The Vlasov-Poisson-Boltzmann/Landau system with polynomial perturbation near Maxwellian

Chuqi Cao, Dingqun Deng, Xingyu Li

THE VLASOV-POISSON-BOLTZMANN/LANDAU SYSTEM WITH POLYNOMIAL PERTURBATION NEAR MAXWELLIAN

CHUQI CAO, DINGQUN DENG AND XINGYU LI

Abstract. In this work, we consider the Vlasov-Poisson-Boltzmann system without angular cutoff and the Vlasov-Poisson-Landau system with Coulomb potential near a global Maxwellian μ in torus or union of cubes. We establish the global existence, uniqueness and large time behavior for solutions in a polynomial-weighted Sobolev space $H^2_{x,v}(\langle v\rangle^k)$ for some constant $k > 0$. For the domain union of cubes, We will consider the specular-reflection boundary condition and its high-order compatible specular boundary condition. The proof is based on extra dissipation term generated from improved semigroup method including electrostatic field with the help of macroscopic estimates.

Contents

1. Introduction

1.1. **Models and equations.** We consider the Vlasov-Poisson-Boltzmann (VPB) and Vlasov-Poisson-Landau (VPL) systems describing the motion of plasma particles of two species in

2010 *Mathematics Subject Classification.* 35A23; 35J05; 35Q20.
Key words and phrases. Vlasov-Poisson-Boltzmann system, Vlasov-Poisson-Landau system, global existence, polynomial weighted space.

bounded domain $\Omega \subseteq \mathbb{R}^3$ (cf. [41, 44]):

$$\begin{aligned} &\partial_t F_+ + v\cdot\nabla_x F_+ - \nabla_x\phi\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_-,F_+),\\ &\partial_t F_- + v\cdot\nabla_x F_- + \nabla_x\phi\cdot\nabla_v F_- = Q(F_+,F_-) + Q(F_-,F_-),\\ &F(0) = F_0, \quad E(0) = E_0, \end{aligned} \tag{1.1}$$

where $E = -\nabla_x\phi$ is the self-consistent electrostatic field satisfying

$$-\Delta_x\phi = \int_{\mathbb{R}^3}(F_+ - F_-)\,dv, \qquad \int_\Omega \phi(t,x)dx = 0, \tag{1.2}$$

and $F(t,x,v) \geq 0$ is a distribution function of particles at time $t > 0$ with position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$.

For the case of Vlasov-Poisson-Boltzmann system, the collision operator Q is a bilinear operator which acts only on the velocity variable v given by

$$Q(G,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(G'_*F' - G_*F)d\sigma dv_*.$$

Here we use the standard notation $F = F(v)$, $G_* = G(v_*)$, $F' = F(v')$, $G'_* = G(v'_*)$, where v', v'_* are velocities of two particles after collision given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

This representation follows from physical law of elastic collision:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The nonnegative function $B(v-v_*,\sigma)$ is called the Boltzmann collision kernel. It depends only on relative velocity $|v-v_*|$ and the deviation angle θ through

$$\cos\theta \overset{\text{def}}{=} \frac{v-v_*}{|v-v_*|}\cdot\sigma.$$

In the present work, we consider the non-cutoff kernel B as the followings.

(**A1**). The Boltzmann kernel B takes the form

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\frac{v-v_*}{|v-v_*|}\cdot\sigma),$$

where b is a nonnegative function.

(**A2**). The angular function $b(\cos\theta)$ is not locally integrable and it satisfies

$$\mathcal{K}\theta^{-1-2s} \leq \sin\theta b(\cos\theta) \leq \mathcal{K}^{-1}\theta^{-1-2s}, \text{ with } 0 < s < 1,$$

for some constant $\mathcal{K} > 0$.

(**A3**). The parameter γ and s satisfy the condition

$$-3 < \gamma \leq 1, \quad 1/2 \leq s \leq 1, \quad \gamma + 2s > -1.$$

(**A4**). Without loss of generality, we may assume that $B(v-v_*,\sigma)$ is supported in the set $0 \leq \theta \leq \pi/2$, i.e. $\frac{v-v_*}{|v-v_*|}\cdot\sigma \geq 0$. Otherwise B can be replaced by its symmetrized form:

$$\overline{B}(v-v_*,\sigma) = |v-v_*|^\gamma\Big(b(\frac{v-v_*}{|v-v_*|}\cdot\sigma) + b(\frac{v-v_*}{|v-v_*|}\cdot(-\sigma))\Big)1_{\frac{v-v_*}{|v-v_*|}\cdot\sigma\geq 0},$$

where 1_A is the indicator function of the set A.

Remark 1.1. *For inverse repulsive potential, it holds that* $\gamma = \frac{p-5}{p-1}$ *and* $s = \frac{1}{p-1}$ *with* $p > 2$. *It is easy to check that* $\gamma + 2s > -1$ *is satisfied for the full range of the inverse power law model. Generally, the case* $\gamma > 0$, $\gamma = 0$, *and* $\gamma < 0$ *correspond to so-called hard, Maxwellian, and soft potentials respectively.*

For the case of Vlasov-Poisson-Landau system, Q is the Landau collision operator given by

$$\begin{aligned} Q(g,f)(v) &= \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v-v_*)(g(v_*)\nabla_v f(v) - f(v)\nabla_{v_*} g(v_*))dv_* \\ &= \partial_i \int_{\mathbb{R}^3} \phi^{ij}(v-v_*)(g_*\partial_j f - f\partial_j g_*)dv_*, \end{aligned}$$

with

$$\begin{aligned} \Phi(u) &= |u|^{\gamma+2}\left(I - \frac{u\otimes u}{|u|^2}\right), \quad -3 \le \gamma \le 1, \\ \phi^{ij}(v) &= |v|^{\gamma+2}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right), \quad -3 \le \gamma \le 1, \end{aligned} \tag{1.3}$$

hereafter we use the convention of summation for repeated indices, and the derivatives are in the velocity variable $\partial_i = \partial_{v_i}$. We will use the notations

$$g_* = g(v_*), \quad f = f(v), \quad \partial_j g_* = \partial_{v_{*j}} g(v_*), \quad \partial_j f = \partial_{v_j} f(v).$$

We also mention that

$$\begin{aligned} &\partial_j \phi^{ij}(v) = -2|v|^\gamma v_i, \\ &\partial_{ij}\phi^{ij}(v) = \begin{cases} -2(\gamma+3)|v|^\gamma, & \text{if } -3 < \gamma \le 1, \\ -8\pi\delta_0, & \text{if } \gamma = -3, \end{cases} \end{aligned}$$

where δ_0 is the Dirac measure.

It's convenient to call it *hard potential* if $\gamma \ge 0$, and *soft potential* if $\gamma \in [-3, 0)$.

1.2. **Reformulation.** We reformulate the Vlasov-Poisson-Boltzmann/Landau system near a global Maxwellian. For simplicity, we assume the initial data F_0 is normalized such that the equilibrium associated to the equation (1.1) will be the standard Gaussian function, i.e.

$$\mu(v) \overset{\text{def}}{=} (2\pi)^{-3/2} e^{-|v|^2/2},$$

which enjoys the same mass, momentum and energy as F_0. In the perturbation framework, we denote $F = [F_+, F_-]$ and let $f = [f_+, f_-]$ satisfies

$$F_\pm = \mu + f_\pm.$$

Then system (1.1) and (1.2) become

$$\begin{cases} \partial_t f_\pm + v\cdot\nabla_x f_\pm \mp \nabla_x\phi\cdot\nabla_v f_\pm \pm \nabla_x\phi\cdot v\mu \\ \qquad\qquad = Q(f_\pm + f_\mp, \mu) + Q(2\mu + f_\pm + f_\mp, f_\pm), \\ -\Delta_x\phi = \displaystyle\int_{\mathbb{R}^3}(f_+ - f_-)dv, \quad \int_\Omega \phi(x)dx = 0, \\ f(0) = f_0, \quad \phi(0) = \phi_0. \end{cases} \tag{1.4}$$

Note that $Q(\mu,\mu) = 0$. We also denote linear operator $L = [L_+, L_-]$ and $\mathcal{L} = [\mathcal{L}_+, \mathcal{L}_-]$ by

$$L_\pm f = 2Q(\mu, f_\pm) + Q(f_\pm + f_\mp, \mu),$$

and

$$\mathcal{L}f = -v\cdot\nabla_x f \mp \nabla_x\phi\cdot v\mu + Lf, \tag{1.5}$$

where $\phi(x)$ is solved by the second equation of (1.4). The kernel of L on $L^2_v \times L^2_v$ is the span of $\{[1,0]\mu, [0,1]\mu, [1,1]v\mu, [1,1]|v|^2\mu\}$ (cf. [41]) and we define the projection of $L^2_v \times L^2_v$ onto $\ker L$ by

$$\mathbf{P}f = \Big(a_+(t,x)[1,0] + a_-(t,x)[0,1] + v\cdot b(t,x)[1,1] + (|v|^2-3)c(t,x)[1,1]\Big)\mu, \tag{1.6}$$

or equivalently by

$$\mathbf{P}_\pm f = \Big(a_\pm(t,x) + v\cdot b(t,x) + (|v|^2-3)c(t,x)\Big)\mu,$$

where function $a_\pm, b, c$ are given by

$$\begin{aligned} a_\pm &= \int_{\mathbb{R}^3} f_\pm dv, \quad b_j = \frac{1}{2}\int_{\mathbb{R}^3} v_i(f_+ + f_-)dv, \\ c &= \int_{\mathbb{R}^3} \frac{|v|^2-3}{12}(f_+ + f_-)dv. \end{aligned} \tag{1.7}$$

Taking inner produce of the first equation of (1.4) with 1 over $\mathbb{R}^3_v$, we have the continuity equation

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_\pm(v)dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f_\pm(v)dv = 0. \tag{1.8}$$

1.3. **Domains.** In this paper, we will consider two kinds of bounded domain Ω, namely, torus and union of finitely many cubes.

Torus. For the case of torus, we set

$$\Omega = \mathbb{T}^3 = [-\pi, \pi]^3.$$

In this case, the solution (F, ϕ) to the Vlasov-Poisson-Boltzmann/Landau system (1.1) with the initial data F_0 enjoys the conservation of mass, momentum and the energy, i.e.

$$\begin{aligned} &\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} F_+ dvdx = \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} F_- dvdx = 0, \quad \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} v(F_+ + F_-)dvdx = 0, \\ &\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} |v|^2(F_+ + F_-)dvdx + \frac{d}{dt}\int_\Omega |\nabla_x\phi(t,x)|^2 dx = 0. \end{aligned} \tag{1.9}$$

By assuming F_0 have the same mass, total momentum and total energy as μ, the conservation laws (1.9) yield

$$\begin{aligned} &\int_\Omega\int_{\mathbb{R}^3} f_+ dvdx = \int_\Omega\int_{\mathbb{R}^3} f_- dvdx = 0, \\ &\int_\Omega\int_{\mathbb{R}^3} v(f_+ + f_-)dvdx = 0, \\ &\int_\Omega\int_{\mathbb{R}^3} |v|^2(f_+ + f_-)dvdx + \int_\Omega |\nabla_x\phi(t,x)|^2 dx = 0. \end{aligned} \tag{1.10}$$

Union of cubes. The second kind of bounded domain is the union of finitely many cubes:

$$\Omega = \cup_{i=1}^N \Omega_i, \tag{1.11}$$

where $\Omega_i = (a_{i,1}, b_{i,1}) \times (a_{i,2}, b_{i,2}) \times (a_{i,3}, b_{i,3})$ with $a_{i,j}, b_{i,j} \in \mathbb{R}$ such that $a_{i,j} < b_{i,j}$. Then $\partial\Omega = \cup_{i=1}^3 \Gamma_i$ is the union of three kinds of boundary $\Gamma_i (i = 1, 2, 3)$, where Γ_i is orthogonal to axis x_i and is the union of finitely many connected sets. We further assume that Γ_i is of *non-zero* spherical measure. Since the boundary of Γ_i's are of *zero* spherical measure, we don't distinguish Γ_i and the interior of Γ_i.

The unit normal outer vector $n(x)$ exists on $\partial\Omega$ almost everywhere with respect to spherical measure. On the interior of $\Gamma_i (i = 1, 2, 3)$, we have $n(x) = e_i$ or $-e_i$, where e_i is the unit vector with ith-component being 1. We will denote vectors $\tau_1(x), \tau_2(x)$ on boundary $\partial\Omega$ such that $(n(x), \tau_1(x), \tau_2(x))$ forms an unit orthonormal basis for $\mathbb{R}^3$ such that for $j = 1, 2$, $\tau_j = e_k$ or $-e_k$ for some k.

The boundary of the phase space is

$$\gamma := \{(x, v) \in \partial\Omega \times \mathbb{R}^3\}.$$

Denoting $n = n(x)$ to be the outward normal direction at $x \in \partial\Omega$, we decompose γ as

$$\begin{aligned} \gamma_- &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad \text{(the incoming set)}, \\ \gamma_+ &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad \text{(the outgoing set)}, \\ \gamma_0 &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad \text{(the grazing set)}. \end{aligned}$$

Correspondingly, we assume that $F(t,x,v)$ satisfies the *specular-reflection* boundary condition:

$$F(t,x,R_x v) = F(t,x,v), \quad \text{on } \gamma_-,$$

where for $(x,v) \in \gamma$,

$$R_x v = v - 2n(x)(n(x)\cdot v). \tag{1.12}$$

This is equivalent to the *specular reflection* boundary condition for perturbation f:

$$f(t,x,R_x v) = f(t,x,v), \quad \text{on } \gamma_-. \tag{1.13}$$

For the boundary condition of electric potential ϕ, we further assume that

$$\partial_n \phi = 0, \quad \text{on } x \in \partial\Omega. \tag{1.14}$$

In particular, the Poisson equation for potential ϕ is a pure Neumann boundary problem and we require zero-mean condition

$$\int_\Omega \int_{\mathbb{R}^3} (f_+ - f_-)\, dvdx = 0, \qquad \text{for } t \geq 0,$$

to ensure its existence, which follows from the following conservation laws (1.15). Similar to (1.10), it's also well-known that the solution to (1.4) in bounded domain Ω given by (1.11) satisfies the conservation laws on mass and energy. That is, the solution f to (1.4) satisfies the following identities when the initial data f_0 satisfies it:

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^3} f_+(t)\, dvdx = \int_{\Omega\times\mathbb{R}^3} f_-(t)\, dvdx = 0, \\ &\int_{\Omega\times\mathbb{R}^3} (f_+(t) + f_-(t))|v|^2\, dvdx + \int_\Omega |\nabla_x \phi(t,x)|^2\, dx = 0. \end{aligned} \tag{1.15}$$

1.4. **Notations.** Let us first introduce the function spaces and notations. We let the multi-indices α and β be $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, $\beta = [\beta_1, \beta_2, \beta_3]$ and define

$$\partial_\beta^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

If each component of α' is not greater than that of the α's, we denote by $\alpha' \leq \alpha$. $\alpha' < \alpha$ means $\alpha' \leq \alpha$, and $|\alpha'| < |\alpha|$. We write $a \lesssim b$ ($a \gtrsim b$) to indicate that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$ ($a \geq Cb$). We use the notation $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We denote $a \gg b$ if a, b are two constants such that $a > b$ and a is sufficiently large. We denote $C_{a_1,a_2,\cdots,a_n}$ by a constant depending on parameters $a_1, a_2, \cdots, a_n$. Moreover, we use parameter ε to represent different positive numbers much less than 1 and determined in different cases. We use $(f,g) = (f,g)_{L^2_v}$ to denote the inner product of f, g over velocity variable for short and, $(f,g)_{L^2_{x,v}}$ to represent the inner product over both spatial and velocity variables. Also, we write $(f,g)_{L^2_k} = (\langle v\rangle^{2k} f, g)_{L^2_v}$,

$$\|f(\theta)\|_{L^1_\theta} = \int_{\mathbb{S}^2} f(\theta) d\sigma = 2\pi \int_0^\pi f(\theta) \sin\theta d\theta,$$

and for any $p \in [1,\infty]$,

$$\|f(t)\|_{L^p_T} = \|f(t)\|_{L^p([0,T])}.$$

For the linear operator $\mathcal{L}$, we write $S_{\mathcal{L}}$ as the semigroup generated by $\mathcal{L}$. Define the **Japanese brackets** by

$$\langle v\rangle := (1 + |v|^2)^{1/2}.$$

For real numbers m, l, we define the weighted Sobolev norm $\|\cdot\|_{H^m_l}$ by

$$|f|_{H^m_l} = |\langle v\rangle^l \langle D_v\rangle^m f(v)|_{L^2_v}.$$

If $m=0$, we write $L^2_l=H^0_l$. Here $a(D)$ is a pseudo-differential operator with the symbol $a(\xi)$, and is defined by

$$(a(D)f)(v):=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{i(v-u)\xi}a(\xi)f(u)dud\xi.$$

The mixed norm $\|\cdot\|_{H^n_xH^m_l}$ is defined as

$$\|f\|_{H^n_xH^m_l}:=\Big(\int_\Omega\|\langle D_x\rangle^n f(x,\cdot)\|^2_{H^m_l}dx\Big)^{1/2}.$$

If $n=0$, we write $H^0_xH^m_l=L^2_xH^m_l$. The entropy $L\log L$ space is defined as

$$L\log L:=\Big\{f(v):\|f\|_{L\log L}=\int_{\mathbb{R}^3}|f|\log(1+|f|)dv\Big\}.$$

For any $k\in\mathbb{R}$, $\gamma\in(-3,1]$, we define

$$\|f\|^2_{L^2_{k+\gamma/2,*}}:=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mu(v_*)|v-v_*|^\gamma|f(v)|^2\langle v\rangle^{2k}dvdv_*.$$

It is direct to verify that $\|f\|_{L^2_{k+\gamma/2,*}}\sim\|f\|_{L^2_{k+\gamma/2}}$ by noticing

$$\int_{\mathbb{R}^3}\mu(v_*)|v-v_*|^\gamma\langle v\rangle^{2k}dv_*\sim\langle v\rangle^{2k+\gamma}.$$

For Boltzmann case, we denote the dissipation norm $L^2_{D,k}$ by

$$\|f\|_{L^2_{D,k}}=\|f\|_{H^s_{k+\gamma/2}},$$

and $\|f\|_{L^2_D}=\|f\|_{L^2_{D,0}}$.

For Landau case, we denote the anisotropic norm $L^2_D(m)$ by

$$\|f\|_{L^2_D(m)}:=\|f\|_{L^2(m\langle v\rangle^{\gamma/2})}+\|\widetilde{\nabla}_v(mf)\|_{L^2(\langle v\rangle^{\gamma/2})}$$

and, for brevity, we let $L^2_{D,k}:=L^2_D(\langle v\rangle^k)$ and $L^2_D:=L^2_{D,0}$. Here $\widetilde{\nabla}_v$ is the anisotropic gradient given by

$$\widetilde{\nabla}_vf:=P_v\nabla_vf+\langle v\rangle(I-P_v)\nabla_vf,\quad P_v\xi:=\left(\xi\cdot\frac{v}{|v|}\right)\frac{v}{|v|},\quad\forall\,\xi\in\mathbb{R}^3.$$

With multi-indices (α,β), we come to define our weight function with some constant k. For $-3\le\gamma\le1$, we choose the weight function $w(\alpha,\beta)$ as

$$w(\alpha,\beta)=\begin{cases}\langle v\rangle^{k-p|\alpha|-q|\beta|+r}, & q=6s-3(\gamma-1),\quad p=q+\gamma-1,\\ & r=2q,\qquad\text{for Boltzmann case,}\\ \langle v\rangle^{k-p|\alpha|-q|\beta|+r}, & q=3-(\gamma-1),\quad p=3,\\ & r=2q,\qquad\text{for Landau case.}\end{cases}\tag{1.16}$$

Note that $w(\alpha,\beta)\ge1$ for any $|\alpha|+|\beta|\le2$ and any $k\ge0$. For brevity, we write $w(|\alpha|,|\beta|)=w(\alpha,\beta)$ throughout the paper. Noticing $w^2(\alpha,\beta)$ is still of the form $\langle v\rangle^k$ for some $k>0$, we have

$$\nabla_v(w^2(\alpha,\beta))=A_{\alpha,\beta}\frac{v}{\langle v\rangle^2}w^2(\alpha,\beta),\tag{1.17}$$

for some constant $A_{\alpha,\beta}$ depending on α,β,k,γ,s. In this work, we will apply a useful space-velocity weight

$$e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\tag{1.18}$$

to eliminate the dissipation loss term. Next we define some useful norms in our analysis. For this, we denote constants to be chosen in Theorem 3.1:

$$C_{|\alpha|,|\beta|}\gg C_{|\alpha|,|\beta_1|},\ \text{ if }|\beta_1|<|\beta|,\quad C_{|\alpha|+1,|\beta|-1}\gg C_{|\alpha|,|\beta|},\tag{1.19}$$

for any multi-indices α, β. Then for both Boltzmann case and Landau case, we denote energy norms:

$$\|f\|_{X_k}^2 = \sum_{|\alpha|+|\beta|\le 2} C_{|\alpha|,|\beta|} \|e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}} w(\alpha,\beta)\partial_\beta^\alpha f\|_{L_x^2 L_v^2}^2, \tag{1.20}$$

and

$$\|f\|_{Y_k}^2 = \sum_{|\alpha|+|\beta|\le 2} C_{|\alpha|,|\beta|} \|w(\alpha,\beta)\partial_\beta^\alpha f\|_{L_x^2 L_D^2}^2. \tag{1.21}$$

We further define the "instant energy functional" $\mathcal{E}_k(t)$ and "dissipation energy functional" $\mathcal{D}_k(t)$ respectively by

$$\mathcal{E}_k(t) \approx \|f\|_{X_k}^2 + \|\nabla_x \phi\|_{H_x^2}^2, \tag{1.22}$$

and

$$\mathcal{D}_k(t) := \|f\|_{Y_k}^2 + \|\nabla_x \phi\|_{H_x^2}^2, \tag{1.23}$$

where the explicit definition of $\mathcal{E}_k(t)$ is given in (4.83). Moreover, we fix the weight index k_0 by assuming

$$k_0 \ge \begin{cases} 14, & \text{for the Boltzmann case,} \\ 7, & \text{for the Landau case.} \end{cases} \tag{1.24}$$

1.5. **Main results.** We may now state our main results.

Theorem 1.2 (Global existence, uniqueness and large-time decay)**.** *Consider the Cauchy problem (1.4) for Vlasov-Poisson-Boltzmann/Landau system. Suppose $\gamma \in (-3, 1]$, $s \in [\frac{1}{2}, 1)$, $\gamma + 2s > -1$ for the Boltzmann case and $\gamma \in [-3, 1]$ for the Landau case. Let $l = 0$ for the hard potential case and $l > \frac{|\gamma|}{2}$ for the soft potential case. Then there exist constants $\lambda > 0$, $M > 0$ (small) such that, for any*

$$k \ge k_0 + 4 + l \quad \text{for the Landau case and} \quad k \ge k_0 + 4 + l \quad \text{for Boltzmann case,}$$

if the initial data f_0 satisfies $F_0(x, v) = \mu + f_0(x, v) \ge 0$, conservation laws (1.10) for the case of torus (or (1.15) for the case of union of cubes) and

$$\mathcal{E}_{k_0+l}(0) \le M \text{ and } \mathcal{E}_k(0) < +\infty, \tag{1.25}$$

then there exists a unique solution $f(t, x, v)$ to (1.4) satisfying $F(t, x, v) = \mu + f(t, x, v) \ge 0$ such that

$$\sup_{0\le t\le T} \mathcal{E}_k(t) \le 2\mathcal{E}_k(0),$$

for any $T > 0$. Moreover, we have the following large-time asymptotic behavior. If $\gamma \ge 0$, then

$$\mathcal{E}_k(t) \le e^{-\lambda t}\mathcal{E}_k(0).$$

If $\gamma < 0$, then we have

$$\mathcal{E}_{k-l}(t) \lesssim (1 + t)^{-\frac{2l}{|\gamma|}}\mathcal{E}_k(0).$$

Here $\mathcal{E}_k(t)$ is given in (1.22).

Remark 1.3. *(1) We emphasize that for global solutions with polynomial weight, one only need to assume the smallness of initial data with respect to a given norm. Here, we only require X_{k_0+l} to be sufficiently small instead of requiring general X_k to be small.*

(2) Notice that the torus case and union of cubes case are similar and we will do integration by parts carefully when Ω is the union of cubes given by (1.11). The integration-by-parts technique for torus case is trivial.

(3) For the soft potential case $\gamma < 0$ and for the non-cutoff Boltzmann equation, in [8], the authors shows that the sharp decay rate to the non-cutoff Boltzmann equation cannot be faster that $k/|\gamma|$. Therefore when k is sufficient large, the decay rate in Theorem 1.2 more or less can be regarded as the optimal one.

(4)In this paper, we only prove the results for $H^2_{x,v}(\langle v\rangle^k)$, but we believe that similar proof can be made for in high regularity space $H^N_{x,v}(\langle v\rangle^k)$.

1.6. **Strategies and ideas of the proof.** In this subsection, we will give some literature and explain main strategies of the proof for our results.

In what follows we recall some known results on the Landau and Boltzmann equations with a focus on the topics under consideration in this paper, particularly on global existence and large-time behavior of solutions to the spatially inhomogeneous equations in the perturbation framework. For global solutions to the renormalized equation with large initial data, we mention the classical works [6, 20–22, 56, 66, 67]. We mention [13, 14] for the smoothing effect for the Boltzmann equation without cut-off. For the stability of vacuum, see [12, 40, 57] for the Landau, cutoff and non-cutoff Boltzmann equation with moderate soft potential respectively.

For the non-cutoff Boltzmann equation, [50–54, 60] obtains global regularity and long time behavior in a very general assumption by just assuming a uniformly bound in t, x such that

$$0 < m_0 \le M(t,x) \le M_0, \quad E(t,x) \le E_0, \quad H(t,x) \le H_0,$$

for some constant $m_0, M_0, E_0, H_0 > 0$, where

$$M(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv,\ E(t,x) = \int_{\mathbb{R}^3} f(t,x,v)|v|^2 dv,\ H(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\ln f(t,x,v)dv.$$

and for the Landau equation the local Hölder estimate is proved in [32] for and higher regularity of solutions was studied in [49] applying a kinetic version of Schauder estimates.

Then we focus on the results in the perturbation framework. In the near Maxwellian framework, global existence and large-time behavior of solutions to the spatially inhomogeneous equations are proved in [37, 38, 63, 64] for the cutoff Boltzmann equation and in [36] for the Landau equation. For the non-cutoff Boltzmann equation, it is proved in [2–5, 33, 34], see also [27] for a recent work on such topic. We refer to [43, 45, 46] for existence theory in bounded domains. The non-cutoff case in union of cubes is considered in [17]. All these works above are based on the following decomposition

$$\partial_t f + v\cdot\nabla_x f = L_\mu f + \Gamma(f,f), \quad L_\mu f = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f,\mu) + \frac{1}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}f),$$

with

$$\Gamma(g,f) = \frac{1}{2}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g,\sqrt{\mu}f) + \frac{1}{2}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f,\sqrt{\mu}g),$$

which means that the solution is constructed in $\mu^{-1/2}$ weighted space.

For the inhomogeneous Boltzmann/Landau equation with polynomial weight perturbation near Maxwellian, Gualdani-Mischler-Mouhot [35] first prove the global existence and large-time behavior of solutions with polynomial velocity weight for the cutoff Boltzmann equation with hard potential. This method is generalized to the Landau equation in [10, 11]. The case of non-cutoff Boltzmann equation is proved in [7, 48] for the hard potential case and [8] for the soft potential case.

Here we mention former works on the Vlasov-Poisson/Maxwell-Boltzmann/Landau system near Maxwellian. The Vlasov-Poisson/Maxwell-Boltzmann system for the cut-off hard sphere case is proved in [41, 42, 62]; see also [29] for the optimal convergence rate. We refer to [24, 30, 31] for other case with the cut-off assumption. For the Landau equation, it is first proved in a famous paper [44] for the tours case and [65, 68] for the whole space case, see also [25]. For the non-cutoff case, we refer to [26, 69] for Vlasov-Poisson-Boltzmann system and [28] for the Vlasov-Maxwell-Boltzmann system, see also [18] for the regularizing effect. The theory of bounded domain is discussed in [9, 19, 23]. We mention that all the works above are based on the following decomposition

$$\partial_t f_\pm + v\cdot\nabla_x f_\pm \mp \nabla_x\phi\cdot\nabla_v f_\pm \pm \nabla_x\phi\cdot v\sqrt{\mu} \pm \frac{1}{2}\nabla_x\phi\cdot vf = L_{\mu,\pm}f + \Gamma_\pm(f,f),$$

where

$$L_{\mu,\pm}f=\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}(f_\pm+f_\mp),\mu)+2\frac{1}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}f_\pm),\tag{1.26}$$

and

$$\Gamma_\pm(f,f)=\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_\pm,\sqrt{\mu}f_\pm)+\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_\mp,\sqrt{\mu}f_\pm),\tag{1.27}$$

which means the result are in $\mu^{-1/2}$ weighted space. We remark here that to our knowledge, our paper is the first to consider such question in a polynomial weighted space.

Then we give some introduction on the semigroup method by Gualdani-Mischler-Mouhot [35] and we improve the method to macroscopic part $\mathbf{P}f$. The main idea of semigroup method can be expressed briefly as follows: Taking the case $\gamma=0$ and the space $H^2_xL^2_k$ for example, by [58] we have

$$\sum_\pm(\mathcal{L}_\pm f,f_\pm)_{H^2_xL^2(\mu^{-1/2})}\le-\lambda\|f\|_{H^2_xL^2(\mu^{-1/2})},\quad\text{if }\mathbf{P}f=0,$$

for some constant $\lambda>0$. Together with the macroscopic estimates from [19,33] for $\mathbf{P}f$, we can deduce that

$$\|S_{\mathcal{L}}(t)f_0\|_{H^2_xL^2(\mu^{-1/2})}\le e^{-\lambda t}\|f_0\|^2_{H^2_xL^2(\mu^{-1/2})}.$$

For some $M,R>0$, define

$$\begin{aligned}A_\pm&=-v\cdot\nabla_x+L_\pm-M\chi_R,\quad K_1=M\chi_R,\\B_\pm&=-v\cdot\nabla_x+L_\pm,\quad K_2=\pm\mu v\cdot\nabla_x\phi,\end{aligned}$$

where $\chi\in D(\mathbb{R})$ is the truncation function satisfying $1_{[-1,1]}\le\chi\le1_{[-2,2]}$ and we denote $\chi_R(\cdot):=\chi(\cdot/R)$ for $R>0$. By the results in [8] we have

$$\sum_\pm(L_\pm f,f_\pm)_{H^2_xL^2_k}\le-C\|f\|^2_{H^2_xH^s_k}+C_k\|f\|^2_{H^2_xL^2_v},$$

then taking $M,R>0$ large, we have

$$(Af,f)_{H^2_xL^2_k}\le-C\|f\|^2_{H^2_xL^2_k},$$

which implies

$$\|S_A(t)f\|_{H^2_xL^2_k}\le e^{-\lambda t}\|f_0\|^2_{H^2_xL^2_k}.$$

By Duhamel's formula $S_B=S_A+S_B*K_1S_A$ and $S_{\mathcal{L}}=S_B+S_{\mathcal{L}}*K_2S_B$, roughly speaking, we have

$$\begin{aligned}\|S_B(t)\|_{H^2_xL^2_k\to H^2_xL^2_k}\le\|S_A(t)\|_{H^2_xL^2_k\to H^2_xL^2_k}+\int_0^t\|S_B(s)\|_{H^2_xL^2(\mu^{-1/2})\to H^2_xL^2(\mu^{-1/2})}\\\times\|K_1\|_{H^2_xL^2_k\to H^2_xL^2(\mu^{-1/2})}\|S_A(t-s)\|_{H^2_xL^2_k\to H^2_xL^2_k}ds\le Ce^{-\lambda t}.\end{aligned}$$

and hence

$$\begin{aligned}\|S_{\mathcal{L}}(t)\|_{H^2_xL^2_k\to H^2_xL^2_k}\le\|S_B(t)\|_{H^2_xL^2_k\to H^2_xL^2_k}+\int_0^t\|S_L(s)\|_{H^2_xL^2(\mu^{-1/2})\to H^2_xL^2(\mu^{-1/2})}\\\times\|K_2\|_{H^2_xL^2_k\to H^2_xL^2(\mu^{-1/2})}\|S_B(t-s)\|_{H^2_xL^2_k\to H^2_xL^2_k}ds\le Ce^{-\lambda t}.\end{aligned}$$

Thus, the rate of convergence for the linear operator $\mathcal{L}$ is established. To estimate the nonlinear part, we need to define a scalar product by

$$((f,g))_k:=(f,g)_{L^2_k}+\eta\int_0^{+\infty}(S_{\mathcal{L}}(\tau)f,S_{\mathcal{L}}(\tau)g)_{L^2_v}d\tau.$$

Due to the fact that

$$\int_0^{+\infty}(S_{\mathcal{L}}(\tau)\mathcal{L}f,S_{\mathcal{L}}(\tau)f)d\tau=\int_0^{+\infty}\frac{d}{d\tau}\|S_{\mathcal{L}}(\tau)f\|_{L^2}^2d\tau=-\|f\|_{L^2}^2,$$

we deduce that

$$((\mathcal{L}f,f))_k=(\mathcal{L}f,f)_{L^2_k}+\eta\int_0^{\infty}(S_{\mathcal{L}}(\tau)Lf,S_{\mathcal{L}}(\tau)f)_{L^2_v}d\tau\sim\|f\|_{H^s_k}^2.$$

if we choose proper η. This shows that the linear operator $\mathcal{L}$ can still be non-negative in a suitable function space. The estimate for the linear operator $\mathcal{L}$ in this space allows us to prove the global well-posedness by combining the nonlinear estimates.

The semigroup method can help us handle the lower order terms, while it could also produce new bad terms to estimate, which forces us to make the assumption $s\geq 1/2$. Precisely, when applying the semigroup method on the Vlasov-Poisson term $\nabla_x\phi(x)\cdot\nabla_v f$, unlike the usual case, where by integration by parts we easily have

$$(\nabla_x\phi(x)\cdot\nabla_v f,f)=0.$$

For the semigroup term we have

$$\int_0^{\infty}(S_{\mathcal{L}}(\tau)\nabla_x\phi(x)\cdot\nabla_v f,S_{\mathcal{L}}(\tau)f)d\tau,$$

since we are not able to change $S_{\mathcal{L}}(t)$ and ∇_v, integration by parts with respect to ∇_v is not allowed and we have to use upper bound to bound this term. By [8] we have

$$\|S_B(t)f\|_{H^2_xL^2_v}\leq t^{-1/2}e^{-\lambda t}\|f\|_{H^2_xH^{-s}_{12}},$$

using Duhamel's formula as before, similarly we can prove

$$\|S_{\mathcal{L}}(t)f\|_{H^2_xL^2_v}\leq t^{-1/2}e^{-\lambda t}\|f\|_{H^2_xH^{-s}_{12}}.$$

Due to the extra ∇_v term, we need to use

$$\begin{aligned}\|S_{\mathcal{L}}(t)\nabla_x\phi(x)\cdot\nabla_v f\|_{H^2_xL^2_v}&\lesssim t^{-1/2}e^{-\lambda t}\|\nabla_x\phi(x)\|_{H^2_x}\|\nabla_v f\|_{H^2_xH^{-s}_{12}}\\&\lesssim t^{-1/2}e^{-\lambda t}\|\nabla_x\phi(x)\|_{H^2_x}\|f\|_{H^2_xH^{s}_{12}}.\end{aligned}$$

which requires $s\geq 1/2$.

