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P. Degond[(1)], A. Diez[(2)], A. Frouvelle[(3)]

[(1)] Institut de Mathématiques de Toulouse ; UMR5219
Université de Toulouse ; CNRS
UPS, F-31062 Toulouse Cedex 9, France
email: pierre.degond@math.univ-toulouse.fr

[(2)] Department of Mathematics, Imperial College London, South Kensington Campus
London, SW7 2AZ, United Kingdom
email: antoine.diez18@imperial.ac.uk

[(3)] CEREMADE, CNRS, Université Paris-Dauphine
Université PSL, 75016 Paris, France
email: frouvelle@ceremade.dauphine.fr
and
CNRS, Université de Poitiers, UMR 7348
Laboratoire de Mathématiques et Applications (LMA), 86000 Poitiers, France
email: amic.frouvelle@math.univ-poitiers.fr

Abstract

We consider a system of self-propelled agents interacting through body attitude coordination in arbitrary dimension $n \geq 3$. We derive the formal kinetic and hydrodynamic limits for this model. Previous literature was restricted to dimension $n = 3$ only and relied on parametrizations of the rotation group that are only valid in dimension 3. To extend the result to arbitrary dimensions $n \geq 3$, we develop a different strategy based on Lie group representations and the Weyl integration formula. These results open the way to the study of the resulting hydrodynamic model (the "Self-Organized Hydrodynamics for Body orientation (SOHB)") in arbitrary dimensions.

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1 Introduction

Collective dynamics is an ubiquitous phenomenon in the living world at all scales (see e.g. locust swarms [6], wildebeest herds [61], fish schools [53], human crowds [65], bacterial colonies [7], embryogenic cell migration [44], semen [22] or collective motion of subcellular structures [56]). It produces large-scale structures (flocks, herds, coordinated motion) which extend far beyond the individual scale. A lot of effort has been put in understanding how local interactions between individuals can produce coordination over such large scales. Most approaches postulate some interaction rules at the individual level and assess them against experimental observations via large-scale computer simulations.

One of the first and most widely used such models is the so-called three-zone model proposed by I. Aoki [3], where individuals are assumed to attract each other at large distances, repel at short distance and align in the middle range. A simplified version of this model where individuals move at constant speed and are only subject to alignment has then been brought forward by T. Vicsek and his collaborators [64]. Independently, F. Cucker and S. Smale have developed a similar alignment model but with particles with variable speeds [23]. All these models have stimulated an intense literature which is impossible to cite exhaustively (see e.g. [15] for the three-zone model, [19, 21, 27, 28, 38, 41, 43, 46, 52, 63, 66] for the Vicsek model, and [1, 4, 5, 16, 48, 49, 50] for the Cucker-Smale model). Variants or combinations of these different models can be found in [8, 9, 10, 54]. Connections between the Cucker-Smale and the Vicsek models via singular limits has been established in [5, 12, 13].

In these models, agents control their motion via the velocity or velocity director. A question is whether new behavior could be observed if more complex control variables are introduced. Recently, a new model in which the agents control their motion by their body attitude has been proposed in dimension $n = 3$ [24, 26, 29, 30, 31, 40]. In this model, a reference frame describing the body attitude is attached to each agent. The agents move at constant speed in the direction of the first basis vector of the frame. The frame is updated to adjust to the "average" body attitude of the neighbors up to some noise. There are similarities with the Vicsek model except in the alignment process, which operates on the full body frame rather than on the single velocity vector. Consequently, heterogeneities in the body frame distribution affect motion and alter the dynamics compared with the Vicsek model [26]. So far, previous works were restricted to dimension $n = 3$. The present paper is the first investigation of this body attitude coordination model in arbitrary dimensions $n \geq 3$.

Traditionally, there are three levels of modelling for systems of interacting agents. The finer level of description is the "particle" level, by which each agent is followed in the course of time by means of ordinary differential equations or stochastic processes [3, 15, 19, 21, 23, 50, 53, 54, 64]. This is an appealing approach as the behavioral rules can be directed encoded in the equations. Due to the large number of particles, these models are not amenable to theoretical qualitative analyses. Thus, computer experiments are needed but computational efficiency drops dramatically when the number of particles becomes large.

Then, a coarser approach consists of looking at the system in some statistical sense. Specifically, one monitors the evolution of the probability distribution of a representa-

tive particle. This leads to the so-called "kinetic" models, which are partial differential equations posed on high-dimensional space consisting of all the positional and control variables of the particles. This approach is mostly developed for theoretical purposes [4, 8, 16, 24, 27, 38, 41, 43] although some numerical simulations have be attempted in low dimensional cases [46].

Finally the coarsest approach consists of describing the system by average or "hydrodynamic" quantities, such as the mean density, mean velocity, mean orientation, etc. In these hydrodynamic models, the agents control variables have been replaced by their local means which only depend on position. This results in considerable information savings and make the models amenable to both intensive numerical simulations and theoretical qualitative analysis [9, 10, 26, 63].

By going from particle to kinetic then hydrodynamic models, microscopic information is gradually lost. Therefore, the validity of the kinetic and hydrodyanmic models compared with the particle ones is a key question. In model cases, it can be proved that kinetic models are a valid approximation of particle models, when the number of particles tends to infinity and a property named "propagation of chaos" is satisfied (see e.g. [11, 14] for the Vicsek model, [48] for the Cucker-Smale model and [33] for a general framework encompassing the body orientation model). Then, hydrodynamic models can be formally and rigorously derived from kinetic ones when the typical interaction time between the particles is small compared to the duration of the experiment (see [27, 28, 32, 39, 52, 66] for the Vicsek model and [1, 5] for the Cucker-Smale model). References [17, 30, 44, 49] give an overview of the whole process from particles to hydrodynamic equations. More phenomenological approaches can be found in [8, 9, 63].

For the body orientation model, the passage from kinetic to hydrodynamic equations has been realized in [24, 26, 29, 30, 31, 40] **in dimension $n = 3$ only**. Specifically, in [29], the particle dynamics was described by a stochastic differential equation leading to a kinetic equation of Fokker-Planck type. In [31] the representation of three-dimensional rotations by unit quaternions was used. In [30] instead, the particle dynamics was a jump process described by a piecewise deterministic Markov process (PDMP) resulting in a Boltzmann-BGK type kinetic equation. In [24, 40] this kinetic equation was studied from the viewpoint of phase transitions. Numerical comparisons of the PDMP with the hydrodynamic models were performed in [26]. In the present paper, we aim to derive the macroscopic equations corresponding to the same PDMP and Boltzmann-BGK kinetic equation as in [26] but in arbitrary dimensions $n \geq 3$. The passage from dimension 3 to arbitrary dimensions is not a simple exercise. The reason is that all previous results obtained in dimension 3 used a parametrization of the rotation group by means of either the Rodrigues formula or (equivalently) the unit quaternions. This parametrization is specific to dimension $n = 3$ and does not extend to higher dimensions. Thus, it is necessary to use a completely different approach. In this paper, our main tools will be representation theory [42] and the so-called Weyl integration formula [62].

The search for a model in dimension $n > 3$ may appear as a pointless endeavour as collectively moving agents (birds, fish) live in a three-dimensional world. In spite of this, there are advantages in deriving the model in arbitrary dimensions. First, the use of concepts specific to dimension 3 (such as cross product, curl, etc.) may hide the underlying structures of the model. By contrast, writing the model in arbitrary dimension

may help decipher such structures (see [18] for an elaboration of this argument). Finally, these models could potentially apply to more abstract entities, such as data. Collective dynamics models are increasingly used in optimization and data science [20, 55]. The present body attitude coordination model or a variant of it could be used as a building block for treating data structured in rotation matrices.

Previous models involving agents' body attitudes have been proposed to model bird flocks [51]. However, there, the agents' interactions are similar to [3] and the body attitude itself is used to incorporate elements of flight physics in an individual agent's trajectory. The interplay between geometry and collective dynamics has recently appeared in the literature. In [2, 36, 37, 58, 60], collective dynamics models are considered where the particles are located on generic manifolds. Restrictions to specific manifolds are considered. For instance, the Lie group of unitary matrices is considered in [45], more general Lie groups in [47, 57] and the rotation group in 3 dimensions in [35, 59]. However, the target is fairly different from ours. In these cases, the particles move in this manifold and are subjected to generalizations of the Cucker-Smale alignment dynamics (or extensions of the Kuramoto synchronization dynamics). In the present work, particles move in classical linear space but their control variables are changed into objects in a more complex geometric structure.

The outline of this paper is as follows. Section 2 is devoted to the description of the modelling context, namely, the underlying particle model, the derivation of the kinetic model and its hydrodynamic scaling. The main result, namely, the formal convergence of the kinetic model towards the hydrodynamic model, is stated and discussed in Section 3. The proof of the main result occupies Section 4. A conclusion and perspectives are drawn in Section 5. The proofs of several lemmas relying on representation theory are given in appendices. A summary of the results coming from representation theory used in these proofs is given at the beginning of the appendix.

2 Modelling context

2.1 The particle system

Our starting point is the following particle system, which is a generalization of the three-dimensional model presented in [26, 30] to arbitrary dimensions $n \geq 3$. We assume the system composed of N particles occupying spatial locations $X_k(t) \in \mathbb{R}^n$, $k \in \{1, \dots, N\}$ in the course of time $t \geq 0$. Each particle is endowed with a moving direct orthonormal frame $(\omega_1^k(t), \dots, \omega_n^k(t))$ which encodes its body attitude, or local body frame. Equivalently, letting $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$, we denote by $A_k(t)$ the unique rotation in $\mathrm{SO}_n\mathbb{R}$ which maps $(e_1, \dots, e_n)$ to $(\omega_1^k(t), \dots, \omega_n^k(t))$. The dynamics of the particles is a jump process. Between two jumps, the particles move in straight line at uniform speed c_0 (supposed identical for all the particles) in the direction of the first basis vector $\omega_1^k(t) = A_k(t)e_1$ of the local body frame (also referred to as the self-propulsion direction), while the body frame $A_k(t)$ remains unchanged.

Jump times are exponentially distributed: they form an increasing random sequence $T_k^1, T_k^2, \dots$ such that $T_k^{j+1} - T_k^j$ follows a Poisson law with constant parameter $\nu > 0$. At jump times T_k^j, X_k is continuous but A_k experiences a jump between $A_k(T_k^j - 0)$ and

$A_k(T_k^j + 0)$. To define how A_k jumps, we first need to define the neighbor's average frame. Omitting the time variable for simplicity, we first introduce the following average

$$J_k = \frac{1}{N} \sum_{\ell=1}^{N} K(X_k - X_\ell)\, A_\ell, \tag{2.1}$$

where the function K (the sensing function): $\mathbb{R}^n \to [0, \infty)$ is given once for all. For simplicity, we will assume that K is a radial function:

$$K(x) = \frac{1}{R^n} \tilde{K}(\frac{|x|}{R}), \tag{2.2}$$

(where $|x|$ is the euclidean norm of x) for a suitable function $\tilde{K}$: $[0, \infty) \to [0, \infty)$ and a spatial scale parameter $R > 0$ referred to as the sensing radius. A typical example of a function $\tilde{K}$ is the indicator of the interval $[0, 1]$. Since J_k is not a rotation matrix in general, we project it onto $\mathrm{SO}_n\mathbb{R}$ using the formula

$$\Theta_k := \arg\max_{A \in \mathrm{SO}_n\mathbb{R}} A \cdot J_k, \tag{2.3}$$

where, for two matrices A, B in $\mathcal{M}_n(\mathbb{R})$, the space of $n \times n$ matrices with real entries,

$$A \cdot B = \mathrm{Tr}\{A^T B\} = \sum_{i,j=1}^{n} A_{ij} B_{ij}, \tag{2.4}$$

is the standard inner product on $\mathcal{M}_n(\mathbb{R})$, Tr stands for the trace and the exponent T for the matrix transpose. Eq. (2.3) defines Θ_k as the element A that maximizes $A \cdot J_k$ over $\mathrm{SO}_n\mathbb{R}$. Indeed, this element is unique if and only if J_k is not singular (i.e. $\det J_k \neq 0$ where det stands for the determinant), which we will suppose from now on. The set of singular matrices being of zero-measure, it is indeed legitimate to expect that such a singularity will not happen except for a negligible set of initial conditions. Moreover, if $\det J_k > 0$, Θ_k coincides with the unique rotation provided by the polar decomposition of J_k. The rotation Θ_k is what we define as the neighbor's average frame. Now, we can define how $A_k(T_k^j - 0)$ jumps to $A_k(T_k^j + 0)$. We first define $J_k(T_k^j - 0)$ by (2.1) with X_k and X_ℓ taking the values $X_k(T_k^j - 0)$ and $X_\ell(T_k^j - 0)$ (note that in general, for $\ell \neq k$, X_ℓ will be continuous at T_k^j). We deduce $\Theta_k(T_k^j - 0)$ from (2.3). Then $A_k(T_k^j + 0)$ is sampled according to the probability distribution $M_{\Theta_k(T_k^j - 0)}$ on $\mathrm{SO}_n\mathbb{R}$, where, for any $\Theta \in \mathrm{SO}_n\mathbb{R}$, the function M_Θ: $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$ is the von-Mises probability density:

$$M_\Theta(A) = \frac{1}{Z} \exp(\kappa\, \Theta \cdot A), \quad Z = \int_{\mathrm{SO}_n\mathbb{R}} \exp\left(\kappa\, \mathrm{Tr} A\right) dA. \tag{2.5}$$

Here, dA stands for the normalized Haar measure on $\mathrm{SO}_n\mathbb{R}$ (we recall that, for $\mathrm{SO}_n\mathbb{R}$ as for any compact group, the Haar measure is invariant by both left and right translations, by conjugation and by group inversion). The parameter κ is the concentration parameter of the von Mises and plays the role of the inverse of the temperature. It is supposed given and constant. The rotation Θ is called the orientation parameter of the von Mises.

In summary, the particle dynamics is a Piecewise Deterministic Markov Process (PDMP) described as follows.

