

A proof of the Multiplicative 1-2-3 Conjecture^{*}

Julien Bensmail^{1**}, Hervé Hocquard², Dimitri Lajou², and Éric Sopena²

¹ Université Côte d'Azur, CNRS, Inria, I3S, France

² Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400, Talence, France

Abstract. We prove that the product version of the 1-2-3 Conjecture, raised by Skowronek-Kaziów in 2012, is true. Namely, for every connected graph with order at least 3, we can assign labels 1, 2, 3 to the edges so that no two adjacent vertices are incident to the same product of labels.

Keywords: 1-2-3 Conjecture · product version · labels 1, 2, 3

1 Introduction

Let G be a graph. A k -labelling $\ell : E(G) \rightarrow \{1, \dots, k\}$ is an assignment of labels $1, \dots, k$ to the edges of G . From ℓ , we can compute different parameters of interest for all vertices v , such as the *sum* $\sigma_\ell(v)$ of incident labels (being formally $\sigma_\ell(v) = \sum_{u \in N(v)} \ell(uv)$), or similarly the *multiset* $\mu_\ell(v)$ of labels incident to v or the *product* $\rho_\ell(v)$ of labels incident to v . We say that ℓ is *s-proper* if σ_ℓ is a proper vertex-colouring of G , *i.e.*, we have $\sigma_\ell(u) \neq \sigma_\ell(v)$ for every edge $uv \in E(G)$. Similarly, we say that ℓ is *m-proper* and *p-proper*, if μ_ℓ and ρ_ℓ , respectively, form proper vertex-colourings of G .

In the context of so-called *distinguishing labellings*, the goal is generally to not only distinguish vertices within some distance according to some parameter computed from labellings (such as the parameters σ_ℓ , μ_ℓ and ρ_ℓ above, to name a few), but also to construct such k -labellings with k as small as possible. We refer the interested reader to [4], which lists hundreds of labelling techniques.

Regarding s-proper, m-proper and p-proper labellings, which are the main focus in this work, we are thus interested, as mentioned above, in finding such k -labellings with k as small as possible, for a given graph G . In other words, we are interested in the parameters $\chi_S(G)$, $\chi_M(G)$ and $\chi_P(G)$ which denote the smallest $k \geq 1$ such that s-proper, m-proper and p-proper, respectively, k -labellings exist (if any). Actually, through greedy labelling arguments, it can be observed that the only connected graph G for which $\chi_S(G)$, $\chi_M(G)$ or $\chi_P(G)$ is not defined, is K_2 , the complete graph on 2 vertices. Consequently, these three parameters are generally investigated for so-called *nice graphs*, which are those graphs with no connected component isomorphic to K_2 .

^{*} Some proofs in this paper are voluntarily omitted due to space limitation; the interested reader will find them in [3], the full version of the current paper.

^{**} Corresponding author. Email address: julien.bensmail.phd@gmail.com.

S-proper, m-proper and p-proper labellings form a subfield of distinguishing labellings, which has been attracting attention due to the so-called 1-2-3 Conjecture, raised, in [6], by Karoński, Łuczak and Thomason in 2004:

1-2-3 Conjecture (sum version). *If G is a nice graph, then $\chi_S(G) \leq 3$.*

Later on, counterparts of the 1-2-3 Conjecture were raised for m-proper and p-proper labellings. Addario-Berry *et al.* first raised, in 2005, the following in [1]:

1-2-3 Conjecture (multiset version). *If G is a nice graph, then $\chi_M(G) \leq 3$.*

while Skowronek-Kaziów then raised, in 2012, the following in [8]:

1-2-3 Conjecture (product version). *If G is a nice graph, then $\chi_P(G) \leq 3$.*

It is worth mentioning that all three conjectures above, if true, would be tight, as attested for instance by complete graphs. Note also that the multiset version of the 1-2-3 Conjecture is, out of the three variants, the easiest one in a sense, as every s-proper or p-proper labelling is also m-proper (thus, proving the sum or product variant of the 1-2-3 Conjecture would prove the multiset one).

To date, the best result towards the sum version of the 1-2-3 Conjecture, proved by Kalkowski, Karoński and Pfender in [5], is that $\chi_S(G) \leq 5$ holds for every nice graph G . Another significant result is due to Przybyło, who recently proved in [7] that even $\chi_S(G) \leq 4$ holds for every nice regular graph G . Karoński, Łuczak and Thomason themselves also proved in [6] that $\chi_S(G) \leq 3$ holds for nice 3-colourable graphs. Regarding the multiset version, for long the best result was the one proved by Addario-Berry, Aldred, Dalal and Reed in [1], stating that $\chi_M(G) \leq 4$ holds for every nice graph G . Building on that result, Skowronek-Kaziów later proved in [8] that $\chi_P(G) \leq 4$ holds for every nice graph G . She also proved that $\chi_P(G) \leq 3$ holds for every nice 3-colourable graph G .