Since the ∂_{x_i} commutes with $\mathcal{L}$, while ∂_{v_i} does not commute with $\mathcal{L}$, to avoid computing the commutator $[S_{\mathcal{L}}(t),\nabla_v]$, we will apply the semigroup term only to $\partial^\alpha f$ without taking any velocity derivative. To avoid taking velocity derivative on the semigroup term, we split the proof into 2 steps. First we prove a rough estimate in the space for $\partial^\alpha_\beta f$ which writes:

$$\begin{aligned}&\partial_t\Big(\frac{1}{2}\|f\|_{X_k}^2+\sum_{|\alpha|\leq 2}C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2\Big)+\frac{\gamma_1}{4}\|f\|_{Y_k}^2\\&\qquad\lesssim\|f\|_{H^2_xL^2_v}^2+C\|\partial_t\phi\|_{L^\infty_x}\|f\|_{X_k}^2+\|\nabla_x\phi\|_{H^2_x}^2+\big(\sqrt{\mathcal{E}_{k_0}(t)}+\mathcal{E}_{k_0}(t)\big)\mathcal{D}_k(t).\quad(1.28)\end{aligned}$$

We briefly explain why in the right hand side, the linear term is $\|f\|^2_{H^2_xL^2_v}$ instead of $\|f\|^2_{H^2_{x,v}}$. For the term $\nabla_x\phi\cdot v\mu$, using integration by parts, we have

$$(\partial^\alpha_\beta(\nabla_x\phi\cdot v\mu),\partial^\alpha_\beta f w^2(\alpha,\beta))=((-1)^{|\beta|}\nabla_x\partial^\alpha\phi\cdot\partial_\beta(v\mu w^2(\alpha,\beta)),\partial^\alpha f)\lesssim C\|f\|^2_{H^2_xL^2_{10}}.$$

Only $H^2_xL^2_{10}$ norm of f is needed. For the Lf term, first we have

$$\sum_{|\alpha|+|\beta|\leq 2}\sum_{\pm}(L_\pm f,f_\pm)_{L^2_xL^2_k}\leq -C\sum_{|\alpha|+|\beta|\leq 2}\|f\|^2_{H^2_xH^s_k}+C_k\sum_{|\alpha|+|\beta|\leq 2}\|f\|^2_{H^2_{x,v}}.$$

We can use interpolation to estimate the $\|f\|^2_{H^2_{x,v}}$ term. More precisely, if $|\beta|\geq 1$ we have

$$C_k\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}}\leq C\|\partial^\alpha f\|^2_{L^2_xH^{|\beta|}_v}\leq\frac{\gamma_1}{24}\|\partial^\alpha f\|^2_{L^2_xH^{|\beta|+s}_v}+C_k\|\partial^\alpha f\|^2_{L^2_{x,v}}\leq\frac{\gamma_1}{24}\|f\|^2_{Y_k}+C_k\|\partial^\alpha f\|^2_{L^2_{x,v}}.$$

Since only $H_x^2L_v^2$ term on the right hand side, by adding a $H_x^2L_v^2$ estimate to (1.28) we can prove there exists functional $\mathcal{E}_1(t)$ satisfying

$$\mathcal{E}_1(t) \approx \|f\|_{X_k}^2 + \|\nabla_x\phi\|_{H_x^3}^2,$$

such that

$$\partial_t\mathcal{E}_1(t) + \lambda\eta\|f\|_{Y_k}^2 + \lambda\sum_{|\alpha|\le 2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L_x^2L_D^2}^2 \lesssim \|f\|_{X_{k_0}}\|f\|_{X_k}^2$$
$$+ C_{\eta,k}\|\{\mathbf{I}-\mathbf{P}\}f\|_{H_x^2L_v^2}^2 + \eta\|\nabla_x\phi\|_{H_x^2}^2 + \eta\|\mathbf{P}f\|_{H_x^2L_v^2}^2 + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t))\mathcal{D}_k(t).$$

Thus we are able to estimate $C_{\eta,k}\|\{\mathbf{I}-\mathbf{P}\}f\|_{H_x^2L_v^2}^2$ by the semigroup method and estimate $\eta\|\nabla_x\phi\|_{H_x^2}^2 + \eta\|\mathbf{P}f\|_{H_x^2L_v^2}^2$ by macroscopic estimate.

1.6.1. *Comments on the weight* (1.18). The polynomial perturbation can be seen as a change of variable $F = \mu + \langle v\rangle^{-k}f_\pm$ for (1.1) for some constant $k > 0$. Then we have

$$\partial_t f_\pm + v\cdot\nabla_x f_\pm \mp \nabla_x\phi\cdot\nabla_v f_\pm \pm \nabla_x\phi\cdot v\mu\langle v\rangle^{2k} \pm k\nabla_x\phi\cdot\frac{v}{\langle v\rangle^2}f = L_{k,\pm}f + \Gamma_{k,\pm}(f,f),$$

with

$$L_{k,\pm}f = \langle v\rangle^k Q(\langle v\rangle^{-k}(f_\pm + f_\mp),\mu) + 2\langle v\rangle^k Q(\mu,\langle v\rangle^{-k}f_\pm),$$

and

$$\Gamma_{k,\pm}(f,f) = \langle v\rangle^k Q(\langle v\rangle^{-k}(f_\pm + f_\mp),\langle v\rangle^{-k}f_\pm).$$

In order to deal with the extra term $\pm k\nabla_x\phi\cdot\frac{v}{\langle v\rangle^2}f$, we follow the idea of [44] and introduce the corresponding weight function

$$\exp\Big\{\frac{\pm k\phi}{\langle v\rangle^2}\Big\}, \tag{1.29}$$

which satisfies

$$v\cdot\nabla_x(e^{\frac{\pm k\phi}{\langle v\rangle^2}}f) = e^{\frac{\pm k\phi}{\langle v\rangle^2}}(v\cdot\nabla_x f \pm k\nabla_x\phi\cdot\frac{v}{\langle v\rangle^2}f).$$

Such choice of weight function allows us to absorb the extra term by taking integration by parts as

$$\Big(\pm k\nabla_x\phi\cdot\frac{v}{\langle v\rangle^2}f, e^{\frac{\pm k\phi}{\langle v\rangle^2}}f\Big)_{L_x^2L_v^2} + \Big(v\cdot\nabla_x f_\pm, e^{\frac{\pm k\phi}{\langle v\rangle^2}}f\Big)_{L_x^2L_v^2} = 0.$$

Unlike [44], where the weight function is $e^{-\phi}$, our weight function potential (1.29) also depends on the velocity variable. Fortunately, new error contributions are of the type $(\nabla_x\phi\cdot v\mu, (e^{\frac{\pm k\phi}{\langle v\rangle^2}} - 1)\langle v\rangle^{2k}f)$, which can be controlled if ϕ is small and its velocity derivative can also be controlled if ϕ is bounded.

1.6.2. *Comments on the weight function* $w(\alpha,\beta)$. The weight $w(\alpha,\beta)$ was firstly introduced in [63, 64] to deal with the time decay of the Boltzmann equation for soft potentials on torus, and it was later used in [21, 44] to investigate the Vlasov-Poisson/Maxwell-Boltzmann/Landau system. The choice of $w(\alpha,\beta)$ should depend on the weighted estimates for the term $v\cdot\nabla_x f$ and $\nabla_x\phi\cdot\nabla_v f$. For $\nabla_x\phi\cdot\nabla_v f$, one has to bound

$$(\nabla_x\partial^{\alpha_1}\phi\cdot\nabla_v\partial_\beta^{\alpha-\alpha_1}f, \partial_\beta^\alpha f w^2(\alpha,\beta))$$
$$\lesssim |\nabla_x\partial^{\alpha_1}\phi|\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial_\beta^{\alpha-\alpha_1}f w(\alpha,\beta)\|_{L_v^2}\|\langle v\rangle^{\frac{\gamma}{2}}\partial_\beta^\alpha f w(\alpha,\beta)\|_{L_v^2}$$

For the Landau case we will use

$$\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial_\beta^{\alpha-\alpha_1}f w(\alpha,\beta)\|_{L_x^2L_v^2} \lesssim \|\partial_\beta^{\alpha-\alpha_1}f w(|\alpha|-1,\beta)\|_{L_x^2H^1_{\gamma/2}}$$

where we need

$$w(\alpha,\beta) \le w(|\alpha|-1,|\beta|)\langle v\rangle^\gamma, \quad \forall\ |\alpha|\ge 1, |\beta|\ge 0.$$

For the Boltzmann case, similarly we use the following interpolation

$$\|f\|_{H^1_v} \lesssim \|f\langle v\rangle^k\|_{H^s_v} + \|f\langle v\rangle^{-ks/(1-s)}\|_{H^{1+s}_v}.$$

which requires

$$w(\alpha,\beta) \le w(|\alpha|-1,|\beta|)^s w(|\alpha|-1,|\beta|+1)^{1-s}\langle v\rangle^\gamma, \quad \forall\ |\alpha|\ge 1, |\beta|\ge 0.$$

That is, fewer derivatives of f should require stronger weights. Since higher derivatives are associated with weaker velocity weights, more careful analysis is needed for spatial Sobolev imbedding to close the energy estimate. Such a cascade of weights takes advantage of the crucial feature of the Landau operator: a weak gain of ∇_v (for the Boltzmann operator it is a weak gain of $(-\Delta_v)^{\frac{s}{2}}$). For the term $v\cdot\nabla_x f$, one has to bound

$$(\partial^{\alpha+e_i}_{\beta-e_i} f, \partial^\alpha_\beta w^2(\alpha,\beta)) \lesssim \|\partial^{\alpha+e_i}_{\beta-e_i} f w(|\alpha|+1,|\beta|-1)\|_{L^2_{\gamma/2}} \|\partial^\alpha_\beta f w(|\alpha|,|\beta|)\|_{L^2_{\gamma/2}}$$

where we need

$$w(\alpha,\beta) \le \langle v\rangle^\gamma w(|\alpha|+1,|\beta|-1).$$

so the $w(\alpha,\beta)$ is taken such that the requirements above holds.

1.6.3. *Difference between the polynomial case and the exponential weight case.* The main difference is that in the exponential case, $L_\mu f$ and $\Gamma(f,f)$ defined in (1.26) and (1.27) are symmetric while in the polynomial case, they are not. Such asymmetry will cause two problems. The first one is the occurrence of an extra $\mathbf{P}f$ term in the linearized equation. More precisely, if we denote

$$\mathbf{P}_\mu f = a_\pm(t,x)\sqrt{\mu} + b(t,x)\cdot v\sqrt{\mu} + c(t,x)|v|^2\sqrt{\mu}.$$

Then by $L_\mu \mathbf{P}_\mu f = 0$ and L_μ is symmetric, we have

$$(L_\mu f, f) = (L_\mu(\mathbf{P}_\mu f + \{\mathbf{I}-\mathbf{P}_\mu\}f), \mathbf{P}_\mu f + \{\mathbf{I}-\mathbf{P}_\mu\}f) = (L_\mu(\{\mathbf{I}-\mathbf{P}_\mu\}f), \{\mathbf{I}-\mathbf{P}_\mu\}f).$$

Only the term with $\{\mathbf{I}-\mathbf{P}_\mu\}f$ remains. While for the polynomial weight case, since L is not symmetric, one merely has

$$(Lf,f) = (L\{\mathbf{I}-\mathbf{P}\}f, f) = (L\{\mathbf{I}-\mathbf{P}\}f, \mathbf{P}f) + (L\{\mathbf{I}-\mathbf{P}\}f, \{\mathbf{I}-\mathbf{P}\}f).$$

An extra term $(L\{\mathbf{I}-\mathbf{P}\}f, \mathbf{P}f)$ occurs. Fortunately, we can bound it by

$$(L\{\mathbf{I}-\mathbf{P}\}f, \mathbf{P}f) \le \epsilon\|\mathbf{P}f\|^2_{L^2_{x,v}} + \epsilon\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x L^2_k} + C_\epsilon\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_{x,v}}.$$

The $\mathbf{P}f$ term can be estimated by macroscopic estimate and the last term can be estimated by semigroup method. Such asymmetry also makes it impossible to extend the result to $x\in\mathbb{R}^3$ using existing techniques. In former works [62, 65] for the whole space $x\in\mathbb{R}^3$, the following fact is used. Since Γ is symmetric, for any smooth functions f,g,h

$$(\Gamma(g,f), \mathbf{P}_\mu h) = 0.$$

Thus in [62, 65], defining the dissipation rate functional D(t) by

$$D(t) = \sum_{|\alpha|\le K} \|\nabla_x\partial^\alpha\phi\|^2_{L^2_x} + \sum_{1\le|\alpha|\le K} \|\mathbf{P}_\mu\partial^\alpha f\|^2_{L^2_{x,v}} + \sum_{0\le|\alpha|+|\beta|\le K} \|w(\alpha,\beta)\{\mathbf{I}-\mathbf{P}_\mu\}\partial^\alpha_\beta f\partial^\alpha f\|^2_{Y_k},$$

the estimate of $\mathbf{P}_\mu f$ is avoided. While for the polynomial case, since

$$(Q(f,g), \mu\phi(v)) \ne 0, \quad \phi(v) = 1, v, |v|^2,$$

such fact does not hold anymore. So the method in [62, 65] can not work for the polynomial case, we have to assume Ω is a bounded domain and use the Poincaré's inequality to estimate $\mathbf{P}f$ by $\nabla_x(\mathbf{P}f)$. How to extend the result to the whole space case remains open.

1.7. **Organization of the paper.** In Section 2, we first recall some basic properties of the non-cutoff Boltzmann/Landau collision operator and then compute the basic estimates for the Vlasov-Poisson-Boltzmann/Landau system. We prove the local existence and uniqueness in Section 3 and obtain corresponding global results and decay of solutions in Section 4.

2. Basic estimates

In this section, we recall some important results on non-cutoff Boltzmann equation and Landau equation and prove several weighted estimates for the Vlasov-Poisson-Landau/Boltzmann equation.

2.1. **Preliminaries on the Boltzmann/Landau equation.** Before introducing the upper and lower bound for the non-cutoff Boltzmann and Landau collision operator, we first state the following lemmas.

Lemma 2.1. *Let $\gamma \in (-2,0)$. Then for any function f, we have*

$$\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f(v_*)|dv_* \lesssim \|f\|_{L^1}^{1+\frac{2\gamma}{3}}\|f\|_{L^2}^{-\frac{2\gamma}{3}} \quad \textit{when} \quad \gamma\in(-\frac{3}{2},0), \tag{2.1}$$

and

$$\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f(v_*)|dv_* \lesssim \|f\|_{L^1}^{1+\frac{\gamma}{2}}\|f\|_{L^3}^{-\frac{\gamma}{2}} \quad \textit{when} \quad \gamma\in(-2,0). \tag{2.2}$$

Proof. Assume f does not equal to 0 otherwise the estimate is trivial. Let $\lambda>0$ be a constant to be determined. We divide the integral into two regions $|v-v_*|\le\lambda$ and $|v-v_*|>\lambda$ we have

$$\int_{\mathbb{R}^3}|v-v_*|^\gamma|f|(v_*)dv_* \le \int_{|v-v_*|\le\lambda}|v-v_*|^\gamma|f|(v_*)dv_* + \int_{|v-v_*|>\lambda}|v-v_*|^\gamma|f|(v_*)dv_*.$$

The second part is bounded by

$$\int_{|v-v_*|>\lambda}|v-v_*|^\gamma|f|(v_*)dv_* \lesssim \lambda^\gamma\|f\|_{L^1}.$$

For the first part we divided it into three cases, for $\gamma\in(-\frac{3}{2},0)$, by Cauchy-Schwarz inequality we have

$$\int_{|v-v_*|\le\lambda}|v-v_*|^\gamma|f|(v_*)dv_* \le \left(\int_{|v-v_*|\le\lambda}|v-v_*|^{2\gamma}dv_*\right)^{\frac12}\|f\|_{L^2} \lesssim \lambda^{\frac32+\gamma}\|f\|_{L^2}.$$

For $\gamma\in(-2,0)$, by Hölder's inequality we have

$$\int_{|v-v_*|\le\lambda}|v-v_*|^\gamma|f|(v_*)dv_* \le \left(\int_{|v-v_*|\le\lambda}|v-v_*|^{\frac32\gamma}dv_*\right)^{\frac23}\|f\|_{L^3} \lesssim \lambda^{2+\gamma}\|f\|_{L^3}.$$

so the proof is ended by taking $\lambda=\|f\|_{L^1}^{\frac23}\|f\|_{L^2}^{-\frac23}, \|f\|_{L^1}^{\frac12}\|f\|_{L^3}^{-\frac12}$ respectively. □

Lemma 2.2. *For any $\gamma\in[-3,1]$ and any functions f,g,h, we have*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{\gamma+2}f(v_*)g(v)h(v)dv_*dv \lesssim \|f\|_{L^2_5}\|g\|_{L^2}\|h\|_{L^2_3}, \tag{2.3}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{\gamma+1}f(v_*)g(v)h(v)dv_*dv \lesssim \|f\|_{L^2_5}\|g\|_{L^2}\|h\|_{H^1_3}, \tag{2.4}$$

and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{\gamma+1}f(v_*)g(v)h(v)dv_*dv \lesssim \|f\|_{L^2_5}\|g\|_{H^1_{\gamma/2+1}}\|h\|_{L^2_{\gamma/2}}. \tag{2.5}$$

If $\gamma\in(-3,1]$, then we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{\gamma}f(v_*)g^2(v)dv_*dv \lesssim \|f\|_{L^2_5}\|g\|^2_{H^1_{\gamma/2}}. \tag{2.6}$$

Proof. If $-2 \le \gamma \le 1$, we deduce that

$$\int_{\mathbb{R}^3} |v-v_*|^{\gamma+2} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L^1_3} \|g\|_{L^2} \|h\|_{L^2_3} \lesssim \|f\|_{L^2_5} \|g\|_{L^2} \|h\|_{L^2_3}.$$

If $-3 \le \gamma < -2$, by (2.1), we have

$$\sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+2} f(v_*) dv_* \lesssim \|f\|_{L^2_2},$$

which implies that

$$\int_{\mathbb{R}^3} |v-v_*|^{\gamma+2} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L^2_2} \int_{\mathbb{R}^3} |g(v)h(v)| dv \lesssim \|f\|_{L^2_2} \|g\|_{L^2} \|h\|_{L^2}.$$

This proves (2.3). For (2.4), if $\gamma \ge -2$, then it reduce to the (2.3). So we only need to consider the case $\gamma \in [-3, -2)$, in which case we have $-3 < \gamma + \frac{1}{2} < 0$ and $-2 < \gamma + \frac{3}{2} < 0$. Then we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+\frac{1}{2}} \frac{1}{\langle v_* \rangle^4} |g(v)|^2 dv dv_* \lesssim \|g\|^2_{L^2}.$$

Applying (2.2) to h, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+\frac{3}{2}} \langle v_* \rangle^4 |f(v_*)|^2 |h(v)|^2 dv dv_* \\ &\qquad \lesssim \|f\|^2_{L^2_2} \||h|^2\|_{L^1}^{1-\frac{\gamma+3/2}{2}} \||h|^2\|_{L^3}^{\frac{\gamma+3/2}{2}} \lesssim \|f\|^2_{L^2_2} \|h\|^2_{H^1}. \end{aligned}$$

Here we apply Sobolev embedding $\|\cdot\|_{L^6} \lesssim \|\cdot\|_{H^1}$. Thus,

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+1} |f(v_*)||g(v)||h(v)| dv dv_* \\ &\le \Big(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+\frac{3}{2}} \langle v_* \rangle^4 |f(v_*)|^2 |h(v)|^2 dv dv_* \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v-v_*|^{\gamma+\frac{1}{2}}}{\langle v_* \rangle^4} |g(v)|^2 dv dv_* \Big)^{\frac{1}{2}} \\ &\le \|f\|_{L^2_2} \|g\|_{L^2} \|h\|_{H^1}. \end{aligned}$$

The proof of (2.4) is then finished. Now we come to (2.5). If $-1 \le \gamma \le 1$, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv &\lesssim \|f\|_{L^1_{\gamma+1}} \int_{\mathbb{R}^3} \langle v \rangle^{\gamma+1} |g(v)h(v)| dv \\ &\lesssim \|f\|_{L^2_4} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}}. \end{aligned}$$

If $-3 \le \gamma < -1$, we have

$$\langle v \rangle^{-(\gamma+1)} \lesssim \langle v_* \rangle^{-(\gamma+1)} \langle v - v_* \rangle^{-(\gamma+1)},$$

which implies

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv \\ &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle v - v_* \rangle^{-(\gamma+1)} |v-v_*|^{\gamma+1} f(v_*) \langle v_* \rangle^{-(\gamma+1)} |g(v)h(v)| \langle v \rangle^{\gamma+1} dv_* dv \\ &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v-v_*|^{\gamma+1}) f(v_*) \langle v_* \rangle^{-(\gamma+1)} |g(v)h(v)| \langle v \rangle^{\gamma+1} dv_* dv := T_1 + T_2. \end{aligned}$$

It is easily seen that

$$T_1 \lesssim \|f\|_{L^1_{-(\gamma+1)}} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}} \lesssim \|f\|_{L^2_5} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}},$$

If $\gamma \in [-2,-1)$, we have $\gamma+1 \in (-\frac{3}{2},0)$ and by (2.1),

$$\begin{aligned} T_2 &\lesssim \sup_{v\in\mathbb{R}^d}\int_{\mathbb{R}^3}|v-v_*|^{\gamma+1}|f(v_*)|\langle v_*\rangle^{-(\gamma+1)}dv_*\int_{\mathbb{R}^3}|g(v)h(v)|\langle v\rangle^{\gamma+1}dv \\ &\lesssim \|f\|_{L^2_5}\|g\|_{H^1_{\gamma/2+1}}\|h\|_{L^2_{\gamma/2}}. \end{aligned}$$

If $\gamma \in [-3,-2)$, by Hardy-Littlewood-Sobolev inequality (cf. [61, Theorem 1.1, pp. 119]) and Hölder's inequality, we have

$$\begin{aligned} T_2 &\lesssim \Big\|\int_{\mathbb{R}^3}|v-v_*|^{\gamma+1}f(v_*)\langle v_*\rangle^{-(\gamma+1)}dv_*\Big\|_{L^6_v}\|hg\langle v\rangle^{\gamma+1}\|_{L^{\frac{6}{5}}} \\ &\lesssim \|f\|_{L^p_{-(\gamma+1)}}\|g\|_{L^3_{\gamma/2+1}}\|h\|_{L^2_{\gamma/2}} \lesssim \|f\|_{L^2_5}\|g\|_{H^1_{\gamma/2+1}}\|h\|_{L^2_{\gamma/2}}, \end{aligned}$$

with $p=\frac{6}{9+2\gamma}\in[1,2]$. The proof for (2.5) is thus finished. Finally we come to prove (2.6). For the case $\gamma\in[-2,1]$, (2.6) can be seen as a special case of (2.5). We focus on the case $\gamma\in(-3,-2)$. Similarly to (2.1), we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{\gamma}|f(v_*)||g(v)|^2dv_*dv \\ \lesssim &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|v-v_*|^{\gamma})|f(v_*)|\langle v_*\rangle^{-\gamma}|g(v)|^2\langle v\rangle^{\gamma}dv_*dv := I_1+I_2. \end{aligned}$$

It is seen that

$$I_1 \lesssim \|f\|_{L^1_{-\gamma}}\|g\|^2_{L^2_{\gamma/2}} \lesssim \|f\|_{L^2_5}\|g\|^2_{L^2_{\gamma/2}}.$$

For I_2, by Hardy-Littlewood-Sobolev inequality (cf. [61, Theorem 1.1, pp. 119]), we have

$$\begin{aligned} I_2 &\lesssim \|f\|_{L^2_{-\gamma}}\Big\|\int_{\mathbb{R}^3}|v-v_*|^{\gamma}|g(v)|^2\langle v\rangle^{\gamma}dv\Big\|_{L^2_{v_*}} \\ &\lesssim \|f\|_{L^2_{-\gamma}}\||g|^2\langle v\rangle^{\gamma}\|_{L^p} \lesssim \|f\|_{L^2_5}\|g\|^2_{L^{2p}_{\gamma/2}} \lesssim \|f\|_{L^2_5}\|g\|^2_{H^1_{\gamma/2}}, \end{aligned}$$

where $p=\frac{6}{9+2\gamma}$ satisfies $2p\in[2,6]$. This gives (2.6) and the proof of Theorem 2.2 is finished. □

We also recall some basic interpolation on x.

Lemma 2.3. *For any function f and constant $k\in\mathbb{R}$, we have*

$$\|fg\|_{H^2_x} \lesssim \min_{c+d=2}\{\|f\|_{H^c_x}\|g\|_{H^d_x}\}.$$

More precisely,

$$\|f\|_{L^\infty_x L^2_v} \lesssim \|f\langle v\rangle^k\|_{H^1_x L^2_v} + \|f\langle v\rangle^{-k}\|_{H^2_x L^2_v}, \tag{2.7}$$

$$\|f\|_{L^3_x L^2_v} \lesssim \|f\langle v\rangle^k\|_{L^2_x L^2_v} + \|f\langle v\rangle^{-k}\|_{H^1_x L^2_v}, \tag{2.8}$$

and for any constant $s\in(0,1)$

$$\|f\|_{H^1_v} \lesssim \|f\langle v\rangle^k\|_{H^s_v} + \|f\langle v\rangle^{-ks/(1-s)}\|_{H^{1+s}_v}. \tag{2.9}$$

Proof. Firstly, we use extension theorem [61, Thoerem VI.5, pp. 181] to extend function f in domain Ω with Lipschitz boundary to a function Ef in $\mathbb{R}^3$ such that $Ef=f$ in Ω and

$$\|Ef\|_{H^k(\mathbb{R}^3)} \lesssim \|f\|_{H^k(\Omega)},$$

for any $k\geq 0$. By Gagliardo–Nirenberg interpolation inequality on $\mathbb{R}$ and $\mathbb{R}^2$ (cf. [55, Theorem 12.83] and [59, Page 125]), we obtain

$$\|Ef\|_{L^\infty_{x_1}(\mathbb{R})} \lesssim \|\partial_{x_1}Ef\|^{1/2}_{L^2_{x_1}(\mathbb{R})}\|Ef\|^{1/2}_{L^2_{x_1}(\mathbb{R})},$$

and

$$\|Ef\|_{L^\infty_{x_2,x_3}(\mathbb{R}^2)} \lesssim \|\nabla_{x_2,x_3}Ef\|_{L^2_{x_2,x_3}(\mathbb{R}^2)}.$$

Combining the above estimates, we have

$$\|f\|_{L^\infty_x(\Omega)} \lesssim \big\|\|Ef\|_{L^\infty_{x_1}(\mathbb{R})}\big\|_{L^\infty_{x_2,x_3}(\mathbb{R}^2)} \lesssim \|f\|^{1/2}_{H^2_x(\Omega)}\|f\|^{1/2}_{H^1_x(\Omega)},$$

which implies (2.7). Also, by Gagliardo-Nirenberg interpolation inequality on $\mathbb{R}^3$, we have

$$\begin{aligned}\|Ef\|_{L^3_x(\mathbb{R}^3)} &\lesssim \|Ef\|^{1/2}_{L^2_x(\mathbb{R}^3)}\|\nabla_x Ef\|^{1/2}_{L^2_x(\mathbb{R}^3)}\\ &\lesssim \|f\|^{1/2}_{L^2_x(\Omega)}\|f\|^{1/2}_{H^1_x(\Omega)},\end{aligned}$$

which gives (2.8). For (2.9), by Young's inequality,

$$\langle\eta\rangle \lesssim \langle\eta\rangle^s\langle v\rangle^k + \langle\eta\rangle^{1+s}\langle v\rangle^{-\frac{ks}{1-s}}$$

and hence $\langle\eta\rangle$ is a symbol in $S(\langle\eta\rangle^s\langle v\rangle^k+\langle\eta\rangle^{1+s}\langle v\rangle^{-\frac{ks}{1-s}})$ (cf. [16]), where η is the Fourier variable of v. Then by [16, Lemma 2.3 and Corollary 2.5], we have

$$\|f\|_{H^1_v} \lesssim \|f\langle v\rangle^k\|_{H^s} + \|f\langle v\rangle^{-ks/(1-s)}\|_{H^{1+s}}.$$

This completes the Lemma. □

2.1.1. *Upper and lower bound for the non-cutoff Boltzmann operator.* In this section, we consider the Boltzmann collision operator $Q(f,g)$.

Lemma 2.4 ([47], Theorem 1.1)**.** *Suppose $\gamma\in(-3,1], s\in(0,1), \gamma+2s>-1$. Let $w_1,w_2\in\mathbb{R}$, $a,b\in[0,2s]$ with $w_1+w_2=\gamma+2s$ and $a+b=2s$. Then for any functions f,g,h we have*
(1) if $\gamma+2s>0$, then

$$|(Q(g,h),f)_{L^2_v}| \lesssim (\|g\|_{L^1_{\gamma+2s+(-w_1)^+ +(-w_2)^+}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}},$$

(2) if $\gamma+2s=0$, then

$$|(Q(g,h),f)_{L^2_v}| \lesssim (\|g\|_{L^1_{w_3}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}},$$

where $w_3=\max\{\delta,(-w_1)^+ +(-w_2)^+\}$, with $\delta>0$ sufficiently small.
(3) if $-1<\gamma+2s<0$, we have

$$|(Q(g,h),f)_{L^2_v}| \lesssim (\|g\|_{L^1_{w_4}} + \|g\|_{L^2_{-(\gamma+2s)}})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}$$

where $w_4=\max\{-(\gamma+2s),\gamma+2s+(-w_1)^+ +(-w_2)^+\}$.

As a first application we have the following corollaries.

Corollary 2.5. *Suppose $\gamma\in(-3,1], s\in(0,1), \gamma+2s>-1$. For any multi-indices $|\beta|\le 2$, any constant $k\ge 0$ and any functions f,g, we have*

$$(Q(\partial_\beta\mu,f),g\langle v\rangle^{2k}) \le C_k\|f\|_{H^s_{k+\gamma/2+2s}}\|g\|_{H^s_{k+\gamma/2}},$$

for some constant $C_k>0$.

Lemma 2.6 ([8], Lemma 3.3)**.** *Suppose that $-3<\gamma\le 1$. For any $k\ge 14$, and functions g,h, we have*

$$\begin{aligned}&|(Q(h,\mu),g\langle v\rangle^{2k})|\\ &\le \|b(\cos\theta)\sin^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}\|_{L^1_\theta}\|h\|_{L^2_{k+\gamma/2,*}}\|g\|_{L^2_{k+\gamma/2,*}} + C_k\|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}} \qquad (2.10)\\ &\le \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|h\|_{L^2_{k+\gamma/2,*}}\|g\|_{L^2_{k+\gamma/2,*}} + C_k\|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}},\end{aligned}$$

for some constant $C_k>0$. Moreover, for any $|\beta|\le 2$ we have

$$|(Q(h,\partial_\beta\mu),g\langle v\rangle^{2k})| \le C_k\|h\|_{L^2_{k+\gamma/2}}\|g\|_{L^2_{k+\gamma/2}}. \qquad (2.11)$$

Remark 2.7. *In [8, Lemma 3.3], the authors only prove the first statement of Lemma 2.6, but the second statement can be proved in the same way by just replacing μ by $\partial_\beta\mu$.*

Theorem 2.8 ([8], Theorem 3.1). *Suppose that $-3<\gamma\le 1, s\in(0,1), \gamma+2s>-1$, $k\ge 14$ and $G=\mu+g\ge 0$. If there exists $A_1, A_2>0$ such that*

$$G\ge 0,\quad \|G\|_{L^1}\ge A_1,\quad \|G\|_{L^1_2}+\|G\|_{L\log L}\le A_2,$$

then

$$\begin{aligned}(Q(G,f),f\langle v\rangle^{2k}) &\le -\frac{1}{8}\|b(\cos\theta)(1-\cos^{2k-3-\gamma}\frac{\theta}{2})\|_{L^1_\theta}\|f\|^2_{L^2_{k+\gamma/2,*}}-2\gamma_1\|f\|^2_{H^s_{k+\gamma/2}}\\ &+C_k\|f\|^2_{L^2_{k+\gamma/2-1/2}}+C_k\|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2}}\|f\|_{H^s_{k+\gamma/2}}+C_k\|g\|_{L^2_{14}}\|f\|^2_{H^s_{k+\gamma/2}}\\ &\le -\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{L^2_{k+\gamma/2,*}}-\gamma_1\|f\|^2_{H^s_{k+\gamma/2}}+C_k\|f\|^2_{L^2}\\ &+C_k\|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2}}\|f\|_{H^s_{k+\gamma/2}}+C_k\|g\|_{L^2_{14}}\|f\|^2_{H^s_{k+\gamma/2}},\end{aligned}\tag{2.12}$$

for some constants $\gamma_1, C_k>0$.

Remark 2.9. *In [8, Theorem 3.1], the authors prove the result for $k\ge 22$. However, the case $k\ge 14$ can also be obtained by applying the same technique. Note that whenever $\|g\|_{L^1_v}$ is small, we have $\|\mu+g\|_{L^1_v}\ge 1-\|g\|_{L^1_v}>A_1$. Moreover, we couldn't obtain a better estimate for the term $(Q(f,g),g\langle v\rangle^{2k})$ as the Landau case (see (2.19)) and thus we should put the term $\mu+g\ge 0$ together.*

Lemma 2.10 ([8], Lemma 3.4 + Lemma 2.4). *Suppose $\gamma\in(-3,1], s\in(0,1), \gamma+2s>-1$. For any functions f,g,h and $k\ge 14$, we have*

$$\begin{aligned}(Q(f,g),h\langle v\rangle^{2k})\lesssim \|f\|_{L^2_{14}}\min\{&\|g\|_{H^s_{k+\gamma/2}}\|h\|_{H^s_{k+\gamma/2+2s}},\ \|g\|_{H^s_{k+\gamma/2+2s}}\|h\|_{H^s_{k+\gamma/2}}\}\\ &+\|g\|_{L^2_{14}}\|f\|_{H^s_{k+\gamma/2}}\|h\|_{H^s_{k+\gamma/2}}.\end{aligned}\tag{2.13}$$

In particular, by duality we have

$$\|Q(f,g)\|_{H^{-s}_{k-\gamma/2}}\lesssim \|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2+2s}}+\|g\|_{L^2_{14}}\|f\|_{H^s_{k+\gamma/2}}.\tag{2.14}$$

2.1.2. *Upper and lower bound for Landau operator.* Next, we give some results about the Landau collision operator $Q(f,g)$.

Lemma 2.11. *Denote $m=\langle v\rangle^k$ with $k\ge 7$. For any $-3\le\gamma\le 1$, any functions f,g and $|\beta|\le 2$, we have*

$$|(Q(f,\partial_\beta\mu),g\langle v\rangle^{2k})|\le C_k\|f\|_{L^2_5}\|g\|_{L^2_5},\tag{2.15}$$

and

$$(Q(\mu,f),f\langle v\rangle^{2k})\le -\gamma_1\|f\|^2_{L^2_D(m)}+C_k\|f\|^2_{L^2},\tag{2.16}$$

for some constants $\gamma_1, C_k>0$.

