



HAL
open science

TITS-TYPE ALTERNATIVE FOR CERTAIN GROUPS ACTING ON ALGEBRAIC SURFACES

Ivan Arzhantsev, M Zaidenberg

► **To cite this version:**

Ivan Arzhantsev, M Zaidenberg. TITS-TYPE ALTERNATIVE FOR CERTAIN GROUPS ACTING ON ALGEBRAIC SURFACES. 2021. hal-03427074v3

HAL Id: hal-03427074

<https://hal.science/hal-03427074v3>

Preprint submitted on 11 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TITS-TYPE ALTERNATIVE FOR CERTAIN GROUPS ACTING ON ALGEBRAIC SURFACES

IVAN ARZHANTSEV AND MIKHAIL ZAIDENBERG

ABSTRACT. According to a theorem of Cantat and Urech, an analog of the classical Tits alternative holds for the group of birational automorphisms of a compact complex Kähler surface. We established in [AZ21] the following Tits-type alternative: if X is a toric affine variety and $G \subset \text{Aut}(X)$ is a subgroup generated by a finite set of unipotent subgroups normalized by the acting torus then either G contains a nonabelian free subgroup or G is a unipotent affine algebraic group. In the present paper we extend the latter result to any group G of automorphisms of a complex affine surface generated by a finite collection of unipotent algebraic subgroups. It occurs that either G contains a nonabelian free subgroup or G is a metabelian unipotent algebraic group.

CONTENTS

Introduction	1
1. Preliminaries	3
1.1. Subgroups of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ and the Tits alternative	3
1.2. Subgroups of the Cremona group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$	4
2. The case of the affine plane	6
3. The case of the projective plane	7
4. The case of an affine surface	10
5. Unipotent group actions and Gizatullin surfaces	11
5.1. Unipotent group actions on affine surfaces	11
5.2. Gizatullin surfaces	12
6. Note added in proof	12
References	14

INTRODUCTION

Let \mathbf{k} be an algebraically closed field of characteristic zero, and let \mathbb{G}_a and \mathbb{G}_m stand for the additive and multiplicative groups of \mathbf{k} , respectively, viewed as algebraic groups. Given an algebraic variety X over \mathbf{k} and a subgroup U of $\text{Aut}(X)$ isomorphic to \mathbb{G}_a and acting regularly on X , U is called a *one-parameter unipotent subgroup* or a \mathbb{G}_a -*subgroup*. Any \mathbb{G}_a -subgroup has the form $U = \exp(t\partial) \subset \text{SAut}(X)$ where $t \in \mathbf{k}$ and ∂ is a locally nilpotent derivation (an *LND* for short) of the structure ring $\mathcal{O}(X)$.

The general linear group $\text{GL}_n(\mathbf{k})$ verifies the Tits alternative [Tit72], that is, any subgroup $G \subset \text{GL}_n(\mathbf{k})$ either contains a nonabelian free subgroup or is a finite extension of a solvable

The first author was supported by the grant RSF-DST 22-41-02019.

2020 *Mathematics Subject Classification*: 14J50, 14R20, 14L30, 14E07, 22F50.

Key words: affine surface, group action, unipotent group, one-parameter subgroup, Cremona group, Tits alternative.

group ¹. In the case of a connected linear algebraic group G defined over \mathbf{k} this alternative follows from the Levi-Malcev theorem. Indeed, consider the Levi decomposition $G = R_{\mathfrak{u}} \rtimes L = R_{\mathfrak{u}} \rtimes (S \cdot T)$ where $R_{\mathfrak{u}}$ is the unipotent radical of G , L is a reductive Levi subgroup, S is a semisimple algebraic group and T is an algebraic torus. We have the alternative: either G is solvable or G contains a semisimple algebraic subgroup S of positive dimension. In the latter case S contains a subgroup isomorphic either to $\mathrm{SL}(2, \mathbf{k})$ or to $\mathrm{PSL}(2, \mathbf{k})$, which in turn contains a nonabelian free subgroup. It follows that G has no nonabelian free subgroup if and only if S is trivial, if and only if G is solvable.

If G has no nontrivial character then $L = S$, that is, $G = R_{\mathfrak{u}} \rtimes S$. In this case G is generated by a finite collection of \mathbb{G}_a -subgroups normalized by a given maximal torus T_{\max} of S . Indeed, this clearly holds for $R_{\mathfrak{u}}$. In turn, S is generated by the finite collection of root \mathbb{G}_a -subgroups with respect to T_{\max} . Such a group G has no nonabelian free subgroup if and only if S is trivial, if and only if G is unipotent. The latter issue holds as well in the following more general setup.

Theorem 0.1 ([AZ21, Theorem 1.1]). *Consider a toric affine variety X over \mathbf{k} . ² Let a subgroup $G = \langle U_1, \dots, U_s \rangle$ of $\mathrm{Aut}(X)$ be generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k normalized by the acting torus. Then either G contains a nonabelian free subgroup or G is a unipotent algebraic group.*

Assuming that $\dim X \geq 2$, X is smooth in codimension 2 and has no nonconstant invertible regular function one can always find a group G as in Theorem 0.1 acting on the smooth locus of X highly transitively, that is, n -transitively for any natural number n [AKZ19, Theorem 1.3]. If G acts doubly transitively then G contains a nonabelian free subgroup, see [AZ21, Corollary 1.2].

The natural question arises whether the assertion of Theorem 0.1 remains true for more general affine varieties, disregarding the existence of a torus action. In Section 4 we give a positive answer for the complex affine surfaces. Our main result (Theorem 4.1) is as follows.

Theorem 0.2. *Let X be an affine algebraic surface over \mathbb{C} and G be a subgroup of $\mathrm{Aut}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . Assume that G contains no nonabelian free subgroup. Then G is a metabelian unipotent affine algebraic group.*

Recall that any abelian group is also metabelian. The proof of Theorem 0.2 uses Propositions 2.1 and 3.1. The former claims the validity of the theorem for the affine plane $X = \mathbb{A}_{\mathbb{C}}^2$. Notice that the conclusion of Theorem 0.2 does not hold any longer for the Cremona group $\mathrm{Bir}(\mathbb{P}_{\mathbf{k}}^2)$, see Example 3.7. In Proposition 3.1 we describe the exceptional cases. The proof of Theorem 0.2 is done in Section 4. In Section 5 we study the actions of unipotent algebraic groups on affine surfaces. Using Theorem 0.2 we deduce a Tits-type alternative for Gizatullin affine surfaces, see Propositions 5.1 and 5.2.

The reason why a solvable subgroup G of $\mathrm{Aut}(X)$ generated by a finite set of unipotent algebraic subgroups is unipotent is clear in the case where G is contained in an affine algebraic group \tilde{G} . Indeed, in this case G is contained in a Borel subgroup B of \tilde{G} and any unipotent subgroup of G is contained in the unipotent radical of B . Hence G itself is unipotent.

¹In [AZ21] we called this property the *enhanced Tits alternative* in order to distinguish from the usage of the term ‘‘Tits alternative’’ applied exceptionally to finitely generated subgroups.

²In [AZ21] X is assumed to be a toric variety with no toric factor. However, the statement obviously holds without the latter assumption.

However, a priori it is not clear why such a subgroup G must be algebraic. The latter holds e.g. if G consists of triangular automorphisms, see [FK18, Proposition 15.2.5], cf also [AS20]. This also happens in the setting of Theorem 0.2. After our paper had been submitted, this was proven to be the case for any affine variety defined over \mathbf{k} , see [KZ22, Theorem B].

The proofs in the present paper rely on several results established for varieties defined over \mathbb{C} , see [Can11, Des07, Des15, Des21, Lam01, Ure21] and Section 1 for an account of the results from these papers used in the proofs. We believe however that the following analog of Theorem 0.2 holds.

Conjecture 0.3. *Let X be an affine algebraic variety defined over an algebraically closed field and let G be a subgroup of $\text{Aut}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . Then either G contains a nonabelian free subgroup, or G is a unipotent affine algebraic group.*

In Section 6 (added in proof) we provide an alternative proof of Theorem 0.2 valid for any algebraically closed field of characteristic zero. This proof applies a recent result from [KZ22] and relies on the Tits alternative for the Cremona group of rank 2 over \mathbf{k} . The proof of the latter is sketched in the book [Lam22] under construction. Therefore, we consider for the moment our alternative proof to be conventional.

