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Abstract

The goal of the paper is a rigorous derivation of a macroscopic traffic flow model with a bifurcation or a local perturbation from a microscopic one. The microscopic model is a simple follow-the-leader with random parameters. The random parameters are used as a statistical description of the road taken by a vehicle and its law of motion. The limit model is a deterministic and scalar Hamilton-Jacobi on a network with a flux limiter, the flux-limiter describing how much the bifurcation or the local perturbation slows down the vehicles. The proof of the existence of this flux limiter—the first one in the context of stochastic homogenization—relies on a concentration inequality and on a delicate derivation of a superadditive inequality.

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Contents

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Introduction

In this paper we study traffic flows models with a bifurcation consisting in a single incoming road which is divided after a junction into several outgoing ones. As a particular case our analysis contains traffic flow models on a single road with a localized perturbation (a bottleneck for instance). There are two main classes of models to describe these situations: microscopic models, which explain how each vehicle behaves in function of the vehicles in front; and macroscopic ones, taking the form of a conservation law in which the main unknown is the density of vehicles on the roads. Our aim is to start from simple microscopic models on a bifurcation (or on a perturbation) and derive from these models continuous ones after scaling. The point is to get a better understanding of the continuous traffic flow models arising as the limit of discrete ones. Indeed there exists many different continuous models of traffic flow on a junction or with a local perturbation in the literature [5, 13, 23, 32, 33, 48] and the relation between these models is not completely clear. If the basic continuous model on a single straight road (the so-called LWR model, from Lighthill and Whitham [38] and Richards [44]) is well understood and justified by micro-macro limits in several contexts [8, 22, 26, 34, 35], there is no consensus for problems with a junction or a bifurcation: the models are only obtained so far by heuristic arguments, with the exception of [28] discussed below. In this paper we show that the continuous model suggested in [31, 36] pops up as the natural limit of follow-the-leader models. The continuous model in [31, 36] takes the form of a flux limited Hamilton-Jacobi equation: it is a kind of integrated form of the basic LWR model outside the junction combined with a "flux limiting condition" on the junction. Our micro-macro derivation holds for a large class of follow-the-leader models, allowing for a possible heterogeneous behavior of the vehicles.

Our starting point is a microscopic model. Before describing it, let us recall that few discrete traffic flow models with a junction or a local perturbation exist in the literature: [25] discusses an interesting leader follower model with a junction including several incoming and outgoing roads: the model we present in the present paper shares similar flavors, but in the much simpler setting of a single incoming road; [5] presents a microscopic model of traffic with a flow limitation at a point and formally justifies the derivation of a conservation law with a discontinuous flux (but leaves the rigorous proof as an open problem); [30] describes a traffic flow model with (deterministic) traffic lights and derives rigorously the continuous model (in terms of a flux limited Hamilton-Jacobi equation on the line). The only model proving micro-macro derivation in the case of a bifurcation is [28]: in [28] there are two outgoing roads and it is assumed (no too realistically) that every second vehicle takes a given road. In this setting the authors show that the convergence of the discrete problem to a flux limited solution of a Hamilton-Jacobi equation on a junction. One of the goals of the present paper is to introduce a more realistic model in which one replaces the deterministic rule of [28] by a random one (e.g., every second vehicle *in average* takes a given outgoing road). The introduction of randomness in traffic flow problems is natural and can be traced back to [21]. The micro-macro derivation of the LWR model from a random one on a single road was established in [19]. Here we prove the corresponding result for a bifurcation.

Short description of the microscopic model. In our discrete model there is one incoming road and K outgoing ones, where $K \in \mathbb{N}$, $K \geqslant 1$. A position on the road is given by a pair (x, k) where x is a real number and k is a label in $\{0, \dots, K\}$. If x is nonpositive, then by convention $k = 0$ and the vehicle is on the incoming road. If x is positive then $k \in \{1, \dots, K\}$ and the vehicle is on the outgoing road k. The junction is an interval around $x = 0$, say, to fix the ideas, $[-R_0, 0]$. The vehicles are labelled by $i \in \mathbb{Z}$. The position of the vehicle labelled i at time t is denoted by $U_i(t)$. The outgoing road the vehicle chooses is fixed from the beginning (independent of time) and denoted by $T_i \in \{1, \dots, K\}$. The motion of the

vehicles is given by a leader-follower model: it satisfies the system of ordinary differential equations

$$\frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t), U_{\ell_i}(t) - U_i(t), U_i(t)), \qquad t \geqslant 0,\ i \in \mathbb{Z}. \tag{1}$$

We assume that all the vehicle are going or have gone through the junction and were ordered before going through the junction: $i+1$ is the label of the vehicle right in front of the vehicle i before this vehicle has gone through the junction. We denote by ℓ_i the label of the first vehicle in front of vehicle i taking the same outgoing road as i (in other words, $\ell_i = \inf\{j > i, T_i = T_j\}$). Each vehicle has a type Z_i encoding, on the one hand, the outgoing road the vehicle is taking or is going to take (namely, $T_i = T(Z_i)$ for a deterministic map $T : \mathcal{Z} \to \{1, \dots, K\}$) and, on the other hand, the "behavior" of the vehicle (for instance, if it is a truck or a race car). The velocity law $V = V_z(e_1, e_2, x)$ depends on the type $z \in \mathcal{Z}$ of the vehicle, the distances e_1 or e_2 to the next vehicle and the position x of the vehicle.

In order to obtain a limit model with a few unknowns and as simple as possible, we do not keep track of all the vehicles of a given type (in contrast with [24]). Instead we prefer a statistical description and assume that the types (Z_i) of the vehicles are random, independent and with the same law (i.i.d.); as a consequence the (T_i) are also i.i.d. In addition, we also suppose that the traffic is homogeneous outside the junction: namely, we assume that, before the junction (i.e., $x \leqslant -R_0$), $V_z(e_1, e_2, x)$ depends only on e_1 and z, i.e., $V_z(e_1, e_2, x) = \tilde{V}_z^0(e_1)$. In the same way, after the junction (i.e., $x \geqslant 0$) we suppose that $V_z(e_1, e_2, x) = \tilde{V}_z^k(e_2)$ depends only on e_2, $k = T(z)$ and z. There are two main reasons to do so: first (and again in contrast with [24]), we will see that these assumptions yield to a relatively simple continuous scalar equations. Second, tracing the type of a vehicle (and even more the road it is going to take later on) seems an impossible task in practice: a statistical description is probably more justified, at least if the structure of the traffic is stable in time.

For later use we denote by $\pi^k := \mathbb{P}[T_i = k]$ the proportion of vehicle taking (or planning to take) road k.

The convergence result and the continuous model. For $\epsilon > 0$, we look at the (scaled) traffic density of vehicles on each road:

$$m^\epsilon(dx, k, t) = \begin{cases} \epsilon \displaystyle\sum_{i\in\mathbb{Z},\ T_i=k} \delta_{\epsilon U_i(t/\epsilon)}(dx) & \text{if } x > 0,\ k \in \{1, \dots, K\} \\ \epsilon \displaystyle\sum_{i\in\mathbb{Z}} \delta_{\epsilon U_i(t/\epsilon)}(dx) & \text{if } x \leqslant 0,\ k = 0 \end{cases}$$

and want to understand the limit, as $\epsilon \to 0$, of m^ϵ. For this it is convenient to integrate in space m^ϵ and look instead at:

$$\nu^\epsilon(x, k, t) = \begin{cases} \epsilon(\pi^k)^{-1} \left(\displaystyle\sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T_i=k} \delta_{\epsilon U_i(t/\epsilon)}((x, +\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{\epsilon U_i(t/\epsilon)}((-\infty, x]) \right) & \\ \qquad\qquad \text{if } x > 0,\ k \in \{1, \dots, K\} & \\ \epsilon \left(\displaystyle\sum_{i\in\mathbb{Z},\ i\leqslant 0} \delta_{U_i(t)}((x, +\infty)) - \sum_{i\in\mathbb{Z},\ i>0} \delta_{U_i(t)}((-\infty, x]) \right) & \text{if } x \leqslant 0,\ k = 0. \end{cases}$$

Note that $\partial_x \nu^\epsilon = -m^\epsilon$ if $x \leqslant 0$ while $\partial_x \nu^\epsilon = -(\pi^k)^{-1} m^\epsilon$ if $x \geqslant 0$ and $k \in \{1, \dots, K\}$. This choice ensures the map ν^ϵ to be "almost continuous" at 0 since the vehicles are split between the K roads after the junction in proportion π^k for the road k. Our main result (Theorem 1.1) roughly states that, under suitable assumptions on V and if $\nu^\epsilon(\cdot, \cdot, 0)$ has a locally uniform (deterministic) limit $\nu_0(\cdot, \cdot)$ at time $t = 0$, then ν^ϵ has a.s. a locally uniform (deterministic) limit ν which is the unique viscosity solution to

$$\begin{cases} \partial_t \nu(x, k, t) + H^k(\partial_x \nu(x, k, t)) = 0 & \text{if } x \neq 0,\ t > 0 \\ \partial_t \nu + \max\{\bar{A}, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} = 0 & \text{at } x = 0 \\ \nu(x, k, 0) = \nu_0(x, k) & \text{for any } x, k. \end{cases} \tag{2}$$

The first equation is a Hamilton-Jacobi (HJ) equation in which the homogenized Hamiltonians $H^k(p)$ can be explicitly computed from the $\tilde{V}^k$. As we explain below it corresponds to an integrated form of

the LWR equation. The second equation describes the behavior of the vehicles at the junction (reduced after scaling to $x = 0$): we explain below the different terms. It roughly says that $\partial_t \nu + \bar{A} = 0$ at $x = 0$ (unless the HJ equation is satisfied at $x = 0$). The real number $\bar{A}$ is the so-called flux limiter. This is the main unknown of the paper. It quantifies how the traffic is slowed down by the junction. We show that

$$A_0 \leqslant \bar{A} \leqslant 0, \text{ where } A_0 := \max_{k\in\{0,\dots,K\}} \min_{p\in\mathbb{R}} H^k(p).$$

When $\bar{A} = A_0$, the flux is not limited at all. If $\bar{A} = 0$, then the traffic is completely stopped by the junction (this does not happen under our assumptions). The existence of $\bar{A}$ is the main point of the paper, which presents the first existence result of a flux limiter in the context of a stochastic homogenization problem. We show that $\bar{A}$ can be computed as follows:

$$\bar{A} = -\lim_{t\to+\infty} \frac{1}{t} \sharp\big\{i \in \mathbb{Z},\ \exists s \in [0,t],\ U_{e,i}(s) = 0\big\},$$

where $\sharp E$ denotes the number of elements of a set E, $e = (e^k)_{k=0,\dots,K}$ is such that $H^k(-1/e^k) = \min_p H^k(p)$ for any $k \in \{0,\dots,K\}$ and $(U_{e,i})$ is the solution to (1) with the "flat" initial condition $U_{e,i}(0) = e^k i$ (where $k = 0$ if $i \leqslant 0$ and $k = T_i$ if $i \geqslant 0$). The quantity $\bar{A}$ can be interpreted as the maximal fraction of vehicles the junction can let pass given an amount of time.

The introduction of Hamilton-Jacobi equations on a junction or stratified domains can be traced back to [1, 2, 3, 10, 16, 18, 37, 45, 46]; a general theory of flux limited solutions was developed in [36] (see also [11]) with, as fundamental result, a comparison theorem; [42, 43] present different arguments for the comparison while [12] proposes a general survey on the topic.

Short discussion of the problem in terms of scalar conservation law. Hamilton-Jacobi equations on a junction and scalar conservation laws with discontinuous coefficients seem intimately connected, although the rigorous relationships between the two notions has not been discussed so far. We do not intend to investigate this point here but only develop formal arguments and postpone a more detailed analysis to future works.

For $k \in \{1,\dots,K\}$, we define the random measures

$$\rho^\epsilon(dx,k,t) = \begin{cases} \epsilon(\pi^k)^{-1} \displaystyle\sum_{i\in\mathbb{Z},\ T_i=k,\ U_i^\epsilon(t/\epsilon)\geqslant 0} \delta_{\epsilon U_i^\epsilon(t/\epsilon)}(dx) & \text{if } x \geqslant 0,\ k \in \{1,\dots,K\} \\ \epsilon \displaystyle\sum_{i\in\mathbb{Z},\ U_i^\epsilon(t/\epsilon)<0} \delta_{\epsilon U_i^\epsilon(t/\epsilon)}(dx) & \text{if } x < 0,\ k = 0 \end{cases}$$

The quantities are the scaled densities of the traffic on each branch of the junction. A elementary computation shows that we have, in the sense of distribution, $\rho^\epsilon = -\partial_x \nu^\epsilon$. According to our main result (Theorem 1.1) ρ^ϵ converges a.s. and in the sense of distribution, to $\rho := -\partial_x \nu$. As ν solves (2) and is Lipschitz continuous, it is known [20] that ρ is, outside the junction, an L^∞ entropy solution of the scalar conservation law

$$\partial_t \rho + \partial_x(f(\rho,x,k)) = 0 \qquad \text{for } x \neq 0, \tag{3}$$

where

$$f(\rho,x,k) = \begin{cases} -H^k(-\rho) & \text{if } x > 0 \text{ and } k \in \{1,\dots,K\}, \\ -H^0(-\rho) & \text{if } x < 0 \text{ and } k = 0, \end{cases}$$

with an initial condition given by $\rho(x,k,0) = -\partial_x \nu_0(x,k)$.

It is well-known [4] that an extra conditions at the junction (depending on the model) is needed to ensure the uniqueness of such a scalar conservation law. The additional equation at $x = 0$ for ν in (2) should lead to this extra condition. It does not seem however obvious how to interpret it in terms of the limit density ρ.

Method of proof. We now describe the method of proof of our main result. As it is quite involve it is convenient for this discussion to reduce drastically the problem by considering the case of a single road on which the vehicle behave in an identical way, expect on a small zone on which they are subject to a

perturbation depending on their type. This situation pops up for instance when the vehicles are slowed down on a small portion of a road by a speed bump to which they may react in a different way depending on their size. The leader-follower model now reads

$$\frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t), U_i(t)) \tag{4}$$

where Z_i is as before the type of the vehicle (supposed to be an i.i.d. random variable) and where $V_z(p,x) = \tilde{V}(p)$ outside the perturbation $[-R_0, 0]$. Following [36] (see also [30, 28]), one expects the limit model to be of the form of a LWR model with a flux limiting condition at the origin. The fundamental diagram outside the perturbation is given by $H(p) = p\tilde{V}(-1/p)$ and the only issue is to compute the flux limiter. In the case of a deterministic model (for instance time periodic, see [30]; or periodic in the type, see [28]) a standard method consists in building a corrector. However in the random setting such a corrector does not necessarily exist: see for instance the discussion in [40]. A standard way to overcome this difficult issue is to identify subadditive quantities [6, 41, 47]. In the case such a quantity is not directly available, a different, more quantitative approach has been developed in [7] and later used in different contexts [17, 27, 39]. As no subadditive quantity seems adequate in our setting we follow this alternative approach.

The starting point is to explore what happens for the "flat" initial condition $U_i(0) = ei$ ($i \in \mathbb{Z}$), where $e > 0$ is such that $H(-1/e) = \min_p H(p)$. In the absence of a perturbation, this initial condition would be a steady state of the problem: namely, $U_i(t) = ei + t\tilde{V}(e)$ solves (4) outside the perturbation. The point is to understand how this steady state solution is modified by the perturbation. For this we introduce the (random) quantity

$$\theta_e(t) = \inf\{i \geqslant 0,\ U_{-i}(t) \leqslant 0\},$$

which corresponds to the number of vehicles having gone through the perturbation at time t. If the problem was unperturbed, one would have simply $\theta_e(t) \simeq t\tilde{V}(e)/e$. To understand if the macroscopic model is affected by the perturbation, one is therefore led to investigate the behavior of $\theta_e(t)/t$ as t tends to infinity. The existence of such a limit is the main difficulty of the work. Indeed, θ_e does not seem to enjoy any obvious sub- or superadditivity property. Following [7] the first step of the proof consists in showing that $\theta_e(t)/t$ is almost deterministic. Namely, we prove that there is a constant C (depending on e) such that, for all $\epsilon \in (0,1]$ and all $t \geqslant C\epsilon^{-1}$,

$$\mathbb{P}\left[|\theta_e(t) - \bar{\theta}_e(t)| \ > \ \epsilon t\right] \leqslant Ct^2 \exp\left\{-\epsilon^2 t/C\right\}.$$

where $\bar{\theta}_e(t) = \mathbb{E}\left[\theta_e(t)\right]$. For this the technique developed in [7] consists in showing that the martingale

$$M_n(t) := \mathbb{E}\left[\ \theta_e(t) \mid \mathcal{F}_n\right] - \mathbb{E}\left[\theta_e(t)\right],$$

(where $(\mathcal{F}_n)$ is the filtration generated by $\{Z_{-n}, Z_{-n+1}, \dots\}$) has bounded increments, coincides with $\theta_e(t)$ for $n = [Ct]$ (where C is a large constant) and then use Azuma's inequality. Although we won't study this model in detail later, it is indeed possible to show in this case that (M_n) has bounded increments by using three facts:

- First $M_0(t) = 0$ since the randomness of $\theta_e(t)$ comes only from Z_i with $i \leqslant 0$ (indeed, $U_i(t) = ei + \tilde{V}(e)t$ for $i \geqslant 0$ is deterministic),
- Second one can show that two subsequent vehicles remain at a distance not larger than e before the perturbation,
- Third one can prove that a vehicle close to the perturbation will cross it in a controlled time.

The next step consists in establishing that $\bar{\theta}_e(t)/t$ has indeed a limit as $t \to +\infty$. The difficult issue is to understand how the profile of the solution $(U_i(t))$ at time t looks like the profile of the $(U_i(0) = ei)$ at time 0. For this one looks at how much $(U_i(t))_{i\in\mathbb{Z}}$ is far from the unperturbed solution $(ei + \tilde{V}(e)t)$: namely one looks at the quantity

$$M_e(t) := \inf_{i\in Z} U_i(t) - ei - \tilde{V}(e)t.$$

If the traffic is slowed down by the perturbation, this quantity is expected to be nonpositive and to decrease in time. An almost finite speed of propagation argument (Lemma 2.6) shows that, far from the perturbation, the solution is almost given by the steady state and therefore the infimum in $M_e(t)$ (if negative) cannot be achieved by large values of $|i|$. So there is a minimum point i_0 for $M_e(t)$. By the envelop theorem one expects that

$$0 > \frac{d}{dt}M_e(t) = \frac{d}{dt}U_{i_0}(t) - \tilde{V}(e) = V_{Z_i}(U_{i_0+1} - U_{i_0}(t), U_{i_0}(t)) - \tilde{V}(e).$$

By minimality of i_0, one also has

$$U_i(t) - U_{i_0}(t) \geqslant e(i - i_0) \qquad \forall i \in \mathbb{Z}. \tag{5}$$

The two inequalities above imply that $U_{i_0}(t) \in [-R_0, 0]$ because otherwise one would have, as $\tilde{V}$ is nondecreasing and (5) holds,

$$0 > \frac{d}{dt}M_e(t) = \tilde{V}(U_{i_0+1} - U_{i_0}(t)) - \tilde{V}(e) \geqslant \tilde{V}(e) - \tilde{V}(e) = 0.$$

The fact that $U_{i_0}(t) \in [-R_0, 0]$ then implies that i_0 is close to $-\theta_e(t)$ and thus that

$$M_e(t) = U_{i_0}(t) - ei_0 - \tilde{V}(e)t = e\theta_e(t) - \tilde{V}(e)t + O(1).$$

On the other hand, by (5), one has

$$U_i(t) \geqslant U_{i_0}(t) + e(i - i_0) \geqslant e(i - i_0 - R_0/e) \qquad \forall i \in \mathbb{Z}.$$

Setting $i_1 = i_0 + R_0/e$, we obtain by comparison that the solution U_i at time $t + s$ is above the solution starting from $e(i - i_1)$:

$$U_i^{\omega}(t + s) \geqslant U_{i-i_1}^{\tau_{i_1}\omega}(s) \qquad \forall i \in \mathbb{Z},$$

(the shift in the ω is due to the fact that one has to shift also the types of the vehicles). By the concentration inequality and the fact that $i_0 \approx -\theta_e(t) \approx -\bar{\theta}_e(t)$, this implies that

$$\bar{\theta}_e(t + s) \geqslant \bar{\theta}_e(s) + \bar{\theta}_e(t) - C.$$

Fekete's Lemma then implies that $\bar{\theta}_e(t)/t$ has a limit and therefore that $\theta_e(t)/t$ has a limit. One can also prove that this limit gives the value $\bar{A}$ of the flux limiter.

The proof in the general case (a bifurcation with several outgoing roads) follows the same lines but is much more involved. Many arguments described above are no longer valid. For instance, it is no longer true that $M_0(t)$ vanishes, because, as the distribution of the vehicles at initial time on the outgoing roads is random, $\theta_e(t)$ actually depends on the behavior of all the vehicles. We overcome this issue by using the approximate finite speed of propagation. Second, the distance between two subsequent vehicle can be arbitrarily large: this is already true at the initial time on the outgoing roads. In addition, because the vehicles have different types, the maximal speed of a leader can be larger than the maximal speed of its follower. We show however that this distance is controlled by the distance to the first "slow" vehicle in front of the leader (Lemma 2.5). The main consequence of this is that $M_n(t)$ cannot have bounded increments (in contrast with [7] for instance; see however [17]); one has to rely on more refined concentration inequalities. Finally the presence of a bifurcation (instead of a perturbation) makes the proof of the superadditive inequality much trickier: it actually relies on the delicate construction of a corrector outside the junction (Subsection 3.3).

Organization of the paper

In the first part we explain the problem and the notation, introduce the standing assumption and state the main result (Theorem 1.1). In the second part we give several facts which are valid for any solution of the system: an estimate of the distance to the next vehicle (Lemma 2.5) and an approximation finite speed

of propagation (Lemma 2.6). In the third part we study the time function $\theta_e(t)/t$, show its concentration (Theorem 3.4) and prove its convergence (Theorem 3.12). In the last part, we derive from this the behavior of the solution starting from the flat initial datum (Lemma 4.3), infer definition of the flux limiter $\bar{A}$ and finally show the homogenization result.

Throughout the paper, the letter C denote a deterministic constant which may change from line to line and which depends on the data but not on time.

1 The main result

1.1 Statement of the problem

We consider the system

$$\frac{d}{dt}U_i^\omega(t) = V_{Z_i^\omega}(U_{i+1}^\omega(t) - U_i^\omega(t), U_{\ell_i^\omega}^\omega(t) - U_i^\omega(t), U_i^\omega(t)) \qquad i \in \mathbb{Z},\ t \geqslant 0, \tag{6}$$

where $V : \mathcal{Z} \times \mathbb{R}^3 \to \mathbb{R}_+$ is Lipschitz continuous in the three last variables (uniformly in the $z-$variable), nondecreasing with respect to the two middle ones and bounded by $\|V\|_\infty$. The type of the vehicle $i \in \mathbb{Z}$ is the random variable Z_i in $\mathcal{Z}$. We assume that $\mathcal{Z}$ is a finite set and that the $(Z_i)_{i\in\mathbb{Z}}$ are i.i.d.

There is a single incoming road and K outgoing roads (where $K \in \mathbb{N}\backslash\{0\}$). The junction $\mathcal{R}$ is given by

$$\mathcal{R} = \bigcup_{k=0}^{K} \mathcal{R}^k, \qquad \mathcal{R}^0 = (-\infty, 0] \times \{0\},\ \mathcal{R}^k = [0, +\infty) \times \{k\} \text{ for } k \in \{1, \dots, K\}.$$

We also denote by $\overset{o}{\mathcal{R}}$ the interior of the roads:

$$\overset{o}{\mathcal{R}} = \bigcup_{k=0}^{K} \overset{o}{\mathcal{R}}^k, \qquad \overset{o}{\mathcal{R}}^0 = (-\infty, 0) \times \{0\},\ \overset{o}{\mathcal{R}}^k = (0, +\infty) \times \{k\} \text{ for } k \in \{1, \dots, K\}.$$

The outgoing road chosen by a vehicle is determined by its type z and is given by the map $T : \mathcal{Z} \to \{1, \dots, K\}$. We set

$$T_i^\omega = T(Z_i^\omega) \text{ and } \ell_i^\omega = \inf\{j > i,\ T_j^\omega = T_i^\omega\} \qquad \forall i \in \mathbb{Z}.$$

The vehicle ℓ_i is the first vehicle in front of i which takes the same outgoing road as i. As the vehicles with the same outgoing road remain ordered, ℓ_i does not depend on time. For $k \in \{1, \dots, K\}$, let $\pi^k = \mathbb{P}[T_0 = k]$ be the probability for a vehicle to take the outgoing road k. By convention, we set $\pi^0 = 1$. *Without loss of generality, we assume throughout the paper that* $\pi^k \in (0, 1]$ *for any* $k \in \{1, \dots, K\}$. In this case ℓ_i is well-defined a.s. since, $\mathbb{P}-$a.s., $\{j > i,\ T_j^\omega = T_i^\omega\}$ is nonempty.

The bifurcation is supposed to be at $x = 0$. We assume that the equation is homogeneous outside a transition zone $[-R_0, 0]$ near the bifurcation: namely we suppose the existence of $R_0 > 0$ and of $\tilde{V}^0, \dots, \tilde{V}^K : [0, +\infty) \to [0, +\infty)$ such that

$$V_z(e_1, e_2, x) = \begin{cases} \tilde{V}_z^0(e_1) & \text{if } x \leqslant -R_0 \\ \tilde{V}_z^k(e_2) & \text{if } x \geqslant 0 \text{ and } T(z) = k. \end{cases}$$

The meaning of this assumption is that, if the position $U_i(t)$ of a vehicle i at time t is not in the interval $(-R_0, 0)$, the velocity of this vehicle is determined by its type and by the distance to the vehicle in front of it (which has label $i + 1$ if $U_i(t) \leqslant -R_0$ and ℓ_i if $U_i(t) \geqslant 0$). It is only when the vehicle is in the transition zone $[-R_0, 0]$ that its velocity also depends possibly on its position and on the vehicles in front; for instance it may slow down to prepare the change of road.

The problem as stated above contains the following particular cases:

- Problem on a single road with a perturbation: in this case there is a single outgoing road and the vehicles solve the simpler system

$$\frac{d}{dt}U_i^\omega(t) = V_{Z_i^\omega}(U_{i+1}^\omega(t) - U_i^\omega(t), U_i^\omega(t)) \qquad i \in \mathbb{Z},\ t \geqslant 0,$$

  where $V_z(e_1, x) = \tilde{V}_z(e_1)$ does not depend on x if $x \notin [-R_0, 0]$.

- Problem in which the type is only the choice of the outgoing road: in this case the system is still of the form (6) but one has $\mathcal{Z} = \{1, \dots, K\}$, $T(z) = z$ and the $\tilde{V}_z^k$ do not depend on z. There is still a transition zone $[-R_0, 0]$ on which the velocity of the vehicle i passes from a dependence to the distance to the vehicle right in front (with label $i+1$) to the distance to the vehicle going on the same outgoing road (with label ℓ_i).

If the proof of homogenization would be somewhat simpler in the first case (as described in the introduction), the second case contains already (almost) all the difficulties we will meet below.

The goal of the paper is to understand the behavior of the solution on large scale of time and space: namely, the behavior of $(x,t) \to \epsilon U_{[x/\epsilon]}(t/\epsilon)$, (where $[y]$ is the integer part of the real number y).

Notation: Throughout the paper, $\Omega := \mathcal{Z}^{\mathbb{Z}}$ is endowed with the product $\sigma-$field $\mathcal{F}$ and with the product probability measure $\mathbb{P}$. We denote by $\tau : \mathbb{Z} \times \Omega \to \Omega$ the shift map defined by

$$(\tau_n \omega)_i = \omega_{i+n}, \qquad \forall \omega = (\omega_i)_{i\in\mathbb{Z}} \in \Omega,\ \forall n \in \mathbb{Z}.$$

We set $Z_i^\omega = \omega_i$ for $\omega = (\omega_i) \in \Omega$ and $i \in \mathbb{Z}$. As $\mathbb{P}$ is the product measure on Ω, this means that the $(Z_i)_i \in \mathbb{Z}$ are i.i.d. We note for later use that $Z_{i-n}^{\tau_n\omega} = Z_i^\omega$ while $\ell_i^{\tau_n\omega} = \ell_{i+n}^\omega - n$ for any $n, i \in \mathbb{Z}$.

For $x, y \in \mathbb{R}$, we denote by $[x]$ the integer part of x, set $(x)_+ = \max\{0, x\}$, $(x)_- = \max\{0, -x\}$, $x \wedge y = \min\{x, y\}$. If $E \in \mathcal{F}$, then $E^c = \Omega \backslash E$.

1.2 Assumptions

Let us state our standing assumptions on V_z:

(H_1) For any $z \in \mathcal{Z}$, the map $(e_1, e_2, x) \to V_z(e_1, e_2, x)$ is Lipschitz continuous from $\mathbb{R}_+^2 \times \mathbb{R}$ to $\mathbb{R}_+$ and nondecreasing with respect to the first two variables;

(H_2) There exists $e_{\max} > \Delta_{\min} > 0$ and $0 < R_2 < R_1 < R_0$, with $R_0 > e_{\max}$, such that for any $z \in \mathcal{Z}$, for any $(e_1, e_2, x) \in \mathbb{R}_+^2 \times \mathbb{R}$,

  (i) $V_z(e_1, e_2, x) = 0$ if ($e_1 \leqslant \Delta_{\min}$ and $x \leqslant -R_2$) or if ($e_2 \leqslant \Delta_{\min}$ and $x \geqslant -R_1$),

  (ii) $V_z(e, e_2, x) = V_z(e_{\max}, e_2, x)$ and $V_z(e_1, e, x) = V_z(e_1, e_{\max}, x)$ if $e \geqslant e_{\max}$;

(H_3) There exists $\tilde{V}^0, \dots, \tilde{V}^K : [0, +\infty) \to [0, +\infty)$ such that

$$V_z(e_1, e_2, x) = \begin{cases} \tilde{V}_z^0(e_1) & \text{if } x \leqslant -R_0 \\ \tilde{V}_z^k(e_2) & \text{if } x \geqslant 0 \text{ and } T(z) = k. \end{cases}$$

(H_4) For any $z \in \mathcal{Z}$ and any $k \in \{0, \dots, K\}$, there exists $h_{\max,z}^k \in (\Delta_{\min}, e_{\max}]$ such that $p \to \tilde{V}_z^k(p)$ is increasing and concave in $[\Delta_{\min}, h_{\max,z}^k]$ and constant on $[h_{\max,z}^k, +\infty)$;

(H_5) There exists $\kappa > 0$ such that, for any $z \in \mathcal{Z}$,

  (i) $V_z(e_1, e_2, x) = \tilde{V}_z^0(e_1)$ if $e_1 \leqslant e_2$, $x \leqslant -R_2$ and $V_z(e_1, e_2, x) \leqslant \kappa$,

  (ii) $\partial_x V_z(e_1, e_2, x) \geqslant 0$ if $x \in [-R_1, 0]$ and $V_z(e_1, e_2, x) \leqslant \kappa$,

  (iii) $V_z(e_1, e_2, x) > 0$ if $e_1 \wedge e_2 > \Delta_{\min}$.

Note that, by assumption (H_2), we have $\tilde V_z^k(e)=0$ if $e\leqslant \Delta_{\min}$ and $\tilde V_z^k(e)=\tilde V_z^k(e_{\max})$ if $e\geqslant e_{\max}$.

Some comment on the assumption are now in order. The assumption that $\mathcal{Z}$ is finite is useful throughout the proofs but could be relaxed; as this would introduce an extra layer of technicalities, we prefer to keep this condition for simplicity. Assumption (H_2) is standard in the analysis of leader-follower models. The existence of $\Delta_{\min}$ prevents vehicles to collide (and could correspond to the size of the smallest vehicle for instance). The existence of $e_{\max}$ just says that the vehicles do not take into account the vehicles too far ahead. Assumption $R_0>e_{\max}$ can be made without loss of generality. Assumption (H_3) means that the roads are homogeneous outside the bifurcation. This formalizes the fact that we concentrate here on a single bifurcation. Assumption (H_4) is also standard in the analysis of leader-follower models. There is one restriction though: the minimal distance such that the velocity has to be positive (i.e., here $\Delta_{\min}$) has to be the same for all vehicle and is not allowed to depend on the type of the vehicle; this restriction is related to the last (and technical) assumption (H_5). Assumption (H_5) has to do with the behavior of vehicles with slow velocity on the junction and ensures that the vehicles starting with a flat initial condition $(U_i(0):=e^k i)_{i\in\mathbb{Z}}$ (where $k=0$ if $i\leqslant 0$ and $k=T_i$ if $i\geqslant 0$ and e^k is such that $H^k(-1/e^k)=\min_p H^k(p)$) have a velocity bounded below by a positive constant independent of time and position (Lemma 3.1). This last property is instrumental throughout the proofs. Assumption (H_5), without being unrealistic, is a little restrictive, but we do not know if it is possible to relax it. We illustrate these assumptions by an example.