Proof. The first estimate follows from [11, Lemma 2.5] and in [10, Lemma 2.12] with μ replacing by $\partial_\beta\mu$ if necessary. The second estimate follows from [11, Lemma 2.3] and [10, Lemma 2.7]. Note that the work [10] is for the case $\gamma\in[-2,1]$ and [11] is for $\gamma\in[-3,-2)$. □

Lemma 2.12 ([10], Lemma 3.5 and [11], Lemma 4.3). *For any $-3\le\gamma\le 1$, $k\ge 7$ and any functions f,g,h, we have*

$$|(Q(f,g),h\langle v\rangle^{2k})|\le C\|f\|_{L^2_7}\min\{\|g\|_{L^2_{D,k+1}}\|h\|_{L^2_{D,k}},\ \|g\|_{L^2_{D,k}}\|h\|_{L^2_{D,k+1}}\}.\tag{2.17}$$

By duality, we have

$$\|Q(f,g)\|_{H^{-1}_{k-\gamma/2}}\lesssim \|f\|_{L^2_7}\|g\|_{L^2_{D,k+1}}.\tag{2.18}$$

When $h=g$, we have a better estimate:

$$(Q(f,g),g\langle v\rangle^{2k})\le C\|f\|_{L^2_7}\|g\|^2_{L^2_{D,k}}.\tag{2.19}$$

Proof. The proof is similar to [10, Lemma 3.5] and [11, Lemma 4.3] with a little modification. Denoting $m = \langle v\rangle^k$, we have

$$\begin{aligned}(Q(f,g),hm^2) &= \int_{\mathbb{R}^3} \partial_j\{(\phi^{ij} * f)\partial_i g - (\partial_i\phi^{ij} * f)g\}hm^2dv\\ &= -\int_{\mathbb{R}^3}(\phi^{ij} * f)\partial_i g\partial_j hm^2dv - \int_{\mathbb{R}^3}(\phi^{ij} * f)\partial_i gh\partial_j m^2dv\\ &+ \int_{\mathbb{R}^3}(\partial_i\phi^{ij} * f)g\partial_j hm^2dv + \int_{\mathbb{R}^3}(\partial_i\phi^{ij} * f)gh\partial_j m^2dv := T_1+T_2+T_3+T_4.\end{aligned}$$

For the T_1, T_2 term, by [10, Lemma 3.5] and [11, Lemma 4.3], we have

$$|T_1| + |T_2| \le C\|f\|_{L^2_7}\|g\|_{L^2_{D,k}}\|h\|_{L^2_{D,k}}.$$

Now we give a better estimate for the T_3, T_4 term. Since $\partial_i\phi^{ij}(v-v_*) \lesssim |v-v_*|^{\gamma+1}$, $\partial_j m^2 \le C\langle v\rangle^{-1}m^2$, for the term T_3, T_4, by (2.5), we have

$$\begin{aligned}|T_3| &\le C\|f\|_{L^2_7}\min\{\|g\|_{L^2_{D,k+1}}\|h\|_{L^2_{D,k}}, \|g\|_{L^2_{D,k}}\|h\|_{L^2_{D,k+1}}\},\\ |T_4| &\le C\|f\|_{L^2_7}\|g\|_{L^2_{D,k}}\|h\|_{L^2_{D,k}},\end{aligned}$$

and (2.17) is thus proved. For the case $g = h$, we give a better estimate for T_3. In fact,

$$\begin{aligned}T_3 &= \int_{\mathbb{R}^3}(\partial_i\phi^{ij} * f)g\partial_j gm^2dv\\ &= -\frac{1}{2}\int_{\mathbb{R}^3}(\partial_{ij}\phi^{ij} * f)g^2m^2dv - \frac{1}{2}\int_{\mathbb{R}^3}(\partial_i\phi^{ij} * f)\partial_j m^2g^2dv := T_{31}+T_{32}.\end{aligned}$$

For the T_{32} term, similar to the T_4 term, by (2.5), we have

$$T_{32} \le C\|f\|_{L^2_7}\|g\|^2_{L^2_{D,k}}.$$

For the T_{31} term, if $\gamma = -3$, then $\partial_{ij}\phi^{ij} = -8\pi\delta_0$. It is easily seen that

$$T_{31} \le C\int_{\mathbb{R}^3}|f|g^2m^2dv \le C\|f\|_{L^2_{-\gamma}}\|g^2\|_{L^2_{2k+\gamma}} \le C\|f\|_{L^2_5}\|g\|^2_{L^4_{k+\gamma/2}} \le C\|f\|_{L^2_5}\|g\|^2_{L^2_{D,k}}.$$

For the case $\gamma \in [0,1]$, we have

$$T_{31} \le C\int_{\mathbb{R}^3}|v-v_*|^\gamma|f_*|g^2m^2dvdv_* \le C\|f\|_{L^1_\gamma}\|g\|^2_{L^2_{k+\gamma/2}} \le C\|f\|_{L^2_5}\|g\|^2_{L^2_{D,k}}.$$

The case $\gamma \in (-3,1]$ follows from (2.6). Collecting the above estimates on T_{31} and T_{32}, we obtain (2.19) and complete the proof of Lemma 2.12. □

For the Landau operator, we need another upper bound which writes

Lemma 2.13. *For the Landau operator Q, for any $\gamma \in [-3,1]$ and any function f,g,h, we have*

$$|(Q(f,g),h)| \le \|f\|_{L^2_5}\|g\|_{L^2}\|h\|_{H^2_5}, \quad |(Q(f,g),h)| \le \|f\|_{L^2_5}\|g\|_{L^2_{10}}\|h\|_{H^2_{-5}},$$

Proof. By integration by parts we easily have

$$\begin{aligned}(Q(f,g),h) &= \int_{\mathbb{R}^3}\Big(-(\phi^{ij} * f)\partial_j g\partial_i h + (\partial_j\phi^{ij} * f)g\partial_i h\Big)dv\\ &= \int_{\mathbb{R}^3}\Big((\phi^{ij} * f)g\partial_i\partial_j h + 2(\partial_j\phi^{ij} * f)g\partial_i h\Big)dv,\end{aligned}$$

then by the homogeneity of ϕ^{ij} and $\partial_j\phi^{ij}$, we have

$$|\phi^{ij}(v-v_*)| \lesssim |v-v_*|^{\gamma+2}, \quad |\partial_j\phi^{ij}(v-v_*)| \lesssim |v-v_*|^{\gamma+1}.$$

So the theorem follows directly from (2.3) and (2.4). □

2.2. **High-order specular boundary conditions.** When Ω is union of cubes given by (1.11), we give the high-order compatible specular boundary condition from [19, Lemma 3.1 and 3.2]. Although the proof in [19] is given for exponential decay perturbations, similar calculations can be applied to polynomial perturbations. Here we provide the proof for the sake of completeness.

Lemma 2.14. *Let (f,ϕ) be the solution to (1.4) with boundary conditions (1.13) and (1.14). Fix $i\in\{1,2,3\}$, $x\in\Gamma_i$ and $v\cdot n(x)\neq 0$. Then we have the following identities on boundary Γ_i:*

$$f(x,v)=f(x,R_xv), \tag{2.20}$$

and

$$\begin{aligned}\partial_{\tau_j}f(x,R_xv)&=\partial_{\tau_j}f(x,v),\\ \partial_{\tau_j\tau_k}f(x,R_xv)&=\partial_{\tau_j\tau_k}f(x,v),\end{aligned} \tag{2.21}$$

for $j,k=1,2$, where $\{\tau_j\}_{j=1,2}=\{e_k\}_{k\neq i}$ are the two direction other than e_i. For the normal derivatives, we have

$$\begin{aligned}\partial_n f(x,R_xv)&=-\partial_n f(x,v),\\ \partial_{\tau_j}\partial_n f(x,R_xv)&=-\partial_{\tau_j}\partial_n f(x,v),\end{aligned} \tag{2.22}$$

for $j=1,2$, and

$$\partial_n^2 f(x,R_xv)=\partial_n^2 f(x,v). \tag{2.23}$$

Proof. Note that R_xv maps γ_- onto γ_+. Then it's direct to obtain (2.20) from (1.13). On Γ_i, $\partial_{\tau_j(x)}$ $(j=1,2)$ is the derivative with direction lies in Γ_i, where $\tau_1(x)$, $\tau_2(x)$ are tangent vector on Γ_i. Then we can obtain (2.21) by taking tangent derivatives on (2.20). For normal derivatives, we will apply the equation (1.4). We claim that

$$\begin{aligned}\nabla_x\phi\cdot\nabla_v f_\pm(R_xv)&=\nabla_x\phi\cdot\nabla_v f_\pm(v),\\ \nabla_x\phi\cdot R_xv\mu(R_xv)&=\nabla_x\phi\cdot v\mu,\end{aligned} \tag{2.24}$$

and

$$Q(f,g)(R_xv)=Q(f(R_xv),g(R_xv)), \tag{2.25}$$

on $n(x)\cdot v\neq 0$, for any x belongs to the interior of Γ_i. Indeed, by (1.14), we have $\partial_{x_i}\phi=0$ on Γ_i. Notice that R_xv sends v_i to $-v_i$ and preserve the other components on Γ_i. For $j=1,2,3$ such that $j\neq i$, we have $\partial_{v_j}f_\pm(R_xv)=\partial_{v_j}f_\pm(v)$ on Γ_i. Thus, on Γ_i, we have

$$\nabla_x\phi\cdot\nabla_v f_\pm(R_xv)=\sum_{j\neq i}\partial_{x_j}\phi\,\partial_{v_j}f_\pm(R_xv)=\nabla_x\phi\cdot\nabla_v f_\pm(v).$$

and

$$\nabla_x\phi\cdot R_xv\mu(R_xv)=\sum_{j\neq i}\partial_{x_j}\phi\,v_j\mu(v)=\nabla_x\phi\cdot v\mu.$$

Next we prove (2.25). For the Boltzmann case, we apply the Carleman representation as in [1, Appendix] to find that

$$\begin{aligned}Q(f,g)(R_xv)=&\int_{\mathbb{R}^3_h}\int_{E_{0,h}}\tilde b(\alpha,h)\mathbf{1}_{|\alpha|\geq|h|}\frac{|\alpha+h|^{\gamma+1+2s}}{|h|^{3+2s}}\\ &\qquad\times\big(f(R_xv+\alpha)g(R_xv-h)-f(R_xv+\alpha-h)g(R_xv)\big)\,d\alpha dh\\ =&\,Q(f(R_xv),g(R_xv)),\end{aligned}$$

where we use change of variable $(\alpha,h)\mapsto(R_x\alpha,R_xh)$ and $E_{0,h}$ is the hyper-plane orthogonal to h containing the origin.

For Landau case, we can write

$$Q(f,g)=\partial_{v_j}\Big[\Big\{\phi^{jk}*f\Big\}\partial_{v_k}g\Big]-\partial_{v_j}\Big[\Big\{\phi^{jk}*\partial_{v_k}f\Big\}g\Big]. \tag{2.26}$$

Notice that $\partial_{v_i} f(R_x v) = -\partial_{v_i}(f(R_x v))$ and $\partial_{v_j} f(R_x v) = \partial_{v_j}(f(R_x v))$ on Γ_i, for $j \neq i$. Then on Γ_i, we have

$$
\begin{aligned}
&\sum_{j,k=1}^{3} \partial_{v_j}\Big[\Big\{\phi^{jk} * f\Big\}\partial_{v_k} g\Big](R_x v)\\
&= \sum_{k=1}^{3} \partial_{v_i}\Big[-\Big\{\phi^{ik} * f\Big\}(R_x v)\partial_{v_k} g(R_x v)\Big] + \sum_{j\neq i}\sum_{k=1}^{3} \partial_{v_j}\Big[\Big\{\phi^{jk} * f\Big\}(R_x v)\partial_{v_k} g(R_x v)\Big]\\
&= \partial_{v_i}\Big[\Big\{\phi^{ii} * [f(R_x v)]\Big\}\partial_{v_i}(g(R_x v))\Big] + \sum_{k\neq i} \partial_{v_i}\Big[\Big\{\phi^{ik} * [f(R_x v)]\Big\}\partial_{v_k}(g(R_x v))\Big]\\
&\quad + \sum_{j\neq i} \partial_{v_j}\Big[\Big\{\phi^{ji} * [f(R_x v)]\Big\}\partial_{v_i}(g(R_x v))\Big] + \sum_{j\neq i, k\neq i} \partial_{v_j}\Big[\Big\{\phi^{jk} * [f(R_x v)]\Big\}\partial_{v_k} g(R_x v)\Big]\\
&= \sum_{j,k=1}^{3} \partial_{v_j}\Big[\Big\{\phi^{jk} * [f(R_x v)]\Big\}\partial_{v_k}(g(R_x v))\Big],
\end{aligned}
$$

where we apply (1.3) to deduce that $\phi^{ik}(R_x v) = -\phi^{ik}(v)$, $\phi^{ji}(R_x v) = -\phi^{ji}(v)$ when $k \neq i$, $j \neq i$. Similar calculation can be applied to the second term of (2.26) and we obtain (2.25) for Landau case. This completes the claim (2.24) and (2.25).

Noticing that

$$v \cdot \nabla_x f = v \cdot n(x)\partial_n f + v \cdot \tau_1(x)\partial_{\tau_1} f + v \cdot \tau_2(x)\partial_{\tau_2} f,$$

we can rewrite (1.4) as

$$
\begin{aligned}
v \cdot n(x)\partial_n f_\pm = &-v \cdot \tau_1(x)\partial_{\tau_1} f_\pm - v \cdot \tau_2(x)\partial_{\tau_2} f_\pm - \partial_t f_\pm\\
&\pm \nabla_x \phi \cdot \nabla_v f_\pm \mp \nabla_x \phi \cdot v\mu + Q(2\mu + f_\pm + f_\mp, \mu + f_\pm).
\end{aligned}
$$

Applying (3.3) and (2.24), we can obtain that on $\partial\Omega$,

$$R_x v \cdot n(x)\partial_n f_\pm(R_x v) = v \cdot n(x)\partial_n f_\pm(v).$$

Since $R_x v \cdot n(x) = -v \cdot n(x)$, this implies (2.22) by taking tangent derivative. Apply ∂_n to (1.4) twice and rewrite it to be

$$
\begin{aligned}
v \cdot n\partial_n\partial_n f_\pm = &-v \cdot \tau_1(x)\partial_{\tau_1}\partial_n f_\pm - v \cdot \tau_2(x)\partial_{\tau_2}\partial_n f_\pm - \partial_t\partial_n f_\pm\\
&+ \pm\partial_n(\nabla_x\phi \cdot \nabla_v f_\pm) \mp \partial_n\nabla_x\phi \cdot v\mu + L_\pm\partial_n f + Q(f_\pm + f_\mp, f_\pm). \quad (2.27)
\end{aligned}
$$

Here, by taking tangent derivatives on (1.14), we have $\partial_n\partial_{x_j}\phi = 0$ on Γ_i for $j \neq i$ and hence,

$$\partial_n\nabla_x\phi \cdot R_x v\mu(R_x v) = \partial_n\partial_{x_i}\phi(R_x v)_i\mu(v) = -\partial_n\partial_{x_i}\phi v_i\mu.$$

Similarly, we have on Γ_i that

$$
\begin{aligned}
&\pm\partial_n(\nabla_x\phi \cdot \nabla_v f_\pm) + \partial_n Q(f_\pm + f_\mp, f_\pm)\\
&= \pm\partial_n\nabla_x\phi \cdot \nabla_v f_\pm \pm \nabla_x\phi \cdot \partial_n\nabla_v f_\pm + Q(\partial_n f_\pm + \partial_n f_\mp, f_\pm) + Q(f_\pm + f_\mp, \partial_n f_\pm)\\
&= \pm\partial_n\partial_{x_i}\phi\,\partial_{v_i} f_\pm \pm \sum_{j\neq i} \partial_{x_j}\phi\,\partial_n\partial_{v_j} f_\pm + Q(\partial_n f_\pm + \partial_n f_\mp, f_\pm) + Q(f_\pm + f_\mp, \partial_n f_\pm).
\end{aligned}
$$

Together with (2.22) and (2.25), we know that on Γ_i,

$$
\begin{aligned}
&\pm\partial_n(\nabla_x\phi \cdot \nabla_v f_\pm)(R_x v) + \partial_n Q(f_\pm + f_\mp, f_\pm)(R_x v)\\
&= \pm\partial_n(\nabla_x\phi \cdot \nabla_v f_\pm)(v) + \partial_n Q(f_\pm + f_\mp, f_\pm)(v).
\end{aligned}
$$

Combining the above identities and (2.27), we have

$$R_x v \cdot n(x)\partial_n^2 f(x, R_x v) = -v \cdot n(x)\partial_n^2 f(x, v).$$

This gives (2.23) by taking tangent derivatives. This completes the proof of Lemma 2.14. □

As a corollary, by definition (1.7), we have the following boundary values for $[a_\pm, b, c]$.

Lemma 2.15. *Let (f,ϕ) be the solution to (1.4) with boundary conditions (1.13) and (1.14). Define $[a_\pm, b, c]$ by (1.7). For $i=1,2,3$ and any $x\in\Gamma_i$, we have*

$$\partial_{x_i} c(x) = \partial_{x_i} a_\pm(x) = \partial_{x_i} b_j(x) = \partial_{x_i x_i} b_i(x) = b_i(x) = 0, \tag{2.28}$$

for $j\neq i$.

Proof. Fix $x\in\Gamma_i$. Notice that on the boundary of union of cubes, we have $\partial_n f = \partial_{x_i} f$ or $-\partial_{x_i} f$. Then by (2.22) and change of variable $v\mapsto R_x v$, we have on Γ_i that

$$\begin{aligned}\partial_{x_i} c &= \frac{1}{12}\int_{\mathbb{R}^3} \partial_{x_i}\big(f_+(x,R_x v)+f_-(x,R_x v)\big)|R_x v|^2\,dv\\ &= -\frac{1}{12}\int_{\mathbb{R}^3}\partial_{x_i}\big(f_+(x,v)+f_-(x,v)\big)|v|^2\,dv = 0.\end{aligned}$$

Similarly, on interior of Γ_i, we have

$$\partial_{x_i} a_\pm = \int_{\mathbb{R}^3}\partial_{x_i} f_\pm(x,R_x v)\,dv = -\int_{\mathbb{R}^3}\partial_{x_i} f_\pm(x,v)\,dv = 0,$$

and for $j\neq i$,

$$\begin{aligned}\partial_{x_i} b_j &= \frac{1}{2}\int_{\mathbb{R}^3}\partial_{x_i}\big(f_+(x,R_x v)+f_-(x,R_x v)\big)(R_x v)_j\,dv\\ &= -\frac{1}{2}\int_{\mathbb{R}^3}\partial_{x_i}\big(f_+(x,v)+f_-(x,v)\big)v_j\,dv = 0.\end{aligned}$$

On Γ_i, we have $(R_x v)_i = -v_i$ and hence by (2.20) and (2.23), we have

$$\begin{aligned} b_i(x) &= \frac{1}{2}\int_{\mathbb{R}^3}\big(f_+(x,R_x v)+f_-(x,R_x v)\big)(R_x v)_i\,dv\\ &= -\frac{1}{2}\int_{\mathbb{R}^3}\big(f_+(x,v)+f_-(x,v)\big)v_i\,dv = 0,\end{aligned}$$

and

$$\begin{aligned}\partial_{x_i x_i} b_i &= \frac{1}{2}\int_{\mathbb{R}^3}\big(\partial_{x_i x_i} f_+(x,R_x v)+\partial_{x_i x_i} f_-(x,R_x v)\big)(R_x v)_i\,dv\\ &= -\frac{1}{2}\int_{\mathbb{R}^3}\big(\partial_{x_i x_i} f_+(x,v)+\partial_{x_i x_i} f_-(x,v)\big)v_i\,dv = 0.\end{aligned}$$

This completes the proof. □

2.3. **Estimates on the weight.** Recall the weight function $w = w(|\alpha|,|\beta|) = w(\alpha,\beta)$ is defined in (1.16). We have the following properties of w.

Lemma 2.16. *Assume $-3<\gamma\le 1$, $1/2\le s<1, \gamma+2s>-1$ for Boltzmann case and $-3\le\gamma\le 1$ for Landau case. For any multi-indices α,β and non-negative integer k, $w(\alpha,\beta)$ satisfies the following properties.*

- *For $|\alpha_1|<|\alpha|$, $|\beta_1|<|\beta|$, we have*

$$w(\alpha,\beta)\langle v\rangle^{6s}\le w(\alpha,\beta_1),\quad w(\alpha,\beta)\langle v\rangle^{6s}\le w(\alpha_1,\beta), \tag{2.29}$$

for the Boltzmann case, and

$$w(\alpha,\beta)\langle v\rangle^{3}\le w(\alpha,\beta_1),\quad w(\alpha,\beta)\langle v\rangle^{3}\le w(\alpha_1,\beta), \tag{2.30}$$

for the Landau case.

- *For both Boltzmann and Landau case, and any $|\alpha|\ge 0$, $|\beta|\ge 1$, we have*

$$w(\alpha,\beta)\le\langle v\rangle^{\gamma-1} w(|\alpha|+1,|\beta|-1). \tag{2.31}$$

For any $|\alpha|+|\beta|\le 2$,

$$\langle v\rangle^k\le w(\alpha,\beta). \tag{2.32}$$

• *For the Boltzmann case and any* $|\alpha| \geq 1$, $|\beta| \geq 0$, *we have*

$$w(\alpha,\beta) \leq w(|\alpha|-1,|\beta|)^s w(|\alpha|-1,|\beta|+1)^{1-s}\langle v\rangle^\gamma. \tag{2.33}$$

Proof. Recall that $q = 6s - 3(\gamma{-}1)$, $p = q + \gamma{-}1$, $r = 2q + 6$, for Boltzmann case, and $q = 3 - (\gamma{-}1)$, $p = 3$, $r = 2q + 6$ for the Landau case. The estimate (2.29) and (2.30) is just from the fact that $-3 \leq \gamma \leq 1$ and

$$p = -3(\gamma{-}1) + 6s + \gamma{-}1 \geq 6s, \quad q \geq 6s,$$

for the Boltzmann case and $p, q \geq 3$ for the Landau case. The estimate (2.31) follows from the fact that

$$k - p|\alpha| - q|\beta| + r \leq k - p(|\alpha|+1) - q(|\beta|-1) + r + \gamma{-}1 \quad \Longleftrightarrow \quad q - p + \gamma{-}1 \geq 0,$$

and we conclude (2.31) from the definition of p, q. From $p \leq q$ and

$$\langle v\rangle^k = w(0,2) = \min_{|\alpha|+|\beta|\leq 2}\{w(\alpha,\beta)\},$$

we obtain (2.32). (2.33) is equivalent to

$$\begin{aligned} &k - p|\alpha| - q|\beta| + r \\ &\qquad \leq (k - p(|\alpha|-1) - q|\beta| + r)s + (k - p(|\alpha|-1) - q(|\beta|+1) + r)(1-s) + \gamma, \end{aligned}$$

which is equivalent to

$$0 \leq ps + p(1-s) - q(1-s) + \gamma \iff 0 \leq qs + (p-q) + \gamma = (6s - 3(\gamma{-}1))s + \gamma{-}1 + \gamma.$$

We thus conclude (2.33) from the definition of q, $1/2 \leq s < 1$ and $-3 < \gamma \leq 1$. □

We can directly deduce the following Corollary from Lemma 2.16.

Corollary 2.17. *Assume the same conditions as in Lemma 2.16. Then*

$$\max_{|\alpha|+|\beta|=1} w^2(\alpha,\beta)\langle v\rangle^\kappa \leq w(0,0)w(1,0), \tag{2.34}$$

and

$$\max_{|\alpha|+|\beta|=2} w^2(\alpha,\beta)\langle v\rangle^\kappa \leq w(1,0)w(2,0), \qquad \max_{\substack{|\alpha|+|\beta|=2\\ |\beta|\geq 1}} w^2(\alpha,\beta)\langle v\rangle^\kappa \leq w(0,1)w(1,1), \tag{2.35}$$

as well as

$$\max_{|\alpha|+|\beta|=2} w^2(\alpha,\beta)\langle v\rangle^\kappa \leq w^2(2,0)\langle v\rangle^\kappa \leq w(1,0)^{4/5}w(2,0)^{6/5}, \tag{2.36}$$

where $\kappa = 4s$ *for Boltzmann case,* $\kappa = 2$ *for Landau case.*

2.4. **Weighted estimates.** Next we come to the nonlinear term for the Vlasov-Poisson term. Noticing $w^2(\alpha,\beta)$ is still of the form $\langle v\rangle^k$ for some $k > 0$, we have

$$\nabla_v(w^2(\alpha,\beta)) = A_{\alpha,\beta}\frac{v}{\langle v\rangle^2}w^2(\alpha,\beta), \tag{2.37}$$

for some constant $A_{\alpha,\beta}$ depending on α, β, k, γ. Then we can apply the space-velocity weight $e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}$. To the end of this section, we will assume

$$\|\phi\|_{L^\infty_x} \leq C < +\infty, \quad \|\psi\|_{L^\infty_x} \leq C < +\infty. \tag{2.38}$$

and hence,

$$\left|e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\right| \leq C < \infty, \quad \left|e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\right| \leq C < \infty. \tag{2.39}$$

Lemma 2.18. *Suppose that $-3 \le \gamma \le 1$ for Landau case and $-3 < \gamma \le 1$ for Boltzmann case. For $|\alpha| + |\beta| \le 2$ and any functions $f_\pm$, for any function ϕ satisfies (2.38), we have*

$$\Big|\big(v\cdot\nabla_x\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta)\big)_{L^2_{x,v}} \mp (\nabla_x\phi\cdot\nabla_v\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}\Big| \lesssim \|\nabla_x\phi\|_{H^2_x}\|f\|^2_{Y_k}, \quad (2.40)$$

and for $|\beta| > 0$,

$$\sum_{|\beta_1|=1}|(\partial_{\beta_1}v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1}f, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f w^2(\alpha,\beta))_{L^2_{x,v}}| \le C\sum_{\substack{|\beta_2|=|\beta|-1\\ |\alpha_2|=|\alpha|+1}}\|\partial^{\alpha_2}_{\beta_2}fw(\alpha_2,\beta_2)\|_{L^2_xL^2_{\gamma/2}}\|\partial^\alpha_\beta fw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}. \quad (2.41)$$

For $|\alpha_1| \ge 1$, we have

$$|(\nabla_x\partial^{\alpha_1}\phi\cdot\nabla_v\partial_\beta^{\alpha-\alpha_1}f, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f w^2(\alpha,\beta))_{L^2_{x,v}}| \lesssim \|\nabla_x\phi\|_{H^3_x}\|f\|^2_{Y_k} \quad (2.42)$$

where $A_{\alpha,\beta}$ is given by (2.37), for some constant $C_k > 0$ for both Landau and Boltzmann case.

Proof. For (2.40), taking integration by parts with respect to ∇_x, we have

$$\big(v\cdot\nabla_x\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta)\big)_{L^2_{x,v}} \pm \frac{A_{\alpha,\beta}}{2}\big(\frac{v\cdot\nabla_x\phi}{\langle v\rangle^2}\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta)\big)_{L^2_{x,v}} = \int_{\partial\Omega}\int_{\mathbb{R}^3} v\cdot n(x)|e^{\frac{\pm A_{\alpha,\beta}\phi}{2\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w(\alpha,\beta)|^2\,dvdS(x).$$

For the torus case, the right hand side is equal to zero. For the case of union of cubes, we apply Lemma 2.14 to obtain $|\partial_\beta^\alpha f_\pm(v)| = |\partial_\beta^\alpha f_\pm(R_xv)|$ for $x\in\partial\Omega$ and R_x is given by (1.12). Thus, by change of variable $v\mapsto R_xv$,

$$\begin{aligned}&\int_{\partial\Omega}\int_{\mathbb{R}^3} v\cdot n(x)|e^{\frac{\pm A_{\alpha,\beta}\phi}{2\langle v\rangle^2}}\partial_\beta^\alpha f_\pm(v)w(\alpha,\beta)|^2\,dvdS(x)\\ =&\int_{\partial\Omega}\int_{\mathbb{R}^3} R_xv\cdot n(x)|e^{\frac{\pm A_{\alpha,\beta}\phi}{2\langle R_xv\rangle^2}}\partial_\beta^\alpha f_\pm(R_xv)w(\alpha,\beta)|^2\,dvdS(x)\\ =&-\int_{\partial\Omega}\int_{\mathbb{R}^3} v\cdot n(x)|e^{\frac{\pm A_{\alpha,\beta}\phi}{2\langle v\rangle^2}}\partial_\beta^\alpha f_\pm(v)w(\alpha,\beta)|^2\,dvdS(x)=0,\end{aligned} \quad (2.43)$$

where $R_xv\cdot n(x) = -v\cdot n(x)$. Taking integration by parts with respect to ∇_v and using (2.37), we have

$$\begin{aligned}&\mp(\nabla_x\phi\cdot\nabla_v\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}\\ =&\mp(\frac{A_{\alpha,\beta}v\cdot\nabla_x\phi\,\phi}{\langle v\rangle^4}\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}\\ &\pm\frac{1}{2}(\frac{A_{\alpha,\beta}v\cdot\nabla_x\phi}{\langle v\rangle^2}\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}.\end{aligned}$$

Noticing $|\frac{v}{\langle v\rangle^4}| \le \langle v\rangle^\gamma$ for $-3\le\gamma\le 1$ and taking summation of the above two estimates, we have

$$\begin{aligned}&\Big|\big(v\cdot\nabla_x\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta)\big)_{L^2_{x,v}} \mp (\nabla_x\phi\cdot\nabla_v\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}\Big|\\ \lesssim&\|\nabla_x\phi\|_{L^\infty_x}\|\langle v\rangle^{\frac{\gamma}{2}}\partial_\beta^\alpha f_\pm w(\alpha,\beta)\|_{L^2_{x,v}}\|\langle v\rangle^{\frac{\gamma}{2}}\partial_\beta^\alpha f_\pm w(\alpha,\beta)\|_{L^2_{x,v}}\\ \lesssim&\|\nabla_x\phi\|_{H^2_x}\|f\|^2_{Y_k}.\end{aligned}$$

This yields (2.40). Here we used the fact (2.38) and its consequence (2.39).

Notice that

$$\partial_i v\cdot\nabla_x\partial^\alpha_{\beta-e_i}f=\partial^{\alpha+e_i}_{\beta-e_i}f,\quad\forall\ i=1,2,3,$$

where e_i is the unit vector with i-th component being 1. Then (2.41) follows from (2.31) and (2.39).

When $1\le|\alpha_1|\le 2$, we have

$$\begin{aligned}
&(\nabla_x\partial^{\alpha_1}\phi\cdot\nabla_v\partial^{\alpha-\alpha_1}_\beta f,e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}\\
\lesssim&\Big(\sum_{|\alpha_1|=1}\|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x}\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xL^2_v}\\
&+\sum_{|\alpha_1|=2}\|\nabla_x\partial^{\alpha_1}\phi\|_{L^3_x}\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^6_xL^2_v}\Big)\|\langle v\rangle^{\frac{\gamma}{2}}\partial^\alpha_\beta fw(\alpha,\beta)\|_{L^2_xL^2_v}\\
\lesssim&\|\nabla_x\phi\|_{H^3_x}\sum_{|\alpha_1|=1}\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xL^2_v}\|f\|_{Y_k}.
\end{aligned}\tag{2.44}$$

For the Boltzmann case, (2.33) implies

$$w(\alpha,\beta)\le w(|\alpha|-1,|\beta|)^sw(|\alpha|-1,|\beta|+1)^{1-s}\langle v\rangle^\gamma,\quad\forall\ |\alpha|\ge1,|\beta|\ge0.$$

Together with (2.9), we have for $|\alpha_1|=1$ that

$$\begin{aligned}
&\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xL^2_v}\le\|\langle v\rangle^{-\frac{\gamma}{2}}\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xH^1_v}\\
\lesssim&\|\partial^{\alpha-\alpha_1}_\beta fw(|\alpha|-1,\beta)\|_{L^2_xH^s_{\gamma/2}}+\|\partial^{\alpha-\alpha_1}_\beta fw(|\alpha|-1,|\beta|+1)\|_{L^2_xH^{1+s}_{\gamma/2}}\lesssim\|f\|_{Y_k}.
\end{aligned}$$

For the Landau case, it follows from (2.30) that

$$w(\alpha,\beta)\le w(|\alpha|-1,|\beta|)\langle v\rangle^\gamma,\quad\forall\ |\alpha|\ge1,|\beta|\ge0.$$

Thus we have

$$\begin{aligned}
\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xL^2_v}&\le\|\langle v\rangle^{-\frac{\gamma}{2}}\partial^{\alpha-\alpha_1}_\beta fw(\alpha,\beta)\|_{L^2_xH^1_v}\\
&\lesssim\|\partial^{\alpha-\alpha_1}_\beta fw(|\alpha|-1,\beta)\|_{L^2_xL^2_D}\lesssim\|f\|_{Y_k}.
\end{aligned}$$

The proof of (2.42) is finished by plugging the above estimates into (2.44). □

Next, we compute the linear part for Poisson term.

Lemma 2.19. *For $|\alpha|+|\beta|\le2$, any functions $f_\pm,\phi$ satisfying (1.4) and any ψ satisfies (2.38), there exists $C_k>0$ such that*

$$|(\partial^\alpha_\beta(\nabla_x\phi\cdot v\mu),e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha_\beta f_\pm w^2(\alpha,\beta))_{L^2_{x,v}}|\le C_k\|\nabla_x\phi\|_{H^2_x}\|f\|_{H^2_xL^2_v}.\tag{2.45}$$

When $|\beta|=0$, we have

$$\begin{aligned}
\sum_\pm\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha f_\pm w^2(\alpha,\beta)\big)_{L^2_{x,v}}&\ge C_\alpha\partial_t\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x}\\
&-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_5}-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k},
\end{aligned}\tag{2.46}$$

for some $C_\alpha>0$.

Proof. From integration by parts about ∂_β, we have

$$\begin{aligned}
&(\partial^\alpha_\beta(\nabla_x\phi\cdot v\mu),e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}\\
=&(\nabla_x\partial^\alpha\phi\cdot\partial_\beta(v\mu)w^2(\alpha,\beta),e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha_\beta f)_{L^2_{x,v}}\\
=&\big((-1)^{|\beta|}\nabla_x\partial^\alpha\phi\cdot\partial_\beta\big(\partial_\beta(v\mu)e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}w^2(\alpha,\beta)\big),\partial^\alpha f\big)_{L^2_{x,v}}.
\end{aligned}\tag{2.47}$$

By Fourier transform, Cauchy-Schwarz inequality and using the exponential decay of μ, it is direct to show that (2.47) is bounded by

$$\|\nabla_x\partial^\alpha\phi\|_{L^2_x}\Big\|\int_{\mathbb{R}^d}\partial^\alpha f\partial_\beta\big(\partial_\beta(v\mu)e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}w^2(\alpha,\beta)\big)dv\Big\|_{L^2_x}\lesssim\|\nabla_x\phi\|_{H^2_x}\|f\|_{H^2_xL^2_v}.$$

When $\beta=0$, we temporarily define the projection

$$Pf=\Big(\int_{\mathbb{R}^3}f(u)\,du+v\cdot\int_{\mathbb{R}^3}uf(u)du+(|v|^2-3)\int_{\mathbb{R}^3}\frac{|u|^2-3}{6}f(u)du\Big)\mu.$$

Then we split $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}=1+e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1$ and $f_\pm=Pf+\{I-P\}f$ to obtain

$$\begin{aligned}\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}&=\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,w^2(\alpha,0)P\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,w^2(\alpha,0)\{I-P\}\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1)w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}.\end{aligned}\tag{2.48}$$

Then one can check that

$$\{I-P\}f_\pm=\{I-P\}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f.\tag{2.49}$$

For the first right hand term of (2.48), we have from (1.8) and $(1.4)_2$ that

$$\begin{aligned}&\sum_\pm\pm\big(\partial^\alpha\nabla_x\phi\cdot v\mu,w^2(\alpha,0)P\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&=\sum_{i=1}^3\Big(\partial^\alpha\nabla_x\phi\cdot v\mu,w^2(\alpha,0)v_i\mu\int_{\mathbb{R}^3}\frac{1}{2}v_i'\partial^\alpha(f_+-f_-)(v')\,dv'\Big)_{L^2_{x,v}}\\&=2C_\alpha\int_\Omega\int_{\mathbb{R}^3}\nabla_x\partial^\alpha\phi\cdot v\partial^\alpha(f_+-f_-)(v)\,dvdx\\&=-2C_\alpha\int_\Omega\partial^\alpha\phi\,\nabla_x\cdot\int_{\mathbb{R}^3}v\partial^\alpha(f_+-f_-)(v)\,dvdx\\&=-2C_\alpha\int_\Omega\partial^\alpha\phi\,\partial_t\Delta_x\phi dx=C_\alpha\,\partial_t\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x},\end{aligned}\tag{2.50}$$

for some constant $C_{\alpha,\beta}$ depending only on α,β. Here we can directly take integration by parts for the case of torus. When Ω is the union of cubes, we need to verify the zero boundary values as the following. Fix $i=1,2,3$. If $\alpha_i=1$, then $\partial_{x_i}\phi=0$ on Γ_i. If $\alpha_i=0,2$, then by Lemma 2.14 and change of variable $v\mapsto R_xv$, we have on Γ_i that

$$\int_{\mathbb{R}^3}v_i\partial^\alpha(f_+-f_-)(v)\,dv=\int_{\mathbb{R}^3}(R_xv)_i\partial^\alpha(f_+-f_-)(R_xv)\,dv=-\int_{\mathbb{R}^3}v_i\partial^\alpha(f_+-f_-)(v)\,dv=0.$$

Note that tangent derivatives doesn't affect the zero boundary values. This completes the integration by parts in (2.50). For the second right hand term of (2.48), noticing exponentially velocity decay in μ and using (2.49), we estimate it by

$$\big|\big(\partial^\alpha\nabla_x\phi\cdot v\mu,w^2(\alpha,\beta)\{I-P\}\partial^\alpha f_\pm\big)_{L^2_{x,v}}\big|\lesssim\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|_{L^2_xL^2_5}.$$

For the third term in (2.48), noticing

$$\big|e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1\big|\lesssim A_{\alpha,\beta}\|\psi\|_{L^\infty}\lesssim A_{\alpha,\beta}\|\nabla_x\psi\|_{H^1_x},$$

we have

$$\big|\big(\partial^\alpha\nabla_x\phi\cdot v\mu,(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1)w^2(\alpha,\beta)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\big|\lesssim\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k}.$$

The above four estimates imply (2.46) and we conclude Lemma 2.19.