Acknowledgments. We are grateful to Yuri Prokhorov, Christian Urech and Stéphane Lamy for useful remarks and indications in the literature. Our thanks also go to the anonymous reviewer for several suggestions that have improved the article.

1. PRELIMINARIES

1.1. **Subgroups of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ and the Tits alternative.** Recall the following facts.

Theorem 1.1 (Rentschler [Ren68]). *Any \mathbb{G}_a -subgroup of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ is conjugate to a subgroup of triangular transformations*

$$(x, y) \mapsto (x + \tau p(y), y), \quad \tau \in \mathbf{k}$$

generated by a triangular LND $\partial = p(y)\partial/\partial x$ for some $p \in \mathbf{k}[y]$.

For the following theorem see e.g. [Jun42, vdK53, Nag72, Kam75, Wri75, Wri78].

Theorem 1.2 (Jung-van der Kulk). *There is a free amalgamated product decomposition $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2) = A *_C B$ where $C = A \cap B$, $A = \text{Aff}(\mathbb{A}_{\mathbf{k}}^2)$ is the affine group of the affine plane and $\text{Jonq}(\mathbb{A}_{\mathbf{k}}^2)$ is the group of de Jonquières transformations*

$$(x, y) \mapsto (\alpha x + p(y), \beta y + \gamma) \quad \text{with } \alpha, \beta \in \mathbf{k}^*, \gamma \in \mathbf{k}.$$

Notice that Theorem 1.1 can be deduced from Theorem 1.2. Indeed, any algebraic subgroup of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ has bounded degree, hence also bounded length [Wri79, 1.10]. However, any subgroup of bounded length of an amalgamated product $A *_C B$ is conjugate to a subgroup of one of the factors A and B [Ser80, Theorem 8]; cf. [Kam79, Corollary 4.2 and Theorem 4.3] and [Wri79, Propositions 0.35 and 1.11] for alternative arguments.

The abelian subgroups of an amalgamated product of two groups are described as follows.

Theorem 1.3 (Moldavanskii [Mol67]). *Consider a free product with amalgamated subgroup $F = A *_C B$. Then for any abelian subgroup G of F precisely one of the following holds:*

- (1) G is conjugate in F to a subgroup of either A or B ;
- (2) $G = \cup_i H_i$ where $H_0 \subset H_1 \subset \dots \subset H_i \dots$ is an infinite non-stationary chain of subgroups such that each H_i is conjugate in F to a subgroup of C while G is not conjugate to any subgroup of A or B ;
- (3) $G = H \times \langle g \rangle$ where H is conjugate to a subgroup of C and $g \in G$ is an element of infinite order non-conjugate to any element of A or B .

See [Wri79] for an alternative approach based on the Bass-Serre theory [Bas76, Ser80]. This approach is also used in [Lam01] and [Lam22] to obtain the following result.

Theorem 1.4 ([Lam01], [Lam22, Proposition 16.14]). *The group $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ verifies the Tits alternative.*

For $\mathbf{k} = \mathbb{C}$ a more detailed analysis is available, see [Lam01, Theorem 2.4, Propositions 3.12 and 4.10]. In fact, this can be extended to general fields, see Remark 6.5.

Theorem 1.5 (Lamy). *For any subgroup G of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ one of the following holds:*

- (1) *either G is conjugate to a subgroup of $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2)$ or $\text{Jonq}(\mathbb{A}_{\mathbb{C}}^2)$, or G belongs to one of the following types:*
- (2) *G is an abelian torsion group;*
- (3) *G is a solvable group which contains a subgroup of finite index isomorphic to \mathbb{Z} ;*
- (4) *G contains a nonabelian free subgroup.*

Let us remind that the de Jonquières group $\text{Jonq}(\mathbb{A}_{\mathbf{k}}^2)$ is solvable, see e.g. [Lam01, Corollary 2.5]. Notice that an abelian subgroup $G \subset \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ of type (2) resp. (3) above fits in case (2) resp. (3) of Theorem 1.3.

1.2. Subgroups of the Cremona group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. More generally, we have the following theorem.

Theorem 1.6 (Cantat-Urech [Can11, Ure21]). *Let X be a compact complex Kählerian surface. Then the group $\text{Bir}(X)$ verifies the Tits alternative.*

The birational actions of algebraic groups were considered e.g. by Weil [Wei55], Rosenlicht [Ros56], Matsumura [Mats63], Grothendieck [Gro67, §20.5] and Demazure [Dem70, §1]. We address [Ser10], [BF13] and [Bla17] for a recent treatment. Recall the following definitions.

Definition 1.7. Let X, Z be algebraic varieties over \mathbf{k} . A mapping $f: Z \rightarrow \text{Bir}(X)$ is said to be a *morphism* if the following holds:

- the induced map

$$Z \times X \rightarrow Z \times X, \quad (z, x) \mapsto (z, f(z)(x))$$

is a birational transformation which induces a biregular isomorphism $U \xrightarrow{\cong} V$ between some dense open subsets $U, V \subset Z \times X$ whose projections to Z are surjective.

Definition 1.8. Let G be an algebraic group over \mathbf{k} . A *birational action* of G on an algebraic variety X over \mathbf{k} is a morphism $G \rightarrow \text{Bir}(X)$ which is also a group homomorphism. A subgroup H of $\text{Bir}(X)$ is called *algebraic* if H is the image of some algebraic group G under a homomorphism $G \rightarrow \text{Bir}(X)$ which is a morphism. If $H \neq 1$ is algebraic and $G = \mathbb{G}_a$ then H is called a \mathbb{G}_a -*subgroup* of $\text{Bir}(X)$.

Recall that any algebraic subgroup of $\text{Bir}(\mathbb{P}_{\mathbf{k}}^n)$ is a linear group [BF13, Remark 2.21]. The next lemma follows directly from the definition of algebraic group action.

Lemma 1.9 ([Dem70, §1]; see also [Bla17, Remark 2.4]). *Let X and Y be algebraic varieties. Then any birational map $X \rightarrow Y$ induces a group isomorphism $\text{Bir}(X) \cong \text{Bir}(Y)$ and conjugates birational group actions on X to birational group actions on Y .*

For the next theorem see [Wei55, Theorem on p. 375], [Ros56, Theorem 1], see also [Zai95, Theorem 4.6], [Bla17, Theorem 2.17], [Kra18, Theorem 1] and [Bri22].

Theorem 1.10. *Let X be an algebraic variety, G be an algebraic group and $G \rightarrow \text{Bir}(X)$ be a birational group action. Then, there exists a birational map $X \rightarrow Y$ where Y is an algebraic variety that conjugates this action to a biregular group action of G on Y .*

Next we turn to classification of nilpotent and solvable subgroups of the complex Cremona group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$.

Theorem 1.11 (Déserti [Des07]). *Let G be a nilpotent subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. Then G is either a torsion group or a virtually metabelian group.*

The next theorem is a reformulation of [Des21, Theorem 8.49], which in turn integrates the results of [Des15] and [Ure21, Theorem 7.1] based in particular on [Can11, Proposition 6.14 and Theorem 7.7].

Theorem 1.12 (Déserti–Urech). *Let G be a solvable subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. Then one of the following holds:*

- (1) G is conjugate to a subgroup of $\text{PGL}(2, \mathbb{C}(t)) \rtimes \text{PGL}(2, \mathbb{C})$;
- (2) G is conjugate to a subgroup of an affine algebraic group defined over \mathbb{C} ;
- (3) G is conjugate to a subgroup of the normalizer $N(T) \cong \mathbb{G}_m^2 \rtimes \text{GL}(2, \mathbb{Z})$ of the standard 2-torus $T \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$;
- (4) Up to passing possibly to a subgroup of index 2, G is isomorphic to $H \rtimes \mathbb{Z}$ for a finite subgroup H of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$.