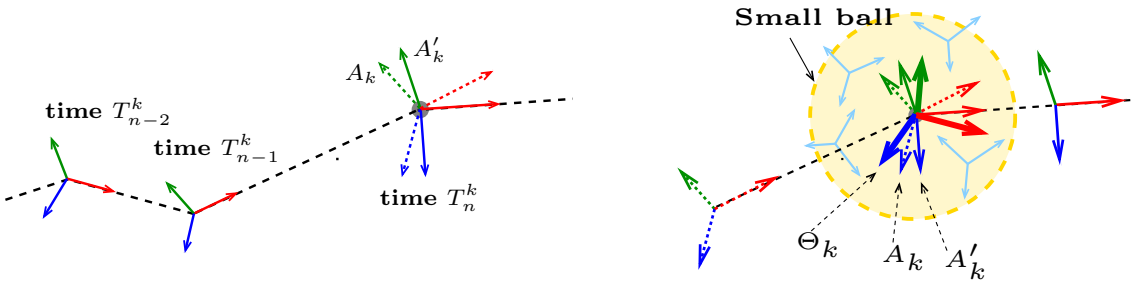
- Between jump times i.e. for all $t \in [T_k^j, T_k^{j+1})$, we have

$$X_k(t) = X_k(T_k^j) + c_0\,(t - T_k^j)\,A_k(t)\,e_1, \quad A_k(t) = A_k(T_k^j + 0). \tag{2.6}$$

- At T_k^j, X_k is continuous and A_k jumps from $A_k(T_k^j - 0)$ to $A_k(T_k^j + 0)$, where $A_k(T_k^j + 0)$ is drawn from a von Mises distribution, i.e.

$$A_k(T_k^j + 0) \sim M_{\Theta_k(T_k^j - 0)}. \tag{2.7}$$

We refer to Fig. 1 for a graphical depiction of the particle dynamics. Ref. [26] provides a link to numerical simulations of this model in dimension $n = 3$ made on graphics processor units (GPU) using the SiSyPHE package [34].



(a) Particle jump process

(b) Detail of a jump

Figure 1: (a) Graphical representation of the particle jump process: between the jump times T_n^k, the body frame (represented by red, green and blue arrows) remains constant and the particle trajectory (the dashed black line) is a straight line directed by the first basis vector of the body frame (in red). At jump times, the body frame experiences a sudden change while the trajectory line is broken and a different straight line begins. (b) Detail of a jump: the body frame jumps from its value A_k (depicted by thin dashed arrows) to a new value A'_k (depicted by thin solid arrows) sampled from a von Mises distribution whose orientation parameter is the frame Θ_k (depicted by thick arrows). The orientation parameter is obtained by averaging the body frame of the neighboring particles (in blue), i.e. those contained in a small ball (in yellow) centered at the jump location.

Remark 2.1. *The normalization constant (aka the partition function) Z in* (2.5) *should be defined by*

$$Z = \int_{\mathrm{SO}_n\mathbb{R}} \exp\left(\kappa \Theta \cdot A\right) dA.$$

However, we can use the change of variables $A' = \Theta^T A$ and the left invariance of the Haar measure to derive the expression of Z in (2.5)*, which incidentally shows that the normalization constant of M_Θ does not depend on Θ.*

2.2 The kinetic equation

A kinetic equation can formally be derived in the limit $N \to \infty$. To justify this, let us temporarily modify the particle model by changing (2.7) into

$$A_k(T_k^j + 0) \sim M_{J_k(T_k^j - 0)}. \tag{2.8}$$

Indeed, we note from (2.5) that the von Mises distribution is well-defined for any matrix Θ and not just for $\Theta \in \mathrm{SO}_n\mathbb{R}$. By this modification, we skip the normalization step and we do not need to assume that J_k is non-singular. In this case, it has been shown in [33] that, under some hypotheses on the initial conditions, the particle system is well-defined for all time and that the random measure

$$\mu^N(x, A, t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{(X_k(t), A_k(t))}(x, A),$$

where $\delta_{(X_k(t),A_k(t))}(x, A)$ stands for the Dirac delta in $\mathbb{R}^N \times \mathrm{SO}_n\mathbb{R}$ located at $(X_k(t), A_k(t))$, converges as $N \to \infty$ in weak sense to a deterministic absolutely continuous measure $f(x, A, t)$ which satisfies a kinetic equation.

Although the extension of this result to the present case is still an open (and likely difficult) problem, we will assume that this result is still true. In this case, the equation satisfied by f is given by the following kinetic model

$$\partial_t f + (c_0 A e_1 \cdot \nabla_x) f = \nu \left(\tilde{\rho}_f M_{\tilde{\Theta}_f} - f \right). \tag{2.9}$$

We remind that Ae_1 is the first column vector of the matrix A. For a given distribution function $f = f(x, A)$, the function $\tilde{\rho}_f \colon \mathbb{R}^n \to [0, \infty)$ is the non-local density

$$\tilde{\rho}_f(x) = \int_{\mathbb{R}^n \times \mathrm{SO}_n\mathbb{R}} K(x - y)\, f(y, A)\, dy\, dA.$$

The function $\tilde{\Theta}_f \colon \mathbb{R}^n \to \mathrm{SO}_n\mathbb{R}$ is defined from (2.3) where J_k is replaced by $\tilde{J}_f$ given by

$$\tilde{J}_f(x) = \int_{\mathbb{R}^n \times \mathrm{SO}_n\mathbb{R}} K(x - y)\, f(y, A)\, A\, dy\, dA.$$

Again, if $\tilde{J}_f$ has non-zero determinant, which we will suppose throughout this paper, $\tilde{\Theta}_f$ is uniquely defined and if $\tilde{J}_f$ has positive determinant, $\tilde{\Theta}_f$ coincides with the orthogonal matrix involved in the polar decomposition of $\tilde{J}_f$. Of course, if f also depends on t, $\tilde{\rho}_f$ and $\tilde{\Theta}_f$ also do. Note that, in $Ae_1 \cdot \nabla_x$, the dot simply means the inner product of two vectors of $\mathbb{R}^n$, namely Ae_1 and the differential symbol ∇_x. The same notation will be used for inner products of vectors and matrices as the context will always be clear.

Remark 2.2. *If we had used the modification* (2.8)*, $\tilde{\Theta}_f$ in* (2.9) *would have been replaced by $\tilde{J}_f$. In that case, the result of [33] would apply and we would have the rigorous convergence of μ^N to f as $N \to \infty$. However, the resulting macroscopic model would be more complicated. This is why we choose to work with the normalized model in spite of the absence of rigorous justification of the kinetic model.*

2.3 Non-dimensionalization and scaling

We now define a time scale t_0 and the associated space scale $x_0 = c_0 t_0$. We introduce two dimensionless parameters

$$\bar{\nu} = \nu t_0, \qquad \bar{R} = \frac{R}{x_0},$$

and define dimensionless variables $x' = x/x_0$, $t' = t/t_0$, and functions $f'(x', A, t') = x_0^n f(x, A, t)$, $\tilde{\rho}'_{f'} = x_0^n \tilde{\rho}_f(x,t)$, $\tilde{J}'_{f'} = x_0^n \tilde{J}_f(x,t)$. In these new variables, Eq. (2.9) is written (dropping the primes for simplicity):

$$\partial_t f + (Ae_1 \cdot \nabla_x) f = \bar{\nu} \left(\tilde{\rho}_f M_{\tilde{\Theta}_f} - f \right). \tag{2.10}$$

with, using (2.2)

$$\begin{aligned} \tilde{\rho}_f(x) &= \int_{\mathbb{R}^n \times \mathrm{SO}_n\mathbb{R}} \frac{1}{\bar{R}^n} \tilde{K}\big(\frac{|x-y|}{\bar{R}}\big) f(y,A)\, dy\, dA, \\ \tilde{J}_f(x) &= \int_{\mathbb{R}^n \times \mathrm{SO}_n\mathbb{R}} \frac{1}{\bar{R}^n} \tilde{K}\big(\frac{|x-y|}{\bar{R}}\big) f(y,A)\, A\, dy\, dA. \end{aligned}$$

We now make the following hydrodynamic scaling assumptions:

$$\frac{1}{\bar{\nu}} = \bar{R} = \varepsilon \ll 1.$$

The parameter $\varepsilon > 0$ is at the same time a dimensionless measure of the relaxation speed of f towards the local equilibrium $\rho_f M_{\Theta_f}$ (analog of the Knusdsen number in rarefied gases) and the typical scale of the interaction radius. An easy Taylor expansion and the fact that the sensing kernel is rotationally symmetric show that

$$\tilde{\rho}_f = \rho_f + \mathcal{O}(\varepsilon^2), \quad \tilde{J}_f = J_f + \mathcal{O}(\varepsilon^2), \quad \tilde{\Theta}_f = \Theta_f + \mathcal{O}(\varepsilon^2),$$

with

$$\rho_f(x) = \int_{\mathrm{SO}_n\mathbb{R}} f(x,A)\, dA, \qquad J_f(x) = \int_{\mathrm{SO}_n\mathbb{R}} f(x,A)\, A\, dA, \tag{2.11}$$

and $\Theta_f(x)$ deduced from $J_f(x)$ by (2.3) (replacing J_k by $J_f(x)$). From (2.10), we get

$$\partial_t f + (Ae_1 \cdot \nabla_x) f = \frac{1}{\varepsilon} \left(\rho_f M_{\Theta_f} - f \right) + \mathcal{O}(\varepsilon),$$

and we drop the $\mathcal{O}(\varepsilon)$ remainder as it has no contribution to the result we are aiming at.

Denoting the unknown by $f^\varepsilon(x, A, t)$ to highlight its dependence on ε, we are finally led to the following perturbation problem

$$\partial_t f^\varepsilon + (Ae_1 \cdot \nabla_x) f^\varepsilon = \frac{1}{\varepsilon} \big(\rho_{f^\varepsilon} M_{\Theta_{f^\varepsilon}} - f^\varepsilon \big), \tag{2.12}$$

with ρ_f and J_f given by (2.11). In this paper, we address the problem of deriving the formal limit $\varepsilon \to 0$ of model (2.12)

3 Main result and discussion

We first introduce some notations. For two vectors $X = (X_i)_{i=1,\dots,n}$ and $Y = (Y_i)_{i=1,\dots,n}$, $X \wedge Y$ and $\nabla_x \wedge X$ denote the antisymmetric matrices:

$$(X \wedge Y)_{ij} = X_i Y_j - X_j Y_i, \quad (\nabla_x \wedge X)_{ij} = \partial_{x_i} X_j - \partial_{x_j} X_i,$$

(note that $X \wedge Y$ is the exterior product of X and Y and, if X is identified with a 1-form through the euclidean structure of $\mathbb{R}^n$, $\nabla_x \wedge X$ is the exterior derivative of X).

For $k \in \mathbb{Z}$ and $p \in \mathbb{N} \setminus \{0\}$, we define the functions $C_{2p}^{(k)}$ and $C_{2p+1}^{(k)}$: $\mathbb{R}^p \to \mathbb{R}$ by

$$C_{2p}^{(k)}(\theta_1, \dots, \theta_p) = 2(\cos k\theta_1 + \dots + \cos k\theta_p), \tag{3.1}$$

$$C_{2p+1}^{(k)}(\theta_1, \dots, \theta_p) = 2(\cos k\theta_1 + \dots + \cos k\theta_p) + 1. \tag{3.2}$$

We also define u_{2p} and u_{2p+1}: $\mathbb{R}^p \to \mathbb{R}$ by

$$u_{2p}(\theta_1, \dots, \theta_p) = \frac{2^{(p-1)^2}}{p!} \prod_{1 \le j < k \le p} \big(\cos\theta_j - \cos\theta_k\big)^2, \tag{3.3}$$

$$u_{2p+1}(\theta_1, \dots, \theta_p) = \frac{2^{p(p-1)}}{p!} \prod_{1 \le j < k \le p} \big(\cos\theta_j - \cos\theta_k\big)^2 \prod_{j=1}^{p} \big(1 - \cos\theta_j\big). \tag{3.4}$$

The goal of this paper is to prove the formal theorem:

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \ge 3$. We assume that (2.12) has a smooth solution f^ε which converges as $\varepsilon \to 0$ as smoothly as needed towards a smooth function f. Then,*

$$f = \rho M_\Theta, \tag{3.5}$$

where $\rho = \rho(x,t)$ and $\Theta = \Theta(x,t)$ are functions from $\mathbb{R}^n \times (0,\infty)$ to $(0,\infty)$ and $\mathrm{SO}_n\mathbb{R}$ respectively which are solutions of the following system:

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega_1) = 0, \tag{3.6}$$

$$\rho\big(\partial_t + c_2\, \Omega_1 \cdot \nabla_x\big)\Theta = \mathbb{W}\Theta, \tag{3.7}$$

where

$$\mathbb{W} = F \wedge \Omega_1 - c_4\, \rho\, \nabla_x \wedge \Omega_1, \tag{3.8}$$

$$F = -c_3\, \nabla_x \rho - c_4\, \rho\, r, \tag{3.9}$$

$$r = \sum_{k=1}^{n} (\nabla_x \cdot \Omega_k)\Omega_k, \tag{3.10}$$

with $\Omega_k = \Theta e_k$, $k = 1, \dots, n$. The constants c_i, $i = 1, \dots, 4$ are expressed as follows. Let

$p \in \mathbb{N}$ such that $n = 2p$ or $n = 2p+1$. We have

$$c_1 = \frac{1}{n} \frac{\displaystyle\int_{[0,2\pi]^p} C_n^{(1)} \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}{\displaystyle\int_{[0,2\pi]^p} \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}, \tag{3.11}$$

$$c_2 = \frac{\displaystyle\int_{[0,2\pi]^p} \left(2C_n^{(3)} - nC_n^{(1)} C_n^{(2)} + (n^2-2)C_n^{(1)}\right) \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}{n(n-2)(n+2) \displaystyle\int_{[0,2\pi]^p} \left(1 - \frac{1}{n} C_n^{(2)}\right) \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}, \tag{3.12}$$

$$c_3 = \frac{1}{2\kappa}, \tag{3.13}$$

$$c_4 = \frac{\displaystyle\int_{[0,2\pi]^p} \left(C_n^{(3)} - \frac{2}{n} C_n^{(1)} C_n^{(2)} + C_n^{(1)}\right) \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}{2(n-2)(n+2) \displaystyle\int_{[0,2\pi]^p} \left(1 - \frac{1}{n} C_n^{(2)}\right) \exp\left(\kappa C_n^{(1)}\right) u_n \, d\tilde{\theta}_p}, \tag{3.14}$$

with the notation $d\tilde{\theta}_p = d\theta_1 \dots d\theta_p$. Furthermore, The function c_1: $[0,\infty) \to \mathbb{R}$, $\kappa \mapsto c_1(\kappa)$ is non-decreasing and satisfies $c_1(0) = 0$ and $\lim_{\kappa\to\infty} c_1(\kappa) = 1$. Thus, it is a bijection from $[0,\infty)$ to $[0,1)$.