□

In [8], one only need to compute x derivative term, but for our Vlasov-Poisson system we also need estimates about v derivative term.

Lemma 2.20. *Suppose that $\gamma \in (-3,1], s \in (0,1), \gamma + 2s > -1$ for Boltzmann case and $\gamma \in [-3,1]$ for Landau case. For any $|\alpha| \geq 0$, $|\beta| \leq 2$, $k \geq 7$, there exist constants $C_k > 0$ such that, for any functions f, g, ψ, we have*

$$\begin{aligned}
&|(\partial_\beta^\alpha Q(f,\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}| \\
\leq& \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}} \\
&+ C_k\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}} \\
&+ C_k\sum_{\beta_1<\beta}\|\partial_{\beta_1}^\alpha f w(\alpha,\beta_1)\|_{L^2_x L^2_{\gamma/2}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}},
\end{aligned}$$

for the Boltzmann case, and

$$|(\partial_\beta^\alpha Q(f,\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}| \leq C_k\sum_{\beta_1\leq\beta}\|\partial_{\beta_1}^\alpha f\|_{L^2_x L^2_5}\|\partial_\beta^\alpha g\|_{L^2_x L^2_5},$$

for the Landau case.

Proof. For both cases, it's easily seen that

$$(\partial_\beta^\alpha Q(f,\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}} = \sum_{\alpha_1\leq\alpha}(Q(\partial_{\beta_1}^\alpha f, \partial_{\beta-\beta_1}\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}.$$

For the Boltzmann case, we split it into two cases: $\beta_1 = \beta$ and $\beta_1 < \beta$. For the case $\beta_1 = \beta$, by (2.10) and integration about x, we have

$$\begin{aligned}
&|(Q(\partial_\beta^\alpha f,\mu), \partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}| \\
\leq&\|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}} \\
&+ C_k\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}.
\end{aligned}$$

Notice that

$$\left|e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\right| \leq A_{\alpha,\beta}\|\psi\|_{L^\infty_x}\langle v\rangle^{-2} \lesssim A_{\alpha,\beta}\|\nabla_x\psi\|_{H^1_x}\langle v\rangle^{-2}. \tag{2.51}$$

By (2.11) we have

$$\begin{aligned}
&|(Q(\partial_\beta^\alpha f,\mu), (e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1)\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}| \\
\leq& C_k\|\partial_\beta^\alpha f w(\alpha,\beta)\langle v\rangle^{-1}\|_{L^2_x L^2_{\gamma/2}}\|(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1)\partial_\beta^\alpha g w(\alpha,\beta)\langle v\rangle\|_{L^2_x L^2_{\gamma/2}} \\
\leq& C_k\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}.
\end{aligned}$$

For $\beta_1 < \beta$, by (2.11), (2.39) and integration about x, we have

$$\begin{aligned}
&|(Q(\partial_{\beta_1}^\alpha f, \partial_{\beta-\beta_1}\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha g w^2(\alpha,\beta))_{L^2_{x,v}}| \\
&\qquad\qquad \leq C_k\sum_{\beta_1<\beta}\|\partial_{\beta_1}^\alpha f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}}\|\partial_\beta^\alpha g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}}.
\end{aligned}$$

So the proof for the Boltzmann case is completed by gathering the three terms. The Landau case can be prove similarly by taking integration in (2.15). □

The next two coercive estimates play a key role. For the Boltzmann case, we have the following.

Lemma 2.21. *Suppose that $-3<\gamma\le 1, s\in(0,1), \gamma+2s>-1$ and $k\ge 14$. For any function f,g, let $G=\mu+g\ge 0$ satisfies*

$$\|G\|_{L^1}\ge A_1,\quad \|G\|_{L^1_2}+\|G\|_{L\log L}\le A_2.$$

for some generic constant A_1, A_2. Then for Q as the Boltzmann collision operator, and any $|\alpha|+|\beta|\le 2$, we have

$$\begin{aligned}
&(\partial^\alpha_\beta Q(\mu+g,f),\partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}\\
\le& -\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xL^2_{\gamma/2,*}}-\gamma_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xH^s_{\gamma/2}}\\
&+C_k\|\langle v\rangle^{14}f\|_{H^2_{x,v}}\|g\|_{Y_k}\|f\|_{Y_k}+C_k\|\langle v\rangle^{14}g\|_{H^2_{x,v}}\|f\|^2_{Y_k}\\
&+C_k\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}}+C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1}f w(\alpha,\beta_1)\|_{H^s_{\gamma/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{H^s_{\gamma/2}},
\end{aligned}\tag{2.52}$$

for some constant $\gamma_1, C_k\ge 0$, where $\|\cdot\|_{Y_k}$ is given in (1.21). Moreover, suppose ψ satisfies (2.38), we have

$$\begin{aligned}
\big|\big(\partial^\alpha_\beta Q(g,f),\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1\big)\partial^\alpha_\beta h w^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|&\le C_k\|\langle v\rangle^{14}g\|_{H^2_{x,v}}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k}\|h\|_{Y_k}\\
&\quad+C_k\|\langle v\rangle^{14}f\|_{H^2_{x,v}}\|\nabla_x\psi\|_{H^1_x}\|g\|_{Y_k}\|h\|_{Y_k},
\end{aligned}\tag{2.53}$$

and

$$\begin{aligned}
&\big|\big(\partial^\alpha_\beta Q(g,f),\partial^\alpha_\beta h w^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|+\big|\big(\partial^\alpha_\beta Q(g,f),e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha_\beta h w^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|\\
&\le C_k\|\langle v\rangle^{14}g\|_{H^2_{x,v}}\min\{\|\langle v\rangle^{2s}f\|_{Y_k}\|h\|_{Y_k},\|f\|_{Y_k}\|\langle v\rangle^{2s}h\|_{Y_k}\}+C_k\|\langle v\rangle^{14}f\|_{H^2_{x,v}}\|g\|_{Y_k}\|h\|_{Y_k}.
\end{aligned}\tag{2.54}$$

Remark 2.22. *To apply Lemma 2.21, one can easily check $\|G\|_{L^1_v}=\|\mu+g\|_{L^1_v}\ge A_1$ when assuming $\|g\|_{L^1_v}$ is small enough.*

Proof. First we have

$$(\partial^\alpha_\beta Q(G,f),\partial^\alpha f w^2(\alpha,\beta))_{L^2_{x,v}}=\sum_{\alpha_1\le\alpha,\beta_1\le\beta}(Q(\partial^{\alpha_1}_{\beta_1}G,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f),\partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}.$$

We split it into several cases. For the case $\alpha_1=\beta_1=0$, after integrating about x in Theorem 2.8, we have

$$\begin{aligned}
&(Q(G,\partial^\alpha_\beta f),\partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}\\
\le& -\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xL^2_{\gamma/2,*}}-\gamma_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xH^s_{\gamma/2}}+C_k\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}}\\
&+\int_\Omega C_k\|\partial^\alpha_\beta f\|_{L^2_{14}}\|g w(\alpha,\beta)\|_{H^s_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}}+C_k\|g\|_{L^2_{14}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{H^s_{\gamma/2}}dx.
\end{aligned}$$

For the case $|\beta_1|>0,\alpha_1=0$, using $\partial_{\beta_1}G=\partial_{\beta_1}\mu+\partial_{\beta_1}g$, we split it into two parts. By (2.29), if $|\beta_1|>0$, we have

$$w(\alpha,\beta)\langle v\rangle^{2s}\le w(\alpha,\beta-\beta_1).$$

Then by Corollary 2.5, we have

$$\begin{aligned}
&|(Q(\partial_{\beta_1}\mu,\partial^\alpha_{\beta-\beta_1}f),\partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}|\\
\le& C_k\|\partial^\alpha_{\beta-\beta_1}f w(\alpha,\beta)\|_{L^2_xH^s_{\gamma/2+2s}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_xH^s_{\gamma/2}}\\
\le& C_k\|\partial^\alpha_{\beta-\beta_1}f w(\alpha,\beta-\beta_1)\|_{L^2_xH^s_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_xH^s_{\gamma/2}}.
\end{aligned}$$

For the $\partial_{\beta_1} g$ term, by Lemma 2.10, we have

$$\begin{aligned}
&|(Q(\partial_{\beta_1} g, \partial^\alpha_{\beta-\beta_1} f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}| \\
\le &\int_\Omega C_k \|\partial^\alpha_{\beta-\beta_1} f\|_{L^2_{14}} \|\partial_{\beta_1} g w(\alpha,\beta)\|_{H^s_{\gamma/2}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} \\
&+ C_k \|\partial_{\beta_1} g\|_{L^2_{14}} \|\partial^\alpha_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx,
\end{aligned}$$

For $|\alpha_1| > 0$, since $\partial^{\alpha_1}\mu = 0$, we have

$$(Q(\partial^{\alpha_1}_{\beta_1} G, \partial^{\alpha-\alpha_1}_{\beta-\beta_1} f), \partial^\alpha_\beta f\langle v\rangle^{2k})_{L^2_{x,v}} = (Q(\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha-\alpha_1}_{\beta-\beta_1} f), \partial^\alpha_\beta f\langle v\rangle^{2k})_{L^2_{x,v}}.$$

By Lemma 2.10 we have

$$\begin{aligned}
&|(Q(\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha-\alpha_1}_{\beta-\beta_1} f), \partial^\alpha_\beta f\langle v\rangle^{2k})_{L^2_{x,v}}| \\
\le &\int_\Omega C_k \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f\|_{L^2_{14}} \|\partial^{\alpha_1}_{\beta_1} g w(\alpha,\beta)\|_{H^s_{\gamma/2}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_{\gamma/2}} dx \\
&+ C_k \int_\Omega \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx.
\end{aligned}$$

Gathering all the terms, we have

$$\begin{aligned}
&(\partial^\alpha_\beta Q(\mu+g, f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}} \\
\le &-\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta} \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} + C_k \|\partial^\alpha_\beta f\|^2_{L^2_{x,v}} - \gamma_1 \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x H^s_{\gamma/2}} \\
&+ C_k \sum_{|\beta_1|<|\beta|} \|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{L^2_x H^s_{\gamma/2}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \\
&+ C_k \int_\Omega \sum_{\alpha_1\le\alpha,\beta_1\le\beta} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f\|_{L^2_{14}} \|\partial^{\alpha_1}_{\beta_1} g w(\alpha,\beta)\|_{H^s_{\gamma/2}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx \\
&+ C_k \int_\Omega \sum_{\alpha_1\le\alpha,\beta_1\le\beta,|\alpha_1|+|\beta_1|>0} \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx \\
&+ C_k \int_\Omega \|g\|_{L^2_{14}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{H^s_{\gamma/2}} dx.
\end{aligned} \tag{2.55}$$

Now we only need to prove that

$$\int_\Omega \|g\|_{L^2_{14}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{H^s_{\gamma/2}} dx \lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|f\|^2_{Y_k}, \tag{2.56}$$

and for all $\alpha_1 \le \alpha$, $\beta_1 \le \beta$, $|\alpha_1| + |\beta_1| > 0$,

$$\int_\Omega \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx \lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|f\|^2_{Y_k}. \tag{2.57}$$

The fifth terms on the right hand side of (2.55) follows similarly by changing the order f and g. First, for the case $|\alpha_1| = |\beta_1| = 0$, we have

$$\begin{aligned}
\int_\Omega \|g\|_{L^2_{14}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{H^s_{\gamma/2}} dx &\lesssim \|g\|_{L^\infty_x L^2_{14}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x H^s_{\gamma/2}} \\
&\lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|f\|^2_{Y_k}.
\end{aligned}$$

This gives (2.56). For (2.57), we split it into two cases: $|\alpha_1| + |\beta_1| = 1$ and $|\alpha_1| + |\beta_1| = 2$. For the case $|\alpha_1| + |\beta_1| = 1$, by $\|fg\|_{L^2_x} \le \|f\|_{L^6_x} \|g\|_{L^3_x}$, we have

$$\begin{aligned}
&\int_\Omega \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{H^s_{\gamma/2}} dx \\
\lesssim &\|\partial^{\alpha_1}_{\beta_1} g\|_{L^6_x L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{L^3_x H^s_{\gamma/2+2s}} \|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \\
\lesssim &\|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{L^3_x H^s_{\gamma/2+2s}} \|f\|_{Y_k}.
\end{aligned} \tag{2.58}$$

We again split it into two parts $|\alpha|+|\beta|=1$ and $|\alpha|+|\beta|=2$. For the case $|\alpha_1|+|\beta_1|=|\alpha|+|\beta|=1$, we have $|\alpha-\alpha_1|=|\beta-\beta_1|=0$. Then by (2.8) and (2.34), we have

$$\|fw(\alpha,\beta)\|_{L^3_x H^s_{\gamma/2+2s}} \lesssim \|fw(0,0)\|_{L^2_x H^s_{\gamma/2}} + \|fw(1,0)\|_{H^1_x H^s_{\gamma/2}} \lesssim \|f\|_{Y_k}.$$

We then consider the case $|\alpha_1|+|\beta_1|=1$, $|\alpha|+|\beta|=2$. This time we have $|\alpha-\alpha_1|=1$, $|\beta-\beta_1|=0$ or $|\alpha-\alpha_1|=0$, $|\beta-\beta_1|=1$. For the first case, by (2.35),

$$\|\partial^{\alpha-\alpha_1} fw(\alpha,\beta)\|_{L^3_x H^s_{\gamma/2+2s}} \le \|fw(1,0)\|_{H^1_x H^s_{\gamma/2}} + \|fw(2,0)\|_{H^2_x H^s_{\gamma/2}} \lesssim \|f\|_{Y_k}.$$

For the second case, $|\beta|\ge 1$ and by (2.35), we obtain

$$\|\partial_{\beta-\beta_1} fw(\alpha,\beta)\|_{L^3_x H^s_{\gamma/2+2s}} \le \|fw(0,1)\|_{L^2_x H^{1+s}_{\gamma/2}} + \|fw(1,1)\|_{H^1_x H^{1+s}_{\gamma/2}} \lesssim \|f\|_{Y_k}.$$

For the case $|\alpha_1|+|\beta_1|=2$, it is easily seen that this time $|\alpha|+|\beta|=2$ and $|\alpha-\alpha_1|=|\beta-\beta_1|=0$. So we have

$$\begin{aligned}
&\int_\Omega \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{H^s_{\gamma/2+2s}} \|\partial^{\alpha}_{\beta} fw(\alpha,\beta)\|_{H^s_{\gamma/2}} dx \\
&\lesssim \|\partial^{\alpha}_{\beta} g\|_{L^2_x L^2_{14}} \|fw(\alpha,\beta)\|_{L^\infty_x H^s_{\gamma/2+2s}} \|\partial^{\alpha}_{\beta} fw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \\
&\lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|fw(\alpha,\beta)\|_{L^\infty_x H^s_{\gamma/2+2s}} \|f\|_{Y_k}.
\end{aligned} \tag{2.59}$$

By (2.7) with suitable k and (2.36), we have

$$\|fw(\alpha,\beta)\|_{L^\infty_x H^s_{\gamma/2+2s}} \le \|fw(1,0)\|_{H^1_x H^s_{\gamma/2}} + \|fw(2,0)\|_{H^2_x H^s_{\gamma/2}} \lesssim \|f\|_{Y_k}.$$

These estimates implies (2.57). Substituting (2.56) and (2.57) into (2.55), we obtain (2.52). For (2.53), by (2.13) and (2.51), we have

$$\begin{aligned}
&\big|\big(\partial^{\alpha}_{\beta} Q(g,f), \big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial^{\alpha}_{\beta} hw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big| \\
&\lesssim \sum_{\substack{\alpha_1\le\alpha\\ \beta_1\le\beta}} \Big(\min\Big\{ \big\| \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{H^s_{\gamma/2}} \big\|_{L^2_x} \|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\langle v\rangle^{2s}\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}}, \\
&\qquad + \big\| \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_{14}} \|\langle v\rangle^{2s}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{H^s_{\gamma/2}} \big\|_{L^2_x} \|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \Big\} \\
&\qquad + \big\| \|\partial^{\alpha_1}_{\beta_1} f\|_{L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} gw(\alpha,\beta)\|_{H^s_{\gamma/2}} \big\|_{L^2_x} \|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \Big).
\end{aligned} \tag{2.60}$$

Applying $L^\infty - L^2$ and $L^3 - L^6$ Hölder's inequality and using the first term in the minimum, the first right-hand term of (2.60) can be estimated as

$$\begin{aligned}
&\Big(\sum_{|\alpha_1|+|\beta_1|=0} \|\partial^{\alpha_1}_{\beta_1} g\|_{L^\infty_x L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{L^2 H^s_{\gamma/2}} \\
&\quad + \sum_{|\alpha_1|+|\beta_1|=1} \|\partial^{\alpha_1}_{\beta_1} g\|_{L^3_x L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{L^6_x H^s_{\gamma/2}} \\
&\quad + \sum_{|\alpha_1|+|\beta_1|=2} \|\partial^{\alpha_1}_{\beta_1} g\|_{L^2_x L^2_{14}} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} fw(\alpha,\beta)\|_{L^\infty_x H^s_{\gamma/2}} \Big) A_{\alpha,\beta} \|\nabla_x\psi\|_{H^1_x} \|\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \\
&\lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|\nabla_x\psi\|_{H^1_x} \|f\|_{Y_k} \|\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}}.
\end{aligned} \tag{2.61}$$

The second right-hand term of (2.60) can be estimated similarly and then we deduce that

$$\begin{aligned}
&\big|\big(\partial^{\alpha}_{\beta} Q(g,f), \big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial^{\alpha}_{\beta} hw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big| \\
&\lesssim \|\langle v\rangle^{14} g\|_{H^2_{x,v}} \|\nabla_x\psi\|_{H^1_x} \|f\|_{Y_k} \|\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}} \\
&\quad + \|\langle v\rangle^{14} f\|_{H^2_{x,v}} \|\nabla_x\psi\|_{H^1_x} \|g\|_{Y_k} \|\partial^{\alpha}_{\beta} hw(\alpha,\beta)\|_{L^2_x H^s_{\gamma/2}}.
\end{aligned}$$

The proof of (2.54) is similar by replacing the term $\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1\big)$ by 1 and $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}$. Note that we keep the minimum in (2.60) this time, and the proof is omitted for brevity. This completes the proof of Lemma 2.21. □

We can also prove the similar result for the Landau case.

Lemma 2.23. *Let $-3\le\gamma\le 1$, Q be the Landau collision operator and $k\ge 7$. There exists a constant $C_k>0$ such that for any function f,g,h, and any function ψ satisfies (2.38), we have the followings.*

(1) For any $|\alpha|+|\beta|\le 2$,

$$\begin{aligned}(\partial^\alpha_\beta Q(\mu,f),\partial^\alpha_\beta fw^2(\alpha,\beta))\le&-\gamma_1\|\partial^\alpha_\beta fw(\alpha,\beta)\|^2_{L^2_xL^2_D}+C_k\|\partial^\alpha_\beta f\|^2_{L^2_xL^2_v}\\&+C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1}fw(\alpha,\beta_1)\|_{L^2_xL^2_D}\|\partial^\alpha_\beta gw(\alpha,\beta)\|_{L^2_xL^2_D}.\end{aligned}\tag{2.62}$$

(2) For any $|\alpha|+|\beta|\le 2$, we have

$$\big|\big(\partial^\alpha_\beta Q(g,f),\partial^\alpha_\beta fw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|\le C_k\|g\langle v\rangle^7\|_{H^2_{x,v}}\|f\|^2_{Y_k},\tag{2.63}$$

$$\big|\big(\partial^\alpha_\beta Q(g,f),\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1\big)\partial^\alpha_\beta hw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|\le C_k\|g\langle v\rangle^7\|_{H^2_{x,v}}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k}\|h\|_{Y_k},\tag{2.64}$$

and

$$\begin{aligned}&\big|\big(\partial^\alpha_\beta Q(g,f),\partial^\alpha_\beta hw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|+\big|\big(\partial^\alpha_\beta Q(g,f),e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial^\alpha_\beta hw^2(\alpha,\beta)\big)_{L^2_{x,v}}\big|\\&\qquad\le C_k\|g\langle v\rangle^7\|_{H^2_{x,v}}\min\{\|f\|_{Y_k}\|\langle v\rangle h\|_{Y_k},\|\langle v\rangle f\|_{Y_k}\|h\|_{Y_k}\}.\end{aligned}\tag{2.65}$$

where $\|f\|^2_{Y_k}$ is given in (1.21).

Proof. Notice that

$$(\partial^\alpha_\beta Q(\mu,f),\partial^\alpha fw^2(\alpha,\beta))_{L^2_{x,v}}=\sum_{\alpha_1\le\alpha,\beta_1\le\beta}(Q(\partial_{\beta_1}\mu,\partial^\alpha_{\beta-\beta_1}f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}.$$

We split it into two cases: $\beta_1=0$ and $|\beta_1|>0$. If $\beta_1=0$, after integrating about x in (2.16), we have

$$(Q(\mu,\partial^\alpha_\beta f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}\le-\gamma_1\|\partial^\alpha_\beta fw(\alpha,\beta)\|^2_{L^2_xL^2_D}+C_k\|\partial^\alpha_\beta f\|^2_{L^2_xL^2_v}.$$

For the case $|\beta_1|>0$, by (2.30), if $|\beta_1|>0$, then

$$w(\alpha,\beta)\langle v\rangle\le w(\alpha,\beta-\beta_1).$$

Then by Lemma 2.12 we have

$$\begin{aligned}|(Q(\partial_{\beta_1}\mu,\partial^\alpha_{\beta-\beta_1}f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}|&\le C_k\|\partial^\alpha_{\beta-\beta_1}fw(\alpha,\beta)\|_{L^2_xL^2_{D,1}}\|\partial^\alpha_\beta gw(\alpha,\beta)\|_{L^2_xL^2_D}\\&\le C_k\|\partial^\alpha_{\beta-\beta_1}fw(\alpha,\beta-\beta_1)\|_{L^2_xL^2_D}\|\partial^\alpha_\beta gw(\alpha,\beta)\|_{L^2_xL^2_D},\end{aligned}$$

So (2.62) is thus finished by gathering the above two terms together. Next, for (2.63), notice that

$$(\partial^\alpha_\beta Q(g,f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}=\sum_{\alpha_1\le\alpha,\beta_1\le\beta}(Q(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}.$$

We again split it into two cases: $|\alpha_1|=|\beta_1|=0$ and $|\alpha_1|+|\beta_1|>0$. For the case $|\alpha_1|=|\beta_1|=0$, by (2.17) and (2.51), we have

$$\begin{aligned}|(Q(g,\partial^\alpha_\beta f),\partial^\alpha_\beta fw^2(\alpha,\beta))_{L^2_{x,v}}|&\le\int_\Omega C_k\|g\|_{L^2_7}\|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1\big)\langle v\rangle\partial^\alpha_\beta fw(\alpha,\beta)\|^2_{L^2_D}dx\\&\le C_k\|g\|_{L^\infty_xL^2_7}\|\nabla_x\psi\|_{H^1_x}\|\partial^\alpha_\beta fw(\alpha,\beta)\|^2_{L^2_xL^2_D}\\&\le C_k\|g\langle v\rangle^7\|_{H^2_{x,v}}\|\nabla_x\psi\|_{H^1_x}\|f\|^2_{Y_k}.\end{aligned}$$

For $\alpha_1 \le \alpha$, $\beta_1 \le \beta$, $|\alpha_1|+|\beta_1|>0$, by (2.17), we have

$$|(Q(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f), \partial_\beta^\alpha f w^2(\alpha,\beta))_{L^2_{x,v}}| \\ \le \int_\Omega C_k \|\partial_{\beta_1}^{\alpha_1} g\|_{L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_{D,1}} \|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_D} dx.$$

We split it into two cases, $|\alpha_1|+|\beta_1|=1$ and $|\alpha_1|+|\beta_1|=2$. For the case $|\alpha_1|+|\beta_1|=1$, by $\|fg\|_{L^2_x} \le \|f\|_{L^6_x}\|g\|_{L^3_x}$ and similarly as (2.58), we have

$$\int_\Omega \|\partial_{\beta_1}^{\alpha_1} g\|_{L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_{D,1}} \|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_D} dx \\ \lesssim \|\langle v\rangle^7 g\|_{H^2_{x,v}} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^3_x L^2_{D,1}} \|f\|_{Y_k}.$$

We again split it into two parts $|\alpha|+|\beta|=1$ and $|\alpha|+|\beta|=2$. For the case $|\alpha_1|+|\beta_1| = |\alpha|+|\beta|=1$, we have $|\alpha-\alpha_1|=|\beta-\beta_1|=0$. By (2.34) and (2.8) we obtain that

$$\|f w(\alpha,\beta)\|_{L^3_x L^2_{D,1}} \lesssim \|f w(0,0)\|_{L^2_x L^2_D} + \|f w(1,0)\|_{H^1_x L^2_D} \lesssim \|f\|_{Y_k}.$$

For the case $|\alpha|+|\beta|=2$, either $|\alpha-\alpha_1|=1, |\beta-\beta_1|=0$ or $|\alpha-\alpha_1|=0, |\beta-\beta_1|=1$. For the first case, by (2.35) and (2.8), we have

$$\|\partial^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^3_x L^2_{D,1}} \le \|f w(1,0)\|_{H^1_x L^2_D} + \|f w(2,0)\|_{H^2_x L^2_D} \lesssim \|f\|_{Y_k}.$$

For the second case, this time $|\beta| \ge 1$. By (2.35) and (2.8), we have

$$\|\partial_{\beta-\beta_1} f w(\alpha,\beta)\|_{L^3_x L^2_{D,1}} \le \|\nabla_v f w(0,1)\|_{L^2_x L^2_D} + \|\nabla_v f w(1,1)\|_{H^1_x L^2_D} \lesssim \|f\|_{Y_k}.$$

For the case $|\alpha_1|+|\beta_1|=2$, we have obviously $|\alpha|+|\beta|=2$ and $|\alpha-\alpha_1|=|\beta-\beta_1|=0$. Similar to (2.59), we have

$$\int_\Omega \|\partial_{\beta_1}^{\alpha_1} g\|_{L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_{D,1}} \|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_D} dx \lesssim \|\langle v\rangle^7 g\|_{H^2_{x,v}} \|f w(\alpha,\beta)\|_{L^\infty_x L^2_{D,1}} \|f\|_{Y_k}.$$

We obtain again from (2.7) and (2.36) that

$$\|f w(\alpha,\beta)\|_{L^\infty_x L^2_{D,1}} \le \|f w(1,0)\|_{H^1_x L^2_D} + \|f w(2,0)\|_{H^2_x L^2_D} \lesssim \|f\|_{Y_k}.$$

The proof of (2.63) is done after gathering the above terms. For (2.64), by (2.17) and (2.51), we have

$$\begin{aligned}
&\big|\big(\partial_\beta^\alpha Q(g,f), \big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial_\beta^\alpha h w^2(\alpha,\beta)\big)_{L^2_{x,v}}\big| \\
&\lesssim \sum_{\alpha_1\le\alpha,\,\beta\le\beta_1} \int_\Omega \|\partial_{\beta_1}^{\alpha_1} g\|_{L^2_7} \min\Big\{ \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_D} \big\|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\langle v\rangle \partial_\beta^\alpha h w(\alpha,\beta)\big\|_{L^2_D}, \quad (2.66) \\
&\qquad\qquad \|\langle v\rangle \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_D} \big\|\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)\partial_\beta^\alpha h w(\alpha,\beta)\big\|_{L^2_D}\Big\} \\
&\lesssim \Big(\sum_{|\alpha_1|+|\beta_1|=0} \|\partial_{\beta_1}^{\alpha_1} g\|_{L^\infty_x L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^2_x L^2_D} \\
&\quad + \sum_{|\alpha_1|+|\beta_1|=1} \|\partial_{\beta_1}^{\alpha_1} g\|_{L^3_x L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^6_x L^2_D} \\
&\quad + \sum_{|\alpha_1|+|\beta_1|=2} \|\partial_{\beta_1}^{\alpha_1} g\|_{L^2_x L^2_7} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha,\beta)\|_{L^\infty_x L^2_D} \Big) A_{\alpha,\beta} \|\nabla_x \psi\|_{H^1_x} \|\partial_\beta^\alpha h w(\alpha,\beta)\|_{L^2_x L^2_D} \\
&\lesssim \|\langle v\rangle^7 g\|_{H^2_{x,v}} \|\nabla_x \psi\|_{H^1_x} \|f\|_{Y_k} \|h\|_{Y_k},
\end{aligned}$$

where we apply $L^\infty - L^2$ and $L^3 - L^6$ Hölder's inequality. The proof of (2.65) is similar by replacing the term $\big(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} - 1\big)$ by 1 and $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}$. Note that this time we keep the minimum in (2.66), and we omit the proof for brevity. This completes the proof of Lemma 2.23.

□

3. Local solutions

The main goal of this section is prove the local-in-time existence and uniqueness of the equation (1.4). Define the series of functions $f^n := F^n - \mu$ by $f^0 = 0$ and

$$\begin{cases} \partial_t f_\pm^{n+1} + v\cdot\nabla_x f_\pm^{n+1} \mp \nabla_x\phi^n\cdot\nabla_v f_\pm^{n+1} \pm \nabla_x\phi^{n+1}\cdot v\mu \\ \qquad = Q(f_\pm^n + f_\mp^n, \mu) + Q(\mu + f_\pm^n, f_\pm^{n+1}) + Q(\mu + f_\mp^n, f_\pm^{n+1}), \\ -\Delta_x\phi^{n+1} = \displaystyle\int_{\mathbb{R}^3} (f_+^{n+1} - f_-^{n+1})dv, \quad \int_\Omega \phi^{n+1}(x)dx = 0, \\ f^{n+1}(0) = f_0^{n+1}, \quad E^{n+1}(0) = E_0^{n+1}. \end{cases} \tag{3.1}$$

If Ω is the union of cubes given by (1.11), then we further assume

$$f(t,x,R_x v) = f(t,x,v), \quad \text{on } \gamma_-,$$
$$\partial_n\phi = 0, \quad \text{on } \partial\Omega.$$

For the Boltzmann case, we further define

$$\|f\|_{\bar{Y}_k} := \sum_{|\alpha|+|\beta|\le 2} C_{|\alpha|,|\beta|}\|\partial_\beta^\alpha f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}}. \tag{3.2}$$

In order to solve (3.1), we first consider the following linear problem.

Lemma 3.1. *Suppose that $-3 < \gamma \le 1$, $\gamma + 2s > -1$ for Boltzmann case and $-3 \le \gamma \le 1$ for Landau case. Let k_0 be given in (1.24). There exist constants $\tau_0, T_0 > 0$ such that, for all $k \ge k_0$ (and let k sufficiently large for Boltzmann case), if $f_0 \in X_k$, $g_\pm \in L^\infty_{T_0}X_k \cap L^2_{T_0}Y_k$, $\psi \in L^\infty_{T_0}H^3_x$ satisfying $\mu + f_0 \ge 0$, $\mu + g \ge 0$ and*

$$\|f_0\|_{X_{k_0}} + \|\nabla_x\psi\|_{L^\infty_{T_0}H^3_x} + \|\partial_t\psi\|_{L^\infty_{T_0}L^\infty_x} + \|g\|_{L^\infty_{T_0}X_{k_0}} \le \tau_0, \tag{3.3}$$

suppose also ϕ, ψ satisfies (2.38), then the Cauchy problem

$$\begin{cases} \partial_t f_\pm + v\cdot\nabla_x f_\pm \mp \nabla_x\psi\cdot\nabla_v f_\pm \pm \nabla_x\phi\cdot v\mu \\ \qquad = Q(g_\pm + g_\mp, \mu) + Q(\mu + g_\pm, f_\pm) + Q(\mu + g_\mp, f_\pm), \\ -\Delta_x\phi = \displaystyle\int_{\mathbb{R}^3}(f_+ - f_-)dv, \quad \int_\Omega \phi(x)dx = 0, \\ f(0) = f_0, \quad E(0) = E_0. \end{cases} \tag{3.4}$$

admits a weak solution $f(t,x,v)$, $\phi(t,x)$ in $[0,T_0]$ satisfying

$$\mu + f \ge 0, \quad f \in L^\infty_{T_0}X_k,$$

with the energy bound

$$\|f\|^2_{L^\infty_{T_0}X_k} + \|\nabla_x\phi\|^2_{L^\infty_{T_0}H^2_x} + \|\partial_t\phi\|_{L^\infty_{T_0}L^\infty_x} + \|f\|^2_{L^2_{T_0}Y_k} \le M\|f_0\|^2_{X_k} + \frac{1}{2}\big(\|g\|^2_{L^2_{T_0}Y_k} + \|\nabla_x\psi\|^2_{L^\infty_{T_0}H^1_x}\big), \tag{3.5}$$

for some generic constant $M > 0$. If Ω is the union of cubes given by (1.11), then we consider (3.4) combined with

$$f(t,x,R_x v) = f(t,x,v), \quad \text{on } \gamma_-,$$
$$\partial_n\phi = 0, \quad \text{on } \partial\Omega.$$

Proof. For $|\alpha| + |\beta| \le 2$, we apply ∂_β^α to (3.4) and obtain

$$\begin{aligned} &\partial_t\partial_\beta^\alpha f_\pm + \partial_\beta^\alpha(v\cdot\nabla_x f_\pm) \mp \partial_\beta^\alpha(\nabla_x\psi\cdot\nabla_v f_\pm) \pm \partial_\beta^\alpha(\nabla_x\phi\cdot v\mu) \\ &\qquad = \partial_\beta^\alpha Q(g_\pm + g_\mp, \mu) + \partial_\beta^\alpha Q(\mu + g_\pm, f_\pm) + \partial_\beta^\alpha Q(\mu + g_\mp, f_\pm). \end{aligned} \tag{3.6}$$

Thus, taking inner product of (3.6) with $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm$ over $\Omega\times\mathbb{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|e^{\frac{\pm A_{\alpha,\beta}\psi}{2\langle v\rangle^2}}\partial_\beta^\alpha f_\pm\|_{L^2_{x,v}}^2+\big(A_{\alpha,\beta}\partial_t\psi\langle v\rangle^{-2}\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&+\big(v\cdot\nabla_x\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&+\sum_{|\beta_1|=1}(\partial_{\beta_1}v\cdot\nabla_x\partial_{\beta-\beta_1}^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm)_{L^2_{x,v}}\\
&\mp\big(\nabla_x\psi\cdot\nabla_v\partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&\mp\sum_{\substack{\alpha_1\le\alpha\\|\alpha_1|\ge1}}\big(\partial^{\alpha_1}\nabla_x\psi\cdot\nabla_v\partial_\beta^{\alpha-\alpha_1} f_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&\pm\big(\partial_\beta^\alpha(\nabla_x\phi\cdot v\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
=&\big(\partial_\beta^\alpha Q(g_\pm+g_\mp,\mu), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&+\big(\partial_\beta^\alpha Q(\mu+g_\pm,f_\pm)+\partial_\beta^\alpha Q(\mu+g_\mp,f_\pm), w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}\\
&+\big(\partial_\beta^\alpha Q(\mu+g_\pm,f_\pm)+\partial_\beta^\alpha Q(\mu+g_\mp,f_\pm), (e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}-1) w^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{x,v}}.
\end{aligned}
\tag{3.7}
$$