Remark 1.13. By Theorem 1.11 the derived length of any connected nilpotent subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is at most 2. The maximal derived length of a solvable subgroup $G \subset \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ ($G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$, respectively) equals 5 [FP18, Proposition 3.14] (is bounded above by 8, respectively [Ure21, Theorem 1.10]). The derived length of a solvable closed connected subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is at most 4 and this bound is reached for the Borel subgroup $\mathcal{B}_2 \cong \text{Aff}_1(\mathbb{C}(y)) \rtimes \text{Aff}_1(\mathbb{C})$ [FH21, Proposition 4 and Lemma 52]. See also [Cor13, Proposition 2.2] for an example of a solvable 3-generated subgroup $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ of derived length 3 which admits no faithful linear representation.

For any $n \geq 1$ there is a nilpotent 2-generated torsion-free subgroup G_n of $\text{Bir}(\mathbb{P}_{\mathbf{k}}^2)$ of nilpotency class n [Cor13, Lemma 2.1]. Furthermore, the unipotent algebraic subgroup G of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ generated by the \mathbb{G}_a -subgroups

$$U_1 = \{(x, y) \mapsto (x + s, y)\} \quad \text{and} \quad U_2 = \{(x, y) \mapsto (x, y + \tau x^n)\}, \quad s, \tau \in \mathbb{C}$$

has nilpotency class n .

2. THE CASE OF THE AFFINE PLANE

In this section we prove Theorem 0.2 in the particular case $X = \mathbb{A}_{\mathbb{C}}^2$. In section 4 we reduce the general case to this one.

Proposition 2.1. *Let G be a subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k defined over \mathbb{C} . Assume that G contains no nonabelian free subgroup. Then G is a unipotent affine algebraic group conjugate in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ to a subgroup of the metabelian unipotent de Jonquières group*

$$\text{jonq}(\mathbb{A}_{\mathbb{C}}^2) = \{(x, y) \mapsto (x + p(y), y + \gamma) \mid p \in \mathbb{C}[y], \gamma \in \mathbb{C}\}.$$

Proof. Under our assumption G is solvable, see Corollary 1.4. So, it is of one of types (1)-(3) in Theorem 1.5. Since G is a connected subgroup of the ind-group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ it cannot be of type (2) or (3). Hence, G is of type (1), that is, G is conjugate to a subgroup of one of the groups $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2)$ and $\text{Jonq}(\mathbb{A}_{\mathbb{C}}^2)$.

Let G be conjugate to a subgroup of $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2)$. Then G is a connected solvable algebraic group. Hence G is contained in a Borel subgroup B of $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2)$. The unipotent subgroups U_1, \dots, U_k are contained in the unipotent radical U of B , hence also $G \subset U$. Consider the standard faithful representation $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2) \rightarrow \text{GL}(3, \mathbb{C})$. Since the Borel subgroups of $\text{Aff}(\mathbb{A}_{\mathbb{C}}^2)$ are conjugated we may suppose that the image of G in $\text{GL}(3, \mathbb{C})$ consists of upper triangular matrices of order 3 with unit diagonal. In other words, G is an algebraic subgroup of the metabelian unipotent affine algebraic group

$$(x, y) \mapsto (x + ay + b, y + \gamma) \quad \text{where } a, b, \gamma \in \mathbb{C}.$$

In this case G is a metabelian unipotent algebraic group (of dimension at most 3).

Let now G be conjugate to a subgroup of $\text{Jonq}(\mathbb{A}_{\mathbb{C}}^2)$. We may assume that $G \subset \text{Jonq}(\mathbb{A}_{\mathbb{C}}^2)$. There is a decomposition into a semidirect product

$$\text{Jonq}(\mathbb{A}_{\mathbb{C}}^2) = \text{jonq}(\mathbb{A}_{\mathbb{C}}^2) \rtimes T$$

where $T = \{(x, y) \mapsto (\alpha x, \beta y) \mid \alpha, \beta \in \mathbb{C}^*\}$ is the maximal torus. Since any homomorphism $U_i \rightarrow T$ is trivial U_i is contained in the unipotent radical $\text{jonq}(\mathbb{A}_{\mathbb{C}}^2)$ of $\text{Jonq}(\mathbb{A}_{\mathbb{C}}^2)$ for $i = 1, \dots, k$. Hence also $G \subset \text{jonq}(\mathbb{A}_{\mathbb{C}}^2)$. There is the ascending filtration

$$\text{jonq}(\mathbb{A}_{\mathbb{C}}^2) = \bigcup_{d=1}^{\infty} \text{jonq}_d(\mathbb{A}_{\mathbb{C}}^2) \quad \text{where } \text{jonq}_d(\mathbb{A}_{\mathbb{C}}^2) = \{f \in \text{jonq}(\mathbb{A}_{\mathbb{C}}^2) \mid \deg(f) \leq d\}$$

is a metabelian unipotent affine algebraic group. This defines on $\text{jonq}(\mathbb{A}_{\mathbb{C}}^2)$ a structure of a nested ind-group. So $U_i \subset \text{jonq}_d(\mathbb{A}_{\mathbb{C}}^2)$ for all $i = 1, \dots, k$ and for some $d \gg 1$. Therefore, $G \subset \text{jonq}_d(\mathbb{A}_{\mathbb{C}}^2)$ is a metabelian unipotent affine algebraic group. \square

Remark 2.2. The validity of the Tits alternative for the group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$ is still an open question. It holds, however, in the tame subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$, see [LP22]. See also [Pop17, Corollary 2] (cf. also [DF91, Corollary 3.2]) for the following partial result.

Every connected solvable affine algebraic subgroup G of $\text{Aut}(\mathbb{A}_{\mathbb{k}}^3)$ is rationally triangulable, that is, conjugate in $\text{Bir}(\mathbb{A}_{\mathbb{k}}^3)$ to a subgroup of the de Jonquières group $\text{Jonq}(\mathbb{A}_{\mathbb{k}}^3)$ of triangular automorphisms of $\mathbb{A}_{\mathbb{k}}^3$.

It follows that the derived length of G is at most 4. Indeed, the derived length of $\text{Jonq}(\mathbb{A}_{\mathbb{k}}^3)$ equals 4 [FP18, Lemma 3.2].

3. THE CASE OF THE PROJECTIVE PLANE

In this section we prove the following proposition.

Proposition 3.1. *Let X be an algebraic surface over \mathbb{C} and let G be a subgroup of $\text{Bir}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . Assume that G contains no non-abelian free subgroup. Then G is a solvable group. Furthermore, one of the following holds:*

- (1) G is a metabelian unipotent affine algebraic group;
- (2) X is rational, G preserves a pencil of rational curves on X and contains a normal subgroup N such that $G/N \cong \mathbb{G}_a$.

Recall [Des21, Theorem 1.1] that, given a pencil Λ of rational curves on $\mathbb{P}_{\mathbf{k}}^2$, the subgroup of the Cremona group $\text{Bir}(\mathbb{P}_{\mathbf{k}}^2)$ consisting of transformations preserving Λ is conjugate to the de Jonquieres subgroup J of the Cremona group $\text{Bir}(\mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1) \cong \text{Bir}(\mathbb{P}_{\mathbf{k}}^2)$. One has

$$J = \text{PGL}(2, \mathbf{k}(t)) \rtimes \text{PGL}(2, \mathbf{k})$$

where $\text{PGL}(2, \mathbf{k})$ acts on $\mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1$ via automorphisms of the second factor and acts on $\text{PGL}(2, \mathbf{k}(t))$ via the natural $\text{PGL}(2, \mathbf{k})$ -action on the rational function field $\mathbf{k}(t)$.

The proof of Proposition 3.1 is preceded by the following lemmas.

Lemma 3.2. *Let G be a solvable subgroup of the subgroup $\text{PGL}(2, \mathbf{k}(t)) \subset J$ generated by a finite collection of \mathbb{G}_a -subgroups $U_1, \dots, U_k \subset \text{Bir}(\mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1)$ defined over \mathbf{k} . Then G is an abelian subgroup of the unipotent radical $R_u(\tilde{B})$ where \tilde{B} is a Borel subgroup of $\text{PGL}(2, \mathbf{k}(t))$.*

Proof. Let \bar{G} be the minimal algebraic subgroup of $\text{PGL}(2, \mathbf{k}(t))$ defined over $\mathbf{k}(t)$ which contains G . It is well known that \bar{G} is solvable; see e.g. [FP18, Lemma 2.3]. It is also connected. Indeed, the component group of \bar{G} is finite. Since the \mathbb{G}_a -subgroups U_i are infinitely divisible they are contained in the neutral component \bar{G}^0 of \bar{G} . Hence $\bar{G} = \bar{G}^0$ is connected.