There is an alternate expression of (3.7) which requires the introduction of additional notations. Suppose A, B and C are three smooth vector fields $\mathbb{R}^n \to \mathbb{R}$. We define

$$\delta(A,B,C) = \big((A \cdot \nabla_x)B\big) \cdot C + \big((B \cdot \nabla_x)C\big) \cdot A + \big((C \cdot \nabla_x)A\big) \cdot B, \tag{3.15}$$

and

$$\Delta_{ijk} = \delta(\Omega_i, \Omega_j, \Omega_k). \tag{3.16}$$

It is easy to check that Δ_{ijk} is antisymmetric by permutations of the indices (i,j,k). Then, we define the following antisymmetric matrix field:

$$\mathbb{A} = \sum_{k,\ell=1}^{n} \Delta_{1k\ell} \, \Omega_k \otimes \Omega_\ell, \tag{3.17}$$

where $\otimes$ denotes the tensor product of two vectors. The matrix $\mathbb{A}$ is just the matrix with entries $\Delta_{1k\ell}$ in the basis $(\Omega_1, \dots, \Omega_n)$. We note that

$$\mathbb{A}\Omega_1 = \sum_{k,\ell=1}^{n} \Delta_{1k\ell} \, \Omega_k \, (\Omega_\ell \cdot \Omega_1) = \sum_{k=1}^{n} \Delta_{1k1} \, \Omega_k = 0, \tag{3.18}$$

by the antisymmetry of Δ_{ijk}. Finally, we define

$$\tilde{\mathbb{W}} = F \wedge \Omega_1 + c_4 \rho \mathbb{A}. \tag{3.19}$$

Then, we have the following proposition, whose proof can be found in Appendix 10.

Proposition 3.2. *Eq.* (3.7) *is equivalent to*

$$\rho\big(\partial_t + (c_2 - c_4)\,\Omega_1 \cdot \nabla_x\big)\Theta = \tilde{\mathbb{W}}\Theta. \tag{3.20}$$

Eq. (3.20) can be expanded into equations for the basis vectors Ω_j. We have:

$$\rho\big(\partial_t + (c_2 - c_4)\,\Omega_1 \cdot \nabla_x\big)\Omega_1 = -c_3 P_{\Omega_1^\perp}\nabla_x\rho - c_4\rho\sum_{k\neq 1}(\nabla_x \cdot \Omega_k)\Omega_k, \tag{3.21}$$

$$\begin{aligned}\rho\big(\partial_t + (c_2 - c_4)\,\Omega_1 \cdot \nabla_x\big)\Omega_j = \big(c_3(\Omega_j \cdot \nabla_x)\rho + c_4\rho(\nabla_x \cdot \Omega_j)\big)\Omega_1 \\ -c_4\rho\sum_{k\notin\{1,j\}}\delta(\Omega_1,\Omega_j,\Omega_k)\Omega_k, \qquad j = 2,\dots,n,\end{aligned} \tag{3.22}$$

where $P_{\Omega_1^\perp} = \mathrm{Id} - \Omega_1 \otimes \Omega_1$ is the orthogonal projection onto $\{\Omega_1\}^\perp$.

Below and in Appendix 10, we show that, in dimension $n = 3$, this system coincides with the so-called "Self-Organized Hydrodynamics for Body-orientation (SOHB)" derived earlier [24, 29]. This model is a nonlinear nonconservative hyperbolic system [25] composed of two equations, a mass conservation (or continuity) equation (3.6) for the density ρ and an evolution equation (3.20) for the mean body orientation Θ.

The continuity equation (3.6) shows that the fluid velocity is $c_1\Omega_1$, i.e. the fluid flows in the direction of the first basis vector of the mean body orientation matrix Θ, at a speed equal to c_1. As $c_1 \in [0, 1]$, the fluid speed is slower than the particle speed. This is due to the dispersion of the particle velocities Ae_1 around the mean velocity Ω_1 and the fact that the norm of the average is less than the average of the norms. This speed reduction is all the greater than the noise intensity κ^{-1} is greater, as the increase of c_1 with κ shows (c_1 is an order parameter [24]). This type of continuity equation is classical in collective dynamics, and has been also found e.g. in the Vicsek model [32].

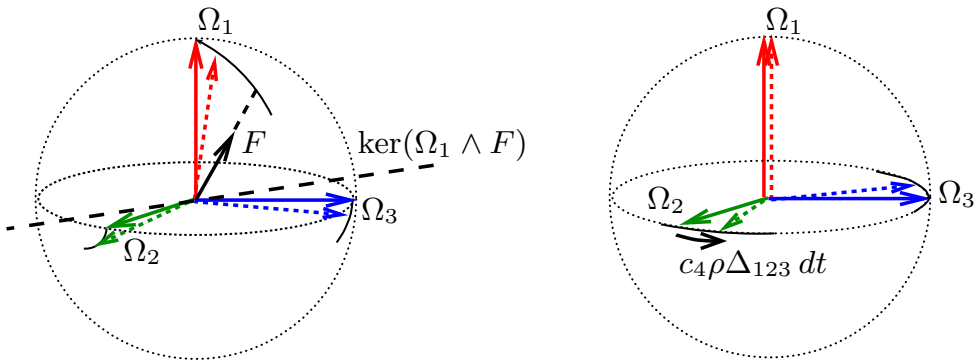
The left-hand side of (3.7) is a convective derivative along the vector field $(c_2 - c_4)\Omega_1$. As $c_2 - c_4 \neq c_1$, this convective derivative is not a material derivative, i.e. it is not the derivative of Θ when a fluid element is followed in its motion. Rather, one needs to move at speed $c_2 - c_4$ different from the fluid speed c_1 along the velocity direction Ω_1 to follow the evolution of Θ. Along this motion, Θ can be seen as a moving frame. So, its (convective) derivative, by classical rigid-body dynamics, is obtained by multiplying it on the left by an antisymmetric matrix, which is the matrix $\tilde{\mathbb{W}}$.

This matrix has two components. The first one is $F \wedge \Omega_1$. It describes an infinitesimal rotation in the plane $\mathrm{Span}\{F, \Omega_1\}$ which tends to align Ω_1 with F. In the orthogonal supplement $(\mathrm{Span}\{F, \Omega_1\})^\perp$ of the frame, this component of $\tilde{\mathbb{W}}$ does not produce any motion. By (3.9), the vector F itself has two components. The first one $-c_3\nabla_x\rho$ tends to turn the fluid velocity away from large density regions. It has the same effect as the pressure gradient in classical fluid dynamics. The second component of F is proportional to the vector r. Its effect is to deviate the fluid in the presence of spatial gradients of the mean body attitude, an effect which has no counterpart in classical fluid dynamics. Novel effects brought by this additional term have been investigated in 3D in [26].

The second component of $\tilde{\mathbb{W}}$ is proportional to $\mathbb{A}$. The important feature of $\mathbb{A}$ is (3.18) which says that the self-propulsion direction Ω_1 is not modified by $\mathbb{A}$. Hence $\mathbb{A}$ describes

an infinitesimal rotation of the frame about the self-propulsion direction Ω_1. Thus, (3.19) can be seen as the decomposition of the infinitesimal rotation $\tilde{W}$ into a component that fixes Ω_1 and is independent of the force F (that proportional to $\mathbb{A}$) and its complement (proportional to $F \wedge \Omega_1$) which embodies the effect of F. Again, the frame rotation about Ω_1 has no counterpart in classical fluid dynamics. Note that this decomposition has been made possible thanks to the choice of the "right" convection velocity $c_2 - c_4$. Expression (3.8) for $\mathbb{W}$ does not have such a simple interpretation, which indicates that the speed c_2 used in (3.7) is not the most natural choice for the convection speed of Θ. Also, note that a single coefficient c_4 controls the two effects of the gradients of body attitude (that proportional to $r \wedge \Omega_1$ and that proportional to $\mathbb{A}$), suggesting that these two effects have a common microscopic origin.

The effects of the two components of $\tilde{\mathbb{W}}$ are depicted in Fig. 2.



(a) Action of $F \wedge \Omega_1$ (b) Action of $\mathbb{A}$

Figure 2: The effects of the two components of $\tilde{\mathbb{W}}$ on the motion of the frame, represented in dimension $n = 3$ by red, green and blue arrows for Ω_1 (the self-propulsion direction), Ω_2 and Ω_3 respectively. The solid arrows represent the frame at a time t and the dashed arrows, at a later time $t + \delta t$ with $\delta t \ll 1$. (a) Action of $F \wedge \Omega_1$ (assuming $\mathbb{A} = 0$): the force F is the black arrow. The frame motion is depicted by the segment of circles joining the ends of the solid arrows and the dashed arrows. The frame rotates in the plane $\mathrm{Span}\{F, \Omega_1\}$ to align Ω_1 with F. The space $(\mathrm{Span}\{F, \Omega_1\})^\perp$ is the kernel of $F \wedge \Omega_1$ (the dashed line) so, this space is kept fixed in the frame motion. (b) Action of $\mathbb{A}$ (assuming $F = 0$): the frame rotates about Ω_1 which remains fixed. Vector Ω_2 moves in the plane $\mathrm{Span}\{\Omega_2, \Omega_3\}$ with angular speed $c_4\rho\Delta_{123}$ (and similarly for Ω_3). The displacement of the end of the arrow representing Ω_2 is depicted by the black arrow.

Let us now focus on the dimension $n = 3$ case and compare the results of the present paper with [26]. Indeed, the two papers feature the same individual-based model as a starting point, but for the restriction to dimension $n = 3$ in [26]. The notations of [26] are slightly modified to fit with the notations used in the present paper. First, we introduce some notations specific to the dimension $n = 3$. For a vector $w \in \mathbb{R}^3$, $[w]_\times$ denotes the antisymmetric matrix associated to the map $\mathbb{R}^3 \ni x \mapsto w \times x \in \mathbb{R}^3$. For two vector fields F and G, we have $F \wedge G = -[F \times G]_\times$, where $F \times G$ stands for the vector product of F

and G. Introducing $\tilde{\delta} =: \Delta_{123}$, we have from (3.17):

$$\mathbb{A} = \tilde{\delta}(\Omega_2 \otimes \Omega_3 - \Omega_3 \otimes \Omega_2) = -\tilde{\delta}[\Omega_1]_\times.$$

Thus, (3.19) leads to

$$\tilde{\mathbb{W}} = -[\tilde{w}]_\times, \qquad \tilde{w} = F \times \Omega_1 + \tilde{\delta}\Omega_1. \tag{3.23}$$

Then, we can check that Eqs. (3.6), (3.20) together with (3.23), (3.9) and (3.10) are identical with Eqs. (20)-(22) of [26] (note that in [26] the constant c_2 is what we call $c_2 - c_4$). Thus, our result and that of [26] are compatible, provided we show that the expressions (3.11)-(3.14) of the constants c_1 to c_4 in dimension $n = 3$ are the same as those found in [26]. This is indeed the case and the computation is given in Appendix 10.

4 Proof of Theorem 3.1

4.1 Limit $\varepsilon \to 0$

We first need the following Lemma whose proof is given in Appendix 7. It uses the following notation:

$$\langle f(A)\rangle_{g(A)} = \frac{\int_{\mathrm{SO}_n\mathbb{R}} f(A)g(A)\,dA}{\int_{\mathrm{SO}_n\mathbb{R}} g(A)\,dA},$$

where f and g are two functions $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$, $g \geq 0$ and $g \not\equiv 0$.

Lemma 4.1. *We have*

$$\int_{\mathrm{SO}_n\mathbb{R}} A\, M_\Theta(A)\, dA = c_1\, \Theta, \quad c_1 = \Big\langle \frac{\mathrm{Tr}A}{n} \Big\rangle_{\exp(\kappa \mathrm{Tr}A)}. \tag{4.1}$$

c_1 is a non-decreasing and satisfies $c_1(0) = 0$ and $\lim_{\kappa\to\infty} c_1(\kappa) = 1$.

Next, for f: $\mathrm{SO}_n\mathbb{R} \to [0,\infty)$ we define the collision operator

$$Q(f) = \rho_f M_{\Theta_f} - f,$$

The following lemma gives the equilibria of Q:

Lemma 4.2. *For functions f: $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$, we have*

$$Q(f) = 0 \iff \exists(\rho,\Theta) \in [0,\infty) \times \mathrm{SO}_n\mathbb{R} \text{ such that } f = \rho M_\Theta.$$

Proof. The left-to-right implication is clear. Conversely, let $f = \rho M_\Theta$. We show that $\rho_f = \rho$ and $\Theta_f = \Theta$. The first equality is clear since M_Θ is a probability density. Then, by Lemma 4.1, we have $J_f = \rho\, c_1 \Theta$ and since $\rho\, c_1 \geq 0$, by the uniqueness of the polar decomposition, Θ is the orthogonal factor in the polar decomposition of J_f and thus, we get $\Theta_f = \Theta$, which shows that $Q(f) = 0$. ∎

Proposition 4.3. *The functions ρ and Θ involved in* (3.5) *satisfy the following equations:*

$$\int_{\mathrm{SO}_n\mathbb{R}} (\partial_t + Ae_1 \cdot \nabla_x)(\rho M_\Theta)\, dA = 0, \tag{4.2}$$

$$\int_{\mathrm{SO}_n\mathbb{R}} (\partial_t + Ae_1 \cdot \nabla_x)(\rho M_\Theta)\, (A\Theta^T - \Theta A^T)\, dA = 0, \tag{4.3}$$

Proof. We clearly have

$$\int_{\mathrm{SO}_n\mathbb{R}} Q(f)\, dA = 0, \ \text{ for all functions } \ f. \tag{4.4}$$

Thus, integrating (2.12) with respect to A we get

$$\int_{\mathrm{SO}_n\mathbb{R}} (\partial_t + Ae_1 \cdot \nabla_x) f^\varepsilon\, dA = 0.$$

Letting $\varepsilon \to 0$ and using (3.5) leads to (4.2).