We denote the second to tenth terms in (3.7) by I_1 to I_9 and estimate them one by one. For I_1, we make a rough estimate:

$$
|I_1|\lesssim\|\partial_t\psi\|_{L^\infty_x}\|f\|_{X_k}^2. \tag{3.8}
$$

Notice that $\|\psi\|_{L^\infty_x}\lesssim\|\nabla_x\psi\|_{H^1_x}\le\tau_0$, which verify (2.38) and we can apply estimates in Section 2.4. By (2.40), we deduce that

$$
|I_2+I_4|\lesssim\|\nabla_x\psi\|_{H^2_x}\|f\|_{Y_k}^2.
$$

Applying (2.41), when $|\beta|\ge1$, we have

$$
|I_3|\lesssim C_\eta\sum_{\substack{|\beta_2|=|\beta|-1\\|\alpha_2|=|\alpha|+1}}\|\partial_{\beta_2}^{\alpha_2} f w(\alpha_2,\beta_2)\|_{L^2_xL^2_{\gamma/2}}^2+\eta\|\partial_\beta^\alpha f w(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}^2,
$$

for any $\eta>0$. By (2.42), we have

$$
|I_5|\lesssim\|\nabla_x\psi\|_{H^3_x}\|f\|_{Y_k}^2.
$$

When $|\beta|=0$, using (2.46), we have

$$
\begin{aligned}
I_6&\ge C_\alpha\partial_t\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_5}-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k}\\
&\ge C_\alpha\partial_t\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-\eta\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-\eta\|w(\alpha,0)\partial^\alpha f\|_{L^2_xL^2_{\gamma/2}}^2\\
&\quad-C_\eta\|\partial^\alpha f\|_{L^2_{x,v}}^2-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k},
\end{aligned}
$$

for any $\eta>0$. When $|\beta|\ge1$, we have from (2.45) that

$$
|I_6|\le C_k\|\nabla_x\phi\|_{H^2_x}\|f\|_{H^2_xL^2_v}. \tag{3.9}
$$

For I_7, since $\theta\in[0,\frac{\pi}{2}]$, we have $\sin^2\frac{\theta}{2}\le\frac{1}{2}$ and hence if $k\ge14$ we have

$$
\frac{1}{2}\|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_k}^2\le\frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_k}^2. \tag{3.10}
$$

applying Lemma 2.20, we have for any $\eta > 0$ that

$$\begin{aligned}
|I_7| \le\ & \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}} \\
& + \eta\|f\|^2_{Y_k} + \eta\|g\|^2_{Y_k} + C_\eta\|f\|_{H^2_{x,v}} \\
& + C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} g w(\alpha,\beta_1)\|_{L^2_x L^2_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}} \\
\le\ & \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} \\
& + \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta g w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} + \eta\|f\|^2_{Y_k} + \eta\|g\|^2_{Y_k} + C_\eta\|f\|_{H^2_{x,v}} \\
& + C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} g w(\alpha,\beta_1)\|_{L^2_x L^2_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}},
\end{aligned}$$

for Boltzmann case. For Landau case, it follows from Lemma 2.20 that

$$|I_7| \le C_k\sum_{|\alpha|+|\beta|\le 2}\|\partial^\alpha_\beta g\|_{L^2_x L^2_5}\|\partial^\alpha_\beta f\|_{L^2_x L^2_5} \le \eta\|f\|^2_{Y_k} + \eta\|g\|^2_{Y_k} + C_\eta\|f\|^2_{H^2_{x,v}},$$

For I_8, we deduce from (2.52) that

$$\begin{aligned}
\sum_{\pm} I_8 \le\ & -\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} - \gamma_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x H^s_{\gamma/2}} \\
& + C_k\|\langle v\rangle^{14} f\|_{H^2_{x,v}}\|g\|_{Y_k}\|f\|_{Y_k} + C_k\|\langle v\rangle^{14} g\|_{H^2_{x,v}}\|f\|^2_{Y_k} \\
& + C_k\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}} + C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{H^s_{\gamma/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{H^s_{\gamma/2}},
\end{aligned}$$

for Boltzmann case. Note that and from (2.62) and (2.63),

$$\begin{aligned}
\sum_{\pm} I_8 \le\ & -\gamma_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_D} + C_k\|\partial^\alpha_\beta f\|^2_{L^2_x L^2_v} \\
& + C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{L^2_x L^2_D}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_D} + C_k\|g\langle v\rangle^7\|_{H^2_{x,v}}\|f\|^2_{Y_k},
\end{aligned}$$

for Landau case, with some constants $\gamma_1, C_k > 0$. For I_9, we have from (2.53) that

$$|I_9| \le C_k\big(\|\langle v\rangle^{14} g\|_{H^2_{x,v}} + C\big)\|\nabla_x\psi\|_{H^1_x}\|f\|^2_{Y_k} + \|\langle v\rangle^{14} f\|_{H^2_{x,v}}\|\nabla_x\psi\|_{H^1_x}\big(\|g\|_{Y_k} + C\big)\|f\|_{Y_k},$$

for Boltzmann case and, from (2.64) that

$$|I_9| \le C_k\big(\|\langle v\rangle^7 g\|_{H^2_{x,v}} + C\big)\|\nabla_x\psi\|_{H^1_x}\|f\|^2_{Y_k},$$

for Landau case. Combining the above estimates, we take summation $\sum_{|\alpha|+|\beta|\le 2,\pm} C_{|\alpha|,|\beta|}\times$(3.7) with $0 < \eta \ll 1$ and constants $C_{|\alpha|,|\beta|}$ satisfying (1.19) to deduce that for the Boltzmann case:

$$\begin{aligned}
& \frac{1}{2}\partial_t\|f\|^2_{X_k} + \partial_t\sum_{|\alpha|\le 2} C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x} + \frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{Y_k} + \gamma_1\|f\|^2_{Y_k} \\
\le\ & \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{Y_k} + \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|_{L^1_\theta}\|g\|^2_{Y_k} + \eta\|g\|^2_{Y_k} + C_{\eta,k}\|f\|^2_{H^2_{x,v}} \\
& + C\|\partial_t\psi\|_{L^\infty_x}\|f\|^2_{X_k} + C_k\Big(\|\nabla_x\psi\|_{H^3_x}\|f\|^2_{Y_k} + \|\nabla_x\phi\|^2_{H^2_x} + \|f\|_{X_{14}}\|g\|_{Y_k}\|f\|_{Y_k} + \|g\|_{X_{14}}\|f\|^2_{Y_k} \\
& + \big(\|g\|_{X_{14}} + C\big)\|\nabla_x\psi\|_{H^1_x}\|f\|^2_{Y_k} + \|f\|_{X_{14}}\|\nabla_x\psi\|_{H^1_x}\big(\|g\|_{Y_k} + C\big)\|f\|_{Y_k}\Big),
\end{aligned} \tag{3.11}$$

and for the Landau case:

$$\frac{1}{2}\partial_t\|f\|^2_{X_k} + \partial_t\sum_{|\alpha|\le 2} C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x} + +\gamma_1\|f\|^2_{Y_k}$$

$$\le \eta\|g\|_{Y_k}^2 + C_{\eta,k}\|f\|_{H^2_{x,v}}^2 + C\|\partial_t\psi\|_{L^\infty_x}\|f\|_{X_k}^2 + C_k\Big(\|\nabla_x\psi\|_{H^3_x}\|f\|_{Y_k}^2$$
$$+ \|\nabla_x\phi\|_{H^2_x}^2\|f\|_{X_7}\|g\|_{Y_k}\|f\|_{Y_k} + \|g\|_{X_7}\|f\|_{Y_k}^2$$
$$+ \big(\|g\|_{X_7}+C\big)\|\nabla_x\psi\|_{H^1_x}\|f\|_{Y_k}^2 + \|f\|_{X_7}\|\nabla_x\psi\|_{H^1_x}\big(\|g\|_{Y_k}+C\big)\|f\|_{Y_k}\Big), \tag{3.12}$$

for some generic constant γ_1 independent of k, where we let $k \ge 14$. Here $\|f\|_{\bar{Y}_k}$ is defined by (3.2) and $\|f\|_{X_k}$ is given by

$$\|f\|_{X_k}^2 := \sum_{|\alpha|+|\beta|\le 2,\,\pm} C_{|\alpha|,|\beta|}\|e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}}\partial_\beta^\alpha f_\pm\|_{L^2_{x,v}}^2.$$

Next we estimate the term $\|\nabla_x^3\phi\|_{L^\infty_T L^2_x}^2$. Using $(3.4)_2$, we have

$$\|\nabla_x^3\phi\|_{L^2_x}^2 \lesssim \|\nabla_x\Delta_x\phi\|_{L^2_x}^2 \lesssim \|\nabla_x f\|_{L^2_x L^2_4}^2 \lesssim \|f\|_{X_{k_0}}^2. \tag{3.13}$$

Here, for the case of union of cubes, we apply embedding [15, Theorem 6.7-5] with boundary values $\partial_{x_i}\phi = 0$ on Γ_i to obtain (3.13). Taking difference of (1.8) over $\pm$, we have

$$\partial_t(a_+ - a_-) = -\nabla_x\cdot\int_{\mathbb{R}^3} v(f_+ - f_-)\,dv,$$

and hence,

$$\|\partial_t\phi\|_{L^\infty_x} \lesssim \|\nabla_x\partial_t\phi\|_{L^2_x}^{1/2}\|\nabla_x\partial_t\phi\|_{H^1_x}^2 \lesssim \|\nabla_x^2\partial_t\phi\|_{L^2_x} \lesssim \|\Delta_x\partial_t\phi\|_{L^2_x}$$
$$\lesssim \|\partial_t(a_+ - a_-)\|_{L^2_x} \lesssim \|\nabla_x f\|_{L^2_x L^2_3} \lesssim \|f\|_{X_{k_0}}. \tag{3.14}$$

Therefore, for both Boltzmann and Landau case, taking integration of (3.11) and (3.12) over $t\in[0,T]$, we have

$$\frac12\|f\|_{L^\infty_T X_k}^2 + \lambda\|\nabla_x\phi\|_{L^\infty_T H^2_x}^2 + \gamma_1\|f\|_{L^2_T Y_k}^2 + \frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{L^2_T\bar{Y}_k}^2$$
$$\le \frac12\|f_0\|_{X_k}^2 + C_{\eta,k}\|f\|_{L^2_T H^2_{x,v}}^2 + C\|\partial_t\psi\|_{L^\infty_T L^\infty_x}\|f\|_{L^2_T X_k}^2 + C_k\Big(\|\nabla_x\phi\|_{L^2_T H^2_x}^2$$
$$+ \big(\|\nabla_x\psi\|_{L^\infty_T H^3_x} + \|g\|_{L^\infty_T X_{k_0}} + C_\eta\|f\|_{L^\infty_T X_{k_0}}^2 + \|g\|_{L^\infty_T X_{k_0}}\|\nabla_x\psi\|_{L^\infty_T H^1_x}\big)\|f\|_{L^2_T Y_k}^2$$
$$+ \eta\|g\|_{L^2_T Y_k}^2 + \|\nabla_x\psi\|_{L^\infty_T H^1_x}\|g\|_{L^2_T Y_k}^2 + \|\nabla_x\psi\|_{L^2_T H^1_x}^2\Big) + \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|g\|_{L^2_T\bar{Y}_k}^2,$$

for the Landau case, taking integration of (3.11) and (3.12) over $t\in[0,T]$ we have

$$\frac12\|f\|_{L^\infty_T X_k}^2 + \lambda\|\nabla_x\phi\|_{L^\infty_T H^2_x}^2 + \gamma_1\|f\|_{L^2_T Y_k}^2$$
$$\le \frac12\|f_0\|_{X_k}^2 + C_{\eta,k}\|f\|_{L^2_T H^2_{x,v}}^2 + C\|\partial_t\psi\|_{L^\infty_T L^\infty_x}\|f\|_{L^2_T X_k}^2 + C_k\Big(\|\nabla_x\phi\|_{L^2_T H^2_x}^2$$
$$+ \big(\|\nabla_x\psi\|_{L^\infty_T H^3_x} + \|g\|_{L^\infty_T X_{k_0}} + C_\eta\|f\|_{L^\infty_T X_{k_0}}^2 + \|g\|_{L^\infty_T X_{k_0}}\|\nabla_x\psi\|_{L^\infty_T H^1_x}\big)\|f\|_{L^2_T Y_k}^2$$
$$+ \eta\|g\|_{L^2_T Y_k}^2 + \|\nabla_x\psi\|_{L^\infty_T H^1_x}\|g\|_{L^2_T Y_k}^2 + \|\nabla_x\psi\|_{L^2_T H^1_x}^2\Big), \tag{3.15}$$

for some generic constant $\lambda > 0$ independent of k,η. Assume the *a priori* assumption corresponding to (3.15) to be

$$\|f\|_{L^\infty_T X_{k_0}}^2 + \|\nabla_x\phi\|_{L^\infty_T H^2_x}^2 \le \varepsilon_0, \tag{3.16}$$

with some small $\varepsilon_0 > 0$. Noticing $\|\cdot\|_{L^2_T} \le T\|\cdot\|_{L^\infty_T}$, choosing $0 < \tau_0 = \tau_0(\eta) < \eta$ in (3.3), $\varepsilon_0 = \varepsilon_0(\eta) > 0$ in (3.16) and $0 < T = T(\eta,k) < \eta$ sufficiently small, for the Landau case, we obtain from (3.15) that

$$\frac14\|f\|_{L^\infty_T X_k}^2 + \frac{\lambda}{2}\|\nabla_x\phi\|_{L^\infty_T H^3_x}^2 + \frac{\gamma_1}{2}\|f\|_{L^2_T Y_k}^2$$
$$\le \frac12\|f_0\|_{X_k}^2 + \eta\|g\|_{L^2_T Y_k}^2 + \|\nabla_x\psi\|_{L^\infty_T H^1_x}\|g\|_{L^2_T Y_k}^2 + \eta\|\nabla_x\psi\|_{L^\infty_T H^1_x}^2,$$

where we used the fact that $\|f\|_{L^2_T H^2_{x,v}} \le T\|f\|_{L^\infty_T H^2_{x,v}}$ and $\|\phi\|_{L^2_T H^2_x} \le T\|\phi\|_{L^\infty_T H^2_x}$. Now we choose $\eta = \eta(\lambda, \gamma_1) > 0$ sufficiently small, which is independent of $k, T, \tau_0, \varepsilon_0$. Then we have

$$\|f\|^2_{L^\infty_T X_k} + \|\nabla_x \phi\|^2_{L^\infty_T H^3_x} + \|f\|^2_{L^2_T Y_k} \le M\|f_0\|^2_{X_k} + \frac{1}{2}\big(\|g\|^2_{L^2_T Y_k} + \|\nabla_x \psi\|^2_{L^\infty_T H^1_x}\big), \tag{3.17}$$

for some generic constant $M > 0$ depends only on λ, γ_1. Choosing τ_0 in (3.3) small enough and noticing $k \ge k_0$, we close the *a priori* estimate (3.16) and deduce (3.5). Therefore, it's standard to apply the *a priori* arguments to obtain a solution to (3.4) in time interval $[0, T_0]$ for some small $T_0 > 0$. For the Boltzmann case we have

$$\begin{aligned}
&\frac{1}{4}\|f\|^2_{L^\infty_T X_k} + \frac{\lambda}{2}\|\nabla_x \phi\|^2_{L^\infty_T H^3_x} + \frac{\gamma_1}{2}\|f\|^2_{L^2_T Y_k} + \frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{L^2_T \bar{Y}_k} \\
&\le \frac{1}{2}\|f_0\|^2_{X_k} + \frac{1}{32}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|g\|^2_{L^2_T \bar{Y}_k} + \eta\|g\|^2_{L^2_T Y_k} + \|\nabla_x \psi\|^2_{L^\infty_T H^1_x}\|g\|^2_{L^2_T Y_k} + \eta\|\nabla_x \psi\|^2_{L^\infty_T H^1_x}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|f\|^2_{L^\infty_T X_k} + \|\nabla_x \phi\|^2_{L^\infty_T H^3_x} + \|f\|^2_{L^2_T Y_k} + \frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{L^2_T \bar{Y}_k} \\
\le& M\|f_0\|^2_{X_k} + \frac{1}{2}\big(\|g\|^2_{L^2_T Y_k} + \|\nabla_x \psi\|^2_{L^\infty_T H^1_x} + \frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|g\|^2_{L^2_T \bar{Y}_k}\big),
\end{aligned}$$

Thus the Boltzmann case can be proved similarly.

We next prove the non-negativity for Boltzmann case. The Landau case can be proved by using the same method, thus we omit the proof for brevity. Set

$$F := \mu + f, \quad G := \mu + g \ge 0$$

and assume $F|_{t=0} \ge 0$. Then (3.4) becomes

$$\begin{aligned}
&\partial_t F_\pm + v \cdot \nabla_x F_\pm \mp \nabla_x \psi \cdot \nabla_v F_\pm = Q(G_\pm, F_\pm) + Q(G_\mp, F_\pm), \\
&-\Delta_x \phi = \int_{\mathbb{R}^3} (F_+ - F_-)\, dv, \quad \int_\Omega \phi(t,x) dx = 0, \\
&F(0) = F_0 \ge 0, \quad E(0) = E_0.
\end{aligned}$$

Denote $F^\pm := \pm\max\{\pm F, 0\}$ and $\beta(s) = \frac{1}{2}(s^-)^2$ with $s^- := \min\{s, 0\}$. Notice that

$$\frac{d\beta}{ds}(F) = F^-, \quad F^-|_{t=0} = 0,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q(G_\pm, F_\pm) F^-_\pm \langle v\rangle^{2k} dv \\
=& \int_{\mathbb{R}^3} Q(G_\pm, F^-_\pm) F^-_\pm \langle v\rangle^{2k} dv + \int_{\mathbb{R}^3} Q(G_\pm, F^+_\pm) F^-_\pm \langle v\rangle^{2k} dv \\
=& \int_{\mathbb{R}^3} Q(G_\pm, F^-_\pm) F^-_\pm \langle v\rangle^{2k} dv + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B (G_\pm)'_* (F^+_\pm)' F^-_\pm \langle v\rangle^{2k} dv_* dv d\sigma \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B (G_\pm)_* F^+_\pm F^-_\pm \langle v\rangle^{2k} dv_* dv d\sigma \\
\le& \int_{\mathbb{R}^3} Q(G_\pm, F^-_\pm) F^-_\pm \langle v\rangle^{2k} dv.
\end{aligned}$$

Note that $F^+_\pm F^-_\pm = 0$ and $(F^+_\pm)' F^-_\pm \le 0$. Similarly, one can obtain

$$\int_{\mathbb{R}^3} Q(G_\mp, F_\pm) F^-_\pm \langle v\rangle^{2k} dv \le \int_{\mathbb{R}^3} Q(G_\mp, F^-_\pm) F^-_\pm \langle v\rangle^{2k} dv.$$

Then we have

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} \beta(F_\pm)\langle v\rangle^{2k} dv dx = -(v \cdot \nabla_x F_\pm, F^-_\pm \langle v\rangle^{2k})_{L^2_{x,v}} \pm (\nabla_x \psi \cdot \nabla_v F_\pm, F^-_\pm \langle v\rangle^{2k})_{L^2_{x,v}}$$

$$
\begin{aligned}
&+ (Q(G_\pm, F_\pm), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} + (Q(G_\mp, F_\pm), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} \\
&= \pm(\nabla_x \psi \cdot \nabla_v F_\pm, F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} + (Q(G_\pm, F_\pm^-), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} \\
&\quad + (Q(G_\mp, F_\pm^-), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}}. \qquad (3.18)
\end{aligned}
$$

Notice that for the case of union of cubes, similar to (2.43), $(v \cdot \nabla_x F_\pm^-, F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} = 0$ by using change of variable $v \mapsto R_x v$. For the first term we have

$$
\begin{aligned}
(\nabla_x \psi \cdot \nabla_v F_\pm, F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} &= -2k(\nabla_x \psi \cdot v \langle v\rangle^{2k-2}, (F_\pm^-)^2)_{L^2_{x,v}} \\
&\le C_k \|\nabla_x \psi\|_{L^\infty_x} \||v|\langle v\rangle^{k-2} F_\pm^-\|_{L^2_x L^2_v} \|\langle v\rangle^k F_\pm^-\|_{L^2_x L^2_v} \\
&\le C_k \|\nabla_x \psi\|_{H^2_x} \|F_\pm^-\|^2_{L^2_x L^2_k}.
\end{aligned}
$$

For the second term, we deduce from (2.12) that

$$
\begin{aligned}
&(Q(G_\pm, F_\pm^-), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} \\
&\le \int_\Omega -\gamma_1 \|F_\pm^-\|^2_{H^s_{k+\gamma/2}} + C_k \|F_\pm^-\|^2_{L^2} + C_k \|F_\pm^-\|_{L^2_{14}} \|g_\pm\|_{H^s_{k+\gamma/2}} \|F_\pm^-\|_{H^s_{k+\gamma/2}} \\
&\qquad\qquad + C_k \|g_\pm\|_{L^2_{14}} \|F_\pm^-\|^2_{H^s_{k+\gamma/2}} dx \\
&\le \int_\Omega -\big(\frac{\gamma_1}{2} - C_k \|g_\pm\|_{L^2_{14}}\big) \|F_\pm^-\|^2_{H^s_{k+\gamma/2}} + C_k \|F_\pm^-\|^2_{L^2_v} + C_k \|F_\pm^-\|^2_{L^2_{14}} \|g_\pm\|^2_{H^s_{k+\gamma/2}} dx \\
&\le -\big(\frac{\gamma_1}{2} - C_k \|g_\pm\|_{L^\infty_x L^2_{14}}\big) \|F_\pm^-\|^2_{L^2_x H^s_{k+\gamma/2}} + C_k \|F_\pm^-\|^2_{L^2_{x,v}} + C_k \|F_\pm^-\|^2_{L^2_x L^2_{14}} \|g_\pm\|^2_{L^\infty_x H^s_{k+\gamma/2}}.
\end{aligned}
$$

Choosing $\tau_0 > 0$ in (3.3) small enough that

$$
\|g_\pm\|_{L^\infty_x L^2_{14}} \le C \|g\|_{L^\infty_T X_{k_0}} \le C\tau_0 \le \frac{\gamma_1}{2C_k}.
$$

Thus,

$$
(Q(G_\pm, F_\pm^-), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} \le C_k \|F_\pm^-\|^2_{L^2_x L^2_k} + C_k \|F_\pm^-\|^2_{L^2_x L^2_k} \|g_\pm\|^2_{Y_k},
$$

and similarly,

$$
(Q(G_\mp, F_\pm^-), F_\pm^- \langle v\rangle^{2k})_{L^2_{x,v}} \le C_k \|F_\pm^-\|^2_{L^2_x L^2_k} + C_k \|F_\pm^-\|^2_{L^2_x L^2_k} \|g_\pm\|^2_{Y_k}.
$$

Plugging the above estimates into (3.18), we obtain

$$
\frac{d}{dt} \|F_\pm^-\|^2_{L^2_x L^2_k} \le 2C_k (1 + \|g_\pm\|^2_{Y_k} + \|\nabla_x \psi\|_{H^1_x}) \|F_\pm^-\|^2_{L^2_x L^2_k}.
$$

Recalling that $\nabla_x \psi \in L^2_{T_0} H^1_x$ and $g \in L^2_{T_0} Y_k$, the Grönwall's inequality implies $F_\pm^- = 0$ on $[0, T_0]$. We then deduce that $F \ge 0$ on $[0, T_0]$. □

We finish this section by proving the theorem below.

Theorem 3.2. *Suppose that $-3 < \gamma \le 1$ for Boltzmann case and $-3 \le \gamma \le 1$ for Landau case. For any $k \ge k_0 + 2$ (and let k sufficiently large for Boltzmann case), there exist small constants $\varepsilon_0, \tau_0, T_0 > 0$, such that if $f_0 \in X_{k_0}$ satisfies $\mu + f_0 \ge 0$ and*

$$
\mathcal{E}_{k_0}(0) \le \varepsilon_0, \quad \mathcal{E}_k(0) \le A, \qquad (3.19)
$$

for some $A > 0$, then the Cauchy problem

$$
\begin{cases}
\partial_t f_\pm + v \cdot \nabla_x f_\pm \mp \nabla_x \phi \cdot \nabla_v f_\pm \pm \nabla_x \phi \cdot v\mu \\
\qquad = Q(f_\pm + f_\mp, \mu) + Q(\mu + f_\pm, f_\pm) + Q(\mu + g_\mp, f_\pm), \\
-\Delta_x \phi = \displaystyle\int_{\mathbb{R}^3} (f_+ - f_-) dv, \quad \int_\Omega \phi(x) dx = 0, \\
f(0) = f_0, \quad E(0) = E_0, \\
f(t, x, R_x v) = f(t, x, v) \text{ on } \gamma_- \text{ and } \partial_n \phi = 0 \text{ on } \partial\Omega, \text{ if } \Omega \text{ is given by (1.11)},
\end{cases} \qquad (3.20)
$$

admits a unique weak solution $f \in L^\infty_{T_0} X_k$ *satisfying* $\mu + f \geq 0$ *and*

$$\sup_{0\leq t\leq T_0} \mathcal{E}_{k_0}(t) \leq \tau_0, \qquad \sup_{0\leq t\leq T_0} \mathcal{E}_k(t) \leq A. \tag{3.21}$$

Proof. We only prove the Boltzmann case and the Landau case can be proved similarly. We recall the sequence of approximate solutions defined in (3.1) with $f^0 = 0$. Applying Lemma 3.1 by taking $f = f^{n+1}$, $\phi = \phi^{n+1}$, $g = f^n$ and $\psi = \phi^n$, choosing ε_0 in (3.19) such that

$$2\sqrt{M}\varepsilon_0 \leq \tau_0,$$

where M is defined in Lemma 3.1, we have $\mu + f^n \geq 0$ and

$$\|f^n\|_{L^\infty_{T_0} X_k} + \|f^n\|_{L^2_{T_0} Y_k} \leq \tau_0. \tag{3.22}$$

Then there exists a weak limit $f \in L^\infty_{T_0} X_k \cap L^2_{T_0} Y_k$ such that $f_n \rightharpoonup f$ weakly in $L^2_{T_0} Y_k$. Then by (3.22), f satisfies (3.21). We next prove the strong convergence of the sequence of f^n. Define $h^n := f^{n+1} - f^n$. Then we have from (3.1) that

$$\begin{cases} \partial_t h^n_\pm + v\cdot\nabla_x h^n_\pm \mp \nabla_x\phi^n\cdot\nabla_v h^n_\pm \mp \nabla_x(\phi^n-\phi^{n-1})\cdot\nabla_v f^n_\pm \pm \nabla_x(\phi^{n+1}-\phi^n)\cdot v\mu \\ \qquad = Q(h^{n-1}_\pm + h^{n-1}_\mp, \mu+f^n) + Q(\mu+f^n_\pm, h^n_\pm) + Q(\mu+f^n_\mp, h^n_\pm), \\ -\Delta_x\phi^{n+1} = \displaystyle\int_{\mathbb{R}^3} (f^{n+1}_+ - f^{n+1}_-)dv, \quad -\Delta_x\phi^n = \int_{\mathbb{R}^3}(f^n_+ - f^n_-)dv, \quad h^n(0) = 0, \\ f^{n+1}(t,x,R_x v) = f^{n+1}(t,x,v) \text{ on } \gamma_- \text{ and } \partial_n\phi^{n+1} = 0 \text{ on } \partial\Omega, \text{ if } \Omega \text{ is given by (1.11).} \end{cases} \tag{3.23}$$

Let $|\alpha| + |\beta| \leq 2$. We apply similar arguments in Lemma 3.1 by applying ∂^α_β to $(3.23)_1$ and taking inner product of the resultant equation with $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\langle v\rangle^{-4}\partial^\alpha_\beta h^n_\pm$ over $\Omega\times\mathbb{R}^3$. Here we illustrate the different terms:

$$\big(\partial^\alpha_\beta(\nabla_x(\phi^n-\phi^{n-1})\cdot\nabla_v f^n_\pm), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\langle v\rangle^{-4}\partial^\alpha_\beta h^n_\pm\big)_{L^2_{x,v}}, \tag{3.24}$$

and

$$\big(\partial^\alpha_\beta Q(h^{n-1}_\pm + h^{n-1}_\mp, f^n), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\langle v\rangle^{-4}\partial^\alpha_\beta h^n_\pm\big)_{L^2_{x,v}}. \tag{3.25}$$

Since $s \geq \frac{1}{2}$, we estimate (3.24) by

$$\begin{aligned} &\Big|\sum_{\alpha_1\leq\alpha}\big(\partial^{\alpha_1}\nabla_x(\phi^n-\phi^{n-1})\cdot\partial^{\alpha-\alpha_1}_\beta\nabla_v f^n_\pm, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\langle v\rangle^{-4}\partial^\alpha_\beta h^n_\pm\big)_{L^2_{x,v}}\Big| \\ \lesssim &\int_\Omega |\partial^{\alpha_1}\nabla_x(\phi^n-\phi^{n-1})|| \underbrace{\langle v\rangle^{\frac{\gamma}{2}} w(\alpha,\beta)\langle D_v\rangle^{\frac{1}{2}}}_{\in Op(\langle v\rangle^{\gamma/2}\langle\eta\rangle^s w(\alpha,\beta))}\partial^{\alpha-\alpha_1}_\beta f^n_\pm| \\ &\qquad\times|\underbrace{\langle v\rangle^{-\frac{\gamma}{2}} w^{-1}(\alpha,\beta)\langle D_v\rangle^{\frac{1}{2}} e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v\rangle^2}} w^2(\alpha,\beta)\langle v\rangle^{-4}}_{\in Op(\langle v\rangle^{\gamma/2}\langle\eta\rangle^s w(\alpha,\beta))}\partial^\alpha_\beta h^n_\pm|\,dx, \end{aligned}$$

where we let $s = 1$ for the Landau case. Here an operator $T \in Op(m)$ means T is a pseudo-differential operator with symbol in $S(m)$. Then by [16, Lemma 2.3 and Corollary 2.5], we can estimate (3.24) by

$$\begin{aligned} &\sum_{\alpha_1\leq\alpha}\int_\Omega |\partial^{\alpha_1}\nabla_x(\phi^n-\phi^{n-1})||\langle v\rangle^{\frac{\gamma}{2}}\langle D_v\rangle^s w(\alpha,\beta)\partial^{\alpha-\alpha_1}_\beta f^n_\pm||\langle v\rangle^{\frac{\gamma}{2}}\langle D_v\rangle^s w(\alpha,\beta)\partial^\alpha_\beta h^n_\pm|\,dx \\ &\qquad\qquad \lesssim \|\nabla_x(\phi^n-\phi^{n-1})\|_{H^2_x}\|f^n\|_{Y_k}\|h^n\|_{Y_k}. \end{aligned}$$

Applying (2.13), we estimate (3.25) by

$$\sum_{\alpha_1\leq\alpha,\,\beta_1\leq\beta}\int_\Omega \big(\|\partial^{\alpha_1}_{\beta_1} h^{n-1}\|_{L^2_{k_0}}\|w(\alpha,\beta)\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f^n\|_{H^s_{\gamma/2+2s-2}}$$

$$+\|\partial_{\beta_1}^{\alpha_1} f^n\|_{L^2_{k_0}}\|w(\alpha,\beta)\partial_{\beta-\beta_1}^{\alpha-\alpha_1}h^{n-1}\|_{H^s_{\gamma/2-2}}\big)\|w(\alpha,\beta)\partial_\beta^\alpha h^n\|_{H^s_{\gamma/2-2}}dx$$
$$\lesssim \big(\|h^{n-1}\|_{X_{k-2}}\|f^n\|_{Y_k}+\|f^n\|_{X_k}\|h^{n-1}\|_{Y_{k-2}}\big)\|h^n\|_{Y_{k-2}},$$

where we let $k \geq k_0+2$. Noticing we use weight $w(\alpha,\beta)\langle v\rangle^{-4}$ instead of $w(\alpha,\beta)$, applying similar arguments for deriving (3.17) and choosing τ_0 in (3.22) sufficiently small, we obtain

$$\begin{aligned}
&\|h^n\|^2_{L^\infty_T X_{k-2}}+\|\nabla_x(\phi^{n+1}-\phi^n)\|^2_{L^\infty_T H^3_x}+\frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|h_n\|^2_{L^2_T\bar{Y}_{k-2}}\|h^n\|^2_{L^2_T Y_{k-2}}\\
&\quad\leq \frac{1}{2}\big(\|h^{n-1}\|^2_{L^2_T Y_{k-2}}+\|\nabla_x(\phi^n-\phi^{n-1})\|^2_{L^\infty_T H^2_x}+\frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|h_{n-1}\|^2_{L^2_T\bar{Y}_{k-2}}\big)\\
&\leq\cdots\leq \frac{1}{2^{n-1}}\big(\|f^1\|^2_{L^2_T Y_{k-2}}+\|\nabla_x\phi^1\|^2_{L^\infty_T H^2_x}+\frac{1}{12}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f^1\|^2_{L^2_T\bar{Y}_{k-2}}\big)\leq\frac{\tau_0}{4^{n-1}}. \quad (3.26)
\end{aligned}$$

Thus $\{f^n\}$ and $\{\nabla_x\phi^n\}$ are Cauchy sequence in $L^\infty_T X_{k-2}$ and $L^\infty_T H^3_x$ respectively. Then their limit (f,ϕ) solves (3.20) and satisfies (3.21).

For the uniqueness, suppose that there are two solutions (f^1,ϕ^1) and (f^2,ϕ^2) satisfy (3.20) and (3.21). Then $h:=f^1-f^2$ satisfies

$$\begin{cases}
\partial_t h_\pm+v\cdot\nabla_x h_\pm\mp\nabla_x\phi^1\cdot\nabla_v h_\pm\pm\nabla_x(\phi^1-\phi^2)\cdot v\mu\\
\qquad=Q(h_\pm+h_\mp,\mu+f^2)+Q(\mu+f^1_\pm,h_\pm)+Q(\mu+f^1_\mp,h_\pm),\\
-\Delta_x(\phi^1-\phi^2)=\displaystyle\int_{\mathbb{R}^3}(h_+-h_-)dv,\quad h(0)=0,\\
h(t,x,R_x v)=h(t,x,v)\text{ on }\gamma_-\text{ and }\partial_n(\phi^1-\phi^2)=0\text{ on }\partial\Omega,\text{ if }\Omega\text{ is given by (1.11)},
\end{cases}$$

We apply the same method for deriving (3.26) to obtain that $h=0$ and the solution to (3.20) is unique. This completes the proof of Theorem 3.2. □

4. Global regularity

The main goal of this section is to prove the Theorem 1.2. Firstly, we establish the macroscopic estimates.

4.1. Macroscopic estimate.

We recall the operator $\mathbf{P}$ defined in (1.6).

Lemma 4.1. *For any $k\geq 4$, we have*

$$\frac{1}{2}\|f\|^2_{L^2_k}\leq\|\mathbf{P}f\|^2_{L^2_k}+\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2_k}\leq C_k\|f\|^2_{L^2_k},\quad \|\mathbf{P}f\|_{L^2_k}\leq C_k\|f\|_{L^2_4}, \quad (4.1)$$

for some constant $C_k\geq 0$.

Proof. Since $f=\mathbf{P}f+(\mathbf{I}-\mathbf{P})f$, we have

$$|f|^2\leq 2|\mathbf{P}f|^2+2|(\mathbf{I}-\mathbf{P})f|^2.$$

This proves the first part of (4.1). For the second part of (4.1), we have

$$|\{\mathbf{I}-\mathbf{P}\}f|\leq 2|\mathbf{P}f|^2+2|f|^2.$$

For the part $\mathbf{P}f$, by the definition (1.7), we have

$$\|\mathbf{P}f\|_{L^2_k}\leq C\|f\|_{L^1_2}\|\mu\langle v\rangle^2\|_{L^2_k}\leq C_k\|f\|_{L^2_4}.$$

This completes the proof of (4.1). □

It is direct to obtain

$$\|a\|^2_{L^2_x}+\|b\|^2_{L^2_x}+\|c\|^2_{L^2_x}\sim\|\mathbf{P}f\|^2_{L^2_{x,v}}.$$

We also rewrite the equation (1.4) as

$$\begin{cases}
\partial_t f_\pm+v\cdot\nabla_x f_\pm\pm\nabla_x\phi\cdot v\mu-L_\pm f=N_\pm(f),\\
-\Delta_x\phi=\displaystyle\int_{\mathbb{R}^3}(f_+-f_-)dv,
\end{cases} \quad (4.2)$$

with initial data $f(0)=f_0$, $E(0)=E_0$, where $L=[L_+,L_-]$ and $N=[N_+,N_-]$ are given by

$$\begin{aligned} L_\pm f &:= Q(f_\pm+f_\mp,\mu)+2Q(\mu,f_\pm),\\ N_\pm(f) &:= \pm\nabla_x\phi\cdot\nabla_v f_\pm+Q(f_\pm,f_\pm)+Q(f_\mp,f_\pm). \end{aligned} \tag{4.3}$$

Our next goal is to estimate $a(t,x),b(t,x),c(t,x)$ in terms of $\{\mathbf{I}-\mathbf{P}\}f$. In contrast to [39], our P is not symmetric and we can not compare the v_i,v_j terms on both side. The following Lemma gives the macroscopic estimates.