It follows that \bar{G} is contained in a Borel subgroup \tilde{B} of $\text{PGL}(2, \mathbf{k}(t))$. Since every unipotent subgroup U_i of \tilde{B} is contained in the unipotent radical $R_u(\tilde{B}) \cong \mathbb{G}_a(\mathbf{k}(t))$, then so is G . \square

Lemma 3.3. *Let X be a projective variety over \mathbf{k} and G be a solvable subgroup of $\text{Bir}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups $U_1, \dots, U_k \subset \text{Bir}(X)$. Assume that G is conjugate to a subgroup of $\text{Aut}(S)$ where S is a projective variety. Then G is a unipotent affine algebraic group. The same conclusion holds if G is conjugate to a subgroup of an affine algebraic group.*

Proof. Assume G is conjugate to a subgroup of $\text{Aut}(S)$. The group G being connected the image of G in $\text{Aut}(S)$ is contained in the neutral component $\text{Aut}^0(S)$. By the Matsusaka theorem [Mat58] $\text{Aut}^0(S)$ is an algebraic group. According to a Chevalley theorem (see [Ros56, Theorem 16]) there exists a normal connected affine algebraic subgroup H of $\text{Aut}^0(S)$ such that $A = \text{Aut}^0(S)/H$ is an abelian variety. Since any homomorphism $U_i \rightarrow A$ is trivial the image of U_i in $\text{Aut}^0(S)$ is contained in H for $i = 1, \dots, k$. Hence also the image of G is contained in H . Being solvable and connected, the image of G is contained in a Borel subgroup B of H . There is a decomposition $B = U \rtimes T$ where U is the unipotent radical of B and T is a maximal torus of H . Since any morphism $U_i \rightarrow T$ is trivial the image of U_i is contained in U for $i = 1, \dots, k$. Therefore the image of G is a closed subgroup of U , and

so, a unipotent affine algebraic group. The same argument proves the last assertion of the lemma. \square

Remark 3.4. Let X be a normal affine variety over \mathbf{k} and let H_1, \dots, H_k be algebraic subgroups of $\text{Aut}(X)$. If these subgroups are contained in an algebraic subgroup H of $\text{Aut}(X)$ then by the Sumihiro theorem [Sum74, Theorem 3] there exists an H -equivariant completion \bar{X} of X so that the action of H_1, \dots, H_k on X extends to their action on \bar{X} . The proof of Lemma 3.3 shows the partial converse, namely,

Assume that H_1, \dots, H_k are connected and their action on X can be extended to a completion \bar{X} of X . Then $G = \langle H_1, \dots, H_k \rangle$ is a connected linear algebraic group.

Recall that $G \subset \text{Aut}^0(\bar{X})$ is closed, hence an algebraic subgroup, see [Hum75, Proposition 7.5].

Lemma 3.5. *Let C be a curve over \mathbf{k} and U and U' be two different \mathbb{G}_a -subgroups of $\text{Aut}(C)$. Then $C \cong \mathbb{P}_{\mathbf{k}}^1$, the fixed points of U and U' are distinct and $\langle U, U' \rangle \cong \text{PGL}(2, \mathbf{k})$.*

The proof is a simple exercise and will be omitted.

Lemma 3.6. *Let X be a projective surface over \mathbf{k} and G be a subgroup of $\text{Bir}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . If X is irrational then either one of the subgroups $\langle U_i, U_j \rangle$ surjects onto $\text{PGL}(2, \mathbf{k})$ and so contains a nonabelian free subgroup, or G is an abelian unipotent algebraic group. The latter holds indeed provided X contains a dense open G -invariant affine subset.*

Proof. If G is abelian then G is a quotient of the abelian affine algebraic group $U_1 \times \dots \times U_k$, which implies the assertion in this case. Otherwise there is a pair of non-commuting \mathbb{G}_a -subgroups U_i and U_j . If the general orbits of U_i and U_j have distinct closures then for the general point $P \in X$ the rational map

$$\mathbb{A}_{\mathbf{k}}^2 \rightarrow X, \quad (s, t) \mapsto U_i(s)U_j(t)(P)$$

is dominant. So X is unirational. By the Castelnuovo rationality theorem X is rational, contrary to our assumption.

Therefore, all the U_i have the same one-dimensional closures $\bar{\mathcal{O}}$ of general orbits, and these are the ones of G . Let $\tilde{\mathcal{O}}$ be the normalization of $\bar{\mathcal{O}}$. Then $U_i|_{\tilde{\mathcal{O}}}$ and $U_j|_{\tilde{\mathcal{O}}}$ are non-commuting \mathbb{G}_a -subgroups of $\text{Aut}(\tilde{\mathcal{O}})$, cf. [Dem70, Exemple 3]. By Lemma 3.5 $\tilde{\mathcal{O}} \cong \mathbb{P}_{\mathbf{k}}^1$, the fixed points of $U_i|_{\tilde{\mathcal{O}}}$ and $U_j|_{\tilde{\mathcal{O}}}$ are distinct and $\langle U_i|_{\tilde{\mathcal{O}}}, U_j|_{\tilde{\mathcal{O}}} \rangle \cong \text{PGL}(2, \mathbf{k})$. Hence $\langle U_i, U_j \rangle$ surjects onto $\text{PGL}(2, \mathbf{k})$.

Assume now that X contains a dense open G -invariant affine subset Y . Then each U_i is transitive on the general orbit $\mathcal{O} \cong \mathbb{A}_{\mathbf{k}}^1$ of G and the restrictions $U_i|_{\mathcal{O}}$ and $U_j|_{\mathcal{O}}$ commute. Hence also U_i and U_j commute for any i, j . \square

Proof of Proposition 3.1. By Theorem 1.6 $\text{Bir}(X)$ verifies the Tits alternative. Since the connected subgroup $G \subset \text{Bir}(X)$ contains no nonabelian free subgroup it is solvable. If X is irrational then by Lemma 3.6 case (1) of the proposition holds. Assume now that X is a rational surface and $G \subset \text{Bir}(X) \cong \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. By Theorem 1.12 G is one of the groups in (1)-(4) of this theorem. Any \mathbb{G}_a -subgroup U_i is infinitely divisible. Hence it admits no nontrivial homomorphism to the group $\Gamma = H \rtimes \mathbb{Z}$ where H is a finite group. The same holds as well for $G = \langle U_1, \dots, U_k \rangle$. Therefore G cannot be of type (4) in Theorem 1.12. It cannot be of type (3)

either, since otherwise the image of G in $N(T)$ is contained in $T = N(T)^0$, which is impossible. If G is of type (2), that is, G is conjugate to a subgroup of an affine algebraic group, then by Lemma 3.3 G is a unipotent affine algebraic group. By Theorem 1.11 G is metabelian, which corresponds to case (1) of the proposition. It remains to treat the possibility for G to be of type (1) in Theorem 1.12.

Thus, we assume in the sequel that G is contained in the de Jonquère subgroup $J = \mathrm{PGL}(2, \mathbb{C}(t)) \rtimes \mathrm{PGL}(2, \mathbb{C})$ of $\mathrm{Bir}(\mathbb{P}_{\mathbb{C}}^2)$. The latter is equivalent to the fact that G preserves a pencil of rational curves on $\mathbb{P}_{\mathbb{C}}^2$, see e.g. [Des21, p. 4]. Let $\varphi: J \rightarrow \mathrm{PGL}(2, \mathbb{C})$ be the quotient morphism. The connected solvable subgroup $\varphi(G)$ is contained in the Borel subgroup B of $\mathrm{PGL}(2, \mathbb{C})$. For any $i = 1, \dots, k$ the image $\varphi(U_i)$ is contained in the unipotent radical $R_u(B) \cong \mathbb{G}_a$. Thus, we have the exact sequence

$$(1) \quad 1 \rightarrow N = \ker(\varphi) \rightarrow G \xrightarrow{\varphi} \mathbb{G}_a = \mathbb{G}_a(\mathbb{C}).$$

Assume first that $N = G$, that is, $G \subset \mathrm{PGL}(2, \mathbb{C}(t))$. By Lemma 3.2 G is abelian. Hence G is a quotient of $\prod_{i=1}^k U_i$ and once again case (1) of the proposition holds.