Then, we remark that

$$\int_{\mathrm{SO}_n\mathbb{R}} Q(f)\, (A\Theta_f^T - \Theta_f A^T)\, dA = 0, \ \text{ for all functions } \ f.$$

Indeed, by Lemma 4.1, we have

$$\int_{\mathrm{SO}_n\mathbb{R}} \rho_f M_{\Theta_f}\, (A\Theta_f^T - \Theta_f A^T)\, dA = c_1 \rho_f (\Theta_f \Theta_f^T - \Theta_f \Theta_f^T) = 0.$$

Besides, by definition of the polar decomposition, there exists a symmetric matrix S such that $J_f \Theta_f^T = S$. So,

$$\int_{\mathrm{SO}_n\mathbb{R}} f\, (A\Theta_f^T - \Theta_f A^T)\, dA = J_f \Theta_f^T - \Theta_f J_f^T = S - S^T = 0.$$

Thus, multiplying (2.12) by $(A\Theta_{f^\varepsilon}^T - \Theta_{f^\varepsilon} A^T)$ and integrating with respect to A we get

$$\int_{\mathrm{SO}_n\mathbb{R}} (\partial_t + Ae_1 \cdot \nabla_x) f^\varepsilon\, (A\Theta_{f^\varepsilon}^T - \Theta_{f^\varepsilon} A^T)\, dA = 0.$$

Taking the limit $\varepsilon \to 0$ yields (4.3). ∎

Remark 4.1. *In classical kinetic theory [17], a key concept is that of "collision invariant (CI)" which, in the present context, is a function $\psi(A)$ such that*

$$\int_{\mathrm{SO}_n\mathbb{R}} Q(f)\, \psi\, dA = 0, \quad \forall f.$$

It expresses that the quantity ψ is conserved during collisions. The derivation of (4.2) *examplifies how CI can be used to derive macroscopic equations. Indeed, Eq.* (4.4) *expresses that constant functions are CI. However, these are the only CI and we do not have enough conservation relations to determine ρ and Θ. In [32], a new "generalized collision invariant (GCI)" concept was introduced to overcome this problem. The quantity $(A\Theta^T - \Theta A^T)$ used in the derivation of* (4.3) *is precisely a GCI. To make this concept more precise, for any $\Theta \in \mathrm{SO}_n\mathbb{R}$, we introduce a linear collision operator*

$$\mathcal{Q}(f, \Theta) = \rho_f M_\Theta - f.$$

We note that $Q(f) = \mathcal{Q}(f, \Theta_f)$. Then, for any fixed $\Theta \in \mathrm{SO}_n\mathbb{R}$, we say that ψ_Θ is a GCI associated with Θ if and only if the following holds:

$$\int_{\mathrm{SO}_n\mathbb{R}} \mathcal{Q}(f, \Theta)\, \psi_\Theta \, dA = 0, \quad \forall f \text{ such that } J_f\Theta^T \text{ is symmetric},$$

and we immediately see that $(A\Theta^T - \Theta A^T)$ satisfies this requirement. Finally, we have

$$\int_{\mathrm{SO}_n\mathbb{R}} Q(f)\, \psi_{\Theta_f} \, dA = 0, \tag{4.5}$$

by the fact that $J_f\Theta_f^T$ is symmetric. So, multiplying the kinetic equation by $\psi_{\Theta_{f^\varepsilon}}$ and integrating with respect to A cancels the $1/\varepsilon$ singularity and allows to pass to the limit $\varepsilon \to 0$ like in the end of the proof of Prop. 4.3. Eq. (4.5) *shows that ψ_{Θ_f} satisfies a conservation relation like a classical CI, except that it depends on f (a classical CI is independent of f). Because of the very simple relaxation form of the collision operator, the determination of the GCI is explicit. In more complex cases, the GCI are not explicit. For instance, in the Fokker-Planck case, they require the inversion of an elliptic operator [29, 31].*

4.2 The fluid model

We are left to re-write Eqs. (4.2), (4.3) into a nice differential system for ρ and Θ. This is easy for (4.2) as shown by the

Lemma 4.4. *Eq.* (4.2) *is equivalent to the continuity equation* (3.6) *where c_1 is given by* (4.1).

Proof. Since Ae_1 does not depend on x, we can move it inside the space derivative in (4.2) and then, we can interchange the time and space derivatives with the integral over A. Using that M_Θ is a probability density and (4.1), we get (3.6). ∎

Before working out (4.3), we recall the following definitions and facts: Let $\mathcal{S}_n$ and $\mathcal{A}_n$ the sub-spaces of $\mathcal{M}_n$ consisting of symmetric and antisymmetric matrices. These two spaces form orthogonal complements with respect to the inner product (2.4):

$$\mathcal{A}_n \overset{\perp}{\oplus} \mathcal{S}_n = \mathcal{M}_n.$$

Let now $\Theta \in \mathrm{SO}_n\mathbb{R}$ and define the tangent manifold T_Θ to $\mathrm{SO}_n\mathbb{R}$ at Θ. We have

$$T_\Theta = \{P\Theta \mid P \in \mathcal{A}_n\}.$$

The orthogonal projection (with respect to the inner product (2.4)) of $A \in \mathcal{M}_n$ onto T_Θ is given by

$$P_{T_\Theta} A = \frac{A\Theta^T - \Theta A^T}{2}\,\Theta.$$

Likewise, the orthogonal complement $T_\Theta^\perp$ is given by

$$T_\Theta^\perp = \{S\Theta \mid S \in \mathcal{S}_n\},$$

and the orthogonal projection of A onto $T_\Theta^\perp$ is

$$P_{T_\Theta^\perp} A = \frac{A\Theta^T + \Theta A^T}{2}\,\Theta.$$

For a given $A \in \mathrm{SO}_n\mathbb{R}$, we will need the derivative of the function $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$, $\Theta \mapsto M_\Theta(A)$ at Θ. This is a linear map $d_\Theta M_\Theta(A)$ from T_Θ to $\mathbb{R}$ given for any $Q \in T_\Theta$ by

$$d_\Theta M_\Theta(A)(Q) = \kappa\, M_\Theta(A)\, A \cdot Q = \kappa\, M_\Theta(A)\, P_{T_\Theta} A \cdot Q, \tag{4.6}$$

We will finally need the following lemma:

Lemma 4.5. *Let F: $\mathcal{M}_n^2 \to \mathbb{R}$ odd with respect to the first argument. Let g : $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$ invariant by transposition ($g(A^T) = g(A)$, $\forall A \in \mathrm{SO}_n\mathbb{R}$). Then, for any $\Theta \in \mathrm{SO}_n\mathbb{R}$, we have*

$$\int_{\mathrm{SO}_n\mathbb{R}} F(P_{T_\Theta} A, P_{T_\Theta^\perp} A)\, g(A\Theta^T)\, dA = 0.$$

Proof. Changing variable to $A' = \Theta A^T \Theta$ in the integral changes the first argument of F in its opposite while the second argument is unchanged. The factor involving g is unchanged due to the invariance of g by transposition and the Haar measure is unchanged due to its invariance by left and right multiplication and by transposition. Since F is odd with respect to the first argument, the integral must be equal to zero. ∎

With these preliminaries, we remark that, after multiplication on the right by $\Theta/2$, (4.3) can be rewritten:

$$X := \int_{\mathrm{SO}_n\mathbb{R}} (\partial_t + Ae_1 \cdot \nabla_x)(\rho M_\Theta(A))\, P_{T_\Theta} A\, dA = 0. \tag{4.7}$$

Now, we have the

Lemma 4.6. *We have $X = X_1 + X_2$, where*

$$\begin{aligned} X_1 &= \int_{\mathrm{SO}_n\mathbb{R}} \big[(P_{T_\Theta} A)e_1 \cdot \nabla_x \rho + \kappa\rho P_{T_\Theta} A \cdot \partial_t \Theta\big]\, P_{T_\Theta} A\, M_\Theta(A)\, dA, \\ X_2 &= \kappa\rho \int_{\mathrm{SO}_n\mathbb{R}} P_{T_\Theta} A \cdot \big((P_{T_\Theta^\perp} A)e_1 \cdot \nabla_x\big)\Theta\, P_{T_\Theta} A\, M_\Theta(A)\, dA. \end{aligned}$$

Proof. Using (4.6), we get

$$(\partial_t + Ae_1 \cdot \nabla_x)(\rho M_\Theta) = M_\Theta\big[(\partial_t + Ae_1 \cdot \nabla_x)\rho + \kappa\rho P_{T_\Theta} A \cdot [(\partial_t + Ae_1 \cdot \nabla_x)\Theta]\big].$$

Now, using that $A = P_{T_\Theta}A + P_{T_\Theta^\perp}A$, we get $X = X_o + X_e$ with

$$\begin{aligned} X_o &= \int_{\mathrm{SO}_n\mathbb{R}} \big[\partial_t\rho + (P_{T_\Theta^\perp}A)e_1 \cdot \nabla_x\rho + \kappa\rho\, P_{T_\Theta}A \cdot \big((P_{T_\Theta}A)e_1 \cdot \nabla_x\big)\Theta\big] P_{T_\Theta}A\, M_\Theta(A)\, dA, \\ X_e &= \int_{\mathrm{SO}_n\mathbb{R}} \big[(P_{T_\Theta}A)e_1 \cdot \nabla_x\rho + \kappa\rho\, P_{T_\Theta}A \cdot \partial_t\Theta + \kappa\rho\, P_{T_\Theta}A \cdot \big((P_{T_\Theta^\perp}A)e_1 \cdot \nabla_x\big)\Theta\big] P_{T_\Theta}A\, M_\Theta(A)\, dA. \end{aligned}$$

Now, $X_o = 0$ thanks to Lemma 4.5 and $X_e = X_1 + X_2$. ∎

From Θ, we construct a matrix $\mathbb{P}$ and a rank-4 tensor $\mathbb{T}$ as follows:

$$\mathbb{P} = \frac{1}{2}\big((\nabla_x\rho \otimes e_1)\,\Theta^T - \Theta\,(\nabla_x\rho \otimes e_1)^T\big) + \kappa\rho\,\partial_t\Theta\,\Theta^T, \tag{4.8}$$

$$\mathbb{T}_{ijk\ell} = \frac{1}{2}\big(\Theta_{k1}\,\Theta_{jm}\,\partial_{x_\ell}\Theta_{im} + \Theta_{\ell 1}\,\Theta_{jm}\,\partial_{x_k}\Theta_{im}\big), \tag{4.9}$$

where $\otimes$ denotes the tensor product and tensor entries are defined with respect to the canonical basis of $\mathbb{R}^n$. Here and throughout the paper except in the appendices and unless otherwise specified, the repeated index summation convention is used. We note that $\mathbb{P}$ is antisymmetric while $\mathbb{T}$ is antisymmetric with respect to (i,j) and symmetric with respect to (k,ℓ). In other words,

$$\begin{aligned} \mathbb{P} &\in \mathcal{A}_n, \\ \mathbb{T} &\in \mathcal{A}_n \otimes \mathcal{S}_n. \end{aligned}$$

The antisymmetry of $\mathbb{P}$ follows from $\partial_t\Theta \in T_\Theta$. The antisymmetry of $\mathbb{T}$ with respect to (i,j) is a consequence of

$$\partial_{x_\ell}(\Theta_{jm}\Theta_{im}) = \partial_{x_\ell}(\Theta\Theta^T)_{ij} = \partial_{x_\ell}\delta_{ij} = 0. \tag{4.10}$$

The symmetry of $\mathbb{T}$ with respect to (k,ℓ) is by construction. We introduce the following maps:

$$\begin{aligned} L &: \mathcal{A}_n \to \mathcal{A}_n, \quad P \mapsto L(P), \\ B &: \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{S}_n, \quad (P,Q) \mapsto B(P,Q), \end{aligned}$$

given by

$$L(P) = \int_{\mathrm{SO}_n\mathbb{R}} (A \cdot P)\,\frac{A - A^T}{2}\, M_{\mathrm{Id}}(A)\, dA, \tag{4.11}$$

$$B(P_1,P_2) = \int_{\mathrm{SO}_n\mathbb{R}} (A \cdot P_1)\,(A \cdot P_2)\,\frac{A + A^T}{2}\, M_{\mathrm{Id}}(A)\, dA. \tag{4.12}$$

The maps L and B are respectively linear and bilinear symmetric. The following lemma gives alternate expressions for X_1 and X_2:

Lemma 4.7. *We have:*

$$X_1 = L(\mathbb{P})\,\Theta, \tag{4.13}$$

and for any matrix $P \in \mathcal{A}_n$,

$$(X_2\Theta^T)\cdot P = \kappa\rho B_{k\ell}(\mathbb{T}_{..k\ell}, P), \tag{4.14}$$

where $\mathbb{T}_{..k\ell}$ stands for the antisymmetric matrix $(\mathbb{T}_{..k\ell})_{ij} = \mathbb{T}_{ijk\ell}$ and $B_{k\ell}$ is the (k,ℓ) entry of the symmetric matrix $B(\cdot,\cdot)$.

Proof. For any $A \in \mathcal{M}_n$ we have $Ae_1 \cdot \nabla_x\rho = A \cdot (\nabla_x\rho \otimes e_1)$. Thus, with (4.8), we have:

$$\begin{aligned} X_1 &= \int_{\mathrm{SO}_n\mathbb{R}} P_{T_\Theta}A \cdot \left[\nabla_x\rho \otimes e_1 + \kappa\rho\partial_t\Theta\right] P_{T_\Theta}A\, M_\Theta(A)\, dA \\ &= \int_{\mathrm{SO}_n\mathbb{R}} P_{T_\Theta}A \cdot \left[P_{T_\Theta}(\nabla_x\rho \otimes e_1) + \kappa\rho\partial_t\Theta\right] P_{T_\Theta}A\, M_\Theta(A)\, dA \\ &= \int_{\mathrm{SO}_n\mathbb{R}} P_{T_\Theta}A \cdot (\mathbb{P}\Theta)\, P_{T_\Theta}A\, M_\Theta(A)\, dA, \end{aligned}$$

and by the invariance of the matrix inner product by multiplication by rotations on the right and the change of variable $A' = A\Theta^T$, we get expression (4.13) for X_1.