An example. Let $0<r_3<r_2<r_1<r_0$. Fix three smooth and nonincreasing maps $\xi_i:\mathbb{R}\to[0,1]$ such that $\xi_i(x)=1$ for $x\leqslant -r_{i-1}$, $\xi_i(x)=0$ for $x\geqslant -r_i$ $(i=1,2,3)$. Fix also $\bar s>0$ and, for any $z\in\mathcal{Z}$ and $k\in\{0,\dots,K\}$, $W_z^k:[0,+\infty)\to[0,+\infty)$ a smooth and nondecreasing map with $W_z(0)=0$, $W_z(s)=W_z(\bar s)$ for any $s\geqslant\bar s$; we also assume that W^k is increasing on $[0,\bar s]$. Then the map defined by $e_1,e_2\geqslant 0$, $x\in\mathbb{R}$, $z\in\mathcal{Z}$ and $k=T(z)$ by

$$\begin{aligned}V_z(e_1,e_2,x)=&\xi_1(x)W_z^0\big((e_1-\Delta_{\min})_+\big)+(1-\xi_1(x))\xi_2(x)W_z^0\big((e_1\wedge e_2-\Delta_{\min})_+\big)\\&+(1-\xi_2(x))W_z^k\big(\xi_3(x)(e_1\wedge e_2-\Delta_{\min})_+ +(1-\xi_3(x))(e_2-\Delta_{\min})_+\big)\end{aligned}$$

satisfies the required conditions with $R_0=r_0$, $R_1=r_2$, $R_2=r_3$ and $\tilde V_z^k(p)=W_z^k((p-\Delta_{\min})_+)$. The complicated expression of V expresses the transition between a configuration in which the vehicle drives at speed $\tilde V_z^0$ and considers only the vehicle in front on road 0 to a configuration in which it drives at speed $\tilde V_z^k$ and considers only the vehicle in front on road k. Namely, before $-r_0$, the vehicle, driving at speed $\tilde V_z^0$, takes into account only the next vehicle on road 0. Between $-r_0$ and $-r_1$, the vehicle, still driving at speed $\tilde V_z^0$, slows down in order to take also into account the next vehicle on the road k. Between $-r_1$ and $-r_2$, the vehicle adapts its speed to road k (passes from velocity $\tilde V_z^0$ to $\tilde V_z^k$). Between $-r_2$ and $-r_3$, the vehicle, driving at speed $\tilde V_z^k$, looses track of the vehicle which was in front on road 0 and only considers the vehicle in front on road k after 0.

1.3 The homogenized velocities and Hamiltonians.

Let $V_{\max,z}^k:=\tilde V_z^k(h_{\max,z}^k)$. Under assumptions (H1)—(H4), the map $\tilde V_z^k:[\Delta_{\min},h_{\max,z}^k]\to[0,V_{\max,z}^k]$ is increasing and continuous for any $z\in\mathcal{Z}$ and any $k\in\{0,\dots,K\}$. We denote by $(\tilde V_z^k)^{-1}$ its inverse.

Let

$$\bar v^0:=\inf_{z\in\mathcal{Z}}\tilde V_z^0(e_{\max}),\qquad \bar v^k:=\inf_{z\in\mathcal{Z},\,T(z)=k}\tilde V_z^k(e_{\max}).\tag{7}$$

We recall from [19] the definition of the homogenized velocities $\bar V^k$ and homogenized Hamiltonians: $\bar V^0$ is the inverse of the continuous increasing map defined on $(0,\bar v^0)$ by $v\to\mathbb{E}\Big[(\tilde V_{Z_0}^0)^{-1}(v)\Big]$. We note that $\bar V^0$ is defined on $(\Delta_{\min},\mathbb{E}\Big[(\tilde V_{Z_0}^0)^{-1}(\bar v^0)\Big])$. We extend it for any $e\in[0,\Delta_{\min}]$ by $\bar V^0(e)=0$ and for $e\geqslant\mathbb{E}\Big[(\tilde V_{Z_0}^0)^{-1}(\bar v^0)\Big]$ by $\bar V^0(e)=\bar v^0$. In the same way we define $\bar V^k$ as the inverse of the continuous increasing map defined on $(0,\bar v^k)$ by $v\to\mathbb{E}\Big[(\tilde V_{Z_0}^k)^{-1}(v)\mid T_0=k\Big]$. It defines $\bar V^k$ on $(\Delta_{\min},\mathbb{E}\Big[(\tilde V_{Z_0}^k)^{-1}(\bar v^0)\mid T_0=k\Big])$.

We extend it for any $e \in [0, \Delta_{\min}]$ by $\bar{V}^k(e) = 0$ and for any $e \geqslant \mathbb{E}\left[(\tilde{V}^k_{Z_0})^{-1}(\bar{v}^0) \mid T_0 = k\right]$ by $\bar{V}^k(e) = \bar{v}^k$. The maps $\bar{V}^k$ (for $k \in \{0, \dots, K\}$) are continuous and bounded on $[0, +\infty)$.

We set, for any $k \in \{1, \dots K\}$,

$$H^0(p) = p\bar{V}^0(-1/p),\ H^k(p) = p\bar{V}^k(-1/(\pi^k p)),\ p \in (-\infty, 0), \qquad H^0(p) = H^k(p) = 0,\ \forall p \geqslant 0$$

and

$$A_0 = \max_{k \in \{0,\dots,K\}} \min_{p \in \mathbb{R}} H^k(p). \tag{8}$$

By Assumption $(H4)$, for $i \in \{0, \dots, K\}$, H^k is convex in $(-1/(\pi^k \Delta_{\min}), 0)$ (see Lemma A.4).

A last set of notation will be needed in order to define the condition at the junction: for $k \in \{0, \dots, K\}$, we denote by $H^{k,+}$ (resp. $H^{k,-}$) the largest nondecreasing (resp. nonincreasing) map below H^k.

1.4 The main result

The main result of the paper states that the system homogenizes: let $(U_i^{0,\epsilon})_{i \in \mathbb{Z}}$ be a deterministic family of initial conditions satisfying the compatibility condition: for any $i \in \mathbb{Z}$,

$$U^{0,\epsilon}_{i+1} \geqslant U^{0,\epsilon}_i + \Delta_{\min} \text{ if } U^{0,\epsilon}_{i+1} \leqslant -R_2 \qquad \text{and} \qquad U^{0,\epsilon}_{\ell_i} \geqslant U^{0,\epsilon}_i + \Delta_{\min} \text{ for any } i \in \mathbb{Z}. \tag{9}$$

Up to relabel the indices, we also assume that $U^\epsilon_{i,0} \leqslant 0$ iff $i \leqslant 0$. Let U^ϵ be the solution of (6) with initial condition $(U_i^{0,\epsilon})_{i \in \mathbb{Z}}$. Let us define, for $k \in \{1, \dots, K\}$ and $(x, t) \in \mathbb{R} \times [0, +\infty)$,

$$N^{\epsilon,\omega}(x, k, t) = \sum_{i \in \mathbb{Z},\ i \leqslant 0,\ T_i = k} \delta_{U^\epsilon_i(t)}((x, +\infty)) - \sum_{i \in \mathbb{Z},\ i > 0,\ T_i = k} \delta_{U^\epsilon_i(t)}((-\infty, x]). \tag{10}$$

and set for $x \leqslant 0$

$$N^{\epsilon,\omega}(x, 0, t) = \sum_{i \in \mathbb{Z},\ i \leqslant 0} \delta_{U^\epsilon_i(t)}((x, +\infty)).$$

Then we introduce the scaled quantities

$$\nu^{\epsilon,\omega}(x, k, t) = \begin{cases} \epsilon(\pi^k)^{-1} N^{\epsilon,\omega}(x/\epsilon, k, t/\epsilon) & \forall (x, k, t) \in \mathbb{R} \times \{1, \dots, K\} \times [0, +\infty) \\ \epsilon N^{\epsilon,\omega}(x/\epsilon, 0, t/\epsilon) & \forall (x, t) \in (-\infty, 0] \times [0, +\infty) \end{cases} \tag{11}$$

Theorem 1.1. *There is a set Ω_0 of full probability and a constant $\bar{A} < 0$ (the flux limiter) such that, if $(U_i^{0,\epsilon})_{i \in \mathbb{Z}}$ is a family of initial conditions such that the associated scaled function $\nu^\epsilon(\cdot, \cdot, 0)$ defined by (11) (with $t = 0$) converges locally uniformly in $\mathcal{R}$ to a Lipschitz continuous map $\nu_0 : \mathcal{R} \to \mathbb{R}$, then, for any $\omega \in \Omega_0$, ν^ϵ converges locally uniformly in $\mathcal{R} \times [0, +\infty)$ to the unique continuous viscosity solution of the Hamilton-Jacobi equation with flux limiter $\bar{A}$:*

$$\begin{cases} \partial_t \nu + H(\partial_x \nu) = 0 \qquad \text{in } \overset{o}{\mathcal{R}} \times (0, +\infty) \\ \partial_t \nu + \max\{\bar{A}, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} = 0 \text{ at } x = 0 \\ \nu(x, k, 0) = \nu_0(x, k) \qquad \text{in } \mathcal{R}. \end{cases} \tag{12}$$

Let us recall the notion of viscosity solution of (12). For this we define the set of test functions $C^1(\mathcal{R} \times (0, +\infty))$ as the set of continuous maps $\phi : \mathcal{R} \times (0, +\infty) \to \mathbb{R}$ such that the restriction to each branch of $\mathcal{R}$ is of class C^1 on this branch and $\partial_t \phi$ exists and is continuous everywhere. We denote by $\partial_k \phi(0, t)$ its derivative at $x = 0$ on the branch k (namely, $\partial_k \phi(0, t) = \partial_x \phi(0, k, t)$, which is well-defined by continuity).

We say that a map ν is a viscosity solution of (12) if $\nu : \mathcal{R} \times [0, +\infty) \to \mathbb{R}$ is uniformly continuous, and if, for any test function $\phi \in C^1(\mathcal{R} \times (0, +\infty))$ such that $\nu - \phi$ has a local maximum (respectively minimum) at $(\bar{x}, \bar{k}, \bar{t}) \in \mathcal{R} \times (0, +\infty)$ one has

$$\partial_t \phi(\bar{x}, \bar{k}, \bar{t}) + H^{\bar{k}}(\partial_x \phi(\bar{x}, \bar{k}, \bar{t})) \leqslant 0 \qquad \text{if } \bar{x} \neq 0 \qquad (\text{resp. } \geqslant 0)$$

$$\partial_t \phi(0, \bar{t}) + \max\{\bar{A}, H^{0,+}(\partial_0 \phi(0, \bar{t})), H^{1,-}(\partial_1 \phi(0, \bar{t})), \dots, H^{K,-}(\partial_K \phi(0, \bar{t}))\} \leqslant 0 \quad \text{if } \bar{x} = 0 \qquad (\text{resp. } \geqslant 0)$$

2 Properties of the solution

In this section we investigate two important properties of the solution: the distance to the next vehicle and the finite speed of propagation.

Throughout the paper we need to define the solution (U_i) of (6) for a finite number of indices, namely for $i \in \{i_1, \dots, i_0\}$ where $i_0, i_1 \in \mathbb{Z}$, $i_1 < i_0$. We say that $(U_i)_{i\in\{i_1,\dots,i_0\}}$ is a subsolution (respectively a supersolution) of (6) on a time interval $[0,T]$ if, for any $i \in \{i_1, \dots, i_0\}$, the map $t \to U_i(t)$ is nondecreasing, Lipschitz continuous with a Lipschitz constant not larger than $\|V\|_\infty$ and if, for any $i \in \{i_1, \dots, i_0\}$ with $\ell_i \leqslant i_0$,

$$\frac{d}{dt}U_i^\omega(t) \leqslant V_{Z_i^\omega}(U_{i+1}^\omega(t) - U_i^\omega(t), U_{\ell_i^\omega}^\omega(t) - U_i^\omega(t), U_i^\omega(t)) \qquad \forall t \in [0,T]$$

(resp.

$$\frac{d}{dt}U_i^\omega(t) \geqslant V_{Z_i^\omega}(U_{i+1}^\omega(t) - U_i^\omega(t), U_{\ell_i^\omega}^\omega(t) - U_i^\omega(t), U_i^\omega(t)) \qquad \forall t \in [0,T].)$$

2.1 Basic properties

Given an initial condition $(U_i^0)_{i\in\mathbb{Z}}$ satisfying the compatibility condition (9), there exists a unique solution $U = (U_i)_{i\in\mathbb{Z}}$ to (6). Moreover we have the following basic comparison principle: if $(U_i)_{i\in\mathbb{Z}}$ and $(\tilde{U}_i)_{i\in\mathbb{Z}}$ are two solutions of (6) such that $U_i(0) \leqslant \tilde{U}_i(0)$ for any $i \in \mathbb{Z}$, then $U_i(t) \leqslant \tilde{U}_i(t)$ for any $i \in \mathbb{Z}$ and any $t \geqslant 0$. These results are standard and are easy consequences of Lemma 2.6 below.

Lemma 2.1 (Basic ordering)**.** *Let $i_0, i_1 \in \mathbb{Z}$ with $i_1 < i_0$, $T > 0$ and $(U_i)_{i\in\{i_1,\dots,i_0\}}$ be a solution of (6) on the time interval $[0,T]$ with $(U_i(0))$ satisfying the compatibility condition (9). We have $U_i(t) \leqslant U_j(t) - \Delta_{\min}$ for any $i_1 \leqslant i < j \leqslant i_0$ such that $\ell_i \leqslant i_0$ and $\ell_j \leqslant i_0$ and any $t \in [0,T]$ with $U_j(t) \leqslant -R_2$ or $U_i(t) \leqslant -R_2$. In addition, for any $i \in \{i_1, \dots, i_0\}$ with $\ell_i \leqslant i_0$ and any $t \in [0,T]$, $U_i(t) \leqslant U_{\ell_i}(t) - \Delta_{\min}$.*

Recall that R_2 and $\Delta_{\min}$ are defined in Assumption (H_2). Note that, after the junction, the order is not necessarily preserved among the vehicles if they have not the same type.

Remark 2.2. We note for later use that, if $U_j(t) \leqslant -R_2$ and $i < j$, then $U_i(t) \leqslant U_j(t) - (j-i)\Delta_{\min}$.

Proof. To prove the first claim, it is enough to check that $U_{i-1}(t) \leqslant U_i(t) - \Delta_{\min}$ for $t \in [0,T]$ if $U_{i-1}(t) \leqslant -R_2$ or $U_i(t) \leqslant -R_2$. We start with the first case. Assume by contradiction that there exists $i \in \{i_1 + 1, \dots, i_0\}$ and a time $s \in [0,T]$ such that $\delta := \Delta_{\min} - (U_i(s) - U_{i-1}(s)) > 0$ and $U_{i-1}(s) \leqslant -R_2$. Let $\tau > 0$ be the largest time such that $U_i(t) - U_{i-1}(t) \geqslant \Delta_{\min} - \delta$ on $[0,\tau)$. Note that $\tau \leqslant s$, $U_i(\tau) - U_{i-1}(\tau) = \Delta_{\min} - \delta$ and $U_{i-1}(\tau) \leqslant -R_2$. Then, for $\epsilon > 0$ small enough, the map $t \to U_i(t) - U_{i-1}(t) + \epsilon/(\tau - t)$ has a minimum on $[0,\tau)$, which is less than $\Delta_{\min}$ and reached at a time $\bar{t} \in (0,\tau)$. By optimality condition we have

$$\begin{aligned} &V_{Z_i}(U_{i+1}(\bar{t}) - U_i(\bar{t}), U_{\ell_i}(\bar{t}) - U_i(\bar{t}), U_i(\bar{t})) \\ &\quad - V_{Z_{i-1}}(U_i(\bar{t}) - U_{i-1}(\bar{t}), U_{\ell_{i-1}}(\bar{t}) - U_{i-1}(\bar{t}), U_{i-1}(\bar{t})) + \epsilon/(\tau - \bar{t})^2 = 0, \end{aligned}$$

where by Assumption (H_2) the second term vanishes because $U_{i-1}(\bar{t}) \leqslant -R_2$ and $U_i(\bar{t}) - U_{i-1}(\bar{t}) \leqslant \Delta_{\min}$. So there is a contradiction and we have proved that $U_{i-1}(t) \leqslant U_i(t)$ if $U_{i-1}(t) \leqslant -R_2$.

Let us now check that $U_{i-1}(t) + \Delta_{\min} \leqslant U_i(t)$ if $U_i(t) \leqslant -R_2$. Let τ be the first time (if any) such that $U_{i-1}(\tau) = -R_2$. We have just proved that $U_{i-1}(s) + \Delta_{\min} \leqslant U_i(s)$ if $s \in [0,\tau]$. Thus $U_i(\tau) > -R_2$, which implies that $t < \tau$ and proves the claim.

The proof of the second statement is analogous: if there is $i \in \{i_1, \dots, i_0\}$ with $\ell_i \leqslant i_0$, a time $s \in [0,T]$ and $\delta > 0$ such that $U_i(s) > U_{\ell_i}(s) - \Delta_{\min} + \delta$, then we look at the largest time τ such that $U_i(t) < U_{\ell_i}(t) - \Delta_{\min} + \delta$ on $[0,\tau]$. We have $U_i(\tau) = U_{\ell_i}(\tau) - \Delta_{\min} + \delta$ and, by the previous step, we

know that $U_i(\tau) > -R_2$. Let $\bar{t} \in (0,\tau)$ be a minimum point of the map $t \to U_{\ell_i}(t) - U_i(t) + \epsilon/(\tau - t)$ on $[0,\tau)$. For $\epsilon > 0$ small enough, this minimum is less than $\Delta_{\min}$. Then by optimality we have

$$\begin{aligned} &V_{Z_{\ell_i}}(U_{\ell_i+1}(\bar{t}) - U_{\ell_i}(\bar{t}), U_{\ell_{\ell_i}}(\bar{t}) - U_{\ell_i}(\bar{t}), U_{\ell_i}(\bar{t})) \\ &\quad - V_{Z_i}(U_{i+1}(\bar{t}) - U_i(\bar{t}), U_{\ell_i}(\bar{t}) - U_i(\bar{t}), U_i(\bar{t})) + \epsilon/(\tau - \bar{t})^2 = 0, \end{aligned}$$

where the second term vanishes by assumption (H2) because $U_i(\bar{t}) \geqslant -R_2 > -R_1$ and $U_{\ell_i}(\bar{t}) - U_i(\bar{t}) \leqslant \Delta_{\min}$. So there is a contradiction and we have proved that $U_i(t) \leqslant U_{\ell_i}(t) - \Delta_{\min}$. □

2.2 The maximal distance to the next vehicle

In this part we investigate the maximal distance to the next vehicle. This question will play an important role in the proof of Theorem 3.4. Here we work in a deterministic setting: we fix a deterministic sequence $(z_i)_{i\in\mathbb{Z}}$ in $\mathcal{Z}$ such that $\ell_i := \inf\{j \geqslant i+1,\ T(z_j) = T(z_i)\}$ is finite for any $i \in \mathbb{Z}$. We consider the system of ODEs

$$\frac{d}{dt}U_i(t) = V_{z_i}(U_{i+1}(t) - U_i(t), U_{\ell_i}(t) - U_i(t), U_i(t)) \qquad i \in \mathbb{Z},\ t \geqslant 0, \tag{13}$$

Let us introduce some notation for the slow vehicles. Recall that

$$\bar{v}^0 := \inf_{z\in\mathcal{Z}} \tilde{V}^0_z(e_{\max}), \qquad \bar{v}^k := \inf_{z\in\mathcal{Z},\ T(z)=k} \tilde{V}^k_z(e_{\max})$$

and let $z^k_{\min}$ (where $k \in \{0,\dots,K\}$) be elements of $\mathcal{Z}$ such that $\tilde{V}^0_{z^0_{\min}}(e_{\max}) = \bar{v}^0$ and, for $k \geqslant 1$, $T(z^k_{\min}) = k$ and $\tilde{V}^k_{z^k_{\min}}(e_{\max}) = \bar{v}^k$. The types $z^k_{\min}$ correspond to "slow vehicles", in the sense that their maximal velocity is the smallest.

Lemma 2.3. *Let $\bar{t} > 0$, $i_1 \in \mathbb{Z}$. Assume that $T(z_{i_1}) = z^0_{\min}$ and that $(U_i)_{i\in\mathbb{Z}}$ is a solution to (13) on $[0,\bar{t}]$ with an initial condition satisfying the compatibility condition (9). If $U_{i_1}(\bar{t}) \leqslant -R_0$, then for any $i \leqslant i_1$ we have*

$$U_{i_1}(\bar{t}) \leqslant U_i(\bar{t}) + U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i).$$

Proof. We prove by induction on $m \in \mathbb{N}$ that $U_{i_1}(\bar{t}) \leqslant U_{i_1-m}(\bar{t}) + U_{i_1}(0) - U_{i_1-m}(0) + e_{\max}m$. The result is obvious for $m = 0$. Let us assume that it holds for some $m-1$ with $m \geqslant 1$. We set $i := i_1 - m$. We argue by contradiction and assume that there exists $t \in [0,\bar{t}]$ such that $U_{i_1}(t) - U_i(t) > U_{i_1}(0) - U_i(0) + e_{\max}m$. Then for $\epsilon > 0$ small, the maximum of $t \to U_{i_1}(t) - U_i(t) - \epsilon/(\bar{t} - t)$ exists on $[0,\bar{t})$ and is larger than $U_{i_1}(0) - U_i(0) + e_{\max}m$. We denote by t_ϵ the point of maximum and we remark that t_ϵ is positive by definition. By optimality condition, we have

$$\tilde{V}^0_{z_{i_1}}(U_{i_1+1}(t_\epsilon) - U_{i_1}(t_\epsilon)) - \tilde{V}^0_{z_i}(U_{i+1}(t_\epsilon) - U_i(t_\epsilon)) - \epsilon/(\bar{t} - t_\epsilon)^2 = 0. \tag{14}$$

On the other hand, by induction assumption, we have $U_{i_1}(t_\epsilon) - U_{i+1}(t_\epsilon) \leqslant U_{i_1}(0) - U_{i+1}(0) + e_{\max}(m-1)$. So

$$\begin{aligned} U_{i+1}(t_\epsilon) - U_i(t_\epsilon) &= U_{i_1}(t_\epsilon) - U_i(t_\epsilon) - (U_{i_1}(t_\epsilon) - U_{i+1}(t_\epsilon)) \\ &> U_{i_1}(0) - U_i(0) + e_{\max}m - (U_{i_1}(0) - U_{i+1}(0) + e_{\max}(m-1)) \geqslant e_{\max}, \end{aligned}$$

so that, by assumption (H2)

$$\tilde{V}^0_{z_i}(U_{i+1}(t_\epsilon) - U_i(t_\epsilon)) = \tilde{V}^0_{z_i}(e_{\max})$$

while, as $z_{i_1} = z^0_{\min}$,

$$\tilde{V}^0_{z_{i_1}}(U_{i_1+1}(t_\epsilon) - U_{i_1}(t_\epsilon)) \leqslant \tilde{V}^0_{z_{i_1}}(e_{\max}) = \bar{v}^0 \leqslant \tilde{V}^0_{z_i}(e_{\max}).$$

This contradicts (14) and proves the result. □

Lemma 2.4. *Let $(U_i)_{i\in\mathbb{Z}}$ be a solution of (13) with an initial condition satisfying the compatibility condition (9). Let $i, i_0 \in \mathbb{Z}$ and $k \in \{1,\dots,K\}$ with $i_0 > i$, $T(z_i) = T(z_{i_0}) = k$ and $z_{i_0} = z^k_{\min}$. If $U_i(s) \geqslant 0$ for some $s \geqslant 0$, then*

$$U_{i_0}(t) \leqslant U_i(t) + U_{i_0}(s) - U_i(s) + e_{\max}\sharp\{j \in \{i+1,\dots,i_0\},\ T(z_j) = T(z_i)\} \qquad \forall t \geqslant s.$$

Proof. We proceed by induction on the value of $n := \sharp\{j \in \{i+1, \dots, i_0\},\ T(z_j) = T(z_i)\}$. Let us fist assume that $n = 1$. Then $i_0 = \ell_i$. We argue by contradiction and assume that there exists $t \geqslant s$ such that $U_{\ell_i}(t) - U_i(t) > e_{\max} + U_{i_0}(s) - U_i(s)$. Then for $\epsilon > 0$ small, the maximum t_ϵ of $t \to U_{\ell_i}(t) - U_i(t) - \epsilon t^2$ exists on $[s, +\infty)$ and is larger than $e_{\max} + U_{\ell_i}(s) - U_i(s)$. Hence t_ϵ is larger than s. By optimality condition, we have (setting $k = T(z_i)$)

$$\tilde{V}^k_{z_{\ell_i}}(U_{\ell_{\ell_i}}(t_\epsilon) - U_{\ell_i}(t_\epsilon)) - \tilde{V}^k_{z_i}(U_{\ell_i}(t_\epsilon) - U_i(t_\epsilon)) - 2\epsilon t_\epsilon = 0. \tag{15}$$

As $U_{\ell_i}(t_\epsilon) - U_i(t_\epsilon) \geqslant e_{\max}$, we get, by the definition of $i_0 = \ell_i$:

$$\tilde{V}^k_{z_i}(U_{\ell_i}(t_\epsilon) - U_i(t_\epsilon)) = \tilde{V}^k_{z_i}(e_{\max}) = -2\epsilon t_\epsilon + \tilde{V}^k_{z^k_{\min}}(U_{\ell_{\ell_i}}(t_\epsilon) - U_{\ell_i}(t_\epsilon)) < \bar{v}^k,$$

which contradicts the definition of $\bar{v}^k$. So the result holds for $n = 1$.

Let us now assume that the result holds for $n-1$ (where $n \geqslant 2$) and let us prove it for n. We argue by contradiction in the same way and suppose that there exists $t \geqslant s$ such that $U_{i_0}(t) - U_i(t) > U_{i_0}(s) - U_i(s) + ne_{\max}$. As above, for $\epsilon > 0$ small, the maximum t_ϵ of $t \to U_{i_0}(t) - U_i(t) - \epsilon t^2$ exists on $[s, +\infty)$ and is larger than $U_{i_0}(s) - U_i(s) + ne_{\max}$. By the induction assumption we have

$$U_{i_0}(\tau) - U_{\ell_i}(\tau) \leqslant U_{i_0}(s) - U_{\ell_i}(s) + (n-1)e_{\max} \qquad \forall \tau \geqslant s.$$

Hence

$$\begin{aligned} U_{\ell_i}(t_\epsilon) - U_i(t_\epsilon) &= U_{i_0}(t_\epsilon) - U_i(t_\epsilon) - (U_{i_0}(t_\epsilon) - U_{\ell_i}(t_\epsilon)) \\ &\geqslant U_{i_0}(s) - U_i(s) + ne_{\max} - (U_{i_0}(s) - U_{\ell_i}(s) + (n-1)e_{\max}) \geqslant e_{\max}. \end{aligned}$$

Moreover, by optimality condition, we have

$$\tilde{V}^k_{z_{i_0}}(U_{\ell_{i_0}}(t_\epsilon) - U_{i_0}(t_\epsilon)) - \tilde{V}^k_{z_i}(U_{\ell_i}(t_\epsilon) - U_i(t_\epsilon)) - 2\epsilon t_\epsilon = 0,$$

which leads to a contradiction as above. □

Lemma 2.5. *Let $(U_i)_{i\in\mathbb{Z}}$ be a solution of (13) with an initial condition satisfying the compatibility condition (9). Assume in addition that there exists $\delta > 0$ such that $(d/dt)U_i(t) \geqslant \delta$ for all $t \geqslant 0$ and all $i \in \mathbb{Z}$. Then, there exists a constant C_0 depending on V and δ only such that, for any $i, i_0 \in \mathbb{Z}$ and $k \in \{1, \dots, K\}$ with $i_0 > i$, $T(z_i) = T(z_{i_0}) = k$ and $z_{i_0} = z^k_{\min}$, we have*

$$U_{i_0}(t) \leqslant U_i(t) + C_0(U_i(0))_- + U_{i_0}(0) - U_i(0) + C_0\sharp\{j \in \{i+1, \dots, i_0\},\ T(z_j) = k\} \qquad \forall t \geqslant 0.$$

If in addition there exists $i_1 \geqslant i_0$ such that $T(z_{i_1}) = z^0_{\min}$ and $U_{i_1}(0) \leqslant -R_0$, then

$$U_{i_0}(t) \leqslant U_i(t) + C_0(1 + U_{i_1}(0) - U_i(0) + i_1 - i) \qquad \forall t \geqslant 0.$$

Proof. If $U_i(0) \geqslant 0$, the first result holds by Lemma 2.4. Let us now assume that $U_i(0) < 0$. We set $n := \sharp\{j \in \{i+1, \dots, i_0\},\ T(z_j) = k\}$. Let $t_i = \inf\{t \geqslant 0,\ U_i(t) = 0\}$. Then, as $(d/dt)U_i(t) \geqslant \delta$, we have $t_i \leqslant \delta^{-1}(-U_i(0))$. By Lemma 2.4, we have

$$U_{i_0}(t) - U_i(t) \leqslant U_{i_0}(t_i) - U_i(t_i) + e_{\max}n \qquad \forall t \geqslant t_i.$$

On the other hand, for $t \in [0, t_i]$, we get

$$U_{i_0}(t) - U_i(t) \leqslant U_{i_0}(0) - U_i(0) + \|V\|_\infty t_i \leqslant U_{i_0}(0) - U_i(0) + C(U_i(0))_-.$$

This proves the first part of the claim.

Assume now that $U_{i_1}(0) \leqslant -R_0$. Let $\bar{t} = \inf\{t \geqslant 0,\ U_{i_1}(t) = -R_0\}$. We know from Lemma 2.3 that

$$U_{i_1}(t) \leqslant U_i(t) + U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i) \qquad \forall t \in [0, \bar{t}].$$

This implies by Lemma 2.1 that

$$U_{i_0}(t) \leqslant U_i(t) + U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i) \qquad \forall t \in [0, \bar{t}]. \tag{16}$$

Let $t_i = \inf\{t \geqslant 0,\ U_i(t) = 0\}$. Then, as

$$U_i(\bar{t}) \geqslant U_{i_1}(\bar{t}) - (U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i)) \geqslant -R_0 - (U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i)),$$

while $U_i(t_i) = 0$, we have, since $(d/dt)U_i(t) \geqslant \delta$,

$$\begin{aligned} t_i - \bar{t} &\leqslant \delta^{-1}(U_i(t_i) - U_i(\bar{t})) \leqslant \delta^{-1}(R_0 + U_{i_1}(0) - U_i(0) + e_{\max}(i_1 - i)) \\ &\leqslant C(1 + U_{i_1}(0) - U_i(0) + i_1 - i). \end{aligned}$$

Note that $U_{i_0}(\bar{t}) \leqslant U_{i_1}(\bar{t}) = -R_0 \leqslant 0$. So, as $U_i(t_i) = 0$,

$$U_{i_0}(t_i) - U_i(t_i) \leqslant U_{i_0}(t_i) - U_{i_0}(\bar{t}) \leqslant \|V\|_\infty (t_i - \bar{t}) \leqslant C(1 + U_{i_1}(0) - U_i(0) + i_1 - i).$$

By Lemma 2.4, we obtain

$$U_{i_0}(t) - U_i(t) \leqslant U_{i_0}(t_i) - U_i(t_i) + e_{\max}(i_0 - i) \leqslant C(1 + U_{i_1}(0) - U_i(0) + i_1 - i) \qquad \forall t \geqslant t_i. \tag{17}$$

Finally, for $t \in [\bar{t}, t_i]$, we have

$$U_{i_0}(t) - U_i(t) \leqslant U_{i_0}(t_i) - U_i(t_i) + \|V\|_\infty (t_i - t) \leqslant C(1 + U_{i_1}(0) - U_i(0) + i_1 - i). \tag{18}$$

Combining (16), (17) and (18) proves the second part of the claim. □

2.3 Approximate speed of propagation

The approximate speed of propagation says that the behavior of a vehicle mostly depends on a finite number of vehicles in front of it. To describe this result we need to introduce a few notation. Given $T \in \mathbb{Z}$, we define by induction

$$J_0^\omega(T) = T, \qquad J_n^\omega(T) = \inf_{k \in \{1,\dots,K\}} \sup\Big\{ i \in \mathbb{Z},\ T_i^\omega = k,\ \ell_i^\omega \leqslant J_{n-1}^\omega(T) \Big\}.$$

We note that the $J_n(T)$ are random and decreasing in n. By construction, if $i \leqslant J_n(T)$, then $i + 1 \leqslant J_{n-1}(T)$ and $\ell_i \leqslant J_{n-1}(T)$.