Let now $N \neq G$. Then φ is surjective, $G/N \cong \mathbb{G}_a$ and we are in case (2) of the proposition. \square

The following example shows that the group G in Proposition 3.1 can be an infinite-dimensional (hence non-algebraic) metabelian group.

Example 3.7. Consider the \mathbb{G}_a -subgroups U_1 and U_2 of $\mathrm{Bir}(\mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1) \cong \mathrm{Bir}(\mathbb{P}_{\mathbf{k}}^2)$ given in affine coordinates $(x, y) \in \mathbb{A}_{\mathbf{k}}^2 \subset \mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1$ by formulas

$$U_1 = \{(x, y) \mapsto (x + \tau, y)\}, \quad U_2 = \{(x, y) \mapsto (x, y + \frac{\tau}{x})\}, \quad \tau \in \mathbf{k}.$$

They are generated by the locally nilpotent derivations

$$\partial_1 = \partial/\partial x \in \mathrm{Der}(\mathbf{k}[x, y]) \quad \text{resp.} \quad \partial_2 = x^{-1}\partial/\partial y \in \mathrm{Der}(\mathbf{k}(x)[y]).$$

The Lie subalgebra $L \subset \mathrm{Der}(\mathbf{k}(x)[y])$ generated by ∂_1 and ∂_2 is two-step solvable. In fact,

$$L := \langle \partial_1, \partial_2 \rangle = \langle L_1, L_2 \rangle \quad \text{where} \quad L_1 = \mathbf{k}\partial_1 \quad \text{and} \quad L_2 = \{f(x^{-1})\partial/\partial y \mid f \in u\mathbf{k}[u]\}$$

are abelian Lie subalgebras. We have $[L, L_2] \subset L_2$ where $L_2 \subset L$ is an abelian ideal. Furthermore, $G = \langle U_1, U_2 \rangle$ is an infinite-dimensional subgroup of the metabelian group $R_u(\tilde{B}) \rtimes R_u(B) \cong \mathbb{G}_a(\mathbf{k}(t)) \rtimes \mathbb{G}_a(\mathbf{k})$, that is, of the semidirect product of unipotent radicals of the Borel subgroups $\tilde{B} \subset \mathrm{PGL}(2, \mathbf{k}(x))$ and $B \subset \mathrm{PGL}(2, \mathbf{k})$. Thus, G is an infinite-dimensional metabelian subgroup of $\mathrm{Bir}(\mathbb{P}_{\mathbf{k}}^2)$.

In the next example we exhibit a \mathbb{G}_a -subgroup U of the semidirect product of the Borel subgroups $\tilde{B} \rtimes B \subset J$ which is not contained in the semidirect product of the unipotent radicals $R_u(\tilde{B}) \rtimes R_u(B)$.

Example 3.8. Consider the \mathbb{G}_a -subgroup $U \subset \tilde{B} \rtimes B \subset J$ acting on $\mathbb{P}_{\mathbf{k}}^1 \times \mathbb{P}_{\mathbf{k}}^1$ via birational transformations

$$(x, y) \mapsto \left(x + \tau, \frac{x + \tau}{x}y + \frac{\tau}{x} \right), \quad \tau \in \mathbf{k}.$$

The infinitesimal generator of U is the derivation

$$\partial = \frac{\partial}{\partial x} + \frac{y+1}{x} \frac{\partial}{\partial y} \quad \text{where} \quad \partial^2(x) = \partial^2(y) = 0.$$

Clearly, $U \notin R_u(\tilde{B}) \rtimes R_u(B) \cong \mathbb{G}_a(\mathbf{k}(x)) \rtimes \mathbb{G}_a(\mathbf{k})$.

Remark 3.9. It is known that any \mathbb{G}_a -subgroup of $\text{Bir}(\mathbb{P}_{\mathbf{k}}^2)$ is conjugate to the subgroup of translations $(x, y) \mapsto (x + \tau, y)$, $\tau \in \mathbf{k}$, see [DF91, Corollary 3.2] and [Pop17, Corollary 5(i)] (for a more general result); cf. also [FH21, Proposition 85].

4. THE CASE OF AN AFFINE SURFACE

The next result is our main theorem.

Theorem 4.1. *Let X be an affine surface over \mathbb{C} and G be a subgroup of $\text{Aut}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . Assume that G contains no nonabelian free subgroup. Then G is a metabelian unipotent affine algebraic group.*

Proof. By Proposition 3.1 the assertion holds unless X is rational, G preserves a pencil Λ of rational curves on X and has a solvable normal subgroup N such that $G/N \cong \mathbb{G}_a$. In this remaining case we consider a functorial resolution of singularities $\tilde{X} \rightarrow X$. It is G -equivariant and the lift to \tilde{X} of any regular action of an algebraic group on X is regular, see [Kol07, §3.4, Proposition 3.9.1 and Theorem 3.36]. Since X is a surface, one can take for such a resolution the normalization followed by the minimal resolution of singularities. Indeed, both of them are functorial.

Thus, we may and will assume that X is a smooth quasi-projective surface and Λ is a G -invariant pencil of non-complete rational curves on X . The fixed components of Λ are G -invariant. Deleting these components we may suppose that the base locus $\text{Bs}(\Lambda)$ is a finite set of points. These points are fixed by the action of G . We can resolve the base points of Λ via a minimal sequence of successive blowups with zero-dimensional centers. The G -action on X lifts to the resolution and the lift of any U_i -action remains regular. Therefore we may assume that Λ is base-points free.

By our assumption, $\varphi: G \rightarrow \mathbb{G}_a$ in (1) is surjective. Hence $\varphi|_{U_l}$ surjects onto \mathbb{G}_a for some $l \in \{1, \dots, k\}$. The effective \mathbb{G}_a -action on the base $\mathbb{P}_{\mathbb{C}}^1$ of Λ has a unique fixed point. Let this point be the point $\infty \in \mathbb{P}^1$. If the corresponding member γ_{∞} of the pencil Λ is a nonzero effective divisor then it is G -invariant. Replacing X by $X \setminus \text{supp}(\gamma_{\infty})$ we may suppose that Λ is parameterized by $\mathbb{A}_{\mathbb{C}}^1 = \mathbb{P}_{\mathbb{C}}^1 \setminus \{\infty\}$. The base point free pencil Λ defines a morphism $\pi: X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ which is G -equivariant with respect to the G -action on X and the $\varphi(G)$ -action on $\mathbb{A}_{\mathbb{C}}^1$. The $\varphi(U_l)$ -action on $\mathbb{A}_{\mathbb{C}}^1$ being free the U_l -action on X is as well free. The general member γ of Λ is a smooth affine curve. The free U_l -action on X yields a trivialization of the fibration $\pi: X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ via the U_l -equivariant isomorphism

$$\mathbb{A}_{\mathbb{C}}^1 \times \gamma \xrightarrow{\cong} X, \quad (\tau, P) \mapsto U_l(\tau)(P), \quad \tau \in \mathbb{C},$$

where U_l acts on $\mathbb{A}_{\mathbb{C}}^1 \times \gamma$ via translations on the first factor. If the affine curve γ is not isomorphic to the affine line then any morphism $\mathbb{A}_{\mathbb{C}}^1 \rightarrow \gamma$ is constant. Hence there is a unique \mathbb{A}^1 -fibration on X , namely, the one coming from $\text{pr}_2: \mathbb{A}_{\mathbb{C}}^1 \times \gamma \rightarrow \gamma$ via the above isomorphism $X \cong \mathbb{A}_{\mathbb{C}}^1 \times \gamma$. It follows that for any $i = 1, \dots, k$ the general orbits of the U_i -action on X are the general fibers of the latter $\mathbb{A}_{\mathbb{C}}^1$ -fibration. In this case U_i and U_j commute for any i, j , hence G is abelian and a quotient of $\prod_{i=1}^k U_i \cong \mathbb{G}_a^k$.

Suppose G is not abelian. Then $\gamma \cong \mathbb{A}_{\mathbb{C}}^1$ and so $X \cong \mathbb{A}_{\mathbb{C}}^2$. Now Proposition 2.1 applies and gives the result. \square

Remarks 4.2. 1. Alternatively, one can use a detailed description of the subgroup of automorphisms of an \mathbb{A}^1 -ruled affine surface which preserve the ruling, see [KPZ17, Theorems 8.13 and 8.25].