We now turn to X_2. First, using the change of variable $A' = A\Theta^T$ and the same invariance of the matrix inner product, we can write

$$X_2 = \frac{\kappa\rho}{8}\int_{\mathrm{SO}_n\mathbb{R}} (A - A^T)\cdot\mathcal{D}(\Theta, (A+A^T)\Theta e_1)\,(A-A^T)\,M_{\mathrm{Id}}(A)\,dA\,\Theta,$$

where, for a vector $w \in \mathbb{R}^n$, we let

$$\mathcal{D}(\Theta, w) = (w\cdot\nabla_x)\Theta\,\Theta^T \in \mathcal{A}_n.$$

We have the following lemma, the proof of which is deferred to the end of the present proof.

Lemma 4.8. *We have*

$$\mathcal{D}(\Theta, w)_{ij} = w_\ell \partial_{x_\ell}\Theta_{im}\,\Theta_{jm}.$$

Thanks to this lemma and to (4.9), we have:

$$\begin{aligned} \mathcal{D}(\Theta, (A+A^T)\Theta e_1)_{ij} &= \big((A+A^T)\Theta e_1\big)_\ell\, \partial_{x_\ell}\Theta_{im}\,\Theta_{jm} \\ &= (A_{\ell k} + A_{k\ell})\,\Theta_{k1}\,\partial_{x_\ell}\Theta_{im}\,\Theta_{jm} = (A_{k\ell}+A_{\ell k})\mathbb{T}_{ijk\ell}, \end{aligned}$$

and thus

$$X_2\,\Theta^T = \frac{\kappa\rho}{8}\mathbb{T}_{ijk\ell}\Big(\int_{\mathrm{SO}_n\mathbb{R}} (A_{ij}-A_{ji})\,(A_{k\ell}+A_{\ell k})\,(A-A^T)\,M_{\mathrm{Id}}(A)\,dA\Big).$$

Taking the matrix inner product with P for any $P \in \mathcal{A}_n$, we get (4.14). ∎

Proof of Lemma 4.8. Define the matrix $\dot\Theta_w$ by $(\dot\Theta_w)_{ij} = (w\cdot\nabla_x)\Theta_{ij}$. Then, we have $(w\cdot\nabla_x)\Theta = P_{T_\Theta}\dot\Theta_w$ and so, with (4.10),

$$\begin{aligned}\mathcal{D}(\Theta,w)_{ij} &= \frac{1}{2}(\dot\Theta_w\,\Theta^T - \Theta\,\dot\Theta_w^T)_{ij}\\ &= \frac{1}{2}\big((w\cdot\nabla_x)\Theta_{im}\,\Theta_{jm} - \Theta_{im}\,(w\cdot\nabla_x)\Theta_{jm}\big) = (w\cdot\nabla_x)\Theta_{im}\,\Theta_{jm},\end{aligned}$$

which shows the lemma. ∎

Now, we express the mappings L and B in the following Lemma, whose proof is given in Sections 8 and 9.

Lemma 4.9. *(i) We have*

$$L(P) = C_2\,P,\ \forall P\in\mathcal{A}_n,\quad \textit{with}\ \ C_2 = \frac{1}{n-1}\Big(1 - \big\langle\frac{\mathrm{Tr}\,A^2}{n}\big\rangle_{\exp(\kappa\mathrm{Tr}A)}\Big). \tag{4.15}$$

(ii) We have

$$B(P,Q) = C_3\,\mathrm{Tr}\,(PQ)\,\mathrm{Id} + C_4\,\Big(\frac{PQ+QP}{2} - \frac{1}{n}\mathrm{Tr}\,(PQ)\,\mathrm{Id}\Big),\ \forall P,\,Q\in\mathcal{A}_n, \tag{4.16}$$

with

$$C_3 = \frac{1}{n-1}\Big\langle\Big(\frac{\mathrm{Tr}A^2}{n}-1\Big)\,\frac{\mathrm{Tr}A}{n}\Big\rangle_{\exp(\kappa\mathrm{Tr}A)}, \tag{4.17}$$

$$C_4 = \frac{2n}{(n-1)(n-2)(n+2)}\Big\langle\frac{\mathrm{Tr}A^3}{n} - 2\frac{\mathrm{Tr}A}{n}\,\frac{\mathrm{Tr}A^2}{n} + \frac{\mathrm{Tr}A}{n}\Big\rangle_{\exp(\kappa\mathrm{Tr}A)}. \tag{4.18}$$

Remark 4.2. *The proof presented in Sections 8 and 9 is based on representation theory. But this lemma can be proved using elementary algebra: Eq.* (4.15) *is proved in [24, Lemma 3.4] in dimension $n\geq 3$, $n\neq 4$; Eq.* (4.16) *can be proved with a similar approach as outlined in Section 9, Remark 9.1.*

From this, we can prove the

Lemma 4.10. *Eq.* (4.3) *is equivalent to* (3.7) *with c_3 given by* (3.13) *and c_2 and c_4 by:*

$$c_2 = \frac{1}{(n-2)(n+2)}\,\frac{\Big\langle 2\frac{\mathrm{Tr}\,A^3}{n} - n^2\,\frac{\mathrm{Tr}\,A}{n}\,\frac{\mathrm{Tr}\,A^2}{n} + (n^2-2)\,\frac{\mathrm{Tr}\,A}{n}\Big\rangle_{e^{\kappa\mathrm{Tr}\,A}}}{\Big\langle 1 - \frac{\mathrm{Tr}\,A^2}{n}\Big\rangle_{e^{\kappa\mathrm{Tr}\,A}}}, \tag{4.19}$$

$$c_4 = \frac{n}{2(n-2)(n+2)}\,\frac{\Big\langle \frac{\mathrm{Tr}\,A^3}{n} - 2\,\frac{\mathrm{Tr}\,A}{n}\,\frac{\mathrm{Tr}\,A^2}{n} + \frac{\mathrm{Tr}\,A}{n}\Big\rangle_{e^{\kappa\mathrm{Tr}\,A}}}{\Big\langle 1 - \frac{\mathrm{Tr}\,A^2}{n}\Big\rangle_{e^{\kappa\mathrm{Tr}\,A}}}. \tag{4.20}$$

Proof. We first note that $(\Omega_k)_\ell = \Theta_{\ell k}$. Thus

$$(\nabla_x \rho \otimes e_1)\,\Theta^T - \Theta\,(\nabla_x \rho \otimes e_1)^T = \nabla_x \rho \wedge \Omega_1.$$

Thus, introducing (4.15) into (4.13) and using (4.8), we get

$$X_1 = C_2\,\mathbb{P}\,\Theta = \kappa C_2\left[\rho \partial_t \Theta + c_3\,(\nabla_x \rho \wedge \Omega_1)\,\Theta\right]. \tag{4.21}$$

Now, combining (4.16) and (4.14), we get for any $P \in \mathcal{A}_n$,

$$(X_2 \Theta^T) \cdot P = \kappa\rho\left[\Big(C_3 - \frac{C_4}{n}\Big)\,\mathrm{Tr}(\mathbb{T}_{\cdot\cdot kk} P) + \frac{C_4}{2}\left(\mathbb{T}_{\cdot\cdot k\ell} P + P \mathbb{T}_{\cdot\cdot k\ell}\right)_{k\ell}\right]. \tag{4.22}$$

Using that P is antisymmetric, we have $\mathrm{Tr}(\mathbb{T}_{\cdot\cdot kk} P) = -\mathbb{T}_{\cdot\cdot kk} \cdot P$ and, using (4.9) together with Lemma 4.8, we get

$$(\mathbb{T}_{\cdot\cdot kk})_{ij} = \Theta_{k1}\,\partial_{x_k}\Theta_{im}\,\Theta_{jm} = \big((\Omega_1 \cdot \nabla_x)\Theta\,\Theta^T\big)_{ij}.$$

This yields

$$\mathrm{Tr}(\mathbb{T}_{\cdot\cdot kk} P) = -\big((\Omega_1 \cdot \nabla_x)\Theta\,\Theta^T\big) \cdot P. \tag{4.23}$$

Similarly, we compute

$$\left(\mathbb{T}_{\cdot\cdot k\ell} P + P \mathbb{T}_{\cdot\cdot k\ell}\right)_{k\ell} = \mathbb{T}_{kmk\ell} P_{m\ell} + P_{km} \mathbb{T}_{m\ell k\ell}.$$

But we have

$$\mathbb{T}_{m\ell k\ell} P_{km} = -\mathbb{T}_{\ell m\ell k} P_{km} = -\mathbb{T}_{k\ell km} P_{m\ell},$$

where the first equality exploits the fact that $\mathbb{T}_{ijk\ell}$ is antisymmetric with respect to (i,j) and symmetric with respect to (k,ℓ), and the second one is just the circular permutation $k \to m \to \ell \to k$. Thus:

$$\left(\mathbb{T}_{\cdot\cdot k\ell} P + P \mathbb{T}_{\cdot\cdot k\ell}\right)_{k\ell} = (\mathbb{T}_{kmk\ell} - \mathbb{T}_{k\ell km}) P_{m\ell} = \mathbb{V}_{m\ell}\,P_{m\ell} = \mathbb{V} \cdot P, \tag{4.24}$$

with

$$\mathbb{V}_{m\ell} = \mathbb{T}_{kmk\ell} - \mathbb{T}_{k\ell km}.$$

Using (4.9), we have

$$2\mathbb{V}_{m\ell} = \Theta_{k1}\,\Theta_{mp}\,\partial_{x_\ell}\Theta_{kp} + \Theta_{\ell 1}\,\Theta_{mp}\,\partial_{x_k}\Theta_{kp} - \Theta_{k1}\,\Theta_{\ell p}\,\partial_{x_m}\Theta_{kp} - \Theta_{m1}\,\Theta_{\ell p}\,\partial_{x_k}\Theta_{kp}. \tag{4.25}$$

The second and fourth terms of (4.25) give

$$\begin{aligned}\Theta_{\ell 1}\,\Theta_{mp}\,\partial_{x_k}\Theta_{kp} - \Theta_{m1}\,\Theta_{\ell p}\,\partial_{x_k}\Theta_{kp} &= \left[(\Omega_1)_\ell\,(\Omega_p)_m - (\Omega_1)_m\,(\Omega_p)_\ell\right]\nabla_x \cdot \Omega_p \\ &= (\Omega_p \wedge \Omega_1)_{m\ell}\,\nabla_x \cdot \Omega_p = (r \wedge \Omega_1)_{m\ell}.\end{aligned}$$

Using a similar computation as (4.10), the first and third terms of (4.25) lead to

$$\begin{aligned}\Theta_{k1}\,\Theta_{mp}\,\partial_{x_\ell}\Theta_{kp} - \Theta_{k1}\,\Theta_{\ell p}\,\partial_{x_m}\Theta_{kp} &= -\Theta_{k1}\,\Theta_{kp}\,(\partial_{x_\ell}\Theta_{mp} - \partial_{x_m}\Theta_{\ell p}) \\ &= -(\partial_{x_\ell}(\Omega_1)_m - \partial_{x_m}(\Omega_1)_\ell) = (\nabla_x \wedge \Omega_1)_{m\ell}.\end{aligned}$$

Collecting these two terms, this gives

$$\mathbb{V} = \frac{1}{2}\big(r \wedge \Omega_1 + \nabla_x \wedge \Omega_1\big). \tag{4.26}$$

Finally, collecting (4.22), (4.23), (4.24) and (4.26), we get

$$(X_2\Theta^T)\cdot P = \kappa\rho\Big[-\Big(C_3 - \frac{C_4}{n}\Big)(\Omega_1\cdot\nabla_x)\Theta\,\Theta^T + \frac{C_4}{4}\,(r\wedge\Omega_1 + \nabla_x\wedge\Omega_1)\Big]\cdot P.$$

Since the matrix inside the bracket is antisymmetric and this identity is valid for any $P \in \mathcal{A}_n$, we get

$$X_2 = \kappa C_2\,\rho\Big[c_2\,(\Omega_1\cdot\nabla_x)\Theta + c_4\,(r\wedge\Omega_1 + \nabla_x\wedge\Omega_1)\,\Theta\Big], \tag{4.27}$$

where

$$c_2 = -\frac{1}{C_2}\Big(C_3 - \frac{C_4}{n}\Big), \qquad c_4 = \frac{C_4}{4C_2}. \tag{4.28}$$

Now, collecting (4.21) and (4.27), we get (3.7), (3.8), (3.9) with c_3 given by (3.13). Finally, inserting (4.15), (4.17) and (4.18) into (4.28), we get (4.19), (4.20), which ends the proof. ∎

Finally, we show that c_1, c_2 and c_4 are given by the expressions (3.11), (3.12) and (3.14) thanks to Weyl's integration formula (see Section 6).

Lemma 4.11. *Let $n \in \mathbb{N}$, $n \geq 3$ and let $p \in \mathbb{N}$ such that $n = 2p$ or $n = 2p+1$. We have c_1, c_2 and c_4 given by* (3.11), (3.12), (3.14) *respectively.*

Proof. We first remark that for any integer k, the function $A \mapsto \mathrm{Tr}A^k$ is a class function. So, we can apply Weyl's integration formula (see Section 6) to all the integrals involved in formulas (4.1), (4.19), (4.20). We note that

$$(R_{\theta_1,\ldots,\theta_p})^k = R_{k\theta_1,\ldots,k\theta_p}, \quad \forall k \in \mathbb{Z},$$

where $R_{\theta_1\ldots\theta_p}$ is defined by (6.2) in the case of $\mathrm{SO}_{2p}\mathbb{R}$ and by (6.3) in the case of $\mathrm{SO}_{2p+1}\mathbb{R}$. We have

$$\mathrm{Tr}\,R_{\theta_1,\ldots,\theta_p} = \begin{cases} 2(\cos\theta_1 + \ldots + \cos\theta_p) & \text{if} \quad n = 2p, \\ 2(\cos\theta_1 + \ldots + \cos\theta_p) + 1 & \text{if} \quad n = 2p+1, \end{cases}$$

so that

$$\mathrm{Tr}\,(R_{\theta_1,\ldots,\theta_p})^k = \begin{cases} C_{2p}^{(k)} & \text{if} \quad n = 2p, \\ C_{2p+1}^{(k)} & \text{if} \quad n = 2p+1. \end{cases}$$

Lemma 4.11 follows immediately. ∎

Collecting Lemmas 4.4, 4.10 and 4.11 shows Theorem 3.1, which ends this section.

