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# THE VLASOV-POISSON-BOLTZMANN/LANDAU SYSTEM WITH POLYNOMIAL PERTURBATION NEAR MAXWELLIAN

CHUQI CAO, DINGQUN DENG AND XINGYU LI

ABSTRACT. In this work, we consider the Vlasov-Poisson-Boltzmann system without angular cutoff and the Vlasov-Poisson-Landau system with Coulomb potential near a global Maxwellian  $\mu$ . We establish the global existence, uniqueness and large time behavior for solutions in a polynomial-weighted Sobolev space  $H_{x,v}^2(\langle v \rangle^k)$  for some constant  $k > 0$ . The proof is based on extra dissipation generated from semigroup method and energy estimates on electrostatic field.

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## 1. INTRODUCTION

1.1. **Models and equations.** We consider the Vlasov-Poisson-Boltzmann (VPB) and Vlasov-Poisson-Landau (VPL) systems describing the motion of plasma particles of two species in torus  $\mathbb{T}^3$  (cf. [35, 37]):

$$\begin{aligned} \partial_t F_+ + v \cdot \nabla_x F_+ - \nabla_x \phi \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_-, F_+), \\ \partial_t F_- + v \cdot \nabla_x F_- + \nabla_x \phi \cdot \nabla_v F_- &= Q(F_+, F_-) + Q(F_-, F_-), \\ F(0) &= F_0, \quad E(0) = E_0, \end{aligned} \tag{1.1}$$

where  $E = -\nabla_x \phi$  is the self-consistent electrostatic field satisfying

$$-\Delta_x \phi = \int_{\mathbb{R}^3} (F_+ - F_-) dv, \quad \int_{\mathbb{T}^3} \phi(t, x) dx = 0, \tag{1.2}$$

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and  $F(t, x, v) \geq 0$  is a distribution function of particles at time  $t > 0$  with position  $x \in \mathbb{T}^3$  and velocity  $v \in \mathbb{R}^3$ .

For Vlasov-Poisson-Boltzmann case, the collision operator  $Q$  is a bilinear operator which acts only on the velocity variable  $v$  given by

$$Q(G, F)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(G'_* F' - G_* F) d\sigma dv_*.$$

Here we use the standard notation  $F = F(v)$ ,  $G_* = G(v_*)$ ,  $F' = F(v')$ ,  $G'_* = G(v'_*)$ , where  $v', v'_*$  are velocities of two particles after collision given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^2.$$

This representation follows from physical law of elastic collision:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The nonnegative function  $B(v - v_*, \sigma)$  is called the Boltzmann collision kernel. It depends only on relative velocity  $|v - v_*|$  and the deviation angle  $\theta$  through

$$\cos \theta \stackrel{\text{def}}{=} \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

In the present work, we consider the non-cutoff kernel  $B$  as the followings.

(A1). The Boltzmann kernel  $B$  takes the form

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right),$$

where  $b$  is a nonnegative function.

(A2). The angular function  $b(\cos \theta)$  is not locally integrable and it satisfies

$$\mathcal{K} \theta^{-1-2s} \leq \sin \theta b(\cos \theta) \leq \mathcal{K}^{-1} \theta^{-1-2s}, \quad \text{with } 0 < s < 1,$$

for some constant  $\mathcal{K} > 0$ .

(A3). The parameter  $\gamma$  and  $s$  satisfy the condition

$$-3 < \gamma \leq 1, \quad 1/2 \leq s \leq 1, \quad \gamma + 2s > -1.$$

(A4). Without loss of generality, we may assume that  $B(v - v_*, \sigma)$  is supported in the set  $0 \leq \theta \leq \pi/2$ , i.e.  $\frac{v - v_*}{|v - v_*|} \cdot \sigma \geq 0$ . Otherwise  $B$  can be replaced by its symmetrized form:

$$\bar{B}(v - v_*, \sigma) = |v - v_*|^\gamma \left( b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) + b\left(\frac{v - v_*}{|v - v_*|} \cdot (-\sigma)\right) \right) 1_{\frac{v - v_*}{|v - v_*|} \cdot \sigma \geq 0},$$

where  $1_A$  is the indicator function of the set  $A$ .

For the case of Vlasov-Poisson-Landau system,  $Q$  is the Landau collision operator given by

$$\begin{aligned} Q(g, f)(v) &= \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v - v_*) (g(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} g(v_*)) dv_* \\ &= \partial_i \int_{\mathbb{R}^3} \phi^{ij}(v - v_*) (g_* \partial_j f - f \partial_j g_*) dv_*, \end{aligned}$$

with

$$\begin{aligned} \Phi(u) &= |u|^{\gamma+2} \left( I - \frac{u \otimes u}{|u|^2} \right), \quad -3 \leq \gamma \leq 1, \\ \phi_{ij}(v) &= |v|^{\gamma+2} \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad -3 \leq \gamma \leq 1, \end{aligned}$$

Hereafter we use the convention of summation for repeated indices, and the derivatives are in the velocity variable  $\partial_i = \partial_{v_i}$ . We will use the notations  $g_* = g(v_*)$ ,  $f = f(v)$ ,  $\partial_j g_* = \partial_{v_{*j}} g(v_*)$ ,

$\partial_j f = \partial_{v_j} f(v)$ . We also mention that

$$\begin{aligned}\partial_j \phi^{ij}(v) &= -2|v|^\gamma v_i, \\ \partial_{ij} \phi^{ij}(v) &= \begin{cases} -2(\gamma+3)|v|^\gamma, & \text{if } -3 < \gamma \leq 1, \\ -8\pi\delta_0, & \text{if } \gamma = -3, \end{cases}\end{aligned}$$

where  $\delta_0$  is the Dirac measure.

It's convenient to call it *hard potential* if  $\gamma \geq 0$ , and *soft potential* if  $\gamma \in [-3, 0)$ .

**1.2. Reformulation.** We recall some basic facts and reformulate the Vlasov-Poisson-Boltzmann and Vlasov-Poisson-Landau system.

• **Conservation Law.** If  $F$  is the solution to the Vlasov-Poisson-Boltzmann/Landau system (1.1) with the initial data  $F_0$ , then it enjoys the conservation of mass, momentum and the energy, i.e.

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} F_+ dv dx &= \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} F_- dv dx = 0, & \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v(F_+ + F_-) dv dx &= 0, \\ \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |v|^2(F_+ + F_-) dv dx &+ \frac{d}{dt} \int_{\mathbb{T}^3} |E(t, x)|^2 dx = 0,\end{aligned}\tag{1.3}$$

and the continuity equation

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} F_\pm dv + \nabla_x \cdot \int_{\mathbb{R}^3} v F_\pm dv = 0.$$

For simplicity, we assume the initial data  $F_0$  is normalized such that the equilibrium associated to the equation (1.1) will be the standard Gaussian function, i.e.

$$\mu(v) \stackrel{\text{def}}{=} (2\pi)^{-3/2} e^{-|v|^2/2},$$

which enjoys the same mass, momentum and energy as  $F_0$ .

• **Perturbation Equation.** In the perturbation framework, we denote  $F = [F_+, F_-]$  and let  $f = [f_+, f_-]$  satisfies

$$F_\pm = \mu + f_\pm.$$

Then system (1.1) and (1.2) become

$$\begin{cases} \partial_t f_\pm + v \cdot \nabla_x f_\pm \mp \nabla_x \phi \cdot \nabla_v f_\pm \pm \nabla_x \phi \cdot v \mu \\ \quad = Q(f_\pm + f_\mp, \mu) + Q(2\mu + f_\pm + f_\mp, f_\pm), \\ -\Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) dv, & \int_{\mathbb{T}^3} \phi(x) dx = 0, \\ f(0) = f_0, & E(0) = E_0. \end{cases}\tag{1.4}$$

Note that  $Q(\mu, \mu) = 0$ . We also denote linear operator  $L = [L_+, L_-]$  and  $\mathcal{L} = [\mathcal{L}_+, \mathcal{L}_-]$  by

$$L_\pm f = 2Q(\mu, f_\pm) + Q(f_\pm + f_\mp, \mu),$$

and

$$\mathcal{L} = -v \cdot \nabla_x + L.\tag{1.5}$$

By assuming  $F_0$  have the same mass, total momentum and total energy as  $\mu$ , the conservation laws (1.3) yields

$$\begin{aligned}\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_+ dv dx &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_- dv dx = 0, \\ \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v(f_+ + f_-) dv dx &= 0, \\ \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |v|^2(f_+ + f_-) dv dx &+ \int_{\mathbb{T}^3} |\nabla_x \phi(t, x)|^2 dx = 0\end{aligned}\tag{1.6}$$

and the continuity equation turns to

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f_{\pm}(v) dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f_{\pm}(v) dv = 0. \quad (1.7)$$

The kernel of  $L$  on  $L_v^2 \times L_v^2$  is the span of  $\{[1, 0]\mu, [0, 1]\mu, [1, 1]v\mu, [1, 1]|v|^2\mu\}$  (cf. [35]) and we define the projection of  $L_v^2 \times L_v^2$  onto  $\ker L$  by

$$\mathbf{P}f = \left( a_+(t, x)[1, 0] + a_-(t, x)[0, 1] + v \cdot b(t, x)[1, 1] + (|v|^2 - 3)c(t, x)[1, 1] \right) \mu, \quad (1.8)$$

or equivalently by

$$\mathbf{P}_{\pm}f = \left( a_{\pm}(t, x) + v \cdot b(t, x) + (|v|^2 - 3)c(t, x) \right) \mu,$$

where function  $a_{\pm}, b, c$  are given by

$$\begin{aligned} a_{\pm} &= \int_{\mathbb{R}^3} f_{\pm} dv, & b_j &= \frac{1}{2} \int_{\mathbb{R}^3} v_j (f_+ + f_-) dv, \\ c &= \int_{\mathbb{R}^3} \frac{|v|^2 - 3}{12} (f_+ + f_-) dv. \end{aligned} \quad (1.9)$$

To apply the damping method, we further define the projection  $\mathbf{\Pi} = [\mathbf{\Pi}_+, \mathbf{\Pi}_-]$  over  $\mathbb{T}^3 \times \mathbb{R}^3$  as

$$\begin{aligned} \mathbf{\Pi}_{\pm}f &= \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_{\pm}(u) dudx + v \cdot \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \frac{u}{2} (f_+ + f_-)(u) dudx \right. \\ &\quad \left. + (|v|^2 - 3) \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \frac{|u|^2 - 3}{12} (f_+ + f_-)(u) dudx \right) \mu. \end{aligned}$$

Notice that  $\mathbf{\Pi}\{\mathbf{I} - \mathbf{P}\}f = 0$  and hence,  $\{\mathbf{I} - \mathbf{P}\}f = \{\mathbf{I} - \mathbf{\Pi}\}\{\mathbf{I} - \mathbf{P}\}f$ .

**1.3. Notations.** Let us first introduce the function spaces and notations. We let the multi-indices  $\alpha$  and  $\beta$  be  $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ ,  $\beta = [\beta_1, \beta_2, \beta_3]$  and define

$$\partial_{\beta}^{\alpha} := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

If each component of  $\alpha'$  is not greater than that of the  $\alpha$ 's, we denote by  $\alpha' \leq \alpha$ .  $\alpha' < \alpha$  means  $\alpha' \leq \alpha$ , and  $|\alpha'| < |\alpha|$ . We write  $a \lesssim b$  ( $a \gtrsim b$ ) indicate that there is a uniform constant  $C$ , which may be different on different lines, such that  $a \leq Cb$  ( $a \geq Cb$ ). We use the notation  $a \sim b$  if  $a \lesssim b$  and  $b \lesssim a$ . We denote constants  $a \gg b$  is  $a > b$  is sufficiently large. We denote  $C_{a_1, a_2, \dots, a_n}$  by a constant depending on parameters  $a_1, a_2, \dots, a_n$ . Moreover, we use parameter  $\varepsilon$  to represent different positive numbers much less than 1 and determined in different cases. We use  $(f, g) = (f, g)_{L_v^2}$  to denote the inner product of  $f, g$  over velocity variable for short and,  $(f, g)_{L_{x,v}^2}$  to represent the inner product over both spatial and velocity variables. Also, we write  $(f, g)_{L_k^2} = (\langle v \rangle^{2k} f, g)_{L_v^2}$ ,

$$\|f(\theta)\|_{L_{\theta}^1} = \int_{\mathbb{S}^2} f(\theta) d\sigma = 2\pi \int_0^{\pi} f(\theta) \sin \theta d\theta,$$

and for any  $p \in [1, \infty]$ ,

$$\|f(t)\|_{L_T^p} = \|f(t)\|_{L^p([0, T])}.$$

For the linear operator  $L$ , we write  $S_L$  as the semigroup generated by  $L$ . Define the **Japanese brackets** by

$$\langle v \rangle := (1 + |v|^2)^{1/2}.$$

For real numbers  $m, l$ , we define the weighted Sobolev norm  $\|\cdot\|_{H_l^m}$  by

$$|f|_{H_l^m} = |\langle v \rangle^l \langle D_v \rangle^m f(v)|_{L_v^2}.$$

If  $m = 0$ , we write  $L_l^2 = H_l^0$ . Here  $a(D)$  is a pseudo-differential operator with the symbol  $a(\xi)$ , and is defined by

$$(a(D)f)(v) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(v-u)\xi} a(\xi) f(u) du d\xi.$$

The mixed norm  $\|\cdot\|_{H_x^n H_l^m}$  is defined as

$$\|f\|_{H_x^n H_l^m} := \left( \int_{\mathbb{T}^3} \|\langle D_x \rangle^n f(x, \cdot)\|_{H_l^m}^2 dx \right)^{1/2}.$$

If  $n = 0$ , we write  $H_x^0 H_l^m = L_x^2 H_l^m$ . The entropy  $L \log L$  space is defined as

$$L \log L := \left\{ f(v) : \|f\|_{L \log L} = \int_{\mathbb{R}^3} |f| \log(1 + |f|) dv \right\}.$$

For any  $k \in \mathbb{R}$ ,  $\gamma \in (-3, 1]$ , we define

$$\|f\|_{L_{k+\gamma/2,*}^2}^2 := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mu(v_*) |v - v_*|^\gamma |f(v)|^2 \langle v \rangle^{2k} dv dv_*.$$

It is direct to verify that  $\|f\|_{L_{k+\gamma/2,*}^2} \sim \|f\|_{L_{k+\gamma/2}^2}$  by noticing  $\int_{\mathbb{R}^3} \mu(v_*) |v - v_*|^\gamma \langle v \rangle^{2k} dv_* \sim \langle v \rangle^{\gamma+2s}$ .

For Boltzmann case, we denote the dissipation norm  $L_{D,k}^2$  by

$$\|f\|_{L_{D,k}^2} = \|f\|_{H_{k+\gamma/2}^s},$$

and  $\|f\|_{L_D^2} = \|f\|_{L_{D,0}^2}$ .

For Landau case, we denote the anisotropic norm  $L_D^2(m)$  by

$$\|f\|_{L_D^2(m)} := \|f\|_{L^2(m\langle v \rangle^{\gamma/2})} + \|\tilde{\nabla}_v(mf)\|_{L^2(\langle v \rangle^{\gamma/2})}$$

and, for brevity, we let  $L_{D,k}^2 := L_D^2(\langle v \rangle^k)$  and  $L_D^2 := L_{D,0}^2$ . Here  $\tilde{\nabla}_v$  is the anisotropic gradient given by

$$\tilde{\nabla}_v f := P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f, \quad P_v \xi := \left( \xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}, \quad \forall \xi \in \mathbb{R}^3.$$

With multi-indices  $(\alpha, \beta)$ , we come to define our weight function with some constant  $k$ . For  $-3 \leq \gamma \leq 1$ , we choose the weight function  $w(\alpha, \beta)$  as

$$w(\alpha, \beta) = \begin{cases} \langle v \rangle^{k-p|\alpha|-q|\beta|+r}, & q = 6s - 3(\gamma-1), \quad p = q + \gamma - 1, \\ & r = 2q, \quad \text{for Boltzmann case,} \\ \langle v \rangle^{k-p|\alpha|-q|\beta|+r}, & q = 3 - (\gamma-1), \quad p = 3, \\ & r = 2q, \quad \text{for Landau case.} \end{cases} \quad (1.10)$$

Note that  $w(\alpha, \beta) \geq 1$  for any  $|\alpha| + |\beta| \leq 2$  and any  $k \geq 0$ . For brevity, we write  $w(|\alpha|, |\beta|) = w(\alpha, \beta)$  throughout the paper. Noticing  $w^2(\alpha, \beta)$  is still of the form  $\langle v \rangle^k$  for some  $k > 0$ , we have

$$\nabla_v(w^2(\alpha, \beta)) = A_{\alpha,\beta} \frac{v}{\langle v \rangle^2} w^2(\alpha, \beta), \quad (1.11)$$

for some constant  $A_{\alpha,\beta}$  depending on  $\alpha, \beta, k, \gamma$ . In this work, we will apply a useful space-velocity weight

$$e^{\frac{\pm A_{\alpha,\beta} \phi}{\langle v \rangle^2}}$$

to eliminate the dissipation loss term. Next we define some useful norms in our analysis. For this, we denote constants to be chosen in Theorem 3.1:

$$C_{|\alpha|,|\beta|} \gg C_{|\alpha|,|\beta_1|}, \quad \text{if } |\beta_1| < |\beta|, \quad C_{|\alpha|+1,|\beta|-1} \gg C_{|\alpha|,|\beta|}, \quad (1.12)$$

for any multi-indices  $\alpha, \beta$ . Then for both Boltzmann case and Landau case, we denote energy norms:

$$\|f\|_{X_k} = \sum_{|\alpha|+|\beta|\leq 2} C_{|\alpha|,|\beta|} \|e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} w(\alpha, \beta) \partial_\beta^\alpha f\|_{L_x^2 L_D^2}^2, \quad (1.13)$$

and

$$\|f\|_{Y_k} = \sum_{|\alpha|+|\beta|\leq 2} \|w(\alpha, \beta) \partial_\beta^\alpha f\|_{L_x^2 L_D^2}^2. \quad (1.14)$$

We further define the ‘‘instant energy functional’’  $\mathcal{E}_k(t)$  and ‘‘dissipation energy functional’’  $\mathcal{D}_k(t)$  respectively by

$$\mathcal{E}_k(t) \approx \|f\|_{X_k} + \|\nabla_x \phi\|_{H_x^3}^2, \quad (1.15)$$

and

$$\mathcal{D}_k(t) := \|f\|_{Y_k} + \|\nabla_x \phi\|_{H_x^3}^2, \quad (1.16)$$

where the explicit definition of  $\mathcal{E}_k(t)$  is given in (4.48). Moreover, we fix the weight index  $k_0$  by assuming

$$k_0 \geq \begin{cases} 17, & \text{for the Boltzmann case,} \\ 10, & \text{for the Landau case.} \end{cases} \quad (1.17)$$

**1.4. Main results.** We may now state our main results.

**Theorem 1.1** (Global existence, uniqueness and large-time decay). *Consider the Cauchy problem (1.4) for Vlasov-Poisson-Boltzmann/Landau system. Suppose  $\gamma \in (-3, 1]$ ,  $s \in [\frac{1}{2}, 1)$ ,  $\gamma + 2s > -1$  for the Boltzmann case and  $\gamma \in [-3, 1]$  for the Landau case. Let  $l = 0$  for the hard potential case and  $l > \frac{|\gamma|}{2}$  for the soft potential case. Then there exist constants  $\lambda > 0$ ,  $M > 0$  (small),  $k_1 \geq k_0$  (large) such that, for any*

$$k \geq k_0 + l \text{ for the Landau case and } k \geq k_1 + l \text{ for Boltzmann case,}$$

*if the initial data  $f_0$  satisfies  $F_0(x, v) = \mu + f_0(x, v) \geq 0$ , conservation laws (1.6) and*

$$\|f_0\|_{X_{k_0+l}} \leq M \text{ and } \|f_0\|_{X_k} < +\infty, \quad (1.18)$$

*then there exists a unique solution  $f(t, x, v)$  to (1.4) satisfying  $F(t, x, v) = \mu + f(t, x, v) \geq 0$  such that*

$$\sup_{0 \leq t \leq T} \mathcal{E}_k(t) \leq 2\mathcal{E}_k(0),$$

*for any  $T > 0$ . Moreover, we have the following large-time asymptotic behavior. If  $\gamma \geq 0$ , then*

$$\mathcal{E}_k(t) \leq e^{-\lambda t} \mathcal{E}_k(0).$$

*If  $\gamma < 0$ , then we have*

$$\mathcal{E}_{k-l}(t) \lesssim (1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(0).$$

*Here  $\mathcal{E}_k(t)$  and  $\mathcal{D}_k(t)$  are given in (1.15) and (1.16) respectively.*

**Remark 1.2.** *We emphasize that for global solutions with polynomial weight, one only need to assume the smallness of initial data with respect to a given norm. Here, we only require  $X_{k_0}$  to be sufficiently small instead of requiring general  $X_k$  to be small.*

**1.5. Strategies and ideas of the proof.** In this subsection, we will explain main strategies and ideas of the proof for our results.

In what follows we recall some known results on the Landau and Boltzmann equations with a focus on the topics under consideration in this paper, particularly on global existence and large-time behavior of solutions to the spatially inhomogeneous equations in the perturbation framework. For global solutions to the renormalized equation with large initial data, we mention the classical works by [5, 15–17, 46, 59, 60]. We mention [12, 13] for the best regularity results available for the Boltzmann equation without cut-off. For the stability of vacuum, see [11, 34, 47] for the Landau, cutoff and non-cutoff Boltzmann equation with moderate soft potential respectively.

For the non-cutoff Boltzmann equation, [41–45, 53] obtains global regularity and long time behavior in a very general assumption by just assuming a uniformly bound in  $t, x$  such that

$$0 < m_0 \leq M(t, x) \leq M_0, \quad E(t, x) \leq E_0, \quad H(t, x) \leq H_0$$

for some constant  $m_0, M_0, E_0, H_0$ , where

$$\begin{aligned} M(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) dv, \\ E(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) |v|^2 dv, \\ H(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) \ln f(t, x, v) dv \end{aligned}$$

and for the Landau equation the local Hölder estimate is proved in [26] for and higher regularity of solutions was studied in [40] applying a kinetic version of Schauder estimates.

Then we focus on the results in the perturbation framework. In the near Maxwellian framework, a key point is to characterize the dissipation property in the  $L^2$  norm in  $v$  for the linearized operator and further control the trilinear term in an appropriate way. More precisely, for the non-cutoff Boltzmann equation, the corresponding linearized operator  $L_\mu$  is self-adjoint and has the null space:

$$L_\mu f = \frac{1}{\sqrt{\mu}} (Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)), \quad \text{Null}(L_\mu) = \text{Span}\{\sqrt{\mu}, \sqrt{\mu}v, \sqrt{\mu}|v|^2\}.$$

and having the coercivity property

$$(f, L_\mu f) \leq -C(\|f\|_{H_{\gamma/2}^s}^2 + \|f\|_{L_{s+\gamma/2}^2}^2), \quad f \in \text{Null}(L_\mu).$$

such coercivity property is essential in the near Maxwellian framework. Without angular cutoff assumption, the spectrum of the linearized operator around a global Maxwellian was studied by Pao [52] in the 1970s. This result was then improved by Mouhot [49] and later by Mouhot-Strain [51].

In the near Maxwellian framework, global existence and large-time behavior of solutions to the spatially inhomogeneous equations are proved in Guo [31, 32] and Strian-Guo [56, 57] for the cutoff Boltzmann equation and in [30] for the Landau equation. For the non-cutoff Boltzmann equation, it is first proved in Grassman-Strain [27, 28] and Alexandre-Morimoto-Ukai-Xu-Yang [1–4] independently, see also [21] for a recent work on such topic. All these works above are based on the following decomposition

$$\partial_t f + v \cdot \nabla_x f = L_\mu f + \Gamma(f, f), \quad L_\mu f = \mu^{-1/2} Q(\mu^{1/2} f, \mu) + \mu^{-1/2} Q(\mu, \mu^{1/2} f)$$

with

$$\Gamma(g, f) = \frac{1}{2} \mu^{-1/2} Q(\mu^{1/2} g, \mu^{1/2} f) + \frac{1}{2} \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g)$$

which means that the solution is constructed in  $\mu^{-1/2}$  weighted space.

For the inhomogeneous Boltzmann/Landau equation with polynomial weight perturbation near Maxwellian, Gualdani-Mischler-Mouhot [29] first prove the global existence and large-time



behavior of solutions with polynomial velocity weight for the cutoff Boltzmann equation with hard potential, this method is generalized to the Landau equation in [9, 10]. The case of non-cutoff Boltzmann equation with hard potential is proved in [6, 39]. The case of cutoff Boltzmann with soft potential is proved in [7], and the non-cutoff Boltzmann equation with soft potential is proved in [8].

Here we mention former works on the Vlasov-Poisson/Maxwell-Boltzmann/Landau system near Maxwellian. The Vlasov-Poisson/Maxwell-Boltzmann system for the cut-off hard sphere case is first proved by Guo [35, 36] in the torus, and in the whole space by [55]; see also [23] for the optimal convergence rate. For other case with the cut-off assumption, we refer to [24, 25] for the proof of Vlasov-Poisson case and [18] for the Vlasov-Maxwell case. For the Landau equation, it is first proved in a famous paper [37] for the torus case and [58, 61] for the whole space case, see [19] for another proof using time-dependent weight function. For the non-cutoff case and the Vlasov-Poisson-Boltzmann system, the soft potential case is proved [20] and the hard potential case is proved in [62] and both paper is based on the estimate by [27, 28]. For the Vlasov-Maxwell-Boltzmann system we refer to [22] for an example. We mention that all the works above are base on the following decomposition

$$\partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \mp \nabla_x \phi \cdot \nabla_v f_{\pm} \pm \nabla_x \phi \cdot v \mu^{1/2} \pm \frac{1}{2} \nabla_x \phi \cdot v f = L_{\mu, \pm} f + \Gamma_{\pm}(f, f),$$

where

$$L_{\mu, \pm} f = \mu^{-1/2} Q(\mu^{1/2}(f_{\pm} + f_{\mp}), \mu) + 2\mu^{-1/2} Q(\mu, \mu^{1/2} f_{\pm}),$$

and

$$\Gamma_{\pm}(f, f) = \mu^{-1/2} Q(\mu^{1/2} f_{\pm}, \mu^{1/2} f_{\pm}) + \mu^{-1/2} Q(\mu^{1/2} f_{\mp}, \mu^{1/2} f_{\pm}),$$

which means the result are in  $\mu^{-1/2}$  weighted space, we remark here that our paper is the first to consider such question in a polynomial weighted space.