Lemma 2.6 (Approximate finite speed of propagation on the junction)**.** *Fix $i_0 \in \mathbb{Z}$, $L \in \mathbb{N}$, $T \geqslant 0$ and $E \in \mathcal{F}$ an event with a positive probability. Assume that, in E, $(U_i)_{i \in \{i_0,\dots,i_0+L\}}$ is a (non decreasing) subsolution while $(\tilde{U}_i)_{i \in \{i_0,\dots,i_0+L\}}$ is a supersolution of the system*

$$\frac{d}{dt} U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t), U_{\ell_i}(t) - U_i(t), U_i(t))$$

for $i = i_0, \dots, J_1(i_0 + L)$ and $t \in [0, T]$. Suppose in addition that, in E, $U_i(0) \leqslant \tilde{U}_i(0)$ for $i \in \{i_0, \dots, i_0 + L\}$. Then, for all $n \in \mathbb{Z}$, $n \geqslant 1$, for all $\omega \in E$ and for all $i \in \{i_0, \dots, J_n(i_0 + L)\}$,

$$U_i(t) \leqslant \tilde{U}_i(t) + C\ 2^{-n} e^{\beta t} \qquad \forall t \in [0, T],$$

where, $\beta = \gamma + 2C_1$, with $\gamma := \sup_{z \in \mathcal{Z}} (\|\partial_1 V_z\|_\infty + \|\partial_2 V_z\|_\infty)$ and $C_1 := \sup_{z \in \mathcal{Z}} \|\partial_x V_z\|_\infty$, and where C depend on β and on $\|V\|_\infty$ only.

Proof. We work in E all along the proof. Let us set $W_i(t) = \tilde{U}_i(t) - U_i(t)$ for $i = i_0, \dots, J_1(i_0 + L)$. Since we work with sub and super-solution, we extend the velocity $V_z(p, x)$ by 0 if $p \leqslant 0$. Let n_0 be the largest integer such that $J_{n_0}(i_0 + L) \geqslant i_0$. We define, for $n \in \{1, \dots, n_0\}$,

$$M_n = \sup_{i \in \{i_0,\dots,J_n(i_0+L)\}} \sup_{s \in [0,T]} e^{-\beta s} [W_i(s)]_-.$$

Note that, as $W_i(0) \geqslant 0$ and $(d/dt)W_i(s) \leqslant \|V\|_\infty$, we have

$$M_1 \leqslant \sup_{i\in\{i_0,\dots,i_0+L\}} \sup_{s\in[0,T]} e^{-\beta s}\|V\|_\infty s \leqslant C,$$

where C depends on β and $\|V\|_\infty$ only. The main step consists in showing that, for all $n \in \{2,\dots,n_0\}$,

$$M_n \leqslant \frac{1}{2}M_{n-1}. \tag{19}$$

Fix $n \in \{2,\dots,n_0\}$ and $i \in \{i_0,\dots,J_n(i_0+L)\}$. We have, for $t \in [0,T]$,

$$\frac{d}{dt}W_i(t) \geqslant V_{Z_i}(\tilde U_{i+1}(t)-\tilde U_i(t),\tilde U_{\ell_i}(t)-\tilde U_i(t),\tilde U_i(t)) - V_{Z_i}(U_{i+1}(t)-U_i(t),U_{\ell_i}(t)-U_i(t),U_i(t)).$$

So

$$\frac{d}{dt}W_i(t) \geqslant A_{i,1}(t)(W_{i+1}(t)-W_i(t)) + A_{i,2}(t)(W_{\ell_i}(t)-W_i(t)) + B_i(t)W_i(t)$$

where

$$A_{i,1}(t) := \int_0^1 \partial_1 V_{Z_i}(w_i(\tau))d\tau,\ A_{i,2}(t) := \int_0^1 \partial_2 V_{Z_i}(w_i(\tau))d\tau,\ B_i(t) := \int_0^1 \partial_3 V_{Z_i}(w_i(t))d\tau,$$

with $w_i(t) = (w_{i,1}(\tau), w_{2,i}(t), w_{3,i}(t))$,

$$w_{i,1}(\tau) = (1-\tau)(\tilde U_{i+1}(t)-\tilde U_i(t)) + \tau(U_{i+1}(t)-U_i(t)),$$
$$w_{2,i}(t) = (1-\tau)(\tilde U_{\ell_i}(t)-\tilde U_i(t)) + \tau(U_{\ell_i}(t)-U_i(t)) \text{ and } w_{3,i}(t) = (1-\tau)\tilde U_i(t) + \tau U_i(t).$$

We note for later use that $0 \leqslant A_{i,1} \leqslant \gamma$, $0 \leqslant A_{i,2} \leqslant \gamma$ and $|B_i| \leqslant C_1$. Setting $A_i = A_{i,1} + A_{i,2}$, we find

$$\begin{aligned} W_i(t) \geqslant & W_i(0)\exp\left\{-\int_0^t (A_i-B_i)(s)ds\right\} + \int_0^t A_{i,1}(s)\exp\left\{-\int_s^t (A_i-B_i)(\tau)d\tau\right\} W_{i+1}(s)ds \\ & + \int_0^t A_{i,2}(s)\exp\left\{-\int_s^t (A_i-B_i)(\tau)d\tau\right\} W_{\ell_i}(s)ds. \end{aligned}$$

As $W_i(0) \geqslant 0$ and $A_i \geqslant 0$, we infer that

$$\begin{aligned} [W_i(t)]_- \leqslant & \int_0^t A_{i,1}(s)\exp\left\{-\int_s^t (A_i-B_i)(\tau)d\tau\right\} [W_{i+1}(s)]_- ds \\ & + \int_0^t A_{i,2}(s)\exp\left\{-\int_s^t (A_i-B_i)(\tau)d\tau\right\} [W_{\ell_i}(s)]_- ds \\ \leqslant & \int_0^t A_i(s)\exp\left\{-\int_s^t (A_i-B_i)(\tau)d\tau + \beta s\right\} e^{-\beta s}([W_{i+1}(s)]_- \vee [W_{\ell_i}(s)]_-)ds \\ \leqslant & \sup_{s\in[0,t]} (e^{-\beta s}([W_{i+1}(s)]_- \vee [W_{\ell_i}(s)]_-) \int_0^t A_i(s)\exp\left\{-\int_s^t (A_i-|B_i|)(\tau)d\tau + \beta s\right\} ds. \end{aligned}$$

We now estimate the last term in the inequality above. After an integration by part, we have, since $\frac{\beta}{2} \geqslant C_1 \geqslant |B_i|$ and $0 \leqslant A_i \leqslant \gamma$,

$$\begin{aligned} & \int_0^t A_i(s)\exp\left\{-\int_s^t (A_i-|B_i|)(\tau)d\tau + \beta s\right\} ds \\ & = \int_0^t A_i(s)\exp\left\{-\int_s^t A_i(\tau)d\tau\right\}\exp\left\{\int_s^t |B_i|(\tau)d\tau + \beta s\right\} ds \\ & = \left[\exp\left\{-\int_s^t (A_i-|B_i|)(\tau)d\tau + \beta s\right\}\right]_0^t - \int_0^t \exp\left\{-\int_s^t (A_i-|B_i|)(\tau)d\tau + \beta s\right\}(-|B_i(s)|+\beta)ds \\ & \leqslant e^{\beta t} - \exp\left\{-\int_0^t (A_i-|B_i|)(\tau)d\tau\right\} - \int_0^t \exp\left\{\gamma(s-t)+\beta s\right\}(-C_1+\beta)ds \\ & \leqslant e^{\beta t} - e^{-\gamma t} - e^{-\gamma t}\int_0^t \exp\left\{(\gamma+\beta)s\right\}(-C_1+\beta)ds. \end{aligned}$$

So

$$\int_0^t A_i(s)\exp\left\{-\int_s^t (A_i-|B_i|)(\tau)d\tau+\beta s\right\}ds$$
$$\leqslant e^{\beta t}-e^{-\gamma t}-\frac{\beta-C_1}{\gamma+\beta}(e^{\beta t}-e^{-\gamma t})\leqslant\frac{\gamma+C_1}{\gamma+\beta}e^{\beta t}=\frac{e^{\beta t}}{2},$$

by the choice of $\beta=\gamma+2C_1$. This shows that

$$\sup_{s\in[0,T]}e^{-\beta s}[W_i(s)]_-\leqslant\frac{1}{2}\sup_{s\in[0,T]}e^{-\beta s}([W_{i+1}(s)]_-\vee[W_{\ell_i}(s)]_-).$$

As $i\leqslant J_n(i_0+L)$, $i+1$ and ℓ_i belong to $\{i_0,\dots,J_{n-1}(i_0+L)\}$. So the right-hand side is less that $M_{n-1}/2$. Taking the supremum over all $i\in\{i_0,\dots,J_n(i_0+L)\}$ gives (19).

By induction, we obtain that, for all $n\leqslant n_0$,

$$M_n\leqslant 2^{-(n-1)}M_1\leqslant C2^{-n},$$

from which we derive the result. □

Next we investigate the behavior of J_n for large values of n:

Lemma 2.7. *There exists a constant $\alpha>0$ such that, for any $T\in\mathbb{Z}$, any $\epsilon\in(0,1]$ and any $n\in\mathbb{N}$,*

$$\mathbb{P}\left[|J_n(T)-T+\alpha n|\geqslant\epsilon n\right]\leqslant 2\exp\{-\epsilon^2 n/C\}.$$

Proof. Let $T\in\mathbb{Z}$ and $X_n:=J_n^\omega(T)$, $n\in\mathbb{N}$. Then $(X_{n+1}-X_n)$ is a family of i.i.d. nonpositive random variables with law given by (for all $m\in\mathbb{N}$)

$$\mathbb{P}[X_1-X_0<-m]\leqslant\mathbb{P}\Big[\exists k\in\{1,\dots,K\},\sharp\{i\in\{T-m,\dots,T\},\ T_i=k\}\leqslant 1\ \Big]\leqslant\sum_{k=1}^K m(1-\pi^k)^{m-1}$$
$$\leqslant Km\pi^{m-1}\leqslant C(\frac{1+\pi}{2})^m,$$

where $\pi=\max_k(1-\pi^k)<1$ and C depends on π and K only. Therefore X_1-X_0 satisfies Bernstein's conditions: there exist positive numbers ν and c (depending on π only) such that $\mathbb{E}\left[|X_1-X_0|^2\right]\leqslant\nu$ and $\mathbb{E}\left[|X_1-X_0|^q\right]\leqslant q!\nu c^{q-2}$ for any integer $q\geqslant 3$ (see Subsection A.1 in the Appendix). Let us set $\alpha=\mathbb{E}\left[X_0-X_1\right]>0$. From Bernstein's Inequality (Corollary 2.11 in [14])

$$\mathbb{P}\left[|J_n(T)-T+\alpha n|\geqslant x\right]\leqslant 2\exp\{-x^2/(C(n+x))\}$$

for some constant C depending on π only. This implies the result for $\epsilon\in(0,1]$. □

3 The time function

The goal of Sections 3 is to build the flux limiter at the junction. For doing so we consider the solution of our system starting with a "flat initial condition" and look at the time it takes to reach 0 from a position (far) on the ingoing road.

Let us fix from now on $e=(e^k)_{k=0,\dots K}$ such that $H^k(-1/e^k)=\min_p H^k(p)$. We note for later use that $e^k>\pi^k\Delta_{\min}$, where $\Delta_{\min}$ is defined in Assumption (H2). We define $(U^\omega_{e,i})_{i\in\mathbb{Z}}$ as the solution of (6) with initial condition defined for any $i\in\mathbb{Z}$ by

$$U^\omega_{e,i}=\begin{cases}e^0 i & \text{if } i\leqslant 0\\ e^k i & \text{if } i\geqslant 0 \text{ and } T_i=k.\end{cases}$$

Then we set

$$\theta^\omega_e(t)=\inf\{i\geqslant 0,\ U^\omega_{e,-i}(t)\leqslant 0\}.$$

The quantity $\theta^\omega_e(t)$ is the number of vehicles having gone through 0 at time t. The goal of the section is to show, by using a concentration inequality, that $\theta_e(t)/t$ has a.s. a deterministic limit as $t\to+\infty$.

3.1 Preliminaries

Let us collect some basic facts on the $(U^\omega_{e,i})_{i\in\mathbb{Z}}$ and on θ_e. Let us set $v^k_e = \bar{V}^k(e^k/\pi^k)$ for $k \in \{0,\dots,K\}$ (where $\pi^0 = 1$ by convention).

Lemma 3.1. *We have* $\min_{k\in\{0,\dots,K\}} v^k_e > 0$.

Proof. As $e > \pi\Delta_{\min}$ and $\mathcal{Z}$ is finite, assumption (H_5) implies the existence of $C > 1$ such that $\tilde{V}^0_z(e^0) \geqslant C^{-1}$ for any $z \in \mathcal{Z}$. Thus $(\bar{V}^0)^{-1}(C^{-1}) = \mathbb{E}[(\tilde{V}^0_{Z_0})^{-1}(C^{-1})] \leqslant e^0$, which shows that $v^0_e = \bar{V}^0(e^0) \geqslant C^{-1}$. The proof for v^k_e (for $k = 1,\dots,K$) works in the same way. □

Lemma 3.2. *There exists* $\delta > 0$ *such that*

$$\frac{d}{dt}U_{e,i}(t) \geqslant \delta \qquad \forall t \geqslant 0,\ \forall i \in \mathbb{Z}.$$

Proof. Recall the definition of κ in assumption (H_5). Let us set $e^{\min} = \min_{k=0,\dots,K} e^k$ and

$$\delta := \min\{\kappa\ ,\ \min_{k=0,\dots,K} v^k_e\ ,\ \min_{x\in\mathbb{R},\ z\in\mathcal{Z}} V_z(e^{\min}, e^{\min}, x)\ ,\ \min_{x\in\mathbb{R},\ z\in\mathcal{Z}} V_z(R_1 - R_2 + \Delta_{\min}, R_1 - R_2 + \Delta_{\min}, x)\}.$$

By Lemma 3.1, the fact that $e^{\min} > \Delta_{\min}$ and assumption (H_5-(iii)) (combined with the fact that $\mathcal{Z}$ is finite and that V is independent of x for $x \notin [-R_0, 0]$), we have that $\delta > 0$. Fix $n \in \mathbb{N}$ large (say, $n \geqslant R_0/e^0$) and let U^n be the solution to (6) with initial condition defined by for any $i \in \mathbb{Z}$, $|i| \leqslant n$ by

$$U^n_i(0) = \begin{cases} e^0 i & \text{if } -n \leqslant i \leqslant 0 \\ e^k i & \text{if } 0 \leqslant i \leqslant n \text{ and } k = T_i. \end{cases}$$

If $|i| > n$ we define $U^n_i(0)$, by induction by setting, if $i < -n$,

$$\tilde{V}^0_{Z_{i-1}}(U^n_i(0) - U^n_{i-1}(0)) = v^0_e$$

and, if $\ell_i > n$ and $k = T_i$,

$$\tilde{V}^k_{Z_i}(U^n_{\ell_i}(0) - U^n_i(0)) = v^k_e.$$

Then U^n converges locally uniformly on $\mathbb{Z}\times[0,+\infty)$ to U_e as $n \to +\infty$. We are going to show that the claim holds for U^n, which implies the claim for U_e.

Let us first note that the claim holds for $t = 0$. Indeed, by definition of $U^n_i(0)$, we have, if $-n \leqslant i < 0$,

$$\frac{d}{dt}U^n_i(0) = V_{Z_i}(e^0, U^n_{\ell_i}(0) - U^n_i(0), e^0 i) \geqslant V_{Z_i}(e^{\min}, e^{\min}, ei) \geqslant \delta,$$

while, if $i \geqslant 0$ and $k = T_i$, then

$$\frac{d}{dt}U^n_i(0) = \tilde{V}^k_{Z_i}(U^n_{\ell_i}(0) - U^n_i(0)) = \begin{cases} \tilde{V}^k_{Z_i}((\ell_i - i)e^k) \geqslant \delta & \text{if } \ell_i \leqslant n, \\ v^k_e \geqslant \delta & \text{otherwise.} \end{cases}$$

Finally, if $i < -n$, then

$$\frac{d}{dt}U^n_i(0) = \tilde{V}^0_{Z_i}(U^n_{i+1}(0) - U^n_i(0)) = v^0_e \geqslant \delta.$$

So we have proved that, in any case, $\frac{d}{dt}U^n_i(0) \geqslant \delta$.

By Lipschitz continuity in time of $I(t) := \inf_{i\in\mathbb{Z}} \frac{d}{dt}\ U^n_i(t)$, we have that $I(t) \geqslant \delta/2$ for $t \geqslant 0$ small. Let $[0,T]$ be an interval on which this inequality holds. We are going to show that actually $I(t) \geqslant \delta$ on $[0,T]$, which is enough to prove the claim. For this we argue by contradiction and assume that there is $(\bar{t}, j) \in [0,T]\times\mathbb{Z}$ such that $\frac{d}{dt}U^n_j(\bar{t}) < \delta$. Then, for $\epsilon > 0$ small enough,

$$I := \inf_{t\in[0,\bar{t}),\ i\in\mathbb{Z}} \frac{d}{dt}U^n_i(t) + \frac{\epsilon}{\bar{t} - t}$$

is less than δ. In the next step we show that the infimum is actually a minimum. For this we first note that $\bar{U}_i^n(t) := U_i^n(0) + v_e^0 t$ for $i \leqslant -N$ and $t \in [0,T]$ is a solution to (6) for N large enough depending on T and n. So, by the approximate finite speed of propagation (Lemma 2.6) applied to U_i^n and $\bar{U}_i^n$ for $i \leqslant -N$ and $t \in [0,T]$, we have for any $t \in [0,T]$

$$\lim_{i\to-\infty} U_i^n(t) - U_i^n(0) = tv_e^0,$$

so that

$$\lim_{i\to-\infty} \frac{d}{dt}U_i^n(t) = \lim_{i\to-\infty} \tilde{V}_{Z_i}^0(U_{i+1}^n(t) - U_i^n(t)) = \lim_{i\to-\infty} \tilde{V}_{Z_i}^0(U_{i+1}^n(0) - U_i^n(0)) = v_e^0 \geqslant \delta.$$

On the other hand, by the construction of U^n, we have for $i > n$ and $T_i = k$ that $U_i^n(t) = U_i^n(0) + tv_e^k$, so that

$$\lim_{i\to+\infty,\ T_i=k} \frac{d}{dt}U_i^n(t) = v_e^k \geqslant \delta.$$

This shows that the infimum in the definition of I is a minimum: let (t_0, i_0) be a minimum point such that i_0 is maximal. Recalling that $I(0) \geqslant \delta$, we have $t_0 > 0$. By the optimality of (t_0, i_0) and the maximality of i_0, we have

$$\frac{d}{dt}U_{i_0+1}^n(t_0) > \frac{d}{dt}U_{i_0}^n(t_0) \text{ and } \frac{d}{dt}U_{\ell_{i_0}}^n(t_0) > \frac{d}{dt}U_{i_0}^n(t_0). \tag{20}$$

By optimality of $t_0 > 0$, we also have (omitting the dependence of $V_{Z_{i_0}}$ with respect to its parameters to simplify the notation)

$$0 = \partial_{e_1}V_{Z_{i_0}}\left(\frac{d}{dt}U_{i_0+1}^n(t_0) - \frac{d}{dt}U_{i_0}^n(t_0)\right) + \partial_{e_2}V_{Z_{i_0}}\left(\frac{d}{dt}U_{\ell_{i_0}}^n(t_0) - \frac{d}{dt}U_{i_0}^n(t_0)\right) + \partial_x V_{Z_{i_0}}\frac{d}{dt}U_{i_0}^n(t_0) + \frac{\epsilon}{(\bar{t}-t_0)^2}.$$

By (20), all the terms are nonnegative except perhaps $\partial_x V_{Z_{i_0}}$. By assumption (H_5-(ii)), we cannot have $U_{i_0}^n(t_0) \geqslant -R_1$ since in this case $\partial_x V_{Z_{i_0}} \geqslant 0$. So $U_{i_0}^n(t_0) < -R_1$. Now, if $U_{\ell_{i_0}}^n(t_0) \geqslant U_{i_0+1}^n(t_0)$, then by assumption ($H_5$-(i)), $V_{Z_{i_0}}$ does not depend on x and therefore $\partial_x V_{Z_{i_0}} = 0$. Thus $U_{\ell_{i_0}}^n(t_0) < U_{i_0+1}^n(t_0)$. By Lemma 2.1, with $i = i_0 + 1$ and $j = \ell_{i_0} > i_0 + 1$, this implies that $U_{\ell_{i_0}}^n(t_0) \geqslant -R_2 + \Delta_{\min}$ and, hence, $U_{i_0+1}^n(t_0) \geqslant -R_2 + \Delta_{\min}$. Thus

$$\frac{d}{dt}U_{i_0}^n(t_0) \geqslant V_{Z_{i_0}}(R_1 - R_2 + \Delta_{\min}, R_1 - R_2 + \Delta_{\min}, U_{i_0}^n(t_0)) \geqslant \delta,$$

which is again impossible. □

A immediate consequence of Lemma 3.2 is that $U_{e,i}(t) \to +\infty$ as $t \to +\infty$ and therefore $\theta_e(t) \to +\infty$ as $t \to +\infty$. Next we show a bound from above for θ_e.

Lemma 3.3. *There exists a constant $C_\theta > 0$ such that*

$$0 \leqslant \theta_e(t) - \theta_e(s) \leqslant C_\theta(t - s + 1) \qquad \forall 0 \leqslant s \leqslant t.$$

Moreover, for $t \geqslant 0$,

$$\theta_e(t) \leqslant C_\theta t.$$

Proof. Let us first prove the second statement. For this we note that, for any $i \geqslant \|V\|_\infty t/e^0$, we have

$$U_{e,-i}(t) \leqslant -e^0 i + \|V\|_\infty t \leqslant 0.$$

So $\theta_e(t) \leqslant \|V\|_\infty t/e^0$, which proves the claim.

We now prove the first statement. Let $i_0 := -\theta_e(s)$. Then $U_{e,i_0}(s) \leqslant 0$. Let δ be the constant given by Lemma 3.2. Assume first that $s \geqslant R_2/\delta$. Then, by Lemma 3.2, $U_{e,i_0}(s - R_2/\delta) \leqslant U_{e,i_0}(s) - R_2 \leqslant -R_2$. Recalling Remark 2.2 after Lemma 2.1 we have, for any $i \leqslant i_0 - \|V\|_\infty(t - s + R_2/\delta)/\Delta_{\min}$,

$$U_{e,i}(t) \leqslant U_{e,i}(s - R_2/\delta) + \|V\|_\infty(t - s + R_2/\delta) \leqslant U_{e,i_0}(s - R_2/\delta) - \Delta_{\min}(i_0 - i) + \|V\|_\infty(t - s + R_2/\delta) \leqslant -R_2 < 0.$$

So, for any $i \leqslant i_0 - \|V\|_\infty(t-s+R_2/\delta)/\Delta_{\min}$, we obtain $U_{e,i}(t) < 0$. This shows that

$$\theta_e(t) \leqslant -i_0 + \|V\|_\infty(t-s+R_2/\delta)/\Delta_{\min} \leqslant \theta_e(s) + C(t-s+1).$$

If $t \leqslant R_2/\delta$, the conclusion obviously holds. Finally, if $s \leqslant R_2/\delta$ and $t > R_2/\delta$, then, by the previous inequality and the first part of the proof, we have

$$\begin{aligned}\theta_e(t) \leqslant & \theta_e(R_2/\delta) + C(t - R_2/\delta + 1)\\ \leqslant & \|V\|_\infty R_2/(\delta e^0) + C(t-s+1)\\ \leqslant & \theta_e(s) + \|V\|_\infty R_2/(\delta e^0) + C(t-s+1)\\ \leqslant & \theta_e(s) + C'(t-s+1).\end{aligned}$$

□

3.2 A concentration inequality

In this section, we prove a concentration inequality for

$$\theta_e(t) = \inf\{i \geqslant 0,\ U_{e,-i}(t) \leqslant 0\}.$$

Theorem 3.4. *There is a constant $C > 0$ such that for any $\epsilon \in (0, C^{-1}]$ and any $t \geqslant C\epsilon^{-1}$,*

$$\mathbb{P}\left[\,|\ \theta_e(t) - \mathbb{E}[\theta_e(t)]|\ \geqslant \epsilon t\right] \leqslant C\exp\{-\epsilon^2 t/C\}.$$

The proof requires several steps. The first issue is that $\theta_e(t)$ depends a priori on all the Z_i, even for any $i \geqslant 0$ large. In order to reduce this dependence, we introduce the auxiliary quantity $\theta_e^m(t)$ defined, for any $m \in \mathbb{N}$ large (say $m \geqslant 2$), by

$$\theta_e^m(t) = \inf\{i \geqslant 0,\ U^m_{e,-i}(t) \leqslant 0\}$$

where the $(U^m_{e,i})_{i\in\mathbb{Z}}$ is the solution of (6) where the sequences Z_i and the initial condition $U_e(0)$ are replaced into Z_i^m and $U_e^m(0)$ defined as follows:

$$Z_i^m := \begin{cases} Z_i & \text{if } i \leqslant m-1\\ z^k_{\min} & \text{if } i \geqslant m \text{ and } T_i = k\end{cases}$$

and

$$U^m_{e,i}(0) = \begin{cases} U_{e,i}(0) & \text{if } i \leqslant m-1\\ e^{T_i}(m-1) + e^{T_i}\sharp\{j \in \{m,\dots,i\},\ T_j = T_i\} & \text{if } i \geqslant m.\end{cases}$$

Let us recall that the $z^k_{\min}$ are introduced at the beginning of Subsection 2.2. Before proceeding, let us collect several important properties of the $U^m_{e,i}$.

Lemma 3.5. *1. For each $i \in \mathbb{Z}$ with $i \leqslant m-1$, $U^m_{e,i}$ is $\sigma\{Z_j,\ j \in \{i,\dots,m-1\}\}-$measurable.*

2. There exists $\delta > 0$ such that $(d/dt)U^m_{e,i}(t) \geqslant \delta$ for any $i \in \mathbb{Z}$ and any $t \geqslant 0$.

3. There exists a constant $C > 0$ such that

$$0 \leqslant \theta_e^m(t) - \theta_e^m(s) \leqslant C(t-s+1) \qquad \text{and} \qquad \theta_e^m(t) \leqslant Ct \qquad \forall 0 \leqslant s \leqslant t.$$

4. Setting $\sigma_i := \inf\{j \geqslant i+1,\ Z_j = z^{T_i}_{\min}\}$ and $\sigma_i^0 = \inf\{j \geqslant i+1,\ Z_j = z^0_{\min}\}$, we have

$$U^m_{e,\ell_i}(t) \leqslant U^m_{e,i}(t) + C(1 + \sigma^0_{\sigma_i} \wedge m - i) \qquad \forall i \leqslant m-1,\ \forall t \geqslant 0. \tag{21}$$

Proof. 1) Let us first check that $U^m_{e,i}(t) = U^m_{e,i}(0) + \tilde{V}^k_{z^k_{\min}}(e^k)t$ for any $i \geqslant m$ with $T_i = k$. Indeed, for such any $i \geqslant m$, we have

$$\frac{d}{dt}U^m_{e,i}(t) = \tilde{V}^k_{z^k_{\min}}(e^k)$$

while

$$\begin{aligned} V_{Z^m_i}(U^m_{e,i+1}(t) - U^m_{e,i}(t), U^m_{e,\ell_i}(t) - U^m_{e,i}(t), U^m_{e,i}(t)) &= \tilde{V}^k_{z^k_{\min}}(U^m_{e,\ell_i}(t) - U^m_{e,i}(t)) \\ &= \tilde{V}^k_{z^k_{\min}}(U^m_{e,\ell_i}(0) - U^m_{e,i}(0)) = \tilde{V}^k_{z^k_{\min}}(e^k). \end{aligned}$$

The measurability of $U^m_{e,i}$ for $i \leqslant m-1$ can then be checked by backward induction. For $i = m-1$, $U^m_{e,m-1}$ solves (if we set $T_i = k$),

$$\frac{d}{dt}U^m_{e,m-1}(t) = \tilde{V}^k_{Z_i}(U^m_{\ell_i}(t) - U^m_{e,i}(t)),\ t \geqslant 0, \qquad U^m_{e,m-1}(0) = e^k(m-1).$$

Given Z_i, the above equation has deterministic coefficients since $U^m_{\ell_i}(t) = e^k(m-1) + \tilde{V}^k_{z^k_{\min}}(e^k)t$. For $i \leqslant m-2$, it can be proved by induction in the same way that $U^m_{e,i}$ satisfies an ODE with coefficients which are $\sigma\{Z_j,\ j \in \{i, \dots, m-1\}\}-$measurable.

2) & 3) The existence of $\delta > 0$ such that $(d/dt)U^m_{e,i}(t) \geqslant \delta$ is a consequence of Lemma 3.2 (which is a deterministic statement). In the same way, the estimate on θ^m_e is an application of Lemma 3.3.

4) Let us finally check (21). For this we use Lemma 2.5. Fix $i \leqslant m-1$ and let $k = T_i$, $i_0 := \inf\{j \geqslant i+1,\ Z^m_j = z^k_{\min}\}$ and $i_1 = \inf\{j \geqslant i_0,\ Z^m_j = z^0_{\min}\}$ (if this exists). Note that $i_0 = \sigma_i \wedge (\inf\{j \geqslant m, T_j = k\})$ and $i_1 = \sigma^0_{\sigma_i}$ if $\sigma^0_{\sigma_i} < m$. If $e^0 i_1 \leqslant -R_0$, then by the second part of Lemma 2.5 we have

$$U^m_{e,\ell_i}(t) \leqslant U^m_{e,i_0}(t) \leqslant U^m_{e,i}(t) + C_0(1 + U^m_{i_1}(0) - U^m_i(0) + i_1 - i) \leqslant U^m_{e,i}(t) + C(1 + i_1 - i) \qquad \forall t \geqslant 0.$$

As $i_1 \leqslant 0 < m-1$ we also have $i_1 = \sigma^0_{\sigma_i} < m$, which proves the inequality in this case.

Let us now assume that $e^0 i_1 > -R_0$. According to the first part of Lemma 2.5 we have

$$U^m_{e,\ell_i}(t) \leqslant U^m_{e,i_0}(t) \leqslant U^m_{e,i}(t) + C_0(e^0 i)_- + U^m_{e,i_0}(0) - U^m_{e,i}(0) + C_0 \sharp\{j \in \{i+1, \dots, i_0\},\ T(Z^m_j) = k\}.$$

As $e^0 i_1 > -R_0$, we have

$$(e^0 i)_- \leqslant e^0(i_1 \wedge m - i) + R_0 \leqslant e^0(\sigma^0_{\sigma_i} \wedge m - i) + R_0.$$

On the other hand, by the construction of the $U^m_{e,j}(0)$, we have $U^m_{e,i_0}(0) = (m \wedge \sigma_i)e^k$. Finally, as by definition of the Z^m_j and of i_0 we also have (with $k := T_i$)

$$\sharp\{j \in \{i+1, \dots, i_0\},\ T(Z^m_j) = k\} \leqslant \sharp\{j \in \{i+1, \dots, m-1\},\ T(Z^m_j) = k\} + 1 \leqslant \sigma_i \wedge m - i + 1.$$

This shows that

$$U^m_{e,\ell_i}(t) \leqslant U^m_{e,i}(t) + C(\sigma^0_{\sigma_i} \wedge m - i + 1) \qquad \forall t \geqslant 0.$$

□

We now note that θ_e and θ^m_e are close.

Lemma 3.6. *There exists a constant $C > 0$ such that, for any $t \geqslant C$ and if $m = [Ct]$,*

$$\mathbb{P}\left[|\theta_e(t) - \theta^m_e(t)\ | > C\right] \leqslant C \exp\{-t/C\}.$$

Proof. Note that $(U_{e,i})$ and $(U^m_{e,i})$ solve the same equation for $i \leqslant m-1$ with the same initial condition. Lemma 2.6 on the approximate speed of propagation then states that there exists constants $C > 0$ and $\beta > 0$ such that, for all $n \in \mathbb{N}$, $n \geqslant 1$ and $i \leqslant J_n(m-1)$,

$$|U^m_{e,i}(s) - U_{e,i}(s)| \leqslant C\ 2^{-n} e^{\beta s} \qquad \forall s \geqslant 0.$$

Fix $t > 0$ and let us choose $n = [\beta t/\ln(2)] + 1$ and $m = [(1+\alpha)n]$ where $\alpha > 0$ is defined in Lemma 2.7. In the event $\{J_n(m-1) \geqslant 0\}$, we have

$$|U^m_{e,i}(s) - U_{e,i}(s)| \leqslant C \qquad \forall s \in [0,t],\ \forall i \leqslant 0.$$

Let us check that this inequality implies in the event $\{J_n(m-1) \geqslant 0\}$ that

$$|\theta^m_e(t) - \theta_e(t)| \leqslant C. \tag{22}$$

Indeed, if $U^m_{e,i}(t) \leqslant 0$, then $U_{e,i}(t) \leqslant C$. Assume $t \geqslant C\delta^{-1}$ where δ is defined in Lemma 3.2. Then Lemma 3.2 implies that $U_{e,i}(t - C\delta^{-1}) \leqslant 0$. This shows that

$$\theta_e(t - C\delta^{-1}) \leqslant \theta^m_e(t),$$

and thus, by Lemma 3.3, that

$$\theta_e(t) - C' \leqslant \theta^m_e(t),$$

for some new constant C'. If $t < C\delta^{-1}$, then, by Lemma 3.3,

$$\theta_e(t) \leqslant C' \leqslant \theta^m_e(t) + C'.$$

Therefore $\theta_e(t) - \theta^m_e(t) \leqslant C'$ in any case. The inequality

$$\theta^m_e(t) - C' \leqslant \theta_e(t)$$

can be checked in the same way, by using points 2) and 3) of Lemma 3.5. This proves (22).