5 Conclusion and perspectives

In this paper, we have derived the hydrodynamic limit of a kinetic model of self-propelled agents interacting through body attitude coordination in arbitrary dimension $n \geq 3$. Previous literature was restricted to dimension $n = 3$. In arbitrary dimension, the derivation uses Lie group representations and the Weyl integration formula. The obtained hydrodynamic model is structurally identical to that obtained in dimension 3 (and referred to as the "Self-Organized Hydrodynamics for Body orientation (SOHB)") but the constants involved have expressions that depend on the dimension. Future work will be concerned with existence of solutions for the SOHB model, rigorous convergence from the kinetic to the SOHB model and derivation of explicit solutions. We will also investigate the situation where the particle dynamics are described by stochastic differential equations instead of PDMP as considered here. In this case, the resulting kinetic model involves a Fokker-Planck operator for which the generalized collision invariants are still unknown. As we have seen, knowing the expression of the GCI is crucial to get an explicit expression of the coefficients of the hydrodynamic model. The numerical resolution of the SOHB model has not been undertaken yet and will require the design of appropriate numerical schemes. Finally, the qualitative properties of the solutions of the SOHB model, and particularly, their topology, need to be further investigated.

Appendices

In the forthcoming sections, we assume $n \geq 3$. The proofs contained in these appendices rely on results from representation theory. We start with recalling a few useful results from this theory. We refer to [42] for the terminology and notations. In all sections of this appendix, the repeated index summation convention is **not** used.

6 Short summary of useful results from representation theory

Let G be a Lie group. A representation of G on the vector space $V = \mathbb{R}^n$ or $\mathbb{C}^n$ is a group morphism: $G \to \mathrm{GL}(V)$ into the group of automorphisms of V. Likewise, if $\mathfrak{g}$ is a Lie algebra, a representation of $\mathfrak{g}$ is a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the space of endomorphisms of V. Here, we will be mostly concerned with the representations of $\mathrm{SO}_n\mathbb{R}$ which acts on $\mathcal{M}_n$ by conjugation, i.e. the action of $R \in \mathrm{SO}_n\mathbb{R}$ sends $M \in \mathcal{M}_n$ to RMR^T. The reason is that our objects of study have remarkable transforms under this action. There is a strong connection between representations of a Lie group and representations of its Lie algebra. Lie algebras have a rich structure and one starts by constructing representations of a Lie algebra before lifting them to the Lie groups that have this Lie algebra in common. We note that the base field is a representation of G or $\mathfrak{g}$. For instance, it sends all the elements of the group to the identity. This is called the trivial representation. If G is a matrix group, i.e. a subgroup of $\mathrm{GL}(V)$, then the mapping ρ: $G \to \mathrm{GL}(V)$ such that $\rho(g) = g$ is also a representation called the standard representation.

A representation is said irreducible if it has no proper subspace which is left invariant by the representation. In good cases (which include those we will consider), any representation can be decomposed into the direct sum of irreducible representations, making irreducible representations the building blocks of the theory. The reason why irreducible representations are so appealing is the so called Schur Lemma:

Lemma 6.1 (Schur Lemma)**.** *Let V and W be two irreducible complex representations of a group G and let T: $V \mapsto W$ be a map of representations, i.e. a linear map which commutes with the representations (one also says, a map which intertwins the representations). Then, there exists $C \in \mathbb{C}$ such that $T = C\,\mathrm{Id}$. Furthermore, $C = 0$ if the two representations are not isomorphic.*

One should be careful that the result does not hold in these terms for real representations: if the two representations are not isomorphic T is still 0 but if the two representations are isomorphic, then T is an isomorphism, but we cannot say that $T = C\,\mathrm{Id}$, except in some special cases. Anyhow, intertwinning maps of irreducible representations have a very simple structure and we want to exploit this structure in the results below. So, for a given representation, we want to find its decomposition in irreducible representations.

The theory starts to construct the finite-dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ as subspaces of tensor products $V^{\otimes d}$ of the standard representation $V = \mathbb{C}^n$. These representations are in bijective correspondence to conjugacy classes

of the symmetric group $\mathfrak{S}_d$, themselves in bijective correspondence to partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ of d, i.e. such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0$ and $\lambda_1 + \ldots + \lambda_{n-1} = d$. The irreducible representation associated with λ is called the Schur functor (or Weyl module) $\mathbb{S}_\lambda(V)$ [42, §15.3].

Among remarkable irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ are the symmetric and exterior powers of V [42, Appendix B]. The symmetric power $\mathrm{Sym}^d(V)$ is the space of symmetric tensors. For $v_1, \ldots, v_d \in V$, we denote by $v_1 \circ \ldots \circ v_d = \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$, the symmetric product of $v_1, \ldots, v_d$. We have $\mathrm{Sym}^d(V) = \mathrm{Span}\{v_1 \circ \ldots \circ v_d \,|\, v_1, \ldots, v_d \in V\} \subset V^{\otimes d}$. Likewise, the exterior power $\Lambda^d(V)$ is the space of antisymmetric tensors. For $v_1, \ldots, v_d \in V$, we denote by $v_1 \wedge \ldots \wedge v_d = \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$, the exterior product of $v_1, \ldots, v_d$, where $\varepsilon(\sigma)$ is the signature of the permutation σ. We have $\Lambda^d(V) = \mathrm{Span}\{v_1 \wedge \ldots \wedge v_d \,|\, v_1, \ldots, v_d \in V\} \subset V^{\otimes d}$. $\mathrm{Sym}^d(V)$ and $\Lambda^d(V)$ are irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ which correspond to $\mathbb{S}_\lambda(V)$ for the partitions $\lambda = (d, 0, \ldots, 0)$ (abbreviated by $\lambda = d$) and $\lambda = (1, \ldots, 1)$ (d times), [42, §15.3].

It is possible to pass from $\mathfrak{sl}_n\mathbb{C}$ to $\mathfrak{so}_n\mathbb{C}$ (for $n \geq 3$) by means of the Weyl construction [42, §19.5]. Let $I = (p, q)$ with $p < q$ be any pair of integers in $\{1, \ldots, d\}$. The contraction Φ_I is the mapping

$$\Phi_I : \left\{ \begin{array}{lcl} V^{\otimes d} & \to & V^{\otimes (d-2)} \\ v_1 \otimes \ldots \otimes v_d & \to & (v_p \cdot v_q)\, v_1 \otimes \ldots \otimes \hat{v}_p \otimes \ldots \otimes \hat{v}_q \otimes \ldots \otimes v_d, \end{array} \right.$$

where the hat means that the corresponding factor is removed. $V^{[d]}$ denotes the intersection of the kernels of such contractions over all pairs I of indices. Then, $\mathbb{S}_{[\lambda]}(V) = \mathbb{S}_\lambda(V) \cap V^{[d]}$, when it is not $\{0\}$, is an irreducible representation of $\mathrm{O}_n\mathbb{C}$. For two associated partitions in the sense of Weyl, λ and μ, $\mathbb{S}_{[\lambda]}(V)$ and $\mathbb{S}_{[\mu]}(V)$ are isomorphic as irreducible representations of $\mathfrak{so}_n\mathbb{C}$ (see [42, §19.5] for the definition of associated partitions). Furthermore, if λ is associated to itself, $\mathbb{S}_{[\lambda]}(V)$ is not irreducible on $\mathfrak{so}_n\mathbb{C}$ but decomposes into the sum of two non-isomorphic irreducible representations of the same dimension. We will encounter this case in Section 8 with $\Lambda^2(\mathbb{C}^4)$ and in Section 9 with $\Lambda^4(\mathbb{C}^8)$.

The representations $\mathbb{S}_{[\lambda]}(V)$ when they are irreducible on $\mathfrak{so}_n\mathbb{C}$, lift to irreducible representations of $\mathrm{SO}_n\mathbb{C}$ [42, Proposition 23.13 (iii)]. Note that not all irreducible representations of $\mathfrak{so}_n\mathbb{C}$ are obtained this way: to complete the list one needs to introduce the spin representations but they do not lift to representations of $\mathrm{SO}_n\mathbb{C}$ [42, Proposition 23.13 (iii)] and will be ignored here. The complex irreducible representations of $\mathrm{SO}_n\mathbb{C}$ and $\mathrm{SO}_n\mathbb{R}$ are the same (see [42, §26.1, Section "Real groups"] for details). Finally complex irreducible representations of $\mathrm{SO}_n\mathbb{R}$ will give rise to real ones (in other words, these complex representations are complexifications of real representations) in mostly all cases. The only troublesome cases are $n = 2p$ even with either p odd and $\lambda_n \neq 0$ or $p \equiv 2 \bmod 4$ and λ_{n-1} odd [42, Prop. 26.26 and 26.27] and we will not meet them.

Let now G be a compact Lie group (such as $\mathrm{SO}_n\mathbb{R}$) endowed with its Haar measure dg and let ρ: $V \to \mathrm{GL}(V)$ be a representation of G. The character of V, denoted by χ_V, is the mapping $G \ni g \mapsto \chi_V(g) = \mathrm{Tr}\rho(g) \in \mathbb{C}$. If V and W are irreducible, we have Schur's orthogonality relation [42, §26.2]:

$$\int_G \chi_V(g)\, \bar{\chi}_W(g)\, dg = \left\{ \begin{array}{ll} 1 & \text{if } V \text{ and } W \text{ are isomorphic,} \\ 0 & \text{otherwise.} \end{array} \right. \tag{6.1}$$

The final result of representation theory that we will need is the Weyl integration formula. For $\mathrm{SO}_n\mathbb{R}$, its statement is given in [62, Theorems IX.9.4 & IX.9.5]. We first introduce some notations. For $\theta \in \mathbb{R}$, we define the planar rotation matrix R_θ by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For $(\theta_1, \dots, \theta_p) \in \mathbb{R}^p$, $R_{\theta_1,\dots,\theta_p}$ denotes the following matrix defined by blocks:

- in the case $n = 2p$, $p \geq 2$,

$$R_{\theta_1,\dots,\theta_p} = \begin{pmatrix} R_{\theta_1} & & & 0 \\ & R_{\theta_2} & & \\ & & \ddots & \\ 0 & & & R_{\theta_p} \end{pmatrix} \in \mathrm{SO}_{2p}\mathbb{R}, \tag{6.2}$$

- in the case $n = 2p+1$, $p \geq 1$,

$$R_{\theta_1,\dots,\theta_p} = \begin{pmatrix} R_{\theta_1} & & & 0 & 0 \\ & R_{\theta_2} & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & & R_{\theta_p} & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \in \mathrm{SO}_{2p+1}\mathbb{R}. \tag{6.3}$$

We define a maximal torus T of $\mathrm{SO}_{2p}\mathbb{R}$ or $\mathrm{SO}_{2p+1}\mathbb{R}$ by

$$T = \{R_{\theta_1\dots\theta_p} \mid (\theta_1, \dots, \theta_p) \in \mathbb{R}^p\}.$$

The maximal torus T is an abelian subgroup of $\mathrm{SO}_{2p}\mathbb{R}$ or $\mathrm{SO}_{2p+1}\mathbb{R}$ isomorphic to the p-dimensional torus. We recall that any element of $\mathrm{SO}_{2p}\mathbb{R}$ or $\mathrm{SO}_{2p+1}\mathbb{R}$ is conjugate to a (non-unique) element of T, i.e. $\forall A \in \mathrm{SO}_{2p}\mathbb{R}$, $\exists B \in T$, $\exists U \in \mathrm{SO}_{2p}\mathbb{R}$ such that $A = UBU^T$ (and similarly with $2p+1$). We also recall that a class function f on a group G is a function that only depends on the conjugation class, i.e. a function that takes the same value on two conjugate elements of the group.

Proposition 6.2 (Weyl integration formula ([62], Theorems IX.9.4 & IX.9.5))**.**
Let $n \in \mathbb{N}$, $n \geq 3$. Let $p \in \mathbb{N}$ such that $n = 2p$ or $n = 2p+1$. For any integrable class function f on $\mathrm{SO}_n\mathbb{R}$, we have

$$\int_{\mathrm{SO}_n\mathbb{R}} f(A)\, dA = \frac{1}{(2\pi)^p} \int_{[0,2\pi]^p} f(R_{\theta_1\dots\theta_p})\, u_n(\theta_1, \dots, \theta_p)\, d\theta_1 \dots d\theta_p, \tag{6.4}$$

where u_n: $\mathbb{R}^p \to \mathbb{R}$ is defined by (3.3) *(in the case $n = 2p$) or* (3.4) *(if $n = 2p+1$).*

Remark 6.1. *(i) There is a typo in [62, Theorems IX.9.4]. The normalization is not correct as can be realized by taking $f = 1$ and $p = 1$. It would correspond to taking the constant in* (3.4) *equal to $\frac{2^{p(p-2)}}{p!}$. We have carefully redone the computation and* (3.4) *can be easily checked for $p = 1$ and $p = 2$.*
(ii) Taking $f = 1$ in (6.4) *leads to $(2\pi)^{-p} \int_{[0,1]^p} u_n\, d\theta_1 \dots d\theta_p = 1$. A direct proof does not seem obvious.*

7 Proof of Lemma 4.1

Changing variable to $A' = \Theta^T A$ in the integral appearing in (4.1) yields

$$\int_{\mathrm{SO}_n\mathbb{R}} A\, M_\Theta(A)\, dA = \Theta \int_{\mathrm{SO}_n\mathbb{R}} A\, M_{\mathrm{Id}}(A)\, dA.$$

We will prove that for any g: $\mathrm{SO}_n\mathbb{R} \to \mathbb{R}$, $A \mapsto g(A)$ invariant by conjugation $g(UAU^T) = g(A)$, $\forall A$, $U \in \mathrm{SO}_n\mathbb{R}$ (i.e. g is a class function) and by transposition $g(A^T) = g(A)$, we have

$$\int_{\mathrm{SO}_n\mathbb{R}} A\, g(A)\, dA = C_1\, \mathrm{Id}, \quad \text{with} \quad C_1 = \frac{1}{n} \int_{\mathrm{SO}_n\mathbb{R}} \mathrm{Tr}A\, g(A)\, dA. \tag{7.1}$$

Obviously, $g(A) = M_{\mathrm{Id}}(A) = Z^{-1} \exp(\mathrm{Tr}A)$ is a class function invariant by transposition and (7.1) directly implies Lemma 4.1. First, we remark that the second formula of (7.1) is a direct consequence of the first one by taking the trace. We now show (7.1).