2. As a curiosity, let us observe that any group G generated by a finite collection of \mathbb{G}_a -subgroups and acting on an affine surface X over \mathbb{C} with a non-closed orbit must contain a nonabelian free subgroup. Indeed, otherwise by Theorem 4.1 G is unipotent and then any G -orbit is closed in X according to a theorem of Rosenlicht. As a simple example consider the group $G = \mathrm{SL}_2(\mathbb{C})$ acting on the affine plane. It is generated by the \mathbb{G}_a -subgroups

$$U_1 = \{(x, y) \mapsto (x + sy, y)\} \quad \text{and} \quad U_2 = \{(x, y) \mapsto (x, y + tx)\}, \quad s, t \in \mathbb{C}.$$

The origin of $\mathbb{A}_{\mathbb{C}}^2$ is a unique closed G -orbit and its complement is an open orbit of G .

However, the above condition is not necessary in order that G contains a nonabelian free subgroup. Indeed, adding a third \mathbb{G}_a -subgroup of shifts $U_3 = \{(x, y) \mapsto (x + r, y), r \in \mathbb{C}\}$ one gets a group $\langle U_1, U_2, U_3 \rangle$ acting transitively on the plane and containing a nonabelian free subgroup.

5. UNIPOTENT GROUP ACTIONS AND GIZATULLIN SURFACES

5.1. Unipotent group actions on affine surfaces.

Proposition 5.1. *Let X be an affine surface over \mathbf{k} and U be a unipotent algebraic subgroup of $\mathrm{Aut}(X)$. Then one of the following holds:*

- (i) *U has an open orbit in X . In this case $X \cong \mathbb{A}_{\mathbf{k}}^2$, U is transitive on $\mathbb{A}_{\mathbf{k}}^2$ and is conjugate to a subgroup of the unipotent de Jonquières subgroup $\mathrm{Jonq}(\mathbb{A}_{\mathbf{k}}^2)$.*
- (ii) *The general orbits of U on X are one-dimensional. In this case U is abelian.*

For any natural number $n \geq 2$ there exists a unipotent subgroup $U \subset \mathrm{Aut}(X)$ of dimension $\dim U = n$ satisfying (i), respectively, (ii).

Proof. In case (i) the open orbit of the unipotent algebraic group U is closed in X , hence coincides with X and is isomorphic to the affine plane $\mathbb{A}_{\mathbf{k}}^2$. By Serre's theorem [Ser80, Theorem 8] the algebraic group U is conjugate to a subgroup of either $\mathrm{Aff}(\mathbb{A}_{\mathbf{k}}^2)$ or $\mathrm{Jonq}(\mathbb{A}_{\mathbf{k}}^2)$. In any case, being unipotent U is conjugate to a subgroup of $\mathrm{Jonq}(\mathbb{A}_{\mathbf{k}}^2)$. It remains to notice that the unipotent algebraic group $G = \langle U_1, U_2 \rangle$ in Remark 1.13 has dimension $n + 2$ and acts transitively on the plane $\mathbb{A}_{\mathbf{k}}^2$.

In case (ii) consider a \mathbb{G}_a -subgroup U_0 of U , and let $\partial_0 \in \mathrm{lie}(U_0)$ be the locally nilpotent derivation generating U_0 . The Lie algebra $\mathrm{lie}(U)$ consists of regular vector fields on X tangent to the orbits of U . Any such derivation has the form $f\partial_0$ where f is a U -invariant rational function on X . It follows that $\mathrm{lie}(U)$, and as consequence also U , are abelian.

The last assertion in this case can be confirmed as follows. Starting with a \mathbb{G}_a -subgroup U_0 of $\mathrm{Aut}(X)$ and a nonconstant U_0 -invariant function $f \in \mathcal{O}(X)$ and adding the \mathbb{G}_a -subgroups $U_i = \exp(f^i \partial_0)$, $i = 1, \dots, n - 1$ one obtains an n -dimensional abelian unipotent subgroup $U \subset \mathrm{Aut}(X)$ where

$$U = \prod_{i=0}^{n-1} U_i = \{\exp(p(f)\partial_0) \mid p \in \mathbf{k}[t], \deg(p) \leq n - 1\}.$$

□

5.2. Gizatullin surfaces. We recall a classification of normal affine surfaces with respect to the ‘size’ of the collection of \mathbb{G}_a -subgroups of their automorphism groups, see e.g. [KPZ17, Section 5]. Given such a surface X over \mathbf{k} we let $\text{SAut}(X)$ be the subgroup of $\text{Aut}(X)$ generated by all the \mathbb{G}_a -subgroups. The *Makar-Limanov invariant* $\text{ML}(X)$ measures the complexity of the action of the group $\text{SAut}(X)$ on X . More precisely, we say that X is of class ML_i where $i \in \{0, 1, 2\}$ if the codimension in X of the general $\text{SAut}(X)$ -orbit is equal to i . The surface X is said to be *rigid* if it is of class ML_2 , that is, does not admit any nontrivial \mathbb{G}_a -action. If X is of class ML_1 then all the effective \mathbb{G}_a -actions on X have the same general orbits contained in the fibers of an $\mathbb{A}_{\mathbf{k}}^1$ -fibration $X \rightarrow B$ over a smooth affine curve B . In this case, under a mild addition assumption, $\text{SAut}(X)$ is an abelian nested ind-group [KPZ17, Section 6]. A simple example is provided by the surface $X = (\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}) \times \mathbb{A}_{\mathbf{k}}^1$. Indeed, this surface carries the unique $\mathbb{A}_{\mathbf{k}}^1$ -fibration $\text{pr}_1: X \rightarrow \mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$.

The most interesting class is the class ML_0 . It consists of the Gizatullin surfaces. A normal affine surface X over \mathbf{k} non-isomorphic to $(\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}) \times \mathbb{A}_{\mathbf{k}}^1$ is called a *Gizatullin surface* if X can be embedded into a complete surface \bar{X} by adding a reduced boundary divisor $D = \bar{X} \setminus X$ which is a chain of smooth rational curves with simple normal crossings. By the Gizatullin theorem [Giz71, Theorems 2 and 3] a normal affine surface X is Gizatullin if and only if $\text{Aut}(X)$ contains two \mathbb{G}_a -subgroups U_1, U_2 whose general orbits are different, in other words, if the group $\text{SAut}(X)$ acts on X with an open orbit, that is, $\text{ML}(X) = \mathbf{k}$. Given a Gizatullin surface X the group $\text{SAut}(X)$ acts infinitely transitively on its open orbit [AFKKZ13, Example 2.3]. For any natural number $n > 1$ there exists a smooth Gizatullin surface X carrying an n -parameter family of $\mathbb{A}_{\mathbf{k}}^1$ -fibrations over $\mathbb{A}_{\mathbf{k}}^1$ such that any two of them are not equivalent under the $\text{Aut}(X)$ -action on X [FKZ11, Example 6.3.21]. Such a surface X also carries an n -parameter family of \mathbb{G}_a -subgroups acting along these $\mathbb{A}_{\mathbf{k}}^1$ -fibrations and pairwise non-conjugate in $\text{Aut}(X)$. Notice that for $X = \mathbb{A}_{\mathbf{k}}^2$ any two $\mathbb{A}_{\mathbf{k}}^1$ -fibrations over $\mathbb{A}_{\mathbf{k}}^1$ are equivalent up to the $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ -action on $\mathbb{A}_{\mathbf{k}}^2$; this follows e.g. from Rentchler’s Theorem 1.1.

Using Theorem 0.2 and Proposition 5.1 we deduce the following result.

Proposition 5.2. *Let X be a Gizatullin surface over \mathbb{C} non-isomorphic to $\mathbb{A}_{\mathbb{C}}^2$ and let G be a subgroup of $\text{Aut}(X)$ generated by a collection of \mathbb{G}_a -subgroups. Then the following Tits-type alternative holds: either G is abelian and acts on X with one-dimensional general orbits, or G contains a nonabelian free subgroup and acts with an open orbit on X .*

Proof. If G acts on X with one-dimensional general orbits then it preserves fiberwise an $\mathbb{A}_{\mathbb{C}}^1$ -fibration $X \rightarrow \mathbb{A}_{\mathbb{C}}^1$. Hence any two \mathbb{G}_a -subgroups of G commute and so G is abelian.