As (22) holds in the event $\{J_n(m-1) \geqslant 0\}$, we get

$$\mathbb{P}\left[|\theta_e(t) - \theta^m_e(t)| \ > C\right] \leqslant \mathbb{P}\left[J_n(m-1) < 0\right] \leqslant \mathbb{P}\left[J_n(m) < 0\right].$$

Recalling the choice of m and Lemma 2.7 (with $\epsilon = 1$) we have

$$\mathbb{P}\left[J_n(m) < 0\right] \leqslant \mathbb{P}\left[J_n(m) - m + \alpha n < -m + \alpha n + 1\right] \leqslant \mathbb{P}\left[J_n(m) - m + \alpha n < -n\right] \leqslant C\exp\{-t/C\}.$$

This gives the result. □

The key step of the proof of Theorem 3.4 consists in establishing a concentration inequality for $\theta^m_e(t)$. To do so, let us set, for $n \in \mathbb{N}$, $\mathcal{F}_{m,n} = \sigma\{Z_i,\ i \in \{m-n, \dots, m-1\}\ \}$ if $n \geqslant 1$ and $\mathcal{F}_{m,0} = \{\varnothing, \Omega\}$. We also set

$$M_n(t) = \mathbb{E}\left[\theta^m_e(t) \mid \mathcal{F}_{m,n}\right] - \mathbb{E}\left[\theta^m_e(t)\right].$$

Note that $(M_n(t))$ is a martingale with $M_0(t) = 0$.

As, by Lemma 3.5, $\theta^m_e(t) \leqslant Ct$ and $\{\theta^m_e(t) \leqslant n\} = \{U^m_{-n}(t) \leqslant 0\}$ is $\mathcal{F}_{m,m+n}-$measurable for any $n \in \mathbb{N}$, we have that $M_n(t) = \theta^m_e(t) - \mathbb{E}[\theta^m_e(t)]$ for $n \geqslant \bar{n} := [Ct]$ for C large enough.

The next step is instrumental and consists in estimating $|M_{n+1}(t) - M_n(t)|$.

Lemma 3.7. *For any $n \in \mathbb{N}$,*

$$|M_{n+1}(t) - M_n(t)| \leqslant C(1 + \sigma^0_{\sigma_{m-n}} \wedge m - (m-n)), \tag{23}$$

where σ^0 and σ are defined in Lemma 3.5.

Proof. Let us first remark that, for any $n \geqslant 0$, $\theta^m_e(t)\mathbf{1}_{\{\theta^m_e(t) \leqslant (m-n-1)_-\}}$ is $\mathcal{F}_{m,n}-$measurable. Hence

$$|M_{n+1}(t) - M_n(t)| = \left|\mathbb{E}\left[\theta^m_e(t) \mid \mathcal{F}_{m,n+1}\right] - \mathbb{E}\left[\theta^m_e(t) \mid \mathcal{F}_{m,n}\right]\right| \mathbf{1}_{\{\theta^m_e(t) > (m-n-1)_-\}}.$$

In the next steps, we work in $\{U^m_{m-n}(t) > 0\}$. Let us introduce some notation. Given $n \in \mathbb{N}$ and a continuous componentwise nondecreasing map $x = (x^1, \dots, x^K) : [0, +\infty) \to \mathbb{R}^K$, we denote by $\hat{U}^{n,x} = (\hat{U}^{n,x,\omega}_i)_{i \leqslant m-n-1}$ the solution to

$$\frac{d}{dt}\hat{U}^{n,x,\omega}_i(\tau)$$
$$= \begin{cases} V_{Z^\omega_i}(\hat{U}^{n,x,\omega}_{i+1}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), \hat{U}^{n,x,\omega}_{l^\omega_i}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), \hat{U}^{n,x,\omega}_i(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \leqslant m-n-1, \\ V_{Z^\omega_i}(\hat{U}^{n,x,\omega}_{i+1}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), x^{T_i}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), \hat{U}^{n,x,\omega}_i(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \geqslant m-n, \\ V_{Z^\omega_i}(x^{k_0}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), x^{T_i}(\tau) - \hat{U}^{n,x,\omega}_i(\tau), \hat{U}^{n,x,\omega}_i(\tau)) & \text{if } i = m-n-1, \end{cases}$$

with for $i \leqslant m-n-1$

$$\hat{U}^{n,x,\omega}_i(0) = \begin{cases} e^0 i & \text{if } i \leqslant 0 \\ e^k i & \text{if } i \geqslant 0 \text{ and } k = T_i \end{cases}$$

where $k_0 \in \{1, \dots, K\}$ is such that $x^{k_0}(0) \leqslant x^k(0)$ for any $k \in \{1, \dots, K\}$ (if there are several minimizers of $x^k(0)$ we choose the smallest one). An important property of the $(\hat{U}^{n,x,\omega}_i)_{i \leqslant m-n-1}$ is that they depend on $\{Z_i,\ i \leqslant m-n-1\}$ only. We also define

$$\hat{\theta}^{n,x,\omega}(t) = \inf\left\{ i \geqslant (m-n-1)_-,\ \hat{U}^{n,x,\omega}_{-i}(t) \leqslant 0 \right\}.$$

We note that $U^{m,\omega}_{e,i}(\tau) = \hat{U}_i^{n,\left(U^{m,\omega}_{e,l^{m-n,\omega}(k)}\right)_{k=1,\dots,K},\omega}(\tau)$ for any $i \leqslant m-n-1$ and $\tau \geqslant 0$ where

$$l^{i,\omega}(k) = \inf\{j \geqslant i,\ T^\omega_j = k\}. \tag{24}$$

Moreover, $\theta^{m,\omega}_e(t) = \hat{\theta}_e^{n,\left(U^{m,\omega}_{e,l^{m-n,\omega}(k)}\right)_{k=1,\dots,K},\omega}(t)$ in $\{U^{m,\omega}_{e,m-n}(t) > 0\}$. As $\hat{\theta}^{n,x}(t)$ depends only on $\{Z_i,\ i \leqslant m-n-1\}$ while the $U^{m,\omega}_{e,l^{m-n,\omega}(k)}$ are $\mathcal{F}_{m,n}-$measurable, we have, in $\{U^{m,\omega}_{e,m-n}(t) > 0\}$,

$$\mathbb{E}\left[\theta^m_e(t) \mid \mathcal{F}_{m,n}\right] = \mathbb{E}\left[\hat{\theta}^{n,x}(t)\right]_{x=\left(\hat{U}^{m,\omega}_{e,l^{m-n,\omega}(k)}\right)_{k=1,\dots,K}}.$$

In the same way, we have

$$\mathbb{E}\left[\theta^m_e(t) \mid \mathcal{F}_{m,n+1}\right] \leqslant 1 + \mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right]_{\tilde{x}=\left(\hat{U}^{m,\omega}_{l^{m-n-1,\omega}(k)}\right)_{k=1,\dots,K}}.$$

So

$$|M_{n+1}(t) - M_n(t)| \leqslant \tag{25}$$
$$1 + \left|\mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right]_{\tilde{x}=\left(U^{m,\omega}_{l^{m-n-1,\omega}(k)}\right)_{k=1,\dots,K}} - \mathbb{E}\left[\hat{\theta}^{n,x}(t)\right]_{x=\left(U^{m,\omega}_{l^{m-n,\omega}(k)}\right)_{k=1,\dots,K}}\right| \mathbf{1}_{\{\theta^m_e(t) > (m-n-1)_-\}}.$$

Next we estimate the difference between $\mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right]$ and $\mathbb{E}\left[\hat{\theta}^{n,x}(t)\right]$ when $\tilde{x} = \left(U^{m,\omega}_{l^{m-n-1,\omega}(k)}\right)_{k=1,\dots,K}$ and $x = \left(U^{m,\omega}_{l^{m-n,\omega}(k)}\right)_{k=1,\dots,K}$ (recall that we work in $\{U^m_{m-n}(t) > 0\}$). For this we fix two C^1 maps $x, \tilde{x} : [0, \infty) \to \mathbb{R}^K$ such that there exists $k_0 \in \{1, \dots, K\}$ and $\gamma \geqslant 1$ with, for any $k \in \{1, \dots, K\}$,

$$\begin{aligned} &(d/d\tau)x^k(\tau) \geqslant \delta \ \text{ and } (d/d\tau)\tilde{x}^k(\tau) \geqslant \delta, \qquad x^k = \tilde{x}^k \text{ if } k \neq k_0 \\ &\text{and } \ -\gamma + x^{k_0}(\tau) \leqslant \tilde{x}^{k_0}(\tau) \leqslant x^{k_0}(\tau) \qquad \forall \tau \geqslant 0. \end{aligned} \tag{26}$$

Note that the conditions above are satisfied by $\tilde{x} = (U^{m,\omega}_{l^{m-n-1,\omega}(k)})_{k=1,\dots,K}$ and $x = (U^{m,\omega}_{l^{m-n,\omega}(k)})_{k=1,\dots,K}$ with $\gamma = C(1 + \sigma^0_{\sigma_{m-n}} \wedge m - (m-n))$ and $k_0 = T_{m-n}$ thanks to Lemma 3.5. Note also that the $\hat{U}^{n,x,\omega}_i$ and $\hat{\theta}^{n,x,\omega}$ satisfy the same conclusion as U_e and θ_e in Lemmas 3.2 (with the same constant δ) and 3.3, the proof being the same.

The main part of the proof consists in showing that, under (26),

$$\left|\mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right] - \mathbb{E}\left[\hat{\theta}^{n,x}(t)\right]\right| \leqslant C(1+\gamma). \tag{27}$$

For this we first check that

$$\mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right] \leqslant \mathbb{E}\left[\hat{\theta}^{n+1,x}(t)\right] \leqslant \mathbb{E}\left[\hat{\theta}^{n,x}(t)\right] + 1. \tag{28}$$

The first inequality holds by comparison, which implies from assumption (26) that $\hat{U}_i^{n+1,\tilde{x}}(s) \leqslant \hat{U}_i^{n+1,x}(s)$ for any $s \geqslant 0$ and any $i \leqslant m-n-2$. For the second inequality, let us set $W_i^\omega(s) = \hat{U}_{i-1}^{n+1,x,\tau_1\omega}(s)$ for $i \leqslant m-n$ and $s \geqslant 0$. Then W_i^ω solves (since $Z_{i-1}^{\tau_1\omega} = Z_i^\omega$, $T_{i-1}^{\tau_1\omega} = T_i^\omega$ and $\ell_{i-1}^{\tau_1\omega} = \ell_i^\omega - 1$)

$$\begin{aligned}
&\frac{d}{dt}W_i^\omega(\tau) \\
&= \begin{cases} V_{Z_i^\omega}(W_{i+1}^\omega(\tau) - W_i^\omega(\tau), W_{l_i^\omega}^\omega(\tau) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \leqslant m-n-1, \\ V_{Z_i^\omega}(W_{i+1}^\omega(\tau) - W_i^\omega(\tau), x^{T_i}(\tau) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \geqslant m-n, \\ V_{Z_i^\omega}(x^{k_0}(\tau) - W_i^\omega(\tau), x^{T_i}(\tau) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i = m-n-1, \end{cases}
\end{aligned}$$

$$W_i^\omega(0) = \begin{cases} e^0(i-1) & \text{if } i \leqslant 1 \\ e^k(i-1) & \text{if } i \geqslant 1 \text{ and } T_{i-1}^{\tau_1\omega} = T_i^\omega = k \end{cases} \leqslant \hat{U}_i^{n,x,\omega}(0) \text{ for } i \leqslant m-n-1,$$

Therefore by comparison

$$\hat{U}_{i-1}^{n+1,x,\tau_1\omega}(\tau) = W_i^\omega(\tau) \leqslant \hat{U}_i^{n,x,\omega}(\tau) \qquad \forall \tau \geqslant 0,\ \forall i \leqslant m-n-1,$$

which implies that

$$\hat{\theta}^{n+1,x,\tau_1\omega}(t) \leqslant \hat{\theta}^{n,x,\omega}(t) + 1$$

and gives the second inequality in (28) after taking expectation.

Using (26), the definitions of $\hat{U}_i^{n+1,\tilde{x},\omega}(0)$ and $\hat{U}_i^{n,x,\omega}(0)$ and the fact that $(d/dt)\hat{U}_i^{n+1,\tilde{x},\omega}(t) \geqslant \delta > 0$ (since the $\hat{U}^{n,x}$ satisfy the conclusion of Lemma 3.2), we get that there exists $C_1 > 0$ large (but which does not dependent on m, n, ω and i) such that

$$\hat{U}_{i-1}^{n+1,\tilde{x},\tau_1\omega}(C_1\gamma) \geqslant \hat{U}_i^{n,x,\omega}(0) \qquad \forall i \leqslant m-n-1, \tag{29}$$

and

$$\tilde{x}^k(t + C_1\gamma) \geqslant x^k(t) \qquad \forall t \geqslant 0,\ \forall k \in \{1,\dots,K\}.$$

We claim that, for some constant $C_2 > 0$,

$$\mathbb{E}\left[\hat{\theta}^{n,x}(t)\right] \leqslant \mathbb{E}\left[\hat{\theta}^{n+1,\tilde{x}}(t)\right] + C_2\gamma. \tag{30}$$

To prove (30) let $W_i^\omega(t) = \hat{U}_{i-1}^{n+1,\tilde{x},\tau_1\omega}(t + C_1\gamma)$ for $i \leqslant m-n-1$. For $i \leqslant m-n-1$, W_i solves, exactly as above,

$$\begin{aligned}
&\frac{d}{dt}W_i^\omega(\tau) \\
&= \begin{cases} V_{Z_i^\omega}(W_{i+1}^\omega(\tau) - W_i^\omega(\tau), W_{l_i^\omega}^\omega(\tau) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \leqslant m-n-1, \\ V_{Z_i^\omega}(W_{i+1}^\omega(\tau) - W_i^\omega(\tau), \tilde{x}^{T_i}(\tau + C_1\gamma) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i \leqslant m-n-2,\ \ell_i \geqslant m-n, \\ V_{Z_i^\omega}(\tilde{x}^{k_0}(\tau + C_1\gamma) - W_i^\omega(\tau), \tilde{x}^{T_i}(\tau + C_1\gamma) - W_i^\omega(\tau), W_i^\omega(\tau)) & \text{if } i = m-n-1, \end{cases}
\end{aligned}$$

$$W_i^\omega(0) = \hat{U}_{i-1}^{n+1,\tilde{x},\tau_1\omega}(C_1\gamma) \geqslant \hat{U}_i^{n,x,\omega}(0) \text{ for } i \leqslant m-n-1.$$

As $\tilde{x}^k(\tau + C_1\gamma) \geqslant x^k(\tau)$ and as V_z is increasing in the first two variables, we obtain by comparison that

$$\hat{U}_{i-1}^{n+1,\tilde{x},\tau_1\omega}(t + C_1\gamma) = W_i^\omega(t) \geqslant \hat{U}_i^{n,x,\omega}(t) \qquad \forall t \geqslant 0,\ \forall i \leqslant m-n-1.$$

Therefore

$$1 + \hat{\theta}^{n+1,\tilde{x},\tau_1\omega}(t + C_1\gamma) \geqslant \hat{\theta}^{n,x,\omega}(t).$$

Using Lemma 3.3 (which holds for $\hat{\theta}$ as explained above), we obtain

$$C_2\gamma + \hat{\theta}^{n+1,\tilde{x},\tau_1\omega}(t) \geqslant \hat{\theta}^{n,\omega}(t),$$

which implies (30) after taking expectation.

Combining (28) and (30) gives (27). Then recalling (25), we obtain (23). □

Let us set

$$\xi_i(t) = |M_{i+1}(t) - M_i(t)|.$$

and

$$[M]_n = \sum_{i=1}^{n} (\xi_i(t))^2, \qquad \langle M\rangle_n = \sum_{i=1}^{n} \mathbb{E}\left[(\xi_i(t))^2 \mid \mathcal{F}_{m,i}\right].$$

Following [15, Theorem 2.1], the following concentration inequality holds:

$$\mathbb{P}\left[|M_n| \geqslant x,\ [M]_n + \langle M\rangle_n \leqslant y\right] \leqslant 2\exp\{-x^2/(2y)\}.$$

This implies that

$$\mathbb{P}\left[|M_n| \geqslant x\right] \leqslant 2\exp\{-x^2/(2y)\} + \mathbb{P}\left[[M]_n + \langle M\rangle_n > y\right]. \tag{31}$$

Lemma 3.8. *There exists a constant $C > 0$ such that*

$$\mathbb{P}\left[[M]_n + \langle M\rangle_n > Cn\right] \leqslant \exp\{-\frac{n}{C}\}.$$

Proof. In view of Lemma 3.7 we have

$$|\xi_i(t)| \leqslant C(1 + \sigma^0_{\sigma_{m-i}} \wedge m - (m-i)).$$

We first replace the right-hand side by a more suitable random variable. For $k \in \{1,\dots,K\}$, let $\sigma_i^k := \inf\{j \geqslant i+1,\ T_j = k,\ Z_j = z^k_{\min}\}$. Note that σ_i^k is independent of $\{Z_j,\ j \leqslant i\}$ and that $\sup_k \sigma_i^k \geqslant \sigma_i$. Then

$$|\xi_i(t)| \leqslant C\sum_{k=1}^{K}(1 + \sigma^0_{\sigma^k_{m-i}} \wedge m - (m-i)).$$

As $\sigma^0_{\sigma^k_{m-i}} \wedge m$ is $\mathcal{F}_{m,i}-$measurable, we have

$$\begin{aligned}
[M]_n + \langle M\rangle_n &\leqslant Cn + C\sum_{k=1}^{K}\sum_{i=1}^{n}(\sigma^0_{\sigma^k_{m-i}} \wedge m - (m-i))^2 \\
&\leqslant Cn + C\sum_{k=1}^{K}\sum_{i=1}^{n}(\sigma^0_{\sigma^k_{m-i}} \wedge m - \sigma^k_{m-i} \wedge m)^2 + C\sum_{k=1}^{K}\sum_{i=1}^{n}(\sigma^k_{m-i} \wedge m - (m-i))^2.
\end{aligned}$$

For $k \in \{0,\dots,K\}$, let us define by induction

$$s_0^k = \inf\{j \geqslant m,\ T_j = k,\ Z_j = z^k_{\min}\},\ s^k_{i+1} = \sup\{j < s_i^k,\ T_j = k,\ Z_j = z^k_{\min}\}.$$

Note that the $(s_i^k - s^k_{i+1})_{i\geqslant 0}$ are i.i.d. and that, by definition, for any $j \in \{s^k_{i+1},\dots,s_i^k - 1\}$, one has $\sigma_j^k = s_i^k$. Therefore

$$\sum_{r=s^k_{i+1}}^{s_i^k-1}(\sigma_r^k - r)^2 = \sum_{r=s^k_{i+1}}^{s_i^k-1}(s_i^k - r)^2 \leqslant C(s_i^k - s^k_{i+1})^3.$$

As $s_0^k \geqslant m$ while $s_n^k \leqslant m-n$, this shows that

$$\sum_{i=1}^{n}(\sigma^k_{m-i} \wedge m - (m-i))^2 \leqslant \sum_{i=m-n}^{m-1}(\sigma_i^k - i)^2 \leqslant \sum_{j=0}^{n-1}\sum_{r=s^k_{j+1}}^{s_j^k-1}(\sigma_r^k - r)^2 \leqslant C\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3.$$

On the other hand,

$$\begin{aligned}
&\sum_{i=1}^{n}(\sigma^0_{\sigma^k_{m-i}} \wedge m - \sigma^k_{m-i} \wedge m)^2 \leqslant \sum_{i=1}^{n}(\sigma^0_{\sigma^k_{m-i}\wedge m} - \sigma^k_{m-i}\wedge m)^2 = \sum_{i=1}^{n}\sum_{j,\ s_j^k \leqslant m}(\sigma^0_{s_j^k} - s_j^k)^2 \mathbf{1}_{s_j^k = \sigma^k_{m-i}} \\
&\leqslant \sum_{j=1}^{n}(\sigma^0_{s_j^k} - s_j^k)^2(s_j^k - s_{j-1}^k) \leqslant \sum_{j=1}^{n}(\sigma^0_{s_j^k} - s_j^k)^4 + \sum_{j=1}^{n}(s_j^k - s_{j-1}^k)^2 \\
&\leqslant \sum_{i=s_n^k}^{m-1}(\sigma_i^0 - i)^4 + \sum_{j=1}^{n-1}(s_j^k - s_{j-1}^k)^3.
\end{aligned}$$

The first term in the right-hand side can be treated as above and we obtain:

$$[M]_n + \langle M\rangle_n \leqslant Cn + C\sum_{k=1}^{K}\left(\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 + \sum_{j=0}^{m-s_n^k-1}(s_j^0 - s_{j+1}^0)^5\right).$$

Therefore

$$\begin{aligned}
\mathbb{P}\left[[M]_n + \langle M\rangle_n > y\right] &\leqslant \sum_{k=1}^{K}\mathbb{P}\left[\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 + \sum_{j=0}^{m-s_n^k-1}(s_j^0 - s_{j+1}^0)^5 > (KC)^{-1}(y-Cn)\right] \\
&\leqslant \sum_{k=1}^{K}\mathbb{P}\left[\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 > (2KC)^{-1}(y-Cn)\right] \\
&\quad + \sum_{k=1}^{K}\mathbb{P}\left[\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 \leqslant (2KC)^{-1}(y-Cn)\ ,\ \sum_{j=0}^{m-s_n^k-1}(s_j^0 - s_{j+1}^0)^5 > (2KC)^{-1}(y-Cn)\right]. \quad (32)
\end{aligned}$$

Let $X_j^k = s_j^k - s_{j+1}^k$ (for $k \in \{0,\dots,K\}$). Then the $(X_j^k)_{j=0,\dots,n}$ are i.i.d. and X_0^k follows a geometric law of parameter $p^k := \mathbb{P}[Z_0 = z^k_{\min}]$ which has exponential moments. In particular $(X_0^k)^3$ satisfies Bernstein's condition: there exists $c^k > 0$ such that, for any $k \in \{0,\dots,K\}$ and $p \geqslant 2$,

$$\mathbb{E}\left[\left||X_0^k| - \mathbb{E}\left[|X_0^k|\right]\right|^p\right] \leqslant \frac{p!(c^k)^{p-2}}{2}v^k \qquad \text{where } v^k := Var(|X_0^k|^3).$$

From Bernstein's Theorem (Corollary 2.11 in [14]) we have

$$\mathbb{P}\left[\sum_{j=0}^{n}(s_j^k - s_{j+1}^k)^3 > \mathbb{E}\left[|X_0^k|^3\right]n + x\right] \leqslant \exp\left\{-\frac{x^2}{2(nv^k + xc^k)}\right\}.$$

This allows to handle the first terms in the right-hand side of (32). As for the second term, we note that, in the event $\{\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 \leqslant (2KC)^{-1}(y-Cn)\}$, we have by Hölder's inequality:

$$s_n^k = s_0^k + \sum_{i=0}^{n-1}(s_{i+1}^k - s_i^k) \geqslant m - n^{2/3}\left(\sum_{i=0}^{n-1}|X_i^k|^3\right)^{1/3} \geqslant m - C_1 n^{2/3}(y-Cn)^{1/3}.$$

Hence

$$\mathbb{P}\left[\sum_{j=0}^{n-1}(s_j^k - s_{j+1}^k)^3 \leqslant (2KC)^{-1}(y - Cn)\,,\ \sum_{j=0}^{m-s_n^k-1}(s_j^0 - s_{j+1}^0)^5 > (2KC)^{-1}(y - Cn)\right]$$
$$\leqslant \mathbb{P}\left[\sum_{j=0}^{[C_1 n^{2/3}(y-Cn)^{1/3}]+1}(s_j^0 - s_{j+1}^0)^5 > (2KC)^{-1}(y - Cn)\right].$$

We can again use Bernstein's inequality to handle this later term: we have, for some constant $c^0 > 0$ and $v^0 := Var(|X_0^0|^5)$,

$$\mathbb{P}\left[\sum_{j=0}^{n}(s_j^0 - s_{j+1}^0)^5 > \mathbb{E}\left[|X_0^k|^5\right] n + x\right] \leqslant \exp\left\{-\frac{x^2}{2(nv^0 + xc^0)}\right\}.$$

So choosing $y = C_2 n$ for a sufficiently large constant C_2 in (32), we obtain that

$$\mathbb{P}\left[[M]_n + \langle M\rangle_n > C_2 n\right] \leqslant 2K \exp\left\{-\frac{n}{C_2}\right\}.$$

□

Proof of Theorem 3.4. Coming back to (31) and using Lemma 3.8, we find

$$\mathbb{P}\left[|M_n| \geqslant x\right] \leqslant 2\exp\{-x^2/(Cn)\} + \exp\{-\frac{n}{C}\}.$$

Recalling that $M_{\bar{n}} = \theta_e^m(t) - \mathbb{E}[\theta_e^m(t)]$ for $\bar{n} = Ct$, we obtain therefore (for $n = \bar{n}$, $x = \epsilon t$ and $\epsilon \in (0, 1]$)

$$\mathbb{P}\left[|\theta_e^m(t) - \mathbb{E}\left[\theta_e^m(t)\right]| \geqslant \epsilon t\right] \leqslant C \exp\left\{-\frac{\epsilon^2 t}{C}\right\}.$$

Then we use Lemma 3.6 to get the concentration inequality for $\theta_e(t)$. □

3.3 A corrector outside the junction

In this part we build a random sequence $(W_i)_{i\in\mathbb{Z}}$ which plays the role of a corrector for large values of $|i|$. We first use $(W_i)_{i\in\mathbb{Z}}$ in this section to investigate the behavior of $U_{e,i}(t)$ for large $|i|$. The main role of the $(W_i)_{i\in\mathbb{Z}}$ will be however in the next section where the property of being a kind of corrector for large values of $|i|$ will be used in a crucial way.

We recall that $e = (e^k)_{k=0,\dots,K}$ is such that $H^k(-1/e^k) = \min_p H^k(p)$. This implies in particular that $v_e^k = \bar{V}^k(e^k/\pi^k) < \bar{v}^k$ for all $k \in \{0, \dots, K\}$, where the $\bar{v}^k$ are defined in (7). We define $(W_i)_{i\in\mathbb{Z}}$ as follows: we set $W_0^\omega = 0$ and define W_i^ω for $i \leqslant -1$ by backward induction by setting:

$$\tilde{V}^0_{Z_i^\omega}(W_{i+1}^\omega - W_i^\omega) = v_e^0 \qquad \text{for } i < 0.$$

For $i \geqslant 1$, define W_i^ω by forward induction by setting, if $\ell_i^{\omega,-1} := \sup\{j < i,\ T_j^\omega = T_i^\omega\}$,

$$W_i^\omega = 0 \text{ if } i \geqslant 1 \text{ and } \ell_i^{\omega,-1} < 0$$

and

$$\tilde{V}^k_{Z^\omega_{\ell_i^{\omega,-1}}}(W_i^\omega - W^\omega_{\ell_i^{\omega,-1}}) = v_e^k, \qquad \text{where } T_i^\omega = k, \qquad \text{if } i \geqslant 1 \text{ and } \ell_i^{\omega,-1} \geqslant 0.$$

By the definition of $e_{\max}$, we have $0 < W_{i+1}^\omega - W_i^\omega \leqslant e_{\max}$ if $i + 1 \leqslant 0$ while $0 < W_{\ell_i}^\omega - W_i^\omega \leqslant e_{\max}$ if $\ell_i \geqslant 1$. We now collect several properties of the sequence (W_i).

Lemma 3.9. *We have*

$$|W_i - W_j| \leqslant e_{\max}|i - j|$$

if $T_i = T_j$ *or if* $i \wedge j \leqslant 0$.

Proof. If $i \leqslant -1$, then by the definition of W_i one has $0 \leqslant W_{i+1} - W_i \leqslant e_{\max}$. So the claim holds if $i \leqslant 0$ and $j \leqslant 0$. Let us now assume that $j \geqslant 0$ and let us set $k = T_j$. If $\ell_j^{-1} < 0$, then $W_j = 0$ and

$$|W_j - W_{\ell_j^{-1}}| = |W_{\ell_j^{-1}}| \leqslant -e_{\max}\ell_j^{-1} \leqslant e_{\max}(j - \ell_j^{-1}).$$

If $\ell_j^{-1} \geqslant 0$, then by the definition of W_j one has $|W_j - W_{\ell_j^{-1}}| \leqslant e_{\max}$. By induction, this implies that $|W_j - W_i| \leqslant e_{\max}|j - i|$ if $T_i = T_j$ and $i \geqslant 0$, $j \geqslant 0$. The claim for $i \geqslant 0$ and $j \leqslant 0$ follows easily. □

Lemma 3.10. *There exists $C > 0$ such that, for any $\epsilon \in (0, 1]$, for any $i_0, i \in \mathbb{Z}$, one has*

$$\mathbb{P}\left[|W_{i-i_0}^{\tau_{i_0}\cdot} - e^0(i - i_0)| > \epsilon|i - i_0|\right] \leqslant C\exp\{-\epsilon^2|i - i_0|/C\} \qquad \text{if } i \leqslant i_0 - C\epsilon^{-1}$$

and

$$\mathbb{P}\left[|W_{i-i_0}^{\tau_{i_0}\cdot} - e^{T_i^{\tau_{i_0}\cdot}}(i - i_0)| > \epsilon|i - i_0|\right] \leqslant C\exp\{-\epsilon^2|i - i_0|/C\} \qquad \text{if } i \geqslant i_0 + C\epsilon^{-1}.$$

Proof. Fix first $i_0 = 0$. For $i \leqslant 0$, the proof is a straightforward application of Hoeffding's inequality ([14, Theorem 2.8]) combined with the property that, by the definition of $\bar{V}^0$, $\mathbb{E}\left[(\tilde{V}_{Z_i}^0)^{-1}(v_e^0)\right] = e^0$.

