \int_x^1 \frac{v^{-p}}{\sqrt{1-v^2}} dv,$$

is the Lebesgue density of  $\mu_p$  which we have already seen in Lemma 4.1.(ii), and  $\zeta'_p$  is the odd function given for  $x > 1$  by

$$(33) \quad \begin{aligned} \zeta'_p(x) &:= \frac{d}{dx} \left( -\int \log|x-y| \mu_p(dy) + \frac{1}{2} V_p(x) \right) \\ &= px^{p-1} \int_1^x \frac{v^{-p}}{\sqrt{v^2-1}} dv, \end{aligned}$$

see [36, pp. 240-241]. We further observe the similarity between (32) and (33): applying the change of variable  $t \leftarrow (1-v)/x$ , we have, for every  $0 < x < 1$ ,

$$\begin{aligned} \pi f_p(1-x) &= \frac{p(1-x)^{p-1}}{\pi} \int_{1-x}^1 \frac{u^{-p}}{\sqrt{1-u^2}} du \\ &= p(1-x)^{p-1} \sqrt{x} \int_0^1 \frac{(1-xt)^{-p}}{\sqrt{2-xt}} \frac{dt}{\sqrt{t}} \\ &= \sqrt{x} w_p(-x), \end{aligned}$$

where we have set

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<sup>5</sup>Strictly speaking, [8, Theorem 1] does not exactly state that the CLT holds as long as the solution  $\psi$  exists and is of class  $\mathcal{C}^3$ . This seems however to be the case after a careful inspection of the proof, which, as summarized in the proof of Proposition 4.3, can be carried out more straightforwardly in our setting.

$$(34) \quad w_p(u) := p(1+u)^{p-1} \int_0^1 \frac{(1+tu)^{-p}}{\sqrt{2+tu}} \frac{dt}{\sqrt{t}}, \quad u \geq -1.$$

Similarly, we can see that  $\zeta'_p(1+x) = \sqrt{x} w_p(x)$  holds for all  $x > 0$ . Because  $\psi_p$  is odd, we conclude that the solution  $\psi_p \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  to the equation (30) is more simply expressed by

$$(35) \quad \psi_p(x) = -\frac{x\sqrt{1+|x|}}{2w_p(|x|-1)}, \quad x \in \mathbb{R}.$$

We denote by  $\lceil p \rceil$  the smallest integer greater than or equal to  $p$ .

**Lemma 5.3** (Regularity of  $\psi_p$ ). *For every  $p > 1$ , the function  $\psi_p$  is of class  $\mathcal{C}^{\lceil p \rceil - 1}$ . In particular, it is of class  $\mathcal{C}^3$  when  $p > 3$ .*

*Proof.* Since  $\psi_p$  is odd, it suffices to check the regularity on  $[0, \infty)$ . It is plain that the function  $w_p$  in (34) does not vanish on  $(-1, \infty)$  and, by differentiation under the integral sign, that it is of class  $\mathcal{C}^\infty$  there. Given (35), it is then clear that  $\psi_p$  is of class  $\mathcal{C}^\infty$  on  $(0, \infty)$ . It remains to check that  $\psi_p$  has  $\mathcal{C}^{\lceil p \rceil - 1}$ -regularity at  $0^+$ .

Because of the expression (31) and of  $x \mapsto \sqrt{1-x^2}$  being smooth at 0, the regularity of  $\psi_p$  at 0 is equivalent to that of  $\bar{\psi}_p(x) := x/f_p(x)$ . For  $0 < x < y < 1$ , the Taylor series

$$(1-v^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{\binom{2n}{n}}{4^n} v^{2n}$$

is uniformly convergent on  $[x, y]$ , and expanding in (32) gives

$$f_p(x) = \frac{px^{p-1}}{\pi} \lim_{y \uparrow 1} \int_x^y \frac{v^{-p}}{\sqrt{1-v^2}} dv = \frac{p}{\pi} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} x^{p-1} \int_x^1 v^{2n-p} dv,$$

by the monotone convergence theorem. The latter integral equals  $-\log x$  if  $p = 2n + 1$ , and  $(x^{2n+1-p} - 1)/(p - 2n - 1)$  otherwise. In any case, there exist constants  $A, B \in \mathbb{R}$  (possibly zero) such that

$$g(x) := f_p(x) + Ax^{p-1} \log x + Bx^{p-1} = \frac{p}{\pi} \sum_{\substack{n \geq 0 \\ 2n+1 \neq p}} \frac{\binom{2n}{n}}{4^n(p-2n-1)} x^{2n}$$

is decomposed as a power series with radius 1, so  $g$  defines a function of class  $\mathcal{C}^\infty$  at 0. Because, for  $p > 1$ ,  $Ax^{p-1} \log x + Bx^{p-1}$  has  $\mathcal{C}^{\lceil p \rceil - 2}$ -regularity at  $0^+$ ,  $f_p$  is therefore of class  $\mathcal{C}^{\lceil p \rceil - 2}$  at  $0^+$ . Then so is  $1/f_p$  since  $f_p(0) = g(0) = p\pi/(p-1) \neq 0$ , and we conclude by Leibniz formula that  $\bar{\psi}_p$  (and thus  $\psi_p$ ) is of class  $\mathcal{C}^{\lceil p \rceil - 1}$  at  $0^+$ . ■

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