Otherwise G contains two non-commuting \mathbb{G}_a -subgroups U_1 and U_2 which act on X with different general orbits. The subgroup $U = \langle U_1, U_2 \rangle$ of G acts on X with an open orbit. If U is unipotent then by Proposition 5.1 $X \cong \mathbb{A}_{\mathbb{C}}^2$ contrary to our assumption. Therefore, by Theorem 0.2 U contains a nonabelian free subgroup. \square

6. NOTE ADDED IN PROOF

We provide here an alternative (though less elementary) proof of Theorem 4.1 which runs over more general fields.

Theorem 6.1. *Fix an algebraically closed field \mathbf{k} of characteristic zero. Assume that the Tits alternative for finitely generated subgroups holds in $\text{Bir}(\mathbb{P}_{\mathbf{k}}^2)$. Let X be an affine surface over*

\mathbf{k} and G be a subgroup of $\text{Aut}(X)$ generated by a finite collection of \mathbb{G}_a -subgroups U_1, \dots, U_k . If G contains no nonabelian free subgroup then G is a metabelian unipotent affine algebraic group.

In the proof of Theorem 6.1 we use the following recent result, see [KZ22, Theorem B].

Theorem 6.2. *Let X be an affine algebraic variety defined over an algebraically closed field \mathbf{k} of characteristic zero. Let $G \subset \text{Aut}(X)$ be a subgroup generated by a finite family U_1, \dots, U_n of connected algebraic subgroups. If G is solvable, then G is an algebraic group. If in addition the U_i are unipotent, then G is a unipotent algebraic group.*

Proof of Theorem 6.1. If X is irrational then by Lemma 3.6 G is an abelian unipotent algebraic group. Suppose further that X is rational and that G contains no nonabelian free subgroup. For any nonunit element $u_i \in U_i$ the subgroup $\langle u_i \rangle$ is Zariski dense in U_i . Hence the finitely generated subgroup $H = \langle u_1, \dots, u_n \rangle$ of G is dense in G for the Zariski ind-topology on $\text{Aut}(X)$, see [FK18, Definition 1.1.3 and Theorem 5.1.1]. Since H contains no nonabelian free subgroup, by our assumption it is virtually solvable. Let $H_0 \subset H$ be a solvable subgroup of finite index m . Then $u_i^{m_i} \in H_0$ for some $m_i \in \{1, \dots, m\}$. It follows that $H_0 \cap U_i$ is Zariski dense in U_i , and so H_0 is Zariski dense in G . We conclude that $G = \overline{H_0}$ is solvable.

Notice that X and the U_i are defined over a subfield $\mathbf{k}_0 \subset \mathbf{k}$ which is a finite extension of \mathbb{Q} . Fix an embedding $\mathbf{k}_0 \hookrightarrow \mathbb{C}$ and consider the complex models $X_{\mathbb{C}}$ of X and $U_i(\mathbb{C}) \cong \mathbb{G}_a(\mathbb{C})$ of the U_i where $U_i(\mathbb{C})$ acts regularly on $X_{\mathbb{C}}$. Consider also the subgroup $G(\mathbb{C}) = \langle U_1(\mathbb{C}), \dots, U_n(\mathbb{C}) \rangle$ of $\text{Aut}(X_{\mathbb{C}})$. By Theorem 6.2 G is a unipotent algebraic group. Hence $G(\mathbb{C})$ is a unipotent algebraic group acting on $X_{\mathbb{C}}$. Since $G(\mathbb{C}) \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is a nilpotent subgroup, by Theorem 1.11 $G(\mathbb{C})$ is a metabelian connected algebraic group. Then $G \subset G(\mathbb{C})$ is also metabelian. \square

The same arguments prove the following multidimensional analog of Theorem 6.1.

Theorem 6.3. *Let X be an affine algebraic variety defined over an algebraically closed field \mathbf{k} of characteristic zero, and let a subgroup $H \subset \text{Aut}(X)$ verify the Tits alternative for finitely generated subgroups. Let $G \subset H$ be a subgroup generated by a finite family U_1, \dots, U_n of unipotent algebraic subgroups. If G contains no nonabelian free subgroup then G is a unipotent algebraic group.*

Using this theorem and [LP22, Main theorem] resp. [BFL14, Theorem C] we come to the following

Corollary 6.4. *The conclusion of Theorem 6.3 holds for*

- $X = \mathbb{A}_{\mathbf{k}}^3$ and $H = \text{Tame}(\mathbb{A}_{\mathbf{k}}^3)$ being the tame subgroup of $\text{Aut}(X)$;
- $X = \text{SL}(2, \mathbb{C})$ viewed as an affine threefold and $H = \text{Tame}(\text{SL}(2, \mathbb{C}))$ being the tame subgroup of $\text{Aut}(X)$.

Remark 6.5. According to [Lam22, Theorem 16.2] the group $\text{Bir}(\mathbb{P}_k^2)$ satisfies the Tits alternative for any field k of characteristic zero. In particular, the conclusion of Theorem 6.1 holds over any algebraically closed field of characteristic zero. Besides, this gives a hope that also Theorem 1.12 and Proposition 3.1 hold in this generality.

The positive characteristic case of Conjecture 0.3 was added by a suggestion of the anonymous referee. A later discussion with Stéphane Lamy gives hope that the algebraic closedness assumption of the base field is also superfluous. This is supported by the fact that the Tits

alternative for finitely generated subgroups holds for $\text{Aut}(\mathbb{A}_k^2)$ over any prime characteristic field k , see [Lam22, Proposition 16.16]. Moreover, an analog of Theorem 1.5 also holds for $\text{Aut}(\mathbb{A}_k^2)$ over any field k according to [Lam22, Proposition 7.49].

Theorem 16.25 in [Lam22] claims that the Tits alternative for finitely generated subgroups holds in the Cremona group $\text{Bir}(\mathbb{P}_k^2)$ over a prime characteristic field k . The proof of this theorem in [Lam22] is under construction, with some arguments just sketched. See also the announcement [Can18, Theorem 6.1] with a reference to [Can11]; however, [Can11] deals exceptionally with complex surfaces and uses transcendental tools. Notice that there exists a classification of maximal algebraic subgroups of $\text{Bir}(\mathbb{P}^2)$ over a perfect field; see [SZ21].

Our proof of Theorem 4.1 uses the Hironaka resolution of singularities. It is known to hold for excellent schemes of dimension 2 defined over a prime characteristic field, see e.g. [Lip78]. Besides, we exploit the expression of a \mathbb{G}_a -subgroup via a locally nilpotent derivation, see e.g. Example 3.7 and the proof of Proposition 5.1. Such an expression is not available in positive characteristic.