Let us now investigate the case $i \geqslant 0$. For $k \in \{1, \dots, K\}$, let $s_0^k = \inf\{i \geqslant 0,\ T_i = k\}$ and let us define by induction $s_{i+1}^k = \inf\{i \geqslant s_i^k + 1,\ T_i = k\}$. Then

$$W_{s_i^k} = \sum_{j=0}^{i} (\tilde{V}_{Z_{s_j^k}}^k)^{-1}(v_e^k),$$

where the $(\tilde{V}_{Z_{s_j^k}}^k)^{-1}(v_e^k)$ are i.i.d. with the same law as $(\tilde{V}_{Z_0}^k)^{-1}(v_e^k)$ given $T_0 = k$, which is bounded by $e_{\max}$. Recall that $\mathbb{E}\left[(\tilde{V}_{Z_0}^k)^{-1}(v_e^k) \mid T_0 = k\right] = e^k/\pi^k$. So, by Hoeffding's inequality,

$$\mathbb{P}\left[|W_{s_i^k} - ie^k/\pi^k| > x\right] \leqslant 2\exp\{-x^2/(2ie_{\max})\}.$$

Since we also have by Bernstein's inequality ([14, Corollary 2.11]):

$$\mathbb{P}\left[|s_i^k - (\pi^k)^{-1}i| > \epsilon i\right] \leqslant 2\exp\{-\epsilon^2 i/C\},$$

we can infer that, for any $i \geqslant C\epsilon^{-1}$ and setting $j_k = [\pi^k i]$,

$$\begin{aligned}
\mathbb{P}\left[|W_i - e^{T_i}i| > \epsilon i\right] &\leqslant \sum_{k=1}^{K} \mathbb{P}\left[|W_i - e^k i| > \epsilon i,\ T_i = k,\ |i - s_{j_k}^k| \leqslant \epsilon i/(2e_{\max})\right] + \mathbb{P}\left[|i - s_{j_k}^k| > \epsilon i/(2e_{\max})\right] \\
&\leqslant \sum_{k=1}^{K} \mathbb{P}\left[|W_{s_{j_k}^k} - e^k j_k/\pi^k| > \epsilon i/2 - |j_k/\pi^k - i|\right] + \mathbb{P}\left[|j_k/\pi^k - s_{j_k}^k| > \epsilon i/(2e_{\max}) - |j_k/\pi^k - i|\right] \\
&\leqslant C\exp\{-\epsilon^2 i/C\}.
\end{aligned}$$

We now address the case $i_0 \neq 0$. We note that $i \to W_{i-i_0}^{\tau_{i_0}\omega}$ can be built exactly as W_i^ω except that the origin is i_0 and ω is shifted by τ_{i_0}. Thus we have in the same way

$$\mathbb{P}\left[|W_{i-i_0}^{\tau_{i_0}\cdot} - e^0(i - i_0)| > \epsilon|i - i_0|\right] \leqslant C\exp\{-\epsilon^2|i - i_0|/C\} \qquad \forall i \in \mathbb{Z},\ i \leqslant i_0 - C\epsilon^{-1}$$

and

$$\mathbb{P}\left[|W_{i-i_0}^{\tau_{i_0}\cdot} - e^{T_i^{\tau_{i_0}\cdot}}(i - i_0)| > \epsilon|i - i_0|\right] \leqslant C\exp\{-\epsilon^2|i - i_0|/C\} \qquad \forall i \in \mathbb{Z},\ i \geqslant i_0 + C\epsilon^{-1}.$$

□

Fix $\bar{C}$ large and to be chosen below, $\epsilon > 0$, $T \geqslant 1$ with $\epsilon T \geqslant C$, for C large enough. We define the event

$$E_{\epsilon,T} := \Big\{ \sup_{|i|\leqslant \bar{C}T,\ |i_0|\leqslant C_\theta T} |W^{\tau_{i_0}\cdot}_{i-i_0} - e^0(i-i_0)\mathbf{1}_{i\leqslant i_0} - e^{T_i^{\tau_{i_0}\cdot}}(i-i_0)\mathbf{1}_{i>i_0}| \leqslant \epsilon T,$$
$$J_{[\bar{C}T/(2\alpha)]}([\bar{C}T]) \geqslant \bar{C}T/2,\ J_{[\bar{C}T/(8\alpha)]}([-\bar{C}T/4]) \geqslant -\bar{C}T/2 \Big\}, \tag{33}$$

where $J_n(T)$ and α are defined in Lemma 2.7. By Lemma 2.7, Lemma 3.9 and Lemma 3.10 we have

$$\mathbb{P}\left[E^c_{\epsilon,T}\right] \leqslant CT^2 \exp\{-\epsilon^2 T/C\}, \tag{34}$$

where $C = C(\bar{C})$. We assume that $\bar{C}$ is so large that

$$C_\theta \leqslant \bar{C}/10 \text{ and } \bar{C} \geqslant 16\beta\alpha/\ln(2), \tag{35}$$

where β is given in the approximate finite speed of propagation (Lemma 2.6) and C_θ is defined in Lemma 3.3.

Lemma 3.11. *If $\bar{C}$ is large enough we have, for $T \geqslant \bar{C}\epsilon^{-1}$ and in $E_{\epsilon,T}$,*

$$\left|e^{T_i} i + v_e^{T_i} s - U_{e,i}(s)\right| \leqslant 3\epsilon T \qquad \forall s \in [0,2T],\ \forall i \in [2(\min_k e^k)^{-1}\epsilon T, \bar{C}T/2] \cap \mathbb{Z} \tag{36}$$

and

$$\left|e^0 i + v_e^0 s - U_{e,i}(s)\right| \leqslant 3\epsilon T \qquad \forall s \in [0,2T],\ \forall i \in [-\bar{C}T, -\bar{C}T/2)] \cap \mathbb{Z}. \tag{37}$$

Proof. Let us first note that for $i \in [2(\min_k e^k)^{-1}\epsilon T, \bar{C}T] \cap \mathbb{Z}$, since we are in $E_{\epsilon,T}$, $W_i \geqslant e^{T_i} i - \epsilon T \geqslant \epsilon T > 0$. Then, the maps $s \to U_{e,i}(s)$ and $s \to W_i^1(s) := W_i - \epsilon T + v_e^{T_i} s$ solve (6) with an initial condition which satisfies, since we are in $E_{\epsilon,T}$, $0 \leqslant W_i^1(0) \leqslant U_{e,i}(0)$. So, by the approximate finite speed of propagation (Lemma 2.6) we have

$$W_i^1(s) \leqslant U_{e,i}(s) + 2^{-n} e^{\beta s} \qquad \forall s \geqslant 0,\ \forall n \in \mathbb{N},\ \forall i \in [2(\min_k e^k)^{-1}\epsilon T, J_n([\bar{C}T])] \cap \mathbb{Z}.$$

Choosing $n = [\bar{C}T/(2\alpha)]$, we obtain, if $\bar{C}$ is large enough (depending on β only) and since we are in $E_{\epsilon,T}$,

$$e^{T_i} i - 2\epsilon T + v_e^{T_i} s \leqslant W_i^1(s) \leqslant U_{e,i}(s) + 1 \qquad \forall s \in [0,2T],\ \forall i \in [2(\min_k e^k)^{-1}\epsilon T, \bar{C}T/2] \cap \mathbb{Z}.$$

Replacing W^1 by $W_i^2(s) := W_i + \epsilon T + v_e^k s$ gives the opposite inequality. Thus (36) holds.

To obtain (37), we note that, for $i \in [-\bar{C}T, -\bar{C}T/2] \cap \mathbb{Z}$, the maps $s \to U_{e,i}(s)$ and $s \to W_i^1(s) := W_i - \epsilon T + v_e^0 s$ solve (6) on the time interval $[0, \bar{C}T/(4\|V\|_\infty)]$ with an initial condition which satisfies, since we are in $E_{\epsilon,T}$, $W_i^1(0) \leqslant U_{e,i}(0)$. So we have as above

$$W_i^1(s) \leqslant U_{e,i}(s) + 2^{-n} e^{\beta s} \qquad \forall s \in [0, \bar{C}T/(4\|V\|_\infty)],\ \forall i \in \{-\bar{C}T, \dots, J_n([-\bar{C}T/4])\}.$$

We choose $n = [\bar{C}T/(8\alpha)]$ and get, for $\bar{C}$ large enough (depending on β and $\|V\|_\infty$), since we are in $E_{\epsilon,T}$,

$$W_i^1(s) \leqslant U_{e,i}(s) + 1 \qquad \forall s \in [0,2T],\ \forall i \in [-\bar{C}T, \dots, -\bar{C}T/2)] \cap \mathbb{Z}.$$

Arguing as above we get (37). □

3.4 A superadditive quantity

The aim of this section is to investigate the existence of a limit for $\theta_e(t)/t$ as $t \to +\infty$. For doing so, we introduce new notation. Fix $\tilde{h} > 0$ such that

$$\tilde{h} < \min_{k\in\{0,\dots,K\}} \frac{v_e^k}{e^k} = \min_{k\in\{0,\dots,K\}} \frac{\bar{V}^k(e^k/\pi^k)}{e^k} = -\max_{k\in\{0,\dots,K\}} H^k(-1/e^k) = -A_0, \tag{38}$$

where A_0 is defined in (8). Let us define

$$\bar{\theta}_e(s) = \mathbb{E}\left[\theta_e(s)\right], \qquad \bar{M}_{e,\tilde{h}}(t) = \inf_{s\in[0,t]} \bar{\theta}_e(s) - \tilde{h}s.$$

We note that the quantity $\bar{M}_{e,\tilde{h}}$ is nonpositive, nonincreasing in t and in $\tilde{h}$. The main result of this section is the following:

Theorem 3.12. *The limit $k_{e,\tilde h}$ of $\bar M_{e,\tilde h}(t)/t$, as $t \to +\infty$ exists, is nonpositive and nonincreasing with respect to $\tilde h$. Let us set*

$$k_e := \inf_{0<\tilde h<-A_0} k_{e,\tilde h} = \lim_{\tilde h \to -A_0} k_{e,\tilde h}.$$

If $k_e < 0$, then the limit $\bar\vartheta_e$ of $\theta_e(t)/t$, as $t \to +\infty$, exists almost surely, is deterministic and satisfies $\bar\vartheta_e < -A_0$.

If $k_e = 0$, then the limit $\liminf_{t\to+\infty} \theta_e(t)/t$ is deterministic and is not smaller than $-A_0$.

To prove the result, we are going to show that $\bar M_{e,\tilde h}$ is almost superadditive (Lemma 3.16), which implies that $\bar M_{e,\tilde h}(t)/t$ has a limit $k_{e,\tilde h}$ as $t \to +\infty$ (Lemma 3.17). This, in turn, will show the existence of a limit for $\bar\theta_e(t)/t$ if $k_{e,\tilde h} < 0$ and thus, by the variance estimate, the a.s. limit of $\theta_e(t)/t$ (Lemma 3.18).

The proof of the superadditivity of $\bar M_{e,\tilde h}$ is intricate and requires the introduction of several additional quantities. Let $\xi^\omega : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ be a measurable map which is smooth, uniformly Lipschitz continuous and increasing in the x variable, with inverse also uniformly Lipschitz continuous, and such that

$$\xi_e^\omega(x,i) = \begin{cases} x/e^0 & \text{if } x \leqslant -\min(R_0, e^0), \\ x/e^k & \text{if } x \geqslant 0,\ T_i = k, \end{cases} \quad \text{and} \qquad \left|\xi_e^\omega(x,i) - \frac{x}{e^0}\mathbf{1}_{x\leqslant 0} - \frac{x}{e^k}\mathbf{1}_{\{x\geqslant 0,\ T_i^\omega = k\}}\right| \leqslant 1.$$

Since the inverse of ξ_e^ω is uniformly Lipschitz continuous, we have in particular, if $x \geqslant y$

$$\xi_e^\omega(x,i) - \xi_e^\omega(y,i) \geqslant C^{-1}(x-y). \tag{39}$$

For $0 < s \leqslant T$, let

$$M_{e,T}^\omega(s) = M_{e,\tilde h,T}^\omega(s) = \inf_{i\in\mathbb{Z}\cap[-\bar CT,\bar CT/2],\ \tau\in[0,s]} \xi_e^\omega(U_{e,i}^\omega(\tau), i) - i - \tilde h\tau.$$

We also set, for any $i_0 \in \mathbb{Z} \cap [-C_\theta T, 0]$,

$$\tilde M_{e,T}^{i_0,\omega}(s) = \tilde M_{e,\tilde h,T}^{i_0,\omega}(s) = \inf_{\tau\in[0,s],\ i\in\mathbb{Z}\cap[-\bar CT,\bar CT/2]} \xi_e^\omega(U_{e,i}^\omega(\tau), i) - \xi_e^\omega(W_{i-i_0}^{\tau_{i_0}\omega}, i-i_0) - i_0 - \tilde h\tau.$$

Note that $M_{e,T}^\omega(0) = 0$ and that $M_{e,T}^\omega$ is nonpositive. We will prove below that $M_{e,T}$ and $\tilde M_{e,T}$ are good approximations of $\bar M_{e,\tilde h}$.

Let us introduce the event

$$\tilde E_{\epsilon,T} = E_{\epsilon,T} \cap \left\{ \sup_{s\leqslant 2T} \left|\theta_e(s) - \bar\theta_e(s)\right| \leqslant \epsilon T, \quad \sup_{i\leqslant 2e_{\max}^{-1}\epsilon T} \ell_i \leqslant \bar CT/2, \quad J_{[\bar CT/(16\alpha)]+1}([\bar CT/2]) \geqslant 0 \right\}, \tag{40}$$

where $E_{\epsilon,T}$ is defined in (33). Recalling Lemma 2.7, Theorem 3.4 and (34) we have

$$\mathbb{P}\left[\tilde E_{\epsilon,T}^c\right] \leqslant CT^2 \exp\{-\epsilon^2 T/C\}. \tag{41}$$

Lemma 3.13. *In $\tilde E_{\epsilon,T}$ and for $i_0 \in \mathbb{Z} \cap [-C_\theta T, 0]$, we have, for $s \in [0,T]$,*

$$M_{e,T}^\omega(s) \leqslant \bar M_{e,\tilde h}(s) + \epsilon T, \qquad \left|M_{e,T}^\omega(s) - \tilde M_{e,T}^{i_0,\omega}(s)\right| \leqslant C\epsilon T,$$

where C depends only on the Lipschitz constant of $\xi = \xi_e^\omega(x,i)$ with respect to x.

Proof. Let $0 \leqslant \tau \leqslant s \leqslant T$ and choose $i = -\theta_e^\omega(\tau)$ in the definition of $M_{e,T}^\omega(s)$. By (35) we know that $i = -\theta_e^\omega(\tau) \in \mathbb{Z} \cap [-\bar CT, 0]$. Then, as we are in $\tilde E_{\epsilon,T}$, we have

$$M_{e,T}^\omega(s) \leqslant 0 + \theta_e^\omega(\tau) - \tilde h\tau \leqslant \bar\theta_e(\tau) - \tilde h\tau + \epsilon T.$$

Taking the infimum over $\tau \in [0,s]$ gives the first inequality. For the second one, let us recall that, since we are in $E_{\epsilon,T}$ defined in (33), we have for any $i \in \mathbb{Z} \cap [-\bar CT, \bar CT]$ and $i_0 \in \mathbb{Z} \cap [-C_\theta T, 0]$

$$|W_{i-i_0}^{\tau_{i_0}\omega} - e^0(i-i_0)\mathbf{1}_{i\leqslant i_0} - e^{T_i^{\tau_{i_0}\omega}}(i-i_0)\mathbf{1}_{i>i_0}| \leqslant \epsilon T.$$

As $x \to \xi_e^\omega(x,i)$ is uniformly Lipschitz continuous with Lipschitz constant C and $T_i^{\tau_{i_0}\omega} = \tau^\omega_{i-i_0}$ we have

$$\begin{aligned}
&\left|\xi_e^\omega(W^{\tau_{i_0}\omega}_{i-i_0}, i-i_0) - (i-i_0)\right| \\
&\quad \leqslant \left|\xi_e^\omega(W^{\tau_{i_0}\omega}_{i-i_0}, i-i_0) - \xi_e^\omega(e^0(i-i_0)\mathbf{1}_{i\leqslant i_0} - e^{T_i^{\tau_{i_0}\omega}}(i-i_0)\mathbf{1}_{i>i_0}, i-i_0)\right| + 1 \\
&\quad \leqslant C\epsilon T + 1.
\end{aligned}$$

This implies the second inequality. □

We now use in a crucial way the construction of (W_i) to obtain the key property of $\tilde M_{e,T}$:

Lemma 3.14. *Let ϵ, T be such that $\epsilon T \geqslant \bar C$ and fix $C_1 > 0$, $\omega \in \tilde E_{\epsilon,T}$, $\bar s \in (0,T]$ and set $i_0 = -\theta_e^\omega(\bar s)$. Assume that $\tilde M^{i_0,\omega}_{e,T}(\tilde s) < 0$ and that $\frac{d}{ds}\tilde M^{i_0,\omega}_{e,T}(\tilde s) < 0$ for some $\tilde s \in (0,T)$ with $|\tilde s - \bar s| \leqslant C_1\epsilon^{1/2}T$. Then there exists i_1 such that $(\tilde s, i_1)$ is a minimum point in the definition of $\tilde M^{i_0,\omega}_{e,T}(\tilde s)$ and there exists large constants c_1 and C (depending on C_1 but not on ω, ϵ, T, $\bar s$ or $\tilde s$) with the property that, if $\tilde s \geqslant c_1\epsilon T$, then we have $|U^\omega_{i_1}(\tilde s)| \leqslant C\epsilon^{1/2}T$, $|i_1 - i_0| \leqslant C\epsilon^{1/2}T$ and*

$$U^\omega_{e,i}(\tilde s) \geqslant e^0(i-i_0)\mathbf{1}_{i\leqslant i_0} + e^{T_i^\omega}(i-i_0)\mathbf{1}_{i>i_0} - C\epsilon^{1/2}T \qquad \forall i \in \mathbb{Z} \cap [-\bar CT, \bar CT/2]. \tag{42}$$

Proof. As $\frac{d}{ds}\tilde M^{i_0,\omega}_{e,T}(\tilde s) < 0$ there exists $i_1 \in \mathbb{Z} \cap [-\bar CT, \bar CT/2]$ such that $(\tilde s, i_1)$ is a minimum point in the definition of $\tilde M^{i_0,\omega}_{e,T}(\tilde s)$.

By the envelope theorem, we have

$$0 > \frac{d}{ds}\tilde M^{i_0,\omega}_{e,T}(\tilde s) = \partial_x\xi_e^\omega(U^\omega_{e,i_1}(\tilde s), i_1)\; V_{Z^\omega_{i_1}}(U^\omega_{e,i_1+1}(\tilde s) - U^\omega_{e,i_1}(\tilde s), U^\omega_{e,\ell^\omega_{i_1}}(\tilde s) - U^\omega_{e,i_1}(\tilde s), U^\omega_{e,i_1}(\tilde s)) - \tilde h. \tag{43}$$

On the other hand, by the optimality of i_1, we have

$$\xi_e^\omega(U^\omega_{e,i}(\tilde s), i) - \xi_e^\omega(U^\omega_{e,i_1}(\tilde s), i_1) \geqslant \xi_e^\omega(W^{\tau_{i_0}\omega}_{i-i_0}, i-i_0) - \xi_e^\omega(W^{\tau_{i_0}\omega}_{i_1-i_0}, i_1-i_0) \qquad \forall i \in \mathbb{Z} \cap [-\bar CT, \bar CT/2]. \tag{44}$$

We first claim that $i_1 \geqslant -\bar CT/2$. Indeed, since we are in $E_{\epsilon,T}$, inequality (37) in Lemma 3.11 implies that, for any $i \in [-\bar CT, -\bar CT/2]$ (which implies that $i \leqslant C_\theta t \leqslant i_0$, $U^\omega_{e,i}(\tilde s) \leqslant -R_0$ and $W^{\tau_{i_0}\omega}_{i-i_0} \leqslant -R_0$ if $\bar C$ is big enough),

$$\begin{aligned}
\xi_e^\omega(U^\omega_{e,i}(\tilde s), i) - \xi_e^\omega(W^{\tau_{i_0}\omega}_{i-i_0}, i-i_0) - i_0 - \tilde h\tau &\geqslant (e^0)^{-1}(e^0 i + v_e^0\tilde s) - (e^0)^{-1}(e^0(i-i_0)) - i_0 - \tilde h\tilde s - 4\epsilon(e^0)^{-1}T \\
&\geqslant ((e^0)^{-1}v_e^0 - \tilde h)\tilde s - 4\epsilon(e^0)^{-1}T > 0
\end{aligned}$$

since $(e^0)^{-1}v_e^0 > \tilde h$ and $\tilde s > c_1\epsilon T$ where c_1 is large enough. This shows that $i_1 \geqslant -\bar CT/2$ because $\tilde M^{i_0,\omega}_{e,T}(\tilde s) < 0$.

In the same way, we have $i_1 \leqslant 2(\min_k e^k)^{-1}\epsilon T$. Indeed, for $i \in [2(\min_k e^k)^{-1}\epsilon T, \bar CT/2]$, by (36) in Lemma 3.11 and for $k = T_i^\omega$, we have

$$\begin{aligned}
\xi_e^\omega(U^\omega_{e,i}(\tilde s), i) - \xi_e^\omega(W^{\tau_{i_0}\omega}_{i-i_0}, i-i_0) - i_0 - \tilde h\tilde s &\geqslant (e^k)^{-1}(e^k i + v_e^k\tilde s) - (e^k)^{-1}(e^k(i-i_0)) - i_0 - \tilde h\tilde s - 4\epsilon(e^k)^{-1}T \\
&> ((e^k)^{-1}v_e^k - \tilde h)\tilde s - 4\epsilon(e^k)^{-1}T > 0.
\end{aligned}$$

Since $\tilde M^{i_0,\omega}_{e,T}(\tilde s) < 0$, this shows that $i_1 \leqslant 2(\min_k e^k)^{-1}\epsilon T$. Recalling the definition of $\tilde E_{\epsilon,T}$, we also have therefore that $\ell_{i_1} \leqslant \bar CT/2$.

We now prove that $|U^\omega_{i_1}(\tilde s)| \leqslant C\epsilon^{1/2}T$. By contradiction, assume first that $U^\omega_{e,i_1}(\tilde s) \geqslant C\epsilon^{1/2}T$, where C is to be chosen below. Then, as $|\tilde s - \bar s| \leqslant C_1\epsilon^{1/2}T$ we obtain $U^\omega_{e,i_1}(\bar s) > 0$ for C large enough. Since $i_0 = -\theta_e^\omega(\bar s)$, we get that $i_1 \geqslant i_0 + 1$. Using successively that $U^\omega_{e,i_1}(\tilde s) > 0$ (for the first equality), (44) and the fact that $\ell_{i_1} \leqslant \bar CT/2$ and that $\ell^\omega_{i_1} - i_0 = \ell^{\tau_{i_0}\omega}_{i_1-i_0}$ (for the inequality), and the definition of the (W_i) (for the last equality), we have, for $k = T^\omega_{i_1}$,

$$\begin{aligned}
&\partial_x\xi_e^\omega(U^\omega_{e,i_1}(\tilde s), i_1)\; V_{Z^\omega_{i_1}}(U^\omega_{e,i_1+1}(\tilde s) - U^\omega_{e,i_1}(\tilde s), U^\omega_{e,\ell^\omega_{i_1}}(\tilde s) - U^\omega_{e,i_1}(\tilde s), U^\omega_{e,i_1}(\tilde s)) \\
&\quad = (e^k)^{-1}V^k_{Z^{\tau_{i_0}\omega}_{i_1-i_0}}(U^\omega_{e,\ell^\omega_{i_1}}(\tilde s) - U^\omega_{e,i_1}(\tilde s)) \\
&\quad \geqslant (e^k)^{-1}V^k_{Z^{\tau_{i_0}\omega}_{i_1-i_0}}(W^{\tau_{i_0}\omega}_{\ell^{\tau_{i_0}\omega}_{i_1-i_0}} - W^{\tau_{i_0}\omega}_{i_1-i_0}) = (e^k)^{-1}v_e^k > \tilde h,
\end{aligned}$$

which contradicts (43). Assume now that $U^\omega_{e,i_1}(\tilde{s}) \leqslant (-C'\epsilon^{1/2}T) \wedge (-R_0)$. Then as before, we have $i_1 + 1 \leqslant i_0$ if C' is large enough and we get

$$\begin{aligned}
&\partial_x \xi^\omega_e(U^\omega_{e,i_1}(\tilde{s}), i_1)\ V_{Z^\omega_{i_1}}(U^\omega_{e,i_1+1}(\tilde{s}) - U^\omega_{e,i_1}(\tilde{s}), U^\omega_{e,\ell^\omega_{i_1}}(\tilde{s}) - U^\omega_{e,i_1}(\tilde{s}), U^\omega_{e,i_1}(\tilde{s})) \\
&\quad = (e^0)^{-1}\ V^0_{Z^\omega_{i_1}}(U^\omega_{e,i_1+1}(\tilde{s}) - U^\omega_{e,i_1}(\tilde{s})) \\
&\quad \geqslant (e^0)^{-1} V^0_{Z^{\tau_{i_0}\omega}_{i_1-i_0}}(W^{\tau_{i_0}\omega}_{i_1+1-i_0} - W^{\tau_{i_0}\omega}_{i_1-i_0}) = (e^0)^{-1} v^0_e > \tilde{h}.
\end{aligned}$$

This gives again a contradiction with (43) and show that $|U^\omega_{i_1}(\tilde{s})| \leqslant C\epsilon^{1/2}T$ for C large enough.

We now claim that this inequality and the fact that $i_0 = -\theta^\omega_e(\bar{s})$ imply that $|i_0 - i_1| \leqslant C\epsilon^{1/2}T$: indeed, let s be such that $U^\omega_{i_1}(s) = 0$ (if it exists, otherwise, we set $s = 0$). Then, by Lemma 3.2, we have

$$|U^\omega_{e,i_1}(\tilde{s})| \geqslant |U^\omega_{e,i_1}(\tilde{s}) - U^\omega_{e,i_1}(s)| \geqslant \delta|\tilde{s} - s|,$$

so that $|\tilde{s} - s| \leqslant \delta^{-1}C'\epsilon^{1/2}T$. If $i_1 \leqslant 0$, by the definition of s, $\theta^\omega_e(s) = -i_1$, we get, recalling Lemma 3.3:

$$|i_0 - i_1| = |\theta^\omega_e(s) - \theta^\omega_e(\bar{s})| \leqslant C_\theta(|s - \bar{s}| + 1) \leqslant C_\theta(|s - \tilde{s}| + |\tilde{s} - \bar{s}| + 1) \leqslant C\epsilon^{1/2}T.$$

If $i_1 \geqslant 0$, we get in the same way $|i_0| \leqslant C\epsilon^{1/2}T$ and so $|i_0 - i_1| \leqslant C\epsilon^{1/2}T$. By the choice of $\bar{C}$ in (35), we have that $i_0 \in [-\bar{C}T/10, 0]$. Thus, for ϵ small enough, we obtain also $i_1 \in [-\bar{C}T, \bar{C}T/2]$. Coming back to (44) we obtain therefore, using the facts that ξ is uniformly Lipschitz continuous, $|U^\omega_{i_1}(\tilde{s})| \leqslant C\epsilon^{1/2}T$ $|i_0 - i_1| \leqslant C\epsilon^{1/2}T$ and the fact that we are in $E_{\epsilon,T}$,

$$\xi^\omega_e(U^\omega_{e,i}(\tilde{s}), i) \geqslant \xi^\omega_e(W^{\tau_{i_0}\omega}_{i-i_0}, i - i_0) - C\epsilon^{1/2}T \qquad \forall i \in \mathbb{Z} \cap [-\bar{C}T, \bar{C}T/2].$$

Since the inverse of ξ is increasing and uniformly Lipschitz continuous, we get

$$U^\omega_{e,i}(\tilde{s}) \geqslant W^{\tau_{i_0}\omega}_{i-i_0} - C\epsilon^{1/2}T \qquad \forall i \in \mathbb{Z} \cap [-\bar{C}T, \bar{C}T/2].$$

Recalling that $\omega \in E_{\epsilon,T}$, we find (42). □

Next we show that we can bound from below $M^\omega_{e,T}(t)$ by $\bar{M}_{e,\tilde{h}}(t)$:

Lemma 3.15. *Let ϵ, T be such that $\epsilon T \geqslant \bar{C}$. There exists a constant C such that, in $\tilde{E}_{\epsilon,T}$ and for $t \in [0,T]$, we have*

$$M^\omega_{e,T}(t) \geqslant \bar{M}_{e,\tilde{h}}(t) - C\epsilon^{1/2}T.$$

Proof. Let

$$\bar{s} = \sup\Big\{s \in [0,t],\ M^\omega_{e,T}(s) \geqslant M^\omega_{e,T}(t) + 4\epsilon^{1/2}(2T - s)\Big\}.$$

If there is no such a $s \in [0,t]$, then $M^\omega_{e,T}(t) \geqslant -8\epsilon^{1/2}T$ since $M^\omega_{e,T}(0) = 0$. So we have $M^\omega_{e,T}(t) \geqslant -C\epsilon^{1/2}T$ while $\bar{M}_{e,\tilde{h}}(t) \leqslant 0$, which proves the result in this case. In the same way, if $\bar{s} \leqslant c_1\epsilon T$ (for c_1 to be chosen below), then, since $s \to M^\omega_{e,T}(s)$ is Lipschitz continuous and $M^\omega_{e,T}(0) = 0$, $M^\omega_{e,T}(\bar{s}) \geqslant -C\bar{s} \geqslant -Cc_1\epsilon T$ and by the definition of $\bar{s}$ we have $M^\omega_{e,T}(\bar{s}) = M^\omega_{e,T}(t) + 4\epsilon^{1/2}(2T - \bar{s})$. So $M^\omega_{e,T}(t) \geqslant -C\epsilon^{1/2}T$ while $\bar{M}_{e,\tilde{h}}(t) \leqslant 0$, which proves the result also in this case.

Assume now that $\bar{s}$ exists and satisfies $\bar{s} > c_1\epsilon T$. We also suppose in a first step that $\bar{s} + C'\epsilon^{1/2}T \leqslant t$ with $C' = C/2$ where C is given by Lemma 3.13. Note that $M^\omega_{e,T}(\bar{s}) = M^\omega_{e,T}(t) + 4\epsilon^{1/2}(2T - \bar{s})$. Let $i_0 = -\theta^\omega_e(\bar{s})$. Then, recalling Lemma 3.13 and the definition of $\bar{s}$, we have

$$\begin{aligned}
\tilde{M}^{i_0,\omega}_{e,T}(\bar{s}) &\geqslant M^\omega_{e,T}(\bar{s}) - C\epsilon T = M^\omega_{e,T}(t) + 4\epsilon^{1/2}(2T - \bar{s}) - C\epsilon T \\
&> M^\omega_{e,T}(\bar{s} + C'\epsilon^{1/2}T) - 4\epsilon^{1/2}(2T - \bar{s} - C'\epsilon^{1/2}T) + 4\epsilon^{1/2}(2T - \bar{s}) - C\epsilon T \\
&= M^\omega_{e,T}(\bar{s} + \epsilon^{1/2}T) + C\epsilon T \\
&> \tilde{M}^{i_0,\omega}_{e,T}(\bar{s} + \epsilon^{1/2}T).
\end{aligned}$$

So there exists $\tilde{s} \in [\bar{s}, \bar{s} + \epsilon^{1/2}T]$ such that $(d/ds)\tilde{M}^{i_0,\omega}_{e,T}(\tilde{s}) < 0$. Note that $|\tilde{s} - \bar{s}| \leqslant C\epsilon^{1/2}T$. Let $(\tilde{s}, i_1)$ (where $i_1 \in \mathbb{Z} \cap [-\bar{C}T, \bar{C}T/2]$) be a minimum point in the definition of $\tilde{M}^{i_0,\omega}_{e,T}(\tilde{s})$. From Lemma 3.14 we know that $|U^{\omega}_{e,i_1}(\tilde{s})| \leqslant C\epsilon^{1/2}T$ and $|i_1 - i_0| \leqslant C\epsilon^{1/2}T$ for a large constant C. Then we have (since we are in $\tilde{E}_{\epsilon,T}$ and by Lemma 3.13),

$$\begin{aligned}
M^{\omega}_{e,T}(t) &\geqslant M^{\omega}_{e,T}(\bar{s}) - C\epsilon^{1/2}T \geqslant \tilde{M}^{i_0,\omega}_{e,T}(\tilde{s}) - C\epsilon^{1/2}T \\
&= \xi^{\omega}_e(U^{\omega}_{e,i_1}(\tilde{s}), i_1) - \xi^{\omega}_e(W^{\tau_{i_0}\omega}_{i_1-i_0}, i_1 - i_0) - i_0 - \tilde{h}\tilde{s} - C\epsilon^{1/2}T \\
&\geqslant -C\epsilon^{1/2}T - i_0 - \tilde{h}\bar{s} \geqslant -C\epsilon^{1/2}T + \theta^{\omega}_e(\bar{s}) - \tilde{h}\bar{s} \\
&\geqslant -C\epsilon^{1/2}T + \bar{\theta}_e(\bar{s}) - \tilde{h}\bar{s} \geqslant -C\epsilon^{1/2}T + \bar{M}_{e,\tilde{h}}(t).
\end{aligned}$$

This proves the result.

The case where $\bar{s}$ satisfies $\bar{s} > c_1\epsilon T$ and $\bar{s} + C'\epsilon^{1/2}T > t$ can be treated in a similar way, by noticing in a first step that

$$\tilde{M}^{i_0,\omega}_{e,T}(\bar{s}) > \tilde{M}^{i_0,\omega}_{e,T}(t)$$

and concluding as in the previous case that there exists a minimizer i_1 for $\tilde{M}^{i_0,\omega}_{e,T}(\tilde{s})$ (for some $\tilde{s} \in [\bar{s}, t]$ and thus such that $|\tilde{s} - \bar{s}| \leqslant C't\epsilon^{1/2}T$ since $0 \leqslant t - \bar{s} \leqslant C'\epsilon^{1/2}T$) such that $|U^{\omega}_{i_1}(\tilde{s})| \leqslant C\epsilon^{1/2}T$ and $|i_0 - i_1| \leqslant C\epsilon^{1/2}T$. We can then complete the proof as above. □

In the next step we show that $\bar{M}_{e,\tilde{h}}(t)$ is almost superadditive.