Lemma 4.2. *For both Landau and Boltzmann case, suppose f solves (4.2). For any integer $m\geq 0$, there exists function $G=G(t)$ satisfying*

$$G(t)\lesssim\sum_{|\alpha|\leq 1}\|\partial^\alpha f\|_{L^2_xL^2_{10}}\|\partial^\alpha\nabla_x\mathbf{P}f\|_{L^2_{x,v}}, \tag{4.4}$$

such that

$$\begin{aligned} &\partial_t G(t)+\lambda\|[a_+,a_-,b,c]\|^2_{H^2_x}+\lambda\|\nabla_x\phi\|^2_{H^2_x}\\ &\qquad\lesssim\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_{10}}+\|N_\parallel\|^2_{H^1_xL^2_v}+\|\nabla_x\phi\|^4_{L^2_x}, \end{aligned} \tag{4.5}$$

Here $N_\parallel$ is the inner product of $N(f)$ with some linear combination of $(1,v_i,v_iv_j,v_i^2,v_i|v|^2)$ over $v\in\mathbb{R}^3$.

Proof. Here we only consider the non-cutoff Boltzmann case and the Landau case is similar. The proof follows the idea in [39, Section 6]. Splitting $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$, we rewrite (4.2) to be

$$\begin{aligned} &\big(\partial_t a_\pm+\partial_t b\cdot v+\partial_t c(|v|^2-3)\big)\mu+v\cdot\nabla_x\big(a_\pm+b\cdot v+c(|v|^2-3)\big)\mu\pm\nabla_x\phi\cdot v\mu\\ &\qquad=-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f). \end{aligned} \tag{4.6}$$

Taking the inner product of (4.6) with $1,v_i^2,|v|^2,v_iv_j$, $i,j=1,2,3$, $i\neq j$ over $v\in\mathbb{R}^3$, we have

$$\begin{aligned} \partial_t a_\pm+\nabla_x\cdot b &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),1)_{L^2_v},\\ \partial_t a_\pm+2\partial_t c+2\partial_{x_i}b_i+\nabla_x\cdot b &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),v_i^2)_{L^2_v},\\ 3\partial_t a_\pm+6\partial_t c+5\nabla_x\cdot b &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),|v|^2)_{L^2_v}. \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \partial_{x_j}b_i+\partial_{x_i}b_j &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f,v_iv_j)_{L^2_v}+(N_\pm(f),v_iv_j)_{L^2_v}\\ &=:\gamma_{1ij}+\gamma_{2ij},\quad i\neq j. \end{aligned} \tag{4.8}$$

By taking the subtraction of $(4.7)_2$ and $(4.7)_1$, we have

$$\begin{aligned} \partial_t c+\partial_{x_i}b_i &= \Big(-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f,\frac{|v_i|^2-1}{2}\Big)_{L^2_v}+\Big(N_\pm(f),\frac{|v_i|^2-1}{2}\Big)_{L^2_v}\\ &=:\gamma_{1i}+\gamma_{2i}. \end{aligned} \tag{4.9}$$

Taking the inner product of (4.6) with v, $v|v|^2$ over $v\in\mathbb{R}^3$, we have

$$\begin{aligned} \partial_t b+\nabla_x(a_\pm\pm\phi)+2\nabla_x c &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),v)_{L^2_v},\\ 5\partial_t b+5\nabla_x(a_\pm\pm\phi)+20\nabla_x c &= (-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),v|v|^2)_{L^2_v}. \end{aligned} \tag{4.10}$$

The two identities in (4.10) implies

$$\nabla_x c=\Big(-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),\frac{v(|v|^2-5)}{10}\Big)_{L^2_v}, \tag{4.11}$$

and

$$\nabla_x a_\pm=-\partial_t b\mp\nabla_x\phi+\Big(-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),\frac{v(10-|v|^2)}{5}\Big)_{L^2_v}. \tag{4.12}$$

Here we use the fact that

$$\int_{\mathbb{R}^3} |v|^2 \mu dv = 3, \quad \int_{\mathbb{R}^3} |v|^4 \mu dv = 15, \quad \int_{\mathbb{R}^3} |v|^6 \mu dv = 105.$$

By the definitions (1.6) of $\mathbf{P}f$ and (4.3) of L, we have for $\psi(v) = 1, v_i, |v|^2$ that

$$(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, \psi(v)) = 0, \quad (L_\pm\{\mathbf{I} - \mathbf{P}\}f, \psi(v)) = 0,$$

and

$$(\partial_t\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, \psi(v)) = 0, \quad (v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, 1) = 0.$$

Thus the first, third identities in (4.7) and first identities in (4.10) (i.e. inner product with $1, v_i, |v|^2$) become

$$\begin{aligned} \partial_t a_\pm + \nabla_x \cdot b &= (N_\pm(f), 1)_{L^2_v}, \\ \partial_t b + \nabla_x a_\pm + 2\nabla_x c \pm \nabla_x \phi &= -(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, v)_{L^2_v} + (N_\pm(f), v)_{L^2_v}, \\ 3\partial_t a_\pm + 6\partial_t c + 5\nabla_x \cdot b &= -(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, |v|^2)_{L^2_v} + (N_\pm(f), |v|^2)_{L^2_v}. \end{aligned} \tag{4.13}$$

Combining $(4.13)_1$ and $(4.13)_3$, we have

$$\partial_t c + \frac{1}{3}\nabla_x \cdot b = -\frac{1}{6}(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, |v|^2)_{L^2_v} + \Big(N_\pm(f), \frac{|v|^2 - 3}{6}\Big)_{L^2_v}. \tag{4.14}$$

For brevity, we define

$$\xi_a = \frac{v(10 - |v|^2)}{5}, \quad \xi_{bi} = \frac{2v_i^2 - 5}{2}, \quad \xi_c = \frac{v(|v|^2 - 5)}{10}. \tag{4.15}$$

It follows that

$$\begin{aligned} |(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, \xi_a)| + |(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, \xi_{bi})| + |(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, v_i v_j)| \\ + |(v \cdot \nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f, \xi_c)| \le \|\nabla_x\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f\|_{L^2_6}. \end{aligned}$$

For the Boltzmann case, by Lemma 2.4, we have

$$\begin{aligned} |(Q(\mu, f), \xi_a)| &\le \|\mu\|_{L^2_{10}} \|f\|_{L^2_{10}} \|\xi_a\|_{H^{2s}_{-7}} \le C\|f\|_{L^2_{10}}, \\ |(Q(f, \mu), \xi_a)| &\le \|f\|_{L^2_{10}} \|\mu\|_{L^2_{10}} \|\xi_a\|_{H^{2s}_{-7}} \le C\|f\|_{L^2_{10}}. \end{aligned}$$

For the Landau case, by Lemma 2.13, we have

$$\begin{aligned} |(Q(\mu, f), \xi_a)| &\le \|\mu\|_{L^2_{10}} \|f\|_{L^2_{10}} \|\xi_a\|_{H^2_{-5}} \le C\|f\|_{L^2_{10}}, \\ |(Q(f, \mu), \xi_a)| &\le \|f\|_{L^2_{10}} \|\mu\|_{L^2_{10}} \|\xi_a\|_{H^2_{-5}} \le C\|f\|_{L^2_{10}}. \end{aligned}$$

Similar arguments can be carried on inner product with ξ_{bi}, $v_i v_j$, ξ_c. Recalling the definition (4.3) of $L = [L_+, L_-]$, we have

$$|(Lf, \xi_a)_{L^2_v}| + |(Lf, \xi_{bi})_{L^2_v}| + |(Lf, v_i v_j)_{L^2_v}| + |(Lf, \xi_c)_{L^2_v}| \lesssim \|f\|_{L^2_{10}}. \tag{4.16}$$

Step 1: Computation of $\nabla_x \partial^\alpha c$. Let $|\alpha| \le 2$. Applying ∂^α on both side of (4.11) and taking inner product with $\nabla_x \partial^\alpha c$ over $x \in \Omega$, we deduce that

$$\begin{aligned} \|\nabla_x \partial^\alpha c\|^2_{L^2_x} &= -\int_\Omega ((\partial_t + v \cdot \nabla_x)\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f, \xi_c)_{L^2_v} \cdot \nabla_x \partial^\alpha c \, dx \\ &\quad - \int_\Omega (L_\pm\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f, \xi_c)_{L^2_v} \cdot \nabla_x \partial^\alpha c \, dx + \int_\Omega (\partial^\alpha N_\pm(f), \xi_c)_{L^2_v} \cdot \nabla_x \partial^\alpha c \, dx \\ &\le -\frac{d}{dt}\int_\Omega (\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f, \xi_c)_{L^2_v} \cdot \nabla_x \partial^\alpha c \, dx \\ &\quad - \int_\Omega (\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\nabla_x \partial^\alpha f, \xi_c)_{L^2_v} \cdot \partial_t \partial^\alpha c \, dx \\ &\quad + C\|\nabla_x \partial^\alpha c\|_{L^2_x} \big(\|\partial^\alpha\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f\|_{H^1_x L^2_{10}} + \|\partial^\alpha N_\|\|_{L^2_x}\big). \end{aligned} \tag{4.17}$$

Here, for the torus case, one can take integration by parts directly. For the union of cubes case, we have from boundary (1.7) that $\partial_{x_i} c = 0$ on Γ_i. On the other hand, when $\alpha_i = 0, 2$, we have from Lemma 2.14 and definition (4.15) of ξ_c that for $x \in \Gamma_i$,

$$\begin{aligned}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f, \xi_c)_{L^2_v} &= (\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f(R_x v), \xi_c(R_x v))_{L^2_v} \\ &= -(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f(v), \xi_c(v))_{L^2_v} = 0.\end{aligned}$$

Note that $\partial^\alpha \mathbf{P}_\pm f(R_x v) = \partial^\alpha \mathbf{P}_\pm f(v)$ by using (2.28). This implies the integration by parts about ∇_x in (4.17). For the term $\partial_t \partial^\alpha c$ in (4.17), it follows from (4.14) that

$$\partial_t \partial^\alpha c = \frac{1}{3}\nabla_x \partial^\alpha b - \frac{1}{6}(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f, |v|^2)_{L^2_v} + \Big(\partial^\alpha N_\pm(f), \frac{|v|^2 - 3}{6}\Big)_{L^2_v}.$$

By Cauchy-Schwarz inequality, for any small $\varepsilon > 0$, we have

$$\begin{aligned}&-\int_\Omega (\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\nabla_x \partial^\alpha f, \xi_c)_{L^2_v} \cdot \partial_t \partial^\alpha c \, dx \\ &\qquad \le \varepsilon \|\nabla_x \partial^\alpha b\|^2 + \frac{C}{\varepsilon}\|\nabla_x \partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|^2_{L^2_x L^2_{10}} + \varepsilon \|\partial^\alpha N_\parallel\|^2_{L^2_x}.\end{aligned}$$

Plugging this into (4.17), we deduce that

$$\begin{aligned}\|\nabla_x \partial^\alpha c\|^2 \le -\frac{d}{dt}\int_\Omega (\{\mathbf{I}_\pm - \mathbf{P}_\pm\}\partial^\alpha f, \xi_c)_{L^2_v} \cdot \nabla_x \partial^\alpha c \, dx + \varepsilon \|\nabla_x \partial^\alpha b\|^2 + \varepsilon \|\nabla_x \partial^\alpha c\|^2 \\ + \frac{C}{\varepsilon}\big(\|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|^2_{H^1_x L^2_{10}} + \|\partial^\alpha N_\parallel\|^2_{L^2_x}\big). \quad (4.18)\end{aligned}$$

Step 2: Computation of $\nabla_x \partial^\alpha b$. For fixed i, we use (4.8), (4.9) to compute

$$\begin{aligned}\Delta_x b_i &= \sum_{j \ne i} \partial_{x_j x_j} b_i + \partial_{x_i x_i} b_i \\ &= \sum_{j \ne i} \big(-\partial_{x_i x_j} b_j + \partial_{x_j}(\gamma_{1ij} + \gamma_{2ij})\big) + \partial_{x_i}(\gamma_{1i} + \gamma_{2i}) - \partial_t \partial_{x_i} c \\ &= \sum_{j \ne i} (\partial_t \partial_{x_i} c - \partial_{x_i}(\gamma_{1j} + \gamma_{2j})) + \sum_{j \ne i} \partial_{x_j}(\gamma_{1ij} + \gamma_{2ij}) + \partial_{x_i}(\gamma_{1i} + \gamma_{2i}) - \partial_t \partial_{x_i} c \\ &= \partial_t \partial_{x_i} c + \sum_{j \ne i} \big(\partial_{x_j}(\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i}(\gamma_{1j} + \gamma_{2j})\big) + \partial_{x_i}(\gamma_{1i} + \gamma_{2i}) \\ &= \sum_{j \ne i} \big(\partial_{x_j}(\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i}(\gamma_{1j} + \gamma_{2j})\big) - \partial_{x_i x_i} b_i + 2\partial_{x_i}(\gamma_{1i} + \gamma_{2i}). \quad (4.19)\end{aligned}$$

We can rewrite the linear terms $\sum_{j \ne i} \big(\partial_{x_j}\gamma_{1ij} - \partial_{x_i}\gamma_{1j}\big) + 2\partial_{x_i}\gamma_{1i}$ including γ_{1i}, γ_{1j}, γ_{1ij} as the linear combination of

$$\partial_{x_j}\left(-((\partial_t + v \cdot \nabla_x)\{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \xi_{ij})_{L^2_v} - (L_\pm \{\mathbf{I} - \mathbf{P}\} f, \xi_{ij})_{L^2_v}\right),$$

where ξ_{ij} are certain linear combinations of the basis $\{1, v_i v_j, |v_i|^2\}^3_{i,j=1}$. Similar to (4.16), we have

$$|(Lf, \xi_{ij})_{L^2_v}| \lesssim \|f\|_{L^2_{10}}.$$

Note that

$$-(\Delta_x \partial^\alpha b_i, \partial^\alpha b_i)_{L^2_x} = \|\nabla_x \partial^\alpha b_i\|^2_{L^2_x}, \quad (\partial_{x_i x_i} \partial^\alpha b_i, \partial^\alpha b_i)_{L^2_x} = -\|\partial_{x_i} \partial^\alpha b_i\|^2_{L^2_x},$$

which is trivial for the torus case and follows from (2.28) for the union of cubes case. Applying ∂^α on both side of (4.19), taking inner product with $\partial^\alpha b_i$ over $x \in \Omega$ and summation over $i = 1, 2, 3$, we deduce that

$$\|\nabla_x \partial^\alpha b\|^2_{L^2_x} = \sum_i \Big(-\int_\Omega \sum_{j \ne i} \big(\partial_{x_j} \partial^\alpha (\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i} \partial^\alpha (\gamma_{1j} + \gamma_{2j})\big) \cdot \partial^\alpha b_i \, dx \Big)$$

$$
\begin{aligned}
&-\sum_i \Big(\int_\Omega |\partial_{x_i}\partial^\alpha b_i|^2 dx - 2\int_\Omega \partial_{x_i}\partial^\alpha(\gamma_{1i}+\gamma_{2i})\cdot\partial^\alpha b_i\, dx\Big)\\
&\le \sum_{i,j}\frac{d}{dt}\int_\Omega (\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial_{x_j}\partial^\alpha f,\xi_{ij})_{L^2_v}\cdot\partial^\alpha b_i\, dx\\
&\quad -\sum_{i,j}\int_\Omega (\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial_{x_j}\partial^\alpha f,\xi_{ij})_{L^2_v}\cdot\partial_t\partial^\alpha b_i\, dx\\
&\quad + C\|\nabla_x\partial^\alpha b\|\big(\|\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|_{H^1_xL^2_{10}}+\|\partial^\alpha N_\|\|_{L^2_x}\big). \qquad (4.20)
\end{aligned}
$$

Here we used integration by parts on ∂_{x_i} and ∂_{x_j}. For the case of union of cubes, we need the following boundary values to complete the integration by parts. For $i=1,2,3$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3)$. If $\alpha_i=0$ or 2, then it follows from (2.28) that $\partial^\alpha b_i=0$ on Γ_i. If $\alpha_i=1$, then applying Lemma 2.14 and change of variable $v\mapsto R_x v$, we can obtain from (4.9) that

$$\partial^\alpha\gamma_{1j}=0,\quad \text{on } \Gamma_i \text{ for } j=1,2,3.$$

Similarly, for $j\neq i$, if $\alpha_j=1$, then $\partial^\alpha b_i=0$ on Γ_j. If $\alpha_j=0$ or 2, then it follows from (4.8) that

$$\partial^\alpha\gamma_{1ij}=0,\quad \text{on } \Gamma_j \text{ for } j\neq i.$$

Next we calculate the term $\partial_t\partial^\alpha b_i$. Applying ∂^α to $(4.13)_2$ yields

$$\partial_t\partial^\alpha b_i=-\nabla_{x_i}\partial^\alpha a_\pm-2\nabla_{x_i}\partial^\alpha c\mp\nabla_{x_i}\partial^\alpha\phi-(v\cdot\nabla_x\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f+\partial^\alpha N_\pm(f),v_i)_{L^2_v}.$$

Then by Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
-\int_\Omega (\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial_{x_j}\partial^\alpha f,\xi_{ij})_{L^2_v}\cdot\partial_t\partial^\alpha b_i\, dx &\le \varepsilon\big(\|\nabla_x\partial^\alpha a_\pm\|^2_{L^2_x}+\|\nabla_x\partial^\alpha c\|^2_{L^2_x}+\|\nabla_x\partial^\alpha\phi\|^2_{L^2_x}\big)\\
&\quad+\varepsilon\|\partial^\alpha N_\|\|^2_{L^2_x}+\frac{C}{\varepsilon}\|\nabla_x\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|^2_{L^2_xL^2_{10}},
\end{aligned}
$$

and (4.20) becomes

$$
\begin{aligned}
\|\nabla_x\partial^\alpha b\|^2 &\le -\sum_{i,j}\frac{d}{dt}\int_\Omega(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_{ij})_{L^2_v}\cdot\partial_{x_j}\partial^\alpha b_i\, dx+\varepsilon\|\nabla_x\partial^\alpha\phi\|^2_{L^2_x}\\
&\quad+\varepsilon\|\nabla_x\partial^\alpha a_\pm\|^2_{L^2_x}+\varepsilon\|\nabla_x\partial^\alpha c\|^2_{L^2_x}+\frac{C}{\varepsilon}\big(\|\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|^2_{H^1_xL^2_{10}}+\|\partial^\alpha N_\|\|^2_{L^2_x}\big), \qquad (4.21)
\end{aligned}
$$

where we choose ε small enough.

Step 3: Computation of $\nabla_x\partial^\alpha a$. Taking the addition and difference of (4.12) over $\pm$, it yields that

$$\nabla_x(a_++a_-)=-2\partial_t b+\sum_\pm\big(-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),\xi_a\big)_{L^2_v}, \qquad (4.22)$$

and

$$\nabla_x(a_+-a_-)=-2\nabla_x\phi+\sum_\pm\pm\big(-(\partial_t+v\cdot\nabla_x)\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f-L_\pm\{\mathbf{I}-\mathbf{P}\}f+N_\pm(f),\xi_a\big)_{L^2_v}. \qquad (4.23)$$

Applying ∂^α derivative on both side of (4.22) and taking inner product with $\nabla_x\partial^\alpha(a_++a_-)$ over Ω, we deduce that

$$
\begin{aligned}
\|\nabla_x\partial^\alpha(a_++a_-)\|^2_{L^2_x} &= -\int_\Omega\partial_t\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx\\
&\quad-\int_\Omega\big(v\cdot\nabla_x\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f+L_\pm\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f+\partial^\alpha N_\pm(f),\xi_a\big)_{L^2_v}\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx. \qquad (4.24)
\end{aligned}
$$

By integration by parts about ∂_t and ∇_x, we have

$$\begin{aligned}
&-\int_\Omega \partial_t\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx\\
=&-\frac{d}{dt}\int_\Omega\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx\\
&-\int_\Omega\nabla_x\cdot\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\,\partial_t\partial^\alpha(a_++a_-)\,dx. \qquad (4.25)
\end{aligned}$$

To apply the integration by parts for the union of cubes case, we write the following boundary values by using (2.28). If $\alpha_i=1$, then $\partial^\alpha(a_++a_-)=0$ on Γ_i. If $\alpha_i=0,2$, then $\partial^\alpha b_i=0$ on Γ_i and by change of variable $v\mapsto R_xv$ and the definition (4.15) of ξ_a, we have

$$\begin{aligned}
(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}&=(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f(R_xv),\xi_a(R_xv))_{L^2_v}\\
&=-(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}=0.
\end{aligned}$$

The time derivative and tangent derivatives doesn't affect the zero boundary values. This completes the integration by parts in (4.25). In view of (4.25) and (4.24), applying $(4.13)_1$, (4.16) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|\nabla_x\partial^\alpha(a_++a_-)\|^2+\frac{d}{dt}\sum_\pm\int_\Omega\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx\\
&\qquad\lesssim\|\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|_{H^1_xL^2_{10}}+\|\partial^\alpha N_\parallel\|^2_{L^2_x}+\|\nabla_x\partial^\alpha b\|^2_{L^2_x}. \qquad (4.26)
\end{aligned}$$

Similarly, noticing that

$$\big(\nabla_x\partial^\alpha\phi,\nabla_x\partial^\alpha(a_+-a_-)\big)_{L^2_x}=-\big(\Delta_x\partial^\alpha\phi,\partial^\alpha(a_+-a_-)\big)_{L^2_x}=\|\Delta_x\partial^\alpha\phi\|^2_{L^2_x},$$

we apply ∂^α to (4.23) and take inner product with $\nabla_x\partial^\alpha(a_+-a_-)$ over Ω to deduce that

$$\begin{aligned}
&\|\nabla_x\partial^\alpha(a_+-a_-)\|^2_{L^2_x}+2\|\Delta_x\partial^\alpha\phi\|^2_{L^2_x}+\frac{d}{dt}\int_\Omega\sum_\pm\mp(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\cdot\nabla_x\partial^\alpha(a_+-a_-)\,dx\\
&\qquad\lesssim\|\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\|_{H^1_xL^2_{10}}+\|\partial^\alpha N_\parallel\|^2_{L^2_x}+\|\nabla_x\partial^\alpha b\|^2_{L^2_x}. \qquad (4.27)
\end{aligned}$$

Notice that

$$|a_++a_-|^2+|a_+-a_-|^2=2|a_+|^2+2|a_-|^2.$$

For the torus case, by Sobolev inequality, we have

$$\|\nabla_x\phi\|^2\lesssim\|\nabla_x^2\phi\|^2=\|\Delta_x\phi\|^2_{L^2_x}. \qquad (4.28)$$

For the zeroth order, we apply the Poincaré inequality to $a_\pm,b,c$ in x and use the conservation law (1.10) to deduce that

$$\begin{aligned}
\|a_\pm\|^2_{L^2_x}&\le C\left(\int_\Omega a_\pm(t,x)dx\right)^2+C\|\nabla_xa_\pm\|^2_{L^2_x}\lesssim\|\nabla_xa_\pm\|^2_{L^2_x},\\
\|b\|^2_{L^2_x}&\le C\left(\int_\Omega b(t,x)dx\right)^2+C\|\nabla_xb\|^2_{L^2_x}\lesssim\|\nabla_xb\|^2_{L^2_x}, \qquad (4.29)\\
\|c\|^2_{L^2_x}&\le C\left(\int_\Omega c(t,x)dx\right)^2+C\|\nabla_xc\|^2_{L^2_x}\lesssim\|\nabla_x\phi\|^4_{L^2_x}+\|\nabla_xc\|^2_{L^2_x}.
\end{aligned}$$

For the union of cubes case, it follows from (1.14) that $\partial_{x_i}\phi=0$ on Γ_i. Then by Sobolev embedding (cf. [15, Theorem 6.7-5]), we also have (4.28). The second equality in (4.28) can be obtained by using integration by parts. For the zeroth order, the conservation laws (1.15) give the conservation on mass and energy and hence, the estimates for $a_\pm,c$ in (4.29) still hold. For the estimate on b, we have from (2.28) that $b_i=0$ on Γ_i for $i=1,2,3$. Then by Sobolev embedding (cf. [15, Theorem 6.7-5]), we obtain

$$\|b\|_{L^2_x}\lesssim\|\nabla_xb\|_{L^2_x}.$$

In view of these facts, we take linear combination $\lambda\times(4.27)+\lambda\times(4.26)+(4.18)+(4.21)$ and summation over $|\alpha|\le 1$ and $\pm$ with sufficiently small $\lambda,\varepsilon>0$ to deduce that

$$\partial_t G(t)+\lambda\sum_{|\alpha|\le 2}\|\partial^\alpha[a_+,a_-,b,c]\|^2_{L^2_x}+\lambda\|\nabla_x\phi\|^2_{H^2_x}\lesssim\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_{10}}+\|N_\parallel\|^2_{H^1_x}+\|\nabla_x\phi\|^4_{L^2_x},$$

where

$$\begin{aligned}G=\sum_{\pm}\sum_{|\alpha|\le 1}\Big(&\int_\Omega(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_c)_{L^2_v}\cdot\nabla_x\partial^\alpha c\,dx\\&+\sum_{i,j}\int_\Omega(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_{ij})_{L^2_v}\cdot\partial_{x_j}\partial^\alpha b_i\,dx\\&+\lambda\int_\Omega\Big(\partial^\alpha b+(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\Big)\cdot\nabla_x\partial^\alpha(a_++a_-)\,dx\\&\mp\lambda\int_\Omega(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}\partial^\alpha f,\xi_a)_{L^2_v}\cdot\nabla_x\partial^\alpha(a_+-a_-)\,dx\Big).\end{aligned}$$

This implies (4.5) and the estimate (4.4) can be directly deduced by Cauchy-Schwarz inequality. □

4.2. **Estimates about the energy.** In this section, we give the energy estimates on (1.4).

Lemma 4.3. *Let $T>0$, $\eta>0$ and $k\ge k_0$. For the Boltzmann case, we further assume $k=k(\eta)>0$ sufficiently large. Suppose $[f,\phi]$ is the solution to equation (1.4) satisfying $\mu+f\ge 0$ and*

$$\mathcal{E}(t)\le C.$$

There exists functional $\mathcal{E}_1(t)$ satisfying

$$\mathcal{E}_1(t)\approx\|f\|^2_{X_k}+\|\nabla_x\phi\|^2_{H^3_x},\tag{4.30}$$

such that, for any $T>0$ and sufficiently large $k\ge 15$, we have

$$\begin{aligned}&\partial_t\mathcal{E}_1(t)+\lambda\eta\|f\|^2_{Y_k}+\lambda\sum_{|\alpha|\le 2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_D}\lesssim\|f\|_{X_{k_0}}\|f\|^2_{X_k}\\&\quad+C_{\eta,k}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v}+\eta\|\nabla_x\phi\|^2_{H^2_x}+\eta\|\mathbf{P}f\|^2_{H^2_xL^2_v}+C(\sqrt{\mathcal{E}_{k_0}(t)}+\mathcal{E}_{k_0}(t))\mathcal{D}_k(t).\end{aligned}\tag{4.31}$$

for some generic constant $\lambda>0$ and any sufficiently small $\eta>0$.

Proof. We proceed by considering spatial derivatives and mixed derivatives. Notice that

$$\|\phi\|_{L^\infty_x}\lesssim\|\nabla_x\phi\|_{H^1_x}\lesssim\mathcal{E}(t)\le C.$$

which verify (2.38) and we can apply estimates in Section 2.4.

Step 1. Estimates with mixed derivatives. We first consider the Boltzmann case. Recall that $A_{\alpha,\beta}$ is given in (1.17). For any $|\alpha|+|\beta|\le 2$, we apply ∂^α_β to (1.4), take inner product with $e^{\frac{\pm A_{\alpha,\beta}\phi}{\langle v\rangle^2}}w^2(\alpha,\beta)\partial^\alpha_\beta f_\pm$ over $\Omega\times\mathbb{R}^3$. Similar to the calculation from (3.7) to (3.11), i.e. replacing (g,ψ) by (f,ϕ) in (3.11), applying (3.10), (3.13), (3.14) and choosing $\eta<\frac{\gamma_1}{4}$, we have

$$\begin{aligned}&\partial_t\Big(\frac{1}{2}\|f\|^2_{X_k}+\sum_{|\alpha|\le 2}C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x}\Big)+\frac{\gamma_1}{2}\|f\|^2_{Y_k}\\&\le C_k\|f\|^2_{H^2_{x,v}}+C\|\partial_t\phi\|_{L^\infty_x}\|f\|^2_{X_k}+C_k\Big(\|\nabla_x\phi\|_{H^3_x}\|f\|^2_{Y_k}+\|\nabla_x\phi\|^2_{H^2_x}+\|f\|_{X_{k_0}}\|f\|^2_{Y_k}\\&\quad+\big(\|f\|_{X_{k_0}}+C\big)\|\nabla_x\phi\|_{H^1_x}\|f\|^2_{Y_k}+\|f\|_{X_{k_0}}\|\nabla_x\phi\|_{H^1_x}\big(\|f\|_{Y_k}+C\big)\|f\|_{Y_k}\Big),\end{aligned}\tag{4.32}$$

for some generic constant γ_1 independent of k. Here C_α and $C_{|\alpha|,0}$ are given in (2.46) and (1.19) respectively. If $|\beta|\geq 1$, by interpolation we have

$$C_k\|\partial_\beta^\alpha f\|_{L^2_{x,v}}^2\leq C\|\partial^\alpha f\|_{L^2_xH_v^{|\beta|}}^2\leq\frac{\gamma_1}{24}\|\partial^\alpha f\|_{L^2_xH_v^{|\beta|+s}}^2+C_k\|\partial^\alpha f\|_{L^2_{x,v}}^2\leq\frac{\gamma_1}{24}\|f\|_{Y_k}^2+C_k\|\partial^\alpha f\|_{L^2_{x,v}}^2. \tag{4.33}$$

Also, by Sobolev embedding or Poincaré's inequality,

$$\|\nabla_x\phi\|_{H^2_x}\lesssim\|\nabla_x^2\phi\|_{H^1_x}=\|\Delta_x\phi\|_{H^1_x}\lesssim\|f\|_{H^1_xL^2_4}\lesssim\eta\|f\|_{Y_k}+C_\eta\|f\|_{H^1_xL^2_v}.$$

In view of the above two estimates, choosing $\eta>0$ small enough, (4.32) implies that

$$\begin{aligned}&\partial_t\Big(\frac{1}{2}\|f\|_{X_k}^2+\sum_{|\alpha|\leq 2}C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2\Big)+\frac{\gamma_1}{4}\|f\|_{Y_k}^2\\&\qquad\lesssim\|f\|_{H^2_xL^2_v}^2+C\|\partial_t\phi\|_{L^\infty_x}\|f\|_{X_k}^2+\|\nabla_x\phi\|_{H^2_x}^2+\big(\sqrt{\mathcal{E}_{k_0}(t)}+\mathcal{E}_{k_0}(t)\big)\mathcal{D}_k(t).\end{aligned} \tag{4.34}$$

For Landau case, one can apply similar calculation as (3.12) instead of (3.11) to derive (4.34) by using the same technique and we omit the details for brevity.