REFERENCES

- [AFKKZ13] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg. *Flexible varieties and automorphism groups*. Duke Math. J. 162 (2013), 767–823.
- [AKZ19] I. Arzhantsev, K. Kuyumzhiyan and M. Zaidenberg. *Infinite transitivity, finite generation, and Demazure roots*. Adv. Math. 351 (2019), 1–32.
- [AS20] I. Arzhantsev and K. Shakhmatov. *Some finiteness results on triangular automorphisms*. Results Math. 77:2 (2022), article 75.
- [AZ21] I. Arzhantsev and M. Zaidenberg. *Tits-type alternative for groups acting on toric affine varieties*. Int. Math. Res. Not. IMRN 2022 (2022), no. 11, 8162–8195.
- [Bas76] H. Bass. *Some remarks on group actions on trees*. Comm. Algebra 4 (1976), 1091–1126.
- [BFL14] C. Bisi, J.-Ph. Furter, and S. Lamy. *The tame automorphism group of an affine quadric threefold acting on a square complex*. J. Ec. polytech. Math. 1 (2014), 161–223.
- [Bla17] J. Blanc. *Algebraic structures of groups of birational transformations*. Proc. Sympos. Pure Math. 94 (2017), 17–30.
- [BF13] J. Blanc and J.-P. Furter. *Topologies and structures of the Cremona groups*. Ann. of Math. (2), 178 (2013), 1173–1198.
- [Bri22] M. Brion. *On models of algebraic group actions*. arXiv:2202.04352 (2022).
- [Can11] S. Cantat. *Sur les groupes de transformations birationnelles des surfaces*. Ann. of Math. (2), 174 (2011), 299–340.
- [Can18] S. Cantat. *The Cremona group*. Algebraic geometry: Salt Lake City 2015. 2015 summer research institute in algebraic geometry, University of Utah, Salt Lake City, UT, USA, July 13–31, 2015. Proceedings. Part 1. T. de Fernex et al. (eds.), 101–142. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute. Proc. Symp. Pure Math. 97:1, (2018).
- [Cor13] Y. Cornulier. *Nonlinearity of some subgroups of the planar Cremona group*. Unpublished manuscript, 2013. Available at: <http://www.normalesup.org/~cornulier/crelin.pdf>.
- [Dem70] M. Demazure. *Sous-groupes algébriques de rang maximum du groupe de Cremona*. Ann. Sci. Ecole Norm. Sup. (4) 3 (1970), 507–588.
- [Des07] J. Déserti. *Sur les sous-groupes nilpotents du groupe de Cremona*. Bull. Braz. Math. Soc. (N.S.) 38(3) (2007), 377–388.
- [Des15] J. Déserti. *On solvable subgroups of the Cremona group*. Illinois J. Math. 59 (2) (2015) 345–358.
- [Des21] J. Déserti. *The Cremona group and its subgroups*. Math. Surveys Monogr. 252. Amer. Math. Soc., Providence, Rhode Island, 2021.
- [DF91] J. S. Deveney and D. R. Finston. *Rationally triangulable automorphisms*. J. Pure Appl. Algebra 72 (1991), 1–4.

- [FKZ11] H. Flenner, S. Kaliman and M. Zaidenberg. *Smooth affine surfaces with non-unique \mathbb{C}^* -actions*. J. Algebraic Geom. 20 (2011), 329–398.
- [FH21] J.-Ph. Furter and I. Hedén. *Borel subgroups of the plane Cremona group*. arXiv:2107.14353 (2021).
- [FK18] J.-Ph. Furter and H. Kraft. *On the geometry of automorphism groups of affine varieties*. arXiv:1809.04175 (2018), 179 pp.
- [FP18] J.-Ph. Furter and P.-M. Poloni. *On the maximality of the triangular subgroup*. Ann. Inst. Fourier 68 (2018), 393–421.
- [Giz71] M. H. Gizatullin. *Quasihomogeneous affine surfaces*. Mathematics of the USSR-Izvestiya 5:5 (1971), 1057–1081.
- [Gro67] A. Grothendieck. *Étude locale des schémas et des morphismes de schémas. EGA IV*. Publ. math. I.H.É.S. 32 (1967), 5–361.
- [Hum75] J. E. Humphreys. *Linear Algebraic Groups*. Grad. Texts in Math. 21. Springer-Verlag, New York Heidelberg Berlin 1975.
- [Jun42] H. W. E. Jung. *Über ganze birationale Transformationen der Ebene*. J. Reine Angew. Math. 184 (1942), 161–174.
- [Kam75] T. Kambayashi. *On the absence of nontrivial separable forms of the affine plane*. J. Algebra 35 (1975), 449–456.
- [Kam79] T. Kambayashi. *Automorphism group of a polynomial ring and algebraic group action on an affine space*. J. Algebra 60 (1979), 439–451.
- [Kol07] J. Kollár. *Lectures on resolution of singularities*. Ann. of Math. Stud. Princeton University Press. Princeton and Oxford, 2007.
- [KPZ17] S. Kovalenko, A. Perepechko and M. Zaidenberg. *On automorphism groups of affine surfaces*. In: Algebraic Varieties and Automorphism Groups, 207–286. Adv. Stud. Pure Math. 75. Mathematical Society of Japan, 2017.
- [Kra18] H. Kraft. *Regularization of rational group actions*. arXiv:1808.08729 (2018).
- [KZ22] H. Kraft and M. Zaidenberg. *Algebraically generated groups and their Lie algebras*. arXiv:2203.11356 (2022).
- [Lam01] S. Lamy. *L’alternative de Tits pour $\text{Aut}(\mathbb{C}^2)$* . J. Algebra 239 (2001), 413–437.
- [Lam22] S. Lamy. *The Cremona group*. Preliminary version, August 09, 2022, available on: <https://www.math.univ-toulouse.fr/~slamy/blog/cremona.html>
- [LP22] S. Lamy and P. Przytycki. *Tits alternative for the 3-dimensional tame automorphism group*. arXiv:2206.05611 (2022).
- [Lip78] J. Lipman. *Desingularization of two-dimensional schemes*. Ann. of Math. (2) 107 (1978), 151–207.
- [Mats63] H. Matsumura. *On algebraic groups of birational transformations*. Lincei. Rend. Sc. fis. mat. e nat. 34 (1963).
- [Mat58] T. Matsusaka. *Polarized varieties, fields of moduli and generalized Kummer varieties of polarized abelian varieties*. Amer. J. Math. 80 (1958), 45–82.
- [Mol67] D. I. Moldavanskii. *Certain subgroups of groups with a single defining relation*. Sibirsk. Mat. Zh. 8 (1967), 1370–1384. English translation in: Soviet Math. J. 8 (1967), 1039–1048.
- [Nag72] M. Nagata. *On automorphism group of $\mathbf{k}[x, y]$* . Lectures in Mathematics. Kyoto Univ., No. 5 (1972), Kinokuniya Book-Store Co., Tokyo, 1972.
- [Pop17] V. L. Popov. *Bass’ triangulability problem*. In: Algebraic Varieties and Automorphism Groups, 425–441. Adv. Stud. Pure Math. 75. Mathematical Society of Japan, 2017.
- [Ren68] R. Rentschler. *Opérations du groupe additif sur le plane affine*. C. R. Math. Acad. Sci. Paris 267 (1968), 384–387.
- [Ros56] M. Rosenlicht. *Some basic theorems on algebraic groups*. Amer. J. Math. 78 (1956), 401–443.
- [SZ21] J. Schneider and S. Zimmermann. *Algebraic subgroups of the plane Cremona group over a perfect field*. Épijournal Géom. Algébrique 5 (2021), <https://doi.org/10.46298/epiga.2021.6715>
- [Ser80] J.-P. Serre. *Trees*. Springer-Verlag, Berlin Heidelberg New York 1980.
- [Ser10] J.-P. Serre. *Le groupe de Cremona et ses sous-groupes finis*. Séminaire Bourbaki 2008/2009. Astérisque 332 (2010), Exp. No. 1000, vii, 75–100.
- [Sum74] H. Sumihiro. *Equivariant completion*. J. Math. Kyoto Univ. 14 (1974), 1–28.

- [Tit72] J. Tits. *Free subgroups in linear groups*. J. Algebra 20 (1972), 250–270.
- [Ure21] Ch. Urech. *Subgroups of elliptic elements of the Cremona group*. J. Reine Angew. Math. 770 (2021), 27–57.
- [vdK53] W. van der Kulk. *On polynomial rings in two variables*. Nieuw Arch. Wiskd. (5) 1 (1953), 33–41.
- [Wei55] A. Weil. *On algebraic groups of transformations*. Amer. J. Math. 77 (1955), 355–391.
- [Wri75] D. Wright. *Algebras which resemble symmetric algebra*. Ph. D. thesis, Columbia University, 1975.
- [Wri78] D. Wright. *The amalgamated free product structure of $\mathrm{GL}_2(\mathbf{k}[x_1, \dots, x_n])$ and the weak Jacobian theorem for two variables*. J. Pure Appl. Algebra 12 (1978), 235–251.
- [Wri79] D. Wright. *Abelian subgroups of $\mathrm{Aut}(\mathbf{k}[x, y])$ and applications to actions on the affine plane*. Illinois J. Math. 23 (1979), 579–635.
- [Zai95] D. Zaitsev. *Regularization of birational group operations in the sense of Weil*. J. Lie Theory 5 (1995), 207–224.

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, FACULTY OF COMPUTER SCIENCE,
 POKROVSKY BOULEVARD 11, MOSCOW, 109028 RUSSIA
Email address: arjantsev@hse.ru

INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ GRENOBLE ALPES,
 CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE
Email address: Mikhail.Zaidenberg@univ-grenoble-alpes.fr