Lemma 3.16. *There is a constant C such that, for any $t \geqslant C$ and any $h \in [0, t]$,*

$$\bar{M}_{e,\tilde{h}}(t + h) \geqslant \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(t) - C(1 + (\ln(t))^{1/8}t^{7/8}). \tag{45}$$

Proof. Fix $0 \leqslant h \leqslant t \leqslant T := 2t + 1$ and $\epsilon > 0$ small enough so that $t \geqslant c_1\epsilon T$, where c_1 is as in Lemma 3.14. We also assume that t is large enough so that $\epsilon T \geqslant \bar{C}$. Let consider

$$\bar{t} = \inf\{s \in [t, t + h],\ \bar{M}_{e,\tilde{h}}(s) < \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}s\}$$

if there is some $s \in [t, t + h]$ such that $\bar{M}_{e,\tilde{h}}(s) < \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}s$ and set $\bar{t} = t + h$ otherwise. If $\bar{t} = t + h$, since $\bar{M}_{e,\tilde{h}}(h) \leqslant 0$, we have

$$\bar{M}_{e,\tilde{h}}(t + h) \geqslant \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}(t + h) \geqslant \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}(t + h).$$

Our aim is to show that a similar inequality holds if $\bar{t} < t + h$.

Let us first consider the case where $\bar{t} < t + h$ and $\bar{t} - \gamma T > t$, where $\gamma = C'\epsilon^{1/4}$ for some large constant C' to be chosen below. Then $\bar{M}_{e,\tilde{h}}(\bar{t}) = \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}\bar{t}$. In $\tilde{E}_{\epsilon,T}$, we have by Lemma 3.13 (for the first inequality) and Lemma 3.15 (for the second one)

$$M^{\omega}_{e,T}(\bar{t}) \leqslant \bar{M}_{e,\tilde{h}}(\bar{t}) + \epsilon T = \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}\bar{t} + \epsilon T \leqslant M^{\omega}_{e,T}(t) + C\epsilon^{1/2}T - \epsilon^{1/4}\bar{t}. \tag{46}$$

As $\bar{t} - \gamma T \in (t, t + h)$, we have, by the definition of $\bar{t}$,

$$\bar{M}_{e,\tilde{h}}(\bar{t} - \gamma T) \geqslant \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}(\bar{t} - \gamma T),$$

so that

$$\begin{aligned}
M^{\omega}_{e}(\bar{t} - \gamma T) &\geqslant \bar{M}_{e,\tilde{h}}(\bar{t} - \gamma T) - C\epsilon^{1/2}T \geqslant \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}(\bar{t} - \gamma T) - C\epsilon^{1/2}T \\
&\geqslant M^{\omega}_{e,T}(t) - \epsilon^{1/4}\bar{t} + \epsilon^{1/4}\gamma T - C\epsilon^{1/2}T \\
&\geqslant M^{\omega}_{e,T}(\bar{t}) - C\epsilon^{1/2}T + \epsilon^{1/4}\gamma T \qquad \text{(by (46))} \\
&\geqslant M^{\omega}_{e,T}(\bar{t}) + C^{-1}\epsilon^{1/4}T,
\end{aligned}$$

if we choose $\gamma = C'\epsilon^{1/4}$ for C' large enough and independent of t and T. Let us set $i_0 = -\theta_e^\omega(\bar{t})$. For ϵ small enough, the inequality above implies (by Lemma 3.13) that

$$\tilde{M}_{e,T}^{i_0,\omega}(\bar{t}-\gamma t) > \tilde{M}_{e,T}^{i_0,\omega}(\bar{t}).$$

So there exists $\tilde{t} \in [\bar{t}-\gamma t, \bar{t}]$ such that $\tilde{M}_{e,T}^{i_0,\omega}(\tilde{t}) < 0$ and $(d/dt)\tilde{M}_{e,T}^{i_0,\omega}(\tilde{t}) < 0$. Note that $|\tilde{t}-\bar{t}| \leqslant \gamma t = C'\epsilon^{1/4}t$ and that $\tilde{t} \geqslant t \geqslant c_1\epsilon T$. According to Lemma 3.14, there exists i_1 such that $(\tilde{t}, i_1)$ is a minimizer of $\tilde{M}_{e,T}^{i_0,\omega}(\tilde{t})$, and i_1 satisfies $|U_{e,i_1}^\omega(\tilde{t})| \leqslant C\epsilon^{1/4}T$, $|i_1 - i_0| \leqslant C\epsilon^{1/4}T$ and

$$U_{e,i}^\omega(\tilde{t}) \geqslant e^0(i-i_0)\mathbf{1}_{i\leqslant i_0} + e^{T_i^\omega}(i-i_0)\mathbf{1}_{i>i_0} - C\epsilon^{1/4}T \qquad \forall i \in \mathbb{Z}\cap[-\bar{C}T, \bar{C}T/2].$$

Let us set $j_0 := -\bar{\theta}_e(\bar{t}) + [C''\epsilon^{1/4}T]$, where C'' is a large constant. Using the definition of ξ_e (for the first line and the last line), (39) (for the second line) and the fact that we are in $\tilde{E}_{\epsilon,T}$ (for the third line), we have, if C'' is large enough and for any $i \in \mathbb{Z}\cap[-\bar{C}T, \bar{C}T/2]$,

$$\begin{aligned}
U_{e,i}^\omega(\tilde{t}) \geqslant& \xi_e^\omega(i-i_0, i) - C\epsilon^{1/4}T - 1\\
\geqslant& \xi_e^\omega(i-j_0, i) + C^{-1}(j_0 - i_0) - C\epsilon^{1/4}T\\
\geqslant& \xi_e^\omega(i-j_0, i) + C^{-1}(\theta_e^\omega(\bar{t}) - \bar{\theta}_e(\bar{t}) + C''\epsilon^{1/4}T - 1) - C\epsilon^{1/4}T\\
\geqslant& e^0(i-j_0)\mathbf{1}_{i\leqslant j_0} + e^{T_i^\omega}(i-j_0)\mathbf{1}_{i>j_0}.
\end{aligned}$$

As the solution starting from $e^0(i-j_0)\mathbf{1}_{i\leqslant j_0} + e^{T_i^\omega}(i-j_0)\mathbf{1}_{i>j_0}$ is $U_{e,i-j_0}^{\tau_{j_0}\omega}$, the approximate finite speed of propagation then implies for all $n \geqslant 1$

$$U_{e,i}^\omega(\tilde{t}+s) \geqslant U_{e,i-j_0}^{\tau_{j_0}\omega}(s) - 2^{-n}e^{\beta s} \qquad \forall s \geqslant 0,\ \forall i \in \mathbb{Z}\cap[-\bar{C}T, J_n([\bar{C}T/2])].$$

Choosing $n = [\bar{C}T/(16\alpha)] + 1$ (with α as in Lemma 2.7), we obtain by the choice of $\bar{C}$ in (35) (which ensures that $-n\ln(2) + \beta T \leqslant 0$) since we are in $\tilde{E}_{\epsilon,T}$ (where $J_n([\bar{C}T/2]) \geqslant 0$):

$$U_{e,i}^\omega(\tilde{t}+s) \geqslant U_{e,i-j_0}^{\tau_{j_0}\omega}(s) - 1 \geqslant U_{e,i-j_0}^{\tau_{j_0}\omega}(s-\delta^{-1}) \qquad \forall s \in [\delta^{-1}, T],\ \forall i \in \mathbb{Z}\cap[-\bar{C}T, 0],$$

where δ is given by Lemma 3.2. Hence, in $\tilde{E}_{\epsilon,T}$, and for $s \in [\delta^{-1}, h]$:

$$\theta_e^\omega(\tilde{t}+s) \geqslant \theta_e^{\tau_{j_0}\omega}(s-\delta^{-1}) - j_0.$$

As $|\tilde{t}-\bar{t}| \leqslant \gamma t = C'\epsilon^{1/4}t$ and by the definition of j_0 we get, in view of Lemma 3.3:

$$\theta_e^\omega(\bar{t}+s) \geqslant \theta_e^{\tau_{j_0}\omega}(s) + \bar{\theta}_e(\bar{t}) - C\epsilon^{1/4}T \qquad \forall s \in [0, h].$$

Recalling the bounds on θ_e and on $\mathbb{P}[E_{\epsilon,T}^c]$ in (41) we obtain by taking expectation and for $s \in [0, h]$:

$$\bar{\theta}_e(\bar{t}+s) \geqslant \bar{\theta}_e(s) + \bar{\theta}_e(\bar{t}) - CT\epsilon^{1/4} - CT\mathbb{P}[E_{\epsilon,T}^c] \geqslant \bar{\theta}_e(s) + \bar{\theta}_e(\bar{t}) - C(\epsilon^{1/4}T + T^3\exp\{-\epsilon^2 T/C\}).$$

Thus, using that $t - \bar{t} \leqslant 0$,

$$\begin{aligned}
\inf_{s\in[0,h+t-\bar{t}]} \bar{\theta}_e(\bar{t}+s) - \tilde{h}(\bar{t}+s) \geqslant& \bar{M}_{e,\tilde{h}}(h+t-\bar{t}) + \bar{\theta}_e(\bar{t}) - \tilde{h}\bar{t} - C(\epsilon^{1/4}T + T^3\exp\{-\epsilon^2 T/C\})\\
\geqslant& \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(\bar{t}) - C(\epsilon^{1/4}T + T^3\exp\{-\epsilon^2 T/C\})\\
\geqslant& \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(t) - C(\epsilon^{1/4}T + T^3\exp\{-\epsilon^2 T/C\}),
\end{aligned}$$

where the last inequality comes from the fact that $\bar{M}_{e,\tilde{h}}(\bar{t}) = \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}\bar{t}$. On the other hand, picking $\omega \in \tilde{E}_{\epsilon,T}$ and using successively Lemma 3.13, inequality $|\tilde{t}-\bar{t}| \leqslant C'\epsilon^{1/4}t$, the definition of $\tilde{t}$ and the fact that $|U_{e,i_1}^\omega(\tilde{t})| \leqslant C\epsilon^{1/4}T$ and that $|i_1 - i_0| \leqslant C\epsilon^{1/4}T$ with $i_0 = -\theta_e^\omega(\bar{t})$, we obtain

$$\begin{aligned}
\bar{M}_{e,\tilde{h}}(\bar{t}) \geqslant& M_{e,T}^\omega(\bar{t}) - \epsilon T \geqslant \tilde{M}_{e,T}^{i_0,\omega}(\bar{t}) - C\epsilon T \geqslant \tilde{M}_{e,T}^{i_0,\omega}(\tilde{t}) - C\epsilon^{1/4}T\\
=& \xi_e^\omega(U_{e,i_1}^\omega(\tilde{t}), i_1) - \xi_e^\omega(W_{i_1-i_0}^{\tau_{i_0}\omega}, i_1 - i_0) - i_0 - \tilde{h}\tilde{t} - C\epsilon^{1/4}T\\
\geqslant& -i_1 - C\epsilon T - \tilde{h}\bar{t} - C\epsilon^{1/4}T \geqslant \theta_e^\omega(\bar{t}) - \tilde{h}\bar{t} - C\epsilon^{1/4}T \geqslant \bar{\theta}_e(\bar{t}) - \tilde{h}\bar{t} - C\epsilon^{1/4}T.
\end{aligned}$$

Therefore

$$\bar{M}_{e,\tilde{h}}(t+h) = \inf\{\bar{M}_{e,\tilde{h}}(\bar{t}) \ , \ \inf_{s\in[0,h+t-\bar{t}]} \bar{\theta}_e(\bar{t}+s) - \tilde{h}(\bar{t}+s)\} \geqslant \inf_{s\in[0,h+t-\bar{t}]} \bar{\theta}_e(\bar{t}+s) - \tilde{h}(\bar{t}+s) - C\epsilon^{1/4}T.$$

So we have obtained the following inequality:

$$\bar{M}_{e,\tilde{h}}(t+h) \geqslant \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(t) - C(\epsilon^{1/4}T + T^3\exp\{-\epsilon^2 T/C\}). \tag{47}$$

In order to handle the case where $\bar{t} < t+h$ and $\bar{t} - \gamma T \leqslant t$, we note that, by the choice of T we have $\bar{t} \geqslant t \geqslant T/3$. Then, for $\epsilon > 0$ small enough, we have in $\tilde{E}_{\epsilon,T}$:

$$\begin{aligned} M^{\omega}_{e,T}(\bar{t}) &\leqslant \bar{M}_{e,\tilde{h}}(\bar{t}) + C\epsilon T = \bar{M}_{e,\tilde{h}}(t) - \epsilon^{1/4}\bar{t} + C\epsilon T \\ &\leqslant M^{\omega}_{e,T}(t) - \epsilon^{1/4}T/3 + C\epsilon^{1/2}T < M^{\omega}_{e,T}(t). \end{aligned}$$

Thus we can find $\tilde{t} \in [t,\bar{t}]$ such that $\tilde{M}^{i_0,\omega}_{e,T}(\tilde{t}) < 0$ and $(d/dt)\tilde{M}^{i_0,\omega}_{e,T}(\tilde{t}) < 0$. Note also that $|\bar{t}-\tilde{t}| \leqslant C'\epsilon^{1/4}T$ because $\bar{t} - \gamma T \leqslant t$. Then we can complete the proof as in the previous case.

We now know that (47) always holds. If we choose $\epsilon = (3C\ln(T)/T)^{1/2}$ (where C is the constant in (47) and where this choice is possible since then $t \geqslant \bar{C}\epsilon^{-1}$ for ϵ small), then, as $T = 2t+1$, (47) becomes

$$\bar{M}_{e,\tilde{h}}(t+h) \geqslant \bar{M}_{e,\tilde{h}}(h) + \bar{M}_{e,\tilde{h}}(t) - C(1 + (\ln(t))^{1/8}t^{7/8}).$$

This holds for any $0 \leqslant h \leqslant t$ with $t \geqslant C$ for some large constant C. □

The main consequence of the previous lemma is the following:

Lemma 3.17. *The limit $k_{e,\tilde{h}} := \lim_{t\to+\infty} \bar{M}_{e,\tilde{h}}(t)/t$ exists and is nonpositive as $t \to +\infty$. If $k_{e,\tilde{h}} < 0$, then $\mathbb{E}[\theta_e(t)]/t$ has a limit as $t \to +\infty$ given by $(k_{e,\tilde{h}} + \tilde{h})$. If $k_{e,\tilde{h}} = 0$, then $\liminf_{t\to+\infty} \frac{\mathbb{E}[\theta_e(t)]}{t} \geqslant \tilde{h}$.*

Proof. As $\bar{M}_{e,\tilde{h}}(t)$ satisfies the almost superadditivity property (45), the limit $k_{e,\tilde{h}} := \lim_{t\to+\infty} \bar{M}_{e,\tilde{h}}(t)/t$ exists. If $k_{e,\tilde{h}} = 0$, then, as, by the definition of $\bar{M}_{e,\tilde{h}}(t)$, we have $\bar{M}_{e,\tilde{h}}(t) \leqslant \bar{\theta}_e(t) - \tilde{h}t$ and we obtain

$$\liminf_{t\to+\infty} \bar{\theta}_e(t)/t \geqslant \tilde{h}.$$

Let us now assume that $k_{e,\tilde{h}} < 0$. By the definition of $\bar{M}_{e,\tilde{h}}(t)$ we know that

$$\liminf_{t\to+\infty} \frac{\bar{\theta}_e(t)}{t} \geqslant \liminf_{t\to+\infty} \frac{\bar{M}_{e,\tilde{h}}(t) + \tilde{h}t}{t} = k_{e,\tilde{h}} + \tilde{h}.$$

To prove the opposite inequality, let $\epsilon \in (0, |k_{e,\tilde{h}}|/2)$ and $T > 0$ be such that $|\bar{M}_{e,\tilde{h}}(t)/t - k_{e,\tilde{h}}| < \epsilon$ for any $t \geqslant T$. Fix $t \geqslant T$ and let us define $\bar{t} = \sup\{s \geqslant t,\ \bar{M}_{e,\tilde{h}}(t) = \bar{M}_{e,\tilde{h}}(s)\} \in [t,+\infty]$. Then, for $s \in [t,\bar{t})$, we have, since $s \geqslant T$,

$$-\epsilon \leqslant \frac{\bar{M}_{e,\tilde{h}}(t)}{t} - k_{e,\tilde{h}} = \frac{\bar{M}_{e,\tilde{h}}(s)}{t} - k_{e,\tilde{h}} = \frac{s}{t}\frac{\bar{M}_{e,\tilde{h}}(s)}{s} - k_{e,\tilde{h}} \leqslant \frac{s}{t}(k_{e,\tilde{h}} + \epsilon) - k_{e,\tilde{h}}.$$

So $s \leqslant t(k_{e,\tilde{h}} - \epsilon)/(k_{e,\tilde{h}} + \epsilon) \leqslant (1 + C\epsilon)t$. In particular, $\bar{t}$ is finite and satisfies $\bar{t} \leqslant t(1 + C\epsilon)$. Note that, by the definitions of $\bar{t}$ and of $\bar{M}_{e,\tilde{h}}(t)$, we have $\bar{\theta}_e(\bar{t}) - \tilde{h}\bar{t} = \bar{M}_{e,\tilde{h}}(\bar{t})$. Therefore, as $\bar{\theta}_e$ is nonnegative and nondecreasing,

$$\frac{\bar{\theta}_e(t)}{t} \leqslant \frac{\bar{\theta}_e(\bar{t})}{t} \leqslant (1+C\epsilon)\frac{\bar{\theta}_e(\bar{t})}{\bar{t}} = (1+C\epsilon)\frac{\bar{M}_{e,\tilde{h}}(\bar{t}) + \tilde{h}\bar{t}}{\bar{t}} \leqslant (1+C\epsilon)(k_{e,\tilde{h}} + \epsilon + \tilde{h}).$$

So

$$\limsup_{t\to+\infty} \frac{\bar{\theta}_e(t)}{t} \leqslant (1+C\epsilon)(k_{e,\tilde{h}} + \epsilon + \tilde{h}),$$

which proves that $\bar{\theta}_e(t)/t$ converges to $k_{e,\tilde{h}} + \tilde{h}$ since ϵ is arbitrary. □

As a consequence, we have

Lemma 3.18. *Let $\tilde{h} < -A_0$. Assume that $k_{e,\tilde{h}} < 0$, where $k_{e,\tilde{h}}$ is defined in Lemma 3.17. Then the limit $\bar{\vartheta}_e$ of $\theta_e(t)/t$ exists a.s. as $t \to +\infty$ and satisfies $\bar{\vartheta}_e < -A_0$.*

If $k_{e,\tilde{h}} = 0$ for any $\tilde{h} < -A_0$, then $\liminf_{t\to+\infty} \theta_e(t)/t$ is deterministic and is not smaller than $-A_0$.

Proof. Assume that $k_{e,\tilde{h}} < 0$. Let us first check that, a.s., $\theta_e(t)/t$ converges to the limit $\bar{\vartheta}_e$ of $\bar{\theta}_e(t)/t = \mathbb{E}[\theta_e(t)]/t$ given by Lemma 3.17 and which satisfies $\bar{\vartheta}_e < -A_0$. This is a classical consequence of the variance estimate in Theorem 3.4. Fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $|\bar{\theta}_e(n)/n - \bar{\vartheta}_e| \leqslant \epsilon$ for any $n \in \mathbb{N}$, $n \geqslant N$. By the variance estimate, we have

$$\mathbb{P}\left[\left|\frac{\theta_e(n)}{n} - \bar{\vartheta}_e\right| > 2\epsilon\right] \leqslant \mathbb{P}\left[\left|\frac{\theta_e(n)}{n} - \frac{\bar{\theta}_e(n)}{n}\right| > \epsilon\right] \leqslant 2\exp\{-\epsilon^2 n/C\},$$

where the right-hand side is summable. So by the Borel-Cantelli Lemma we have, a.s.

$$\bar{\vartheta}_e - 2\epsilon \leqslant \liminf_n \frac{\theta_e(n)}{n} \leqslant \limsup_n \frac{\theta_e(n)}{n} \leqslant \bar{\vartheta}_e + 2\epsilon.$$

As ϵ is arbitrary, this implies the a.s. convergence of $(\theta_e(n)/n)$ to $\bar{\vartheta}_e$. The convergence of $(\theta_e(t)/t)$ to $\bar{\vartheta}_e$ as the continuous variable t tends to infinity comes directly from the regularity in time of θ_e (Lemma 3.3). The proof in the case $k_{e,\tilde{h}} = 0$ goes exactly along the same lines. □

Proof of Theorem 3.12. It is a straightforward application of the previous lemmas. □

4 Definition of the flux limiter and homogenization

We recall that $e = (e^k)_{k=0,\dots,K}$ is such that $H^k(-1/e^k) = \min_p H^k(p)$. The aim of this part is to define the flux limiter $\bar{A}$, building on the construction of $\bar{\vartheta}_e$ in the previous section. For this we introduce new notation. Recall that $A_0 = \max_{k\in\{0,\dots,K\}} \min_{p\in\mathbb{R}} H^k(p)$. Given $A \in [A_0, 0)$ and $k \in \{0, \dots, K\}$, we denote by $p_A^{k,+}$ (respectively $p_A^{k,-}$) the largest (resp. the smallest) solution to $H^k(p) = A$ and set

$$\phi_A(x,k) = \begin{cases} p_A^{0,-}x & \text{if } x \leqslant 0,\ k \in \{0,\dots,K\} \\ p_A^{k,+}x & \text{if } x \geqslant 0,\ k \in \{1,\dots,K\} \end{cases} \tag{48}$$

and

$$\psi_A(y,k) = \phi_A^{-1}(-y,k) = \begin{cases} (-p_A^{0,-})^{-1}y & \text{if } y \leqslant 0,\ k \in \{0,\dots,K\} \\ (-p_A^{k,+})^{-1}y & \text{if } y \geqslant 0,\ k \in \{1,\dots,K\}. \end{cases} \tag{49}$$

We note for later use that if $A_1 < A_2$, then $\phi_{A_1} < \phi_{A_2}$ and $\psi_{A_1} < \psi_{A_2}$ in $\mathbb{R}\backslash\{0\}$.

We define the flux limiter $\bar{A}$ as

$$\bar{A} := \begin{cases} -\bar{\vartheta}_e & \text{if } k_e < 0 \\ A_0 & \text{otherwise} \end{cases}$$

with k_e and $\bar{\vartheta}_e$ defined by Theorem 3.12. Note that if $k_e < 0$, then, by Theorem 3.12,

$$\bar{A} = -\bar{\vartheta}_e > \max_{k\in\{0,\dots,K\}} H^k(-1/e^k) = A_0.$$

4.1 The limit of N_e and U_e

We recall that U_e is the solution of (6) with initial condition $U^\omega_{e,i}(0) = e^0 i\mathbf{1}_{i\leqslant 0} + e^{T_i^\omega} i\mathbf{1}_{i\geqslant 0}$. Let us define, for $k \in \{1,\dots,K\}$ and $(x,t) \in \mathbb{R}\times[0,+\infty)$,

$$N_e^\omega(x,k,t) = \sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T_i=k} \delta_{U^\omega_{e,i}(t)}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{U^\omega_{e,i}(t)}((-\infty,x]) \tag{50}$$

and, for $(x,t) \in (-\infty,0] \times [0,+\infty)$,

$$N_e^\omega(x,0,t) = \sum_{k=1}^K N_e^k(x,t) = \sum_{i\in\mathbb{Z},\ i\leqslant 0} \delta_{U_{e,i}^\omega(t)}((x,+\infty)).$$

We set

$$n_e^{\epsilon,\omega}(x,k,t) = \epsilon N_e^\omega(x/\epsilon,k,t/\epsilon)$$

and

$$\nu_e^{\omega,\epsilon}(x,k,t) = \begin{cases} n_e^{\omega,\epsilon}(x,0,t) & \text{if } k=0 \text{ and } x \leqslant 0, \\ (\pi^k)^{-1} n_e^{\omega,\epsilon}(x,k,t) & \text{if } k \in \{1,\dots,K\} \text{ and } x \in \mathbb{R}. \end{cases}$$

Lemma 4.1. *There is a set Ω_0 of full probability such that, for all $\omega \in \Omega_0$,*

$$\lim_{\epsilon\to 0} \nu_e^{\omega,\epsilon}(x,k,0) = \begin{cases} -x/e^0 & \text{if } (x,k) \in (-\infty,0] \times \{0,\dots,K\}, \\ -x/e^k & \text{if } (x,k) \in [0,+\infty) \times \{1,\dots,K\}, \end{cases}$$

and the limit is locally uniform in x.

Proof. The result is obvious if $k=0$ since $U_{e,i}(0) = e^0 i$. We have, for $k \in \{1,\dots,K\}$ and $x \leqslant 0$,

$$\begin{aligned} N_e^\omega(x,k,0) &= \sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T_i=k} \delta_{e^0 i}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{e^k i}((-\infty,x]) \\ &= \sharp\{i\in\mathbb{Z},\ x/e^0 < i \leqslant 0,\ T_i = k\} \end{aligned}$$

(where $\sharp E$ is the cardinal of a set E). So, by the law of large number, we have, a.s. and locally uniformly in $(-\infty,0]$,

$$n_e^{\epsilon,\omega}(x,k,0) = \epsilon\sharp\{i\in\mathbb{Z},\ x/(\epsilon e^0) < i \leqslant 0,\ T_i = k\} \to -\pi^k x/e^0.$$

This proves the locally uniform convergence for $\nu_e^{\epsilon,\omega}(x,k,0)$ if $k \in \{1,\dots,K\}$ on $(-\infty,0]$.

The proof for $k \in \{1,\dots,K\}$ and $x > 0$ is similar: since

$$\begin{aligned} N_e^\omega(x,k,0) &= \sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T_i=k} \delta_{e^0 i}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{e^k i}((-\infty,x]) \\ &= -\sharp\{i\in\mathbb{Z},\ 0 < i < x/e^k,\ T_i = k\}, \end{aligned}$$

we have as above that

$$n_e^{\epsilon,\omega}(x,k,0) = -\epsilon\sharp\{i\in\mathbb{Z},\ 0 < i < x/(\epsilon e^k),\ T_i = k\} \to -\pi^k x/e^k,$$

so that $\nu_e^{\epsilon,\omega}(x,k,0)$ converges a.s. and locally uniformly to $-x/e^k$. □

We now want to study the convergence of ν_e^ϵ as $\epsilon \to 0$. Let us first give a regularity result for ν_e^ϵ. This result will be proved later in a more general setting (see Lemma 4.8).

Lemma 4.2. *The function ν_e^ϵ satisfies, for any $k \in \{0,\dots,K\}$,*

$$|\nu_e^\epsilon(x,k,t) - \nu_e^\epsilon(y,k,s)| \leqslant C(|x-y| + |t-s| + \epsilon),$$

for a constant C depending on $\Delta_{\min}$, (π^k) and $\|V\|_\infty$ only.

Lemma 4.3. *Let e be such that $H^k(-1/e^k) = \min_p H^k(p)$ and assume that $k_e < 0$, where k_e is defined in Theorem 3.12. There exists a set Ω_0 of full probability such that, for all $\omega \in \Omega_0$, $\nu_e^{\omega,\epsilon}$ converges locally uniformly to ν_e which satisfies*

$$\nu_e(x,0,t) = \nu_e(x,1,t) = \cdots = \nu_e(x,K,t) \qquad \text{in } (-\infty,0] \times [0,+\infty)$$

and is given by

$$\nu_e(x,k,t) = \min\left\{\phi_{\bar A}(x,k) - \bar A t\ ,\ -x/e^k - tH^k(-1/e^k)\right\} \qquad \forall (x,k,t) \in \mathcal{R} \times [0,+\infty). \tag{51}$$

Proof. Let us denote by Ω_1 the intersection of the set of full probability measure given in Lemma 4.1 and in Lemma A.3. Let Ω_0 be the set of $\omega \in \Omega_1$ such that $\theta_e^\omega(t)/t$ converges to $\bar{\vartheta}_e$ as $t \to +\infty$. By Lemma 3.18 we know that $\mathbb{P}[\Omega_0] = 1$. Fix now $\omega \in \Omega_0$. By Lemma 4.2, we can consider a locally uniform limit w, up to a subsequence, of $n_e^{\omega,\epsilon}(\cdot, 0, \cdot)$ in $(-\infty, 0] \times [0, +\infty)$. Then, by standard homogenization [19] (see Subsection A.3 in Appendix), w solves

$$\begin{cases} \partial_t w + H^0(\partial_x w) = 0 \text{ in } (-\infty, 0) \times (0, +\infty) \\ w(x, 0) = -x/e^0 \text{ in } (-\infty, 0]. \end{cases} \tag{52}$$

Moreover, by the definition of θ_e, we have $n^{\omega,\epsilon}(0, 0, t) = \epsilon\theta_e^\omega(t/\epsilon)$. Therefore

$$w(0, t) = \bar{\vartheta}_e t \qquad \forall t \geqslant 0. \tag{53}$$

The solution to (52)-(53) is unique and given by

$$w(x, t) := \min\{-x/e^0 - H^0(-1/e^0)t \ , \ p_{\bar{A}}^{0,-} x - \bar{A}t\} \text{ in } (-\infty, 0) \times (0, +\infty), \tag{54}$$

since $\bar{A} > H^0(-1/e^0)$ and H^0 is convex (see Lemma A.2). Therefore the whole sequence $n_e^{\omega,\epsilon}(\cdot, 0, \cdot)$ converges to w locally uniformly on $(-\infty, 0] \times [0, +\infty)$ as $\epsilon \to 0$ for any $\omega \in \Omega_0$. We set $n_e(x, 0, t) = \nu_e(x, 0, t) := w(x, t)$.

Let us now fix $\omega \in \Omega_0$, $y < 0$ and set $i_\epsilon = [y/(e^0\epsilon)]$. Our next step is to show that, if $t < y/(e^0\bar{A})$, then

$$u(y, t) := \lim_{\epsilon \to 0} \epsilon U_{e,i_\epsilon}^\omega(t/\epsilon) = \min\{y - H^0(-1/e^0)e^0 t \ , \ (-p_{\bar{A}}^{0,-})^{-1}(y/e^0 - \bar{A}t)\}. \tag{55}$$

Indeed, as the map $t \to \epsilon U_{e,i_\epsilon}^\omega(t/\epsilon)$ is uniformly Lipschitz continuous with $\epsilon U_{e,i_\epsilon}^\omega(0) \to y$ as $\epsilon \to 0$, we can find a subsequence which converges to some map $t \to u(y, t)$. Assume that $U_{e,i_\epsilon}^\omega(t/\epsilon) \leqslant -R_0$ for any ϵ small enough. Then for any $i \in \mathbb{Z}$ with $i \leqslant 0$, one has thanks to Lemma 2.1:

$$U_{e,i_\epsilon}^\omega(t/\epsilon) \leqslant U_{e,i}^\omega(t/\epsilon) \qquad \text{if and only if} \qquad i \geqslant i_\epsilon.$$

Therefore

$$N_e^\omega(U_{e,i_\epsilon}^\omega(t/\epsilon), 0, t/\epsilon) = -i_\epsilon.$$

Multiplying by ϵ and letting $\epsilon \to 0$, we find

$$w(u(y, t), t) = -y/e^0$$

and thus, by (54),

$$u(y, t) = w^{-1}(-y/e^0, t) = \min\{y - H^0(-1/e^0)e^0 t \ , \ (-p_{\bar{A}}^{0,-})^{-1}(y/e^0 - \bar{A}t)\}.$$

This holds if $U_{e,i_\epsilon}^\omega(t/\epsilon) \leqslant -R_0$ for any ϵ small enough, which is ensured by the condition $t < y/(e^0\bar{A})$ and ϵ small enough thanks to the equality above. Our proof of (55) is then complete since the limit is independent of the subsequence.

We now turn to the proof of the convergence of $n_e^{\omega,\epsilon}(\cdot, k, \cdot)$ in $(-\infty, 0] \times [0, +\infty)$. Fix $\omega \in \Omega_0$, $x < 0$ and $t \geqslant 0$. The map $y \mapsto \min\{y - H^0(-1/e^0)e^0 t \ , \ (-p_{\bar{A}}^{0,-})^{-1}(y/e^0 - \bar{A}t)$ being increasing in $(-\infty, t/(e^0\bar{A}))$ and being equal to 0 for $y = t/(e^0\bar{A})$, there exists y with $t < y/(e^0\bar{A})$ and such that

$$u(y, t) = x.$$

We set $i_\epsilon = [y/(e^0\epsilon)]$. Assume also that $n_e^{\omega,\epsilon}(\cdot, k, \cdot)$ converges up to a subsequence to $n_e(\cdot, k, \cdot)$. By the same argument as above, we have

$$N_e^\omega(U_{e,i_\epsilon}^\omega(t/\epsilon), k, t/\epsilon) = \sharp\{i \in \{i_\epsilon, \dots, 0\}, \ T_i = k\}.$$

We multiply by ϵ and let $\epsilon \to 0$. Recalling Lemma 4.1 and the previous step, we get

$$n_e(u(y, t), k, t) = -\pi^k y/e^0 = \pi^k w(u(y, t), t).$$

By definition of y, this shows that $n_e(x,k,t) = \pi^k w(x,t) = \pi^k n_e(x,0,t)$ as well as the equality $\nu_e(x,k,t) = \nu_e(x,0,t)$ for any $x < 0$ and $t \geqslant 0$. By continuity of $\nu_e(\cdot,k,\cdot)$ (this is a direct consequence of Lemma 4.2), we also get the result for $x = 0$.

