



**HAL**  
open science

## A Murnaghan-Nakayama rule for Grothendieck polynomials of Grassmannian type

Khanh Khanh Nguyen Duc Nguyen Duc, Dang Tuan Hiep, Tran Ha Son, Do  
Le Hai Thuy

► **To cite this version:**

Khanh Khanh Nguyen Duc Nguyen Duc, Dang Tuan Hiep, Tran Ha Son, Do Le Hai Thuy. A Murnaghan-Nakayama rule for Grothendieck polynomials of Grassmannian type. 2021. hal-03427454

**HAL Id: hal-03427454**

**<https://hal.science/hal-03427454>**

Preprint submitted on 13 Nov 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Murnaghan-Nakayama rule for Grothendieck polynomials of Grassmannian type

Dang Tuan Hiep, Nguyen Duc Khanh, Tran Ha Son, Do Le Hai Thuy

## Abstract

We consider the Grothendieck polynomials appearing in the K-theory of Grassmannians, which are analogs of Schur polynomials. This paper aims to establish a version of the Murnaghan-Nakayama rule for Grothendieck polynomials of the Grassmannian type. This rule allows us to express the product of a Grothendieck polynomial with a power sum symmetric polynomial into a linear combination of other Grothendieck polynomials.

*2020 Mathematics Subject Classification.* 05E05, 14M15, 19E08.

*Key words and phrases.* Murnaghan-Nakayama rule, K-theory of Grassmannians, Grothendieck polynomials of Grassmannian type.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
2.1	Symmetric polynomials . . . . .	3
2.2	Partitions, diagrams and binary tableaux . . . . .	4
2.3	Grothendieck polynomials of Grassmannian type . . . . .	5
<b>3</b>	<b>Proof of the main theorem</b>	<b>5</b>

## 1 Introduction

The K-theory of flag varieties was studied by Kostant and Kumar [KK87], and by Demazure [Dem74]. In [Las07], [LS82], Lascoux and Schutzenberger introduced the Grothendieck polynomials as representatives for the structure sheaves of the Schubert varieties in a flag variety. For any permutation  $w \in \bigcup_{m \geq 1} S_m$ , the Grothendieck polynomial  $\mathfrak{G}_w := \mathfrak{G}_w(x_1, x_2, \dots)$  is defined by isobaric divided difference operators. Fomin and Kirilov found combinatorial insight of these polynomials in [FK94], [FK96].

Let  $s_\lambda$  be the Schur function associated with a partition  $\lambda$ , and  $p_k$  be the power-sum symmetric functions of degree  $k$  [Mac91]. The classical Murnaghan-Nakayama rule describes the decomposition of the product  $s_\lambda p_k$  to the sum of Schur functions [Mac91] as follows. We have

$$s_\lambda p_k = \sum_{\mu} (-1)^{r(\mu/\lambda)+1} s_{\mu},$$

where the sum runs over all partitions  $\mu$  such that  $\mu/\lambda$  is a ribbon of size  $k$ ,  $r(\mu/\lambda)$  is the number of rows of skew shape  $\mu/\lambda$ .

The classical Murnaghan-Nakayama rule plays an important role in the representation theory of the symmetric groups. It gave a formula for the character table (see [Mur37], [Nak40a], [Nak40b]). For this reason, many extensions and generalizations of classical Murnaghan-Nakayama rule were studied. Indeed, a version for non-commutative symmetric functions is given by Fomin and Green in [FG98] (it led to formulas for characters of representations associated with stable Schubert and Grothendieck polynomials). A skew version and its generalization of multiplication with quantum power sum function are given by [AM11], [Kon12]. A version for noncommutative Schur functions can be found in [Tew16]. A plethystic version is given by [Wil16]. A version for loop Schur functions is given by [Ros14] (it provides a fundamental step in the orbifold Gromov-Witten/Donaldson-Thomas correspondence in [RZ13]). A version in the cohomology of an affine Grassmannian can be found in [BSZ11]. An extended version to Schubert polynomials and the quantum cohomology of Grassmannians can be found in [MS18].