Using that g and the Haar measure are invariant by transposition, (7.1) is equivalent to saying that

$$\int_{\mathrm{SO}_n\mathbb{R}} \big(\frac{A + A^T}{2} - \frac{\mathrm{Tr}A}{n}\, \mathrm{Id}\big)\, g(A)\, dA = 0. \tag{7.2}$$

Let $\mathcal{S}_n^0$ be the subspace of $\mathcal{S}_n$ consisting of trace-free matrices and define the mapping K: $\mathcal{S}_n^0 \to \mathbb{R}$, $S \mapsto K(S)$ by

$$K(S) = \int_{\mathrm{SO}_n\mathbb{R}} A \cdot S\, g(A)\, dA.$$

Eq. (7.2) is equivalent to saying that $K = 0$. Let $U \in \mathrm{SO}_n\mathbb{R}$. Using the change of variables $A' = U^T A U$ and the invariance of g and of the Haar measure by conjugation, we have

$$K(USU^T) = K(S). \tag{7.3}$$

The space $\mathcal{S}_n^0$ is an irreducible representation of $\mathrm{SO}_n\mathbb{R}$ (see Lemma 7.1 below) and $\mathbb{R}$ is the trivial representation, which is also irreducible. Formula (7.3) says that the linear map K intertwins these two irreducible representations. Since they are not isomorphic (they do not have the same dimension), by Schur's Lemma (see [42, Lemma 1.7] or Section 6), K must be identically 0.

Specifying $g = M_{\mathrm{Id}}$, we now prove the properties of c_1 stated in the second part of Lemma 4.1. First, we show that $c_1(0) = 0$. Indeed,

$$c_1(0) = \frac{1}{n} \int_{\mathrm{SO}_n\mathbb{R}} \mathrm{Tr}A\, dA.$$

The function $\mathrm{SO}_n\mathbb{R} \ni A \mapsto \mathrm{Tr}\, A \in \mathbb{R}$ is the character of the standard representation, while the function $\mathrm{SO}_n\mathbb{R} \ni A \mapsto 1 \in \mathbb{R}$ is the character of the trivial representations. Since both are irreducible, by (6.1), we get $c_1(0) = 0$. Now, we show that c_1 is nondecreasing. By differentiating (4.1) with respect to κ, we get $n\frac{dc_1}{d\kappa} = \frac{N}{D}$ with

$$D = \Big(\int_{\mathrm{SO}_n\mathbb{R}} \mathrm{Tr}A\, e^{\kappa \mathrm{Tr}A}\, dA \Big)^2 > 0,$$

and

$$\begin{aligned} N &= \int_{(\mathrm{SO}_n\mathbb{R})^2} e^{\kappa \mathrm{Tr}A}\, e^{\kappa \mathrm{Tr}A'} \Big[(\mathrm{Tr}\, A)^2 - \mathrm{Tr}A\, \mathrm{Tr}A' \Big]\, dA\, dA' \\ &= \frac{1}{2} \int_{(\mathrm{SO}_n\mathbb{R})^2} e^{\kappa \mathrm{Tr}A}\, e^{\kappa \mathrm{Tr}A'} \Big[\mathrm{Tr}A - \mathrm{Tr}A' \Big]^2\, dA\, dA' \geq 0. \end{aligned}$$

We finally show that $c_1(\kappa) \to 1$ as $\kappa \to \infty$. This is a classical method of concentration of measure. Define the probability measures

$$d\mu_\kappa(A) = \frac{e^{\kappa \mathrm{Tr}A}\, dA}{\int_{\mathrm{SO}_n\mathbb{R}} e^{\kappa \mathrm{Tr}A}\, dA}.$$

Since $\mathrm{SO}_n\mathbb{R}$ is compact, the family $\{\mu_\kappa\}_{\kappa \in [0,\infty)}$ is tight, so there is a sequence $(\kappa_n)_{n\geq 1}$, $\kappa_n \to \infty$ and a probability measure μ on $\mathrm{SO}_n\mathbb{R}$ such that μ_{κ_n} converges weakly to μ, i.e. for any measurable subset S of $\mathrm{SO}_n\mathbb{R}$, $\mu_{\kappa_n}(S) \to \mu(S)$. Now, the support of μ is reduced to the singleton $\{\mathrm{Id}\}$, so that it is the Dirac delta $\mu = \delta_{\mathrm{Id}}$. Indeed, we show that for any $B \in \mathrm{SO}_n\mathbb{R}$, $B \neq \mathrm{Id}$, there exists an open set W containing B such that $\mu(W) = 0$. First, we note that since $B \neq \mathrm{Id}$, then, $\mathrm{Tr}B < n = \mathrm{Tr}\,\mathrm{Id}$. This is because B is conjugate to one of the matrices $R_{\theta_1,\ldots,\theta_p}$ and that $\mathrm{Tr}\, R_{\theta_1,\ldots,\theta_p} \leq n$ with equality if and only if $\cos\theta_1 = \ldots = \cos\theta_p = 1$, i.e. $R_{\theta_1,\ldots,\theta_p} = \mathrm{Id}$. Let c, c' be two constants such that $\frac{n+\mathrm{Tr}B}{2} < c < c' < n$ and define $W = \mathrm{Tr}^{-1}\big((-\infty, c)\big)$ and $W' = \mathrm{Tr}^{-1}\big((c', \infty)\big)$. Then,

$$\begin{aligned} \mu_\kappa(W) &= \frac{\int_W e^{\kappa \mathrm{Tr}A}\, dA}{\int_{\mathrm{SO}_n\mathbb{R}} e^{\kappa \mathrm{Tr}A}\, dA} \leq \frac{\int_W e^{\kappa \mathrm{Tr}A}\, dA}{\int_{W'} e^{\kappa \mathrm{Tr}A}\, dA} \\ &\leq \frac{e^{\kappa c} m(W)}{e^{\kappa c'} m(W')} \to 0 \text{ as } \kappa \to \infty, \end{aligned}$$

where $m(W)$, $m(W')$ denote the Haar measures of W and W'. Now, since the limit of all convergent subsequences is δ_{Id}, the whole family μ_κ converges weakly to δ_{Id}. In particular, since $c_1(\kappa) = \frac{1}{n}\langle \mu_\kappa, \mathrm{Tr}A\rangle$, this implies that

$$\lim_{\kappa\to\infty} c_1(\kappa) = \frac{1}{n}\langle \delta_{\mathrm{Id}}, \mathrm{Tr}A\rangle = \frac{1}{n}\mathrm{Tr}\,\mathrm{Id} = 1.$$

This ends the proof of Lemma 4.1. ∎

Remark 7.1. *To explore the properties of c_1, one could also use* (3.11). *From $C^{(1)}_{2p} \leq 2p$ and $C^{(1)}_{2p+1} \leq 2p+1$ it follows that $c_1 \leq 1$. The other properties of c_1 are easy to prove in the case $n = 2p$. For instance, to prove $c_1 \geq 0$ it is enough to show that the numerator of* (3.11) *is nonnegative. Then, we note that $C^{(1)}_{2p}$ is changed in its opposite when $(\theta_1, \ldots, \theta_i, \ldots, \theta_p)$ is changed into $(-\theta_1, \ldots, -\theta_i, \ldots, -\theta_p)$, while u_{2p} is unchanged. Consequently, defining $D = (C^{(1)}_{2p})^{-1}\big((0,\infty)\big)$, we can write*

$$\int_{[0,\pi]^p} C^{(1)}_{2p} \exp\big(\kappa C^{(1)}_{2p}\big)\, u_{2p}\, d\tilde{\theta}_p = 2\int_D C^{(1)}_{2p} \sinh\big(\kappa C^{(1)}_{2p}\big)\, u_{2p}\, d\tilde{\theta}_p \geq 0.$$

The same method would permit to show that $\kappa \mapsto c_{2p}(\kappa)$ is non decreasing and that $c_{2p}(0) = 0$. This is entirely different in the case $n = 2p+1$. For instance, that $c_{2p+1}(0) = 0$ amounts to the identity

$$\int_{[0,\pi]^p} \big(2(\cos\theta_1 + \ldots + \cos\theta_p) + 1\big) \exp\Big(\kappa\big(2(\cos\theta_1 + \ldots + \cos\theta_p) + 1\big)\Big) \times \prod_{1\le j<k\le p} \big(\cos\theta_j - \cos\theta_k\big)^2 \prod_{j=1}^{p} \big(1 - \cos\theta_j\big)\, d\theta_1 \ldots d\theta_p = 0,$$

which does not look obvious to show directly. Similarly, the positivity of $c_{2p+1}(\kappa)$ or that $\kappa \mapsto c_{2p+1}(\kappa)$ is increasing are not obvious as well.

Lemma 7.1. *The space $\mathcal{S}_n^0$ is an irreducible representation of $\mathrm{SO}_n\mathbb{R}$.*

Proof. This fact is classical but we sketch it here as a warm-up for the use of the concepts of Section 6. The space of symmetric matrices with complex entries is isomorphic to $\mathrm{Sym}^2(V)$ with $V = \mathbb{C}^n$, which is the Weyl module $\mathbb{S}_2(V)$ and thus, an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$. We use Weyl's construction using the contractions (see Section 6 or [42, §19.5]), to find its associated irreducible representations over $\mathfrak{so}_n\mathbb{C}$. There is only one contraction

$$\Phi : \left\{ \begin{array}{lcl} \mathrm{Sym}^2(V) & \to & \mathbb{C} \\ v_1 \circ v_2 & \to & 2(v_1 \cdot v_2). \end{array} \right. \tag{7.4}$$

The kernel of this map,

$$\mathbb{S}_{[2]}(V) = \mathrm{Span}\{v_1 \circ v_2 \mid v_1,\, v_2 \in V,\ \text{such that } v_1 \cdot v_2 = 0\}. \tag{7.5}$$

is an irreducible representation of $\mathfrak{so}_n\mathbb{C}$. In terms of matrices, $\mathbb{S}_{[2]}(V)$ is nothing but the space of trace-free symmetric matrices with complex entries. Thus, $\mathbb{S}_{[2]}(V)$ is a complex irreducible representation of $\mathrm{SO}_n\mathbb{R}$ and is of real type (see Section 6) so its real part $\mathcal{S}_n^0$ is an irreducible real representation of $\mathrm{SO}_n\mathbb{R}$. ∎

8 Proof of Lemma 4.9 (i)

Again, we show that (4.15) is true in the more general case where M_{Id} is replaced by any class function g (but there is no need to suppose that g is invariant by transposition). By linearity, we extend L given by (4.11) to a mapping $\tilde{L}$: $\tilde{\mathcal{A}}_n \to \tilde{\mathcal{A}}_n$, where $\tilde{\mathcal{A}}_n$ is the complexification of $\mathcal{A}_n$, i.e. the space of antisymmetric matrices with complex entries. Thus, $\tilde{L}(P + iQ) = L(P) + iL(Q)$, $\forall P, Q \in \mathcal{A}_n$. The space $\tilde{\mathcal{A}}_n$ is isomorphic to the exterior square $\Lambda^2(V)$ with $V = \mathbb{C}^n$.

(i) Case $n \neq 4$. In this case $\Lambda^2(V)$ is an irreducible representation of $\mathfrak{so}_n\mathbb{C}$ [42, Theorems 19.2 and 19.14]. It lifts into an irreducible representation of $\mathrm{SO}_n\mathbb{C}$ and consequently, of $\mathrm{SO}_n\mathbb{R}$. Now, $\tilde{L}$ is a mapping from $\Lambda^2(V)$ to itself which intertwins the two representations (i.e. $\tilde{L}(UPU^T) = U\tilde{L}(P)U^T$, $\forall U \in \mathrm{SO}_n\mathbb{R}$, by the same method as in Section 7). By

Schur's Lemma [42, Lemma 1.7], there exists $C_2 \in \mathbb{C}$ such that $\tilde{L}(P+iQ) = C_2(P+iQ)$, $\forall P,\, Q \in \mathcal{A}_n$. Taking $Q = 0$, we get

$$L = C_2 \operatorname{Id}_{\Lambda^2(V)}, \tag{8.1}$$

and since $L(P)$ has real entries, we have $C_2 \in \mathbb{R}$.

We now show the second formula of (4.15). Taking the matrix inner product of $L(P) = C_2\, P$ with P leads to

$$\int_{\mathrm{SO}_n\mathbb{R}} (A \cdot P)^2\, g(A)\, dA = C_2 P \cdot P,$$

and taking $P = e_i \wedge e_j$ for $i \neq j$ gives

$$\int_{\mathrm{SO}_n\mathbb{R}} (A_{ij} - A_{ji})^2\, g(A)\, dA = 2C_2(\delta_{ii}\, \delta_{jj} - \delta_{ij}\, \delta_{ji}),$$

where $A = (A_{ij})_{i,j=1,\dots,n}$. We note that the formula is still valid for $i = j$. Summing over i, j, expanding the square, and noting that

$$\sum_{i,j} A_{ij}^2 = \operatorname{Tr}(A^T A) = \operatorname{Tr} \operatorname{Id} = n, \quad \sum_{i,j} A_{ij} A_{ji} = \operatorname{Tr} A^2, \tag{8.2}$$

we arrive at the second formula of (4.15) when $g = M_{\mathrm{Id}}$.