Then we give some introduction on the semigroup method, which is first initiated in [50] and extended in an abstract setting in a famous work by Gualdani-Mischler-Mouhot [29]; see also its application in kinetic Fokker-Planck equation in [48]. The main idea of semigroup method can be expressed briefly as follows: Taking the homogeneous Boltzmann equation case  $\gamma = 0$  for example, by [51] we have

$$\sum_{\pm} (L_{\pm} f, f_{\pm})_{L^2(\mu^{-1/2})} \leq -\lambda \|f\|_{L^2(\mu^{-1/2})}, \quad \text{if } \mathbf{P}f = 0,$$

for some constant  $\lambda > 0$ , which implies

$$\|S_{\mathcal{L}}(t)f_0\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda t} \|f_0\|_{L^2(\mu^{-1/2})}^2, \quad \text{if } \mathbf{P}f_0 = 0.$$

Define

$$A = M\chi R, \quad B = L - A$$

where  $\chi \in D(\mathbb{R})$  is the truncation function satisfying  $1_{[-1,1]} \leq \chi \leq 1_{[-2,2]}$  and we denote  $\chi_a(\cdot) := \chi(\cdot/a)$  for  $a > 0$ . Thus, if we can prove that

$$\sum_{\pm} (L_{\pm} f, f_{\pm})_{L_k^2} \leq -C \|f\|_{L_k^2}^2 + C_k \|f\|_{L_v^2}^2,$$

then taking  $M, R > 0$  large we have

$$(Bf, f)_{L_k^2} \leq -C \|f\|_{L_k^2}^2, \quad \|Af\|_{L^2(\mu^{-1/2})} \leq C \|f\|_{L_k^2},$$

which implies

$$\|S_B(t)f\|_{L_k^2} \leq e^{-\lambda t} \|f_0\|_{L_k^2}^2.$$

By Duhamel's formula  $S_L = S_B + S_L * AS_B$ , we have

$$\|S_{\mathcal{L}}(t)\|_{L_k^2 \rightarrow L_k^2} \leq \|S_B(t)\|_{L_k^2 \rightarrow L_k^2} + \int_0^t \|S_L(s)\|_{L^2(\mu^{-1/2}) \rightarrow L^2(\mu^{-1/2})}$$

$$\times \|A\|_{L_k^2 \rightarrow L^2(\mu^{-1/2})} \|S_B(t-s)\|_{L_k^2 \rightarrow L_k^2} ds \leq C e^{-\lambda t}.$$

Thus, the rate of convergence for the linear operator  $\mathcal{L}$  is established. Define a scalar product as

$$((f, g))_k = (f, g)_{L_k^2} + C_0 \int_0^{+\infty} (S_{\mathcal{L}}(\tau)f, S_{\mathcal{L}}(\tau)g) d\tau.$$

Since

$$\int_0^{+\infty} (S_{\mathcal{L}}(\tau)f, S_{\mathcal{L}}(\tau)f) d\tau \leq \|f\|_{L_k^2}^2 \int_0^{+\infty} e^{-2\lambda\tau} d\tau \leq C \|f\|_{L_k^2}^2$$

we know that  $((\cdot, \cdot))$  is an equivalent norm to  $L_k^2$ . Moreover,

$$\int_0^{+\infty} (S_{\mathcal{L}}(\tau)\mathcal{L}f, S_{\mathcal{L}}(\tau)f) d\tau = \int_0^{+\infty} \frac{d}{d\tau} \|S_{\mathcal{L}}(\tau)f\|_{L^2}^2 d\tau = \lim_{\tau \rightarrow \infty} \|S_{\mathcal{L}}(\tau)f\|_{L^2}^2 - \|f\|_{L^2}^2 = -\|f\|_{L^2}^2.$$

Then we deduce that

$$\begin{aligned} ((\mathcal{L}f, f)) &= (\mathcal{L}f, f)_{L_k^2} + C_0 \int_0^{+\infty} (S_{\mathcal{L}}(\tau)\mathcal{L}f, S_{\mathcal{L}}(\tau)f) d\tau \\ &= -C_1 \|f\|_{L_k^2} + (C_2 - C_0) \|f\|_{L^2} \leq -C_1 \|f\|_{L_k^2}, \end{aligned}$$

if we choose a suitable large  $C_0 > 0$ . The estimate for the linear operator  $\mathcal{L}$  in this equivalent space allows us to conclude the full convergence by combining the nonlinear estimates. Here we give the example based on the exponential convergence, which is for hard potential. For soft potentials, similar proof can be made by replace  $e^{-\lambda t}$  by polynomial or sub-exponential rate of convergence and using interpolation between norms.

We mention here that all these calculations on the semigroup method will only works for the Boltzmann equation in the sense  $f = \{\mathbf{I} - \Pi\}f$ . The conservation laws of the Boltzmann equation ensure that  $\mathbf{P}f_0 = 0$  implies  $\mathbf{P}f = 0$ . But for the Vlasov-Poisson-Boltzmann system, the conservation law is different and  $\mathbf{P}f_0 = 0$  can not implies  $\mathbf{P}f = 0$ . So we rather work on the estimate of  $\{\mathbf{I} - \mathbf{P}\}f$  term. Note that in our setting  $\{\mathbf{I} - \mathbf{P}\}f = \{\mathbf{I} - \Pi\}\{\mathbf{I} - \mathbf{P}\}f$  and we can apply the semigroup method.

The polynomial perturbation can be seen as change of variable  $F = \mu + \langle v \rangle^k f_{\pm}$  for (1.1). Then

$$\partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \mp \nabla_x \phi \cdot \nabla_v f_{\pm} \pm \nabla_x \phi \cdot v \mu \langle v \rangle^{2k} \pm k \nabla_x \phi \cdot \frac{v}{\langle v \rangle^2} f = L_{k, \pm} f + \Gamma_{k, \pm}(f, f),$$

with

$$L_{k, \pm} f = \langle v \rangle^k Q(\langle v \rangle^{-k}(f_{\pm} + f_{\mp}), \mu) + 2 \langle v \rangle^k Q(\mu, \langle v \rangle^{-k} f_{\pm}),$$

and

$$\Gamma(f, f) = \langle v \rangle^k Q(\langle v \rangle^{-k}(f_{\pm} + f_{\mp}), \langle v \rangle^{-k} f_{\pm}).$$

In order to deal with the extra term  $\pm k \nabla_x \phi \cdot \frac{v}{\langle v \rangle^2} f$ , we follow the idea of [37] and introduce the corresponding weight function

$$\exp \left\{ \frac{\pm k \phi}{\langle v \rangle^2} \right\}.$$

This allows us to absorb the extra term by taking integration by parts as

$$\left( \pm k \nabla_x \phi \cdot \frac{v}{\langle v \rangle^2} f, e^{\frac{\pm k \phi}{\langle v \rangle^2}} f \right)_{L_v^2} + \left( v \cdot \nabla_x f_{\pm}, e^{\frac{\pm k \phi}{\langle v \rangle^2}} f \right)_{L_v^2} = 0.$$

We next explain that why do we need the condition  $s \geq 1/2$  for the Boltzmann case. This is because the norm

$$\int_0^{+\infty} (S_{\mathcal{L}}(\tau)f, S_{\mathcal{L}}(\tau)f) d\tau$$

will produce new bad terms when applying on the Vlasov-Poisson term  $\nabla_x \phi(x) \cdot \nabla_v f$ :

$$\int_0^\infty (S_{\mathcal{L}}(\tau) \nabla_x \phi(x) \cdot \nabla_v f, S_{\mathcal{L}}(\tau) f) d\tau,$$

in which case we cannot take integration by parts with respect to  $\nabla_v$ . Due to the extra  $\nabla_v$  term, we need to use

$$\|S_{\mathcal{L}}(t) \nabla_x \phi(x) \cdot \nabla_v f\|_{L_v^2} \lesssim t^{-1/2} e^{-\lambda t} |\nabla_x \phi(x)| \|\nabla_v f\|_{H_{12}^{-s}} \lesssim t^{-1/2} e^{-\lambda t} |\nabla_x \phi(x)| \|f\|_{H_{12}^s}$$

which requires  $s \geq 1/2$  and

$$\|S_{\mathcal{L}}(t) f\|_{L_v^2} \leq t^{-1/2} e^{-\lambda t} \|f\|_{H_k^{-s}}.$$

In contract to former papers [1–4] and [27, 28], for the non-cutoff Boltzmann equation we do not use anisotropic norm. Instead, we apply the  $H_m^k$  norms. By [2, 38, 56] we have

$$(Q(f, g), h) \leq \|f\|_{L_5^2} \|g\|_{H_{\gamma/2+2s}^s} \|h\|_{H_{\gamma/2}^s}, \quad (Q(g, f), f) \leq -C_g \|f\|_{H_{\gamma/2}^s} + C \|f\|_{L_{\gamma/2}^2}.$$

It means that using isotropic norms will cause a  $2s$  weight loss norm difference in the upper. But by decomposition

$$(\partial_\beta^\alpha Q(\mu + f, f), \partial_\beta^\alpha f) = (Q(\mu + f, \partial_\beta^\alpha f), \partial_\beta^\alpha f) + \sum_{\substack{\alpha_1 \leq \alpha, \beta_1 \leq \beta \\ |\alpha_1| + |\beta_1| < |\alpha| + |\beta|}} (Q(\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f, \partial_{\beta_1}^{\alpha_1} f), \partial_\beta^\alpha f),$$

the loss is only on the second term. As in [56, 57] we apply descent velocity weight  $w(\alpha, \beta)$  with respect to  $\alpha$  and  $\beta$  to absorb the extra  $2s$  loss.

**1.6. Organization of the paper.** In Section 2, we first recall some basic properties of the non-cutoff Boltzmann/Landau collision operator and then compute the basic estimates for the Vlasov-Poisson-Boltzmann/Landau system. We prove the local existence and uniqueness in Section 3 and obtain corresponding global results and decay of solutions in Section 4.

## 2. BASIC ESTIMATES

In this section, we recall some important results on non-cutoff Boltzmann equation and Landau equation.

**2.1. Preliminaries on the Boltzmann/Landau equation.** Before introducing the upper and lower bound for the non-cutoff Boltzmann and Landau collision operator, we first state the following lemmas.

**Lemma 2.1** ([7], Lemma 2.5). *Let  $\nu \in (0, 2)$ . Then for any function  $f$ , we have*

$$\sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{-\nu} |f(v_*)| dv_* \lesssim \|f\|_{L^1}^{1-\frac{2\nu}{3}} \|f\|_{L^2}^{\frac{2\nu}{3}} \quad \text{when } \nu \in (0, \frac{3}{2}), \quad (2.1)$$

and

$$\sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{-\nu} |f(v_*)| dv_* \lesssim \|f\|_{L^1}^{1-\frac{\nu}{2}} \|f\|_{L^3}^{\frac{\nu}{2}} \quad \text{when } \nu \in (0, 2). \quad (2.2)$$

**Lemma 2.2.** *For any  $\gamma \in [-3, 1]$  and any functions  $f, g, h$ , we have*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L_5^2} \|g\|_{L^2} \|h\|_{L_3^2}, \quad (2.3)$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L_5^2} \|g\|_{L^2} \|h\|_{H_3^1}, \quad (2.4)$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L_5^2} \|g\|_{H_{\gamma/2+1}^1} \|h\|_{L_{\gamma/2}^2}. \quad (2.5)$$

If  $\gamma \in (-3, 1]$ , then we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma f(v_*) g^2(v) dv_* dv \lesssim \|f\|_{L^2_5} \|g\|_{H^1_{\gamma/2}}^2. \quad (2.6)$$

*Proof.* If  $-2 \leq \gamma \leq 1$ , we deduce that

$$\int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L^1_3} \|g\|_{L^2} \|h\|_{L^2_3} \lesssim \|f\|_{L^2_5} \|g\|_{L^2} \|h\|_{L^2_3}.$$

If  $-3 \leq \gamma < -2$ , by (2.1), we have

$$\int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} f(v) dv_* \leq \|f\|_{L^2_2},$$

which implies that

$$\int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L^2_2} \int_{\mathbb{R}^3} |g(v) h(v)| dv \lesssim \|f\|_{L^2_2} \|g\|_{L^2} \|h\|_{L^2}.$$

This proves (2.3). For (2.4), if  $\gamma \geq -2$ , then it reduce to the (2.3). So we only need to consider the case  $\gamma \in [-3, -2)$ , in which case we have  $-3 < \gamma + \frac{1}{2} < 0$  and  $-2 < \gamma + \frac{3}{2} < 0$ . Then we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+\frac{1}{2}} \frac{1}{\langle v_* \rangle^4} |g(v)|^2 dv dv_* \leq C \|g\|_{L^2}^2.$$

Applying (2.2) to  $h$ , we have

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+\frac{3}{2}} \langle v_* \rangle^4 |f(v_*)|^2 |h(v)|^2 dv dv_* \\ \lesssim \|f\|_{L^2_2} \| |h|^2 \|_{L^1}^{1-\frac{\gamma+3/2}{2}} \| |h|^2 \|_{L^3}^{\frac{\gamma+3/2}{2}} \lesssim \|f\|_{L^2_2}^2 \|h\|_{H^1}^2. \end{aligned}$$

Here we apply Sobolev embedding  $\|\cdot\|_{L^6} \lesssim \|\cdot\|_{H^1}$ . Thus,

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} |f(v_*)| |g(v)| |h(v)| dv dv_* \\ \leq \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+\frac{3}{2}} \langle v_* \rangle^4 |f(v_*)|^2 |h(v)|^2 dv dv_* \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v - v_*|^{\gamma+\frac{1}{2}}}{\langle v_* \rangle^4} |g(v)|^2 dv dv_* \right)^{\frac{1}{2}} \\ \leq \|f\|_{L^2_2} \|g\|_{L^2} \|h\|_{H^1}. \end{aligned}$$

The proof of (2.4) is then finished. Now we come to (2.5). If  $-1 \leq \gamma \leq 1$ , we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv \lesssim \|f\|_{L^1_{\gamma+1}} \int_{\mathbb{R}^3} \langle v \rangle^{\gamma+1} |g(v) h(v)| dv \\ \lesssim \|f\|_{L^2_4} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}}, \end{aligned}$$

If  $-3 \leq \gamma < -1$ , we have

$$\langle v \rangle^{-(\gamma+1)} \lesssim \langle v_* \rangle^{-(\gamma+1)} \langle v - v_* \rangle^{-(\gamma+1)},$$

which implies

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} f(v_*) g(v) h(v) dv_* dv \\ \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle v - v_* \rangle^{-(\gamma+1)} |v - v_*|^{\gamma+1} f(v_*) \langle v_* \rangle^{-(\gamma+1)} |g(v) h(v)| \langle v \rangle^{\gamma+1} dv_* dv \\ \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v - v_*|^{\gamma+1}) f(v_*) \langle v_* \rangle^{-(\gamma+1)} |g(v) h(v)| \langle v \rangle^{\gamma+1} dv_* dv := T_1 + T_2. \end{aligned}$$

It is easily seen that

$$T_1 \lesssim \|f\|_{L^1_{-(\gamma+1)}} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}} \lesssim \|f\|_{L^2_5} \|g\|_{L^2_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}},$$

If  $\gamma \in [-2, -1)$ , we have  $\gamma + 1 \in (-\frac{3}{2}, 0)$  and by (2.1),

$$\begin{aligned} T_2 &\lesssim \sup_{v \in \mathbb{R}^d} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} |f(v_*)| \langle v_* \rangle^{-(\gamma+1)} dv_* \int_{\mathbb{R}^3} |g(v)h(v)| \langle v \rangle^{\gamma+1} dv \\ &\lesssim \|f\|_{L^2_5} \|g\|_{H^1_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}}. \end{aligned}$$

If  $\gamma \in [-3, -2)$ , by Hardy-Littlewood-Sobolev inequality (cf. [54, Theorem 1.1, pp. 119]) and Hölder's inequality, we have

$$\begin{aligned} T_2 &\lesssim \left\| \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} f(v_*) \langle v_* \rangle^{-(\gamma+1)} dv_* \right\|_{L^6_5} \|hg \langle v \rangle^{\gamma+1}\|_{L^6_5} \\ &\lesssim \|f\|_{L^p_{-(\gamma+1)}} \|g\|_{L^3_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}} \lesssim \|f\|_{L^2_5} \|g\|_{H^1_{\gamma/2+1}} \|h\|_{L^2_{\gamma/2}}, \end{aligned}$$

with  $p = \frac{6}{9+2\gamma} \in [1, 2]$ . The proof for (2.5) is thus finished. Finally we come to prove (2.6). For the case  $\gamma \in [-2, 1]$ , (2.6) can be seen as a special case of (2.5). We focus on the case  $\gamma \in (-3, -2)$ , similarly to (2.1), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma |f(v_*)| |g(v)|^2 dv_* dv \\ &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v - v_*|^\gamma) |f(v_*)| \langle v_* \rangle^{-\gamma} |g(v)|^2 \langle v \rangle^\gamma dv_* dv := I_1 + I_2. \end{aligned}$$

It is seen that

$$I_1 \lesssim \|f\|_{L^1_{-\gamma}} \|g\|_{L^2_{\gamma/2}}^2 \lesssim \|f\|_{L^2_5} \|g\|_{L^2_{\gamma/2}}^2.$$

For  $I_2$ , by Hardy-Littlewood-Sobolev inequality (cf. [54, Theorem 1.1, pp. 119]), we have

$$\begin{aligned} I_2 &\lesssim \|f\|_{L^2_{-\gamma}} \left\| \int_{\mathbb{R}^3} |v - v_*|^\gamma |g(v)|^2 \langle v \rangle^\gamma dv \right\|_{L^2_{v_*}} \\ &\lesssim \|f\|_{L^2_{-\gamma}} \| |g|^2 \langle v \rangle^\gamma \|_{L^p} \lesssim \|f\|_{L^2_5} \|g\|_{L^2_{\gamma/2}}^{2p} \lesssim \|f\|_{L^2_5} \|g\|_{H^1_{\gamma/2}}^2, \end{aligned}$$

where  $p = \frac{6}{9+2\gamma}$  satisfies  $2p \in [2, 6]$ . This gives (2.6) and the proof of Theorem 2.2 is finished.  $\square$

We also recall some basic interpolation on  $x$ .

**Lemma 2.3.** *For any function  $f$  and constant  $k \in \mathbb{R}$ , we have*

$$\|fg\|_{H^2_x} \lesssim \min_{c+d=2} \{\|f\|_{H^c_x} \|g\|_{H^d_x}\}.$$

Moreover,

$$\|f\|_{L^\infty_x L^2_v} \lesssim \|f \langle v \rangle^{\frac{3k}{2}}\|_{H^1_x L^2_v} + \|f \langle v \rangle^{-k}\|_{H^2_x L^2_v}, \quad (2.7)$$

$$\|f\|_{L^3_x L^2_v} \lesssim \|f \langle v \rangle^k\|_{L^2_x L^2_v} + \|f \langle v \rangle^{-k}\|_{H^1_x L^2_v}, \quad (2.8)$$

and

$$\|f\|_{H^1_v} \lesssim \|f \langle v \rangle^k\|_{H^s} + \|f \langle v \rangle^{-ks/(1-s)}\|_{H^{1+s}}. \quad (2.9)$$

*Proof.* By Fourier transform on  $x$ , i.e.  $\widehat{f} = \int_{\mathbb{T}^3} f(x) e^{-ix \cdot y} dx$ , we have

$$\begin{aligned} \|f\|_{L^\infty_x L^2_v} &\lesssim \|\widehat{f}\|_{L^1_y L^2_v} \lesssim \|\langle y \rangle^{8/5} \widehat{f}(y)\|_{L^2_y L^2_v} \leq \|\langle y \rangle \langle v \rangle^{\frac{3k}{2}} \widehat{f}\|_{L^2_x L^2_v}^{2/5} \|\langle y \rangle^2 \langle v \rangle^{-k} \widehat{f}\|_{L^2_x L^2_v}^{3/5} \\ &\lesssim \|f \langle v \rangle^{\frac{3k}{2}}\|_{H^1_x L^2_v} + \|f \langle v \rangle^{-k}\|_{H^2_x L^2_v}. \end{aligned}$$

Similarly, we have

$$\|f\|_{L^3_x L^2_v} \lesssim \|\widehat{f}\|_{L^{3/2}_y L^2_v} \lesssim \|\langle y \rangle^{2/3} \widehat{f}(y)\|_{L^2_y L^2_v} \lesssim \|f \langle v \rangle^k\|_{L^2_x} + \|f \langle v \rangle^{-k}\|_{H^1_x}.$$

By Young's inequality,  $\langle \eta \rangle \lesssim \langle \eta \rangle^s \langle v \rangle^k + \langle \eta \rangle^{1+s} \langle v \rangle^{-\frac{ks}{1-s}}$  and hence  $\langle \eta \rangle$  is a symbol in  $S(\langle \eta \rangle^s \langle v \rangle^k + \langle \eta \rangle^{1+s} \langle v \rangle^{-\frac{ks}{1-s}})$  (cf. [14]), where  $\eta$  is the Fourier variable of  $v$ . Then by [14, Lemma 2.3 and Corollary 2.5], we have

$$\|f\|_{H_v^1} \lesssim \|f\langle v \rangle^k\|_{H^s} + \|f\langle v \rangle^{-ks/(1-s)}\|_{H^{1+s}}.$$

This completes the Lemma.  $\square$

2.1.1. *Upper and lower bound for the non-cutoff Boltzmann operator.* In this section, we consider the Boltzmann collision operator  $Q(f, g)$ .

**Lemma 2.4** ([38], Theorem 1.1). *Let  $w_1, w_2 \in \mathbb{R}$ ,  $a, b \in [0, 2s]$  with  $w_1 + w_2 = \gamma + 2s$  and  $a + b = 2s$ . Then for any functions  $f, g, h$  we have*

(1) *if  $\gamma + 2s > 0$ , then*

$$|(Q(g, h), f)_{L_v^2}| \lesssim (\|g\|_{L_{\gamma+2s+(-w_1)^++(-w_2)^+}^1} + \|g\|_{L^2}) \|h\|_{H_{w_1}^a} \|f\|_{H_{w_2}^b},$$

(2) *if  $\gamma + 2s = 0$ , then*

$$|(Q(g, h), f)_{L_v^2}| \lesssim (\|g\|_{L_{w_3}^1} + \|g\|_{L^2}) \|h\|_{H_{w_1}^a} \|f\|_{H_{w_2}^b},$$

where  $w_3 = \max\{\delta, (-w_1)^+ + (-w_2)^+\}$ , with  $\delta > 0$  sufficiently small.

(3) *if  $-1 < \gamma + 2s < 0$ , we have*

$$|(Q(g, h), f)_{L_v^2}| \lesssim (\|g\|_{L_{w_4}^1} + \|g\|_{L_{-(\gamma+2s)}^2}) \|h\|_{H_{w_1}^a} \|f\|_{H_{w_2}^b}$$

where  $w_4 = \max\{-\gamma + 2s, \gamma + 2s + (-w_1)^+ + (-w_2)^+\}$ .

As a first application we have the following corollaries.

**Corollary 2.5.** *Suppose  $\gamma > -3$  and  $\gamma + 2s > -1$ . For any multi-indices  $|\beta| \leq 2$ , any constant  $k \geq 0$  and any functions  $f, g$ , we have*

$$(Q(\partial_\beta \mu, f), g\langle v \rangle^{2k}) \leq C_k \|f\|_{H_{k+\gamma/2+2s}^s} \|g\|_{H_{k+\gamma/2}^s},$$

for some constant  $C_k > 0$ .

**Lemma 2.6** ([8], Lemma 3.3). *Suppose that  $-3 < \gamma \leq 1$ . For any  $k \geq 14$ , and functions  $g, h$ , we have*

$$\begin{aligned} & |(Q(h, \mu), g\langle v \rangle^{2k})| \\ & \leq \|b(\cos \theta) \sin^{k-\frac{3+\gamma}{2}}\|_{L_\theta^1} \|h\|_{L_{k+\gamma/2,*}^2} \|g\|_{L_{k+\gamma/2,*}^2} + C_k \|h\|_{L_{k+\gamma/2-1/2}^2} \|g\|_{L_{k+\gamma/2-1/2}^2} \\ & \leq \|b(\cos \theta) \sin^{k-2}\|_{L_\theta^1} \|h\|_{L_{k+\gamma/2,*}^2} \|g\|_{L_{k+\gamma/2,*}^2} + C_k \|h\|_{L_{k+\gamma/2-1/2}^2} \|g\|_{L_{k+\gamma/2-1/2}^2}, \end{aligned} \quad (2.10)$$

for some constant  $C_k > 0$ . Moreover, for any  $|\beta| \leq 2$  we have

$$|(Q(h, \partial_\beta \mu), g\langle v \rangle^{2k})| \leq C_k \|h\|_{L_{k+\gamma/2}^2} \|g\|_{L_{k+\gamma/2}^2}. \quad (2.11)$$

**Remark 2.7.** *In [8, Lemma 3.3], the authors only prove the first statement of Lemma 2.6, but the second statement can be proved in the same way by just replacing  $\mu$  by  $\partial_\beta \mu$ .*

**Theorem 2.8** ([8], Theorem 3.1). *Suppose that  $-3 < \gamma \leq 1$ ,  $\gamma + 2s > -1$ ,  $k \geq 14$  and  $G = \mu + g \geq 0$ . If there exists  $A_1, A_2 > 0$  such that*

$$G \geq 0, \quad \|G\|_{L^1} \geq A_1, \quad \|G\|_{L^2} + \|G\|_{L \log L} \leq A_2,$$

then

$$\begin{aligned}
(Q(G, f), f\langle v \rangle^{2k}) &\leq -\frac{1}{8} \|b(\cos \theta)(1 - \cos^{2k-3-\gamma} \frac{\theta}{2})\|_{L^1_\theta} \|f\|_{L^2_{k+\gamma/2,*}}^2 - \gamma_1 \|f\|_{H^s_{k+\gamma/2}}^2 \\
&\quad + C_k \|f\|_{L^2_{k+\gamma/2-1/2}}^2 + C_k \|f\|_{L^2_{14}} \|g\|_{H^s_{k+\gamma/2}} \|f\|_{H^s_{k+\gamma/2}} + C_k \|g\|_{L^2_{14}} \|f\|_{H^s_{k+\gamma/2}}^2 \\
&\leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L^1_\theta} \|f\|_{L^2_{k+\gamma/2,*}}^2 - \gamma_1 \|f\|_{H^s_{k+\gamma/2}}^2 + C_k \|f\|_{L^2}^2 \\
&\quad + C_k \|f\|_{L^2_{14}} \|g\|_{H^s_{k+\gamma/2}} \|f\|_{H^s_{k+\gamma/2}} + C_k \|g\|_{L^2_{14}} \|f\|_{H^s_{k+\gamma/2}}^2,
\end{aligned} \tag{2.12}$$

for some constants  $\gamma_1, C_k > 0$ .

**Remark 2.9.** In [8, Theorem 3.1], the authors prove the result for  $k \geq 22$ . However, the case  $k \geq 14$  can also be obtained by applying the same technique. Note that whenever  $\|g\|_{L^1_v}$  is small, we have  $\|\mu + g\|_{L^1_v} \geq 1 - \|g\|_{L^1_v} > 0$ . Moreover, we couldn't obtain a better estimate for the term  $(Q(f, g), g\langle v \rangle^{2k})$  as the Landau case (see (2.19)) and thus we should put the term  $\mu + g \geq 0$  together.