In this paper, we restrict our attention on the simplest complex flag variety: the Grassmann variety of  $n$  dimensional subspaces of  $\mathbb{C}^{n+m}$ . The Grothendieck polynomials in this case are indexed by Grassmanian permutations [Buc02]. Namely, let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition of length at most  $n$ . The Grassmannian permutation  $w_\lambda$  of descent  $n$  is defined by  $w_\lambda(i) = i + \lambda_{n+1-i}$  for  $i \in [1, n]$  and  $w_\lambda(i) < w_\lambda(i+1)$  for all  $i \neq n$ . Set  $G_\lambda = \mathfrak{G}_{w_\lambda}$ . There are several new formulas for  $G_\lambda$ , for example, in the terms of set-valued tableaux [Buc02] or Jacobi-Trudy identity [Kir16]. We here recall the Weyl identity given by Ikeda and Naruse [IN13] (see also [IS14]). Namely,

$$G_\lambda = \frac{\det(x_i^{\lambda_j+n-j}(1+\beta x_i)^{j-1})_{n \times n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)},$$

where  $\beta$  is a formal parameter. Remind that, if  $\beta = 0$ , then  $G_\lambda$  is identified with the Schur function  $s_\lambda$ . The products of  $G_\lambda$  with other special symmetric polynomials  $e_k, h_k$  are mentioned in [Len00], in which Lenart studied the Pieri rules of the Grassmannian Grothendieck polynomials. Our work on the product  $G_\lambda p_k$  can be considered as a K-theoretic version of the classical Murnaghan-Nakayama rule.

Let  $\lambda$  and  $\mu$  be partitions of length at most  $n$  and  $\lambda \leq \mu$ . Let  $|\mu/\lambda|, c(\mu/\lambda), r(\mu/\lambda)$  be the size, number of columns, number of rows of the skew shape  $\mu/\lambda$ . We say two boxes in a skew shape are *adjacent* whenever they share an edge, and we say a skew shape  $\mu/\lambda$  is *connected* whenever every pair of its boxes is connected by a sequence of adjacent boxes. A *ribbon* is a connected skew shape with no  $2 \times 2$  square. The main result of this paper is stated as follows.

**Theorem 1.** *For any partition  $\lambda \in \mathcal{P}_n$  and  $k \in \mathbb{N}$ , we have*

$$G_\lambda p_k = \sum_{\mu} (-\beta)^{|\mu/\lambda|-k} (-1)^{k-c(\mu/\lambda)} \binom{r(\mu/\lambda)-1}{k-c(\mu/\lambda)} G_\mu, \quad (1)$$

where the sum runs over all partitions  $\mu \in \mathcal{P}_n$ ,  $\mu \geq \lambda$  such that  $c(\mu/\lambda) \leq k$ ,  $\mu/\lambda$  is connected and contains at least a ribbon of size  $k$  along with its northwest border.

This paper is organized as follows. In Section 2 we recall the basic knowledge related to symmetric polynomials, partitions, diagrams, binary tableaux, Grothendieck polynomials of Grassmannian type. In Section 3 we prove our main result.

## 2 Preliminaries

### 2.1 Symmetric polynomials

A polynomial  $f(x_1, \dots, x_n)$  in  $n$  variables is said to be *symmetric* if for all permutations  $\sigma \in S_n$ , we have

$$f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n).$$

There are some fundamental symmetric polynomials: The  $k$ -th *elementary symmetric polynomial*

$$e_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \dots x_{i_k},$$

the  $k$ -th *complete homogeneous symmetric polynomial*

$$h_k = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \dots x_{i_k},$$

and the  $k$ -th *power sum symmetric polynomial*

$$p_k = \sum_{i=1}^n x_i^k.$$

The next lemma gives us a way to write the power sum symmetric polynomial in terms of elementary and complete homogeneous symmetric polynomials. It is one of the important keys in the proof of Theorem 1.

**Lemma 1.** *Let  $k$  be a positive integer. Then we have*

$$p_k = \sum_{i=0}^{k-1} (-1)^i (k-i) e_i h_{k-i}. \quad (2)$$

*Proof.* Consider the following generating functions

$$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i=1}^n \frac{1}{1 - x_i t},$$

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i=1}^n (1 + x_i t),$$

$$P(t) = \sum_{k \geq 1} p_k t^{k-1} = \sum_{i=1}^n \frac{x_i}{1 - x_i t}.$$

By (2.6), (2.10) in [Mac98], we have

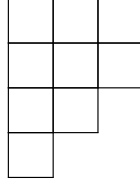
$$P(t) = H'(t)/H(t) = H'(t)E(-t).$$

Compare the coefficients of  $t^{k-1}$  in both sides of the identity, we get the conclusion.  $\square$