Next we turn to the limits for $x \geqslant 0$. Let $\omega \in \Omega_0$ and $\nu_e(\cdot,k,\cdot)$ be a locally uniform limit, up to a subsequence, of $\nu^{\omega,\epsilon}(\cdot,k,\cdot)$ for $k \in \{1,\dots,K\}$, which exists by Lemma 4.2. By the previous arguments, we know that for $x \geqslant 0$

$$\nu_e(x,k,0) = -x/e^k, \qquad \nu_e(0,k,t) = -\bar{A}t. \tag{56}$$

On the other hand, on each branch $\mathcal{R}_k$ the dynamical system corresponds (up to relabelling) to the standard ODE

$$\frac{d}{dt}U_j = V^k_{Z_j}(U_{j+1}(t) - U_j(t))$$

(where the sequence $(Z_j)_{j\in\mathbb{Z},\ T_j=k}$ is defined for indices j such that $T_j = k$). By homogenization (See Subsection A.3 in Appendix) n^k_e solves

$$\partial_t n^k_e + \bar{H}^k(\partial_x n^k_e) = 0 \qquad \text{in } (0,+\infty)\times(0,+\infty),$$

where $\bar{H}^k(p) = p\bar{V}^k(-1/p)$ for any $p < 0$ and $\bar{H}^k(p) = 0$ if $p \geqslant 0$. Hence ν^k_e solves

$$\partial_t \nu^k_e + H^k(\partial_x \nu^k_e) = 0 \qquad \text{in } (0,+\infty)\times(0,+\infty),$$

where H^k is given by $H^k(p) = (\pi^k)^{-1}\bar{H}^k(\pi^k p) = p\bar{V}^k(-1/(\pi^k p))$. Complemented with (56) this system has a unique solution given by

$$\nu_e(x,k,t) = \min\left\{-x/e^k - H^k(-1/e^k)t\ ,\ p^{k,+}_{\bar{A}}x - \bar{A}t\right\}.$$

As before this shows that the whole sequence $\nu^{\omega,\epsilon}(\cdot,k,\cdot)$ converges to ν^k_e given above. □

In the case $k_e = 0$, we have the following result.

Lemma 4.4. *Assume that $e = (e^k)$ is such that $H^k(-1/e^k) = \min_p H^k(p)$ for any $k \in \{0,\dots,K\}$ and assume that $k_e = 0$. Then there exists a set Ω_0 of full probability such that, for all $\omega \in \Omega_0$, $\nu^{\omega,\epsilon}_e$ converges to ν_e which satisfies*

$$\nu_e(x,0,t) = \nu_e(x,1,t) = \cdots = \nu_e(x,K,t) \qquad \text{in } (-\infty,0]\times[0,+\infty)$$

and is given by

$$\nu_e(x,k,t) = \min\left\{\phi_{A_0}(x,k) - A_0 t\ ,\ -x/e^k - tH^k(-1/e^k)\right\} \qquad \forall (x,k,t) \in \mathcal{R}\times[0,+\infty), \tag{57}$$

where A_0 is given by (8).

Proof. Note that with our choice of e we have $A_0 = \max_{k\in\{0,\dots,K\}} H^k(-1/e^k)$. Let $\bar{\theta}_e(t) = \mathbb{E}[\theta_e(t)]$ and $\bar{\theta}^\epsilon_e(t) = \epsilon\bar{\theta}_e(t/\epsilon)$. Then, using Lemma 3.3, $\bar{\theta}^\epsilon_e$ converges, up to a subsequence, to a Lipschitz continuous map $t \to \bar{\vartheta}_e(t)$. From now on we argue along this subsequence and note that it does not depend on ω. According to Lemma 3.18, we have $\bar{\vartheta}_e(t) \geqslant -A_0 t$. We also note that, by Theorem 3.4, $\epsilon\theta_e(t/\epsilon)$ converges a.s. locally uniformly to $\bar{\vartheta}_e(t)$. Let Ω'_0 be the set of $\omega \in \Omega_0$ such that this limit holds (note that this set depends on the subsequence, we will come back to this point at the very end of the proof). Without loss of generality, we also assume that, for any $\omega \in \Omega'_0$,

$$\lim_{s\to-\infty}(-s)^{-1}\sharp\{i \in \mathbb{Z}\cap(s,0],\ T^\omega_i = k\} = \pi^k.$$

Recall that

$$N^\omega_e(0,k,t) = \sharp\{i\in\mathbb{Z},\ i\leqslant 0,\ T^\omega_i = k,\ U^\omega_{e,i}(t) > 0\} = \sharp\{i\in\mathbb{Z}\cap(-\theta^\omega_e(t),0],\ T^\omega_i = k\}.$$

Therefore

$$n^{\omega,\epsilon}(0,k,t) = \epsilon\sharp\{i\in\mathbb{Z}\cap(-\theta^\omega_e(t/\epsilon),0],\ T^\omega_i = k\}.$$

If $\omega \in \Omega_0'$, then we get

$$\lim_{\epsilon \to 0} n^{\omega,\epsilon}(0,k,t) = \bar{\vartheta}_e(t)\pi^k.$$

because $\epsilon\theta_e^\omega(t/\epsilon)$ converges to the deterministic limit $\bar{\vartheta}_e(t)$. So arguing as above, for any $\omega \in \Omega_0'$, we can find a subsequence (subsequence of the previous subsequence and depending this time on ω) such that $\nu_e^{\omega,\epsilon}$ converges locally uniformly to a continuous solution ν_e of

$$\begin{cases} \partial_t \nu_e + H(\partial_x \nu_e) = 0 & \text{in } \overset{o}{\mathcal{R}} \times (0,+\infty) \\ \nu_e(x,k,0) = -x/e^k & \text{in } \mathcal{R} \\ \nu_e(0,k,t) = \bar{\vartheta}_e(t) & \forall k \in \{0,\dots,K\},\ t \geqslant 0. \end{cases} \tag{58}$$

Let $\tilde{\nu}_e$ be the solution of the junction problem without flux limiter:

$$\begin{cases} \partial_t \tilde{\nu}_e + H(\partial_x \tilde{\nu}_e) = 0 & \text{in } \overset{o}{\mathcal{R}} \times (0,+\infty) \\ \tilde{\nu}_e(x,k,0) = -x/e^k & \text{in } \mathcal{R} \\ \partial_t \tilde{\nu}_e + \max\{A_0, H^{0,+}(\partial_0 \tilde{\nu}_e), H^{1,-}(\partial_1 \tilde{\nu}_e), \dots, H^{K,-}(\partial_K \tilde{\nu}_e)) = 0 & \text{at } x = 0. \end{cases} \tag{59}$$

The solution is given by (see Lemma A.1)

$$\tilde{\nu}_e(x,k,t) = \min\left\{\phi_{A_0}(x,k) - A_0 t\ ,\ -x/e^k - tH^k(-1/e^k)\right\}. \tag{60}$$

We know from [36, Theorem 2.7] that ν_e is a subsolution to (59), as it is continuous and satisfies the Hamilton-Jacobi equation in $(\mathbb{R}\backslash\{0\}) \times \{0,\dots,K\} \times (0,+\infty)$. Therefore $\nu_e \leqslant \tilde{\nu}_e$ by comparison [36]. In addition, we get by the definition of A_0 and since $\bar{\vartheta}_e(t) \geqslant -A_0 t$:

$$-A_0 t \leqslant \bar{\vartheta}_e(t) = \nu_e(0,k,t) \leqslant \tilde{\nu}_e(0,k,t) = -A_0 t.$$

This shows that $\nu_e(0,k,t) = \bar{\vartheta}_e(t) = -A_0 t$ for any k.

So we have proved that the whole sequence $\bar{\theta}_e(t)/t$ converges as $t \to +\infty$ to $-A_0 t$. This shows, exactly as in the proof of Lemma 3.18 that $\theta_e^\omega(t)/t$ converges a.s. to $-A_0 t$ as $t \to +\infty$. We can then proceed as in the case $k_e < 0$ and find a set Ω_0 of full probability such that $\nu_e^{\omega,\epsilon}$ converges for any $\omega \in \Omega_0'$ to the unique solution of (58) with $\bar{\vartheta}_e(t) = -A_0 t$, which is nothing but $\tilde{\nu}_e$. This completes the proof of the lemma. □

For later use it will be convenient to rewrite the previous lemma in term of the behavior of the U_e. Let us define, for $k \in \{1,\dots,K\}$ and $x \in \mathbb{R}$

$$[x]_k = [x]_k^\omega = \sup\{i \in \mathbb{Z},\ i \leqslant x,\ T_i = k\}.$$

We note that, a.s.,

$$\lim_{\epsilon \to 0} \epsilon[x/\epsilon]_k = x$$

and that this convergence holds locally uniformly in x.

By the definition of N_e, we have, for any $y < 0$ and $t \geqslant 0$,

$$N_e(U_{e,[y]_k}(t),k,t) = \sharp\left\{j \leqslant 0,\ T_j = k,\ U_{e,j}(t) > U_{e,[y]_k}(t)\right\} = \sharp\left\{j \leqslant 0,\ T_j = k,\ j > [y]_k\right\}.$$

since the order is preserved in time among vehicle with a same type (Lemma 2.1). Therefore

$$\nu_e^\epsilon(\epsilon U_{e,[y/\epsilon]_k}(t/\epsilon),k,t) = \epsilon(\pi^k)^{-1}\sharp\left\{j \leqslant 0,\ T_j = k,\ j > [y/\epsilon]_k\right\}$$

from which we derive that any uniform limit $y(\cdot)$ (up to subsequences) of $t \to \epsilon U_{e,[y/\epsilon]_k}(t/\epsilon)$ satisfies

$$\nu_e(y(t),k,t) = -y.$$

We know by Lemma 4.3 (when $k_e < 0$) or Lemma 4.4 (when $k_e = 0$), that

$$\nu_e(x,k,t) = \min\left\{\phi_{\bar{A}}(x,k) - \bar{A}t\ ,\ -x/e^k - tH^k(-1/(e^k))\right\}, \qquad \forall (x,k) \in \mathcal{R}$$

where $\bar{A} = -\bar{\vartheta}_e$ if $k_e < 0$ and $\bar{A} = A_0$ otherwise. This shows that

$$y(t) = \begin{cases} \min\left\{\psi_{\bar{A}}(y - \bar{A}t, 0)\ ,\ ye^0 - e^0 t H^0(-1/e^0)\right\} & \text{if } y \leqslant \bar{A}t \\ \min\left\{\psi_{\bar{A}}(y - \bar{A}t, k)\ ,\ ye^k - e^k t H^k(-1/e^k)\right\} & \text{if } y \geqslant \bar{A}t. \end{cases}$$

Since this limit is independent of the choice of the subsequence, we have proved the following (the case $y \geqslant 0$ being treated in the same way):

Corollary 4.5. *Let e be such that $H^k(-1/e^k) = \min_p H^k(p)$. For $k \in \{0, \dots, K\}$, let*

$$u_e^\epsilon(y, k, t) := \epsilon U_{e,[y/\epsilon]_k}(t/\epsilon).$$

Then u_e^ϵ converges a.s. and locally uniformly to

$$u_e(y, k, t) := \begin{cases} \min\left\{\psi_{\bar{A}}(y - \bar{A}t, 0)\ ,\ ye^0 - e^0 t H^0(-1/e^0)\right\} & \text{if } y \leqslant \bar{A}t \\ \min\left\{\psi_{\bar{A}}(y - \bar{A}t, k)\ ,\ ye^k - e^k t H^k(-1/e^k)\right\} & \text{if } y \geqslant \bar{A}t. \end{cases} \tag{61}$$

4.2 Comparison principle

An important point in the proof of the homogenization is to explain how the comparison for the solutions U pass to the limit. This is the aim of the following lemma:

Lemma 4.6. *We fix a solution U of*

$$\frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t), U_{\ell_i}(t) - U_i(t), U_i(t)) \qquad i \in \mathbb{Z}$$

and set $u^\epsilon(x, k, t) = \epsilon U_{[x/\epsilon]_k}(t/\epsilon)$. Let e be such that $H^k(-1/e^k) = \min_p H^k(p)$. There exists a constant $C > 1$ and a set Ω_0 of full probability independent of U such that, if $\omega \in \Omega_0$, if u_*^ω is any half relaxed lower limit of $u^{\omega,\epsilon}$ as $\epsilon \to 0^+$ (possibly up to a subsequence) and if there exists $\gamma > 0$, a time $t_0 \geqslant 0$ and $a, b \in \mathbb{R}$ with $b > -t_0$, such that

$$u_*^\omega(x, k, t_0) \geqslant u_e(x + a, k, t_0 + b) \qquad \forall (x, k) \in [-\gamma, \gamma] \times \{1, \dots, K\} \tag{62}$$

and such that the minimum of $(x, k) \to u_*^\omega(x, k, t_0) - u_e(x + a, k, t_0 + b)$ is not reached on $\{-\gamma, \gamma\} \times \{1, \dots, K\}$, then

$$u_*^\omega(x, k, t_0 + s) \geqslant u_e(x + a, k, t_0 + b + s) \qquad \forall (x, k, s) \in [-\gamma/2, \gamma/2] \times \{1, \dots, K\} \times [0, C^{-1}\gamma].$$

In the same way, if $u^{\omega,*}$ is any half relaxed upper limit of some U_i^ω (possibly up to a subsequence) and if there exists $\gamma > 0$, $t_0 \geqslant 0$ and $a, b \in \mathbb{R}$ with $b > -t_0$, such that

$$u^{\omega,*}(x, k, t_0) \leqslant u_e(x + a, k, t_0 + b) \qquad \forall x \in [-\gamma, \gamma] \times \{1, \dots, K\}$$

and such that the maximum of $(x, k) \to u_*^\omega(x, k, t_0) - u_e(x + a, k, t_0 + b)$ is not reached on $\{-\gamma, \gamma\} \times \{1, \dots, K\}$, then

$$u^{\omega,*}(x, k, t_0 + s) \leqslant u_e(x + a, k, t_0 + b + s) \qquad \forall (x, k, t) \in [-\gamma/2, \gamma/2] \times \{1, \dots, K\} \times [0, C^{-1}\gamma].$$

Proof. We only prove the first statement, the proof for the second one being symmetric. Let Ω_0 be the set of ω such that $u_e^{\omega,\epsilon}$ converges to u_e locally uniformly, such that $(\epsilon J_{[\gamma/(3\alpha\epsilon)]}(\epsilon^{-1}\gamma))$ converges to $\gamma - \alpha\gamma/(3\alpha) = 2\gamma/3$ as $\epsilon \to 0$ (see Lemma 2.7) and for which the conclusions of Lemma 4.7 (below) hold.

Since $u_e^{\omega,\epsilon}$ converges locally uniformly to u_e, for any $\eta \in (0, 1)$ small and $M \geqslant 1$ large (to be chosen below), there exists $\epsilon_0 > 0$ such that the set

$$E_\eta := \left\{\omega \in \Omega,\ \sup_{\epsilon \in (0, \epsilon_0)} \|u_e^{\omega,\epsilon} - u_e\|_{L^\infty([-M,M] \times \{1,\dots,k\} \times [0,M])} \leqslant \eta\right\}$$

has a probability larger than $1-\eta$. Let us set $E_\eta(\omega) := \{n \in \mathbb{Z},\ \tau_n\omega \in E_\eta\}$. Let $n_\epsilon := [a/\epsilon]$. By Lemma 4.7 below, there exists $m_{\epsilon,\eta} \geqslant -n_\epsilon$ with $m_{\epsilon,\eta} \in E_\eta(\omega)$ and

$$|m_{\epsilon,\eta} + n_\epsilon| \leqslant C_1(\omega,\eta) + C_2\eta|n_\epsilon|. \tag{63}$$

We will use below that $\epsilon n_\epsilon \to a$ as $\epsilon \to 0$ and therefore that $\epsilon m_{\epsilon,\eta}$ converges to $-a$ as $\epsilon \to 0$ and then $\eta \to 0$.

By (62), the fact that u^ω_* is the half relaxed lower limit of $u^{\omega,\epsilon}$ and by the definition of $m_{\epsilon,\eta}$, there exists $(x_{\epsilon,\eta}, k_{\epsilon,\eta}) \in (-\gamma,\gamma)\times\{1,\dots,K\}$, minimum point of $u^{\epsilon,\omega}(\cdot,\cdot,t_0) - u_e^{\epsilon,\tau_{m_{\epsilon,\eta}}\omega}(\cdot+a,\cdot,t_0+b)$ such that

$$u^{\omega,\epsilon}(x_{\epsilon,\eta},k_{\epsilon,\eta},t_0) - u_e^{\tau_{m_{\epsilon,\eta}}\omega}(x_{\epsilon,\eta}+a,k_{\epsilon,\eta},t_0+b) \to \min$$

as ϵ and η tend to 0, where

$$\min := \min_{(x,k)\in[-\delta,\delta]\times\{0,\dots,K\}} (u^\omega_*(\cdot,\cdot,t_0) - u_e(\cdot+a,\cdot,t_0+b)) \geqslant 0.$$

By minimality of $(x_{\epsilon,\eta}, k_{\epsilon,\eta})$, we have

$$\begin{aligned} &u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x+a,k,t_0+b) - u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x_{\epsilon,\eta}+a,k_{\epsilon,\eta},b) \\ &\quad\leqslant u^{\omega,\epsilon}(x,k,t_0) - u^{\omega,\epsilon}(x_{\epsilon,\eta},k_{\epsilon,\eta},t_0) \text{ for } (x,k)\in[-\gamma,\gamma]\times\{1,\dots,K\}. \end{aligned}$$

As $-\epsilon m_{\epsilon,\eta} \leqslant \epsilon n_\epsilon \leqslant a$ and u_e^ϵ is nondecreasing in the first variable, we obtain also

$$\begin{aligned} &u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x-\epsilon m_{\epsilon,\eta},k,t_0+b) - u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x_{\epsilon,\eta}+a,k_{\epsilon,\eta},b) \\ &\quad\leqslant u^{\omega,\epsilon}(x,k,t_0) - u^{\omega,\epsilon}(x_{\epsilon,\eta},k_{\epsilon,\eta},t_0) \text{ for } (x,k)\in[-\gamma,\gamma]\times\{1,\dots,K\}. \end{aligned}$$

For $i \in [-\epsilon^{-1}\gamma, \epsilon^{-1}\gamma]\cap\mathbb{Z}$, we have, if we set $k := T_i^\omega = T_{i-m_{\epsilon,\eta}}^{\tau_{m_{\epsilon,\eta}}\omega}$ and $x = \epsilon i$, that $i = [x/\epsilon]_k^\omega$ and $i - m_{\epsilon,\eta} = [x/\epsilon - m_{\epsilon,\eta}]_k^{\tau_{m_k}\omega}$ with $x \in [-\gamma,\gamma]$. Therefore we can rewrite the inequality above in terms of U_e and U to get

$$U_{e,i-m_{\epsilon,\eta}}^{\tau_{m_{\epsilon,\eta}}\omega}(\epsilon^{-1}(t_0+b)) - r_{\epsilon,\eta} \leqslant U_i^\omega(\epsilon^{-1}t_0) \text{ for } i\in[-\epsilon^{-1}\gamma,\epsilon^{-1}\gamma]\cap\mathbb{Z}.$$

where $r_{\epsilon,\eta} := \epsilon^{-1}(u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x_{\epsilon,\eta}+a,k_{\epsilon,\eta},b) - u^{\omega,\epsilon}(x_{\epsilon,\eta},k_{\epsilon,\eta},t_0))$. Let us note for later use that, as $\epsilon\to0$, $\epsilon r_{\epsilon,\eta}$ converges to $-\min \leqslant 0$. By Lemma 3.2 and using the fact that $t_0+b>0$ we obtain, from ϵ small enough,

$$U_{e,i-m_{\epsilon,\eta}}^{\tau_{m_{\epsilon,\eta}}\omega}(\epsilon^{-1}(t_0+b) - C(r_{\epsilon,\eta})_+) \leqslant U_i^\omega(\epsilon^{-1}t_0) \text{ for } i\in[-\epsilon^{-1}\gamma,\epsilon^{-1}\gamma]\cap\mathbb{Z}.$$

As $(U_{e,i-m_{\epsilon,\eta}}^{\tau_{m_{\epsilon,\eta}}\omega}(\epsilon^{-1}(t_0+b) - C(r_{\epsilon,\eta})_+ + \cdot))$ and $U_i^\omega(\epsilon^{-1}t_0+\cdot))$ solve equation (6) and can be compared at time 0 for $i\in[-\epsilon^{-1}\gamma,\epsilon^{-1}\gamma-1]\cap\mathbb{Z}$, we obtain by approximate finite speed of propagation (Lemma 2.6) that for any $n\in\mathbb{N}$,

$$U_{e,i-m_{\epsilon,\eta}}^{\tau_{m_{\epsilon,\eta}}\omega}(\epsilon^{-1}(t_0+b) - C(r_{\epsilon,\eta})_+ + s) \leqslant U_i^\omega(\epsilon^{-1}t_0+s) + C2^{-n}e^{\beta s} \text{ for } i\in[-\epsilon^{-1}\gamma, J_n(\epsilon^{-1}\gamma)]\cap\mathbb{Z},\ s\geqslant 0.$$

Coming back to the scaled problem and choosing $n = [\gamma/(3\alpha\epsilon)]$ (where α is defined in Lemma 2.7) and for $s \leqslant \gamma\ln(2)/(3\alpha\beta\epsilon)$, so that $-\ln(2)n + \beta s \leqslant 0$, this implies that

$$\begin{aligned} &u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x-\epsilon m_{\epsilon,\eta},k,b+t_0 - C\epsilon(r_{\epsilon,\eta})_+ + t) \leqslant u^{\omega,\epsilon}(x,k,t_0+t) + C\epsilon \\ &\qquad\text{for } (x,k,t)\in[-\gamma,\epsilon J_{[\gamma/(3\alpha\epsilon)]}(\epsilon^{-1}\gamma)]\times\{1,\dots,K\}\times[0,\gamma/(3\alpha\beta)]. \end{aligned}$$

By the choice of ω, $(\epsilon J_{[\gamma/(3\alpha\epsilon)]}(\epsilon^{-1}\gamma))$ converges to $2\gamma/3$ as $\epsilon\to0$. So, for ϵ small enough, we find

$$\begin{aligned} &u_e^{\tau_{m_{\epsilon,\eta}}\omega,\epsilon}(x-\epsilon m_{\epsilon,\eta},k,b+t_0 - C\epsilon(r_{\epsilon,\eta})_+ + t) \leqslant u^{\omega,\epsilon}(x,k,t_0+t) + C\epsilon \\ &\qquad\text{for } (x,k,t)\in[-\gamma,\gamma/2]\times\{1,\dots,K\}\times[0,C^{-1}\gamma]. \end{aligned}$$

So we obtain, from the definition of E_η and for M large (depending on a and γ only) and for ϵ and η small:

$$u_e(x-\epsilon m_{\epsilon,\eta},k,b+t_0-C\epsilon(r_{\epsilon,\eta})_+ +t) \leqslant u^{\omega,\epsilon}(x,k,t_0+t)+\eta+C\epsilon \text{ for } (x,k,t)\in[-\gamma,\gamma/2]\times\{1,\dots,K\}\times[0,C^{-1}\gamma].$$

Recall that $\epsilon m_{\epsilon,\eta}$ converges to $-a$ while $C\epsilon(r_{\epsilon,\eta})_+$ tends to 0 as ϵ and η tend to 0: we can let $\epsilon\to 0$ (taking the half relaxed limit in the right-hand side) and then $\eta\to 0$ to find:

$$u_e(x+a,k,t_0+b+t)\leqslant u^\omega_*(x,k,t_0+t) \text{ for } (x,k,t)\in[-\gamma,\gamma/2]\times\{1,\dots,K\}\times[0,C^{-1}\gamma].$$

This proves our claim. □

Lemma 4.7. *Let $E\in\mathcal{F}$ be such that $\mathbb{P}[E]>1-\epsilon$ where $\epsilon\in(0,1/16)$. Let $E(\omega)=\{n\in\mathbb{Z},\ \tau_n\omega\in E\}$. There is a set Ω_0 of full probability such that, for any $\omega\in\Omega_0$, there exists $C_1(\omega,\epsilon)$ and C_2 universal such that, for any $n\in\mathbb{Z}$, one can find $m^\pm\in E(\omega)$ with $|n-m^\pm|\leqslant C_1(\omega,\epsilon)+C_2\epsilon|n|$, $m^+\geqslant n$ and $m^-\leqslant n$.*

Proof. By the ergodic theorem, we have

$$\lim_{r\to+\infty}(2r+1)^{-1}\sharp\left(E(\omega)\cap([-r,r]\cap\mathbb{Z})\right)=\mathbb{P}[E]\geqslant 1-\epsilon \qquad \text{a.s..}$$

(where $\sharp(A)$ is the cardinal of A). Let Ω_0 be the set of full probability for which this holds. Fix $\omega\in\Omega_0$ and let $R=R(\omega,\epsilon)$ be such that

$$(2r+1)^{-1}\sharp\left(E(\omega)\cap([-r,r]\cap\mathbb{Z})\right)\geqslant 1-2\epsilon \qquad \forall r\geqslant R(\omega,\epsilon). \tag{64}$$

Fix $n\in\mathbb{Z}$. For simplicity we assume that $n\geqslant 0$ and we look for m^+. The other case can be treated in a symmetric way. Let us choose $r=11+R+[n(1+8\epsilon)]$ and assume that $[n,r]\cap E(\omega)=\varnothing$. Then by (64) we have $r-n\leqslant 2\epsilon(2r+1)$, which implies that (as $\epsilon\leqslant 1/16$)

$$10+R+8\epsilon n\leqslant r-n\leqslant 2\epsilon(22+2R+2n(1+8\epsilon))\leqslant 44/16+R/4+6\epsilon n.$$

This is impossible and therefore there exists $m^+\in E(\omega)\cap[n,r]$. Then $m^+\in E(\omega)$, $m^+\geqslant n$ and $m^+-n\leqslant r-n\leqslant n+C_1(\omega,\epsilon)+C_2\epsilon n$ where $C_1(\omega,\epsilon)=12+R(\omega,\epsilon)$ while $C_2=8$. □

4.3 Proof of the homogenization

From now on we fix Ω_0 such that $\mathbb{P}[\Omega_0]=1$ and such that, for any $\omega\in\Omega_0$, for $e=(e^k)$ such that $H^k(-1/e^k)=\min H^k$, ν^ϵ_e converges locally uniformly to the map ν_e given in Lemma 4.3 or Lemma 4.4. Moreover, we assume that, if $\omega\in\Omega_0$ the conclusions of Lemma 4.6 and of Lemma A.3 holds.

Let $(U^\epsilon_{i,0})$ be a deterministic family of initial conditions satisfying the compatibility condition (9) and assume, up to relabel the indices, that $U^\epsilon_{i,0}\leqslant 0$ if and only if $i\leqslant 0$. Let (U^ϵ_i) be the solution to

$$\frac{d}{dt}U^\epsilon_i(t)=V_{Z_i}(U^\epsilon_{i+1}(t)-U^\epsilon_i(t),U^\epsilon_{\ell_i}(t)-U^\epsilon_i(t),U^\epsilon_i(t)) \qquad i\in\mathbb{Z}$$

with initial condition $U^\epsilon_i(0)=U^\epsilon_{i,0}$, $i\in\mathbb{Z}$.

We set, for $k\in\{1,\dots,K\}$ and $(x,t)\in\mathbb{R}\times[0,+\infty)$,

$$N^\omega(x,k,t)=\sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T^\omega_i=k}\delta_{U^\omega_i(t)}((x,+\infty))-\sum_{i\in\mathbb{Z},\ i>0,\ T^\omega_i=k}\delta_{U^\omega_i(t)}((-\infty,x]) \tag{65}$$

and, for $(x,t)\in(-\infty,0]\times[0,+\infty)$,

$$N^\omega(x,0,t)=\sum_{k=1}^K N^\omega(x,k,t).$$

We set $n^{\omega,\epsilon}(x,k,t)=\epsilon N^\omega(x/\epsilon,k,t/\epsilon)$ and

$$\nu^{\omega,\epsilon}(x,k,t)=\begin{cases} n^{\omega,\epsilon}(x,0,t) & \text{if } k=0 \text{ and } x\leqslant 0,\\ \pi_k^{-1}n^{\omega,\epsilon}(x,k,t) & \text{if } k\in\{1,\dots,K\} \text{ and } x\in\mathbb{R}.\end{cases}$$

Let us first check that ν^ϵ is well defined.

Lemma 4.8. *Let $(U^\epsilon_{i,0})$ and U^ϵ be as above. Then, for any $t \geqslant 0$,*

$$\lim_{i\to\pm\infty} U^\epsilon_i(t) = \pm\infty. \tag{66}$$

Hence ν^ϵ is well-defined and satisfies, for any $k \in \{0,\dots,K\}$,

$$|\nu^\epsilon(x,k,t) - \nu^\epsilon(y,k,s)| \leqslant C(|x-y| + |t-s| + \epsilon),$$

for a constant C depending on $\Delta_{\min}$, π^k and $\|V\|_\infty$ only.

Proof. The compatibility condition (9) implies that (66) holds for $t = 0$. Then it holds for any t since V is bounded. Fix $k \in \{1,\dots,K\}$ and let $x, y \in \mathbb{R}$ with $x < y$. We have

$$\left|\delta_{U^\omega_i(t/\epsilon)}((y/\epsilon,+\infty)) - \delta_{U^\omega_i(t/\epsilon)}((x/\epsilon,+\infty))\right| = \begin{cases} 1 & \text{if } x/\epsilon < U^\omega_i(t/\epsilon) \leqslant y/\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.1 there are at most $[(y-x)/(\epsilon\Delta_{\min})] + 1$ vehicles of the same type in $(x/\epsilon, y/\epsilon]$. Arguing in the same way for the difference

$$\left|\delta_{U^\omega_i(t)}((-\infty,y]) - \delta_{U^\omega_i(t)}((-\infty,x])\right|,$$

we infer that

$$|\nu^\epsilon(x,k,t) - \nu^\epsilon(y,k,t)| \leqslant 2(\pi^k)^{-1}(|x-y|/\Delta_{\min} + \epsilon).$$

Fix now $0 \leqslant s < t$. We have

$$\left|\delta_{U^\omega_i(t/\epsilon)}((x/\epsilon,+\infty)) - \delta_{U^\omega_i(s/\epsilon)}((x/\epsilon,+\infty))\right| = \begin{cases} 1 & \text{if } U^\omega_i(s/\epsilon) \leqslant x/\epsilon < U^\omega_i(t/\epsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Let $i_0 \in \mathbb{Z}$ be the smallest index such that $U^\omega_{i_0}(s/\epsilon) \leqslant x/\epsilon < U^\omega_{i_0}(t/\epsilon)$ and $T_{i_0} = k$ and i_1 be the largest one. Then

$$U^\omega_{i_1}(s/\epsilon) \leqslant x/\epsilon < U^\omega_{i_0}(t/\epsilon) \leqslant U^\omega_{i_0}(s/\epsilon) + \|V\|_\infty(t-s)/\epsilon.$$

Still by Lemma 2.1 we must have $i_1 - i_0 \leqslant [\|V\|_\infty(t-s)/(\epsilon\Delta_{\min})] + 1$, so that

$$|\nu^\epsilon(x,k,t) - \nu^\epsilon(x,k,s)| \leqslant 2(\pi^k)^{-1}(\|V\|_\infty(t-s)/(\Delta_{\min}) + \epsilon).$$

The Lipschitz continuity of $\nu^\epsilon(\cdot,0,\cdot) = \sum_{k=1}^K \pi^k\nu^\epsilon(\cdot,k,\cdot)$ is then immediate. □

We assume that $\nu^{\omega,\epsilon}(\cdot,\cdot,0) \to \nu_0$ locally uniformly, where ν_0 is deterministic. Note that ν_0 is Lipschitz continuous and satisfies $\nu_0(x,0) = \nu_0(x,1) = \cdots = \nu_0(x,k)$ for $x \leqslant 0$. We fix $\omega \in \Omega_0$ and let ν^ω be any uniform limit of $\nu^{\omega,\epsilon}$. We already know (cf. Subsection A.3 in Appendix) that ν^ω satisfies

$$\begin{cases} \partial_t\nu + H(\partial_x\nu) = 0 & \text{in } \overset{o}{\mathcal{R}} \times (0,T) \\ \nu(x,k,0) = \nu_0(x,k) & \text{in } \mathcal{R}. \end{cases}$$

Our aim is to show that ν^ω also satisfies

$$\partial_t\nu + \max\{\bar{A}, H^{0,+}(\partial_0\nu), H^{1,-}(\partial_1\nu),\dots,H^{K,-}(\partial_K\nu))\} = 0 \text{ at } x = 0.$$

We first show that ν^ω is continuous in 0 (and does not depend on k for $x \leqslant 0$).