**Lemma 2.10** ([8], Lemma 3.4 + Lemma 2.4). For any functions  $f, g, h$  and  $k \geq 14$ , we have

$$\begin{aligned}
(Q(f, g), h\langle v \rangle^{2k}) &\lesssim \|f\|_{L^2_{14}} \min\{\|g\|_{H^s_{k+\gamma/2}} \|h\|_{H^s_{k+\gamma/2+2s}}, \|g\|_{H^s_{k+\gamma/2+2s}} \|h\|_{H^s_{k+\gamma/2}}\} \\
&\quad + \|g\|_{L^2_{14}} \|f\|_{H^s_{k+\gamma/2}} \|h\|_{H^s_{k+\gamma/2}}.
\end{aligned} \tag{2.13}$$

In particular, by duality we have

$$\|Q(f, g)\|_{H^{k-\gamma/2}} \lesssim \|f\|_{L^2_{14}} \|g\|_{H^s_{k+\gamma/2+2s}} + \|g\|_{L^2_{14}} \|f\|_{H^s_{k+\gamma/2}}. \tag{2.14}$$

2.1.2. *Upper and lower bound for Landau operator.* In this section, we consider the Landau collision operator  $Q(f, g)$ .

**Lemma 2.11.** Denote  $m = \langle v \rangle^k$  with  $k \geq 7$ . For any  $-3 \leq \gamma \leq 1$ , any functions  $f, g$  and  $|\beta| \leq 2$ , we have

$$|(Q(f, \partial_\beta \mu), g\langle v \rangle^{2k})| \leq C_k \|f\|_{L^2_5} \|g\|_{L^2_5}, \tag{2.15}$$

and

$$(Q(\mu, f), f\langle v \rangle^{2k}) \leq -\gamma_1 \|f\|_{L^2_D(m)}^2 + C_k \|f\|_{L^2}^2, \tag{2.16}$$

for some constants  $\gamma_1, C_k > 0$ .

*Proof.* The first estimate follows from [10, Lemma 2.5] and in [9, Lemma 2.12] with  $\mu$  replacing by  $\partial_\beta \mu$  if necessary. The second estimate follows from [10, Lemma 2.3] and [9, Lemma 2.7]. Note that the work [9] is for the case  $\gamma \in [-2, 1]$  and [10] is for  $\gamma \in [-3, -2]$ .  $\square$

**Lemma 2.12** ([9], Lemma 3.5 and [10], Lemma 4.3). For any  $-3 \leq \gamma \leq 1$ ,  $k \geq 7$  and any functions  $f, g, h$ , we have

$$|(Q(f, g), h\langle v \rangle^{2k})| \leq C \|f\|_{L^2_7} \min\{\|g\|_{L^2_{D,k+1}} \|h\|_{L^2_{D,k}}, \|g\|_{L^2_{D,k}} \|h\|_{L^2_{D,k+1}}\}. \tag{2.17}$$

By duality, we have

$$\|Q(f, g)\|_{H^{k-\gamma/2}} \lesssim \|f\|_{L^2_7} \|g\|_{L^2_{D,k+1}}. \tag{2.18}$$

When  $h = g$ , we have a better estimate:

$$(Q(f, g), g\langle v \rangle^{2k}) \leq C \|f\|_{L^2_7} \|g\|_{L^2_{D,k}}^2. \tag{2.19}$$

*Proof.* The proof is similar to [9, Lemma 3.5] and [10, Lemma 4.3] with a little modification. Denoting  $m = \langle v \rangle^k$ , we have

$$\begin{aligned} (Q(f, g), hm^2) &= \int_{\mathbb{R}^3} \partial_j \{(\phi^{ij} * f) \partial_i g - (\partial_i \phi^{ij} * f) g\} hm^2 dv \\ &= - \int_{\mathbb{R}^3} (\phi^{ij} * f) \partial_i g \partial_i hm^2 dv - \int_{\mathbb{R}^3} (\phi^{ij} * f) \partial_i gh \partial_j m^2 dv \\ &\quad + \int_{\mathbb{R}^3} (\partial_i \phi^{ij} * f) g \partial_j hm^2 dv + \int_{\mathbb{R}^3} (\partial_i \phi^{ij} * f) gh \partial_j m^2 dv := T_1 + T_2 + T_3 + T_4. \end{aligned}$$

For the  $T_1, T_2$  term, by [9, Lemma 3.5] and [10, Lemma 4.3], we have

$$|T_1| + |T_2| \leq C \|f\|_{L^2_\gamma} \|g\|_{L^2_{D,k}} \|h\|_{L^2_{D,k}}.$$

Now we give a better estimate for the  $T_3, T_4$  term. Since  $\partial_j m^2 \leq C \langle v \rangle^{-1} m^2$ , for the term  $T_3, T_4$ , by (2.5), we have

$$\begin{aligned} |T_3| &\leq C \|f\|_{L^2_\gamma} \min\{\|g\|_{L^2_{D,k+1}} \|h\|_{L^2_{D,k}}, \|g\|_{L^2_{D,k}} \|h\|_{L^2_{D,k+1}}\}, \\ |T_4| &\leq C \|f\|_{L^2_\gamma} \|g\|_{L^2_{D,k}} \|h\|_{L^2_{D,k}}, \end{aligned}$$

and (2.17) is thus proved. For the case  $g = h$ , we give a better estimate for  $T_3$ . In fact,

$$\begin{aligned} T_3 &= \int_{\mathbb{R}^3} (\partial_i \phi^{ij} * f) g \partial_j gm^2 dv \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} (\partial_{ij} \phi^{ij} * f) g^2 m^2 dv - \frac{1}{2} \int_{\mathbb{R}^3} (\partial_i \phi^{ij} * f) \partial_j m^2 g^2 dv := T_{31} + T_{32}. \end{aligned}$$

For the  $T_{32}$  term, similar to the  $T_4$  term, by (2.5), we have

$$T_{32} \leq C \|f\|_{L^2_\gamma} \|g\|_{L^2_{D,k}}^2.$$

For the  $T_{31}$  term, if  $\gamma = -3$ , then  $\partial_{ij} \phi^{ij} = -8\pi\delta_0$ . It is easily seen that

$$T_{31} \leq C \int_{\mathbb{R}^3} |f| g^2 m^2 dv \leq C \|f\|_{L^2_{-\gamma}} \|g^2\|_{L^2_{2k+\gamma}} \leq C \|f\|_{L^2_\gamma} \|g\|_{L^4_{k+\gamma/2}}^2 \leq C \|f\|_{L^2_\gamma} \|g\|_{L^2_{D,k}}^2.$$

For the case  $\gamma \in [0, 1]$ , we have

$$T_{31} \leq C \int_{\mathbb{R}^3} |v - v_*|^\gamma |f_*| g^2 m^2 dv dv_* \leq C \|f\|_{L^1_\gamma} \|g\|_{L^2_{k+\gamma/2}}^2 \leq C \|f\|_{L^2_\gamma} \|g\|_{L^2_{D,k}}^2.$$

The case  $\gamma \in (-3, 1]$  follows from (2.6). Collecting the above estimates on  $T_{31}$  and  $T_{32}$ , we obtain (2.19) and complete the proof of Lemma 2.12.  $\square$

For the Landau operator, we need another upper bound which writes

**Lemma 2.13.** *For the Landau operator  $Q$ , any  $\gamma \in [-3, 1]$  and any function  $f, g, h$ , we have*

$$|(Q(f, g), h)| \leq \|f\|_{L^2_\gamma} \|g\|_{L^2} \|h\|_{H^2_\gamma}, \quad |(Q(f, g), h)| \leq \|f\|_{L^2_\gamma} \|g\|_{L^2_{10}} \|h\|_{H^2_{-5}},$$

*Proof.* By integration by parts we easily have

$$\begin{aligned} (Q(f, g), h) &= \int_{\mathbb{R}^3} \left( -(\phi^{ij} * f) \partial_j g \partial_i h + (\partial_j \phi^{ij} * f) g \partial_i h \right) dv \\ &= \int_{\mathbb{R}^3} \left( (\phi^{ij} * f) g \partial_i \partial_j h + 2(\partial_j \phi^{ij} * f) g \partial_i h \right) dv, \end{aligned}$$

Then by the homogeneity of  $\phi^{ij}$  and  $\partial_j \phi^{ij}$ , we have

$$|\phi^{ij}(v - v_*)| \lesssim |v - v_*|^{\gamma+2}, \quad |\partial_j \phi^{ij}(v - v_*)| \lesssim |v - v_*|^{\gamma+1}.$$

So the theorem follows directly from (2.3) and (2.4).  $\square$



**2.2. Estimates on the weight.** For  $|\beta| \geq 1$ , we need to deal with the term  $v \cdot \nabla_x f$ . Recall the weight function  $w = w(|\alpha|, |\beta|) = w(\alpha, \beta)$  is defined in (1.10). We have the following properties of  $w$ .

**Lemma 2.14.** *Assume  $-3 < \gamma \leq 1$ ,  $1/2 \leq s < 1$  for Boltzmann case and  $-3 \leq \gamma \leq 1$  for Landau case. For any multi-indices  $\alpha, \beta$  and non-negative integer  $k$ ,  $w(\alpha, \beta)$  satisfies the following properties.*

- For  $|\alpha_1| < |\alpha|$ ,  $|\beta_1| < |\beta|$ , we have

$$w(\alpha, \beta) \langle v \rangle^{6s} \leq w(\alpha, \beta_1), \quad w(\alpha, \beta) \langle v \rangle^{6s} \leq w(\alpha_1, \beta), \quad (2.20)$$

for the Boltzmann case, and

$$w(\alpha, \beta) \langle v \rangle^3 \leq w(\alpha, \beta_1), \quad w(\alpha, \beta) \langle v \rangle^3 \leq w(\alpha_1, \beta), \quad (2.21)$$

for the Landau case.

- For both Boltzmann and Landau case, and any  $|\alpha| \geq 0$ ,  $|\beta| \geq 1$ , we have

$$w(\alpha, \beta) \leq \langle v \rangle^{\gamma-1} w(|\alpha| + 1, |\beta| - 1). \quad (2.22)$$

- For both Boltzmann and Landau case, and any  $|\alpha| + |\beta| \leq 2$ ,

$$\langle v \rangle^{k+6} \leq w(\alpha, \beta). \quad (2.23)$$

- For the Boltzmann case and any  $|\alpha| \geq 1$ ,  $|\beta| \geq 0$ , we have

$$w(\alpha, \beta) \leq w(|\alpha| - 1, |\beta|)^s w(|\alpha| - 1, |\beta| + 1)^{1-s} \langle v \rangle^\gamma. \quad (2.24)$$

*Proof.* Recall that  $q = 6s - 3(\gamma-1)$ ,  $p = q + \gamma - 1$ ,  $r = 2q + 6$ , for Boltzmann case, and  $q = 3 - (\gamma-1)$ ,  $p = 3$ ,  $r = 2q + 6$  for the Landau case. The estimate (2.20) and (2.21) is just from the fact that  $-3 \leq \gamma \leq 1$  and

$$p = -3(\gamma-1) + 6s + \gamma - 1 \geq 6s, \quad q \geq 6s,$$

for the Boltzmann case and  $p, q \geq 3$  for the Landau case. The estimate (2.22) follows from the fact that

$$k - p|\alpha| - q|\beta| + r \leq k - p(|\alpha| + 1) - q(|\beta| - 1) + r + \gamma - 1 \iff q - p + \gamma - 1 \geq 0,$$

and we conclude (2.22) from the definition of  $p, q$ . From  $p \leq q$  and

$$\langle v \rangle^{k+6} = w(0, 2) = \min_{|\alpha|+|\beta| \leq 2} \{w(\alpha, \beta)\},$$

we obtain (2.23). (2.24) is equivalent to

$$\begin{aligned} k - p|\alpha| - q|\beta| + r \\ \leq (k - p(|\alpha| - 1) - q|\beta| + r)s + (k - p(|\alpha| - 1) - q(|\beta| + 1) + r)(1 - s) + \gamma, \end{aligned}$$

which is equivalent to

$$0 \leq ps + p(1 - s) - q(1 - s) + \gamma \iff 0 \leq qs + (p - q) + \gamma = (6s - 3(\gamma-1))s + \gamma - 1 + \gamma.$$

We thus conclude (2.24) from the definition of  $q$ ,  $1/2 \leq s < 1$  and  $-3 < \gamma \leq 1$ .  $\square$

We can directly deduce the following Corollary from Lemma 2.14.

**Corollary 2.15.** *Assume the same conditions as in Lemma 2.14. Then*

$$\max_{|\alpha|+|\beta|=1} w^2(\alpha, \beta) \langle v \rangle^\kappa \leq w(0, 0)w(1, 0), \quad (2.25)$$

and

$$\max_{|\alpha|+|\beta|=2} w^2(\alpha, \beta) \langle v \rangle^\kappa \leq w(1, 0)w(2, 0), \quad \max_{\substack{|\alpha|+|\beta|=2 \\ |\beta| \geq 1}} w^2(\alpha, \beta) \langle v \rangle^\kappa \leq w(0, 1)w(1, 1), \quad (2.26)$$

as well as

$$\max_{|\alpha|+|\beta|=2} w^2(\alpha, \beta) \langle v \rangle^\kappa \leq w^2(2, 0) \langle v \rangle^\kappa \leq w(1, 0)^{4/5} w(2, 0)^{6/5}, \quad (2.27)$$

where  $\kappa = 4s$  for Boltzmann case,  $\kappa = 2$  for Landau case.

**2.3. Weighted estimates.** Next we come to the nonlinear term for the Vlasov-Poisson term. Noticing  $w^2(\alpha, \beta)$  is still of the form  $\langle v \rangle^k$  for some  $k > 0$ , we have

$$\nabla_v(w^2(\alpha, \beta)) = A_{\alpha, \beta} \frac{v}{\langle v \rangle^2} w^2(\alpha, \beta), \quad (2.28)$$

for some constant  $A_{\alpha, \beta}$  depending on  $\alpha, \beta, k, \gamma$ . Then we can apply the space-velocity weight  $e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}}$ . To the end of this section, we will assume

$$\|\phi\|_{L_x^\infty} \leq C < \infty. \quad (2.29)$$

and hence,

$$\left| e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \right| \leq C < \infty. \quad (2.30)$$

**Lemma 2.16.** *Suppose that  $-3 \leq \gamma \leq 1$  for Landau case and  $-3 < \gamma \leq 1$  for Boltzmann case. For  $|\alpha| + |\beta| \leq 2$  and any functions  $f_\pm, \phi$ , we have*

$$\begin{aligned} & \left| (v \cdot \nabla_x \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \mp (\nabla_x \phi \cdot \nabla_v \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \right| \\ & \lesssim \|\nabla_x \phi\|_{H_x^2} \|f\|_{Y_k}^2, \end{aligned} \quad (2.31)$$

and for  $|\beta| > 0$ ,

$$\begin{aligned} & \sum_{|\beta_1|=1} |(\partial_{\beta_1} v \cdot \nabla_x \partial_{\beta-\beta_1}^\alpha f, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \\ & \leq C \sum_{\substack{|\beta_2|=|\beta|-1 \\ |\alpha_2|=|\alpha|+1}} \|\partial_{\beta_2}^{\alpha_2} f w(\alpha_2, \beta_2)\|_{L_x^2 L_{\gamma/2}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}^2}. \end{aligned} \quad (2.32)$$

For  $|\alpha_1| \geq 1$ , we have

$$|(\nabla_x \partial^{\alpha_1} \phi \cdot \nabla_v \partial_\beta^{\alpha-\alpha_1} f, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \lesssim \|\nabla_x \phi\|_{H_x^3} \|f\|_{Y_k}^2 \quad (2.33)$$

where  $A_{\alpha, \beta}$  is given by (2.28), for some constant  $C_k > 0$  for both Landau and Boltzmann case.

*Proof.* For (2.31), taking integration by parts with respect to  $\nabla_x$ , we have

$$(v \cdot \nabla_x \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \pm \frac{A_{\alpha, \beta}}{2} \left( \frac{v \cdot \nabla_x \phi}{\langle v \rangle^2} \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta) \right)_{L_{x,v}^2} = 0.$$

Taking integration by parts with respect to  $\nabla_v$  and using (2.28), we have

$$\begin{aligned} & \mp (\nabla_x \phi \cdot \nabla_v \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \\ & = \mp \left( \frac{A_{\alpha, \beta} v \cdot \nabla_x \phi}{\langle v \rangle^4} \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta) \right)_{L_{x,v}^2} \\ & \mp \frac{1}{2} \left( \frac{A_{\alpha, \beta} v \cdot \nabla_x \phi}{\langle v \rangle^2} \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta) \right)_{L_{x,v}^2}. \end{aligned}$$

Noticing  $|\frac{v}{\langle v \rangle^4}| \leq \langle v \rangle^\gamma$  for  $-3 \leq \gamma \leq 1$  and taking summation of the above two estimates, we have

$$\begin{aligned} & \left| (v \cdot \nabla_x \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \mp (\nabla_x \phi \cdot \nabla_v \partial_\beta^\alpha f_\pm, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_\beta^\alpha f_\pm w^2(\alpha, \beta))_{L_{x,v}^2} \right| \\ & \lesssim \|\nabla_x \phi\|_{L_x^\infty} \|\langle v \rangle^{\frac{\gamma}{2}} \partial_\beta^\alpha f_\pm w(\alpha, \beta)\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma}{2}} \partial_\beta^\alpha f_\pm w(\alpha, \beta)\|_{L_{x,v}^2} \\ & \lesssim \|\nabla_x \phi\|_{H_x^2} \|f\|_{Y_k}^2. \end{aligned}$$

This yields (2.31). Here we used the fact (2.29) and its consequence (2.30).

Notice that

$$\partial_i v \cdot \nabla_x \partial_{\beta - e_i}^\alpha f = \partial_{\beta - e_i}^{\alpha + e_i} f, \quad \forall i = 1, 2, 3,$$

where  $e_i$  is the unit vector with  $i$ -th component being 1. Then (2.32) follows from (2.22) and (2.30).

When  $1 \leq |\alpha_1| \leq 2$ , we have

$$\begin{aligned} & (\nabla_x \partial^{\alpha_1} \phi \cdot \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f, e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} \partial_{\beta}^{\alpha} f w^2(\alpha, \beta))_{L_{x,v}^2} \\ & \lesssim \left( \sum_{|\alpha_1|=1} \|\nabla_x \partial^{\alpha_1} \phi\|_{L_x^\infty} \|\langle v \rangle^{-\frac{\gamma}{2}} \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 L_v^2} \right. \\ & \quad \left. + \sum_{|\alpha_1|=2} \|\nabla_x \partial^{\alpha_1} \phi\|_{L_x^3} \|\langle v \rangle^{-\frac{\gamma}{2}} \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^6 L_v^2} \right) \|\langle v \rangle^{\frac{\gamma}{2}} \partial_{\beta}^{\alpha} f w(\alpha, \beta)\|_{L_x^2 L_v^2} \\ & \lesssim \|\nabla_x \phi\|_{H_x^3} \sum_{|\alpha_1|=1} \|\langle v \rangle^{-\frac{\gamma}{2}} \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 L_v^2} \|f\|_{Y_k}. \end{aligned} \quad (2.34)$$

For the Boltzmann case, (2.24) implies

$$w(\alpha, \beta) \leq w(|\alpha| - 1, |\beta|)^s w(|\alpha| - 1, |\beta| + 1)^{1-s} \langle v \rangle^\gamma, \quad \forall |\alpha| \geq 1, |\beta| \geq 0.$$

Together with (2.9), we have for  $|\alpha_1| = 1$  that

$$\begin{aligned} & \|\langle v \rangle^{-\frac{\gamma}{2}} \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 L_v^2} \leq \|\partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 H_v^1} \\ & \lesssim \|\partial_{\beta}^{\alpha - \alpha_1} f w(|\alpha| - 1, \beta)\|_{L_x^2 H_{\gamma/2}^s} + \|\partial_{\beta}^{\alpha - \alpha_1} f w(|\alpha| - 1, |\beta| + 1)\|_{L_x^2 H_{\gamma/2}^{1+s}} \lesssim \|f\|_{Y_k}. \end{aligned}$$

For the Landau case, it follows from (2.21) that

$$w(\alpha, \beta) \leq w(|\alpha| - 1, |\beta|) \langle v \rangle^\gamma, \quad \forall |\alpha| \geq 1, |\beta| \geq 0.$$

Thus we have

$$\begin{aligned} \|\langle v \rangle^{-\frac{\gamma}{2}} \nabla_v \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 L_v^2} & \leq \|\langle v \rangle^{-\frac{\gamma}{2}} \partial_{\beta}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{L_x^2 H_v^1} \\ & \lesssim \|\partial_{\beta}^{\alpha - \alpha_1} f w(|\alpha| - 1, \beta)\|_{L_x^2 L_D^2} \lesssim \|f\|_{Y_k}. \end{aligned}$$

The proof of (2.33) is finished by plugging the above estimates into (2.34).  $\square$

Next, we compute the linear part for Poisson term.

**Lemma 2.17.** *For  $|\alpha| + |\beta| \leq 2$ , any functions  $f_{\pm}, \phi$  satisfying (1.4) and any  $\psi$ , there exists  $C_k > 0$  such that*

$$|(\partial_{\beta}^{\alpha} (\nabla_x \phi \cdot v \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_{\beta}^{\alpha} f_{\pm} w^2(\alpha, \beta))_{L_{x,v}^2}| \leq C_k \|\nabla_x \phi\|_{H_x^2} \|f\|_{H_x^2 L_v^2}. \quad (2.35)$$

When  $|\beta| = 0$ , we have

$$\begin{aligned} & \sum_{\pm} \pm (\partial^{\alpha} \nabla_x \phi \cdot v \mu, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial^{\alpha} f_{\pm} w^2(\alpha, \beta))_{L_{x,v}^2} \geq C_{\alpha} \partial_t \|\partial^{\alpha} \nabla_x \phi\|_{L_x^2}^2 \\ & \quad - C \|\partial^{\alpha} \nabla_x \phi\|_{L_x^2} \|\partial^{\alpha} \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2} - C \|\partial^{\alpha} \nabla_x \phi\|_{L_x^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}, \end{aligned} \quad (2.36)$$

for some  $C_{\alpha} > 0$ .

*Proof.* From integration by parts, we have

$$\begin{aligned} & (\partial_{\beta}^{\alpha} (\nabla_x \phi \cdot v \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_{\beta}^{\alpha} f w^2(\alpha, \beta))_{L_{x,v}^2} \\ & = (\nabla_x \partial^{\alpha} \phi \cdot \partial_{\beta} (v \mu) w^2(\alpha, \beta), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_{\beta}^{\alpha} f)_{L_{x,v}^2} \\ & = ((-1)^{|\beta|} \nabla_x \partial^{\alpha} \phi \cdot \partial_{\beta} (\partial_{\beta} (v \mu) e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta)), \partial^{\alpha} f)_{L_{x,v}^2}. \end{aligned} \quad (2.37)$$

By Fourier transform, Cauchy-Schwarz inequality and using the exponential decay of  $\mu$ , it is direct to show that (2.37) is bounded by

$$\|\nabla_x \partial^\alpha \phi\|_{L_x^2} \left\| \int_{\mathbb{R}^d} \partial^\alpha f \partial_\beta (\partial_\beta (v\mu)) e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} w^2(\alpha, \beta) dv \right\|_{L_x^2} \lesssim \|\nabla_x \phi\|_{H_x^2} \|f\|_{H_x^2 L_v^2}.$$

When  $\beta = 0$ , we split  $e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} = 1 + e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1$  and  $f_\pm = \mathbf{P}_\pm f + \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f$  to obtain

$$\begin{aligned} \pm (\partial^\alpha \nabla_x \phi \cdot v\mu, e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L_{x,v}^2} &= \pm (\partial^\alpha \nabla_x \phi \cdot v\mu, w^2(\alpha, 0) P \partial^\alpha f_\pm)_{L_{x,v}^2} \\ &\quad \pm (\partial^\alpha \nabla_x \phi \cdot v\mu, w^2(\alpha, 0) \{I - P\} \partial^\alpha f_\pm)_{L_{x,v}^2} \\ &\quad \pm (\partial^\alpha \nabla_x \phi \cdot v\mu, (e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1) w^2(\alpha, 0) \partial^\alpha f_\pm)_{L_{x,v}^2}. \end{aligned} \quad (2.38)$$

where we temporarily define the projection

$$Pf = \left( \int_{\mathbb{R}^3} f(u) du + v \cdot \int_{\mathbb{R}^3} u f(u) du + (|v|^2 - 3) \int_{\mathbb{R}^3} \frac{|u|^2 - 3}{6} f(u) du \right) \mu.$$

Then one can check that

$$\{I - P\}f_\pm = \{I - P\}\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f. \quad (2.39)$$

For the first right hand term of (2.38), we have from (1.7) and (1.4)<sub>2</sub> that

$$\begin{aligned} &\sum_{\pm} \pm (\partial^\alpha \nabla_x \phi \cdot v\mu, P \partial^\alpha f_\pm)_{L_{x,v}^2} \\ &= \sum_{i=1}^3 \left( \partial^\alpha \nabla_x \phi \cdot v\mu, w^2(\alpha, 0) v_i \mu \int_{\mathbb{R}^3} \frac{1}{2} v'_i \partial^\alpha (f_+ - f_-)(v') dv' \right)_{L_{x,v}^2} \\ &= 2C_\alpha \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla_x \partial^\alpha \phi \cdot v \partial^\alpha (f_+ - f_-)(v) dv dx \\ &= -2C_\alpha \int_{\mathbb{T}^3} \partial^\alpha \phi \nabla_x \cdot \int_{\mathbb{R}^3} v \partial^\alpha (f_+ - f_-)(v) dv dx \\ &= -2C_\alpha \int_{\mathbb{T}^3} \partial^\alpha \phi \partial_t \Delta_x \phi dx = C_\alpha \partial_t \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2, \end{aligned}$$

for some constant  $C_{\alpha,\beta}$  depending only on  $\alpha, \beta$ . For the second right hand term of (2.38), noticing exponentially velocity decay in  $\mu$  and using (2.39), we estimate it by

$$\left| (\partial^\alpha \nabla_x \phi \cdot v\mu, w^2(\alpha, \beta) \{I - P\} \partial^\alpha f_\pm)_{L_{x,v}^2} \right| \lesssim \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\}f\|_{L_x^2 L_v^2}.$$

For the third term in (2.38), noticing

$$\left| e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1 \right| \lesssim A_{\alpha,\beta} \|\psi\|_{L^\infty} \lesssim A_{\alpha,\beta} \|\nabla_x \psi\|_{H_x^1},$$

we have

$$\left| (\partial^\alpha \nabla_x \phi \cdot v\mu, (e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1) w^2(\alpha, \beta) \partial^\alpha f_\pm)_{L_{x,v}^2} \right| \lesssim \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}.$$

The above three estimates imply (2.36) and we conclude Lemma 2.17.  $\square$

In [8], one only need to compute  $x$  derivative term, but for our Vlasov-Poisson system we also need estimates about  $v$  derivative term.