## 2.2 Partitions, diagrams and binary tableaux

A non-negative integer sequence  $\lambda = (\lambda_1, \lambda_2, \dots)$  is called a *partition* if  $\lambda_1 \geq \lambda_2 \geq \dots$ . If  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  with  $\lambda_l > 0$  and  $\sum_{i=1}^l \lambda_i = m$ , we write  $l(\lambda) = l$ ,  $|\lambda| = m$ . We call  $l(\lambda)$  the length, and  $|\lambda|$  the size of the partition  $\lambda$ . Each partition  $\lambda$  is presented by a *Young diagram* that is a collection of boxes such that: The leftmost boxes of each row are in a column, and the numbers of boxes from top row to bottom row are  $\lambda_1, \lambda_2, \dots$ , respectively.

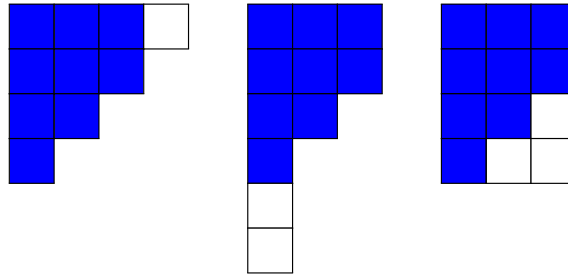
**Example 1.** The Young diagram associated to the partition  $\lambda = (3, 3, 2, 1, 0, 0)$  is



Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\nu = (\nu_1, \nu_2, \dots)$  be partitions. We define the sum of two partitions by  $\lambda + \nu = (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \dots)$ . For a non-negative integer  $n$ , we denote  $\mathcal{P}_n$  the set of all partition of length at most  $n$ . Let  $(1^n)$  be the  $n$ -tuple partition  $(1, \dots, 1)$ , and  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n$ , then we have  $\lambda + (1^n) = (\lambda_1 + 1, \dots, \lambda_n + 1)$ .

Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\mu = (\mu_1, \mu_2, \dots)$  be two partitions. We say that  $\lambda$  is smaller than  $\mu$  if and only if  $\lambda_i \leq \mu_i$  for all  $i$ , and we write  $\lambda \leq \mu$ . In this case, we define the *skew Young diagram*  $\mu/\lambda$  as the result of removing boxes in the Young diagram  $\lambda$  from the Young diagram  $\mu$ . We write  $|\mu/\lambda| = |\mu| - |\lambda|$  for the size, and  $r(\mu/\lambda), c(\mu/\lambda)$  for the number of rows, columns of the skew Young diagram  $\mu/\lambda$  respectively. We say two boxes in a skew shape are *adjacent* whenever they share an edge, and we say a skew shape  $\mu/\lambda$  is *connected* whenever every pair of its boxes is connected by a sequence of adjacent boxes. A *ribbon* is a connected skew shape with no  $2 \times 2$  square. A *binary tableau*  $T$  of skew shape  $\mu/\lambda$  is a result of filling the skew Young diagram  $\mu/\lambda$  by the alphabet  $\{0, 1\}$  such that the entry in the bottom of each column is 1. A binary tableau  $T$  is said to have *content*  $\alpha(T) = (\alpha_0, \alpha_1)$  if  $\alpha_i = \alpha_i(T)$  is the number of entries  $i$  in  $T$ . We write  $\text{sh}(T)$  for the shape of the tableau  $T$ .

**Example 2.** We consider partitions in  $\mathcal{P}_6$ :  $\lambda = (3, 3, 2, 1, 0, 0)$  and  $\mu = (4, 3, 3, 3, 1, 1)$ . Then  $\mu \geq \lambda$  and the skew diagram  $\mu/\lambda$  has  $r(\mu/\lambda) = 5$  rows,  $c(\mu/\lambda) = 4$  columns. In this case  $\mu/\lambda$  is not connected and is not a ribbon. However, it contains ribbons of size 1, 2, 3, for example



The following tableau  $T$  is a binary tableau of skew shape  $\text{sh}(T) = \mu/\lambda$ .

			1
		0	
	1	1	
1			
1			

Here the diagram in blue means the Young diagram  $\lambda$  removed from the Young diagram  $\mu$ . The content of the binary tableau  $T$  is  $\alpha(T) = (1, 5)$ .