Lemma 4.9. *Let ν^ω be any uniform limit (up to subsequences) of $\nu^{\omega,\epsilon}$. Then, for all $t \geqslant 0$ and $x \leqslant 0$*

$$\nu^\omega(x,0,t) = \nu^\omega(x,1,t) = \cdots = \nu^\omega(x,K,t).$$

Proof. Let $x \leqslant 0$ and $i_0^\epsilon,\ i_1^\epsilon \in \mathbb{N}$ be the indices such that $U_{-i_0^\epsilon}(0) \leqslant x/\epsilon < U_{-i_0^\epsilon+1}(0)$ and $U_{-i_1^\epsilon}(t/\epsilon) \leqslant x/\epsilon < U_{-i_1^\epsilon+1}(t/\epsilon)$. We assume in a first step that $\nu^\omega(x,0,t) - \nu^\omega(x,0,0) \geqslant 2c > 0$. Then, for ϵ small enough, $\nu^{\omega,\epsilon}(x,0,t) - \nu^{\omega,\epsilon}(x,0,0) \geqslant c$. As, by assumption, $U^\epsilon_{i,0} \leqslant 0$ if and only if $i \leqslant 0$, this implies that

$$\sum_{i\leqslant 0} \delta_{U_i^\omega(t/\epsilon)}((x/\epsilon,+\infty)) - \sum_{i\leqslant 0} \delta_{U_i^\omega(0)}((x/\epsilon,+\infty)) \geqslant c/\epsilon.$$

Since by Lemma 2.1, the cars remains ordered before 0, we deduce that $i_1^\epsilon - i_0^\epsilon \geqslant c/\epsilon$. Moreover, since $U_{-i_1^\epsilon+1}(0) - x/\epsilon \leqslant U_{-i_1^\epsilon+1}(0) - U_{-i_0^\epsilon}(0) \leqslant \Delta_{\min}(i_0^\epsilon - i_1^\epsilon + 1)$, we get that $i_1^\epsilon - i_0^\epsilon \leqslant \frac{||V||_\infty}{\Delta_{\min}} \frac{t}{\epsilon} + 1$. We then have that $\epsilon(i_1^\epsilon - i_0^\epsilon)$ is bounded and converges, up to a subsequence, to a constant z. Remarking that

$$\nu^{\omega,\epsilon}(x,0,t) - \nu^{\omega,\epsilon}(x,0,0) = \epsilon(i_1^\epsilon - i_0^\epsilon) \to \nu^\omega(x,0,t) - \nu^\omega(x,0,0),$$

we deduce that $z = \nu^\omega(x,0,t) - \nu^\omega(x,0,0)$. Hence, for every $k \in \{1,\dots,K\}$, by the law of large number,

$$\nu^{\omega,\epsilon}(x,k,t) - \nu^{\omega,\epsilon}(x,k,0) = (\pi_k)^{-1}\epsilon\sharp\{i \in \{-\epsilon i_1^\epsilon/\epsilon,\dots,-\epsilon i_0^\epsilon/\epsilon\} \text{ s.t. } T_i^\omega = k\} \to (\pi_k)^{-1}\pi_k z = z.$$

Since $\nu^\omega(x,k,0) = \nu^\omega(x,0,0)$, this implies that $\nu^\omega(x,k,t) = \nu^\omega(x,0,t)$.

Assume now that $\nu^\omega(x,0,t) - \nu^\omega(x,0,0) = 0$. Then

$$0 \leqslant n^{\omega,\epsilon}(x,k,t) - n^{\omega,\epsilon}(x,k,0) \leqslant \nu^{\omega,\epsilon}(x,0,t) - \nu^{\omega,\epsilon}(x,0,0).$$

Sending $\epsilon \to 0$, we deduce that $\nu^\omega(x,k,t) - \nu^\omega(x,k,0) = (\pi_k)^{-1}(n^\omega(x,k,t) - n^\omega(x,k,0)) = 0$ and so $\nu^\omega(x,k,t) = \nu^\omega(x,k,0) = \nu^\omega(x,0,0) = \nu^\omega(x,0,t)$. □

It will be convenient to consider also the limit of $u^{\omega,\epsilon}(y,k,t) := \epsilon U^\omega_{[y/\epsilon]_k}(t/\epsilon)$ along the same subsequence as for $\nu^{\omega,\epsilon}$ for $k \in \{1,\dots,K\}$. Let $u^{\omega,*}$ and u^ω_* be the half-relaxed limits of $u^{\omega,\epsilon}$ (along that same subsequence). As $x \to \nu^\omega(x,k,t)$ is nonincreasing, it has an inverse

$$\begin{aligned}\tilde{u}^\omega(y,k,t) &:= \inf\{x \in \mathbb{R},\ \nu^\omega(x,k,t) < -y\} \qquad (\text{with } \tilde{u}^\omega(y,k,t) = +\infty \text{ if there is no such a } x)\\ &= \sup\{x \in \mathbb{R},\ \nu^\omega(x,k,t) \geqslant -y\}\end{aligned}$$

Note that $\tilde{u}^\omega$ is usc, while its lower semicontinuous envelope is given by

$$\tilde{u}^\omega_*(y,k,t) := \inf\{x \in \mathbb{R},\ \nu^\omega(x,k,t) \leqslant -y\} = \sup\{x \in \mathbb{R},\ \nu^\omega(x,k,t) > -y\}.$$

Lemma 4.10. *We have*

$$u^{\omega,*} \leqslant \tilde{u}^\omega \text{ and } u^\omega_* \geqslant \tilde{u}^\omega_*.$$

Note that, at each point where $\tilde{u}^\omega$ is continuous, we have $u^{\omega,*} = \tilde{u}^\omega = u^\omega_*$.

Proof. We only do the proof of the first inequality, the proof of the other one being similar. Fix (y,k,t) with $k \in \{1,\dots,K\}$ and let $(y_\epsilon,t_\epsilon) \to (y,t)$ be such that $u^{\omega,\epsilon}(y_\epsilon,k,t_\epsilon) \to u^{\omega,*}(y,k,t)$. Let $i_\epsilon := [y_\epsilon/\epsilon]_k$. By the definition of N and the fact that the U_i with $T_i = K$ remain ordered (see Lemma 2.1), we have

$$N^\omega(U^\omega_{i_\epsilon}(t_\epsilon/\epsilon),k,t_\epsilon/\epsilon) = N^\omega(U^\omega_{i_\epsilon}(0),k,0) = \begin{cases} \sharp\{j \in \{i_\epsilon,\dots,-1\},\ T_j = T_{i_\epsilon}\} & \text{if } i_\epsilon \leqslant 0\\ -\sharp\{j \in \{0,\dots,i_\epsilon\},\ T_j = T_{i_\epsilon}\} & \text{if } i_\epsilon > 0.\end{cases}$$

Hence

$$\nu^{\omega,\epsilon}(\epsilon U^\omega_{i_\epsilon}(t_\epsilon/\epsilon),k,t_\epsilon) = \epsilon(\pi^k)^{-1}\begin{cases} \sharp\{j \in \{[y_\epsilon/\epsilon]_k,\dots,-1\},\ T_j = k\} & \text{if } [y_\epsilon/\epsilon]^\omega_k \leqslant 0\\ -\sharp\{j \in \{0,\dots,[y_\epsilon/\epsilon]_k\},\ T_j = k\} & \text{if } [y_\epsilon/\epsilon]^\omega_k > 0.\end{cases},$$

where $\epsilon U^\omega_{i_\epsilon}(t_\epsilon/\epsilon) = \epsilon U^\omega_{[y_\epsilon/\epsilon]_k}(t_\epsilon/\epsilon) = u^{\omega,\epsilon}(y_\epsilon,k,t_\epsilon) \to u^{\omega,*}(y,k,t)$ while $\epsilon i_\epsilon \to y$. So, by uniform convergence of $\nu^{\omega,\epsilon}$, we obtain $\nu^\omega(u^{\omega,*}(y,t),k,t) = -y$. This shows that $\tilde{u}^\omega(y,t) \geqslant u^{\omega,*}(y,t)$. □

Lemma 4.11 (Supersolution at the junction)**.** *Let $\xi : [0,T] \to \mathbb{R}$ be a smooth test function and $A > \bar{A}$ be such that $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_A(x,k)$ has a local minimum on $\mathcal{R} \times (0,+\infty)$ at $(0,t_0)$. Then*

$$\xi'(t_0) + A \geqslant 0.$$

Proof. As $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_A(x,k)$ has a local minimum in $\mathcal{R}\times(0,+\infty)$ at $(0,t_0)$ and $\phi_{\bar A} < \phi_A$ on $\overset{o}{\mathcal{R}}$ with an equality at 0, modifying ξ if necessary, the map $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_{\bar A}(x,k)$ has a strict local minimum $\mathcal{R}\times(0,+\infty)$ at $(0,t_0)$: assuming that $\xi(t_0) = 0$, there exists $\gamma > 0$ such that, for any $(x,k,t) \in \mathcal{R}\times(t_0-\gamma, t_0+\gamma)$ with $(x,t) \neq (0,t_0)$ and $|x| \leqslant \gamma$,

$$\nu^\omega(x,k,t) - \xi(t) - \phi_{\bar A}(x,k) > \nu^\omega(0,t_0), \tag{67}$$

with an equality at $(0,t_0)$. As $\nu^\omega(x,k,t_0) = \nu^\omega(x,0,t_0)$ for $x \leqslant 0$, inequality (67) actually holds for any $(x,k,t) \in (-\gamma,\gamma)\times\{1,\dots,K\}\times(t_0-\gamma,t_0+\gamma)$ with $(x,t) \neq (0,t_0)$.

Let $y_0 = -\nu^\omega(0,t_0)$. By (67), we have that $\nu^\omega(x,k,t_0) > -y_0$ for $x \in (-\gamma,0)$, so that

$$\tilde u^\omega_*(y_0,k,t_0) = \inf\{x \in \mathbb{R},\ \nu^\omega(x,k,t_0) \leqslant -y_0\} = 0.$$

By continuity of ν^ω, there exists $\gamma' \in (0,\gamma)$ such that, if $(y,k,t) \in (y_0-\gamma', y_0+\gamma')\times\{1,\dots,K\}\times(t_0-\gamma', t_0+\gamma')$, then

$$\tilde u^\omega_*(y,k,t) = \inf\{x \in (-\gamma,+\infty),\ \nu^\omega(x,k,t) \leqslant -y\}.$$

Therefore

$$\begin{aligned}
\tilde u^\omega_*(y,k,t) &\geqslant \min\{\gamma\ ,\ \inf\{x\in(-\gamma,\gamma),\ \nu^\omega(x,k,t) \leqslant -y\}\ \} \\
&\geqslant \min\{\gamma\ ,\ \inf\{x\in(-\gamma,\gamma),\ \xi(t) + \phi_{\bar A}(x,k) - y_0 \leqslant -y\}\ \} \\
&\geqslant \min\{\gamma\ ,\ \inf\{x\in\mathbb{R},\ \xi(t) + \phi_{\bar A}(x,k) - y_0 \leqslant -y\}\ \} \ =\ \min\{\gamma\ ,\ \psi_{\bar A}(y - y_0 + \xi(t), k)\}.
\end{aligned}$$

If $(y,k,t) = (y_0,k,t_0)$, then $\psi_{\bar A}(y_0 - y_0 + \xi(t_0),k) = \psi_{\bar A}(0,k) = 0 < \gamma$, so that, reducing γ' if necessary, we get

$$\tilde u^\omega_*(y,k,t) \geqslant \psi_{\bar A}(y - y_0 + \xi(t),k) \qquad \forall (y,k,t) \in (y_0-\gamma', y_0+\gamma')\times\{1,\dots,K\}\times(t_0-\gamma', t_0+\gamma'), \tag{68}$$

In addition, as the inequality in (67) is strict, we have a strict inequality in the above inequality unless $(y,k,t) = (y_0,k,t_0)$. By (61) we have

$$u_e(y,k,t) := \begin{cases} \min\left\{\psi_{\bar A}(y-\bar A t,k)\ ,\ ye^0 - e^0 t H^0(-1/e^0)\right\} & \text{if } y \leqslant \bar A t \\ \min\left\{\psi_{\bar A}(y-\bar A t,k)\ ,\ ye^k - e^k t H^k(-1/e^k)\right\} & \text{if } y \geqslant \bar A t. \end{cases}$$

Let us fix $T > 0$ and set $y_T := \bar A T$. The equality above can be rewritten as

$$\begin{aligned}
&u_e(y + \bar A(t-t_0) + y_T, k, t - t_0 + T) \\
&\quad = \begin{cases} \min\left\{\psi_{\bar A}(y,k)\ ,\ (y + \bar A(t-t_0) + y_T)e^0 - e^0(t - t_0 + T)H^0(-1/e^0)\right\} & \text{if } y \leqslant 0 \\ \min\left\{\psi_{\bar A}(y,k)\ ,\ (y + \bar A(t-t_0) + y_T)e^k - e^k(t - t_0 + T)H^k(-1/e^k)\right\} & \text{if } y \geqslant 0. \end{cases}
\end{aligned}$$

By (68), this implies that, for any $(y,k,t) \in (y_0-\gamma', y_0+\gamma')\times\{1,\dots,K\}\times(t_0-\gamma', t_0+\gamma')$,

$$\tilde u^\omega_*(y,k,t) \geqslant \psi_{\bar A}(y - y_0 + \xi(t),k) \geqslant u_e(y - y_0 + \xi(t) + \bar A(t-t_0) + y_T, k, t - t_0 + T).$$

with a strict inequality if $(y,k,t) \neq (y_0,k,t_0)$.

We apply Lemma 4.6 with initial time $t_0 - \tau$, where $\tau > 0$ is so small that the minimum of the map

$$(y,k) \to u^\omega_*(y,k,t_0-\tau) - u_e(y - y_0 + \xi(t_0-\tau) - \bar A\tau + y_T, k, -\tau + T).$$

is not reached at $y \in \{-\gamma,\gamma\}$: this is possible since this minimum point converges to $(y_0,0)$ as $\tau \to 0+$. Then by Lemma 4.6 we get, if $s \geqslant 0$ and $|y - y_0|$ are small enough (depending on γ' only)

$$u^\omega_*(y,k,t_0 - \tau + s) \geqslant u_e(y - y_0 + \xi(t_0-\tau) - \bar A\tau + y_T, k, -\tau + T + s).$$

For $y = y_0$ and $s = \tau$, we get

$$\begin{aligned}
0 = u^\omega_*(y_0,k,t_0) &\geqslant u_e(\xi(t_0-\tau) - \bar A\tau + y_T, k, T) \\
&= \begin{cases} \min\left\{\psi_{\bar A}(\xi(t_0-\tau) - \bar A\tau + y_T - \bar A T, k)\ ,\ e^0(\xi(t_0-\tau) - \bar A\tau + y_T - TH^0(-1/e^0))\right\} & \text{if } \xi(t_0-\tau) \leqslant \bar A\tau \\ \min\left\{\psi_{\bar A}(\xi(t_0-\tau) - \bar A\tau + y_T - \bar A T, k)\ ,\ e^k(\xi(t_0-\tau) - \bar A\tau + y_T - TH^k(-1/e^k))\right\} & \text{if } \xi(t_0-\tau) \geqslant \bar A\tau \end{cases} \\
&\geqslant \begin{cases} \min\left\{\psi_{\bar A}(\xi(t_0-\tau) - \bar A\tau, k)\ ,\ e^0(\xi(t_0-\tau) - \bar A\tau)\right\} & \text{if } \xi(t_0-\tau) \leqslant \bar A\tau \\ \min\left\{\psi_{\bar A}(\xi(t_0-\tau) - \bar A\tau, k)\ ,\ e^k(\xi(t_0-\tau) - \bar A\tau)\right\} & \text{if } \xi(t_0-\tau) \geqslant \bar A\tau \end{cases}
\end{aligned}$$

because $y_T = \bar{A}T \geqslant T \max_k H^k(-1/e^k)$. Assume now that $\xi'(t_0) < -\bar{A}$. Then, since $\xi(t_0) = 0$ and $\xi'(t_0) < -\bar{A}$, one has $\xi(t_0 - \tau) - \bar{A}\tau > 0$ and thus the right-hand side in the inequality above is positive. This leads to a contradiction and shows that $\xi'(t_0) \geqslant -\bar{A} \geqslant -A$. □

Lemma 4.12 (Subsolution at the junction)**.** *Assume that $\bar{A} > A_0$ and let $A_0 \leqslant A < \bar{A}$ and $\xi : [0,T] \to \mathbb{R}$ be a smooth test function be such that $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_A(x,k)$ has a local maximum on $\mathcal{R} \times (0,+\infty)$ at $(0,t_0)$. Then*

$$\xi'(t_0) + A \leqslant 0.$$

Proof. We argue as in the supersolution case. As $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_A(x,k)$ has a local maximum on $\mathcal{R} \times (0,+\infty)$ at $(0,t_0)$ and $\phi_{\bar{A}} > \phi_A$ on $\overset{o}{\mathcal{R}}$ with an equality at 0, modifying ξ if necessary, the map $(x,k,t) \to \nu^\omega(x,k,t) - \xi(t) - \phi_{\bar{A}}(x,k)$ has a strict local maximum at $(0,t_0)$: assuming that $\xi(t_0) = 0$, there exists $\gamma > 0$ such that, for any $(x,k,t) \in \mathcal{R} \times (t_0 - \gamma, t_0 + \gamma)$ with $(x,t) \neq (0,t_0)$ and $|x| \leqslant \gamma$,

$$\nu^\omega(x,k,t) - \xi(t) - \phi_{\bar{A}}(x) < \nu^\omega(0,t_0), \tag{69}$$

with an equality at $(0,t_0)$. As $\nu^\omega(x,k,t) = \nu^\omega(x,0,t)$ for $x \leqslant 0$, inequality (69) also holds for any $(x,k,t) \in (-\gamma,\gamma) \times \{1,\dots,K\} \times (t_0 - \gamma, t_0 + \gamma)$ with $(x,t) \neq (0,t_0)$.

Let $y_0 = -\nu^\omega(0,t_0)$. By (69), we have that $\nu^\omega(x,k,t_0) < -y_0$ for $x \in (0,\gamma)$, so that

$$\tilde{u}^\omega(y_0,k,t_0) = \sup\{x \in \mathbb{R},\ \nu^\omega(x,k,t_0) \geqslant -y_0\} = 0.$$

By continuity of ν^ω, there exists $\gamma' \in (0,\gamma)$ such that, if $(y,k,t) \in (y_0 - \gamma', y_0 + \gamma') \times \{1,\dots,K\} \times (t_0 - \gamma', t_0 + \gamma')$, then

$$\tilde{u}^\omega(y,k,t) = \sup\{x \in (-\infty,\gamma),\ \nu^\omega(x,k,t) \geqslant -y\}.$$

Therefore

$$\begin{aligned}
\tilde{u}^\omega(y,k,t) &\leqslant \max\{-\gamma\ ,\ \sup\{(-\gamma,\gamma),\ \nu^\omega(x,k,t) \geqslant -y\}\ \} \\
&\leqslant \max\{-\gamma\ ,\ \sup\{x \in (-\gamma,\gamma),\ \xi(t) + \phi_{\bar{A}}(x,k) - y_0 \geqslant -y\}\ \} \\
&\leqslant \max\{-\gamma\ ,\ \sup\{x \in \mathbb{R},\ \xi(t) + \phi_{\bar{A}}(x,k) - y_0 \geqslant -y\}\ \} \ =\ \max\{-\gamma\ ,\ \psi_{\bar{A}}\left(y - y_0 + \xi(t),k\right)\}.
\end{aligned}$$

If $(y,k,t) = (y_0,k,t_0)$, then $\psi_{\bar{A}}\left(y_0 - y_0 + \xi(t_0),k\right) = \psi_{\bar{A}}\left(0,k\right) = 0 > -\gamma$, so that, reducing δ' if necessary, we get

$$\tilde{u}^\omega(y,k,t) \leqslant \psi_{\bar{A}}\left(y - y_0 + \xi(t),k\right) \qquad \forall (y,k,t) \in (y_0 - \gamma', y_0 + \gamma') \times \{1,\dots,K\} \times (t_0 - \gamma', t_0 + \gamma'). \tag{70}$$

In addition, as inequality in (67) is strict, we have a strict inequality in the above inequality unless $(y,k,t) = (y_0,k,t_0)$. By (61) we have

$$u_e(y,k,t) := \begin{cases} \min\left\{\psi_{\bar{A}}(y - \bar{A}t,k)\ ,\ ye^0 - e^0 t H^0(-1/e^0)\right\} & \text{if } y \leqslant \bar{A}t \\ \min\left\{\psi_{\bar{A}}(y - \bar{A}t,k)\ ,\ ye^k - e^k t H^k(-1/e^k)\right\} & \text{if } y \geqslant \bar{A}t. \end{cases}$$

Let us fix $T > 0$ and set $y_T := \bar{A}T$. Note that $u_e(y_T,k,T) = \psi_{\bar{A}}(y_T - \bar{A}T,k) < y_T e^k - e^k T H^k(-1/e^k)$ because $\bar{A} > \max_{k \in \{0,\dots K\}} H^k(-1/e^k)$. So, reducing γ' if necessary, the equality above can be rewritten as

$$u_e(y + \bar{A}t + y_T, t + T) = \psi_{\bar{A}}(y,k) \qquad \forall (y,k,t) \in (-\gamma',\gamma') \times \{1,\dots,K\} \times (-\gamma',\gamma'). \tag{71}$$

By (70), this implies that, for any $(y,k,t) \in (y_0 - \gamma', y_0 + \gamma') \times \{1,\dots,K\} \times (t_0 - \gamma', t_0 + \gamma')$,

$$\tilde{u}^\omega(y,k,t) \leqslant \psi_{\bar{A}}\left(y - y_0 + \xi(t),k\right) = u_e(y - y_0 + \xi(t) + \bar{A}(t - t_0) + y_T, k, t - t_0 + T).$$

By Lemma 4.6, applied at time $t_0 - \tau$, we get, if $s \geqslant 0$ and $|y - y_0|$ are small enough (depending on γ' only)

$$u^\omega(y,k,t_0 - \tau + s) \leqslant u_e(y - y_0 + \xi(t_0 - \tau) - \bar{A}\tau + y_T, k, -\tau + T + s).$$

For $y = y_0$ and $s = \tau$, we get

$$\begin{aligned}
0 = u^\omega(y_0,k_\tau,t_0) &\leqslant u_e(\xi(t_0 - \tau) - \bar{A}\tau + y_T, k_\tau, T) \\
&= \psi_{\bar{A}}(\xi(t_0 - \tau) - \bar{A}\tau + y_T - \bar{A}T, k_\tau) = \psi_{\bar{A}}(\xi(t_0 - \tau) - \bar{A}\tau, k_\tau),
\end{aligned}$$

where the second equality holds because of (71). Then, if $\xi'(t_0) > -\bar{A}$ and since $\xi(t_0) = 0$, one has $\xi(t_0 - \tau) - \bar{A}\tau < 0$ and thus the right-hand side in the inequality above is negative. This leads to a contradiction and shows that $\xi'(t_0) \leqslant -\bar{A} \leqslant -A$. □

Proof of Theorem 1.1. We just have to show that ν^ω satisfies in the viscosity sense

$$\partial_t \nu + \max\{\bar{A}, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} = 0 \text{ at } x = 0.$$

Let $A_n > \bar{A}$ be such that $A_n \to A$. By Lemma 4.11 and [36, Theorem 2.11], ν^ω is a super-solution of

$$\begin{cases} \partial_t \nu + H(\partial_x \nu) = 0 \qquad \text{in } \overset{o}{\mathcal{R}} \times (0,T) \\ \nu(x,k,0) = \nu_0(x,k) \qquad \text{in } \mathcal{R} \\ \partial_t \nu + \max\{A_n, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} = 0 \text{ at } x = 0. \end{cases}$$

By stability [36, Proposition 2.6], we then get that ν^ω satisfies

$$\partial_t \nu + \max\{\bar{A}, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} \geqslant 0 \text{ at } x = 0.$$

We now turn to the sub-solution property. Following [36, Theorem 2.7], ν^ω being continuous and a subsolution of the Hamilton-Jacobi equation in $\overset{o}{\mathcal{R}}$, is a subsolution at $x = 0$ with $A = A_0$. So we can assume from now on that $\bar{A} > A_0$. Arguing as above (taking $A_n < \bar{A}$ with $A_n \to \bar{A}$), we then get that

$$\partial_t \nu + \max\{\bar{A}, H^{0,+}(\partial_0 \nu), H^{1,-}(\partial_1 \nu), \dots, H^{K,-}(\partial_K \nu))\} \leqslant 0 \text{ at } x = 0.$$

□

A Appendix

A.1 Computation for Lemma 2.7

Let $X := X_0 - X_1$. We have $\mathbb{P}[X > m] \leqslant K r^m$ for $m \in \mathbb{N}$ and where $r := (\pi + 1)/2 \in (0,1)$. So, for $q \geqslant 1$,

$$\mathbb{E}[|X|^q] = \sum_{m=0}^{\infty} m^q \mathbb{P}[X = m] \leqslant \sum_{m=1}^{\infty} m^q \mathbb{P}[X \geqslant m] \leqslant \frac{K}{r} \sum_{m=1}^{\infty} m^q r^m.$$

Note that $x \to x^q r^x$ is increasing on $[0, -q/\ln(r)]$ and decreasing on $[-q/\ln(r), +\infty]$. So we can approximate the sum in the right-hand side by

$$\frac{K}{r} \sum_{m=1}^{\infty} m^q r^m \leqslant \frac{2K}{r} \int_0^{\infty} x^q r^x dx = \frac{2K}{r|\ln(r)|^{q+1}} \int_0^{\infty} y^q e^{-y} dy = \frac{2Kq!}{r|\ln(r)|^{q+1}}.$$

A.2 Flux-limited solutions

Lemma A.1. *Assume that $e = (e^k)$ is such that $H^k(-1/e^k) = \min_p H^k(p)$ for any $k \in \{0, \dots, K\}$. The solution of the junction problem without flux limiter:*

$$\begin{cases} \partial_t \tilde{\nu}_e + H(\partial_x \tilde{\nu}_e) = 0 \qquad \text{in } \overset{o}{\mathcal{R}} \times (0, +\infty) \\ \tilde{\nu}_e(x,k,0) = -x/e^k \qquad \text{in } \mathcal{R} \\ \partial_t \tilde{\nu}_e + \max\{A_0, H^{0,+}(\partial_0 \tilde{\nu}_e), H^{1,-}(\partial_1 \tilde{\nu}_e), \dots, H^{K,-}(\partial_K \tilde{\nu}_e)\} = 0 \qquad \text{at } x = 0. \end{cases} \tag{72}$$

is given by

$$\tilde{\nu}_e(x,k,t) = \min\left\{\phi_{A_0}(x,k) - A_0 t \ , \ -x/e^k - t H^k(-1/e^k)\right\}. \tag{73}$$

Proof. By stability of super-solution, we classically have that $\tilde{\nu}_e$ is a super-solution. Let us prove that it is a sub-solution. First remark that for $x \neq 0$, $(x,k,t) \mapsto \phi_{A_0}(x,k) - A_0 t$ and $(x,k,t) \mapsto -x/e^k - tH^k(-1/e^k)$ are (smooth) solutions of the equation. Then, using [9, Theorem 9.2 ii)], $\tilde{\nu}_e$ is a sub-solution. We then have to consider the case $x = 0$. Let ϕ be a test function such that $\tilde{\nu}_e - \phi$ reaches a minimum at $(0, t_0)$. By [36, Theorem 2.7], it is sufficient to take ϕ such that $\phi(x,k,t) = \psi(t) + \phi_{A_0}(x,k)$. Since for x close to 0, we have $\tilde{\nu}_e(x,t,k) = \phi_{A_0}(x,k) - A_0 t$ (because $A_0 \geqslant H^k(-1/e^k)$), we deduce that $t \mapsto -A_0 t - \psi(t)$ reaches a minimum at t_0 and so $\psi'(t) = -A_0$. This implies that

$$\partial_t \phi + \max\{A_0, H^{0,+}(\partial_0 \phi), H^{1,-}(\partial_1 \phi), \dots, H^{K,-}(\partial_K \phi)\} = -A_0 + \max\{A_0, \min_p H^0(p), \dots, \min_p H^K(p)\} = 0$$

and so $\tilde{\nu}_e$ is a sub-solution. Finally, for $t = 0$, since $A_0 \geqslant H^k(-1/e^k)$, we have $\tilde{\nu}_e(x,k,0) = -x/e^k < \phi_{A_0}(x,k)$. □

Lemma A.2. *Assume that $\bar{\vartheta}_e < -H^0(-1/e^0)$ and H^0 is convex. Then, the solution of*

$$\begin{cases} \partial_t w + H^0(\partial_x w) = 0 \text{ in } (-\infty, 0) \times (0, +\infty) \\ w(x,0) = -x/e^0 \text{ in } (-\infty, 0] \\ w(0,t) = \bar{\vartheta}_e t \text{ for } t \geqslant 0. \end{cases} \tag{74}$$

is unique and given by

$$w(x,t) := \min\{-x/e^0 - H^0(-1/e^0)t \ , \ p^{0,-}_{-\bar{\vartheta}_e} x + \bar{\vartheta}_e t\} \text{ in } (-\infty, 0) \times (0, +\infty). \tag{75}$$

Proof. The proof is similar to the one of Lemma A.1. Indeed it is sufficient to remark that, by [36, Proposition 2.12], w is solution of (74) iff w is solution of

$$\begin{cases} \partial_t w + H^0(\partial_x w) = 0 \text{ in } (-\infty, 0) \times (0, +\infty) \\ w(x,0) = -x/e^0 \text{ in } (-\infty, 0] \\ \partial_t w + \max\{A_e, H^{0,+}(\partial w)\} = 0 \qquad \text{at } x = 0, \end{cases} \tag{76}$$

where $A_e = -\vartheta_e$.

□

A.3 Homogenization outside the junction

We consider a family of solutions (U_i^ϵ) of (6) and define ν^ϵ from (U_i^ϵ) as in Section 4. Let us also fix a set $(a,b) \times \{k\} \times [t_0, t_1]$ with $t_0 < t_1$, $a < b < 0$ if $k = 0$ and $b > a > 0$ if $k \in \{1, \dots, K\}$. The following result is an easy adaptation of [19].

Lemma A.3. *There is a set Ω_0 of full probability (independent of U^ϵ) such that, for any $\omega \in \Omega$, if ν^ϵ is bounded above (respectively below) on $(a,b) \times \{k\} \times [t_0, t_1]$, then any half-relaxed upper limit (resp. half-relaxed lower limit) of ν^ϵ as $\epsilon \to 0$ (possibly up to a subsequence) is a viscosity subsolution (resp. supersolution) of the Hamilton-Jacobi equation*

$$\partial_t \nu(\cdot, k, \cdot) + H^k(\partial_x \nu(\cdot, k, \cdot)) = 0 \text{ in } (a,b) \times \{k\} \times [t_0, t_1].$$

A.4 Convexity of the effective Hamiltonians

Lemma A.4. *Assume that the $\tilde{V}_z^0$ are concave on $[\Delta_{\min}, +\infty)$ for any $z \in \mathcal{Z}$. Then $\bar{V}^0$ is also concave in $[\Delta_{\min}, +\infty)$ and H^0 is convex in $[-1/\Delta_{\min}, 0]$. In the same way, the H^k are convex on $[-\pi^k/\Delta_{\min}, 0]$ for any $k \in \{1, \dots, K\}$.*

Proof. Recall that a one-to-one map $\phi : I \to J$ (where I and J are open intervals) is increasing and concave if and only if ϕ^{-1} is increasing and convex. Thus the maps $(\tilde{V}_z^0)^{-1}$ (for $z \in \mathcal{Z}$) are increasing and convex from $(0, \min_{z'} h^0_{\max,z'})$ to $(\Delta_{\min}, \bar{e}^0)$. So $v \to \mathbb{E}\left[(\tilde{V}_z^0)^{-1}(v)\right]$ is also increasing and convex from $(0, \min_{z'} h^0_{\max,z'})$ to $(\Delta_{\min}, \bar{e}^0)$. This shows that its inverse $\bar{V}^0$ is increasing and concave from $(\Delta_{\min}, \bar{e}^0)$ to $(0, \min_{z'} h^0_{\max,z'})$. As $\bar{V}^0$ is continuous and is constant after $\bar{e}^0$, we infer that $\bar{V}^0$ is concave on $[\Delta_{\min}, +\infty)$. Finally, as $H^0(p) = p\bar{V}^0(-1/p)$ on $(-1/\Delta_{\min}, 0)$, H^0 is convex on this interval: indeed, if H^0 and $\bar{V}^0$ are smooth, then $(H^0)''(p) = p^{-3}(\bar{V}^0)''(-1/p) \geqslant 0$; the general case can be treated by approximation. □

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