**Lemma 2.18.** *Suppose that  $\gamma \in (-3, 1]$  for Boltzmann case and  $\gamma \in [-3, 1]$  for Landau case. For any  $|\alpha| \geq 0$ ,  $|\beta| \leq 2$ ,  $k \geq 7$ , there exist constants  $\gamma_3, C_k > 0$  such that, for any functions  $f, g, \psi$ , we have*

$$\begin{aligned} & |(\partial_\beta^\alpha Q(f, \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}| \\ & \leq \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}} \\ & \quad + C_k \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2-1/2}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2-1/2}} \\ & \quad + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{L_x^2 L_{\gamma/2}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}}, \end{aligned}$$

for the Boltzmann case, and

$$|(\partial_\beta^\alpha Q(f, \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}| \leq C_k \sum_{\beta_1 \leq \beta} \|\partial_{\beta_1}^\alpha f\|_{L_x^2 L_\beta^2} \|\partial_\beta^\alpha g\|_{L_x^2 L_\beta^2},$$

for the Landau case.

*Proof.* For the Boltzmann case, it's easily seen that

$$|(\partial_\beta^\alpha Q(f, \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}| \leq \sum_{\alpha_1 \leq \alpha} |(Q(\partial_{\beta_1}^\alpha f, \partial_{\beta-\beta_1} \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}|.$$

We split it into two cases:  $\beta_1 = \beta$  and  $\beta_1 < \beta$ . For the case  $\beta_1 = \beta$ , by (2.10) and integration about  $x$ , we have

$$\begin{aligned} & |(Q(\partial_\beta^\alpha f, \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}| \\ & \leq \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}} \\ & \quad + C_k \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2-1/2}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2-1/2}}. \end{aligned}$$

For  $\beta_1 < \beta$ , by (2.11) and integration about  $x$ , we have

$$\begin{aligned} & |(Q(\partial_{\beta_1}^\alpha f, \partial_{\beta-\beta_1} \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha g w^2(\alpha, \beta))_{L_{x, v}^2}| \\ & \leq C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}}. \end{aligned}$$

So the proof for the Boltzmann case is completed by gathering the two terms. The Landau case can be prove similarly by taking summation in (2.15).  $\square$

The next two coercive estimates play a key role. For the Boltzmann case, we have the following.

**Lemma 2.19.** *Suppose that  $-3 < \gamma \leq 1$ ,  $\gamma + 2s > -1$  and  $k \geq 14$ . Let  $G = \mu + g \geq 0$  satisfies*

$$\|G\|_{L^1} \geq A_1, \quad \|G\|_{L^1_2} + \|G\|_{L \log L} \leq A_2.$$

for some generic constant  $A_1, A_2$ . Then for  $Q$  as the Boltzmann collision operator, and any  $|\alpha| + |\beta| \leq 2$ , we have

$$\begin{aligned} & (\partial_\beta^\alpha Q(\mu + g, f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x, v}^2} \\ & \leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}}^2 - \gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}^2 \\ & \quad + C_k \|\langle v \rangle^{14} f\|_{H_{x, v}^2} \|g\|_{Y_k} \|f\|_{Y_k} + C_k \|\langle v \rangle^{14} g\|_{H_{x, v}^2} \|f\|_{Y_k}^2 \\ & \quad + C_k \|\partial_\beta^\alpha f\|_{L_{x, v}^2}^2 + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{H_{\gamma/2}^s}, \end{aligned} \tag{2.40}$$

for some constant  $\gamma_1, C_k \geq 0$ , where  $\|\cdot\|_{Y_k}$  is given in (1.14). Moreover, we have

$$\begin{aligned} |(\partial_\beta^\alpha Q(g, f), (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| &\leq C_k \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \|h\|_{Y_k} \\ &\quad + C_k \|\langle v \rangle^{14} f\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|g\|_{Y_k} \|h\|_{Y_k}, \end{aligned} \quad (2.41)$$

and

$$\begin{aligned} |(\partial_\beta^\alpha Q(g, f), \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| + |(\partial_\beta^\alpha Q(g, f), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| \\ \leq C_k \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \min\{\|\langle v \rangle^{2s} f\|_{Y_k} \|h\|_{Y_k}, \|f\|_{Y_k} \|\langle v \rangle^{2s} h\|_{Y_k}\} + C_k \|\langle v \rangle^{14} f\|_{H_{x,v}^2} \|g\|_{Y_k} \|h\|_{Y_k}. \end{aligned} \quad (2.42)$$

**Remark 2.20.** To apply Lemma 2.19, one can easily check  $\|G = \mu + g\|_{L_v^1} \geq A_1$  when assuming  $\|g\|_{L_v^1}$  is small enough.

*Proof.* First we have

$$(\partial_\beta^\alpha Q(G, f), \partial^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} = \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} (Q(\partial_{\beta_1}^{\alpha_1} G, \partial_{\beta - \beta_1}^{\alpha - \alpha_1} f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}.$$

We split it into several cases. For the case  $\alpha_1 = \beta_1 = 0$ , after integrating about  $x$  in Theorem 2.8, we have

$$\begin{aligned} & (Q(G, \partial_\beta^\alpha f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} \\ & \leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}}^2 - \gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}^2 + C_k \|\partial_\beta^\alpha f\|_{L_{x,v}^2}^2 \\ & \quad + \int_{\mathbb{T}^3} C_k \|\partial_\beta^\alpha f\|_{L_{14}^2} \|g w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} + C_k \|g\|_{L_{14}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s}^2 dx. \end{aligned}$$

For the case  $|\beta_1| > 0, \alpha_1 = 0$ , using  $\partial_{\beta_1} G = \partial_{\beta_1} \mu + \partial_{\beta_1} g$ , we split it into two parts. By (2.20), if  $|\beta_1| > 0$ , we have

$$w(\alpha, \beta) \langle v \rangle^{2s} \leq w(\alpha, \beta - \beta_1).$$

Then by Corollary 2.5, we have

$$\begin{aligned} & |(Q(\partial_{\beta_1} \mu, \partial_{\beta - \beta_1}^\alpha f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \\ & \leq C_k \|\partial_{\beta - \beta_1}^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\ & \leq C_k \|\partial_{\beta - \beta_1}^\alpha f w(\alpha, \beta - \beta_1)\|_{L_x^2 H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}. \end{aligned}$$

For the  $\partial_{\beta_1} g$  term, by Lemma 2.10, we have

$$\begin{aligned} & |(Q(\partial_{\beta_1} g, \partial_{\beta - \beta_1}^\alpha f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \\ & \leq \int_{\mathbb{T}^3} C_k \|\partial_{\beta - \beta_1}^\alpha f\|_{L_{14}^2} \|\partial_{\beta_1} g w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} \\ & \quad + C_k \|\partial_{\beta_1} g\|_{L_{14}^2} \|\partial_{\beta - \beta_1}^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx, \end{aligned}$$

For  $|\alpha_1| > 0$ , since  $\partial^{\alpha_1} \mu = 0$ , we have

$$(Q(\partial_{\beta_1}^{\alpha_1} G, \partial_{\beta - \beta_1}^{\alpha - \alpha_1} f), \partial_\beta^\alpha f \langle v \rangle^{2k})_{L_{x,v}^2} = (Q(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta - \beta_1}^{\alpha - \alpha_1} f), \partial_\beta^\alpha f \langle v \rangle^{2k})_{L_{x,v}^2}.$$

By Lemma 2.10 we have

$$\begin{aligned} & |(Q(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta - \beta_1}^{\alpha - \alpha_1} f), \partial_\beta^\alpha f \langle v \rangle^{2k})_{L_{x,v}^2}| \\ & \leq \int_{\mathbb{T}^3} C_k \|\partial_{\beta - \beta_1}^{\alpha - \alpha_1} f\|_{L_{14}^2} \|\partial_{\beta_1}^{\alpha_1} g w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_{\gamma/2}^2} dx \\ & \quad + C_k \int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta - \beta_1}^{\alpha - \alpha_1} f w(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx. \end{aligned}$$

Gathering all the terms, we have

$$\begin{aligned}
& (\partial_\beta^\alpha Q(\mu + g, f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} \\
& \leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2,*}^2}^2 + C_k \|\partial_\beta^\alpha f\|_{L_{x,v}^2}^2 - \gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}^2 \\
& + C_k \sum_{|\beta_1| < |\beta|} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{L_x^2 H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\
& + C_k \int_{\mathbb{T}^3} \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f\|_{L_{14}^2} \|\partial_{\beta_1}^{\alpha_1} g w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \\
& + C_k \int_{\mathbb{T}^3} \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta, |\alpha_1| + |\beta_1| > 0} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \\
& + C_k \int_{\mathbb{T}^3} \|g\|_{L_{14}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s}^2 dx. \tag{2.43}
\end{aligned}$$

Now we only need to prove that

$$\int_{\mathbb{T}^3} \|g\|_{L_{14}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \lesssim \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|f\|_{Y_k}^2, \tag{2.44}$$

and for all  $\alpha_1 \leq \alpha$ ,  $\beta_1 \leq \beta$ ,  $|\alpha_1| + |\beta_1| > 0$ ,

$$\int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \lesssim \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|f\|_{Y_k}^2. \tag{2.45}$$

The fifth to seventh terms in (2.43) follows similarly by changing the order  $f$  and  $g$ . First, for the case  $|\alpha_1| = |\beta_1| = 0$ , we have

$$\begin{aligned}
& \int_{\mathbb{T}^3} \|g\|_{L_{14}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \\
& \lesssim \|g\|_{L_x^\infty L_{14}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\
& \lesssim \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|f\|_{Y_k}^2.
\end{aligned}$$

This gives (2.44). For the term (2.45), we split it into two cases:  $|\alpha_1| + |\beta_1| = 1$  and  $|\alpha_1| + |\beta_1| = 2$ . For the case  $|\alpha_1| + |\beta_1| = 1$ , by  $\|fg\|_{L_x^2} \leq \|f\|_{L_x^6} \|g\|_{L_x^3}$ , we have

$$\begin{aligned}
& \int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \\
& \lesssim \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^6 L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_x^3 H_{\gamma/2+2s}^s} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\
& \lesssim \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_x^3 H_{\gamma/2+2s}^s} \|f\|_{Y_k}.
\end{aligned} \tag{2.46}$$

We again split it into two parts  $|\alpha| + |\beta| = 1$  and  $|\alpha| + |\beta| = 2$ . For the case  $|\alpha_1| + |\beta_1| = |\alpha| + |\beta| = 1$ , we have  $|\alpha - \alpha_1| = |\beta - \beta_1| = 0$ . Then by (2.8) together with (2.25), we have

$$\|f w(\alpha, \beta)\|_{L_x^3 H_{\gamma/2+2s}^s} \lesssim \|f w(0, 0)\|_{L_x^2 H_{\gamma/2}^s} + \|f w(1, 0)\|_{H_x^1 H_{\gamma/2}^s} \lesssim \|f\|_{Y_k}.$$

We then consider the case  $|\alpha_1| + |\beta_1| = 1$ ,  $|\alpha| + |\beta| = 2$ . This time we have  $|\alpha - \alpha_1| = 1$ ,  $|\beta - \beta_1| = 0$  or  $|\alpha - \alpha_1| = 0$ ,  $|\beta - \beta_1| = 1$ . For the first case, by (2.26),

$$\|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_x^3 H_{\gamma/2+2s}^s} \leq \|f w(1, 0)\|_{H_x^1 H_{\gamma/2}^s} + \|f w(2, 0)\|_{H_x^2 H_{\gamma/2}^s} \lesssim \|f\|_{Y_k}.$$

For the second case,  $|\beta| \geq 1$  and by (2.26), we obtain

$$\|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_x^3 H_{\gamma/2+2s}^s} \leq \|f w(0, 1)\|_{L_x^2 H_{\gamma/2}^{1+s}} + \|f w(1, 1)\|_{H_x^1 H_{\gamma/2}^{1+s}} \lesssim \|f\|_{Y_k}.$$

For the case  $|\alpha_1| + |\beta_1| = 2$ , it is easily seen that this time  $|\alpha| + |\beta| = 2$  and  $|\alpha - \alpha_1| = |\beta - \beta_1| = 0$ . So we have

$$\begin{aligned} & \int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{H_{\gamma/2+2s}^s} \|\partial_{\beta}^{\alpha} fw(\alpha, \beta)\|_{H_{\gamma/2}^s} dx \\ & \lesssim \|\partial_{\beta}^{\alpha} g\|_{L_x^2 L_{14}^2} \|fw(\alpha, \beta)\|_{L_x^{\infty} H_{\gamma/2+2s}^s} \|\partial_{\beta}^{\alpha} fw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\ & \lesssim \langle v \rangle^{14} g\|_{H_{x,v}^2} \|fw(\alpha, \beta)\|_{L_x^{\infty} H_{\gamma/2+2s}^s} \|f\|_{Y_k}. \end{aligned} \quad (2.47)$$

By (2.7) with suitable  $k$  and (2.27), we have

$$\begin{aligned} \|fw(\alpha, \beta)\|_{L_x^{\infty} H_{\gamma/2+2s}^s} & \leq \|fw(1, 0)^{2/5} w(2, 0)^{3/5}\|_{L_x^{\infty} H_{\gamma/2}^s} \\ & \leq \|fw(1, 0)\|_{H_x^1 H_{\gamma/2}^s} + \|fw(2, 0)\|_{H_x^2 H_{\gamma/2}^s} \lesssim \|f\|_{Y_k}. \end{aligned}$$

These estimates implies (2.45). Substituting (2.44) and (2.45) into (2.43), we obtain (2.40). For (2.41), notice that

$$\left| e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1 \right| \leq A_{\alpha, \beta} \|\psi\|_{L_x^{\infty}} \langle v \rangle^{-2} \lesssim A_{\alpha, \beta} \|\nabla_x \psi\|_{H_x^1} \langle v \rangle^{-2}. \quad (2.48)$$

By (2.13) and (2.48), we have

$$\begin{aligned} & \left| (\partial_{\beta}^{\alpha} Q(g, f), (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw^2(\alpha, \beta))_{L_{x,v}^2} \right| \\ & \lesssim \sum_{\substack{\alpha_1 \leq \alpha \\ \beta_1 \leq \beta}} \left( \min \left\{ \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{H_{\gamma/2}^s} \|L_x^2\| (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \langle v \rangle^{2s} \partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}, \right. \right. \\ & \quad \left. \left. + \|\partial_{\beta_1}^{\alpha_1} g\|_{L_{14}^2} \|\langle v \rangle^{2s} \partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{H_{\gamma/2}^s} \|L_x^2\| (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \right\} \right. \\ & \quad \left. + \|\partial_{\beta_1}^{\alpha_1} f\|_{L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} gw(\alpha, \beta)\|_{H_{\gamma/2}^s} \|L_x^2\| (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \right). \end{aligned} \quad (2.49)$$

Applying  $L^{\infty} - L^2$  and  $L^3 - L^6$  Hölder's inequality, the first right-hand term of (2.49) can be estimated as

$$\begin{aligned} & \left( \sum_{|\alpha_1|+|\beta_1|=0} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^{\infty} L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L^2 H_{\gamma/2}^s} \right. \\ & \quad + \sum_{|\alpha_1|+|\beta_1|=1} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^3 L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^6 H_{\gamma/2}^s} \\ & \quad \left. + \sum_{|\alpha_1|+|\beta_1|=2} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^2 L_{14}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^{\infty} H_{\gamma/2}^s} \right) A_{\alpha, \beta} \|\nabla_x \psi\|_{H_x^1} \|\partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\ & \lesssim \langle v \rangle^{14} g\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \|\partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}. \end{aligned} \quad (2.50)$$

The second right-hand term of (2.49) can be estimated similarly and we deduce that

$$\begin{aligned} & \left| (\partial_{\beta}^{\alpha} Q(g, f), (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw^2(\alpha, \beta))_{L_{x,v}^2} \right| \\ & \lesssim \langle v \rangle^{14} g\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \|\partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s} \\ & \quad + \langle v \rangle^{14} f\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|g\|_{Y_k} \|\partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}. \end{aligned}$$

The proof of (2.42) is similar by replacing the term  $(e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1)$  by 1 and  $e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}}$ . Note that we keep the minimum in (2.49) this time, and the proof is omitted for brevity. This completes the proof of Lemma 2.19.  $\square$

We can also prove the similar result for the Landau case.



**Lemma 2.21.** *Let  $-3 \leq \gamma \leq 1$ ,  $Q$  be the Landau collision operator and  $k \geq 7$ . There exists a constant  $C_k > 0$  such that for any function  $f, g, h, \psi$ , we have the followings.*

(1) For any  $|\alpha| + |\beta| \leq 2$ ,

$$\begin{aligned} (\partial_\beta^\alpha Q(\mu, f), \partial_\beta^\alpha f w^2(\alpha, \beta)) &\leq -\gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_D^2}^2 + C_k \|\partial_\beta^\alpha f\|_{L_x^2 L_D^2}^2 \\ &\quad + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{L_x^2 L_D^2} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_D^2}. \end{aligned} \quad (2.51)$$

(2) For any  $|\alpha| + |\beta| \leq 2$ , we have

$$|(\partial_\beta^\alpha Q(g, f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \leq C_k \|g \langle v \rangle^\gamma\|_{H_{x,v}^2} \|f\|_{Y_k}^2, \quad (2.52)$$

$$|(\partial_\beta^\alpha Q(g, f), (e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1) \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| \leq C_k \|g \langle v \rangle^\gamma\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \|h\|_{Y_k}, \quad (2.53)$$

and

$$\begin{aligned} |(\partial_\beta^\alpha Q(g, f), \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| + |(\partial_\beta^\alpha Q(g, f), e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} \partial_\beta^\alpha h w^2(\alpha, \beta))_{L_{x,v}^2}| \\ \leq C_k \|g \langle v \rangle^\gamma\|_{H_{x,v}^2} \min\{\|f\|_{Y_k} \|\langle v \rangle h\|_{Y_k}, \|\langle v \rangle f\|_{Y_k} \|h\|_{Y_k}\}. \end{aligned} \quad (2.54)$$

where  $\|f\|_{Y_k}^2$  is given in (1.14).

*Proof.* Notice that

$$(\partial_\beta^\alpha Q(\mu, f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} = \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} (Q(\partial_{\beta_1}^{\alpha_1} \mu, \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}.$$

We split it into two cases:  $\beta_1 = 0$  and  $|\beta_1| > 0$ . If  $\beta_1 = 0$ , after integrating about  $x$  in (2.16), we have

$$(Q(\mu, \partial_\beta^\alpha f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} \leq -\gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_D^2}^2 + C_k \|\partial_\beta^\alpha f\|_{L_x^2 L_D^2}^2.$$

For the case  $|\beta_1| > 0$ , by (2.21) we have, if  $|\beta_1| > 0$ ,

$$w(\alpha, \beta) \langle v \rangle \leq w(\alpha, \beta - \beta_1).$$

Then by Lemma 2.12 we have

$$\begin{aligned} |(Q(\partial_{\beta_1}^{\alpha_1} \mu, \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| &\leq C_k \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_x^2 L_{D,1}^2} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_D^2} \\ &\leq C_k \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta_1)\|_{L_x^2 L_D^2} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_D^2}, \end{aligned}$$

So (2.51) is thus finished by gathering the above two terms together. Next, for (2.52), notice that

$$(\partial_\beta^\alpha Q(g, f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2} = \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} (Q(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}.$$

We again split it into two cases:  $|\alpha_1| = |\beta_1| = 0$  and  $|\alpha_1| + |\beta_1| > 0$ . For the case  $|\alpha_1| = |\beta_1| = 0$ , by (2.17) and (2.48), we have

$$\begin{aligned} |(Q(g, \partial_\beta^\alpha f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| &\leq \int_{\mathbb{T}^3} C_k \|g\|_{L_7^2} \|(e^{\frac{\pm A_{\alpha,\beta}\psi}{\langle v \rangle^2}} - 1) \langle v \rangle \partial_\beta^\alpha f w(\alpha, \beta)\|_{L_D^2}^2 dx \\ &\leq C_k \|g\|_{L_x^\infty L_7^2} \|\nabla_x \psi\|_{H_x^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_D^2}^2 \\ &\leq C_k \|g \langle v \rangle^\gamma\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}^2. \end{aligned}$$

For  $\alpha_1 \leq \alpha$ ,  $\beta_1 \leq \beta$ ,  $|\alpha_1| + |\beta_1| > 0$ , by (2.17), we have

$$\begin{aligned} |(Q(\partial_{\beta_1}^{\alpha_1} g, \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f), \partial_\beta^\alpha f w^2(\alpha, \beta))_{L_{x,v}^2}| \\ \leq \int_{\mathbb{T}^3} C_k \|\partial_{\beta_1}^{\alpha_1} g\|_{L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f w(\alpha, \beta)\|_{L_D^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_{D,1}^2} dx. \end{aligned}$$

We split it into two cases,  $|\alpha_1| + |\beta_1| = 1$  and  $|\alpha_1| + |\beta_1| = 2$ . For the case  $|\alpha_1| + |\beta_1| = 1$ , by  $\|fg\|_{L_x^2} \leq \|f\|_{L_x^6} \|g\|_{L_x^3}$  and similarly as (2.46), we have

$$\begin{aligned} \int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_{D,1}^2} \|\partial_{\beta}^{\alpha} fw(\alpha, \beta)\|_{L_D^2} dx \\ \lesssim \|\langle v \rangle^7 g\|_{H_{x,v}^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^3 L_{D,1}^2} \|f\|_{Y_k}. \end{aligned}$$

We again split it into two parts  $|\alpha| + |\beta| = 1$  and  $|\alpha| + |\beta| = 2$ . For the case  $|\alpha_1| + |\beta_1| = |\alpha| + |\beta| = 1$ , we have  $|\alpha - \alpha_1| = |\beta - \beta_1| = 0$ . By (2.25) and (2.8) we obtain that

$$\|fw(\alpha, \beta)\|_{L_x^3 L_{D,1}^2} \lesssim \|fw(0, 0)\|_{L_x^2 L_D^2} + \|fw(1, 0)\|_{H_x^1 L_D^2} \lesssim \|f\|_{Y_k}.$$

For the case  $|\alpha| + |\beta| = 2$ , either  $|\alpha - \alpha_1| = 1, |\beta - \beta_1| = 0$  or  $|\alpha - \alpha_1| = 0, |\beta - \beta_1| = 1$ . For the first case, by (2.26) and (2.8), we have

$$\|\partial^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^3 L_{D,1}^2} \leq \|fw(1, 0)\|_{H_x^1 L_D^2} + \|fw(2, 0)\|_{H_x^2 L_D^2} \lesssim \|f\|_{Y_k}.$$

For the second case, this time  $|\beta| \geq 1$ . By (2.26) and (2.8), we have

$$\|\partial_{\beta-\beta_1} fw(\alpha, \beta)\|_{L_x^3 L_{D,1}^2} \leq \|\nabla_v fw(0, 1)\|_{L_x^2 L_D^2} + \|\nabla_v fw(1, 1)\|_{H_x^1 L_D^2} \lesssim \|f\|_{Y_k}.$$

For the case  $|\alpha_1| + |\beta_1| = 2$ , we have obviously  $|\alpha| + |\beta| = 2$  and  $|\alpha - \alpha_1| = |\beta - \beta_1| = 0$ . Similar to (2.47), we have

$$\int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_{D,1}^2} \|\partial_{\beta}^{\alpha} fw(\alpha, \beta)\|_{L_D^2} dx \lesssim \|\langle v \rangle^7 g\|_{H_{x,v}^2} \|fw(\alpha, \beta)\|_{L_x^\infty L_{D,1}^2} \|f\|_{Y_k}.$$

We obtain again from (2.7) and (2.27) that

$$\|fw(\alpha, \beta)\|_{L_x^\infty L_{D,1}^2} \leq \|fw(1, 0)\|_{H_x^1 L_D^2} + \|fw(2, 0)\|_{H_x^2 L_D^2} \lesssim \|f\|_{Y_k}.$$

The proof of (2.52) is done after gathering the above terms. For (2.53), by (2.17) and (2.48), we have

$$\begin{aligned} & \left| (\partial_{\beta}^{\alpha} Q(g, f), (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw^2(\alpha, \beta)) \right|_{L_{x,v}^2} \\ & \lesssim \sum_{\alpha_1 \leq \alpha, \beta \leq \beta_1} \int_{\mathbb{T}^3} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_7^2} \min \{ \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_D^2} \| (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \langle v \rangle \partial_{\beta}^{\alpha} hw(\alpha, \beta) \|_{L_D^2}, \\ & \qquad \qquad \qquad \|\langle v \rangle \partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_D^2} \| (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) \partial_{\beta}^{\alpha} hw(\alpha, \beta) \|_{L_D^2} \} \\ & \lesssim \left( \sum_{|\alpha_1|+|\beta_1|=0} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^\infty L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L^2 L_D^2} \right. \\ & \quad + \sum_{|\alpha_1|+|\beta_1|=1} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^3 L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^6 L_D^2} \\ & \quad \left. + \sum_{|\alpha_1|+|\beta_1|=2} \|\partial_{\beta_1}^{\alpha_1} g\|_{L_x^2 L_7^2} \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} fw(\alpha, \beta)\|_{L_x^\infty L_D^2} \right) A_{\alpha, \beta} \|\nabla_x \psi\|_{H_x^1} \|\partial_{\beta}^{\alpha} hw(\alpha, \beta)\|_{L_x^2 L_D^2} \\ & \lesssim \|\langle v \rangle^7 g\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \|h\|_{Y_k}, \end{aligned} \quad (2.55)$$

where we apply  $L^\infty - L^2$  and  $L^3 - L^6$  Hölder's inequality. The proof of (2.54) is similar by replacing the term  $(e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1)$  by 1 and  $e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}}$ . Note that this time we keep the minimum in (2.55), and we omit the proof for brevity. This completes the proof of Lemma 2.21.  $\square$

### 3. LOCAL SOLUTIONS

The main goal of this section is prove the local-in-time existence and uniqueness of the equation (1.4). Define the series of functions  $f^n := F^n - \mu$  by  $f^0 = 0$  and

$$\begin{cases} \partial_t f_{\pm}^{n+1} + v \cdot \nabla_x f_{\pm}^{n+1} \mp \nabla_x \phi^n \cdot \nabla_v f_{\pm}^{n+1} \pm \nabla_x \phi^{n+1} \cdot v \mu \\ \quad = Q(f_{\pm}^n + f_{\mp}^n, \mu) + Q(\mu + f_{\pm}^n, f_{\pm}^{n+1}) + Q(\mu + f_{\mp}^n, f_{\pm}^{n+1}), \\ -\Delta_x \phi^{n+1} = \int_{\mathbb{R}^3} (f_+^{n+1} - f_-^{n+1}) dv, \quad \int_{\mathbb{T}^3} \phi^{n+1}(x) dx = 0, \\ f^{n+1}(0) = f_0^{n+1}, \quad E^{n+1}(0) = E_0^{n+1}. \end{cases} \quad (3.1)$$

In order to solve (3.1), we first consider the following linear problem. For the Boltzmann case, we further define

$$\|f\|_{\bar{Y}_k} := \sum_{|\alpha|+|\beta|\leq 2} \|\partial_{\beta}^{\alpha} f w(\alpha, \beta)\|_{L_x^2 L_v^2 \gamma^{1/2, *}}^2. \quad (3.2)$$

**Lemma 3.1.** *Suppose that  $-3 < \gamma \leq 1$  for Boltzmann case and  $-3 \leq \gamma \leq 1$  for Landau case. Let  $k_0$  be given in (1.17). There exist constants  $\tau_0, T_0 > 0$  such that, for all  $k \geq k_0$  (and let  $k$  sufficiently large for Boltzmann case), if  $f_0 \in X_k$ ,  $g_{\pm} \in L_{T_0}^{\infty} X_k \cap L_{T_0}^2 Y_k$ ,  $\psi \in L_{T_0}^{\infty} H_x^3$  satisfying  $\mu + f_0 \geq 0$ ,  $\mu + g \geq 0$  and*

$$\|f_0\|_{X_{k_0}} + \|\nabla_x \psi\|_{L_{T_0}^{\infty} H_x^3} + \|\partial_t \psi\|_{L_{T_0}^{\infty} L_x^{\infty}} + \|g\|_{L_{T_0}^{\infty} X_{k_0}} \leq \tau_0, \quad (3.3)$$

then the Cauchy problem

$$\begin{cases} \partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \mp \nabla_x \psi \cdot \nabla_v f_{\pm} \pm \nabla_x \phi \cdot v \mu \\ \quad = Q(g_{\pm} + g_{\mp}, \mu) + Q(\mu + g_{\pm}, f_{\pm}) + Q(\mu + g_{\mp}, f_{\pm}), \\ -\Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) dv, \quad \int_{\mathbb{T}^3} \phi(x) dx = 0, \\ f(0) = f_0, \quad E(0) = E_0. \end{cases} \quad (3.4)$$

admits a weak solution  $f(t, x, v)$ ,  $\phi(t, x)$  in  $[0, T_0]$  satisfying

$$\mu + f \geq 0, \quad f \in L_{T_0}^{\infty} X_k,$$

with the energy bound

$$\|f\|_{L_{T_0}^{\infty} X_k}^2 + \|\nabla_x \phi\|_{L_{T_0}^{\infty} H_x^3}^2 + \|\partial_t \phi\|_{L_{T_0}^{\infty} L_x^{\infty}}^2 + \|f\|_{L_{T_0}^2 Y_k}^2 \leq C \|f_0\|_{X_k}^2 + \frac{1}{4} (\|g\|_{L_{T_0}^2 Y_k}^2 + \|\nabla_x \psi\|_{L_{T_0}^{\infty} H_x^1}^2), \quad (3.5)$$

for some generic constant  $C > 0$ .