### 2.3 Grothendieck polynomials of Grassmannian type

The K-theory of flag varieties was studied by Kostant and Kumar [KK87], and by Demazure [Dem74]. The Grothendieck polynomials were introduced by Lascoux and Schützenberger as representatives for the structure sheaves of the Schubert varieties in a flag variety (see [Las07], [LS82]). For any permutation  $w \in \bigcup_{m \geq 1} S_m$ , the Grothendieck polynomial  $\mathfrak{G}_w := \mathfrak{G}_w(x_1, x_2, \dots)$  is defined by isobaric divided difference operators (see [Las07], [LS82]). In particular, for Grassmann varieties of  $n$  dimensional subspaces of  $\mathbb{C}^{n+m}$ , the Grothendieck polynomials are indexed by Grassmannian permutations [Buc02]. Namely, let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition of length at most  $n$ . The Grassmannian permutation  $w_\lambda$  of descent  $n$  is defined by  $w_\lambda(i) = i + \lambda_{n+1-i}$  for  $i \in [1, n]$  and  $w_\lambda(i) < w_\lambda(i+1)$  for all  $i \neq n$ . Set  $G_\lambda = \mathfrak{G}_{w_\lambda}$ . By [IN13], the polynomial can be defined by the following bi-alternant formula

$$G_\lambda = \frac{\det(x_i^{\lambda_j + n - j} (1 + \beta x_i)^{j-1})_{n \times n}}{\det(x_i^{n-j})_{n \times n}},$$

where the denominator

$$\det(x_i^{n-j})_{n \times n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

is the well-known Vandermonde determinant.

**Example 3.** For  $n = 2$  and  $\lambda = (2, 1)$ , we have

$$\begin{aligned} G_{(2,1)}(x_1, x_2) &= \frac{\begin{vmatrix} x_1^3 & x_1(1 + \beta x_1) \\ x_2^3 & x_2(1 + \beta x_2) \end{vmatrix}}{x_1 - x_2} \\ &= x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2. \end{aligned}$$

**Remark 1.** For  $\beta = 0$ , the Grothendieck polynomial  $G_\lambda$  is coincide with the Schur polyninomial  $s_\lambda$ .

## 3 Proof of the main theorem

We only need to focus on the case  $\beta \neq 0$ . Indeed, if  $\beta = 0$ , then  $G_\lambda$  is the Schur function associated with partition  $\lambda$ . The Murnaghan-Nakayama rule for Schur polynomials is very well-known (see [Mac91]). When  $\beta \neq 0$ , set

$$\tilde{G}_\lambda(x_1, \dots, x_n) = \beta^{|\lambda|} G_\lambda\left(\frac{x_1}{\beta}, \dots, \frac{x_n}{\beta}\right).$$

Theorem 1 can be reduced to the following theorem.

**Theorem 2.** For any partition  $\lambda \in \mathcal{P}_n$  and  $k \in \mathbb{N}$ , we have

$$\tilde{G}_{\lambda} p_k = \sum_{\mu} (-1)^{|\mu/\lambda| - c(\mu/\lambda)} \binom{r(\mu/\lambda) - 1}{k - c(\mu/\lambda)} \tilde{G}_{\mu}, \quad (3)$$

where the sum runs over all partitions  $\mu \in \mathcal{P}_n$ ,  $\mu \geq \lambda$  such that  $c(\mu/\lambda) \leq k$ ,  $\mu/\lambda$  is connected and contains a ribbon of size  $k$  along with its northwest border.

Before going to the proof, we need to restate the following lemma. It was used in [Len00] as one of the important keys to obtain the Pieri rules for Grothendieck polynomials of Grassmannian type.

**Lemma 2.** [Len00, Theorem 3.2] For any partition  $\lambda \in \mathcal{P}_n$  and  $k \in \mathbb{N}$ , we have

$$\tilde{G}_{\lambda} e_k = \sum_{T: \text{sh}(T) = \mu/\lambda} (-1)^{\alpha_0(T)} \tilde{G}_{\mu}, \quad (4)$$

$$\tilde{G}_{\lambda} h_k = \sum_{T: \text{sh}(T) = \mu/\lambda} (-1)^{\alpha_0(T)} \tilde{G}_{\mu}. \quad (5)$$

The first sum runs over all binary tableaux  $T$  of shape  $\mu/\lambda$  with  $\mu \in \mathcal{P}_n$ ,  $\lambda \leq \mu \leq \lambda + (1^n)$ ,  $\alpha_1(T) = k$ . The second sum runs over all binary tableaux  $T$  of shape  $\mu/\lambda$  with  $\mu \in \mathcal{P}_n$ ,  $\lambda \leq \mu$ ,  $\alpha_1(T) = k$ , no two 1's in the same column.