Step 2. Estimates with spatial derivatives. For any $|\alpha|\leq 2$, we apply ∂^α to (1.4), take inner product with $e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm$ over $\Omega\times\mathbb{R}^3$ to deduce that

$$\begin{aligned}&\frac{1}{2}\partial_t\|e^{\frac{\pm A_{\alpha,0}\phi}{2\langle v\rangle^2}}\partial^\alpha f_\pm\|_{L^2_{x,v}}^2+\big(A_{\alpha,0}\partial_t\phi\langle v\rangle^{-2}\partial^\alpha f_\pm,e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&+\big(v\cdot\nabla_x\partial^\alpha f_\pm,e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&\mp\big(\nabla_x\phi\cdot\nabla_v\partial^\alpha f_\pm,e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&\mp\sum_{\substack{\alpha_1\leq\alpha\\|\alpha_1|\geq 1}}\big(\partial^{\alpha_1}\nabla_x\phi\cdot\nabla_v\partial^{\alpha-\alpha_1}f_\pm,e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&\pm\big(\partial^\alpha(\nabla_x\phi\cdot v\mu),e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}}\\&=\big(\partial^\alpha L_\pm f+\partial^\alpha Q(f_\pm+f_\mp,f_\pm),e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}}w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}},\end{aligned} \tag{4.35}$$

where $L_\pm$ is given in (4.3). We denote the second to seventh terms in (4.35) by J_1 to J_6 and estimate them one by one. Similar to the calculation of I_1 to I_6 from (3.8) to (3.9), i.e. replacing (g,ψ) by (f,ϕ), one can obtain the following estimates by using (2.40) and (2.42):

$$\begin{aligned}|J_1+J_2+J_3+J_4|&\lesssim\big(\|\partial_t\phi\|_{L^\infty_x}+\|\nabla_x\phi\|_{H^3_x}\big)\|f\|_{Y_k}^2\\&\lesssim C\|\partial_t\phi\|_{L^\infty_x}\|f\|_{X_k}^2+\|\nabla_x\phi\|_{H^2_x}\|f\|_{Y_k}^2\\&\lesssim\|f\|_{X_{k_0}}\|f\|_{X_k}^2+\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),\end{aligned}$$

where we apply (3.13) and (3.14) for the terms $\|\nabla_x^3\phi\|_{L^2_x}$ and $\|\partial_t\phi\|_{L^\infty_x}$ respectively. For J_5, using (2.46), we have

$$\begin{aligned}J_5&\geq C_\alpha\partial_t\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_5}-C\|\partial^\alpha\nabla_x\phi\|_{L^2_x}\|\nabla_x\phi\|_{H^1_x}\|f\|_{Y_k}\\&\geq C_\alpha\partial_t\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-\eta\|\partial^\alpha\nabla_x\phi\|_{L^2_x}^2-\eta\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_{\gamma/2}}^2\\&\quad-C_\eta\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_v}^2-C\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),\end{aligned}$$

for any $\eta > 0$. Here C_α is given in (2.46). For the term J_6, we need some different estimates. Noticing $L_\pm(\partial^\alpha f) = 0$, we split J_6 into

$$\begin{aligned}
J_6 &= \big(\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}} \\
&\quad + \big(\partial^\alpha Q(f_\pm + f_\mp, \mathbf{P}_\pm f), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}} \\
&\quad + \big(\partial^\alpha Q(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f + \{\mathbf{I}_\mp - \mathbf{P}_\mp\}f, \mu), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}} \\
&=: J_7 + J_8 + J_9.
\end{aligned}$$

We further split J_7 as

$$\begin{aligned}
J_7 &= \big(\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f), w^2(\alpha,0)\partial^\alpha\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f\big)_{L^2_{x,v}} \\
&\quad + \big(\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f), w^2(\alpha,0)\partial^\alpha\mathbf{P}_\pm f\big)_{L^2_{x,v}} \\
&\quad + \big(\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f), \big(e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} - 1\big)w^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{x,v}} \\
&=: J_{7,1} + J_{7,2} + J_{7,3}.
\end{aligned}$$

For Boltzmann case, we deduce from (2.52) that $J_{7,1}$ satisfies

$$\begin{aligned}
\sum_\pm J_{7,1} &\le -\frac{1}{8}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x L^2_{\gamma/2,*}} \\
&\quad - \frac{\gamma_1}{2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x H^s_{\gamma/2}} + C_k\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t) + C_k\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{x,v}}.
\end{aligned}$$

For $J_{7,2}$ and $J_{7,3}$, for the Boltzmann case, we deduce from Lemma 2.4 that

$$\begin{aligned}
|J_{7,2}| &\lesssim (C + \|\langle v\rangle^{14} f\|_{H^2_x L^2_v})\|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_x L^2_v}\|w^2(\alpha,0)\mathbf{P}f\|_{H^2_x H^{2s}_{\gamma+2s}} \\
&\lesssim \eta\|\mathbf{P}f\|^2_{H^2_x L^2_v} + C_\eta\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_x L^2_v} + C_k\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t).
\end{aligned}$$

It follows from (2.53) that

$$\begin{aligned}
|J_{7,3}| &\lesssim \|\nabla_x\phi\|_{H^1_x}\Big((C + \|\langle v\rangle^{14} f\|_{H^2_{x,v}})\|\{\mathbf{I}-\mathbf{P}\}f\|_{Y_k}\|\mathbf{P}f\|_{Y_k} \\
&\qquad + \|\langle v\rangle^{14}\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_{x,v}}(C + \|f\|_{Y_k})\|\mathbf{P}f\|_{Y_k}\Big) \\
&\lesssim \big(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)\big)\mathcal{D}_k(t).
\end{aligned}$$

For the Landau case, it follows from (2.62) and (2.63) that

$$\sum_\pm J_{7,1} \le -\frac{\gamma_1}{2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x L^2_D} + C_k\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x L^2_v} + C_k\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),$$

with some constants $\gamma_1, C_k > 0$. Applying Lemma 2.13 and (2.64), we have

$$\begin{aligned}
|J_{7,2}| &\lesssim (C + \|f\|_{L^2_5})\|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_x L^2_v}\|\mathbf{P}f\|_{H^2_x H^2_5} \\
&\lesssim \eta\|\mathbf{P}f\|^2_{H^2_x L^2_v} + C_\eta\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_x L^2_v} + C_k\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),
\end{aligned}$$

and

$$|J_{7,3}| \lesssim (C + \|\langle v\rangle^7 f\|_{H^2_{x,v}})\|\nabla_x\phi\|_{H^1_x}\|\{\mathbf{I}-\mathbf{P}\}f\|_{Y_k}\|f\|_{Y_k} \lesssim \big(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)\big)\mathcal{D}_k(t).$$

For J_8, we deduce from (2.54) that for the Boltzmann case:

$$|J_8| \lesssim \|\langle v\rangle^{14} f\|_{H^2_{x,v}}\|\langle v\rangle^{2s}\mathbf{P}f\|_{Y_k}\|f\|_{Y_k} + \|\langle v\rangle^{14}\mathbf{P}f\|_{H^2_{x,v}}\|f\|_{Y_k}\|f\|_{Y_k} \lesssim \sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),$$

and from (2.65) that for the Landau case:

$$|J_8| \lesssim \|f\langle v\rangle^7\|_{H^2_{x,v}}\|\langle v\rangle \mathbf{P}f\|_{Y_k}\|f\|_{Y_k} \lesssim \sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t).$$

Here we let $k \geq k_0$. For J_9, we split it as

$$\begin{aligned} J_9 =& \big(\partial^\alpha Q(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f + \{\mathbf{I}_\mp - \mathbf{P}_\mp\}f, \mu), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} w^2(\alpha,0)\partial^\alpha\{\mathbf{I}_\pm - \mathbf{P}_\pm\}ff_\pm\big)_{L^2_{x,v}} \\ &+ \big(\partial^\alpha Q(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f + \{\mathbf{I}_\mp - \mathbf{P}_\mp\}f, \mu), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v\rangle^2}} w^2(\alpha,0)\partial^\alpha\mathbf{P}_\pm f_\pm\big)_{L^2_{x,v}} \\ =:& J_{9,1} + J_{9,2}. \end{aligned}$$

Then for Boltzmann case, applying Lemma 2.20, we have for any $\eta > 0$ that

$$\begin{aligned} |J_{9,1}| \leq& \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_{\gamma/2,*}}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}fw(\alpha,0)\|_{L^2_xL^2_{\gamma/2,*}} \\ &+ C_k\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_{\gamma/2-1/2}}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}fw(\alpha,0)\|_{L^2_xL^2_{\gamma/2-1/2}} \\ \leq& \frac{1}{16}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_{\gamma/2,*}} \\ &+ \eta\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_{\gamma/2}} + C_{k,\eta}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v}, \end{aligned}$$

For the term $J_{9,2}$, by Lemma 2.4 we have

$$\begin{aligned} |J_{9,2}| \leq& C\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_7}\|\mu\|_{H^{2s}}\|\partial^\alpha\mathbf{P}fw(\alpha,0)\|_{L^2_xL^2_{\gamma+2s}} \\ \leq& \eta\|\partial^\alpha\mathbf{P}f\|^2_{L^2_{x,v}} + \eta\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_{\gamma/2}} + C_{k,\eta}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v}, \end{aligned}$$

For Landau case, we have from Lemma 2.20 that

$$\begin{aligned} |J_9| \leq& C_k\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_xL^2_5}\|\partial^\alpha f\|_{L^2_xL^2_5} \\ \leq& \eta\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_D} + \eta\|\partial^\alpha\mathbf{P}f\|^2_{L^2_{x,v}} + C_\eta\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v}. \end{aligned}$$

Taking summation $\sum_\pm\sum_{|\alpha|\leq 2}$ (4.35), combining the above estimates for J_1 to J_9, applying (3.10) and (4.33), we deduce that for small $\eta > 0$,

$$\begin{aligned} &\partial_t\sum_{|\alpha|\leq 2}\Big(\frac{1}{2}\|e^{\frac{\pm A_{\alpha,0}\phi}{2\langle v\rangle^2}}\partial^\alpha f_\pm\|^2_{L^2_{x,v}} + C_\alpha\|\nabla_x\phi\|^2_{H^2_x}\Big) + \frac{\gamma_1}{3}\sum_{|\alpha|\leq 2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xH^s_{\gamma/2}} \\ &\quad\lesssim \|f\|_{X_{k_0}}\|f\|^2_{X_k} + C_{\eta,k}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v} + \eta\|\nabla_x\phi\|^2_{H^2_x} + \eta\|\mathbf{P}f\|^2_{H^2_xL^2_v} \\ &\qquad + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t))\mathcal{D}_k(t). \quad (4.36) \end{aligned}$$

Here, for the Boltzmann case, we use the fact that

$$\|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta} \leq \frac{1}{16}\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta},$$

which follows from $k \geq k_0$.

Taking linear combination $\eta \times$ (4.34) + (4.36) and applying (3.14), we obtain

$$\begin{aligned} &\partial_t\mathcal{E}_1(t) + \lambda\eta\|f\|^2_{Y_k} + \frac{\gamma_1}{4}\sum_{|\alpha|\leq 2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_D} \\ &\quad\lesssim \|f\|_{X_{k_0}}\|f\|^2_{X_k} + \eta\|f\|^2_{H^2_xL^2_v} + C_{\eta,k}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{H^2_xL^2_v} + \eta\|\nabla_x\phi\|^2_{H^2_x} + \eta\|\mathbf{P}f\|^2_{H^2_xL^2_v} \\ &\qquad + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t))\mathcal{D}_k(t). \quad (4.37) \end{aligned}$$

where $\mathcal{E}_1(t)$ is given by

$$\mathcal{E}_1(t) = \frac{\eta}{2}\|f\|^2_{X_k} + \eta\sum_{|\alpha|\leq 2}C_\alpha C_{|\alpha|,0}\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x}$$

$$+\sum_{\pm}\sum_{|\alpha|\le 2}\Big(\frac{1}{2}\|e^{\frac{\pm A_{\alpha,0}\phi}{2\langle v\rangle^2}}\partial^\alpha f_\pm\|^2_{L^2_{x,v}}+C_\alpha\partial_t\|\nabla_x\phi\|^2_{H^2_x}\Big). \quad (4.38)$$

It's direct to verify that $\mathcal{E}_1(t)$ satisfies (4.30). For the second right-hand term of (4.37), we split it as

$$\eta\|f\|^2_{H^2_xL^2_v}\lesssim \eta\|\mathbf{P}f\|^2_{H^2_xL^2_v}+\eta\sum_{|\alpha|\le 2}\|w(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_xL^2_D}. \quad (4.39)$$

Choosing η sufficiently small, the second right hand term of (4.39) can be absorbed by the left hand side of (4.37) and we obtain (4.31). This completes the proof of Lemma 4.3.

□

4.3. **Recover the energy from semigroup method.** According to Lemma 4.3, we only need to deal with term $\|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_xL^2_v}$ without velocity derivative on the right hand side of (4.31). In order to eliminate this term, we define the semigroup generated by $\mathcal{L}$ given in (1.5) to be $S_{\mathcal{L}}(t)$. Then we first give some estimate on $S_{\mathcal{L}}(t)$, which is the solution operator to equation

$$\begin{aligned}&\partial_t f+v\cdot\nabla_x f\pm\nabla_x\phi\cdot v\mu=L_\pm f,\quad f(0)=f_0,\\ &-\Delta_x\phi=\int_{\mathbb{R}^3}(f_+-f_-)\,dv,\quad \int_\Omega\phi(t,x)dx=0.\end{aligned} \quad (4.40)$$

If the domain Ω is union of cubes given by (1.11), then we further assume

$$\partial_n\phi=0\quad\text{on }\partial\Omega. \quad (4.41)$$

To obtain the estimate of (4.40), we denote

$$-\Delta_x^{-1}\int_{\mathbb{R}^3}(\cdot)_+-(\cdot)_-\,dv$$

to be the solution operator to the second equation of (4.40). Then we define linear operators $A=[A_+,A_-]$, $B=[B_+,B_-]$ and K_1,K_2 as

$$\begin{aligned}A_\pm&=-v\cdot\nabla_x+L_\pm-M\chi_R,\\ K_1&=M\chi_R,\\ B_\pm&=-v\cdot\nabla_x+L_\pm,\\ K_2&=\pm\mu v\cdot\nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}\big((\cdot)_+-(\cdot)_-\big)\,dv,\\ \mathcal{L}&=-v\cdot\nabla_x\pm\mu v\cdot\nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}\big((\cdot)_+-(\cdot)_-\big)\,dv+L_\pm\end{aligned}$$

where $M>0$ is a large constant and $\chi\in D(\mathbb{R})$ is the truncation function satisfying $1_{[-1,1]}\le\chi\le 1_{[-2,2]}$ and we denote $\chi_R(\cdot):=\chi(\cdot/R)$ for $R>0$. For the case of union of cubes, we consider the domain of these operators, i.e. $A,B,\mathcal{L}$, with restriction of specular-reflection boundary condition for f and Neumann boundary condition for ϕ:

$$f(R_xv)=f(v)\text{ on }\gamma_-,\quad \partial_n\Delta_x^{-1}\int_{\mathbb{R}^3}\big(f_+-f_-\big)\,dv=0\text{ on }\partial\Omega.$$

Then we have the following lemma.

Lemma 4.4. *Consider both Boltzmann and Landau case. For $k\ge k_0$, there exists $\varepsilon_0>0$ such that if*

$$\|f\|_{H^2_xL^2_{k_0+4}}\le\varepsilon_0, \quad (4.42)$$

then we have

$$\|S_{\mathcal{L}}(t)f\|_{H^2_xL^2_k}\lesssim\langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k+4}}, \quad (4.43)$$

and

$$\Big(\int_0^\infty \|\langle v\rangle^k S_{\mathcal{L}}(t)f\|^2_{H^2_x L^2_v}\,dt\Big)^{\frac{1}{2}} \lesssim \|f\|_{H^2_x H^{-s}_{k-\gamma/2}}. \tag{4.44}$$

Here we suppose that $s = 1$ for Landau case.

Proof. We will prove (4.43) and (4.44) in two steps. We only prove the Boltzmann case and the Landau case can be proved similar.

Step 1. By Duhamel's principle, we have

$$S_B(t) = S_A(t) + \int_0^t S_B(t-s)K_1 S_A(s)\,ds, \tag{4.45}$$

and

$$S_{\mathcal{L}}(t) = S_B(t) + \int_0^t S_{\mathcal{L}}(t-s)K_2 S_B(s)\,ds. \tag{4.46}$$

Using Lemma 2.6 and Theorem 2.8 with $g = 0$ for the Boltzmann case and Lemma 2.11 for the Landau case, we deduce that for $k \geq k_0$,

$$(-Af, \langle v\rangle^{2k} f)_{L^2_{x,v}} \geq \frac{\gamma_1}{2}\|\langle v\rangle^k f\|^2_{L^2_x L^2_D} \gtrsim \|\langle v\rangle^{k+\frac{\gamma}{2}} f\|^2_{L^2_{x,v}}, \tag{4.47}$$

where k_0 is given in (1.24). Here $(v\cdot\nabla_x\partial^\alpha f, \partial^\alpha f)_{L^2_{x,v}} = 0$ by using change of variable $v \mapsto R_x v$ for the case of union of cubes. Moreover, we use Lemma 2.14 to obtain

$$(v\cdot\nabla_x f, \langle v\rangle^{2k} f)_{L^2_{x,v}} = 0 \tag{4.48}$$

for the case of union of cubes, and by using change of variable $v \mapsto R_x v$. For the case of torus, (4.48) also holds. Then A generates a semigroup on L^2_k such that

$$\|S_A(t)f\|_{L^2_x L^2_k} \leq \|f\|_{L^2_x L^2_k}. \tag{4.49}$$

Also, by definition of semigroup, we have equation

$$\partial_t S_A(t)f - AS_A(t)f = 0. \tag{4.50}$$

Taking inner product of (4.50) with $\langle v\rangle^{2k} S_A(t)f$ over $\Omega\times\mathbb{R}^3$, we have from (4.47) that

$$\frac{1}{2}\partial_t\|\langle v\rangle^k S_A(t)f\|^2_{L^2_{x,v}} + \lambda\|\langle v\rangle^k S_A(t)f\|^2_{L^2_x L^2_D} \leq 0. \tag{4.51}$$

For the hard potential case, i.e. $\gamma \geq 0$, we have

$$\|\langle v\rangle^k S_A(t)f\|_{L^2_x L^2_D} \gtrsim \|\langle v\rangle^k S_A(t)f\|_{L^2_x L^2_v},$$

and hence,

$$\|S_A(t)f\|_{L^2_x L^2_k} \leq e^{-\lambda t}\|f\|_{L^2_x L^2_k}, \tag{4.52}$$

for some $\lambda > 0$. For the soft potential case and $k \geq k_1 \geq k_0$, it follows from (4.51) that

$$\begin{aligned}\partial_t\|S_A(t)f\|^2_{L^2_x L^2_{k_1}} &\lesssim -\|S_A(t)f\|^2_{L^2_x L^2_{k_1+\frac{\gamma}{2}}}\\ &\lesssim -\lambda\langle R\rangle^\gamma\|S_A(t)f\|^2_{L^2_x L^2_{k_1}} + \langle R\rangle^{2(k-k_1)+\gamma}\|S_A(t)f\|^2_{L^2_x L^2_k},\end{aligned}$$

for some $\lambda > 0$. Solving this ODE and using (4.49), we have

$$\begin{aligned}\|S_A(t)f\|^2_{L^2_x L^2_{k_1}} &\lesssim e^{-\lambda\langle R\rangle^\gamma t}\big(\|f\|^2_{L^2_x L^2_{k_1}} + \langle R\rangle^{2(k-k_1)+\gamma}\int_0^t e^{\lambda\langle R\rangle^\gamma s}\,ds\|f\|^2_{L^2_x L^2_k}\big)\\ &\lesssim e^{-\lambda\langle R\rangle^\gamma t}\|f\|^2_{L^2_x L^2_{k_1}} + \langle R\rangle^{2(k-k_1)}\|f\|^2_{L^2_x L^2_k}.\end{aligned}$$

Choosing $\langle R\rangle = \big(\langle t\rangle[\log\langle t\rangle]^{-\frac{2(k-k_0)}{\lambda}}\big)^{-\frac{1}{\gamma}}$, we have

$$\|S_A(t)f\|_{L^2_x L^2_{k_1}} \lesssim \langle t\rangle^{-\frac{k-k_1-1/2}{|\gamma|}}\|f\|_{L^2_x L^2_k}, \tag{4.53}$$

For $k \geq k_1 + 4$, since $|\gamma| \leq 3$, we know that $\langle t\rangle^{-\frac{k-k_1-1/2}{|\gamma|}} \leq \langle t\rangle^{-\frac{7}{6}}$. Together with (4.52), we deduce that

$$\|S_A(t)f\|_{L^2_x L^2_{k_1}} \lesssim \langle t\rangle^{-\frac{7}{6}} \|f\|_{L^2_x L^2_k}, \tag{4.54}$$

For the semigroups $S_B(t)$ and $S_{\mathcal{L}}(t)$ in exponential weighted space, we have

$$\begin{aligned} \partial_t S_B(t)f - BS_B(t)f = 0, \\ \partial_t S_{\mathcal{L}}(t)f - \mathcal{L}S_{\mathcal{L}}(t)f = 0. \end{aligned} \tag{4.55}$$

Taking $H^2_x L^2_v$ inner product with $\mu^{-1}S_B(t)f$ and $\mu^{-1}S_{\mathcal{L}}(t)f$ over $\Omega \times \mathbb{R}^3$ respectively, we have

$$\begin{aligned} &\partial_t \|\mu^{-\frac{1}{2}} S_B(t)f\|^2_{H^2_x L^2_v} + \lambda \|\mu^{-\frac{1}{2}}\{\mathbf{I} - \mathbf{P}\mu^{\frac{1}{2}}\}(S_B(t)f)\|^2_{H^2_x H^s_{\gamma/2}} \leq 0, \\ &\partial_t \|\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t)f\|^2_{H^2_x L^2_v} + \partial_t \|\nabla_x \phi\|^2_{H^2_x} + \lambda \|\mu^{-\frac{1}{2}}\{\mathbf{I} - \mathbf{P}\mu^{\frac{1}{2}}\}(S_{\mathcal{L}}(t)f)\|^2_{H^2_x H^s_{\gamma/2}} \leq 0. \end{aligned} \tag{4.56}$$

where the dissipation rate of $L_\pm$ for exponential perturbation can be found in [33, 36] and ϕ is given by

$$\phi = \Delta_x^{-1} \int_{\mathbb{R}^3} (S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_- \, dv. \tag{4.57}$$

Here, in the second estimate of (4.56), we used the fact that

$$\partial_t \int_{\mathbb{R}^3} S_{\mathcal{L}}(t)f \, dv + \int_{\mathbb{R}^3} v \cdot \nabla_x S_{\mathcal{L}}(t)f \, dv = 0,$$

and thus,

$$\begin{aligned} &\sum_{\pm} \big(\pm \mu v \cdot \nabla_x \partial^\alpha \Delta_x^{-1} \int_{\mathbb{R}^3} (S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_- \, dv, \mu^{-1}\partial^\alpha (S_{\mathcal{L}}(t)f)_\pm \big)_{L^2_{x,v}} \\ &= \big(\partial^\alpha \phi, \int_{\mathbb{R}^3} v \cdot \nabla_x \partial^\alpha ((S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_-) \, dv\big)_{L^2_x} \\ &= \big(\partial^\alpha \phi, -\partial_t \partial^\alpha \int_{\mathbb{R}^3} ((S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_-) \, dv\big)_{L^2_x} \\ &= \frac{1}{2}\partial_t \|\partial^\alpha \nabla_x \phi\|^2_{L^2_x}. \end{aligned}$$

Here we take integration by parts on ∇_x with (4.41) and use boundary condition (1.13) to deduce

$$\int_{\mathbb{R}^3} v((S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_-) \, dv = 0$$

on $\partial\Omega$ for the case of union of cubes.

In order to derive the weighted version of (4.56), taking $H^2_x L^2_v$ inner product of (4.55) with $\langle v\rangle^k \mu^{-1} S_B(t)f$ and $\langle v\rangle^k \mu^{-1} S_{\mathcal{L}}(t)f$ over $\Omega \times \mathbb{R}^3$ respectively, we have

$$\begin{aligned} &\partial_t \|\langle v\rangle^k \mu^{-\frac{1}{2}} S_B(t)f\|^2_{H^2_x L^2_v} + \lambda \|\langle v\rangle^k (\mu^{-\frac{1}{2}} S_B(t)f)\|^2_{H^2_x H^s_{\gamma/2}} \\ &\qquad \leq C_R \|\chi_R \mu^{-\frac{1}{2}} S_B(t)f\|^2_{H^2_x L^2_v}, \\ &\partial_t \|\langle v\rangle^k \mu^{-\frac{1}{2}} S_{\mathcal{L}}(t)f\|^2_{H^2_x L^2_v} + \partial_t \|\nabla_x \phi\|^2_{H^2_x} + \lambda \|\langle v\rangle^k (\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t)f)\|^2_{H^2_x H^s_{\gamma/2}} \\ &\qquad \leq C_\eta \|\nabla_x \phi\|^2_{H^2_x} + C_{\eta,R} \|\chi_R \mu^{-\frac{1}{2}} S_{\mathcal{L}}(t)f\|^2_{H^2_x L^2_v}. \end{aligned} \tag{4.58}$$

By macroscopic estimates from [19, Theorem 3.4] for the case of union of cubes and exponential perturbation, we have

$$\begin{aligned}
&\partial_t \mathcal{E}_{int,B}(t) + \lambda \|\mu^{-\frac{1}{2}} \mathbf{P} \mu^{\frac{1}{2}} S_B(t) f\|^2_{H^2_x H^s_{\gamma/2}} \le C \|\mu^{-\frac{1}{2}} \{\mathbf{I} - \mathbf{P}\mu^{\frac{1}{2}}\} S_B(f)\|^2_{H^2_x L^2_{\gamma/2}}, \\
&\partial_t \mathcal{E}_{int,\mathcal{L}}(t) + \partial_t \|\nabla_x \phi\|^2_{H^2_x} + \lambda \|\mu^{-\frac{1}{2}} \mathbf{P} \mu^{\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x H^s_{\gamma/2}} + \lambda \|\nabla_x \phi\|^2_{H^2_x} \\
&\quad \le C \|\mu^{-\frac{1}{2}} \{\mathbf{I} - \mathbf{P}\mu^{\frac{1}{2}}\} S_{\mathcal{L}}(f)\|^2_{H^2_x L^2_{\gamma/2}} + \|\nabla_x \phi\|^4_{L^2_x},
\end{aligned} \tag{4.59}$$

where ϕ is given by (4.57) and $\mathcal{E}_{int,B}, \mathcal{E}_{int,\mathcal{L}}$ are functional satisfying

$$\mathcal{E}_{int,B}(t) \lesssim \|\mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v}, \quad \mathcal{E}_{int,\mathcal{L}}(t) \lesssim \|\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_v}. \tag{4.60}$$

respectively. Although the macroscopic estimates from [19] is for union of cubes, the case of torus can be similarly derived with simpler calculations; see also [44]. Taking linear combinations (4.56) + $\kappa^2 \times$ (4.58) + $\kappa \times$ (4.59) and assuming the *a priori* assumption

$$\|\nabla_x \phi\|^2_{H^2_x} \le \varepsilon, \tag{4.61}$$

with $\varepsilon > 0$ sufficiently small and ϕ given by (4.57), we have

$$\begin{aligned}
&\partial_t \mathcal{E}_B(t) + \lambda \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x H^s_{\gamma/2}} \le 0, \\
&\partial_t \mathcal{E}_{\mathcal{L}}(t) + \lambda \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x H^s_{\gamma/2}} + \lambda \|\nabla_x \phi\|^2_{H^2_x} \le 0,
\end{aligned} \tag{4.62}$$

where

$$\begin{aligned}
\mathcal{E}_B(t) &= \|\mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} + \kappa^2 \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} + \kappa \mathcal{E}_{int,\mathcal{L}}, \\
\mathcal{E}_{\mathcal{L}}(t) &= \|\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_v} + \kappa^2 \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_v} + \kappa \mathcal{E}_{int,\mathcal{L}} + \|\nabla_x \phi\|^2_{H^2_x}.
\end{aligned}$$

With (4.60), it's direct to check that

$$\begin{aligned}
\mathcal{E}_B(t) &\sim \|\mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} + \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v}, \\
\mathcal{E}_{\mathcal{L}}(t) &\sim \|\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_v} + \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_v} + \|\nabla_x \phi\|^2_{H^2_x}.
\end{aligned}$$

We calculate the first inequality of (4.62) and the second one is similar. When $\gamma \ge 0$, we have

$$\partial_t \mathcal{E}_B(t) + \lambda \|\langle v \rangle^k \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} \le 0,$$

and thus,

$$\|\langle v \rangle^k \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} \le e^{-\lambda t} \|\langle v \rangle^k \mu^{-\frac{1}{2}} f\|^2_{H^2_x L^2_v}.$$

When $\gamma < 0$, we have

$$\partial_t \mathcal{E}_B(t) + \lambda \|\langle v \rangle^{k + \frac{\gamma}{2}} \mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_v} \le 0.$$

Then we apply similar arguments for obtaining (4.53) to deduce that

$$\|\mu^{-\frac{1}{2}} S_B(t) f\|^2_{H^2_x L^2_{k-4}} \lesssim \langle t \rangle^{-\frac{2(3-\frac{1}{2})}{|\gamma|}} \|\mu^{-\frac{1}{2}} f\|^2_{H^2_x L^2_k} \lesssim \langle t \rangle^{-\frac{7}{6}} \|\mu^{-\frac{1}{2}} f\|^2_{H^2_x L^2_k}. \tag{4.63}$$

The estimate for hard potential case is also included in (4.63). Similarly, it follows from the second estimate of (4.62) that

$$\|\mu^{-\frac{1}{2}} S_{\mathcal{L}}(t) f\|^2_{H^2_x L^2_{k-4}} + \|\nabla_x \phi\|^2_{H^2_x} \lesssim \langle t \rangle^{-\frac{7}{6}} \big(\|\mu^{-\frac{1}{2}} f\|^2_{H^2_x L^2_k} + \|\nabla_x \phi|_{t=0}\|^2_{H^2_x} \big). \tag{4.64}$$

Now we turn back to Duhamel's principle (4.45) and (4.46). For any f, we have from (4.45), (4.54) and (4.63) that for $k_1 \ge k_0$,

$$\|S_B(t) f\|_{H^2_x L^2_{k_1}} \lesssim \|S_A(t) f\|_{H^2_x L^2_{k_1}} + \int_0^t \|\mu^{-\frac{1}{2}} S_B(t-s) K_1 S_A(s) f\|_{H^2_x L^2_v} \, ds$$

$$\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\|\mu^{-\frac{1}{2}}K_1S_A(s)f\|_{H^2_xL^2_4}\,ds$$
$$\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\|S_A(s)f\|_{H^2_xL^2_v}\,ds$$
$$\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}}. \tag{4.65}$$

Thus, recalling ϕ is given by (4.57), it follows from (4.46), (4.64) and (4.65) that for $k_1 \geq k_0$,

$$\begin{aligned}\|S_{\mathcal{L}}(t)f\|_{H^2_xL^2_{k_1}} &\lesssim \|S_B(t)f\|_{H^2_xL^2_{k_1}} + \int_0^t \|\mu^{-\frac{1}{2}}S_{\mathcal{L}}(t-s)K_2S_B(s)f\|_{H^2_xL^2_v}\,ds\\ &\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\big(\|\mu^{-\frac{1}{2}}K_2S_B(s)f\|_{H^2_xL^2_4}\\ &\qquad + \|\nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}(K_2S_B(s)f)_+ - (K_2S_B(s)f)_-\,dv\|_{H^2_x}\big)\,ds.\end{aligned} \tag{4.66}$$

Note that by using odd property on μv, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}(K_2S_B(s)f)_+ - (K_2S_B(s)f)_-\,dv\\ &\quad = 2\int_{\mathbb{R}^3}\mu(v)v\cdot\nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}\big((S_B(s)f)_+(u) - (S_B(s)f)_-(u)\big)\,du\,dv = 0.\end{aligned}$$

Also, letting

$$\phi_1 = \Delta_x^{-1}\int_{\mathbb{R}^3}\big((S_B(s)f)_+(u) - (S_B(s)f)_-(u)\big)\,du,$$

when Ω is torus, we know that

$$\|\nabla_x\phi_1\|_{L^2_x} \lesssim \|\Delta_x\phi_1\|_{L^2_x}. \tag{4.67}$$

If Ω is union of cubes given by (1.11), noticing from (4.41) that $\partial_{x_i}\phi_1 = 0$ on Γ_i, by Sobolev inequality [15, Theorem 6.7-5], we also have (4.67). Then we obtain from (4.66) and (4.54) that for $k_1 \geq k_0$,

$$\begin{aligned}\|S_{\mathcal{L}}(t)f\|_{H^2_xL^2_{k_1}} &\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\|\langle v\rangle^4\mu^{\frac{1}{2}}(v)v\cdot\nabla_x\phi_1\|_{H^2_xL^2_v}\,ds\\ &\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\|\Delta_x\phi_1\|_{H^2_xL^2_{k_1+4}}\,ds\\ &\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}} + \int_0^t \langle t-s\rangle^{-\frac{7}{6}}\|S_B(s)f\|_{H^2_xL^2_2}\,ds\\ &\lesssim \langle t\rangle^{-\frac{7}{6}}\|f\|_{H^2_xL^2_{k_1+4}}.\end{aligned} \tag{4.68}$$

Next we check that (4.61) is fulfilled if $\varepsilon_0 > 0$ in (4.42) is small enough. Indeed, similar to (4.67), we have

$$\|\nabla_x\phi\|_{H^2_x} \lesssim \|\Delta_x\phi\|_{H^1_x} = \|\int_{\mathbb{R}^3}(S_{\mathcal{L}}(t)f)_+ - (S_{\mathcal{L}}(t)f)_-\,dv\|_{H^1_x}. \tag{4.69}$$

Then it follows from (4.68) that

$$\|\nabla_x\phi\|_{H^2_x} \lesssim \|S_{\mathcal{L}}(t)f\|_{H^1_xL^2_2} \lesssim \|f\|_{H^2_xL^2_{k_0+4}}.$$

This closes the *a priori* assumption (4.61) and completes the proof of (4.43).

Step 2. For brevity of notations, we let $s = 1$ for the Landau case. In order to obtain (4.44), we consider the dual A_l^* of $A_l = \langle v\rangle^l A(\langle v\rangle^{-l}f)$ for any $l \in \mathbb{R}$. Notice that

$$\big(A_l^*f, \langle v\rangle^{2k}f\big)_{L^2_xL^2_v} = \big(\langle v\rangle^l f, A(\langle v\rangle^{2k-l}f)\big)_{L^2_xL^2_v} = \big(\langle v\rangle^{2l-2k}g, Ag\big)_{L^2_xL^2_v},$$

where we let $g = \langle v\rangle^{2k-l} f$. Then the taking $L^2_x L^2_v$ inner product of

$$\partial_t S_{A_l^*}(t) f - A_l^* S_{A_l^*}(t) f = 0$$

with $\langle v\rangle^{2k} S_{A_l^*}(t) f$, we have

$$\partial_t \|\langle v\rangle^k S_{A_l^*}(t) f\|^2_{L^2_x L^2_v} + \big(A_l^* S_{A_l^*}(t) f, \langle v\rangle^{2k} S_{A_l^*}(t) f\big)_{L^2_x L^2_v} = 0,$$

and hence, for $l - k \geq k_0$,

$$\partial_t \|\langle v\rangle^{l-k} g\|^2_{L^2_x L^2_v} + \big(\langle v\rangle^{2l-2k} g, Ag\big)_{L^2_x L^2_v} = 0,$$

where $g = \langle v\rangle^{2k-l} S_{A_l^*}(t) f$. Taking integration over t, we have

$$\lambda \int_0^\infty \|\langle v\rangle^{l-k} g\|^2_{L^2_x H^s_{\gamma/2}}\, dt \leq \|\langle v\rangle^{l-k} g|_{t=0}\|^2_{L^2_x L^2_v}.$$

That is, for $l - k \geq k_0$,

$$\lambda \int_0^\infty \|\langle v\rangle^{k} S_{A_l^*}(t) f\|^2_{L^2_x H^s_{\gamma/2}}\, dt \leq \|\langle v\rangle^{k} f\|^2_{L^2_x L^2_v}. \tag{4.70}$$

Observe that if f satisfies equation $\partial_t f = Af$ then $g = \langle v\rangle^l f$ satisfies $\partial_t g = A_l g$, and also that $(Af, \langle v\rangle^{2l} f)_{L^2_{x,v}} = (A_l g, g)_{L^2_{x,v}}$. Thus, $\langle v\rangle^l S_A(t) f = S_{A_l}(\langle v\rangle^l f)$. Moreover, by duality, we have

$$(S_{A_l} f, g)_{L^2_{x,v}} = (f, S_{A_l^*} g)_{L^2_{x,v}}.$$

Therefore, for some sequence $\{\varphi_n\}$ in Schwartz space such that $\|\varphi_n\|_{L^2_x L^2_k} \leq 1$, we have from (4.70) that

$$\begin{aligned}
\int_0^\infty \|\langle v\rangle^k S_A(t) f\|^2_{L^2_x L^2_v}\, dt &= \int_0^\infty \lim_{n\to\infty} \big|\big(\langle v\rangle^{2k} S_A(t) f, \varphi_n\big)_{L^2_{x,v}}\big|^2\, dt \\
&= \liminf_{n\to\infty} \int_0^\infty \big|\big(\langle v\rangle^{2k} f, S_{A^*_{2k}}(t)\varphi_n\big)_{L^2_{x,v}}\big|^2\, dt \\
&\leq \liminf_{n\to\infty} \int_0^\infty \|\langle v\rangle^k f\|^2_{L^2_x H^{-s}_{-\gamma/2}} \|\langle v\rangle^k S_{A^*_{2k}}(t)\varphi_n\|^2_{L^2_x H^s_{\gamma/2}}\, dt \\
&\lesssim \|\langle v\rangle^{k-\gamma/2} f\|^2_{L^2_x H^{-s}_v},
\end{aligned} \tag{4.71}$$

for $k \geq k_0$. This is the estimate for $S_A(t)$. To obtain the estimates for $S_B(t)$, it follows from Duhamel's principle (4.45) that

$$\begin{aligned}
\Big(\int_0^\infty \|\langle v\rangle^k S_B(t) f\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12} &\lesssim \Big(\int_0^\infty \|\langle v\rangle^k S_A(t) f\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12} \\
&\quad + \Big(\int_0^\infty \Big\|\langle v\rangle^k \int_0^t S_B(s) K_1 S_A(t-s) f\, ds\Big\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12}.
\end{aligned} \tag{4.72}$$

Using (4.71) and (4.63), the second right-hand term of (4.72) can be estimated by

$$\begin{aligned}
&\int_0^\infty \Big(\int_s^\infty \|\mu^{-\frac12} S_B(s) K_1 S_A(t-s) f\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12} ds \\
&\lesssim \int_0^\infty \Big(\int_0^\infty \langle s\rangle^{-\frac76} \|\langle v\rangle^k \mu^{-\frac12} K_1 S_A(t) f\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12} ds \\
&\lesssim \Big(\int_0^\infty \|S_A(s) f\|^2_{H^2_x L^2_v}\, ds\Big)^{\frac12} \lesssim \|f\|_{H^2_x H^{-s}_{k_0-\gamma/2}}.
\end{aligned}$$

Note that ∇_x commutes with $S_A(t)$. The first term on the right hand side of (4.72) can be estimated by using (4.71). Thus, for $k \geq k_0$, we have from (4.72) that

$$\Big(\int_0^\infty \|\langle v\rangle^k S_B(t) f\|^2_{H^2_x L^2_v}\, dt\Big)^{\frac12} \lesssim \|f\|_{H^2_x H^{-s}_{k-\gamma/2}}. \tag{4.73}$$

Next we apply (4.46), and (4.73) to derive the estimates on $S_{\mathcal{L}}(t)$:

$$\begin{aligned}\Big(\int_0^\infty \|\langle v\rangle^k S_{\mathcal{L}}(t)f\|^2_{H^2_xL^2_v}\,dt\Big)^{\frac12} &\lesssim \Big(\int_0^\infty \|\langle v\rangle^k S_B(t)f\|^2_{H^2_xL^2_v}\,dt\Big)^{\frac12}\\ &\quad+\Big(\int_0^\infty \Big\|\langle v\rangle^k\int_0^t S_{\mathcal{L}}(s)K_2S_B(t-s)f\,ds\Big\|^2_{H^2_xL^2_v}\,dt\Big)^{\frac12}\\ &\lesssim \|f\|_{H^2_xH^{-s}_{k-\gamma/2}}+\int_0^\infty\Big(\int_s^\infty \|\langle v\rangle^k S_{\mathcal{L}}(s)K_2S_B(t-s)f\|^2_{H^2_xL^2_v}\,dt\Big)^{\frac12}ds. \quad (4.74)\end{aligned}$$

Applying (4.64), (4.67) and (4.73), the second right-hand term of (4.74) can be estimated by

$$\begin{aligned}&\int_0^\infty\Big(\int_s^\infty \|\langle v\rangle^k\mu^{-\frac12}S_{\mathcal{L}}(s)K_2S_B(t-s)f\|^2_{H^2_xL^2_v}\,dt\Big)^{\frac12}ds\\ &\lesssim\int_0^\infty\langle s\rangle^{-\frac76}\Big(\int_0^\infty\|\langle v\rangle^{k+4}\mu^{-\frac12}K_2S_B(t)f\|^2_{H^2_xL^2_v}\\ &\qquad+\|\nabla_x\Delta_x^{-1}\underbrace{\int_{\mathbb{R}^3}(K_2S_B(t)f)_+-(K_2S_B(t)f)_-\,dv}_{=0}\|^2_{H^2_x}\,dt\Big)^{\frac12}ds\\ &\lesssim\Big(\int_0^\infty\|\langle v\rangle^{k+4}\mu^{-\frac12}\mu v\cdot\nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}\big((S_B(t)f)_+-(S_B(t)f)_-\big)\,dv\|_{H^2_xL^2_v}\,dt\Big)^{\frac12}\\ &\lesssim\Big(\int_0^\infty\|S_B(t)f\|_{H^2_xL^2_2}\,dt\Big)^{\frac12}\lesssim\|f\|_{H^2_xH^{-s}_{k-\gamma/2}}.\end{aligned}$$

The term with brace equal to zero because $\int_{\mathbb{R}^3}v\mu\,dv=0$. Inserting this into (4.74), we obtain (4.44). This completes the proof of Lemma 4.4.