*Proof.* For  $|\alpha| + |\beta| \leq 2$ , we apply  $\partial_{\beta}^{\alpha}$  to (3.4) to obtain

$$\begin{aligned} \partial_t \partial_{\beta}^{\alpha} f_{\pm} + \partial_{\beta}^{\alpha} (v \cdot \nabla_x f_{\pm}) \mp \partial_{\beta}^{\alpha} (\nabla_x \psi \cdot \nabla_v f_{\pm}) \pm \partial_{\beta}^{\alpha} (\nabla_x \phi \cdot v \mu) \\ = \partial_{\beta}^{\alpha} Q(g_{\pm} + g_{\mp}, \mu) + \partial_{\beta}^{\alpha} Q(\mu + g_{\pm}, f_{\pm}) + \partial_{\beta}^{\alpha} Q(\mu + g_{\mp}, f_{\pm}). \end{aligned} \quad (3.6)$$

Taking inner product of (3.6) with  $e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_{\beta}^{\alpha} f_{\pm}$  over  $\mathbb{T}^3 \times \mathbb{R}^3$ , one has

$$\begin{aligned} & \frac{1}{2} \partial_t \|e^{\frac{\pm A_{\alpha, \beta} \psi}{2\langle v \rangle^2}} \partial_{\beta}^{\alpha} f_{\pm}\|_{L_{x,v}^2}^2 + (A_{\alpha, \beta} \partial_t \psi \langle v \rangle^{-2} \partial_{\beta}^{\alpha} f_{\pm}, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_{\beta}^{\alpha} f_{\pm})_{L_{x,v}^2} \\ & + (v \cdot \nabla_x \partial_{\beta}^{\alpha} f_{\pm}, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_{\beta}^{\alpha} f_{\pm})_{L_{x,v}^2} \\ & + \sum_{|\beta_1|=1} (\partial_{\beta_1} v \cdot \nabla_x \partial_{\beta-\beta_1}^{\alpha} f_{\pm}, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_{\beta}^{\alpha} f_{\pm})_{L_{x,v}^2} \\ & \mp (\nabla_x \psi \cdot \nabla_v \partial_{\beta}^{\alpha} f_{\pm}, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_{\beta}^{\alpha} f_{\pm})_{L_{x,v}^2} \end{aligned}$$

$$\begin{aligned}
& \mp \sum_{\substack{\alpha_1 \leq \alpha \\ |\alpha_1| \geq 1}} (\partial^{\alpha_1} \nabla_x \psi \cdot \nabla_v \partial_\beta^{\alpha - \alpha_1} f_\pm, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm)_{L_{x,v}^2} \\
& \pm (\partial_\beta^\alpha (\nabla_x \phi \cdot v \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm)_{L_{x,v}^2} \\
& = (\partial_\beta^\alpha Q(g_\pm + g_{\mp}, \mu), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm)_{L_{x,v}^2} \\
& \quad + (\partial_\beta^\alpha Q(\mu + g_\pm, f_\pm) + \partial_\beta^\alpha Q(\mu + g_{\mp}, f_\pm), w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm)_{L_{x,v}^2} \\
& \quad + (\partial_\beta^\alpha Q(\mu + g_\pm, f_\pm) + \partial_\beta^\alpha Q(\mu + g_{\mp}, f_\pm), (e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} - 1) w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm)_{L_{x,v}^2}. \tag{3.7}
\end{aligned}$$

We denote the second to tenth terms in (3.7) by  $I_1$  to  $I_9$  and estimate them one by one. For  $I_1$ , we make a rough estimate:

$$|I_1| \lesssim \|\partial_t \psi\|_{L_x^\infty} \|f\|_{X_k}^2. \tag{3.8}$$

Notice that  $\|\psi\|_{L_x^\infty} \lesssim \|\nabla_x \psi\|_{H_x^1} \leq \tau_0$ , which verify (2.29) and we can apply estimates in Section 2.3. By (2.31), we deduce that

$$|I_2 + I_4| \lesssim \|\nabla_x \psi\|_{H_x^2} \|f\|_{Y_k}^2.$$

Applying (2.32), when  $|\beta| \geq 1$ , we have

$$|I_3| \lesssim C_\eta \sum_{\substack{|\beta_2| = |\beta| - 1 \\ |\alpha_2| = |\alpha| + 1}} \|\partial_{\beta_2}^{\alpha_2} f w(\alpha_2, \beta_2)\|_{L_x^2 L_{\gamma/2}^2}^2 + \eta \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}^2}^2,$$

for any  $\eta > 0$ . By (2.33), we have

$$|I_5| \lesssim \|\nabla_x \psi\|_{H_x^3} \|f\|_{Y_k}^2.$$

When  $|\beta| = 0$ , using (2.36), we have

$$\begin{aligned}
I_6 & \geq C_\alpha \partial_t \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - C \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_y^2} - C \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k} \\
& \geq C_\alpha \partial_t \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - \eta \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - \eta \|w(\alpha, 0) \partial^\alpha f\|_{L_x^2 L_{\gamma/2}^2}^2 \\
& \quad - C_\eta \|\partial^\alpha f\|_{L_{x,v}^2}^2 - C \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k},
\end{aligned}$$

for any  $\eta > 0$ . When  $|\beta| \geq 1$ , we have from (2.35) that

$$|I_6| \leq C_k \|\nabla_x \phi\|_{H_x^2} \|f\|_{H_x^2 L_v^2}. \tag{3.9}$$

For  $I_7$ , applying Lemma 2.18, we have for any  $\eta > 0$  that

$$\begin{aligned}
|I_7| & \leq \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}^2} \\
& \quad + \eta \|f\|_{Y_k}^2 + \eta \|g\|_{Y_k}^2 + C_\eta \|f\|_{H_{x,v}^2} \\
& \quad + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha g w(\alpha, \beta_1)\|_{L_x^2 L_{\gamma/2}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}^2} \\
& \leq C_\eta \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}^2}^2 + \eta \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2, *}^2}^2 \\
& \quad + \eta \|f\|_{Y_k}^2 + \eta \|g\|_{Y_k}^2 + C_\eta \|f\|_{H_{x,v}^2} \\
& \quad + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha g w(\alpha, \beta_1)\|_{L_x^2 L_{\gamma/2}^2} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}^2},
\end{aligned}$$

for Boltzmann case. Here we used  $\|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \leq \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \leq C$ , where  $C$  is independent of  $C$ . For Landau case, it follows from Lemma 2.18 that

$$|I_7| \leq C_k \sum_{|\alpha|+|\beta|\leq 2} \|\partial_\beta^\alpha g\|_{L_x^2 L_v^2} \|\partial_\beta^\alpha f\|_{L_x^2 L_v^2} \leq \eta \|f\|_{Y_k}^2 + \eta \|g\|_{Y_k}^2 + C_\eta \|f\|_{H_{x,v}^2}^2,$$

For  $I_8$ , we deduce from (2.40) that

$$\begin{aligned} \sum_{\pm} I_8 &\leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_{\gamma/2}^2}^2 - \gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 H_{\gamma/2}^s}^2 \\ &\quad + C_k \|\langle v \rangle^{14} f\|_{H_{x,v}^2} \|g\|_{Y_k} \|f\|_{Y_k} + C_k \|\langle v \rangle^{14} g\|_{H_{x,v}^2} \|f\|_{Y_k}^2 \\ &\quad + C_k \|\partial_\beta^\alpha f\|_{L_{x,v}^2}^2 + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{H_{\gamma/2}^s} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{H_{\gamma/2}^s}, \end{aligned}$$

for Boltzmann case. Note that and from (2.51) and (2.52) that

$$\begin{aligned} \sum_{\pm} I_8 &\leq -\gamma_1 \|\partial_\beta^\alpha f w(\alpha, \beta)\|_{L_x^2 L_D^2}^2 + C_k \|\partial_\beta^\alpha f\|_{L_x^2 L_v^2}^2 \\ &\quad + C_k \sum_{\beta_1 < \beta} \|\partial_{\beta_1}^\alpha f w(\alpha, \beta_1)\|_{L_x^2 L_D^2} \|\partial_\beta^\alpha g w(\alpha, \beta)\|_{L_x^2 L_D^2} + C_k \|g \langle v \rangle^7\|_{H_{x,v}^2} \|f\|_{Y_k}^2, \end{aligned}$$

for Landau case, with some constants  $\gamma_1, C_k > 0$ . For  $I_9$ , we have from (2.41) that

$$|I_9| \leq C_k (\|\langle v \rangle^{14} g\|_{H_{x,v}^2} + C) \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}^2 + \|\langle v \rangle^{14} f\|_{H_{x,v}^2} \|\nabla_x \psi\|_{H_x^1} (\|g\|_{Y_k} + C) \|f\|_{Y_k},$$

for Boltzmann case and, from (2.53) that

$$|I_9| \leq C_k (\|\langle v \rangle^7 g\|_{H_{x,v}^2} + C) \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}^2,$$

for Landau case. Combining the above estimates, we take summation  $\sum_{|\alpha|+|\beta|\leq 2} C_{|\alpha|,|\beta|} \times (3.7)$  with  $0 < \eta \ll 1$  and constants  $C_{|\alpha|,|\beta|}$  satisfying (1.12) to deduce that for the Boltzmann case:

$$\begin{aligned} &\frac{1}{2} \partial_t \|f\|_{X_k}^2 + \partial_t \sum_{|\alpha|\leq 2} C_\alpha C_{|\alpha|,0} \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 + \frac{1}{16} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|f\|_{Y_k}^2 + \gamma_1 \|f\|_{Y_k}^2 \\ &\leq C_\eta \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|f\|_{Y_k}^2 + \eta \|g\|_{Y_k}^2 + C_{\eta,k} \|f\|_{H_{x,v}^2}^2 + C \|\partial_t \psi\|_{L_x^\infty} \|f\|_{X_k}^2 \\ &\quad + C_k \left( \|\nabla_x \psi\|_{H_x^3} \|f\|_{Y_k}^2 + \|\nabla_x \phi\|_{H_x^2}^2 + \|f\|_{X_{14}} \|g\|_{Y_k} \|f\|_{Y_k} + \|g\|_{X_{14}} \|f\|_{Y_k}^2 \right. \\ &\quad \left. + (\|g\|_{X_{14}} + C) \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}^2 + \|f\|_{X_{14}} \|\nabla_x \psi\|_{H_x^1} (\|g\|_{Y_k} + C) \|f\|_{Y_k} \right), \end{aligned} \quad (3.10)$$

and for the Landau case:

$$\begin{aligned} &\frac{1}{2} \partial_t \|f\|_{X_k}^2 + \partial_t \sum_{|\alpha|\leq 2} C_\alpha C_{|\alpha|,0} \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 + \gamma_1 \|f\|_{Y_k}^2 \\ &\leq \eta \|g\|_{Y_k}^2 + C_{\eta,k} \|f\|_{H_{x,v}^2}^2 + C \|\partial_t \psi\|_{L_x^\infty} \|f\|_{X_k}^2 + C_k \left( \|\nabla_x \psi\|_{H_x^3} \|f\|_{Y_k}^2 \right. \\ &\quad \left. + \|\nabla_x \phi\|_{H_x^2}^2 + \|f\|_{X_7} \|g\|_{Y_k} \|f\|_{Y_k} + \|g\|_{X_7} \|f\|_{Y_k}^2 \right. \\ &\quad \left. + (\|g\|_{X_7} + C) \|\nabla_x \psi\|_{H_x^1} \|f\|_{Y_k}^2 + \|f\|_{X_7} \|\nabla_x \psi\|_{H_x^1} (\|g\|_{Y_k} + C) \|f\|_{Y_k} \right), \end{aligned} \quad (3.11)$$

for some generic constant  $\gamma_1$  independent of  $k$ , where we let  $k \geq 14$ . Here  $\|f\|_{Y_k}$  is defined by (3.2) and  $\|f\|_{X_k}$  is given by

$$\|f\|_{X_k}^2 := \sum_{|\alpha|+|\beta|\leq 2} C_{|\alpha|,|\beta|} e^{\frac{\pm A_{\alpha,\beta} \psi}{\langle v \rangle^2}} \|\partial_\beta^\alpha f_\pm\|_{L_{x,v}^2}^2.$$

Next we will consider the extra term  $\|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_k}^2$  only for the Boltzmann case. Since  $\theta \in [0, \frac{\pi}{2}]$ , we have  $\sin^2\frac{\theta}{2} \leq \frac{1}{2}$  and hence,

$$b(\cos\theta)\sin^2\frac{\theta}{2} \gg b(\cos\theta)\sin^{k-2}\frac{\theta}{2}$$

when  $k \gg 1$ . Thus, choosing  $k = k(\eta)$  sufficiently large (which depends on  $\eta$ ), we have

$$C_\eta \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_k}^2 \leq \frac{1}{32} \|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_k}^2. \quad (3.12)$$

Here we estimate the term  $\|\nabla_x^3\phi\|_{L_T^\infty L_x^2}^2$ . Using (3.4)<sub>2</sub>, we have

$$\|\nabla_x^3\phi\|_{L_x^2}^2 \lesssim \|\nabla_x\Delta_x\phi\|_{L_x^2}^2 \lesssim \|\nabla_x f\|_{L_x^2 L_x^2}^2 \lesssim \|f\|_{X_{k_0}}^2. \quad (3.13)$$

Taking difference of (1.7) over  $\pm$ , we have

$$\partial_t(a_+ - a_-) = -\nabla_x \cdot \int_{\mathbb{R}^3} v(f_+ - f_-) dv,$$

and hence,

$$\begin{aligned} \|\partial_t\phi\|_{L_x^\infty} &\lesssim \|\nabla_x\partial_t\phi\|_{L_x^2}^{1/2} \|\nabla_x\partial_t\phi\|_{H_x^1}^2 \lesssim \|\nabla_x^2\partial_t\phi\|_{L_x^2} \lesssim \|\Delta_x\partial_t\phi\|_{L_x^2} \\ &\lesssim \|\partial_t(a_+ - a_-)\|_{L_x^2} \lesssim \|\nabla_x f\|_{L_x^2 L_x^2} \lesssim \|f\|_{X_{k_0}}. \end{aligned} \quad (3.14)$$

Therefore, for both Boltzmann and Landau case, taking integration of (3.10) over  $t \in [0, T]$ , we have

$$\begin{aligned} &\frac{1}{3} \|f\|_{L_T^\infty X_k}^2 + \lambda \|\nabla_x\phi\|_{L_T^\infty H_x^3}^2 + \gamma_1 \|f\|_{L_T^2 Y_k}^2 \\ &\leq \frac{1}{2} \|f_0\|_{X_k}^2 + C_{\eta,k} \|f\|_{L_T^2 H_{x,v}^2}^2 + C \|\partial_t\psi\|_{L_T^\infty L_x^\infty} \|f\|_{L_T^2 X_k}^2 + C_k \left( \|\nabla_x\phi\|_{L_T^2 H_x^2}^2 \right. \\ &\quad \left. + (\|\nabla_x\psi\|_{L_T^\infty H_x^3} + \|g\|_{L_T^\infty X_{k_0}} + C_\eta \|f\|_{L_T^\infty X_{k_0}} + \|g\|_{L_T^\infty X_{k_0}} \|\nabla_x\psi\|_{L_T^\infty H_x^1}) \|f\|_{L_T^2 Y_k}^2 \right. \\ &\quad \left. + \eta \|g\|_{L_T^2 Y_k}^2 + \|\nabla_x\psi\|_{L_T^\infty H_x^1} \|g\|_{L_T^2 Y_k}^2 + \|\nabla_x\psi\|_{L_T^2 H_x^1}^2 \right), \end{aligned} \quad (3.15)$$

for some generic constant  $\lambda > 0$  independent of  $k, \eta$ . Assume the *a priori* assumption corresponding to (3.15) to be

$$\|f\|_{L_T^\infty X_{k_0}}^2 + \|\nabla_x\phi\|_{L_T^\infty H_x^2}^2 \leq \varepsilon_0, \quad (3.16)$$

with some small  $\varepsilon_0 > 0$ . Noticing  $\|\cdot\|_{L_T^2} \leq T\|\cdot\|_{L_T^\infty}$ , choosing  $0 < \tau_0 = \tau_0(\eta) < \eta$  in (3.3),  $\varepsilon_0 = \varepsilon_0(\eta) > 0$  in (3.16) and  $0 < T = T(\eta, k) < \eta$  sufficiently small, we obtain from (3.15) that

$$\begin{aligned} &\frac{1}{4} \|f\|_{L_T^\infty X_k}^2 + \frac{\lambda}{2} \|\nabla_x\phi\|_{L_T^\infty H_x^3}^2 + \frac{\gamma_1}{2} \|f\|_{L_T^2 Y_k}^2 \\ &\leq \frac{1}{2} \|f_0\|_{X_k}^2 + \eta \|g\|_{L_T^2 Y_k}^2 + \|\nabla_x\psi\|_{L_T^\infty H_x^1} \|g\|_{L_T^2 Y_k}^2 + \eta \|\nabla_x\psi\|_{L_T^\infty H_x^1}^2. \end{aligned}$$

Now we choose  $\eta = \eta(\lambda, \gamma_1) > 0$  sufficiently small, which is independent of  $k, T, \tau_0, \varepsilon_0$ . Then we have

$$\|f\|_{L_T^\infty X_k}^2 + \|\nabla_x\phi\|_{L_T^\infty H_x^3}^2 + \|f\|_{L_T^2 Y_k}^2 \leq M \|f_0\|_{X_k}^2 + \frac{1}{4} (\|g\|_{L_T^2 Y_k}^2 + \|\nabla_x\psi\|_{L_T^\infty H_x^1}^2), \quad (3.17)$$

for some generic constant  $M > 0$  depends only on  $\lambda, \gamma_1$ . Choosing  $\tau_0$  in (3.3) small enough and noticing  $k \geq k_0$ , we close the *a priori* estimate (3.16) and deduce (3.5). Therefore, it's standard to apply the *a priori* arguments to obtain a solution to (3.4) in time interval  $[0, T_0]$  for some small  $T_0 > 0$ .

We next prove the non-negativity for Boltzmann case. The Landau case can be proved by using the same method the we omit the proof for brevity. Set  $F := \mu + f$ ,  $G := \mu + g \geq 0$ . Then (3.4) becomes

$$\begin{aligned} \partial_t F_{\pm} + v \cdot \nabla_x F_{\pm} \mp \nabla_x \psi \cdot \nabla_v F_{\pm} &= Q(G_{\pm}, F_{\pm}) + Q(G_{\mp}, F_{\pm}), \\ -\Delta_x \phi &= \int_{\mathbb{R}^3} (F_+ - F_-) dv, \quad \int_{\mathbb{T}^3} \phi(t, x) dx = 0, \\ F(0) = F_0 &\geq 0, \quad E(0) = E_0. \end{aligned}$$

Denote  $F^{\pm} := \pm \max\{\pm F, 0\}$  and  $\beta(s) = \frac{1}{2}(s^-)^2$  with  $s^- := \min\{s, 0\}$ . Notice that

$$\frac{d\beta}{ds}(F) = F^-, \quad F^-|_{t=0} = 0,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} Q(G_{\pm}, F_{\pm}) F_{\pm}^- \langle v \rangle^{2k} dv \\ &= \int_{\mathbb{R}^3} Q(G_{\pm}, F_{\pm}^-) F_{\pm}^- \langle v \rangle^{2k} dv + \int_{\mathbb{R}^3} Q(G_{\pm}, F_{\pm}^+) F_{\pm}^- \langle v \rangle^{2k} dv \\ &= \int_{\mathbb{R}^3} Q(G_{\pm}, F_{\pm}^-) F_{\pm}^- \langle v \rangle^{2k} dv + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(G_{\pm})'_* (F_{\pm}^+)' F_{\pm}^- \langle v \rangle^{2k} dv_* dv d\sigma \\ &\quad - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(G_{\pm})_* F_{\pm}^+ F_{\pm}^- \langle v \rangle^{2k} dv_* dv d\sigma \\ &\leq \int_{\mathbb{R}^3} Q(G_{\pm}, F_{\pm}^-) F_{\pm}^- \langle v \rangle^{2k} dv. \end{aligned}$$

Note that  $F_{\pm}^+ F_{\pm}^- = 0$  and  $(F_{\pm}^+)' F_{\pm}^- \leq 0$ . Similarly, one can obtain  $\int_{\mathbb{R}^3} Q(G_{\mp}, F_{\pm}) F_{\pm}^- \langle v \rangle^{2k} dv \leq \int_{\mathbb{R}^3} Q(G_{\mp}, F_{\pm}^-) F_{\pm}^- \langle v \rangle^{2k} dv$ . Then we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \beta(F_{\pm}) \langle v \rangle^{2k} dv dx &= -(v \cdot \nabla_x F_{\pm}, F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \pm (\nabla_x \phi \cdot \nabla_v F_{\pm}, F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \\ &\quad + (Q(G_{\pm}, F_{\pm}), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} + (Q(G_{\mp}, F_{\pm}), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \\ &= \pm (\nabla_x \psi \cdot \nabla_v F_{\pm}, F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} + (Q(G_{\pm}, F_{\pm}^-), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \\ &\quad + (Q(G_{\mp}, F_{\pm}^-), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}}. \end{aligned} \tag{3.18}$$

For the first term we have

$$\begin{aligned} (\nabla_x \psi \cdot \nabla_v F_{\pm}, F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} &= -2k (\nabla_x \psi \cdot v \langle v \rangle^{2k-2}, (F_{\pm}^-)^2)_{L^2_{x,v}} \\ &\leq C_k \|\nabla_x \psi\|_{L^{\infty}_x} \| |v| \langle v \rangle^{k-2} F_{\pm}^- \|_{L^2_x L^2_v} \| \langle v \rangle^k F_{\pm}^- \|_{L^2_x L^2_v} \\ &\leq C_k \|\nabla_x \psi\|_{H^1_x} \|F_{\pm}^- \|_{L^2_x L^2_k}. \end{aligned}$$

For the second term, we deduce from (2.12) that

$$\begin{aligned} &(Q(G_{\pm}, F_{\pm}^-), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \\ &\leq \int_{\mathbb{T}^3} -\gamma_1 \|F_{\pm}^- \|_{H^{s_{k+\gamma/2}}}^2 + C_k \|F_{\pm}^- \|_{L^2}^2 + C_k \|F_{\pm}^- \|_{L^2_{14}} \|g_{\pm} \|_{H^{s_{k+\gamma/2}}} \|F_{\pm}^- \|_{H^{s_{k+\gamma/2}}} \\ &\quad + C_k \|g_{\pm} \|_{L^2_{14}} \|F_{\pm}^- \|_{H^{s_{k+\gamma/2}}}^2 dx \\ &\leq \int_{\mathbb{T}^3} -\left(\frac{\gamma_1}{2} - C_k \|g_{\pm} \|_{L^2_{14}}\right) \|F_{\pm}^- \|_{H^{s_{k+\gamma/2}}}^2 + C_k \|F_{\pm}^- \|_{L^2_v}^2 + C_k \|F_{\pm}^- \|_{L^2_{14}}^2 \|g_{\pm} \|_{H^{s_{k+\gamma/2}}}^2 dx \\ &\leq -\left(\frac{\gamma_1}{2} - C_k \|g_{\pm} \|_{L^{\infty}_x L^2_{14}}\right) \|F_{\pm}^- \|_{L^2_x H^{s_{k+\gamma/2}}}^2 + C_k \|F_{\pm}^- \|_{L^2_{x,v}}^2 + C_k \|F_{\pm}^- \|_{L^2_x L^2_{14}}^2 \|g_{\pm} \|_{L^{\infty}_x H^{s_{k+\gamma/2}}}^2. \end{aligned}$$

Choosing  $\tau_0$  small enough that  $\|g_{\pm} \|_{L^{\infty}_x L^2_{14}} \leq C \|g\|_{L^{\infty}_T X_k} \leq C \tau_0 \leq \frac{\gamma_1}{2C_k}$ . Thus,

$$(Q(G_{\pm}, F_{\pm}^-), F_{\pm}^- \langle v \rangle^{2k})_{L^2_{x,v}} \leq C_k \|F_{\pm}^- \|_{L^2_x L^2_k}^2 + C_k \|F_{\pm}^- \|_{L^2_x L^2_k}^2 \|g_{\pm} \|_{Y_k}^2,$$

and similarly,  $(Q(G_{\mp}, F_{\pm}^{-}), F_{\pm}^{-} \langle v \rangle^{2k})_{L_{x,v}^2} \leq C_k \|F_{\pm}^{-}\|_{L_x^2 L_k^2}^2 + C_k \|F_{\pm}^{-}\|_{L_x^2 L_k^2}^2 \|g_{\pm}\|_{Y_k}^2$ . Plugging the above estimates into (3.18), we obtain

$$\frac{d}{dt} \|F_{\pm}^{-}\|_{L_x^2 L_k^2}^2 \leq 2C_k (1 + \|g_{\pm}\|_{Y_k}^2 + \|\nabla_x \psi\|_{H_x^1}) \|F_{\pm}^{-}\|_{L_x^2 L_k^2}^2.$$

Recalling that  $\nabla_x \psi \in L_{T_0}^2 H_x^1$  and  $G \in L_{T_0}^2 Y_k$ , the Grönwall's inequality implies  $F_{\pm}^{-} = 0$  on  $[0, T_0]$ . We then deduce that  $F \geq 0$  on  $[0, T_0]$ .  $\square$

We finish this section by proving the theorem below.