*Proof of Theorem 2.* By Lemmas 1 and 2, we have

$$\tilde{G}_{\lambda} p_k = \sum_{i=0}^{k-1} (-1)^i (k-i) \tilde{G}_{\lambda} e_i h_{k-i} \quad (6)$$

$$= \sum_{i=0}^{k-1} (-1)^i (k-i) \sum_{T_1: \text{sh}(T_1) = \nu/\lambda} (-1)^{\alpha_0(T_1)} \tilde{G}_{\nu} h_{k-i} \quad (7)$$

$$= \sum_{i=0}^{k-1} (-1)^i (k-i) \sum_{T_1: \text{sh}(T_1) = \nu/\lambda} (-1)^{\alpha_0(T_1)} \sum_{T_2: \text{sh}(T_2) = \mu/\nu} (-1)^{\alpha_0(T_2)} \tilde{G}_{\mu} \quad (8)$$

$$= \sum_{\mu} \sum_{T=T_1 \cup T_2: \text{sh}(T) = \mu/\lambda} (-1)^{|\mu/\lambda| - \alpha_1(T_2)} \alpha_1(T_2) \tilde{G}_{\mu}. \quad (9)$$

The last sum runs over binary tableaux  $T = T_1 \cup T_2$  of shape  $\mu/\lambda$ , with  $\mu \in \mathcal{P}_n$ ,  $\mu \geq \lambda$ ,  $\alpha_1(T) = k$ , and  $T_1, T_2$  are of the forms mentioned in the equalities (4), (5), respectively. Fix such a shape  $\mu$  containing  $\lambda$ , we are going to determine the form of  $\mu$  and the coefficient of  $\tilde{G}_{\mu}$  appearing in the decomposition of  $\tilde{G}_{\lambda} p_k$ .

First step: Construct all tableaux  $T$  mentioned in (9). We proceed as follows.

- First, we numbering all boxes in the bottom of each column of the skew diagram  $\mu/\lambda$  by 1. Let  $\mathcal{B}$  be the set of boxes we have created.
- Now, we will choose a subset of boxes in  $\mathcal{B}$  and decide it as the set of boxes in bottom of  $T_2$ , say  $\mathcal{B}_2$ . In fact, we can not choose such subset randomly because its completion in  $\mathcal{B}$ , say  $\mathcal{B}_1$ , will be a subset of boxes in bottom of  $T_1$ . Hence, it must satisfy a strict condition that the boxes in  $\mathcal{B}_1$  is located in the skew diagram  $(\lambda + (1^n))/\lambda$ . So, to choose a subset  $\mathcal{B}_2$ , we should start from choosing  $\mathcal{B}_1$ . Fix a number of entries 1 in  $T_2$ , say  $\alpha_1(T_2) = j$ , we have

- The cardinal of  $\mathcal{B}_1$  is  $c(\mu/\lambda) - j$ .
- The elements in  $\mathcal{B}_1$  are chosen randomly from  $\gamma := \mathcal{B} \cap (\lambda + (1^n))/\lambda$ .

Hence for a fixed  $\alpha_1(T_2) = j$ , the number of choices of  $\mathcal{B}_2$  is equal the number of choices of  $\mathcal{B}_1$  and it is

$$\binom{|\gamma|}{c(\mu/\lambda) - j}. \quad (10)$$

Since  $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$ , we have

$$j = |\mathcal{B}_2| \in [|\mathcal{B} \setminus \mathcal{B}_1|, |\mathcal{B}|] = [c(\mu/\lambda) - |\gamma|, c(\mu/\lambda)]. \quad (11)$$

- Now, the last step to construct tableau  $T$  is locating remaining entries 1 of  $T$  which is not in the bottom  $\mathcal{B}$  in the skew diagram  $(\lambda + (1^n)/\lambda) \cap (\mu/\lambda)$ . We have
  - The number of remaining entries 1 is  $k - c(\mu/\lambda)$ .
  - Such entries 1 are chosen randomly from  $\eta := (\lambda + (1^n)/\lambda) \cap (\mu/\lambda) \setminus \gamma$ .