□

Next we can introduce a norm

$$|\!|\!|f|\!|\!|^2:=\sum_{|\alpha|\le2}\int_0^{+\infty}\|S_{\mathcal{L}}(\tau)\partial^\alpha f\|^2_{L^2_xL^2_v}d\tau, \quad (4.75)$$

where the associate inner product is given by

$$((f,g)):=\int_0^{+\infty}\sum_{|\alpha|\le2}(S_{\mathcal{L}}(\tau)\partial^\alpha f,S_{\mathcal{L}}(\tau)\partial^\alpha g)_{L^2_xL^2_v}d\tau,$$

Note that ∂^α commutes with $\mathcal{L}$ and hence, commutes with $S_{\mathcal{L}}(t)$. Then by (4.43) and (4.44), we have

$$|\!|\!|f|\!|\!|\lesssim\|f\|_{H^2_xL^2_{k_0+4}}, \quad (4.76)$$

and

$$|\!|\!|f|\!|\!|\lesssim\|f\|_{H^2_xH^{-s}_{k_0-\frac{\gamma}{2}}}\lesssim\|f\|_{H^2_xH^{-s}_{k_0+\frac32}} \quad (4.77)$$

where we let $s=1$ for the Landau case. Then we can recover the loss energy by using this norm.

Lemma 4.5. *For both the Boltzmann and Landau case, suppose (f,ϕ) is the solution to the equation (1.4). Then we have*

$$\frac12\partial_t|\!|\!|\partial^\alpha f|\!|\!|^2+\|f\|^2_{H^2_xL^2_v}\lesssim\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t), \quad (4.78)$$

where $|\!|\!|\cdot|\!|\!|$ is given in (4.75), $\mathcal{E}_k(t)$ and $\mathcal{D}_k(t)$ are given in (1.22) and (1.23).

Proof. Recall the first equation of (1.4):

$$\partial_t f_\pm = \underbrace{\pm\nabla_x\phi\cdot\nabla_v f}_{\text{Term 1}} + \underbrace{\mathcal{L}f}_{\text{Term 2}} + \underbrace{Q(f_\pm + f_\mp, f_\pm)}_{\text{Term 3}}, \tag{4.79}$$

where $\mathcal{L}$ is given in (1.5). Then we take the $((\cdot,\cdot))$ inner product of (4.79) with f and compute every term separately. Since ∂_t commute with L, we know that ∂_t commute with $S_{\mathcal{L}}(\tau)$ and hence,

$$((\partial_t\partial^\alpha f, \partial^\alpha f)) = \frac{1}{2}\partial_t\|\partial^\alpha f\|^2.$$

Next we compute the Term 1. Since $s \geq \frac{1}{2}$ implies $1-s \leq s$, by Lemma 2.10 and (4.77), we have

$$\begin{aligned}
&\int_0^{+\infty}\sum_{|\alpha|\leq 2}(S_{\mathcal{L}}(\tau)\partial^\alpha(\nabla_x\phi\cdot\nabla_v f), S_{\mathcal{L}}(\tau)\partial^\alpha f)_{L^2_xL^2_v}d\tau\\
\lesssim &\sum_{|\alpha|\leq 2}\|\partial^\alpha(\nabla_x\phi\cdot\nabla_v f)\|_{L^2_xH^{-s}_{k_0+3/2}}\sum_{|\alpha|\leq 2}\|\partial^\alpha f\|_{L^2_xL^2_{k_0+3/2}}\\
\lesssim &\|\nabla_x\phi\|_{H^2_x}\|f\|_{H^2_xH^{1-s}_{k_0+3/2}}\|f\|_{H^2_xL^2_{k_0+3/2}}\\
\lesssim &\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t),
\end{aligned}$$

for the non-cutoff Boltzmann case, where we let $k \geq k_0 + 3/2$. Similarly, for the Landau case, we have from (4.76) that for $k \geq k_0 + 4$,

$$\begin{aligned}
&\int_0^{+\infty}\sum_{|\alpha|\leq 2}(S_{\mathcal{L}}(\tau)\partial^\alpha(\nabla_x\phi\cdot\nabla_v f), S_{\mathcal{L}}(\tau)\partial^\alpha f)_{L^2_xL^2_v}d\tau\\
\lesssim &\sum_{|\alpha|\leq 2}\|\partial^\alpha(\nabla_x\phi\cdot\nabla_v f)\|_{L^2_xL^2_{k_0+4}}\|f\|_{H^2_xL^2_{k_0+4}}\\
\lesssim &\|\nabla_x\phi\|_{H^2_x}\|f\|_{H^2_xH^1_{k_0+4}}\|f\|_{H^2_xL^2_{k_0+4}}\\
\lesssim &\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t).
\end{aligned}$$

For the term 2 and for both Boltzmann and Landau case, since $\lim_{t\to\infty}\langle t\rangle^{\frac{7}{6}} = 0$ in (4.43), we have

$$\begin{aligned}
&\int_0^{\infty}\sum_{|\alpha|\leq 2}(S_{\mathcal{L}}(\tau)\mathcal{L}\partial^\alpha f, S_{\mathcal{L}}(\tau)\partial^\alpha f)_{L^2_xL^2_v}d\tau\\
= &\int_0^{\infty}\frac{d}{d\tau}\sum_{|\alpha|\leq 2}\|S_{\mathcal{L}}(\tau)\partial^\alpha f\|^2_{L^2_xL^2_v}d\tau\\
= &-\sum_{|\alpha|\leq 2}\|\partial^\alpha f\|^2_{L^2_xL^2_v}.
\end{aligned}$$

Finally, we consider the nonlinear part, i.e. Term 3. For the Boltzmann case, by (2.14), (4.76) and (4.77), we have

$$\begin{aligned}
&\int_0^{\infty}\sum_{|\alpha|\leq 2}(S_{\mathcal{L}}(\tau)\partial^\alpha Q(f_\pm + f_\mp, f_\pm), S_{\mathcal{L}}(\tau)\partial^\alpha f)_{L^2_xL^2_v}d\tau\\
\leq &C_k\sum_{|\alpha|\leq 2}\sum_{\alpha_1\leq\alpha}\|Q(\partial^{\alpha_1}(f_\pm + f_\mp), \partial^{\alpha-\alpha_1}f_\pm)\|_{L^2_xH^{-s}_{k_0-\gamma/2}}\|\partial^\alpha f\|_{L^2_xL^2_{k_0}}\\
\leq &C_k(\|f\|_{H^2_xL^2_{14}}\|f\|_{H^2_xH^s_{k_0+\gamma/2+2s}} + \|f\|_{H^2_xL^2_{14}}\|f\|_{H^2_xH^s_{k_0+\gamma/2}})\|\partial^\alpha f\|_{L^2_xL^2_{k_0}}\\
\leq &C_k\sqrt{\mathcal{E}_{k_0}(t)}\mathcal{D}_k(t).
\end{aligned}$$

Here we let $k_0 \geq 14$, $k \geq k_0 + \gamma/2 + 2s$ and apply similar discussion on α_1 as in (2.61). For the Landau case, by (2.18), (4.76) and (4.77), we have

$$\begin{aligned}
&\int_0^\infty \sum_{|\alpha|\leq 2} (S_{\mathcal{L}}(\tau)\partial^\alpha Q(f_\pm + f_\mp, f_\pm), S_{\mathcal{L}}(\tau)\partial^\alpha f)_{L^2_x L^2_v} d\tau \\
&\leq C_k \sum_{|\alpha|\leq 2} \sum_{\alpha_1 \leq \alpha} \|Q(\partial^{\alpha_1} f_\pm + f_\mp, \partial^{\alpha-\alpha_1} f_\pm)\|_{L^2_x H^{-1}_{k_0+3/2}} \|\partial^\alpha f\|_{L^2_x L^2_{k_0}} \\
&\leq C_k \|f\|_{H^2_x L^2_{k_0+5/2}} \|f\|_{H^2_x L^2_{D,k_0+5/2}} \|\partial^\alpha f\|_{L^2_x L^2_{k_0+5/2}} \\
&\leq C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),
\end{aligned}$$

where $k \geq k_0 + 4$ and we also apply similar discussion on α_1 as in (2.61). Combining the above estimates, the $((\cdot,\cdot))$ inner product of (4.79) with $\partial^\alpha f$ yields

$$\frac{1}{2}\partial_t |||f|||^2 + \sum_{|\alpha|\leq 2} \|\partial^\alpha f\|^2_{L^2_x L^2_v} \lesssim \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t).$$

Then we conclude Lemma 4.5.

□

4.4. **Proof of the main theorem.** We start this subsection by proving the main stability theorem below. Theorem 4.6 and 4.7 together with local existence from Theorem 3.2 imply Theorem 1.2. To prove Theorem 1.2, we assume the *a priori* assumption as

$$\sup_{0\leq t\leq T} \mathcal{E}_{k_0}(t) \leq 2M, \tag{4.80}$$

for the case $\gamma \in [0,1]$ and

$$\sup_{0\leq t\leq T} (1+t)^{\frac{2l}{|\gamma|}} \mathcal{E}_{k_0}(t) \leq CM, \tag{4.81}$$

for the case $\gamma \in [-3,0)$, where $l > \frac{|\gamma|}{2}$ is a constant and we further assume $k \geq k_0 + l$ in this case.

Theorem 4.6. *Assume that f_0 satisfies the conservation laws (1.10) and $F_0 = \mu + f_0 \geq 0$. There exists $k_1 \geq k_0$ such that for*

$$k \geq k_0 \text{ for Landau case and } k \geq k_1 \text{ for Boltzmann case,}$$

there exists a small constant $M \geq 0$ such that if

$$\mathcal{E}_{k_0}(0) \leq M, \quad \mathcal{E}_k(0) < \infty,$$

then there exist a unique global solution $f(t,x,v)$ to the Vlasov-Poisson-Boltzmann/Landau system (1.4) with $F = \mu + f \geq 0$ satisfying

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t), \tag{4.82}$$

for any $T > 0$, for some generic constant $\lambda > 0$.

Proof. We take linear combination $\kappa \times (4.5) + (4.31) + C_0 \times (4.78)$ with $\kappa, C_0 > 0$ to deduce that

$$\begin{aligned}
&\partial_t \mathcal{E}_k(t) + \lambda\kappa \|[a_+, a_-, b, c]\|^2_{H^2_x} + \lambda\kappa \|\nabla_x \phi\|^2_{H^2_x} + \lambda\eta \|f\|^2_{Y_k} \\
&+ C_0 \|f\|^2_{H^2_x L^2_v} + \lambda \sum_{|\alpha|\leq 2} \|w(\alpha,0)\partial^\alpha \{\mathbf{I}-\mathbf{P}\} f\|^2_{L^2_x L^2_D} \\
\lesssim\ & \|f\|_{X_{k_0}} \|f\|^2_{X_k} + \kappa \|\{\mathbf{I}-\mathbf{P}\} f\|^2_{H^2_x L^2_{10}} + \kappa \|N_{\|}\|^2_{H^1_x L^2_v} \\
&+ C_{\eta,k} \|\{\mathbf{I}-\mathbf{P}\} f\|^2_{H^2_x L^2_v} + \eta \|\nabla_x \phi\|^2_{H^2_x} + \eta \|\mathbf{P} f\|^2_{H^2_x L^2_v} + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t),
\end{aligned}$$

for some generic constant $\lambda > 0$, where $\mathcal{E}_k(t)$ is given by

$$\mathcal{E}_k(t) := \kappa G(t) + \mathcal{E}_1(t) + \frac{C_0}{2} \sum_{|\alpha| \le 2} |||\partial^\alpha f|||^2 \tag{4.83}$$

Here $G(t)$, $\mathcal{E}_1(t)$ and $|||\cdot|||$ are given in Lemma 4.2, Lemma 4.3 and (4.75) respectively. Choosing $\kappa > 0$ sufficiently small, and then $\eta > 0$ sufficiently small, and finally $C_0 > 0$ sufficiently large, we have

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t) + \kappa \|N_\parallel\|^2_{H^1_x L^2_v} + (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t), \tag{4.84}$$

for some generic constant $\lambda > 0$, where $\mathcal{D}_k(t)$ is given by (1.23). Then one can check (1.22) by using (4.4), (4.38), (4.76) and (4.77) with sufficiently small κ. Note that $e^{\frac{\pm A_{\alpha,0}\phi}{2\langle v\rangle^2}} \approx 1$ as in (2.39).

Next, recall from Lemma 4.2 that

$$\begin{aligned} N_\parallel &= (\pm \nabla_x \phi \cdot \nabla_v f_\pm)_\parallel + Q(f_\pm + f_\mp, f_\pm)_\parallel \\ &= (\pm \nabla_x \phi \cdot \nabla_v f_\pm, \xi) + (Q(f_\pm + f_\mp, f_\pm), \xi), \end{aligned}$$

where ξ is some linear combination of $1, v_i, v_i v_j, v_i^2, v_i |v|^2$. For $|\alpha| \le 1$, we have

$$\begin{aligned} \|\partial^\alpha (\nabla_x \phi \cdot \nabla_v f, \xi)_{L^2_v}\|_{L^2_x} &\lesssim \sum_{|\alpha_1| = 0} \|\partial^{\alpha_1} \nabla_x \phi\|_{L^\infty_x} \|\partial^{\alpha - \alpha_1} \nabla_v f\|_{L^2_x L^2_5} \\ &\quad + \sum_{|\alpha_1| = 1} \|\partial^{\alpha_1} \nabla_x \phi\|_{L^3_x} \|\partial^{\alpha - \alpha_1} \nabla_v f\|_{L^6_x L^2_5} \\ &\lesssim \|\nabla_x \phi\|_{H^2_x} \|f\|_{H^1_x H^1_5} \lesssim \sqrt{\mathcal{E}_{k_0}(t) \mathcal{D}_k(t)}. \end{aligned} \tag{4.85}$$

For the Boltzmann case, by Lemma 2.4, we apply similar discussion on α_1 as in (4.85) to deduce that

$$\begin{aligned} \|\partial^\alpha (Q(f_\pm + f_\mp, f_\pm), \xi)\|^2_{L^2_{x,v}} &\lesssim \|f\|_{H^2_x L^2_{10}} \|f\|_{H^2_x L^2_{10}} \|\xi\|_{H^{2s}_{-7}} \\ &\lesssim \sqrt{\mathcal{E}_{k_0}(t) \mathcal{D}_k(t)}. \end{aligned}$$

For the Landau case, by (2.17) we have

$$\begin{aligned} \|\partial^\alpha (Q(f_\pm + f_\mp, f_\pm), \xi \langle v\rangle^{-14} \langle v\rangle^{14})\|^2_{L^2_{x,v}} &\lesssim \|f\|_{H^2_x L^2_7} \|f\|_{H^2_x L^2_{D,8}} \|\xi\|_{L^2_{D,-7}} \\ &\lesssim \sqrt{\mathcal{E}_{k_0}(t) \mathcal{D}_k(t)}. \end{aligned}$$

Therefore, we obtain

$$\sum_{|\alpha| \le 1} \|\partial^\alpha N_\parallel\|^2_{L^2_x} \lesssim \mathcal{E}_{k_0}(t) \mathcal{D}_k(t).$$

Then (4.84) implies

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t) + (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t). \tag{4.86}$$

Choosing M in the *a priori* assumption (4.80) and (4.81) sufficiently small, we have from (4.86) that

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t),$$

for some $\lambda > 0$. This concludes Theorem 4.6. □

To conclude Theorem 1.2, we need to prove the large-time behavior as the following.

Theorem 4.7. *Let $l = 0$ for hard potential case and $l > \frac{|\gamma|}{2}$ for soft potential case. Let $k \geq k_0 + 2 + l$ (and let k sufficiently large for Boltzmann case). If the solution (f, ϕ) of (1.4) satisfies*

$$\mathcal{E}_{k_0+2+l}(0) \leq M, \quad \mathcal{E}_k(0) < \infty.$$

Then there exists a constant $\lambda > 0$, such that for any $t > 0$, we have

$$\sup_{0\leq t\leq T} \mathcal{E}_k(t) \leq 2\mathcal{E}(0). \tag{4.87}$$

Moreover, if $0 \leq \gamma \leq 1$, then

$$\mathcal{E}_k(t) \leq e^{-\lambda t}\mathcal{E}_k(0).$$

If $\gamma < 0$, then we have

$$\mathcal{E}_{k-l}(t) \lesssim (1+t)^{-\frac{2l}{|\gamma|}}\mathcal{E}_k(0). \tag{4.88}$$

Proof. If $0 \leq \gamma \leq 1$, noticing that $\|\cdot\|_{L^2_{x,v}} \lesssim \|\cdot\|_{L^2_D}$ and choosing M in (4.80) small enough, we have $\|f\|_{X_{k_0}} \lesssim \mathcal{E}_{k_0} \lesssim M$, and hence, by (4.82) we have

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{E}_k(t) \leq 0.$$

Solving this ODE, we obtain $\mathcal{E}_k(t) \leq e^{-\lambda t}\mathcal{E}_k(0)$. This close the *a priori* assumption (4.80) and conclude the case of hard potential.

Next we assume $\gamma < 0$. For any $l > \frac{|\gamma|}{2}$, we have from (4.81) that

$$\|f\|_{X_{k_0}} \lesssim \mathcal{E}_{k_0}(t) \lesssim M(1+t)^{-\frac{2l}{|\gamma|}}. \tag{4.89}$$

Let $p = \frac{-\gamma+2l}{2l}$ and $p' = \frac{-\gamma+2l}{-\gamma}$. Then by $L^p - L^{p'}$ Hölder's inequality, we have

$$\begin{aligned}
\|w(\alpha,\beta)\partial^\alpha_\beta f\|^2_{L^2_v} &= \int_{\mathbb{R}^3} \langle v\rangle^{\frac{2l\gamma}{-\gamma+2l}} \langle v\rangle^{-\frac{2l\gamma}{-\gamma+2l}} |w(\alpha,\beta)\partial^\alpha_\beta f|^2\, dv \\
&\leq \|\langle v\rangle^{\gamma/2} w(\alpha,\beta)\partial^\alpha_\beta f\|^{\frac{4l}{-\gamma+2l}}_{L^2_v} \|\langle v\rangle^l w(\alpha,\beta)\partial^\alpha_\beta f\|^{\frac{-2\gamma}{-\gamma+2l}}_{L^2_v} \\
&\leq \mathcal{D}_k^{\frac{2l}{-\gamma+2l}} \mathcal{E}_{k+l}^{\frac{-\gamma}{-\gamma+2l}}.
\end{aligned}$$

From definition (1.22) and (1.20), we know that $\mathcal{E}_k^{\frac{-\gamma+2l}{2l}} \mathcal{E}_{k+l}^{\frac{\gamma}{2l}} \leq \mathcal{D}_k$. Then it follows from (4.82) and (4.89) that

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{E}_k^{\frac{-\gamma+2l}{2l}}(t) \Big(\sup_{0\leq t\leq T} \mathcal{E}_{k+l} \Big)^{\frac{\gamma}{2l}} \leq CM(1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(t). \tag{4.90}$$

Neglecting the second left hand term of (4.90), we have

$$\partial_t \mathcal{E}_k(t) \leq CM(1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(t).$$

Taking integration over $t \in [0, T]$ yields

$$\sup_{0\leq t\leq T} \mathcal{E}_k(t) \leq \mathcal{E}_k(0) + CM \int_0^T (1+t)^{-\frac{2l}{|\gamma|}}\, dt \sup_{0\leq t\leq T} \mathcal{E}_k(t).$$

Since $\frac{2l}{|\gamma|} > 1$, choosing $M > 0$ sufficiently small, we have

$$\sup_{0\leq t\leq T} \mathcal{E}_k(t) \leq 2\mathcal{E}_k(0). \tag{4.91}$$

This gives (4.87). Next we solve (4.90) directly. It's direct to obtain that

$$\partial_t(\mathcal{E}_k^{\frac{\gamma}{2l}}(t)) = \frac{\gamma}{2l}\mathcal{E}_k^{\frac{\gamma-2l}{2l}}(t)\partial_t\mathcal{E}_k(t) \geq -\frac{\gamma\lambda}{2l}\Big(\sup_{0\leq t\leq T} \mathcal{E}_{k+l} \Big)^{\frac{\gamma}{2l}} + \frac{CM\gamma}{2l}(1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k^{\frac{\gamma}{2l}}(t),$$

and thus,

$$\partial_t\big(\exp\big\{\frac{CM|\gamma|^2}{2l(-2l+|\gamma|)}(1+t)^{-\frac{2l}{|\gamma|}+1}\big\}\mathcal{E}_k^{\frac{\gamma}{2l}}(t)\big)$$
$$\geq -\frac{\gamma\lambda}{2l}\exp\big\{\frac{CM|\gamma|^2}{2l(-2l+|\gamma|)}(1+t)^{-\frac{2l}{|\gamma|}+1}\big\}\Big(\sup_{0\leq t\leq T}\mathcal{E}_{k+l}\Big)^{\frac{\gamma}{2l}}.$$

Note that $-2l+|\gamma|<0$. Taking integration over $t\in[0,T]$, we have

$$\exp\big\{\frac{CM|\gamma|^2}{2l(-2l+|\gamma|)}(1+t)^{-\frac{2l}{|\gamma|}+1}\big\}\mathcal{E}_k^{\frac{\gamma}{2l}}(t)\geq C_{\gamma,l}\mathcal{E}_k^{\frac{\gamma}{2l}}(0)+C_{\gamma,l}\int_0^t 1\,dt\Big(\sup_{0\leq t\leq T}\mathcal{E}_{k+l}\Big)^{\frac{\gamma}{2l}},$$

and hence,

$$\mathcal{E}_k(t)\leq C_{\gamma,l}(1+t)^{\frac{2l}{\gamma}}\sup_{0\leq t\leq T}\mathcal{E}_{k+l}(t).$$

Replacing k by $k-l\geq k_0$, we apply (4.91) to deduce

$$\mathcal{E}_{k-l}(t)\lesssim(1+t)^{\frac{2l}{\gamma}}\sup_{0\leq t\leq T}\mathcal{E}_k(t)\lesssim(1+t)^{\frac{2l}{\gamma}}\mathcal{E}_k(0).$$

Choosing $k-l=k_0$ and applying (1.25), we close the *a priori* assumption (4.81). Using the standard continuity arguments, we obtain (4.88). This completes the proof of Theorem 4.7. □

Acknowledgments. The first and the second author have been supported by grants from Beijing Institute of Mathematical Sciences and Applications and Yau Mathematical Science Center, Tsinghua University, and the third author has been supported by grants from project ANR-17-CE40-0030 of the French National Research Agency (ANR) and CEREMADE, Université Paris Dauphine.

References

[1] R. Alexandre, F. Hérau, and W.-X. Li, Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff, J. Math. Pures Appl. **126** (2019), 1–71.

[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. *Regularizing effect and local existence for the non-cutoff Boltzmann equation.* Arch. Ration. Mech. Anal., 198(1):39-123, 2010.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, *The Boltzmann equation without angular cutoff in the whole space: I, global existence for soft potential.* Journal of Functional Analysis 262 (2012), no. 3, 915-1010.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. *The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential.* Anal. Appl. (Singap.), 9(2):113-134, 2011.

[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. *Global existence and full regularity of the Boltzmann equation without angular cutoff.* Comm. Math. Phys., 304(2):513-581, 2011.

[6] R. Alexandre and C. Villani. *On the Boltzmann equation for long-range interactions.* Comm. Pure Appl. Math., 55(1):30-70, 2002.

[7] R. Alonso, Y. Morimoto, W. Sun and T. Yang. *Non-cutoff Boltzmann equation with polynomial decay perturbation.* Revista Mathematica Iberoamericana, 37(2021), no. 1, 189-292.

[8] C. Cao., L-B. He and J.Ji. *Propagation of moments and sharp convergence rate for inhomogeneous non-cutoff Boltzmann equation with soft potentials.* arXiv:2204.01394.

[9] Y. Cao, C. Kim, and D. Lee. *Global Strong Solutions of the Vlasov–Poisson–Boltzmann System in Bounded Domains.* Arch. Ration. Mech. Anal., 233(3):1027–1130, apr 2019.

[10] K. Carrapatoso, I. Tristani, and K.-C. Wu. *Cauchy problem and exponential stability for the inhomogeneous Landau equation.* Arch. Ration. Mech. Anal. 221(2016), no.1, 363-418.

[11] K. Carrapatoso and S. Mischler. *Landau equation for very soft and Coulomb potentials near Maxwellians.* Annals of PDE (2017), no. 3, 1-65.

[12] S. Chaturvedi, *Stability of Vacuum for the Boltzmann Equation with Moderately Soft Potentials.* Ann. PDE 7, 15 2021.

[13] Y. Chen and L. He. *Smoothing estimates for Boltzmann equation with full-range interactions: Spatially homogeneous case.* Arch. Ration. Mech. Anal, 201(2):501-548, 2011.

[14] Y. Chen and L. He. *Smoothing estimates for Boltzmann equation with full-range interactions: Spatially inhomogeneous case.* Arch. Ration. Mech. Anal, 203(2012), no.2, 343-377.

[15] P. G. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications: With 401 Problems and 52 Figures*. CAMBRIDGE, 2013.
[16] D.-Q. Deng *Dissipation and Semigroup on H^k_n: Non-cutoff Linearized Boltzmann Operator with Soft Potential*. SIAM J. Math. Anal., 52(3):3093–3113, jan 2020.
[17] D.-Q. Deng *The Landau and Non-cutoff Boltzmann Equation in Union of Cubes. Preprint*. arXiv:2106.00383.
[18] D.-Q. Deng *Regularity of the Vlasov–Poisson–Boltzmann System Without Angular Cutoff*. Comm. Math. Phys., 387(3):1603–1654, sep 2021.
[19] D.-Q. Deng. The non-cutoff Vlasov-Poisson-Boltzmann and Vlasov-Poisson-Landau Systems in bounded domain. *Preprint*. arXiv:2112.02315.
[20] L. Desvillettes and C. Villani. *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation*. Invent. Math., 159(2):245-316, 2005.
[21] R. J. DiPerna and P.-L. Lions. *On the Fokker-Planck-Boltzmann equation*. Comm. Math. Phys., 120(1):1-23, 1988.
[22] R. J. DiPerna and P.-L. Lions. *On the Cauchy problem for Boltzmann equations: global existence and weak stability*. Ann. of Math. (2), 130(2):321-366, 1989.
[23] H. Dong, Y. Guo, and Z. Ouyang. *The Vlasov-Poisson-Landau System with the Specular-Reflection Boundary Condition Preprint*. arXiv:2010.05314..
[24] R. Duan, Y. Lei, T. Yang and Z. Zhao. *The Vlasov-Maxwell-Boltzmann system near Maxwellians in the whole space with very soft potentials*. Comm. Math. Phys, 351(2017), no.1, 95-153.
[25] R. Duan. *Global smooth dynamics of a fully ionized plasma with long-range collisions*. Ann Inst H Poincaré Anal Non Linéaire, 2014, 31: 751-778.
[26] R. Duan, S. Liu. *The Vlasov-Poisson-Boltzmann system without angular cutoff*. Commun. Math. Phys. 324(1), 1-45 (2013).
[27] R. Duan, S. Liu, S. Sakamoto, and R. Strain *Global mild solutions of the Landau and non-cutoff Boltzmann equations* Comm. Pure Appl. Math., 74 (2021), no. 5, 932-1020.
[28] R. Duan, S. Liu, T. Yang, and H. Zhao. *Stability of the nonrelativistic Vlasov-Maxwell-Boltzmann system for angular non-cutoff potentials*. Kinet. Relat. Models, 6(1): 159-204, 2013.
[29] R. Duan and R. M. Strain. *Optimal large-time behavior of the Vlasov-Maxwell-Boltzmann system in the whole space*. Comm. Pure Appl. Math., 24: 1497-1546, 2011.
[30] R. Duan, T. Yang and H. Zhao. *The Vlasov-Poisson-Boltzmann system in the whole space: The hard potential case*. J. Differential Equations, 252 (2012), 6356-6386.
[31] R. Duan, T. Yang and H. Zhao. *The Vlasov-Poisson-Boltzmann system for soft potentials*. Math Models Methods Appl Sci, 23: 979-1028, 2013.
[32] F. Golse, C. Imbert, C. Mouhot and A. F. Vasseur. *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation*. Annali della Scuola Normale Superiore di Pisa 5, 19 (2019), no. 1, 253-295.
[33] P. T. Gressman and R. M. Strain. *Global classical solutions of the Boltzmann equation without angular cut-off*. J. Amer. Math. Soc., 24(3):771-847, 2011.
[34] P. T. Gressman and R. M. Strain. *Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production*. Adv. Math, 227(6):2349-2384, 2011.
[35] M. P. Gualdani, S. Mischler, and C. Mouhot. *Factorization of non-symmetric operators and exponential H-theorem*. Mém. Soc. Math. Fr. (N.S.), (153):137, 2017.
[36] Y. Guo. *The Landau equation in a periodic box*. Comm. Math. Phys., 231(3):391-434, 2002. .
[37] Y. Guo. *Classical solutions to the Boltzmann equation for molecules with an angular cutoff*. Arch. Ration. Mech. Anal., 169(4):305-353, 2003.
[38] Y. Guo. *The Boltzmann equation in the whole space*. Indiana Univ. Math. J., 53(4):1081-1094, 2004.
[39] Y. Guo. *Boltzmann diffusive limit beyond the Navier-Stokes approximation*. Comm. Pure Appl. Math., 59(2006), no.5, 626-687.
[40] Y. Guo. *The Vlasov-Poisson-Boltzmann system near vacuum*. Comm. Math. Phys., 218(2001), no.2, 293-313.
[41] Y. Guo. *The Vlasov-Poisson-Boltzmann system near Maxwellians*. Comm. Pure Appl. Math., 55(2002), no.9, 1104-1135.
[42] Y. Guo. *The Vlasov-Maxwell-Boltzmann system near Maxwellians*. Invent. Math., 153(2003), no.3, 593-630.
[43] Y. Guo. *Decay and Continuity of the Boltzmann Equation in Bounded Domains*. Arch. Ration. Mech. Anal., 197(3):713–809, dec 2009.
[44] Y. Guo. *The Vlasov-Poisson-Landau system in a periodic box*. J. Amer. Math. Soc., 25(2012), no.3, 759-812.
[45] Y. Guo, H. J. Hwang, J. W. Jang, and Z. Ouyang. *The Landau Equation with the Specular Reflection Boundary Condition*. Arch. Ration. Mech. Anal., 236(3):1389–1454, feb 2020.
[46] Y. Guo, C. Kim, D. Tonon, and A. Trescases. *Regularity of the Boltzmann equation in convex domains*. Invent. Math., 207(1):115–290, jul 2016.
[47] L.-B. He. *Sharp bounds for Boltzmann and Landau collision operators*. Annales Scientifiques de l'École Normale Supérieure 51(5), 2018.

[48] F. Hérau, D. Tonon and I. Tristani. *Regularization estimates and Cauchy Theory for inhomogeneous Boltzmann equation for hard potentials without cut-off.* Commun. Math. Phys., 377, 697-771, (2020).
[49] C. Henderson and S. Snelson. *C^∞ smoothing for weak solutions of the inhomogeneous Landau equation.* Arch. Ration. Mech. Anal., 236(1):113-143, 2020.
[50] C. Imbert, C. Mouhot, L. Silvestre. *Decay estimates for large velocities in the Boltzmann equation without cutoff.* Journal de l'École Polytechnique Mathématiques 7, 143-184 (2020)
[51] C. Imbert, C. Mouhot, L. Silvestre. *Gaussian lower bounds for the Boltzmann equation without cut-off.* SIAM J. Math. Anal. 52, no. 3, 2930-2944.
[52] C. Imbert and L. Silvestre. *The weak Harnack inequality for the Boltzmann equation without cut-off.* J. Eur. Math. Soc. 22(2):507-592, 2020.
[53] C. Imbert and L. Silvestre. *Global regularity estimates for the Boltzmann equation without cut-off*, accepted by Journal of the American Mathematical Society.
[54] C. Imbert and L. Silvestre. *The Schauder estimate for kinetic integral equations.* Analysis and PDE, 14(1), 171-204.
[55] G. Leoni. *A first course in Sobolev spaces.* American Mathematical Society, Providence, Rhode Island, 2017.
[56] P.-L. Lions. *On Boltzmann and Landau equations.* Philos. Trans. Roy. Soc. London Ser. A, 346(1679):191-204, 1994.
[57] J. Luk. *Stability of vacuum for the Landau equation with moderately soft potentials.* Ann. PDE, 5(2019), no.1, 11.
[58] C. Mouhot and R. M. Strain. *Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff.* J. Math. Pures Appl. (9), 87(5):515-535, 2007.
[59] L. Nirenberg. *On elliptic partial differential equations.* Ann. Sc. Norm. Super. Pisa Cl. Sci., Ser. 3, 13(2):115–162, 1959.
[60] L. Silvestre. *A new regularization mechanism for the Boltzmann equation without cut-off.* Comm. Math. Phys., 348(1):69-100, 2016.
[61] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* PRINCETON UNIV PR, 1971.
[62] R. M. Strain. *The Vlasov-Maxwell-Boltzmann system in the whole space.* Comm. Math. Phys, 268(2006), no.2, 543-567.
[63] R. M. Strain and Y. Guo. *Almost exponential decay near Maxwellian.* Comm. Partial Differential Equations, 31(1-3):417-429, 2006.
[64] R. M. Strain and Y. Guo. *Exponential decay for soft potentials near Maxwellian.* Arch. Ration. Mech. Anal., 187(2):287-339, 2008.
[65] R. M. Strain and K. Zhu. *The Vlasov-Poisson-Landau system in $\mathbb{R}^3_x$.* Arch. Ration. Mech. Anal., 210(2):615-671, 2013.
[66] C. Villani. *On the Cauchy problem for Landau equation: sequential stability, global existence.* Adv. Differential Equations, 1(5):793-816, 1996.
[67] C. Villani. *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations.* Arch. Ration. Mech. Anal., 143(3): 273-307, 1998.
[68] Y. Wang. *Global solution and time decay of the Vlasov-Poisson-Landau system in R^3_x.* SIAM J. Math Anal, 44(2012), no.5, 3281-3323.
[69] Q. Xiao, L. Xiong, H. Zhao. *The Vlasov-Poisson-Boltzmann system for non-cutoff hard potentials.* Sci China Math, 57: 515-540, 2014.

(Chuqi Cao) Yau Mathematical Science Center and Beijing Institute of Mathematical Sciences and Applications, Tsinghua University, Beijing 100084, P. R. China.
Email address: chuqicao@gmail.com

(Dingqun Deng) Beijing Institute of Mathematical Sciences and Applications and Yau Mathematical Science Center, Tsinghua University, Beijing, People's Republic of China
Email address: dingqun.deng@gmail.com

(Xingyu Li) CEREMADE (CNRS UMR n.7534), PSL research university, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France.
Email address: li@ceremade.dauphine.fr