**Theorem 3.2.** *Suppose that  $-3 < \gamma \leq 1$  for Boltzmann case and  $-3 \leq \gamma \leq 1$  for Landau case. For any  $k \geq k_0$  (and let  $k$  sufficiently large for Boltzmann case), there exist small constants  $\varepsilon_0, \tau_0, T_0 > 0$ , such that if  $f_0 \in X_{k_0}$  satisfies  $\mu + f_0 \geq 0$  and*

$$\|f_0\|_{X_{k_0}} \leq \varepsilon_0, \quad \|f_0\|_{X_k} < \infty, \quad (3.19)$$

then the Cauchy problem

$$\begin{cases} \partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \mp \nabla_x \phi \cdot \nabla_v f_{\pm} \pm \nabla_x \phi \cdot v \mu \\ \quad = Q(f_{\pm} + f_{\mp}, \mu) + Q(\mu + f_{\pm}, f_{\pm}) + Q(\mu + g_{\mp}, f_{\pm}), \\ -\Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) dv, \quad \int_{\mathbb{T}^3} \phi(x) dx = 0, \\ f(0) = f_0, \quad E(0) = E_0. \end{cases} \quad (3.20)$$

admits a unique weak solution  $f \in L_{T_0}^{\infty} X_k$  satisfying  $\mu + f \geq 0$  and

$$\|f\|_{L_{T_0}^{\infty} X_{k_0}} + \|f\|_{L_{T_0}^2 Y_{k_0}} \leq \tau_0. \quad (3.21)$$

*Proof.* We recall the sequence of approximate solutions defined in (3.1) with  $f^0 = 0$ . Applying Lemma 3.1 by taking  $f = f^{n+1}$ ,  $\phi = \phi^{n+1}$ ,  $g = f^n$  and  $\psi = \phi^n$ , choosing  $\varepsilon_0$  in (3.19) such that

$$2\sqrt{M}\varepsilon_0 \leq \tau_0,$$

we have  $\mu + f^n \geq 0$  and

$$\|f^n\|_{L_{T_0}^{\infty} X_k} + \|f^n\|_{L_{T_0}^2 Y_k} \leq \tau_0. \quad (3.22)$$

Then there exists a weak limit  $f \in L_{T_0}^{\infty} X_k \cap L_{T_0}^2 Y_k$  such that  $f_n \rightharpoonup f$  weakly in  $L_{T_0}^2 Y_k$ . Then by (3.22),  $f$  satisfies (3.21). We next prove the strong convergence of the sequence of  $f^n$ . Define  $h^n := f^{n+1} - f^n$ . Then we have from (3.1) that

$$\begin{cases} \partial_t h_{\pm}^n + v \cdot \nabla_x h_{\pm}^n \mp \nabla_x \phi^n \cdot \nabla_v h_{\pm}^n \mp \nabla_x (\phi^n - \phi^{n-1}) \cdot \nabla_v f_{\pm}^n \pm \nabla_x (\phi^{n+1} - \phi^n) \cdot v \mu \\ \quad = Q(h_{\pm}^{n-1} + h_{\mp}^{n-1}, \mu + f^n) + Q(\mu + f_{\pm}^n, h_{\pm}^n) + Q(\mu + f_{\mp}^n, h_{\pm}^n), \\ -\Delta_x \phi^{n+1} = \int_{\mathbb{R}^3} (f_+^{n+1} - f_-^{n+1}) dv, \quad h^n(0) = 0. \end{cases} \quad (3.23)$$

Let  $|\alpha| + |\beta| \leq 2$ . Apply  $\partial_{\beta}^{\alpha}$  to (3.23)<sub>1</sub> and take inner product of the resultant equation with  $e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \langle v \rangle^{-4} \partial_{\beta}^{\alpha} h_{\pm}^n$  over  $\mathbb{T}^3 \times \mathbb{R}^3$ , where

we apply similar arguments in Lemma 3.1 and illustrate the different terms

$$(\partial_{\beta}^{\alpha} (\nabla_x (\phi^n - \phi^{n-1}) \cdot \nabla_v f_{\pm}^n), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \langle v \rangle^{-4} \partial_{\beta}^{\alpha} h_{\pm}^n)_{L_{x,v}^2}, \quad (3.24)$$

and

$$(\partial_{\beta}^{\alpha} Q(h_{\pm}^{n-1} + h_{\mp}^{n-1}, f^n), e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \langle v \rangle^{-4} \partial_{\beta}^{\alpha} h_{\pm}^n)_{L_{x,v}^2}. \quad (3.25)$$

We estimate (3.24) by

$$\left| \sum_{\alpha_1 \leq \alpha} (\partial^{\alpha_1} \nabla_x (\phi^n - \phi^{n-1}) \cdot \partial_{\beta}^{\alpha - \alpha_1} \nabla_v f_{\pm}^n, e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \langle v \rangle^{-4} \partial_{\beta}^{\alpha} h_{\pm}^n)_{L_{x,v}^2} \right|$$



$$\begin{aligned}
&\lesssim \int_{\mathbb{T}^3} |\partial^{\alpha_1} \nabla_x (\phi^n - \phi^{n-1})| \underbrace{\|\langle v \rangle^{\frac{\gamma}{2}} w(\alpha, \beta) \langle D_v \rangle^{\frac{1}{2}} \partial_\beta^{\alpha - \alpha_1} f_\pm^n\|}_{\in Op(\langle v \rangle^{\gamma/2} \langle \eta \rangle^s w(\alpha, \beta))} \\
&\quad \times \underbrace{\|\langle v \rangle^{-\frac{\gamma}{2}} w^{-1}(\alpha, \beta) \langle D_v \rangle^{\frac{1}{2}} e^{\frac{\pm A_{\alpha, \beta} \psi}{\langle v \rangle^2}} w^2(\alpha, \beta) \langle v \rangle^{-4} \partial_\beta^\alpha h_\pm^n\|}_{\in Op(\langle v \rangle^{\gamma/2} \langle \eta \rangle^s w(\alpha, \beta))} dx.
\end{aligned}$$

Here an operator  $T \in Op(m)$  means  $T$  is a pseudo-differential operator with symbol in  $S(m)$ . Then by [14, Lemma 2.3 and Corollary 2.5], we can estimate (3.24) by

$$\begin{aligned}
&\sum_{\alpha_1 \leq \alpha} \int_{\mathbb{T}^3} |\partial^{\alpha_1} \nabla_x (\phi^n - \phi^{n-1})| \|\langle v \rangle^{\frac{\gamma}{2}} \langle D_v \rangle^s w(\alpha, \beta) \partial_\beta^{\alpha - \alpha_1} f_\pm^n\| \|\langle v \rangle^{\frac{\gamma}{2}} \langle D_v \rangle^s w(\alpha, \beta) \partial_\beta^\alpha h_\pm^n\| dx \\
&\lesssim \|\nabla_x (\phi^n - \phi^{n-1})\|_{H_x^2} \|f^n\|_{Y_k} \|h^n\|_{Y_k}.
\end{aligned}$$

Applying (2.13), we estimate (3.25) by

$$\begin{aligned}
&\sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} \int_{\mathbb{T}^3} (\|\partial_{\beta_1}^{\alpha_1} h^{n-1}\|_{L_{k_0}^2} \|w(\alpha, \beta) \partial_{\beta - \beta_1}^{\alpha - \alpha_1} f^n\|_{H_{\gamma/2+2s-2}^s} \\
&\quad + \|\partial_{\beta_1}^{\alpha_1} f^n\|_{L_{k_0}^2} \|w(\alpha, \beta) \partial_{\beta - \beta_1}^{\alpha - \alpha_1} h^{n-1}\|_{H_{\gamma/2-2}^s}) \|w(\alpha, \beta) \partial_\beta^\alpha h^n\|_{H_{\gamma/2-2}^s} dx \\
&\lesssim (\|h^{n-1}\|_{X_{k-2}} \|f^n\|_{Y_k} + \|f^n\|_{X_k} \|h^{n-1}\|_{Y_{k-2}}) \|h^n\|_{Y_{k-2}},
\end{aligned}$$

where we let  $k \geq 16$ . Noticing we use weight  $w(\alpha, \beta) \langle v \rangle^{-4}$  instead of  $w(\alpha, \beta)$ , applying similar arguments for deriving (3.17) and choosing  $\tau_0$  in (3.22) sufficiently small, we obtain

$$\begin{aligned}
&\|h^n\|_{L_T^\infty X_{k-2}}^2 + \|\nabla_x (\phi^{n+1} - \phi^n)\|_{L_T^\infty H_x^3}^2 + \|h^n\|_{L_T^2 Y_{k-2}}^2 \leq \frac{1}{4} (\|h^{n-1}\|_{L_T^2 Y_{k-2}}^2 + \|\nabla_x (\phi^n - \phi^{n-1})\|_{L_T^\infty H_x^2}^2) \\
&\leq \dots \leq \frac{1}{4^{n-1}} (\|f^1\|_{L_T^2 Y_{k-2}}^2 + \|\nabla_x \phi^1\|_{L_T^\infty H_x^2}^2) \leq \frac{\tau_0}{4^{n-1}}. \quad (3.26)
\end{aligned}$$

Thus  $\{f^n\}$  and  $\{\nabla_x \phi^n\}$  are Cauchy sequence in  $L_T^\infty X_{k-2}$  and  $L_T^\infty H_x^3$  respectively. Then their limit  $(f, \phi)$  solves (3.20) and satisfies (3.21).

For the uniqueness, suppose that there are two solutions  $(f^1, \phi^1)$  and  $(f^2, \phi^2)$  satisfy (3.20) and (3.21). Then  $h := f^1 - f^2$  satisfies

$$\begin{cases} \partial_t h_\pm + v \cdot \nabla_x h_\pm \mp \nabla_x \phi^1 \cdot \nabla_v h_\pm \pm \nabla_x (\phi^1 - \phi^2) \cdot v \mu \\ \quad = Q(h_\pm + h_\mp, \mu + f^2) + Q(\mu + f_\pm^1, h_\pm) + Q(\mu + f_\mp^1, h_\pm), \\ -\Delta_x (\phi^1 - \phi^2) = \int_{\mathbb{R}^3} (h_+ - h_-) dv, \quad h(0) = 0. \end{cases}$$

We apply the same method for deriving (3.26) to obtain that  $h = 0$  and the solution to (3.20) is unique. This completes the proof of Theorem 3.2.  $\square$

#### 4. GLOBAL REGULARITY

The main goal of this section is to prove the Theorem 1.1. The proof is based on the following lemmas.

**4.1. Macroscopic equation.** We recall the operator  $\mathbf{P}$  defined in (1.8). Firstly, we establish the macroscopic estimates.

**Lemma 4.1.** *For any  $k \geq 4$ , we have*

$$\frac{1}{2} \|f\|_{L_k^2}^2 \leq \|\mathbf{P}f\|_{L_k^2}^2 + \|(\mathbf{I} - \mathbf{P})f\|_{L_k^2}^2 \leq C_k \|f\|_{L_k^2}^2, \quad \|\mathbf{P}f\|_{L_k^2} \leq C_k \|f\|_{L_4^2}, \quad (4.1)$$

for some constant  $C_k \geq 0$ .

*Proof.* Since  $f = \mathbf{P}f + (\mathbf{I} - \mathbf{P})f$ , we have

$$|f|^2 \leq 2|\mathbf{P}f|^2 + 2|(\mathbf{I} - \mathbf{P})f|^2.$$

This proves the first part of (4.1). For the second part of (4.1), we have

$$|\{\mathbf{I} - \mathbf{P}\}f| \leq 2|\mathbf{P}f|^2 + 2|f|^2.$$

For the part  $\mathbf{P}f$ , by the definition (1.9), we have

$$\|\mathbf{P}f\|_{L_k^2} \leq C\|f\|_{L_2^1}\|\mu\langle v \rangle^2\|_{L_k^2} \leq C_k\|f\|_{L_4^2}.$$

This completes the proof of (4.1).  $\square$

Recall the presentation of  $\mathbf{P}f$  defined in (1.8). It is direct to obtain

$$\|a\|_{L_x^2}^2 + \|b\|_{L_x^2}^2 + \|c\|_{L_x^2}^2 \sim \|\mathbf{P}f\|_{L_{x,v}^2}^2.$$

We also rewrite the equation (1.4) as

$$\begin{cases} \partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \pm \nabla_x \phi \cdot v \mu - L_{\pm} f = N_{\pm}(f), \\ -\Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) dv, \end{cases} \quad (4.2)$$

with initial data  $f(0) = f_0$ ,  $E(0) = E_0$ , where  $L = [L_+, L_-]$  and  $N = [N_+, N_-]$  are given by

$$\begin{aligned} L_{\pm} f &:= Q(f_{\pm} + f_{\mp}, \mu) + 2Q(\mu, f_{\pm}), \\ N_{\pm} &:= \pm \nabla_x \phi \cdot \nabla_v f_{\pm} + Q(f_{\pm}, f_{\pm}) + Q(f_{\mp}, f_{\pm}). \end{aligned} \quad (4.3)$$

Our next goal is to estimate  $a(t, x), b(t, x), c(t, x)$  in terms of  $\{\mathbf{I} - \mathbf{P}\}f$ . In contrast to [33], our  $P$  is not symmetric and we can not compare the  $v_i, v_j$  terms on both side. The following Lemma gives the macroscopic estimates.

**Lemma 4.2.** *For both Landau and Boltzmann case, suppose  $f$  solves (4.2). For any integer  $m \geq 0$ , there exists function  $G = G(t)$  satisfying*

$$G(t) \lesssim \sum_{|\alpha| \leq 1} \|\partial^{\alpha} f\|_{L_x^2 L_{10}^2} \|\partial^{\alpha} \nabla_x \mathbf{P}f\|_{L_{x,v}^2}, \quad (4.4)$$

such that

$$\begin{aligned} \partial_t G(t) + \lambda \| [a_+, a_-, b, c] \|_{H_x^2}^2 + \lambda \| \nabla_x \phi \|_{H_x^3}^2 \\ \lesssim \| \{\mathbf{I} - \mathbf{P}\}f \|_{H_x^2 L_{10}^2}^2 + \| N_{\parallel} \|_{H_x^1 L_v^2}^2 + \| \nabla_x \phi \|_{L_x^2}^4, \end{aligned} \quad (4.5)$$

Here  $N_{\parallel}$  is inner product of  $N(f)$  with some linear combination of  $(1, v_i, v_i v_j, v_i |v|^2)$  over  $v \in \mathbb{R}^3$ .

*Proof.* Here we only consider the non-cutoff Boltzmann case and the Landau case is similar. The proof follows the idea in [33, Section 6]. Splitting  $f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f$ , we rewrite (4.2) to be

$$\begin{aligned} (\partial_t a_{\pm} + \partial_t b \cdot v + \partial_t c(|v|^2 - 3))\mu + v \cdot \nabla_x (a_{\pm} + b \cdot v + c(|v|^2 - 3))\mu \pm \nabla_x \phi \cdot v \mu \\ = -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_{\pm} - \mathbf{P}_{\pm}\}f - L_{\pm} \{\mathbf{I} - \mathbf{P}\}f + N_{\pm}(f). \end{aligned} \quad (4.6)$$

Taking the inner product of (4.6) with  $1, v_i^2, |v|^2, v_i v_j$ ,  $i, j = 1, 2, 3$ ,  $i \neq j$  over  $v \in \mathbb{R}^3$ , we have

$$\begin{aligned} \partial_t a_{\pm} + \nabla_x \cdot b &= -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_{\pm} - \mathbf{P}_{\pm}\}f - L_{\pm} \{\mathbf{I} - \mathbf{P}\}f + N_{\pm}(f), 1)_{L_v^2}, \\ \partial_t a_{\pm} + 2\partial_t c + 2\partial_{x_i} b_i + \nabla_x \cdot b &= -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_{\pm} - \mathbf{P}_{\pm}\}f - L_{\pm} \{\mathbf{I} - \mathbf{P}\}f + N_{\pm}(f), v_i^2)_{L_v^2}, \\ 3\partial_t a_{\pm} + 6\partial_t c + 5\nabla_x \cdot b &= -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_{\pm} - \mathbf{P}_{\pm}\}f - L_{\pm} \{\mathbf{I} - \mathbf{P}\}f + N_{\pm}(f), |v|^2)_{L_v^2}. \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \partial_{x_j} b_i + \partial_{x_i} b_j &= -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_{\pm} - \mathbf{P}_{\pm}\}f - L_{\pm} \{\mathbf{I} - \mathbf{P}\}f, v_i v_j)_{L_v^2} + (N_{\pm}(f), v_i v_j)_{L_v^2} \\ &=: \gamma_{1ij} + \gamma_{2ij}, \quad i \neq j. \end{aligned} \quad (4.8)$$

By taking the subtraction of (4.7)<sub>2</sub> and (4.7)<sub>1</sub>, we have

$$\begin{aligned}\partial_t c + \partial_{x_i} b_i &= \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f, \frac{|v_i|^2 - 1}{2} \right)_{L_v^2} + (N_\pm(f), \frac{|v_i|^2 - 1}{2})_{L_v^2} \\ &=: \gamma_{1i} + \gamma_{2i},\end{aligned}\quad (4.9)$$

Taking the inner product of (4.6) with  $v, v|v|^2$  over  $v \in \mathbb{R}^3$ , we have

$$\begin{aligned}\partial_t b + \nabla_x(a_\pm \pm \phi) + 2\nabla_x c &= \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), v \right)_{L_v^2}, \\ 5\partial_t b + 5\nabla_x(a_\pm \pm \phi) + 20\nabla_x c &= \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), v|v|^2 \right)_{L_v^2}.\end{aligned}\quad (4.10)$$

The two identities in (4.10) implies

$$\nabla_x c = \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), \frac{v(|v|^2 - 5)}{10} \right)_{L_v^2}, \quad (4.11)$$

and

$$\nabla_x a_\pm = -\partial_t b \mp \nabla_x \phi + \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), \frac{v(10 - |v|^2)}{5} \right)_{L_v^2}. \quad (4.12)$$

Here we use the fact that

$$\int_{\mathbb{R}^3} |v|^2 \mu dv = 3, \quad \int_{\mathbb{R}^3} |v|^4 \mu dv = 15, \quad \int_{\mathbb{R}^3} |v|^6 \mu dv = 105.$$

By the definitions (1.8) of  $\mathbf{P}f$  and (4.3) of  $L$ , we have for  $\psi(v) = 1, v_i, |v|^2$  that

$$\left( \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \psi(v) \right) = 0, \quad \left( L_\pm \{\mathbf{I} - \mathbf{P}\} f, \psi(v) \right) = 0,$$

and

$$\left( \partial_t \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \psi(v) \right) = 0, \quad \left( v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, 1 \right) = 0.$$

Thus the first, third identities in (4.7) and first identities in (4.10) (i.e. inner product with  $1, v_i, |v|^2$ ) become

$$\begin{aligned}\partial_t a_\pm + \nabla_x \cdot b &= (N_\pm(f), 1)_{L_v^2} \\ \partial_t b + \nabla_x a_\pm + 2\nabla_x c \pm \nabla_x \phi &= -(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, v)_{L_v^2} + (N_\pm(f), v)_{L_v^2} \\ 3\partial_t a_\pm + 6\partial_t c + 5\nabla_x \cdot b &= -(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, |v|^2)_{L_v^2} + (N_\pm(f), |v|^2)_{L_v^2}.\end{aligned}\quad (4.13)$$

Combining (4.13)<sub>1</sub> and (4.13)<sub>3</sub>, we have

$$\partial_t c + \frac{1}{3} \nabla_x \cdot b = -\frac{1}{6} (v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, |v|^2)_{L_v^2} + \left( N_\pm(f), \frac{|v|^2 - 3}{6} \right)_{L_v^2}. \quad (4.14)$$

For brevity, we define

$$\xi_a = \frac{v(10 - |v|^2)}{5}, \quad \xi_{bi} = \frac{2v_i^2 - 5}{2}, \quad \xi_c = \frac{v(|v|^2 - 5)}{10}.$$

It follows that

$$\begin{aligned}|(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \xi_a)| &+ |(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \xi_{bi})| + |(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, v_i v_j)| \\ &+ |(v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \xi_c)| \leq \|\nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{L_v^2}.\end{aligned}$$

For the Boltzmann case, by Lemma 2.4, we have

$$\begin{aligned}|(Q(\mu, f), \xi_a)| &\leq \|\mu\|_{L_{10}^2} \|f\|_{L_{10}^2} \|\xi_a\|_{H_{-7}^{2s}} \leq C \|f\|_{L_{10}^2}, \\ |(Q(f, \mu), \xi_a)| &\leq \|f\|_{L_{10}^2} \|\mu\|_{L_{10}^2} \|\xi_a\|_{H_{-7}^{2s}} \leq C \|f\|_{L_{10}^2}.\end{aligned}$$

For the Landau case, by Lemma 2.13, we have

$$\begin{aligned}|(Q(\mu, f), \xi_a)| &\leq \|\mu\|_{L_{10}^2} \|f\|_{L_{10}^2} \|\xi_a\|_{H_{-5}^2} \leq C \|f\|_{L_{10}^2}, \\ |(Q(f, \mu), \xi_a)| &\leq \|f\|_{L_{10}^2} \|\mu\|_{L_{10}^2} \|\xi_a\|_{H_{-5}^2} \leq C \|f\|_{L_{10}^2}.\end{aligned}$$

Similar arguments can be carried on inner product with  $\xi_{bi}$ ,  $v_i v_j$ ,  $\xi_c$ . Recalling the definition (4.3) of  $L = [L_+, L_-]$ , we have

$$|(Lf, \xi_a)_{L_v^2}| + |(Lf, \xi_{bi})_{L_v^2}| + |(Lf, v_i v_j)_{L_v^2}| + |(Lf, \xi_c)_{L_v^2}| \lesssim \|f\|_{L_{10}^2}. \quad (4.15)$$

Let  $|\alpha| = m$ . Applying  $\partial^\alpha$  on both side of (4.11) and taking inner product with  $\nabla_x \partial^\alpha c$  over  $x \in \mathbb{T}^3$ , we deduce that

$$\begin{aligned} \|\nabla_x \partial^\alpha c\|_{L_x^2}^2 &= - \int_{\mathbb{T}^3} ((\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c \, dx \\ &\quad - \int_{\mathbb{T}^3} (L_\pm \{\mathbf{I} - \mathbf{P}\} \partial^\alpha f, \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c \, dx + \int_{\mathbb{T}^3} (\partial^\alpha N_\pm(f), \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c \, dx \\ &\leq - \frac{d}{dt} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c \, dx \\ &\quad - \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \nabla_x \partial^\alpha f, \xi_c)_{L_v^2} \cdot \partial_t \partial^\alpha c \, dx \\ &\quad + C \|\nabla_x \partial^\alpha c\|_{L_x^2} (\|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2} + \|\partial^\alpha N_\pm\|_{L_{x,v}^2}). \end{aligned} \quad (4.16)$$

It follows from (4.14) that

$$\partial_t \partial^\alpha c = \frac{1}{3} \nabla_x \partial^\alpha b - \frac{1}{6} (v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, |v|^2)_{L_v^2} + \left( \partial^\alpha N_\pm(f), \frac{|v|^2 - 3}{6} \right)_{L_v^2}.$$

By Cauchy-Schwarz inequality, for any small  $\varepsilon > 0$ , we have

$$\begin{aligned} - \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \nabla_x \partial^\alpha f, \xi_c)_{L_v^2} \cdot \partial_t \partial^\alpha c \, dx \\ \leq \varepsilon \|\nabla_x \partial^\alpha b\|^2 + \frac{C}{\varepsilon} \|\nabla_x \partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{L_x^2 L_{10}^2}^2 + \varepsilon \|\partial^\alpha N_\pm\|_{L_{x,v}^2}^2. \end{aligned}$$

Plugging this into (4.16), we deduce that

$$\begin{aligned} \|\nabla_x \partial^\alpha c\|^2 &\leq - \frac{d}{dt} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c \, dx + \varepsilon \|\nabla_x \partial^\alpha b\|^2 + \varepsilon \|\nabla_x \partial^\alpha c\|^2 \\ &\quad + \frac{C}{\varepsilon} (\|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2}^2 + \|\partial^\alpha N_\pm\|_{L_{x,v}^2}^2). \end{aligned} \quad (4.17)$$

We next compute  $\nabla_x \partial^\alpha b$ . For fixed  $i$ , we use (4.8), (4.9) to compute

$$\begin{aligned} \Delta_x b_i &= \sum_{j \neq i} \partial_{x_j x_j} b_i + \partial_{x_i x_i} b_i \\ &= \sum_{j \neq i} (-\partial_{x_i x_j} b_j + \partial_{x_j} (\gamma_{1ij} + \gamma_{2ij})) + \partial_{x_i} (\gamma_{1i} + \gamma_{2i}) - \partial_t \partial_{x_i} c \\ &= \sum_{j \neq i} (\partial_t \partial_{x_i} c - \partial_{x_i} (\gamma_{1j} + \gamma_{2j})) + \sum_{j \neq i} \partial_{x_j} (\gamma_{1ij} + \gamma_{2ij}) + \partial_{x_i} (\gamma_{1i} + \gamma_{2i}) - \partial_t \partial_{x_i} c \\ &= \partial_t \partial_{x_i} c + \sum_{j \neq i} (\partial_{x_j} (\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i} (\gamma_{1j} + \gamma_{2j})) + \partial_{x_i} (\gamma_{1i} + \gamma_{2i}) \\ &= \sum_{j \neq i} (\partial_{x_j} (\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i} (\gamma_{1j} + \gamma_{2j})) - \partial_{x_i x_i} b_i + 2\partial_{x_i} (\gamma_{1i} + \gamma_{2i}). \end{aligned} \quad (4.18)$$