Hence the number of choice of this step is

$$\binom{|\eta|}{k - c(\mu/\lambda)}. \quad (12)$$

So, we have described a way to construct tableaux  $T$  of given skew shape  $\mu/\lambda$  such that the numbers of entries 1 in  $T_2$  is a fixed number  $j$ .

Second step: Substitute (11), (10), (12) to (9) and simplify it. We have

$$\tilde{G}_{\lambda p_k} = \sum_{\mu} \sum_{j=c(\mu/\lambda)-|\gamma|}^{c(\mu/\lambda)} (-1)^{|\mu/\lambda|-j} j \binom{|\gamma|}{c(\mu/\lambda) - j} \binom{|\eta|}{k - c(\mu/\lambda)} \tilde{G}_{\mu}. \quad (13)$$

Note that the binomial coefficient

$$\binom{|\eta|}{k - c(\mu/\lambda)}$$

depends only on  $\lambda, \mu$  and  $k$ . Thus, in order to simplify the coefficient of  $\tilde{G}_{\mu}$ , we only need to determine the sum

$$\sum_{j=c(\mu/\lambda)-|\gamma|}^{c(\mu/\lambda)} (-1)^{|\mu/\lambda|-j} j \binom{|\gamma|}{c(\mu/\lambda) - j}.$$

Obviously,  $|\gamma| \geq 1$ . We prove the following lemma.

**Lemma 3.** *The sum*

$$\sum_{j=c(\mu/\lambda)-|\gamma|}^{c(\mu/\lambda)} (-1)^{c(\mu/\lambda)-j} j \binom{|\gamma|}{c(\mu/\lambda) - j} \quad (14)$$

is equal to 0 if  $|\gamma| > 1$  and 1 if  $|\gamma| = 1$ .



*Proof.* Set  $i = c(\mu/\lambda) - j$  and  $c = c(\mu/\lambda)$ . Then (14) can be rewritten as

$$\sum_{i=0}^{|\gamma|} (-1)^i (c - i) \binom{|\gamma|}{i} = c \sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} - \sum_{i=0}^{|\gamma|} (-1)^i i \binom{|\gamma|}{i}.$$

Since  $|\gamma| \geq 1$ , then

$$\sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} = (1 - 1)^{|\gamma|} = 0.$$

If  $|\gamma| > 1$ , then

$$\sum_{i=0}^{|\gamma|} (-1)^i i \binom{|\gamma|}{i} = |\gamma| (1 - 1)^{|\gamma|-1} = 0.$$

If  $|\gamma| = 1$ , then

$$\sum_{i=0}^{|\gamma|} (-1)^i i \binom{|\gamma|}{i} = \sum_{i=0}^1 (-1)^i i \binom{1}{i} = -1.$$

We obtain the result as desired.  $\square$

Now, we consider two cases.

- If  $|\gamma| = 1$  then  $|(\lambda + (1^n)/\lambda) \cap (\mu/\lambda)| = r(\mu/\lambda)$  because if not, then  $|\gamma| > 1$ . So  $|\eta| = r(\mu/\lambda) - 1$ . The coefficient of  $\tilde{G}_\mu$  in (13) is

$$(-1)^{|\mu/\lambda| - c(\mu/\lambda)} \binom{r(\mu/\lambda) - 1}{k - c(\mu/\lambda)}.$$

- If  $|\gamma| > 1$ , the coefficient of  $\tilde{G}_\mu$  in (13) is 0.

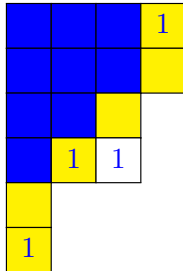
We see that, the condition  $|\gamma| = 1$  is equivalent to the condition  $\mu/\lambda$  is connected. When  $|\gamma| = 1$ , the conditions  $\mu/\lambda$  is a shape of a tableau  $T = T_1 \cup T_2$ , where  $T_1, T_2$  are of form in (4), (5) and  $\alpha_1(T) = k$  are equivalent to the conditions  $c(\mu/\lambda) \leq k$  (entries 1 in bottom  $\mathcal{B}$  is a part of all entries 1 of  $T$ ),  $k - c(\mu/\lambda) \leq r(\mu/\lambda) - 1$  (entries 1 not in bottom  $\mathcal{B}$  can be filled into  $\eta$ ). The last inequality condition can be rewritten as