A breakthrough result was recently obtained by Vučković, as he totally proved the multiset version of the 1-2-3 Conjecture in [9]. Due to connections between m-proper and p-proper 3-labellings, we observed in [2] that this result directly implies that $\chi_P(G) \leq 3$ holds for every nice regular graph G . Inspired by Vučković's proof scheme, we were also able to prove that $\chi_P(G) \leq 3$ holds for nice 4-chromatic graphs G , and to prove related results that are very close to what is stated in the product version of the 1-2-3 Conjecture.

Building on these results, we prove the following throughout this paper.

Theorem 1. *The product version of the 1-2-3 Conjecture is true. That is, every nice graph admits p-proper 3-labellings.*

2 Proof of Theorem 1

Let us start by introducing some terminology and recalling some properties of p-proper labellings, which will be used throughout the proof. Let G be a graph, and ℓ be a 3-labelling of G . For a vertex $v \in V(G)$ and a label $i \in \{1, 2, 3\}$, we

denote by $d_i(v)$ the i -degree of v by ℓ , being the number of edges incident to v that are assigned label i by ℓ . Note then that $\rho_\ell(v) = 2^{d_2(v)}3^{d_3(v)}$. We say that v is *1-monochromatic* if $d_2(v) = d_3(v) = 0$, while we say that v is *2-monochromatic* (*3-monochromatic*, resp.) if $d_2(v) > 0$ and $d_3(v) = 0$ ($d_3(v) > 0$ and $d_2(v) = 0$, resp.). In case v has both 2-degree and 3-degree at least 1, we say that v is *bichromatic*. We also define the $\{2, 3\}$ -degree of v as the sum $d_2(v) + d_3(v)$ of its 2-degree and 3-degree. If v is bichromatic, then its $\{2, 3\}$ -degree is at least 2.

Because ℓ assigns labels 1, 2, 3, and, in particular, because 2 and 3 are coprime, note that, for every edge uv of G , we have $\rho_\ell(u) \neq \rho_\ell(v)$ when u and v have different 2-degrees, 3-degrees, or $\{2, 3\}$ -degrees. In particular, u and v cannot be in conflict, *i.e.*, verify $\rho_\ell(u) = \rho_\ell(v)$, if u and v are i -monochromatic and j -monochromatic for $i \neq j$, or if u is monochromatic while v is bichromatic.

Before going into the proof of Theorem 1, let us start by giving an overview of it. Let G be a nice graph. Our goal is to build a p-proper 3-labelling ℓ of G . We can clearly assume that G is connected. We also set $t = \chi(G)$, where, recall, $\chi(G)$ refers to the chromatic number³ of G . In particular, $t \geq 2$.

In what follows, we construct ℓ through three main steps. First, we need to partition the vertices of G in a way verifying specific cut properties, forming what we call a valid partition of $V(G)$ (see later Definition 1 for a more formal definition). In short, a *valid partition* $\mathcal{V} = (V_1, \dots, V_t)$ is a partition of $V(G)$ into t independent sets V_1, \dots, V_t fulfilling two main properties, being, roughly put, that 1) every vertex v in some part V_i with $i > 1$ has an incident *upward edge* to every part V_j with $j < i$, and 2) for every connected component of $G[V_1 \cup V_2]$ having only one edge, we can freely swap its two vertices in V_1 and V_2 while preserving the main properties of a valid partition.

Once we have this valid partition \mathcal{V} in hand, we can then start constructing ℓ . The main part of the labelling process, Step 2 below, consists in starting from all edges of G being assigned label 1 by ℓ , and then processing the vertices of V_3, \dots, V_t one after another, possibly changing the labels by ℓ assigned to some of their incident edges, so that certain product types are achieved by ρ_ℓ . These desired product types can be achieved due to the many upward edges that some vertices are incident to (in particular, the deeper a vertex lies in \mathcal{V} , the more upward edges it is incident to). The product types we achieve for the vertices depend on the part V_i of \mathcal{V} they belong to. In particular, the modifications we make on ℓ guarantee that all vertices in V_3, \dots, V_t are bichromatic, every two vertices in V_i and V_j with $i, j \in \{3, \dots, t\}$ and $i \neq j$ have different 2-degrees or 3-degrees, all vertices in V_2 are 1-monochromatic or 2-monochromatic, and all vertices in V_1 are 1-monochromatic or 3-monochromatic. By itself, achieving these product types makes ℓ almost p-proper, in the sense that the only possible conflicts are between 1-monochromatic vertices in V_1 and V_2 . An important point also, is that, through these label modifications, we will make sure that all edges of $G[V_1 \cup V_2]$ remain assigned label 1, and no vertex in $V_3 \cup \dots \cup V_t$ has 3-degree 1,

³ Recall that a *proper k -vertex-colouring* of a graph G is a partition (V_1, \dots, V_k) of $V(G)$ where all V_i 's are independent. The *chromatic number* $\chi(G)$ of G is the smallest $k \geq 1$ such that proper k -vertex-colourings of G exist. G is *k -colourable* if $\chi(G) \leq k$.