We can rewrite the linear terms  $\sum_{j \neq i} (\partial_{x_j} \gamma_{1ij} - \partial_{x_i} \gamma_{1j}) + 2\partial_{x_i} \gamma_{1i}$  including  $\gamma_{1i}$ ,  $\gamma_{1j}$ ,  $\gamma_{1ij}$  as the linear combination of

$$\partial_{x_j} (-(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f, \xi_{ij})_{L_v^2} - (L_\pm \{\mathbf{I} - \mathbf{P}\} f, \xi_{ij})_{L_v^2},$$

where  $\xi_{ij}$  are certain linear combinations of the basis  $\{1, v_i v_j, |v_i|^2\}_{i,j=1}^3$ . Similar to (4.15), we have

$$|(Lf, \xi_{ij})_{L_v^2}| \lesssim \|f\|_{L_{10}^2}.$$

Applying  $\partial^\alpha$  on both side of (4.18), taking inner product with  $\partial^\alpha b_i$  over  $x \in \mathbb{T}^3$  and summation over  $i = 1, 2, 3$ , we deduce that

$$\begin{aligned}
\|\nabla_x \partial^\alpha b\|^2 &= \sum_i \left( - \int_{\mathbb{T}^3} \sum_{j \neq i} (\partial_{x_j} \partial^\alpha (\gamma_{1ij} + \gamma_{2ij}) - \partial_{x_i} \partial^\alpha (\gamma_{1j} + \gamma_{2j})) \cdot \partial^\alpha b_i dx \right) \\
&\quad - \sum_i \left( \int_{\mathbb{T}^3} |\partial_{x_i} \partial^\alpha b_i|^2 dx - 2 \int_{\mathbb{T}^3} \partial_{x_i} \partial^\alpha (\gamma_{1i} + \gamma_{2i}) \cdot \partial^\alpha b_i dx \right) \\
&\leq \sum_{i,j} \frac{d}{dt} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial_{x_j} \partial^\alpha f, \xi_{ij})_{L_v^2} \cdot \partial^\alpha b_i dx \\
&\quad - \sum_{i,j} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial_{x_j} \partial^\alpha f, \xi_{ij})_{L_v^2} \cdot \partial_t \partial^\alpha b_i dx \\
&\quad + C \|\nabla_x \partial^\alpha b\| (\|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2} + \|\partial^\alpha N\|_{L_{x,v}^2}). \tag{4.19}
\end{aligned}$$

Applying  $\partial^\alpha$  to (4.13)<sub>2</sub> yields

$$\partial_t \partial^\alpha b_i = -\nabla_{x_i} \partial^\alpha a_\pm - 2\nabla_{x_i} \partial^\alpha c \mp \nabla_{x_i} \partial^\alpha \phi - (v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f + \partial^\alpha N_\pm(f), v_i)_{L_v^2}.$$

Then by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
- \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial_{x_j} \nabla_x \partial^\alpha f, \xi_{ij})_{L_v^2} \cdot \partial_t \partial^\alpha b_i dx &\leq \varepsilon (\|\nabla_x \partial^\alpha a_\pm\|_{L_x^2}^2 + \|\nabla_x \partial^\alpha c\|_{L_x^2}^2 + \|\nabla_x \partial^\alpha \phi\|_{L_x^2}^2) \\
&\quad + \varepsilon \|\partial^\alpha N\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\nabla_x \partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{L_x^2 L_{10}^2}^2,
\end{aligned}$$

and (4.19) becomes

$$\begin{aligned}
\|\nabla_x \partial^\alpha b\|^2 &\leq - \sum_{i,j} \frac{d}{dt} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial_{x_j} \partial^\alpha f, \xi_{ij})_{L_v^2} \cdot \partial_{x_j} \partial^\alpha b_i dx + \varepsilon \|\nabla_x \partial^\alpha \phi\|_{L_x^2}^2 \\
&\quad + \varepsilon \|\nabla_x \partial^\alpha a_\pm\|_{L_x^2}^2 + \varepsilon \|\nabla_x \partial^\alpha c\|_{L_x^2}^2 + \frac{C}{\varepsilon} (\|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2}^2 + \|\partial^\alpha N\|_{L_{x,v}^2}^2), \tag{4.20}
\end{aligned}$$

where we choose  $\varepsilon$  small enough. The final step is to compute  $\nabla_x \partial^\alpha a$ . Taking the addition and difference of (4.12) over  $\pm$  yields

$$\nabla_x (a_+ + a_-) = -2\partial_t b + \sum_{\pm} \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), \xi_a \right)_{L_v^2}, \tag{4.21}$$

and

$$\nabla_x (a_+ - a_-) = -2\nabla_x \phi + \sum_{\pm} \pm \left( -(\partial_t + v \cdot \nabla_x) \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f - L_\pm \{\mathbf{I} - \mathbf{P}\} f + N_\pm(f), \xi_a \right)_{L_v^2}. \tag{4.22}$$

Applying  $\partial^\alpha$  derivative on both side of (4.21) and taking inner product with  $\nabla_x \partial^\alpha (a_+ + a_-)$  over  $\mathbb{T}^3$ , we deduce that

$$\begin{aligned}
\|\nabla_x \partial^\alpha (a_+ + a_-)\|_{L_x^2}^2 &= - \int_{\mathbb{T}^3} \partial_t \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx \\
&\quad - \int_{\mathbb{T}^3} (v \cdot \nabla_x \{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f + L_\pm \{\mathbf{I} - \mathbf{P}\} \partial^\alpha f + \partial^\alpha N_\pm(f), \xi_a)_{L_v^2} \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx. \tag{4.23}
\end{aligned}$$

By integration by parts about  $\partial_t$  and  $\nabla_x$ , we have

$$\begin{aligned}
&- \int_{\mathbb{T}^3} \partial_t \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx \\
&= - \frac{d}{dt} \int_{\mathbb{T}^3} \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx
\end{aligned}$$

$$- \int_{\mathbb{T}^3} \nabla_x \cdot \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \partial_t \partial^\alpha (a_+ + a_-) dx.$$

In view of this and (4.23), applying (4.13)<sub>1</sub>, (4.15) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\nabla_x \partial^\alpha (a_+ + a_-)\|^2 + \frac{d}{dt} \sum_{\pm} \int_{\mathbb{T}^3} \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx \\ \lesssim \|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2} + \|\partial^\alpha N\|_{L_{x,v}^2}^2 + \|\nabla_x \partial^\alpha b\|_{L_x^2}^2. \end{aligned} \quad (4.24)$$

Similarly, noticing that

$$(\nabla_x \partial^\alpha \phi, \nabla_x \partial^\alpha (a_+ - a_-))_{L_x^2} = -(\Delta_x \partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L_x^2} = \|\Delta_x \partial^\alpha \phi\|_{L_x^2}^2,$$

we apply  $\partial^\alpha$  to (4.22) and take inner product with  $\nabla_x \partial^\alpha (a_+ - a_-)$  over  $\mathbb{T}^3$  to deduce that

$$\begin{aligned} \|\nabla_x \partial^\alpha (a_+ - a_-)\|_{L_x^2}^2 + 2\|\Delta_x \partial^\alpha \phi\|_{L_x^2}^2 + \frac{d}{dt} \int_{\mathbb{T}^3} \sum_{\pm} \mp (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \cdot \nabla_x \partial^\alpha (a_+ - a_-) dx \\ \lesssim \|\partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f\|_{H_x^1 L_{10}^2} + \|\partial^\alpha N\|_{L_{x,v}^2}^2 + \|\nabla_x \partial^\alpha b\|_{L_x^2}^2. \end{aligned} \quad (4.25)$$

Notice that  $|a_+ + a_-|^2 + |a_+ - a_-|^2 = 2|a_+|^2 + 2|a_-|^2$  and by Sobolev inequality,

$$\|\nabla_x \partial^\alpha \phi\|^2 \lesssim \|\Delta_x \partial^\alpha \phi\|^2.$$

For the zeroth order, we apply the Poincaré inequality to  $a_\pm, b, c$  in  $x$  and use the conservation law (1.6) to deduce that

$$\begin{aligned} \|a_\pm\|_{L_x^2}^2 &\leq C \left( \int_{\mathbb{T}^3} a_\pm(t, x) dx \right)^2 + C \|\nabla_x a_\pm\|_{L_x^2}^2 \lesssim \|\nabla_x a_\pm\|_{L_x^2}^2, \\ \|b\|_{L_x^2}^2 &\leq C \left( \int_{\mathbb{T}^3} b(t, x) dx \right)^2 + C \|\nabla_x b\|_{L_x^2}^2 \lesssim \|\nabla_x b\|_{L_x^2}^2, \\ \|c\|_{L_x^2}^2 &\leq C \left( \int_{\mathbb{T}^3} c(t, x) dx \right)^2 + C \|\nabla_x c\|_{L_x^2}^2 \lesssim \|\nabla_x \phi\|_{L_x^2}^4 + \|\nabla_x c\|_{L_x^2}^2. \end{aligned}$$

In view of this, we take linear combination  $\lambda \times (4.25) + \lambda \times (4.24) + (4.17) + (4.20)$  and summation over  $|\alpha| \leq 1$  and  $\pm$  with sufficiently small  $\lambda, \varepsilon > 0$  to deduce that

$$\partial_t G(t) + \lambda \sum_{|\alpha| \leq 2} \|\partial^\alpha [a_+, a_-, b, c]\|_{L_x^2}^2 + \lambda \|\nabla_x \phi\|_{H_x^3}^2 \lesssim \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_{10}^2}^2 + \|N\|_{H_x^1 L_v^2}^2 + \|\nabla_x \phi\|_{L_x^2}^4,$$

where

$$\begin{aligned} G &= \sum_{\pm} \sum_{|\alpha| \leq 1} \left( \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_c)_{L_v^2} \cdot \nabla_x \partial^\alpha c dx \right. \\ &\quad + \sum_{i,j} \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial_{x_j} \partial^\alpha f, \xi_{ij})_{L_v^2} \cdot \partial_{x_j} \partial^\alpha b_i dx \\ &\quad + \lambda \int_{\mathbb{T}^3} \left( \partial^\alpha b + (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \right) \cdot \nabla_x \partial^\alpha (a_+ + a_-) dx \\ &\quad \mp \lambda \int_{\mathbb{T}^3} (\{\mathbf{I}_\pm - \mathbf{P}_\pm\} \partial^\alpha f, \xi_a)_{L_v^2} \cdot \nabla_x \partial^\alpha (a_+ - a_-) dx \Big). \end{aligned}$$

This implies (4.5) and the estimate (4.4) can be directly deduced by Cauchy-Schwarz inequality.  $\square$

4.2. **Estimates about the energy.** In this section, we give the energy estimates on (1.4).

**Lemma 4.3.** *Suppose  $[f, \phi]$  is the solution to equation (1.4) satisfying  $\mu + f \geq 0$  and*

$$\mathcal{E}(t) \leq C.$$

*There exists functional  $\mathcal{E}_1(t)$  satisfying*

$$\mathcal{E}_1(t) \approx \|f\|_{X_k}^2 + \|\nabla_x \phi\|_{H_x^3}^2, \quad (4.26)$$

*such that, for any  $T > 0$  and sufficiently large  $k \geq 15$ , we have*

$$\begin{aligned} \partial_t \mathcal{E}_1(t) + \lambda \varepsilon \|f\|_{Y_k}^2 + \lambda \sum_{|\alpha| \leq 2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2 &\lesssim \|f\|_{X_{k_0}} \|f\|_{X_k}^2 \\ &+ C_{\varepsilon, k} \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2 + \varepsilon \|\nabla_x \phi\|_{H_x^2}^2 + \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t). \end{aligned} \quad (4.27)$$

*for some generic constant  $\lambda > 0$  and any sufficiently small  $\varepsilon > 0$ .*

*Proof.* We proceed by considering spatial derivatives and mixed derivatives. Notice that

$$\|\phi\|_{L_x^\infty} \lesssim \|\nabla_x \phi\|_{H_x^1} \lesssim \mathcal{E}(t) \leq C.$$

which verify (2.29) and we can apply estimates in Section 2.3.

**Step 1. Estimates with mixed derivatives.** We first consider the Boltzmann case. Recall that  $A_{\alpha, \beta}$  is given in (1.11). For any  $|\alpha| + |\beta| \leq 2$ , we apply  $\partial_\beta^\alpha$  to (1.4), take inner product with  $e^{\frac{\pm A_{\alpha, \beta} \phi}{\langle v \rangle^2}} w^2(\alpha, \beta) \partial_\beta^\alpha f_\pm$  over  $\mathbb{T}^3 \times \mathbb{R}^3$ . Similar to the calculation from (3.7) to (3.10), i.e. replacing  $(g, \psi)$  by  $(f, \phi)$  in (3.10), applying (3.12), (3.13), (3.14) and choosing  $\eta < \frac{\gamma_1}{4}$ , we have

$$\begin{aligned} &\partial_t \left( \frac{1}{2} \|f\|_{X_k}^2 + \sum_{|\alpha| \leq 2} C_\alpha C_{|\alpha|, 0} \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 \right) + \frac{1}{32} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|f\|_{Y_k}^2 + \frac{\gamma_1}{2} \|f\|_{Y_k}^2 \\ &\leq C_k \|f\|_{H_{x,v}^2}^2 + C \|\partial_t \phi\|_{L_x^\infty} \|f\|_{X_k}^2 + C_k \left( \|\nabla_x \phi\|_{H_x^3} \|f\|_{Y_k}^2 + \|\nabla_x \phi\|_{H_x^2}^2 + \|f\|_{X_{k_0}} \|f\|_{Y_k}^2 \right. \\ &\quad \left. + (\|f\|_{X_{k_0}} + C) \|\nabla_x \phi\|_{H_x^1} \|f\|_{Y_k}^2 + \|f\|_{X_{k_0}} \|\nabla_x \phi\|_{H_x^1} (\|f\|_{Y_k} + C) \|f\|_{Y_k} \right), \end{aligned} \quad (4.28)$$

for some generic constant  $\gamma_1$  independent of  $k$ . Here  $C_\alpha$  and  $C_{|\alpha|, 0}$  are given in (2.36) and (1.12) respectively. If  $|\beta| \geq 1$ , by interpolation we have

$$C_k \|\partial_\beta^\alpha f\|_{L_{x,v}^2}^2 \leq C \|\partial^\alpha f\|_{L_x^2 H_v^{|\beta|}}^2 \leq \frac{\gamma_1}{24} \|\partial^\alpha f\|_{L_x^2 H_v^{|\beta|+s}}^2 + C_k \|\partial^\alpha f\|_{L_{x,v}^2}^2 \leq \frac{\gamma_1}{24} \|f\|_{Y_k}^2 + C_k \|\partial^\alpha f\|_{L_{x,v}^2}^2. \quad (4.29)$$

Also, by Sobolev embedding or Poincaré's inequality,

$$\|\nabla_x \phi\|_{H_x^1} \lesssim \|\nabla_x^2 \phi\|_{L_x^2} = \|\Delta_x \phi\|_{L_x^2} \lesssim \|f\|_{L_x^2 L_v^2} \lesssim \varepsilon \|f\|_{Y_k} + C_\varepsilon \|f\|_{L_{x,v}^2}.$$

In view of the above two estimates, choosing  $\varepsilon > 0$  small enough, (4.28) implies that

$$\begin{aligned} &\partial_t \left( \frac{1}{2} \|f\|_{X_k}^2 + \sum_{|\alpha| \leq 2} C_\alpha C_{|\alpha|, 0} \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 \right) + \frac{\gamma_1}{4} \|f\|_{Y_k}^2 \\ &\lesssim \|f\|_{H_x^2 L_v^2}^2 + C \|\partial_t \phi\|_{L_x^\infty} \|f\|_{X_k}^2 + (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t). \end{aligned} \quad (4.30)$$

For Landau case, one can apply similar calculation as (3.11) instead of (3.10) to derive (4.30) by using the same technique and we omit the details for brevity.

**Step 2. Estimates with spatial derivatives.** For any  $|\alpha| \leq 2$ , we apply  $\partial^\alpha$  to (1.4), take inner product with  $e^{\frac{\pm A_{\alpha, 0} \phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm$  over  $\mathbb{T}^3 \times \mathbb{R}^3$  to deduce that

$$\frac{1}{2} \partial_t \|e^{\frac{\pm A_{\alpha, 0} \phi}{2\langle v \rangle^2}} \partial^\alpha f_\pm\|_{L_{x,v}^2}^2 + (A_{\alpha, 0} \partial_t \langle v \rangle^{-2} \partial^\alpha f_\pm, e^{\frac{\pm A_{\alpha, 0} \phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L_{x,v}^2}$$

$$\begin{aligned}
& + (v \cdot \nabla_x \partial^\alpha f_\pm, e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& \mp (\nabla_x \phi \cdot \nabla_v \partial^\alpha f_\pm, e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& \mp \sum_{\substack{\alpha_1 \leq \alpha \\ |\alpha_1| \geq 1}} (\partial^{\alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha-\alpha_1} f_\pm, e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& \pm (\partial^\alpha (\nabla_x \phi \cdot v \mu), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& = (\partial^\alpha L_\pm f + \partial^\alpha Q(f_\pm + f_\mp, f_\pm), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}}, \tag{4.31}
\end{aligned}$$

where  $L_\pm$  is given in (4.3). We denote the second to seventh terms in (4.31) by  $J_1$  to  $J_6$  and estimate them one by one. Similar to the calculation of  $I_1$  to  $I_6$  from (3.8) to (3.9), i.e. replacing  $(g, \psi)$  by  $(f, \phi)$ , one can obtain the following estimates from (2.31) and (2.33):

$$\begin{aligned}
|J_1 + J_2 + J_3 + J_4| & \lesssim (\|\partial_t \phi\|_{L_x^\infty} + \|\nabla_x \phi\|_{H_x^3}) \|f\|_{Y_k}^2 \\
& \lesssim C \|\partial_t \phi\|_{L_x^\infty} \|f\|_{X_k}^2 + \|\nabla_x \phi\|_{H_x^3} \|f\|_{Y_k}^2 \lesssim \|f\|_{X_{k_0}} \|f\|_{X_k}^2 + \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),
\end{aligned}$$

where we apply (3.13) and (3.14) for the terms  $\|\nabla_x^3 \phi\|_{L_x^2}$  and  $\|\partial_t \phi\|_{L_x^\infty}$  respectively. For  $J_5$ , using (2.36), we have

$$\begin{aligned}
J_5 & \geq C_\alpha \partial_t \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - C \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2} - C \|\partial^\alpha \nabla_x \phi\|_{L_x^2} \|\nabla_x \phi\|_{H_x^1} \|f\|_{Y_k} \\
& \geq C_\alpha \partial_t \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - \varepsilon \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 - \varepsilon \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^{\gamma/2}}^2 \\
& \quad - C_\varepsilon \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2 - C \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),
\end{aligned}$$

for any  $\varepsilon > 0$ . Here  $C_\alpha$  is given in (2.36). For the term  $J_6$ , we need some different estimates. Noticing  $L_\pm(\partial^\alpha f) = 0$ , we split  $J_6$  into

$$\begin{aligned}
J_6 & = (\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& \quad + (\partial^\alpha Q(f_\pm + f_\mp, \mathbf{P}_\pm f), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& \quad + (\partial^\alpha Q(\{\mathbf{I}_\pm - \mathbf{P}_\pm\} f + \{\mathbf{I}_\mp - \mathbf{P}_\mp\} f, \mu), e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& =: J_7 + J_8 + J_9.
\end{aligned}$$

We further split  $J_7$  as

$$\begin{aligned}
J_7 & = (\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f), w^2(\alpha, 0) \partial^\alpha \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f)_{L^2_{x,v}} \\
& \quad + (\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f), w^2(\alpha, 0) \partial^\alpha \mathbf{P}_\pm f)_{L^2_{x,v}} \\
& \quad + (\partial^\alpha Q(2\mu + f_\pm + f_\mp, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f), (e^{\frac{\pm A_{\alpha,0}\phi}{\langle v \rangle^2}} - 1) w^2(\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \\
& =: J_{7,1} + J_{7,2} + J_{7,3}.
\end{aligned}$$

For Boltzmann case, we deduce from (2.40) that  $J_{7,1}$  satisfies

$$\begin{aligned}
\sum_{\pm} J_{7,1} & \leq -\frac{1}{12} \|b(\cos \theta) \sin^2 \frac{\theta}{2}\|_{L_\theta^1} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^{\gamma/2,*}}^2 \\
& \quad - \frac{\gamma_1}{3} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 H_{\gamma/2}^s}^2 + C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t) + C_k \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L^2_{x,v}}^2.
\end{aligned}$$



For  $J_{7,2}$  and  $J_{7,3}$ , for the Boltzmann case, we deduce from Lemma 2.4 that

$$\begin{aligned} |J_{7,2}| &\lesssim (C + \|\langle v \rangle^{14} f\|_{H_x^2 L_v^2}) \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2} \|w^2(\alpha, 0) \mathbf{P} f\|_{H_x^2 H_{\gamma+2s}^2} \\ &\lesssim \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 + C_\varepsilon \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2 + C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t). \end{aligned}$$

It follows from (2.41) that

$$\begin{aligned} |J_{7,3}| &\lesssim \|\nabla_x \phi\|_{H_x^1} \left( (C + \|\langle v \rangle^{14} f\|_{H_{x,v}^2}) \|\{\mathbf{I} - \mathbf{P}\} f\|_{Y_k} \|\mathbf{P} f\|_{Y_k} \right. \\ &\quad \left. + \|\langle v \rangle^{14} \{\mathbf{I} - \mathbf{P}\} f\|_{H_{x,v}^2} (C + \|f\|_{Y_k}) \|\mathbf{P} f\|_{Y_k} \right) \\ &\lesssim (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t). \end{aligned}$$

For the Landau case, it follows from (2.51) and (2.52) that

$$\sum_{\pm} J_{7,1} \leq -\frac{\gamma_1}{2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_D^2}^2 + C_k \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2 + C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),$$

with some constants  $\gamma_1, C_k > 0$ . Applying Lemma 2.13 and (2.53), we have

$$\begin{aligned} |J_{7,2}| &\lesssim (C + \|f\|_{L^2}) \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2} \|\mathbf{P} f\|_{H_x^2 H_5^2} \\ &\lesssim \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 + C_\varepsilon \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2 + C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t), \end{aligned}$$

and

$$|J_{7,3}| \lesssim (C + \|\langle v \rangle^7 f\|_{H_{x,v}^2}) \|\nabla_x \phi\|_{H_x^1} \|\{\mathbf{I} - \mathbf{P}\} f\|_{Y_k} \|f\|_{Y_k} \lesssim (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t).$$

For  $J_8$ , we deduce from (2.42) that for the Boltzmann case:

$$|J_8| \lesssim \|\langle v \rangle^{14} f\|_{H_{x,v}^2} \|\langle v \rangle^{2s} \mathbf{P} f\|_{Y_k} \|f\|_{Y_k} + \|\langle v \rangle^{14} \mathbf{P} f\|_{H_{x,v}^2} \|f\|_{Y_k} \|f\|_{Y_k} \lesssim \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),$$

and from (2.54) that for the Landau case:

$$|J_8| \lesssim \|f\langle v \rangle^7\|_{H_{x,v}^2} \|\langle v \rangle \mathbf{P} f\|_{Y_k} \|f\|_{Y_k} \lesssim \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t).$$

Here we let  $k \geq k_0 \geq 10$  for the Landau case. For  $J_9$ , applying Lemma 2.18, we have for any  $\varepsilon > 0$  that for Boltzmann case:

$$\begin{aligned} |J_9| &\leq \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_{\gamma/2,*}^2} \|\partial^\alpha f w(\alpha, 0)\|_{L_x^2 L_{\gamma/2,*}^2} \\ &\quad + C_k \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_{\gamma/2-1/2}^2} \|\partial^\alpha f w(\alpha, 0)\|_{L_x^2 L_{\gamma/2-1/2}^2} \\ &\leq C_\varepsilon \|b(\cos \theta) \sin^{k-2} \frac{\theta}{2}\|_{L_\theta^1} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_{\gamma/2,*}^2}^2 + \varepsilon \|\partial^\alpha \mathbf{P} f\|_{L_x^2 L_v^2}^2 \\ &\quad + \varepsilon \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_{\gamma/2}^2}^2 + C_{k,\varepsilon} \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2, \end{aligned}$$

and for Landau case:

$$\begin{aligned} |J_9| &\leq C_k \sum_{|\alpha|+|\beta|\leq 2} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_5^2} \|\partial^\beta f\|_{L_x^2 L_5^2} \\ &\leq \varepsilon \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_D^2}^2 + \varepsilon \|\partial^\alpha \mathbf{P} f\|_{L_{x,v}^2}^2 + C_\varepsilon \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2. \end{aligned}$$

Taking summation  $\sum_{\pm} \sum_{|\alpha|\leq 2}$  (4.31), combining the above estimates for  $J_1$  to  $J_9$ , applying (3.12) and (4.29), we deduce for small  $\varepsilon > 0$ ,

$$\begin{aligned} \partial_t \sum_{|\alpha|\leq 2} \left( \frac{1}{2} \|e^{\frac{\pm A_{\alpha,0}\phi}{2\langle v \rangle^2}} \partial^\alpha f_{\pm}\|_{L_{x,v}^2}^2 + C_\alpha \|\nabla_x \phi\|_{H_x^2}^2 \right) + \frac{\gamma_1}{3} \sum_{|\alpha|\leq 2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 H_{\gamma/2}^s}^2 \\ \lesssim \|f\|_{X_{k_0}} \|f\|_{X_k}^2 + C_{\varepsilon,k} \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2 + \varepsilon \|\nabla_x \phi\|_{H_x^2}^2 + \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 \end{aligned}$$

$$+ C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t))\mathcal{D}_k(t). \quad (4.32)$$

Here we let  $k \geq k_0$  (and let  $k$  sufficiently large for Boltzmann case).