$$k \leq c(\mu/\lambda) + r(\mu/\lambda) - 1. \quad (15)$$

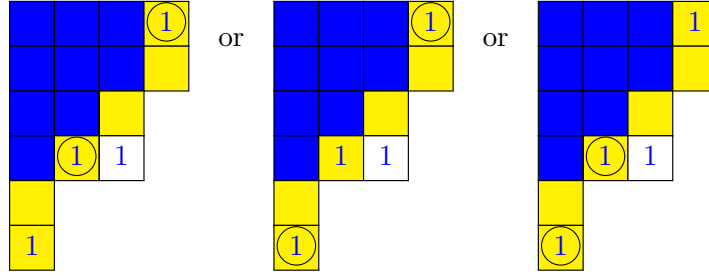
Since  $\mu/\lambda$  is connected, the right hand side of (15) counts the boxes of the maximal ribbon contained in skew shape  $\mu/\lambda$  along with its northwest border. Hence, the conditions of  $\mu$  such that  $\tilde{G}_\mu$  appears in the decomposition of  $\tilde{G}_{\lambda p_k}$  are:  $\mu \in \mathcal{P}_n, \mu \geq \lambda$  such that  $c(\mu/\lambda) \leq k$ ,  $\mu/\lambda$  is connected and contains at least a ribbon of size  $k$  along with its northwest border.  $\square$

The example below visualize the first step: constructing tableaux  $T$ , in the proof of Theorem 2.

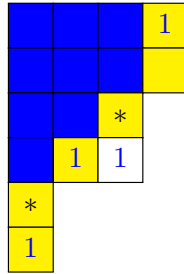
**Example 4.** We continue Example 2. In the picture below,  $\mathcal{B}$  is the set of four boxes  $\boxed{1}$ , and the skew diagram  $(\lambda + (1^n))/\lambda$  is colored in yellow. So  $\gamma$  is the set of three yellow boxes with blue entries 1 inside.



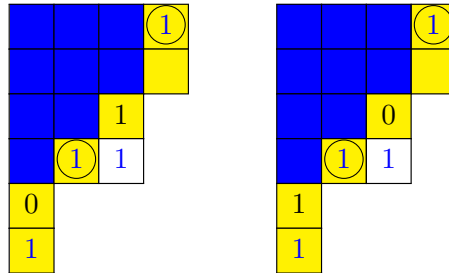
The range of the number of entries 1 in  $T_2$  is  $j \in [1, 4]$ . If we fix  $j = 2$ , then there are three choices of  $\mathcal{B}_1$  (also  $\mathcal{B}_2$ ) as in the picture below (the entries 1 in  $\mathcal{B}_1$  are circled).



The skew shape  $\eta$  contains two boxes where we put \* inside in the picture below.



If  $k = 5$ , then we just need to put only one remaining entry 1 randomly to the boxes marked by \*. The remaining boxes of  $\mu/\lambda$  are numbered by 0. For example, if we fix the first choice of  $\mathcal{B}_1$  in the picture above ( $j = 2$ ), we have two tableaux below (empty yellow boxes are not counted in tableaux).



## Acknowledgements

The first author would like to thank Professor Takeshi Ikeda for introducing Grothendieck polynomials and explaining their importance in the study of K-theory of Grassmannians. The second author would like to express his sincere gratitude for the Visiting Fellowship supported by MathCoRe and Professor Petra Schwer at Otto-von-Guericke University Magdeburg. He would also like to thank Professor Cristian Lenart for his strong encouragement and for taking his time to read the manuscript and give valuable comments.

## References

- [AM11] Sami H. Assaf and Peter R. W. McNamara. A Pieri rule for skew shapes. *Journal of Combinatorial Theory - Series A*, 118(1), 2011.
- [BSZ11] Jason Bandlow, Anne Schilling, and Mike Zabrocki. The Murnaghan–Nakayama rule for k-Schur functions. *Journal of Combinatorial Theory, Series A*, 118(5):1588–1607, 2011.