2-degree at least 2, and odd $\{2, 3\}$ -degree; in last Step 3 below, we will use that last fact to remove remaining conflicts by allowing some vertices of $V_1 \cup V_2$ to become *special*, *i.e.*, make their product realising these exact label conditions.

Step 3 is designed to get rid of the last conflicts between the adjacent 1-monochromatic vertices of V_1 and V_2 without introducing new ones in G . To that end, we will consider the set \mathcal{H} of the connected components of $G[V_1 \cup V_2]$ having conflicting vertices, and, if needed, modify the labels assigned by ℓ to some of their incident edges so that no conflicts remain, and no new conflicts are created in G . To make sure that no new conflicts are created between vertices in $V_1 \cup V_2$ and vertices in $V_3 \cup \dots \cup V_t$, we will modify labels while making sure that all vertices in $V_1 \cup V_2$ are monochromatic or special. An important point also, is that the fixing procedures we introduce require the number of edges in a connected component of \mathcal{H} to be at least 2. Because of that, once Step 2 ends, we must ensure that \mathcal{H} does not contain a connected component with only one edge incident to two 1-monochromatic vertices. To guarantee this, we will also make sure, during Step 2, to modify labels and the partition \mathcal{V} slightly so that \mathcal{H} has no such configuration.

Step 1: Constructing a valid partition

Let $\mathcal{V} = (V_1, \dots, V_t)$ be a partition of $V(G)$ where each V_i is an independent set. Note that such a partition exists, as, for instance, any proper t -vertex-colouring of G forms such a partition of $V(G)$. For every vertex $u \in V_i$, an incident *upward edge* (*downward edge*, resp.) is an edge uv for which v belongs to some V_j with $j < i$ ($j > i$, resp.). Note that all vertices in V_1 have no incident upward edges, while all vertices in V_t have no incident downward edges.

We denote by $M_0(\mathcal{V})$ (also denoted M_0 when the context is clear) the set of isolated edges in the subgraph $G[V_1 \cup V_2]$ of G induced by the vertices of $V_1 \cup V_2$. That is, M_0 contains the edges of the connected components of $G[V_1 \cup V_2]$ that consist in one edge only. To lighten the exposition, whenever referring to the vertices of M_0 , we mean the vertices of G incident to the edges in M_0 .

For an edge $uv \in M_0$ with $u \in V_1$ and $v \in V_2$, *swapping* uv consists in modifying the partition \mathcal{V} by removing u from V_1 (v from V_2 , resp.) and adding it to V_2 (V_1 , resp.). In other words, we exchange the parts to which u and v belong. Note that if V_1 and V_2 are independent sets before the swap, then, because $uv \in M_0$, by definition the resulting new V_1 and V_2 remain independent. Also, the set M_0 is unchanged by the swap operation.

We can now give a formal definition for the notion of valid partition.

Definition 1 (Valid partition). *For a t -colourable graph G , a partition $\mathcal{V} = (V_1, \dots, V_t)$ of $V(G)$ is valid (for G) if \mathcal{V} verifies the following properties.*

- (\mathcal{I}) *Every V_i is an independent set.*
- (\mathcal{P}_1) *Every vertex in some V_i with $i \geq 2$ has a neighbour in V_j for every $j < i$.*
- (\mathcal{S}) *For every sequence $(e_i)_i$ of edges of $M_0(\mathcal{V})$, successively swapping every e_i (in any order) results in a partition \mathcal{V}' verifying Properties (\mathcal{I}) and (\mathcal{P}_1).*

Note that Property (S) implies the following property:

(P₂) Swapping any number of edges of $M_0(\mathcal{V})$ results in a valid partition \mathcal{V}' .

To prove Theorem 1, as mentioned earlier, to start constructing ℓ we need to have a valid partition of G in hand. The following result guarantees its existence.

Lemma 1. *Every nice t -colourable graph G admits a valid partition.*

Proof. For a partition $\mathcal{V} = (V_1, \dots, V_t)$ of $V(G)$ where each V_i is independent (such a partition exists, as attested by any proper t -vertex-colouring of G), set $f(\mathcal{V}) = \sum_{k=1}^t k \cdot |V_k|$. Among all possible \mathcal{V} 's, consider a \mathcal{V} that minimises $f(\mathcal{V})$.