Taking linear combination  $\varepsilon \times (4.30) + (4.32)$  and applying (3.14), we obtain

$$\begin{aligned} \partial_t \mathcal{E}_1(t) + \lambda \varepsilon \|f\|_{Y_k}^2 + \frac{\gamma_1}{4} \sum_{|\alpha| \leq 2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2 \\ \lesssim \|f\|_{X_{k_0}} \|f\|_{X_k}^2 + \varepsilon \|f\|_{H_x^2 L_v^2}^2 + C_{\varepsilon, k} \|\{\mathbf{I} - \mathbf{P}\} f\|_{H_x^2 L_v^2}^2 + \varepsilon \|\nabla_x \phi\|_{H_x^2}^2 + \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 \\ + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t))\mathcal{D}_k(t). \end{aligned} \quad (4.33)$$

where  $\mathcal{E}_1(t)$  is given by

$$\begin{aligned} \mathcal{E}_1(t) = \frac{\varepsilon}{2} \|f\|_{X_k}^2 + \varepsilon \sum_{|\alpha| \leq 2} C_\alpha C_{|\alpha|, 0} \|\partial^\alpha \nabla_x \phi\|_{L_x^2}^2 \\ + \sum_{|\alpha| \leq 2} \left( \frac{1}{2} \|e^{\frac{\pm A_{\alpha, 0} \phi}{2(v)^2}} \partial^\alpha f_\pm\|_{L_{x,v}^2}^2 + C_\alpha \partial_t \|\nabla_x \phi\|_{H_x^2}^2 \right). \end{aligned} \quad (4.34)$$

It's direct to verify that  $\mathcal{E}_1(t)$  satisfies (4.26). For the first right-hand term of (4.33), we split it as

$$\varepsilon \|f\|_{H_x^2 L_v^2}^2 \lesssim \varepsilon \|\mathbf{P} f\|_{H_x^2 L_v^2}^2 + \varepsilon \sum_{|\alpha| \leq 2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2. \quad (4.35)$$

Choosing  $\varepsilon$  sufficiently small, the second right hand term of (4.35) can be absorbed by the left hand side of (4.33) and we obtain (4.27). This completes the proof of Lemma 4.3.  $\square$

According to Lemma 4.3, we only need to deal with term  $\|f\|_{H_x^2 L_v^2}$  without velocity derivative on the right hand side of (4.30). Before going to the future steps, we introduce a norm which is very important in our proof. Recall  $\mathbf{P}$  is defined in (1.8) and  $\mathcal{L}$  is given in (1.5). We define

$$\|f\| := \int_0^{+\infty} \|S_{\mathcal{L}}(\tau) \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2}^2 d\tau, \quad (4.36)$$

where the associate inner product is given by

$$((f, g)) := \int_0^{+\infty} (S_{\mathcal{L}}(\tau) \{\mathbf{I} - \mathbf{P}\} f, S_{\mathcal{L}}(\tau) \{\mathbf{I} - \mathbf{P}\} g)_{L_x^2 L_v^2} d\tau,$$

with  $\eta > 0$  to be determined later. Note that  $\{\mathbf{I} - \mathbf{P}\} f = \{\mathbf{I} - \mathbf{\Pi}\} \{\mathbf{I} - \mathbf{P}\} f$ , then we can apply the following works. By [8, Corollary 3.3], when  $k \geq 14$  we have

$$\|S_{\mathcal{L}}(t) \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2} \leq C e^{-\lambda t} \|\{\mathbf{I} - \mathbf{P}\} f\|_{L_k^2},$$

for  $\gamma \geq 0$  and

$$\|S_{\mathcal{L}}(t) \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2} \leq C \langle t \rangle^{-\frac{k-10}{|\gamma|}} \|\{\mathbf{I} - \mathbf{P}\} f\|_{L_k^2},$$

for  $-3 < \gamma < 0, \gamma + 2s > -1$ . Thus, for both cases, we have

$$\|S_{\mathcal{L}}(t) \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2} \leq \theta(t) \|\{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_k^2}, \quad \int_0^\infty \theta^2(s) ds < C_k, \quad \lim_{t \rightarrow \infty} \theta(t) = 0, \quad (4.37)$$

and by [8] Lemma 5.2 we have

$$\|S_{\mathcal{L}}(t) \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 L_v^2} \leq \theta_1(t) \|\{\mathbf{I} - \mathbf{P}\} f\|_{L_x^2 H_{14}^{-s}}, \quad \int_0^\infty \theta(s) \theta_1(s) ds < C_k, \quad \theta_1(t) = t^{-1/2} \theta(t), \quad (4.38)$$

It follows that

$$\|f\| \lesssim \|f\|_{L_x^2 L_{14}^2}, \quad (4.39)$$

for the Boltzmann case. For the Landau case, when  $k > 10$ , similarly by [10] Theorem 3.5, we have that if  $\gamma \geq 0$ , then

$$\|S_{\mathcal{L}}(t)\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2} \leq C e^{-\lambda t} \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_{10}^2},$$

and if  $-3 \leq \gamma < 0$ , then

$$\|S_{\mathcal{L}}(t)\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2} \leq C \langle t \rangle^{-\frac{k-5}{|\gamma|}} \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_{10}^2},$$

Thus, for both cases with  $k > 10$ , we have

$$\|S_{\mathcal{L}}(t)\{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_v^2} \leq \theta(t) \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_{10}^2}, \quad \int_0^\infty \theta^2(s) ds < C, \quad \lim_{t \rightarrow \infty} \theta(t) = 0, \quad (4.40)$$

and by [10] Theorem 3.5

$$\|S_{\mathcal{L}}(t)\{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_v^2} \leq \theta_1(t) \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 H_7^{-1}}, \quad \int_0^\infty \theta(s) \theta_1(s) ds < C_k, \quad \theta_1(t) = t^{-1/2} \theta(t), \quad (4.41)$$

It follows that

$$\|f\| \lesssim \|f\|_{L_x^2 L_{10}^2}, \quad (4.42)$$

for the Landau case.

The following lemma plays a key role to the proof of our main result.

**Lemma 4.4.** *For both the Boltzmann and Landau case, suppose  $(f, \phi)$  is the solution to the equation (1.4). Then we have*

$$\frac{1}{2} \partial_t \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|^2 + \|\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^2 L_v^2}^2 \lesssim \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t), \quad (4.43)$$

where  $\|\cdot\|$  is given in (4.36),  $\mathcal{E}_k(t)$  and  $\mathcal{D}_k(t)$  is given in (1.15) and (1.16).

*Proof.* Applying  $\partial^\alpha$  on both sides of (1.4), we have

$$\begin{aligned} \partial_t \partial^\alpha f_\pm + \underbrace{v \cdot \nabla_x \partial^\alpha f_\pm}_{\text{Term 1}} \pm \underbrace{\nabla_x \partial^\alpha \phi \cdot v \mu}_{\text{Term 2}} = \underbrace{\pm \nabla_x \phi \cdot \partial^\alpha \nabla_v f}_{\text{Term 3}} \pm \underbrace{\sum_{\alpha_1 < \alpha} C_{\alpha_1}^{\alpha_1} \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \partial^{\alpha_1} \nabla_v f}_{\text{Term 4}} \\ + \underbrace{\partial^\alpha L_\pm f}_{\text{Term 5}} + \underbrace{\partial^\alpha Q(f_\pm + f_\mp, f_\pm)}_{\text{Term 6}}, \end{aligned} \quad (4.44)$$

where  $L_\pm$  is given in (4.3). Then we take the  $((\cdot, \cdot))$  inner product of (4.44) with  $\partial^\alpha f$  compute the every term separately. Since  $\partial_t$  and  $\nabla_x$  commute with  $L$ , we know that  $\partial_t$  commute with  $S_{\mathcal{L}}(t)$  and hence,

$$((\partial_t \partial^\alpha f, \partial^\alpha f)) = \frac{1}{2} \partial_t \|\partial^\alpha f\|^2.$$

Next we compute the Term 2. By the definition of  $\mathbf{P}$ , we have

$$\{\mathbf{I}_\pm - \mathbf{P}_\pm\}(v_i \mu) = 0, \quad i = 1, 2, 3,$$

It follows from (4.36) that

$$((\nabla_x \partial^\alpha \phi \cdot v \mu, \partial^\alpha f)) = \int_0^{+\infty} (S_{\mathcal{L}}(\tau) \{\mathbf{I} - \mathbf{P}\} \nabla_x \partial^\alpha \phi \cdot v \mu, S_{\mathcal{L}}(\tau) \{\mathbf{I} - \mathbf{P}\} \partial^\alpha f)_{L_x^2 L_v^2} d\tau = 0.$$

For Term 3, since  $s \geq \frac{1}{2}$  implies  $1 - s \leq s$ , by Lemma 2.10 and (4.38), we have

$$\begin{aligned}
& \int_0^{+\infty} (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\nabla_x \phi \cdot \partial^\alpha \nabla_v f, S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& \lesssim \|\nabla_x \phi \cdot \partial^\alpha \nabla_v \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 H_{10}^{-s}} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_{12}^2} \int_0^\infty \theta_1(\tau)\theta(\tau) d\tau \\
& \lesssim \|\nabla_x \phi\|_{L_x^\infty} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 H_{10}^{1-s}} \|\partial^\alpha f\|_{L_x^2 L_{\gamma/2+k}^2} \\
& \lesssim \|\nabla_x \phi\|_{H_x^2} \|\partial^\alpha f\|_{L_x^2 H_{k+\gamma/2}^s}.
\end{aligned}$$

for the non-cutoff Boltzmann case. For the Landau case, we have from (4.40) that for  $k \geq 10$ ,

$$\begin{aligned}
& \int_0^{+\infty} (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\nabla_x \phi \cdot \partial^\alpha \nabla_v f, S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& \lesssim \|\nabla_x \phi \cdot \partial^\alpha \nabla_v \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_{10}^2} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_{12}^2} \int_0^\infty \theta^2(\tau) d\tau \\
& \lesssim \|\nabla_x \phi\|_{L_x^\infty} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 H_{10}^1} \|\partial^\alpha f\|_{L_x^2 L_{10}^2} \\
& \lesssim \|\nabla_x \phi\|_{H_x^2} \|\partial^\alpha f\|_{L_x^2 L_{D,k}^2}.
\end{aligned}$$

Then we compute the Term 4. For both Boltzmann and Landau case, we have

$$\begin{aligned}
& \int_0^{+\infty} (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\nabla_x \partial^{\alpha-\alpha_1} \phi \cdot \nabla_v \partial^{\alpha_1} f, S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& \lesssim \left( \sum_{|\alpha_1|=1} \|\nabla_x \partial^{\alpha-\alpha_1} \phi\|_{L_x^\infty} \|\partial^{\alpha_1} \nabla_v \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_{k_0}^2} \right. \\
& \quad \left. + \sum_{|\alpha_1|=0} \|\nabla_x \partial^{\alpha-\alpha_1} \phi\|_{L_x^3} \|\partial^{\alpha_1} \nabla_v \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^6 L_{k_0}^2} \right) \|f\|_{Y_k} \\
& \lesssim \|\nabla_x \phi\|_{H_x^3} \|f\|_{Y_k}^2.
\end{aligned}$$

Next we compute the Term 5. Notice that  $\partial_t S_{\mathcal{L}}(t)f = S_{\mathcal{L}}(t)\mathcal{L}f$ , by using the properties of semigroup. Thus, for the linear part, for both Boltzmann and Landau case, since  $\lim_{t \rightarrow \infty} \theta(t) = 0$  in (4.37), we have

$$\begin{aligned}
& \int_0^\infty (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\mathcal{L}\partial^\alpha f, S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& = \int_0^\infty (S_{\mathcal{L}}(\tau)L\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f, S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& = \int_0^\infty \frac{d}{d\tau} \|S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_x^2 L_v^2}^2 d\tau \\
& = -\|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_x^2 L_v^2}^2.
\end{aligned}$$

Finally, we consider the nonlinear part, i.e. Term 6. For the Boltzmann case, by (2.14), (4.37) and (4.38), we have

$$\begin{aligned}
& \int_0^\infty (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha Q(f_\pm + f_\mp, f_\pm), S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& \leq C_k \sum_{\alpha_1 \leq \alpha} \|\{\mathbf{I} - \mathbf{P}\}Q(\partial^{\alpha_1}(f_\pm + f_\mp), \partial^{\alpha-\alpha_1} f_\pm)\|_{L_x^2 H_{14}^{-s}} \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_x^2 L_{14}^2} \int_0^\infty \theta_1(\tau)\theta(\tau) d\tau \\
& \leq C_k (\|f\|_{H_x^2 L_{14}^2} \|f\|_{H_x^2 H_{17}^s} + \|f\|_{H_x^2 L_{14}^2} \|f\|_{H_x^2 H_{14}^s}) \|\partial^\alpha f\|_{L_x^2 L_{14}^2} \\
& \leq C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t).
\end{aligned}$$

Here we let  $k \geq 17$  and apply similar discussion on  $\alpha_1$  as in (2.50). For the Landau case, by (2.18), (4.40) and (4.41), we have

$$\begin{aligned}
& \int_0^\infty (S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha Q(f_\pm + f_\mp, f_\pm), S_{\mathcal{L}}(\tau)\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f)_{L_x^2 L_v^2} d\tau \\
& \leq C_k \sum_{\alpha_1 \leq \alpha} \|\{\mathbf{I} - \mathbf{P}\}Q(\partial^{\alpha_1} f_\pm + f_\mp, \partial^{\alpha - \alpha_1} f_\pm)\|_{L_x^2 H_7^{-1}} \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_x^2 L_{10}^2} \int_0^\infty \theta_1(\tau)\theta(\tau) d\tau \\
& \leq C_k \|f\|_{H_x^2 L_7^2} \|f\|_{H_x^2 L_{D,10}^2} \|\partial^\alpha f\|_{L_x^2 L_{10}^2} \\
& \leq C_k \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t),
\end{aligned}$$

where we also apply similar discussion on  $\alpha_1$  as in (2.50). Combining the above estimates, the  $((\cdot, \cdot))$  inner product of (4.44) with  $\partial^\alpha f$  yields

$$\frac{1}{2} \partial_t \|\partial^\alpha f\|^2 + \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_x^2 L_v^2}^2 \lesssim \sqrt{\mathcal{E}_{k_0}(t)} \mathcal{D}_k(t).$$

Taking summation over  $|\alpha| \leq 2$ , we conclude Lemma 4.4.  $\square$

**4.3. Proof of the main theorem.** We start this subsection by proving the main stability theorem below. Theorem 4.5 and 4.6 together implies Theorem 1.1. To prove Theorem 1.1, we assume the *a priori* assumption as

$$\sup_{0 \leq t \leq T} \mathcal{E}_{k_0}(t) \leq 2M, \tag{4.45}$$

for the case  $\gamma \in [0, 1]$  and

$$\sup_{0 \leq t \leq T} (1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_{k_0}(t) \leq 2M, \tag{4.46}$$

for the case  $\gamma \in [-3, 0)$ , where  $l > \frac{|\gamma|}{2}$ , we further assume  $k \geq k_0 + l$  in this case.

**Theorem 4.5.** *Let  $k \geq k_0$ . Assume that  $f_0$  satisfies the conservation laws (1.6) and  $F_0 = \mu + f_0 \geq 0$ . There exists a small constant  $M \geq 0$  such that if*

$$\mathcal{E}_{k_0}(0) \leq M, \quad \mathcal{E}_k(0) < \infty,$$

*then there exist a unique global solution  $f(t, x, v)$  to the Vlasov-Poisson-Boltzmann/Landau system (1.4) with  $F = \mu + f \geq 0$  satisfying*

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \leq \|f\|_{X_{k_0}} \mathcal{E}_k(t), \tag{4.47}$$

*for any  $T > 0$ , for some generic constant  $\lambda > 0$ .*

*Proof.* We take linear combination  $\eta \times (4.5) + (4.27) + C_0 \times (4.43)$  with  $\eta, C_0 > 0$  to deduce that

$$\begin{aligned}
& \partial_t \mathcal{E}_k(t) + \lambda \eta \|[a_+, a_-, b, c]\|_{H_x^2}^2 + \lambda \eta \|\nabla_x \phi\|_{H_x^3}^2 + \lambda \varepsilon \|f\|_{Y_k}^2 \\
& + C_0 \|\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^2 L_v^2}^2 + \lambda \sum_{|\alpha| \leq 2} \|w(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_D^2}^2 \\
& \lesssim \|f\|_{X_{k_0}} \|f\|_{X_k}^2 + \eta \|\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^2 L_{10}^2}^2 + \eta \|N\|_{H_x^1 L_v^2}^2 \\
& + C_{\varepsilon, k} \|\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^2 L_v^2}^2 + \varepsilon \|\nabla_x \phi\|_{H_x^2}^2 + \varepsilon \|\mathbf{P}f\|_{H_x^2 L_v^2}^2 + C(\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t),
\end{aligned}$$

for some generic constant  $\lambda > 0$ , where  $\mathcal{E}_k(t)$  is given by

$$\mathcal{E}_k(t) := \eta G(t) + \mathcal{E}_1(t) + \frac{C_0}{2} \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|^2 \tag{4.48}$$

Here  $G(t)$ ,  $\mathcal{E}_1(t)$  and  $\|\cdot\|$  are given in Lemma 4.2, Lemma 4.3 and (4.36) respectively. Choosing  $\eta > 0$  sufficiently small, and then  $\varepsilon > 0$  sufficiently small, and finally  $C_0 > 0$  sufficiently large, we have

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t) + \eta \|N_{\parallel}\|_{H_x^1 L_v^2}^2 + (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t), \quad (4.49)$$

for some generic constant  $\lambda > 0$ , where  $\mathcal{D}_k(t)$  is given by (1.16). Then one can check (1.15) by using (4.4), (4.34), (4.39) and (4.42) with sufficiently small  $\eta$ . Note that  $e^{\frac{\pm A_{\alpha,0}\phi}{2(v)^2}} \approx 1$ .

Next, recall from Lemma 4.2 that

$$\begin{aligned} N_{\parallel} &= (\pm \nabla_x \phi \cdot \nabla_v f_{\pm})_{\parallel} + Q(f_{\pm} + f_{\mp}, f_{\pm})_{\parallel} \\ &= (\pm \nabla_x \phi \cdot \nabla_v f_{\pm}, \xi) + (Q(f_{\pm} + f_{\mp}, f_{\pm}), \xi), \end{aligned}$$

where  $\xi$  is some linear combination of  $1, v_i, v_i v_j, v_i |v|^2$ . For  $|\alpha| \leq 1$ , we have

$$\begin{aligned} \|\partial^{\alpha}(\nabla_x \phi \cdot \nabla_v f, \xi)_{L_v^2}\|_{L_x^2} &\lesssim \sum_{|\alpha_1|=0} \|\partial^{\alpha_1} \nabla_x \phi\|_{L_x^{\infty}} \|\partial^{\alpha-\alpha_1} \nabla_v f\|_{L_x^2 L_v^2} \\ &\quad + \sum_{|\alpha_1|=1} \|\partial^{\alpha_1} \nabla_x \phi\|_{L_x^3} \|\partial^{\alpha-\alpha_1} \nabla_v f\|_{L_x^6 L_v^2} \\ &\lesssim \|\nabla_x \phi\|_{H_x^2} \|f\|_{H_x^1 H_v^1} \lesssim \sqrt{\mathcal{E}_{k_0}(t) \mathcal{D}_k(t)}. \end{aligned} \quad (4.50)$$

For the Boltzmann case, by Lemma 2.4, we apply similar discussion on  $\alpha_1$  as in (4.50) to deduce that

$$\begin{aligned} \|\partial^{\alpha}(Q(f_{\pm} + f_{\mp}, f_{\pm}), \xi)\|_{L_{x,v}^2}^2 &\lesssim \|f\|_{H_x^2 L_{10}^2} \|f\|_{H_x^2 L_{10}^2} \|\xi\|_{H_{-7}^{2s}} \\ &\lesssim \sqrt{\mathcal{E}_{k_0}(t) \mathcal{D}_k(t)}. \end{aligned}$$

Therefore, we obtain

$$\sum_{|\alpha| \leq 1} \|\partial^{\alpha} N_{\parallel}\|_{L_x^2}^2 \lesssim \mathcal{E}_{k_0}(t) \mathcal{D}_k(t).$$

Then (4.49) implies

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t) + (\sqrt{\mathcal{E}_{k_0}(t)} + \mathcal{E}_{k_0}(t)) \mathcal{D}_k(t). \quad (4.51)$$

Choosing  $M$  in the *a priori* assumption (4.45) and (4.46) sufficiently small, we have from (4.51) that

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{D}_k(t) \lesssim \|f\|_{X_{k_0}} \mathcal{E}_k(t),$$

for some  $\lambda > 0$ . This concludes Theorem 4.5.  $\square$

To conclude Theorem 1.1, we need to prove the large-time behavior as the following.

**Theorem 4.6.** *Let  $l = 0$  for hard potential case and  $l > \frac{|\gamma|}{2}$  for soft potential case. Let  $k \geq k_0 + l$  (and let  $k$  sufficiently large for Boltzmann case). If the solution  $(f, \phi)$  of (1.4) satisfies*

$$\mathcal{E}_{k_0+l}(0) \leq M, \quad \mathcal{E}_k(0) < \infty.$$

*Then there exists a constant  $\lambda > 0$ , such that for any  $t > 0$ , we have*

$$\sup_{0 \leq t \leq T} \mathcal{E}_k(t) \leq 2\mathcal{E}(0). \quad (4.52)$$

*Moreover, if  $0 \leq \gamma \leq 1$ , then*

$$\mathcal{E}_k(t) \leq e^{-\lambda t} \mathcal{E}_k(0).$$

*If  $\gamma < 0$ , then we have*

$$\mathcal{E}_{k-l}(t) \lesssim (1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(0). \quad (4.53)$$

*Proof.* If  $0 \leq \gamma \leq 1$ , noticing that  $\|\cdot\|_{L^2_{x,v}} \lesssim \|\cdot\|_{L^2_D}$  and choosing  $M$  in (4.45) small enough, we have  $\|f\|_{X_{k_0}} \lesssim \mathcal{E}_{k_0} \lesssim M$ , and hence, by (4.47) we have

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{E}_k(t) \leq 0.$$

Solving this ODE, we obtain  $\mathcal{E}_k(t) \leq e^{-\lambda t} \mathcal{E}_k(0)$ . This close the *a priori* assumption (4.45) and conclude the case of hard potential.

Next we assume  $\gamma < 0$ . For any  $l > \frac{|\gamma|}{2}$ , we have from (4.46) that

$$\|f\|_{X_{k_0}} \lesssim \mathcal{E}_{k_0}(t) \lesssim M(1+t)^{-\frac{2l}{|\gamma|}}. \quad (4.54)$$

Let  $p = \frac{-\gamma+2l}{2l}$  and  $p' = \frac{-\gamma+2l}{-\gamma}$ . Then by  $L^p - L^{p'}$  Hölder's inequality, we have

$$\begin{aligned} \|w(\alpha, \beta) \partial_\beta^\alpha f\|_{L^2_v}^2 &= \int_{\mathbb{R}^3} \langle v \rangle^{\frac{2l\gamma}{-\gamma+2l}} \langle v \rangle^{-\frac{2l\gamma}{-\gamma+2l}} |w(\alpha, \beta) \partial_\beta^\alpha f|^2 dv \\ &\leq \|\langle v \rangle^{\gamma/2} w(\alpha, \beta) \partial_\beta^\alpha f\|_{L^2_v}^{\frac{4l}{-\gamma+2l}} \|\langle v \rangle^l w(\alpha, \beta) \partial_\beta^\alpha f\|_{L^2_v}^{\frac{-2\gamma}{-\gamma+2l}} \\ &\leq \mathcal{D}_k^{\frac{2l}{-\gamma+2l}} \mathcal{E}_{k+l}^{\frac{-\gamma}{-\gamma+2l}}. \end{aligned}$$

From definition (1.15) and (1.13), we know that  $\mathcal{E}_k^{\frac{-\gamma+2l}{2l}} \mathcal{E}_{k+l}^{\frac{\gamma}{2l}} \leq \mathcal{D}_k$ . Then it follows from (4.47) and (4.54) that

$$\partial_t \mathcal{E}_k(t) + \lambda \mathcal{E}_k^{\frac{-\gamma+2l}{2l}}(t) \left( \sup_{0 \leq t \leq T} \mathcal{E}_{k+l} \right)^{\frac{\gamma}{2l}} \leq CM(1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(t). \quad (4.55)$$

Neglecting the second left hand term of (4.55), we have

$$\partial_t \mathcal{E}_k(t) \leq CM(1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k(t).$$

Taking integration over  $t \in [0, T]$ , we have

$$\sup_{0 \leq t \leq T} \mathcal{E}_k(t) \leq \mathcal{E}_k(0) + CM \int_0^T (1+t)^{-\frac{2l}{|\gamma|}} dt \sup_{0 \leq t \leq T} \mathcal{E}_k(t).$$

Since  $\frac{2l}{|\gamma|} > 1$ , choosing  $M > 0$  sufficiently small, we have

$$\sup_{0 \leq t \leq T} \mathcal{E}_k(t) \leq 2\mathcal{E}_k(0). \quad (4.56)$$

This gives (4.52). Next we solve (4.55) directly. It's direct to obtain that

$$\partial_t (\mathcal{E}_k^{\frac{\gamma}{2l}}(t)) = \frac{\gamma}{2l} \mathcal{E}_k^{\frac{\gamma-2l}{2l}}(t) \partial_t \mathcal{E}_k(t) \geq -\frac{\gamma\lambda}{2l} \left( \sup_{0 \leq t \leq T} \mathcal{E}_{k+l} \right)^{\frac{\gamma}{2l}} + \frac{CM\gamma}{2l} (1+t)^{-\frac{2l}{|\gamma|}} \mathcal{E}_k^{\frac{\gamma}{2l}}(t),$$

and thus,

$$\partial_t \left( \exp \left\{ \frac{-CM\gamma}{2l} (1+t)^{-\frac{2l}{|\gamma|}} \right\} \mathcal{E}_k^{\frac{\gamma}{2l}}(t) \right) \geq -\frac{\gamma\lambda}{2l} \exp \left\{ \frac{-CM\gamma}{2l} (1+t)^{-\frac{2l}{|\gamma|}} \right\} \left( \sup_{0 \leq t \leq T} \mathcal{E}_{k+l} \right)^{\frac{\gamma}{2l}}.$$

Taking integration over  $t \in [0, T]$ , we have

$$\exp \left\{ \frac{-CM\gamma}{2l} (1+t)^{-\frac{2l}{|\gamma|}} \right\} \mathcal{E}_k^{\frac{\gamma}{2l}}(t) \geq C_{\gamma,l} \mathcal{E}_k^{\frac{\gamma}{2l}}(0) + C_{\gamma,l} \int_0^T 1 dt \left( \sup_{0 \leq t \leq T} \mathcal{E}_{k+l} \right)^{\frac{\gamma}{2l}},$$

and hence,

$$\mathcal{E}_k(t) \leq C_{\gamma,l} (1+t)^{\frac{2l}{\gamma}} \sup_{0 \leq t \leq T} \mathcal{E}_{k+l}(t).$$

Replacing  $k$  by  $k-l \geq k_0$ , we have from (4.56) that

$$\mathcal{E}_{k-l}(t) \lesssim (1+t)^{\frac{2l}{\gamma}} \sup_{0 \leq t \leq T} \mathcal{E}_k(t) \lesssim (1+t)^{\frac{2l}{\gamma}} \mathcal{E}_k(0).$$

Choosing  $k - l = k_0$  and applying (1.18), we close the *a priori* assumption (4.46). Using the *a priori* arguments, we obtain (4.53). This completes the proof of Theorem 4.6. □

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