- [Buc02] Anders S. Buch. A Littlewood–Richardson rule for the K-theory of Grassmannians. *Acta mathematica*, 189(1):37–78, 2002.
- [Dem74] Michel Demazure. Désingularisation des variétés de Schubert généralisées. In *Annales scientifiques de l’École normale supérieure*, volume 7, pages 53–88, 1974.
- [FG98] Sergey Fomin and Curtis Green. Noncommutative Schur functions and their applications. *Discrete Mathematics*, 193(1–3):179–200, 1998.
- [FK94] Sergey Fomin and Anatol N Kirillov. Grothendieck polynomials and the Yang–Baxter equation. In *Proc. Formal Power Series and Alg. Comb*, pages 183–190, 1994.
- [FK96] Sergey Fomin and Anatol N Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Mathematics*, 153(1-3):123–143, 1996.
- [IN13] Takeshi Ikeda and Hiroshi Naruse. K-theoretic analogues of factorial Schur P- and Q-functions. *Advances in Mathematics*, 243:22–66, 2013.
- [IS14] Takeshi Ikeda and Tatsushi Shimazaki. A proof of K-theoretic Littlewood–Richardson rules by Bender-Knuth-type involutions. *Mathematical Research Letters*, 21(2):333–339, 2014.
- [Kir16] Anatol N. Kirillov. On some quadratic algebras  $i 1/2$ : combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss–Catalan, universal Tutte and reduced polynomials. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 12:002, 2016.
- [KK87] Bertram Kostant and Shrawan Kumar. T-equivariant K-theory of generalized flag varieties. *Proceedings of the National Academy of Sciences*, 84(13):4351–4354, 1987.
- [Kon12] Matjaž Konvalinka. Skew quantum Murnaghan–Nakayama rule. *Journal of Algebraic Combinatorics*, 35(4):519–545, 2012.
- [Las07] Alain Lascoux. Anneau de Grothendieck de la variété de drapeaux. In *The Grothendieck Festschrift*, pages 1–34. Springer, 2007.
- [Len00] Cristian Lenart. Combinatorial aspects of the K-theory of Grassmannians. *Annals of Combinatorics*, 4(1):67–82, 2000.
- [LS82] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux. *CR Acad. Sci. Paris Sér. I Math*, 295(11):629–633, 1982.
- [Mac91] Ian G. Macdonald. *Notes on Schubert polynomials*, volume 6. Dép. de mathématique et d’informatique, Université du Québec à Montréal, 1991.
- [Mac98] Ian G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.
- [MS18] Andrew Morrison and Frank Sottile. Two Murnaghan–Nakayama rules in Schubert calculus. *Annals of Combinatorics*, 22(2):363–375, 2018.

- [Mur37] Francis D Murnaghan. The characters of the symmetric group. *American Journal of Mathematics*, 59(4):739–753, 1937.
- [Nak40a] Tadasi Nakayama. On some modular properties of irreducible representations of a symmetric group, i. In *Japanese journal of mathematics: transactions and abstracts*, volume 17, pages 165–184. The Mathematical Society of Japan, 1940.
- [Nak40b] Tadasi Nakayama. On some modular properties of irreducible representations of symmetric groups, ii. In *Japanese journal of mathematics: transactions and abstracts*, volume 17, pages 411–423. The Mathematical Society of Japan, 1940.
- [Ros14] Dustin Ross. The loop Murnaghan–Nakayama rule. *Journal of Algebraic Combinatorics*, 39(1):3–15, 2014.
- [RZ13] Dustin Ross and Zhengyu Zong. The gerby Gopakumar–Mariño–Vafa formula. *Geometry & Topology*, 17(5):2935–2976, 2013.
- [Tew16] Vasu Tewari. A Murnaghan–Nakayama rule for noncommutative Schur functions. *European Journal of Combinatorics*, 58:118–143, 2016.
- [Wil16] Mark Wildon. A combinatorial proof of a plethystic Murnaghan–Nakayama rule. *SIAM Journal on Discrete Mathematics*, 30(3):1526–1533, 2016.

Faculty of Mathematics and Computer Science, Da Lat University, 1 Phu Dong Thien Vuong, Da Lat city, Lam Dong, Vietnam  
 E-mail: [hiepdt@dlu.edu.vn](mailto:hiepdt@dlu.edu.vn)

Department of Mathematics, Otto-von-Guericke University of Magdeburg, Germany  
 E-mail: [khanh.mathematic@gmail.com](mailto:khanh.mathematic@gmail.com)

The Education Publisher in Ho Chi Minh city, 231 Nguyen Van Cu, District 5, Ho Chi Minh city, Vietnam  
 E-mail: [tranhason1705@gmail.com](mailto:tranhason1705@gmail.com)

Bao Loc High School for the Gifted, 5 Quang Trung, Bao Loc city, Lam Dong, Vietnam  
 E-mail: [cbl.dolehaithuy@gmail.com](mailto:cbl.dolehaithuy@gmail.com)