Suppose that there is a vertex $u \in V_i$ with $i \geq 2$ for which Property (P₁) does not hold, *i.e.*, there is a $j < i$ such that u has no incident upward edge to V_j . By moving u to V_j , we obtain another partition \mathcal{V}' of $V(G)$ where every part is an independent set. However, note that $f(\mathcal{V}') = f(\mathcal{V}) + j - i < f(\mathcal{V})$, a contradiction to the minimality of \mathcal{V} . From this, we deduce that every partition \mathcal{V} minimising f must verify Property (P₁).

Let now \mathcal{V}' be the partition of $V(G)$ obtained by successively swapping edges of $M_0(\mathcal{V})$. Recall that the swapping operation preserves Property (I) and observe that $f(\mathcal{V}) = f(\mathcal{V}')$. Hence, \mathcal{V}' minimises f and thus verifies Properties (I) and (P₁). Thus Property (S) also holds, and \mathcal{V} is a valid partition of G . \square

From here, we assume that we have a valid partition $\mathcal{V} = (V_1, \dots, V_t)$ of G .

Step 2: Labelling the upward edges of V_3, \dots, V_t

From G and \mathcal{V} , our goal now is to construct a 3-labelling ℓ of G achieving certain properties, the most important of which being that the only possible conflicts are between pairs of vertices of V_1 and V_2 that do not form an edge of M_0 . The following result sums up the exact conditions we want ℓ to fulfil. Recall that a vertex v is special by ℓ , if $d_3(v) = 1$, $d_2(v) \geq 2$ and $d_2(v) + d_3(v)$ is odd. Note that special vertices are bichromatic.

Lemma 2. *For every nice graph G and every valid partition (V_1, \dots, V_t) of G , there exists a 3-labelling ℓ of G such that:*

1. all vertices of V_1 are either 1-monochromatic or 3-monochromatic,
2. all vertices of V_2 are either 1-monochromatic or 2-monochromatic,
3. all vertices of $V_3 \cup \dots \cup V_t$ are bichromatic,
4. no vertex is special,
5. if $u \in V_1$ and $v \in V_2$ are adjacent, then $\ell(uv) = 1$,
6. if two vertices u and v are in conflict, then $u \in V_1$ and $v \in V_2$ (or vice versa), and at least one of u or v has a neighbour w in $V_1 \cup V_2$.

Proof. From now on, we fix the valid partition $\mathcal{V} = (V_1, \dots, V_t)$ of G . During the construction of ℓ , we may have, however, to swap some edges of M_0 , resulting in a different valid partition of G . Abusing the notations, for simplicity

we will still denote by \mathcal{V} any valid partition of G obtained this way, through swapping edges. Recall that valid partitions are closed under swapping edges of M_0 (Property (\mathcal{P}_2) of Definition 1).

Our goal is to design ℓ so that it not only verifies the four colour properties of Items 1 to 4 of the statement, but also achieves the following refined product types, for every vertex v in a part V_i of \mathcal{V} :

- $v \in V_1$: v is 1-monochromatic or 3-monochromatic;
- $v \in V_2$: v is 1-monochromatic or 2-monochromatic;
- $v \in V_3$: v is bichromatic with 2-degree 1 and even $\{2, 3\}$ -degree;
- $v \in V_4$: v is bichromatic with 3-degree 2 and odd $\{2, 3\}$ -degree;
- $v \in V_5$: v is bichromatic with 2-degree 2 and even $\{2, 3\}$ -degree;
- ...
- $v \in V_{2n}$, $n \geq 3$: v is bichromatic with 3-degree n and odd $\{2, 3\}$ -degree;
- $v \in V_{2n+1}$, $n \geq 3$: v is bichromatic with 2-degree n and even $\{2, 3\}$ -degree;
- ...

We start from ℓ assigning label 1 to all edges of G . Let us now describe how to modify ℓ so that the conditions above are met for all vertices. We consider the vertices of V_t, \dots, V_3 following that order, from “bottom to top”, and modify labels assigned to upward edges. An important condition we will maintain, is that every vertex in an odd part V_{2n+1} ($n \geq 0$) has all its incident downward edges (if any) labelled 3 or 1, while every vertex in an even part V_{2n} ($n \geq 1$) has all its incident downward edges (if any) labelled 2 or 1. Note that this is trivially verified for the vertices in V_t , since they have no incident downward edges.

At any point in the process, let M be the set of edges of M_0 for which both ends are 1-monochromatic (initially, $M = M_0$). When treating a vertex $u \in V_3 \cup \dots \cup V_t$, we define M_u as the subset of edges of M having an end that is a neighbour of u . For every edge $e \in M_u$, we choose one end of e that is a neighbour of u and we add it to a set S_u . Note that $|S_u| = |M_u|$. Another goal during the labelling process, to fulfil Item 6, is to label the edges incident to u so that at least one end of every edge in M_u is no longer 1-monochromatic. Note that the set M_u considered when labelling the edges incident to u is not necessarily the set of edges of M_0 incident to a neighbour of u , as, during the whole process, some of these edges might be removed from M when dealing with previous vertices in $V_3 \cup \dots \cup V_t$.

Let us now consider the vertices in V_t, \dots, V_3 one by one, following that order. Let thus $u \in V_i$ be a vertex that has not been treated yet, with $i \geq 3$. Recall that every vertex belonging to some V_j with $j > i$ was treated earlier on, and thus has its desired product. Suppose that $i = 2n$ with $n \geq 2$ ($i = 2n + 1$ with $n \geq 1$, resp.). Recall also that u is assumed to have all its incident downward edges labelled 1 or 2 (3, resp.), due to how vertices in V_j 's with $j > i$ have been treated earlier on, and to have all its incident upward edges labelled 1.

If $M_u \neq \emptyset$, then we swap edges of M_u , if necessary, so that every vertex in S_u belongs to V_2 (V_1 , resp.). This does not invalidate any of our invariants since both ends of an edge in S_u are 1-monochromatic.

In any case, by Property (\mathcal{P}_1), we know that, for every $j < i$, there is a vertex $x_j \in V_j$ which is a neighbour of u . In particular, the vertex x_1 (x_2 , resp.) does not belong to S_u (but may be the other end of an edge in M_u). We label the edges $ux_3, ux_5, \dots, ux_{2n-1}$ with 3 ($ux_4, ux_6, \dots, ux_{2n}$ with 2, resp.). Note that, at this point, $d_3(u) = n - 1$ ($d_2(u) = n - 1$, resp.). To finish dealing with u , we need to distinguish two cases depending on whether M_u is empty or not.

- Suppose first that $M_u = \emptyset$. Label ux_1 with 3 (ux_2 with 2, resp.). Now u has the desired 3-degree (2-degree, resp.). If $i > 3$, then label ux_{i-2} with 2 (3, resp.) so that u is sure to be bichromatic. If $i > 3$ and the $\{2, 3\}$ -degree of u does not have the desired parity, then label ux_2 with 2 (ux_1 with 3, resp.). If $u \in V_3$ and the $\{2, 3\}$ -degree of u is even, then u is already bichromatic since $d_2(u) = 1$. If $u \in V_3$ and the $\{2, 3\}$ -degree of u is odd, then label ux_1 with 3 to adjust the parity of the $\{2, 3\}$ -degree of u and make u bichromatic. In all cases, u gets bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.), which is what is desired for u .
- Suppose now that $M_u \neq \emptyset$. Let $z \in S_u$ and let e be the edge of M_u containing z . For every $w \in S_u \setminus \{z\}$, we label the edge uw with 2 (3, resp.). Then:
 - If $d_2(u) + d_3(u)$ is odd (even, resp.), then label uz with 2 (3, resp.) and ux_1 with 3 (ux_2 with 2, resp.). In this case, every edge in M_u is incident to at least one vertex which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.).
 - If $d_2(u) + d_3(u)$ is even (odd, resp.) and $d_2(u) > 0$ ($d_3(u) > 0$, resp.), then swap e and label uz with 3 (2, resp.). Note that, after the swap of e , we have $z \in V_1$ ($z \in V_2$, resp.). In this case, every edge in M_u is incident to at least one vertex which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.).
 - The last case is when $d_2(u) + d_3(u)$ is even (odd, resp.) and $d_2(u) = 0$ ($d_3(u) = 0$, resp.). If $i > 4$, then we can label ux_{i-2} with 2 (3, resp.) and fall back into one of the previous cases. If $i = 4$, then the only edge labelled 3 is the edge ux_3 which implies that $d_3(u) = 1$, which is impossible since $d_2(u) = 0$ and $d_2(u) + d_3(u)$ is odd. If $i = 3$, then the conditions of this case imply that $d_2(u) = 1$ while every upward edge incident to u is labelled 1 or 3 and similarly for every incident downward edge; this case thus cannot occur.

To finish, we remove the edges of M_u from M since their two ends are not both 1-monochromatic any more.

At the end of this process, all vertices in V_1 are 1-monochromatic or 3-monochromatic, while all vertices in V_2 are 1-monochromatic or 2-monochromatic. Every vertex in $V_3 \cup \dots \cup V_i$ is bichromatic and there are no conflicts involving any pair of these vertices. Indeed if $a \in V_i$ and $b \in V_j$ are adjacent with $i > j \geq 3$, then either i and j do not have the same parity, in which case a and b do not have the same $\{2, 3\}$ -degree; or both i and j are even (odd, resp.) and

$d_3(a) = \frac{i}{2} \neq \frac{j}{2} = d_3(b)$ ($d_2(a) = \frac{i-1}{2} \neq \frac{j-1}{2} = d_2(b)$, resp.). Note also that no vertex in G is special, as special vertices have 3-degree 1, 2-degree at least 2, and odd $\{2, 3\}$ -degree. Also, we did not relabel any edge in the cut (V_1, V_2) .

Finally, suppose that there is a conflict between two vertices u and v . Previous remarks imply that $u \in V_1$ and $v \in V_2$ (or *vice versa*) and that both u and v are 1-monochromatic. If none of u and v has another neighbour w in $V_1 \cup V_2$, then the edge uv belongs to the set M_0 . Since G is nice, one of u or v must have a neighbour z in $V_3 \cup \dots \cup V_t$. Hence $uv \in M_z$. Recall also that we relabelled the edges incident to z in such a way that, for every edge of M_z , at least one incident vertex became 2-monochromatic or 3-monochromatic, a contradiction to the existence of u and v . Hence, all properties of the lemma hold. \square

Step 3: Labelling the edges between V_1 and V_2

From now on, we will modify a 3-labelling ℓ of G obtained by applying Lemma 2. We denote by \mathcal{H} the set of the connected components of $G[V_1 \cup V_2]$ that contain two adjacent vertices $u \in V_1$ and $v \in V_2$ having the same product by ℓ . By Items 1 and 2 of Lemma 2, such u and v are 1-monochromatic. Also, by Item 6 of Lemma 2, recall that every connected component of \mathcal{H} has at least two edges. In what follows, we only relabel edges of some connected components $H \in \mathcal{H}$ with making sure that their vertices (in $V_1 \cup V_2$) are monochromatic or special. This ensures that only vertices of H have their product affected, thus that no new conflicts involving vertices in $V_3 \cup \dots \cup V_t$ are created.

For a subgraph X of $H \in \mathcal{H}$ (possibly $X = H$), if, after having relabelled edges of X , no conflict remains between vertices of X and all vertices of X are either monochromatic or special, then we say that X verifies Property (\mathcal{P}_3) .

Lemma 3. *If we can relabel the edges of every $H \in \mathcal{H}$ so that every H verifies Property (\mathcal{P}_3) , then the resulting 3-labelling is p -proper.*

Proof. This is because if we get rid of all conflicts in \mathcal{H} , then the only possible remaining conflicts are between vertices in $V_1 \cup V_2$ and in $V_3 \cup \dots \cup V_t$. In particular, recall that any two vertices of two distinct connected components $H_1, H_2 \in G[V_1 \cup V_2]$ cannot be adjacent. Note also that, because we only relabelled edges in \mathcal{H} , the vertices in $V_3 \cup \dots \cup V_t$ retain the product types described in Lemma 2. In particular, they remain bichromatic and none of them is special. Thus, they cannot be in conflict with the vertices in $V_1 \cup V_2$. \square

In order to show that we can relabel the edges of every $H \in \mathcal{H}$ so that it fulfils Property (\mathcal{P}_3) , the following result will be particularly handy.

Lemma 4. *For every integer $s \in \{2, 3\}$, every connected bipartite graph H whose edges are labelled 1 or s , and any vertex v in any part $V_i \in \{V_1, V_2\}$ of H , we can relabel the edges of H with 1 and s so that $d_s(u)$ is odd (even, resp.) for every $u \in V_i \setminus \{v\}$, and $d_s(u)$ is even (odd, resp.) for every $u \in V_{3-i}$.*

Proof. As long as H has a vertex u different from v that does not verify the desired condition, apply the following. Choose P any path from u to v , which exists by the connectedness of H . Now follow P from u to v , and change the labels of the traversed edges from 1 to s and *vice versa*. It can be noted that this alters the parity of the s -degrees of u and v , while this does not alter that parity for any of the other vertices of H . Thus, this makes u satisfy the desired condition, while the situation did not change for the other vertices different from u and v . Thus, once this process ends, all vertices of H different from v have their s -degree being as desired by the resulting labelling. \square

We are now ready to treat the connected components $H \in \mathcal{H}$ independently, so that they all meet Property (\mathcal{P}_3) . To ease the reading, we distinguish several cases depending on the types and on the degrees of the vertices that H includes. In each of the successive cases we consider, it is implicitly assumed that H does not meet the conditions of any previous case.

Claim 1. *If $H \in \mathcal{H}$ has a 3-monochromatic vertex $v \in V_1$, or a 1-monochromatic vertex $v_1 \in V_1$ having two 1-monochromatic neighbours $u_1, u_2 \in V_2$ with degree 1 (in H), then we can relabel edges of H so that H verifies Property (\mathcal{P}_3) .*

Proof. Recall that all edges of H are assigned label 1; thus, if a vertex of H is 3-monochromatic, then it must be due to incident downward edges to V_3, \dots, V_t .

If H has a 1-monochromatic vertex $v_1 \in V_1$ that is adjacent to two degree-1 1-monochromatic vertices $u_1, u_2 \in V_2$, then we set $\ell(v_1u_1) = \ell(v_1u_2) = 3$. Note that u_1 and u_2 become 3-monochromatic with 3-degree 1, and are thus no longer in conflict with v_1 , as it becomes 3-monochromatic with 3-degree 2. Note that either we got rid of all conflicts in H and H now verifies Property (\mathcal{P}_3) as desired, or conflicts between other 1-monochromatic vertices of H remain. In the latter case, we continue with the following arguments.

Assume H has remaining conflicts, and that H has a 3-monochromatic vertex $v \in V_1$ (and, due to the previous process, perhaps 3-monochromatic vertices u_1 and u_2 in V_2 , in which case their 3-degree (and degree in H) is precisely 1, while their unique neighbour v in $V_1 \cap V(H)$ is 3-monochromatic with 3-degree 2). Let X be the set of all 3-monochromatic vertices of H belonging to V_1 . Let C_1, \dots, C_q denote the $q \geq 1$ connected components of $H - X$ that do not consist in a 3-monochromatic vertex of V_2 (the vertices u_1 and u_2 we dealt with earlier on). For every C_i , we choose arbitrarily a vertex $x_i \in X$ and a vertex $y_i \in C_i$ such that x_i and y_i are adjacent in H . Note that the vertices of C_i are either 1-monochromatic or 2-monochromatic (in which case they belong to V_2), since all 3-monochromatic vertices of H are part of X (or are the vertices u_1 and u_2 dealt with earlier on, which we have omitted and are not part of the C_i 's).

By Lemma 4, in every C_i we can relabel the edges with 1 and 2 so that all vertices in $(V_2 \cap V(C_i)) \setminus \{y_i\}$ are 2-monochromatic with odd 2-degree, while all vertices in $V_1 \cap V(C_i)$ are 2-monochromatic with even 2-degree or possibly 1-monochromatic if their even 2-degree is 0. In particular, recall that y_i must be 1-monochromatic or 2-monochromatic. If y_i has odd 2-degree, then there are

no conflicts between vertices of C_i . If y_i has even non-zero 2-degree, then we set $\ell(x_i y_i) = 3$, thereby making y_i special.

Let Y be the set of all 1-monochromatic y_i 's having a 1-monochromatic neighbour w_i in C_i . Let H' be the subgraph of H induced by $Y \cup X$. Note that every edge of H' is labelled 1. Let now Q_1, \dots, Q_p denote the connected components of H' and choose $x_k \in X \cap V(Q_k)$ for every $k \in \{1, \dots, p\}$. For every k , we apply Lemma 4 with labels 1 and 3 so that all vertices in $V_2 \cap V(Q_k)$ get 3-monochromatic with odd 3-degree, while all vertices in $V_1 \cap V(Q_k) \setminus \{x_k\}$ get 3-monochromatic with even 3-degree or possibly 1-monochromatic (3-degree 0).

If x_k is involved in a conflict with a vertex $y_i \in V_2 \cap V(Q_k)$, then this is because x_k has odd 3-degree. Then:

- If $\ell(x_k y_i) = 3$, then $d_3(y_i) = d_3(x_k) \geq 3$ since $x_k \in X$ (x_k must thus be incident to at least one other edge labelled 3, either a downward edge to V_3, \dots, V_t or an edge incident to u_1 (and similarly an edge incident to u_2)). We here assign label 1 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i does not change. Note that y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.
- Otherwise, if $\ell(x_k y_i) = 1$, then we assign label 3 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i remains odd and must be at least 3. Again y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.

We claim that we got rid of all conflicts in H . Indeed, consider two adjacent vertices $a \in V_1 \cap V(H)$ and $b \in V_2 \cap V(H)$. Suppose first that a and b belong to some C_i . Note that, with the exception of y_i and maybe of the vertex w_i (if it exists and $y_i \in Y$), every vertex of C_i is 1-monochromatic or 2-monochromatic, the vertices of $V_1 \cap V(C_i)$ having even 2-degree and the vertices of $V_2 \cap V(C_i)$ having odd 2-degree. Thus, no conflict involves two of these vertices. Suppose now that $b = y_i$. If y_i is 2-monochromatic with odd 2-degree, then there is no conflict involving y_i in C_i since all of its neighbours in C_i have even 2-degree. If y_i is special, then it is the only special vertex of C_i , so, here again, it cannot be involved in a conflict. If $y_i \notin Y$ and y_i is 1-monochromatic, then y_i has no other 1-monochromatic neighbour in C_i by definition of Y . If $y_i \in Y$, then y_i is 3-monochromatic with odd 3-degree, the only other possible 3-monochromatic neighbour of y_i in C_i being w_i , but we showed previously that their 3-degrees differ. Thus, in all cases, there cannot be conflicts between vertices of C_i .

We are left with the case where a and b do not belong to the same C_i . In particular, this implies that $a \in X$ and that a is 3-monochromatic. The only possible 3-monochromatic vertices in V_2 are the vertices of Y , which have odd 3-degree, and the 3-monochromatic vertices u_1 and u_2 with 3-degree 1 and degree 1 in H which might have been created at the very beginning of the proof. If $b \in Y$, then, due to the application of Lemma 4 above, the only vertex of X which can have odd 3-degree is some x_k , but for this vertex we either ensured that it was involved in no conflict, or we tweaked the labelling so that it got even 3-degree without modifying the labelling properties obtained through Lemma 4. If b is u_1

or u_2 , then b has only one neighbour v . Note that the edges vu_1 and vu_2 are still labelled 3 as they are not part of the Q_i 's, and, thus, $d_3(b) = 1$ and $d_3(v) \geq 2$. Hence, there is no conflict between vertices of X and other vertices of H . This implies that H verifies Property (\mathcal{P}_3) . \square

We can thus assume that H does not meet any of the conditions in Claim 1. The next step is showing that we can treat H in a similar way, in case H contains a 1-monochromatic vertex $u \in V_2$ with at least two neighbours in H . This can be proved similarly as Claim 1, by investigating the structure of H and making use of Lemma 4 to relabel edges of H in such a way that all remaining conflicts are located in very precise places of H (so that we can then handle them one by one). The formal proof being long, tedious, and in the same vein as that of Claim 1, due to space limitation we omit it from this paper. The interested reader will find the whole proof in [3], the full version of the current paper.

Claim 2. *If H has a 1-monochromatic vertex $u \in V_2$ with at least two neighbours in H , then we can relabel edges of H so that H verifies Property (\mathcal{P}_3) .*

Assuming H does not meet any of the conditions in Claims 1 and 2, final arguments allow to relabel edges of H to get rid of all its conflicts.

Claim 3. *We can relabel edges of H so that it verifies Property (\mathcal{P}_3) .*

Proof. Let $v \in V_1$ and $u \in V_2$ be two adjacent 1-monochromatic vertices of H (which must exist as otherwise H would verify Property (\mathcal{P}_3)). Because H has at least two edges (as otherwise it would belong to M , not to \mathcal{H}), at least one of v and u must have another neighbour in H . Since Claim 2 does not apply, note that u must have degree 1 in H (since all neighbours of u in H must be 1-monochromatic due to Claim 1 not applying). So v is also adjacent to $k \geq 1$ vertices $x_1, \dots, x_k \in V_2$ different from u , which must all be 2-monochromatic (because of incident downward edges to V_3, \dots, V_t ; recall that all edges of H are labelled 1) as otherwise Claim 2 would apply.

Set $H' = H - u$. According to Lemma 4, we can relabel edges in H' with 1 and 2 so that all vertices in $(V_1 \cap V(H')) \setminus \{v\}$ have odd 2-degree, while all vertices in $V_2 \cap V(H')$ have even 2-degree. Recall that u is 1-monochromatic. Thus, if also v is 2-monochromatic with odd 2-degree, then we are done. Assume thus that v is 2-monochromatic with even 2-degree.

- Assume first that the 2-degree of v is even at least 2. In that case, set $\ell(vu) = 3$. This way, u becomes 3-monochromatic, while v becomes special.
- Assume now v is 1-monochromatic. This implies that $\ell(vx_1) = 1$. Change $\ell(vx_1)$ to 3. This way, x_1 becomes special (recall its 2-degree is even and at least 1, due to incident downward edges), while v becomes 3-monochromatic. Note that u remains 1-monochromatic.

In both cases, it can be checked that H now fulfils Property (\mathcal{P}_3) . \square

At this point, we dealt with all connected components of \mathcal{H} , and the resulting labelling ℓ of G is p-proper by Lemma 3. The whole proof is thus complete.

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