(ii) Case $n = 4$. $\Lambda^2(V)$ (with $V = \mathbb{C}^4$) is not an irreducible representation of $\mathfrak{so}_4\mathbb{C}$. It decomposes into the direct sum of two non-isomorphic irreducible representations of $\mathfrak{so}_4\mathbb{C}$ [42, Theorem 19.2 (ii)]:

$$\Lambda^2(V) = \Lambda_+ \oplus \Lambda_-, \tag{8.3}$$

both having dimension 3 (we remind that $\dim \Lambda^2(V) = \binom{4}{2} = 6$). Furthermore, Λ_+ and Λ_- lift into complex irreducible representations of $\mathrm{SO}_4\mathbb{C}$ and thus, of $\mathrm{SO}_4\mathbb{R}$ [42, Proposition 23.13 (iii)]. Let $\mathcal{T}_\pm$: $\Lambda^2(V) \to \Lambda_\pm$ be the projections of $\Lambda^2(V)$ on these two sub-representations. The map $\tilde{L}$ can be decomposed by blocks using (8.3) on both its domain and codomain. Each block being a complex irreducible representation of $\mathrm{SO}_4\mathbb{R}$, we can apply Schur's lemma and conclude that any map between two blocks is equal to 0 if the blocks are not isomorphic and equal to $C \operatorname{Id}$ for some constant $C \in \mathbb{C}$ if the blocks are isomorphic. The pairs of isomorphic blocks are (Λ_+, Λ_+) and (Λ_-, Λ_-). It follows that there exist two constants C_2^+, $C_2^- \in \mathbb{C}$ such that

$$\tilde{L} = C_2^+\, \mathcal{T}_+ + C_2^-\, \mathcal{T}_-, \tag{8.4}$$

We now compute $\mathcal{T}_\pm$. We have an automorphism of $\mathrm{SO}_4\mathbb{R}$ representations α: $\Lambda^2(V) \to \Lambda^2(V)$ given by

$$(\alpha(v_1 \wedge v_2) \cdot v_3 \wedge v_4) = \det(v_1, v_2, v_3, v_4), \quad \forall (v_1, \dots, v_4) \in V^4. \tag{8.5}$$

We recall that the inner product on $\Lambda^2(V)$ is given by $(v_1 \wedge v_2 \cdot v_3 \wedge v_4) = 2[(v_1 \cdot v_3)(v_2 \cdot v_4) - (v_1 \cdot v_4)(v_2 \cdot v_3)]$. A simple computation shows that

$$\alpha(e_i \wedge e_j) = \frac{1}{4} \sum_{k,\ell=1}^{4} \varepsilon_{ijk\ell}\, e_k \wedge e_\ell,$$

where $\varepsilon_{ijk\ell} = 0$ if two or more indices among $\{i, j, k, \ell\}$ are equal, and is the signature of the permutation $1 \to i$, $2 \to j$, $3 \to k$, $4 \to \ell$ otherwise. We note that

$$\sum_{k,\ell=1}^{4} \varepsilon_{ijk\ell}\, \varepsilon_{k\ell mn} = 2(\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm}).$$

It follows that $\alpha^2(e_i \wedge e_j) = \frac{1}{4} e_i \wedge e_j$, hence $(2\alpha)^2 = \mathrm{Id}_{\Lambda^2(V)}$. Since clearly $2\alpha \neq \pm \mathrm{Id}_{\Lambda^2(V)}$, the eigenvalues of 2α are ± 1. The associated eigenspaces are sub-representations of $\Lambda^2(V)$. Since the only sub-representations of $\Lambda^2(V)$ are $\Lambda_\pm$ which are irreducible, these eigenspaces coincide with $\Lambda_\pm$. We let Λ_+ be the eigenspace associated with eigenvalue 1 and Λ_- with eigenvalue -1. Additionally, we see from (8.5) that α is self-adjoint. Thus, Λ_+ and Λ_- are orthogonal. It follows that 2α is the orthogonal reflection of $\Lambda^2(V)$ in the subspace Λ_+. The projections $\mathcal{T}_\pm$ are given by $\mathcal{T}_\pm = \frac{1}{2}\mathrm{Id}_{\Lambda^2(V)} \pm \alpha$. From (8.4), we deduce that

$$\tilde{L} = \frac{C_2^+ + C_2^-}{2}\, \mathrm{Id}_{\Lambda^2(V)} + (C_2^+ - C_2^-)\alpha. \tag{8.6}$$

Now, we additionally assume that g is invariant by all automorphisms of $\mathrm{SO}_4\mathbb{C}$ defined by $A \mapsto UAU^T$ when U ranges in $\mathrm{O}_4\mathbb{C}$. When $U \in \mathrm{O}_4\mathbb{C} \setminus \mathrm{SO}_4\mathbb{C}$, these automorphisms consist of the conjugation by an element U which does not belong to $\mathrm{SO}_4\mathbb{C}$ although the image UAU^T still belongs to $\mathrm{SO}_4\mathbb{C}$, so they are outer automorphisms. We note that $g = M_{\mathrm{Id}}$ satisfies this assumption. Since the Haar measure on $\mathrm{SO}_4\mathbb{C}$ is the restriction to $\mathrm{SO}_4\mathbb{C}$ of the Haar measure of $\mathrm{O}_4\mathbb{C}$ (up to a normalization factor), it is invariant by these outer automorphisms. It follows that $\tilde{L}$ satisfies $\tilde{L}(UPU^T) = U\tilde{L}(P)U^T$, $\forall U \in \mathrm{O}_4\mathbb{R}$, i.e., $\tilde{L}$ is an intertwining map for $\mathrm{O}_4\mathbb{C}$, not only $\mathrm{SO}_4\mathbb{C}$. On the other hand, α is alternating by outer-automorphisms, i.e. it satisfies $\alpha(UPU^T) = \det U\, U\alpha(P)U^T$. This is a consequence of (8.5) and of the fact that $\det(Uv_1, \ldots, Uv_4) = \det U\, \det(v_1, \ldots, v_4)$. Taking the conjugation of (8.6) by $U \in \mathrm{O}_4\mathbb{C} \setminus \mathrm{SO}_4\mathbb{C}$, we get $\tilde{L} = \frac{C_2^+ + C_2^-}{2}\, \mathrm{Id}_{\Lambda^2(V)} - (C_2^+ - C_2^-)\alpha$. Hence, (8.1) follows with $C_2 = C_2^+ = C_2^-$ and the proof can be completed like in Case (i). ∎

9 Proof of Lemma 4.9 (ii)

Like in the previous section, we show (4.16) for any class function g replacing M_{Id}. By the same method as in Section 7, the symmetric bilinear map B: $\mathcal{A}_n \times \mathcal{A}_n \to \mathcal{S}_n$ satisfies

$$B(UP_1U^T, UP_2U^T) = UB(P_1, P_2)U^T, \quad \forall P_1, P_2 \in \mathcal{A}_n, \quad \forall U \in \mathrm{SO}_n\mathbb{R}. \tag{9.1}$$

It can be extended by linearity to the complexifications of $\mathcal{A}_n$ and $\mathcal{S}_n$. These are respectively $\Lambda^2(V)$ and $\mathrm{Sym}^2(V)$ for $V = \mathbb{C}^n$. The extended symmetric bilinear map, denoted by $\breve{B}$: $\Lambda^2(V) \times \Lambda^2(V) \to \mathrm{Sym}^2(V)$, is given by

$$\breve{B}(P_1 + iQ_1, P_2 + iQ_2) = B(P_1, P_2) - B(Q_1, Q_2) + i(B(P_1, Q_2) + B(Q_1, P_2)),$$

for all P_1, P_2, Q_1, $Q_2 \in \mathcal{A}_n$. The extended map $\breve{B}$ still satisfies the invariance property (9.1), now with antisymmetric matrices P_1, P_2 with complex entries. Due to the universal

property of the symmetric product [42, Appendix B], $\breve{B}$ determines a unique linear map $\tilde{B}\colon \mathrm{Sym}^2(\Lambda^2(V)) \to \mathrm{Sym}^2(V)$ given by

$$\tilde{B}\big((v_1 \wedge v_2) \circ (w_1 \wedge w_2)\big) = \breve{B}(v_1 \wedge v_2, w_1 \wedge w_2), \quad \forall v_1,\, v_2,\, w_1,\, w_2 \in V.$$

Both $\mathrm{Sym}^2(\Lambda^2(V))$ and $\mathrm{Sym}^2(V)$ are complex representations of $\mathrm{SO}_n\mathbb{R}$. Furthermore, Eq. (9.1) implies that $\tilde{B}$ intertwins the two representations. So, we are led to find the decompositions of $\mathrm{Sym}^2(\Lambda^2(V))$ and $\mathrm{Sym}^2(V)$ into irreducible representations of $\mathrm{SO}_n\mathbb{R}$.

The decomposition of $\mathrm{Sym}^2(V) = \mathbb{S}_2(V)$ into irreducible representations of $\mathfrak{so}_n\mathbb{C}$ has been initiated in the proof of Lemma 7.1. What is missing is to find the supplementary representation(s) of $\mathbb{S}_{[2]}(V)$ (given by (7.5)) in $\mathrm{Sym}^2(V)$. Using [42, p. 263], we have:

$$\mathrm{Sym}^2(V) = \mathbb{S}_{[2]}(V) \oplus \mathbb{C}\Psi, \quad \Psi =: \sum_{i=1}^{n} e_i \circ e_i, \tag{9.2}$$

The projections $\mathcal{P}_0$ and $\mathcal{P}_1$ of $\mathbb{S}_2(V)$ on $\mathbb{C}\Psi$ and $\mathbb{S}_{[2]}(V)$ are respectively given by

$$\mathcal{P}_0(v \circ w) = \frac{1}{n}(v \cdot w)\Psi, \quad \mathcal{P}_1 = \mathrm{Id}_{\mathbb{S}_2(V)} - \mathcal{P}_0. \tag{9.3}$$

Indeed, using that $\Phi(\Psi) = 2n$, we verify that $\Phi \circ \mathcal{P}_1 = 0$ (where Φ is the contraction (7.4)), showing that $\mathrm{Im}\,\mathcal{P}_1 \subset \mathbb{S}_{[2]}(V)$. In terms of matrices, $\Psi = 2\,\mathrm{Id}$ and (9.3) corresponds to the decomposition

$$S = S_0 + S_1, \quad S_0 = S - \frac{1}{n}\mathrm{Tr}\,S\,\mathrm{Id}, \quad S_1 = \frac{1}{n}\mathrm{Tr}\,S\,\mathrm{Id},$$

of a complex symmetric matrix S into a trace-free symmetric matrix S_0 and a scalar matrix S_1. Of course, $\mathbb{S}_{[2]}(V)$ and $\mathbb{C}\Psi$ lift into complex irreducible representations of $\mathrm{SO}_n\mathbb{C}$ and thus of $\mathrm{SO}_n\mathbb{R}$

Now, we consider $\mathrm{Sym}^2(\Lambda^2(V))$ and first decompose it into irreducible representations of $\mathfrak{sl}_n\mathbb{C}$. We apply Pieri's formula [42, Exercise 6.16] and decompose

$$\mathrm{Sym}^2(\Lambda^2(V)) = \Lambda^4(V) \oplus \mathbb{S}_{(2,2)}(V), \tag{9.4}$$

Being Weyl modules, both are irreducible representations of $\mathfrak{sl}_n\mathbb{C}$. We have the following obvious formulas for the dimensions of $\mathrm{Sym}^2(\Lambda^2(V))$ and $\Lambda^4(V)$:

$$\begin{aligned} \dim \mathrm{Sym}^2(\Lambda^2(V)) &= \frac{1}{2}\frac{n(n-1)}{2}\Big(\frac{n(n-1)}{2}+1\Big) = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n), \\ \dim \Lambda^4(V) &= \binom{n}{4} = \frac{1}{24}(n^4 - 6n^3 + 11n^2 - 6n). \end{aligned}$$

On the other hand, using [42, Theorem 6.3(i)], we have

$$\dim \mathbb{S}_{(2,2)}(V) = \frac{1}{12}n^2(n^2-1), \tag{9.5}$$

and we verify that these formulas are consistent with (9.4).

The space $\mathbb{S}_{(2,2)}(V)$ in (9.4) can be characterized as

$$\mathbb{S}_{(2,2)}(V) = \mathrm{Span}\{(v_1 \wedge v_2) \circ (v_1 \wedge v_3) \mid v_1,\, v_2,\, v_3 \in V\}. \tag{9.6}$$

Indeed, we have a linear map F of $\mathfrak{sl}_n\mathbb{C}$-representations

$$F : \left\{ \begin{array}{ccc} \mathrm{Sym}^2(\Lambda^2(V)) & \to & \Lambda^4(V) \\ (v_1 \wedge v_2) \circ (v_3 \wedge v_4) & \mapsto & v_1 \wedge v_2 \wedge v_3 \wedge v_4, \end{array} \right.$$

which is clearly surjective. Its kernel $\mathrm{Ker}\, F$ is not $\{0\}$ (otherwise, we would have $\mathrm{Sym}^2(\Lambda^2(V)) \approx \Lambda^4(V)$ and this cannot be the case because the dimensions are different) and is a sub-representation of $\mathrm{Sym}^2(\Lambda^2(V))$. Thus, $\mathrm{Ker}\, F = \mathbb{S}_{(2,2)}(V)$. Clearly $F\big((v_1 \wedge v_2) \circ (v_1 \wedge v_3)\big) = 0$, $\forall v_1,\, v_2,\, v_3 \in V$. So, $\Sigma \subset \mathbb{S}_{(2,2)}(V)$, where Σ denotes the right-hand side of (9.6). The space Σ is a sub-representation of $\mathrm{Sym}^2(\Lambda^2(V))$. Finally, $\Sigma \neq \{0\}$ (take two independent vectors v_1 and v_2. Then, $(v_1 \wedge v_2) \circ (v_1 \wedge v_2)$ belongs to Σ and is not 0). Thus, we must have $\Sigma = \mathbb{S}_{(2,2)}(V)$.

Next, we find how an element of $\mathrm{Sym}^2(\Lambda^2(V))$ decomposes along (9.4). We define two endomorphisms $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathrm{Sym}^2(\Lambda^2(V))$ which commute with the action of $\mathfrak{sl}_n\mathbb{C}$. They are defined by their action on a generator $(v_1 \wedge v_2) \circ (v_3 \wedge v_4)$ according to

$$\begin{aligned} \mathcal{T}_1\big((v_1 \wedge v_2) \circ (v_3 \wedge v_4)\big) &= \frac{1}{3}\big[(v_1 \wedge v_2) \circ (v_3 \wedge v_4) - (v_1 \wedge v_3) \circ (v_2 \wedge v_4) \\ &\quad -(v_1 \wedge v_4) \circ (v_3 \wedge v_2)\big] = \frac{1}{3}\, v_1 \wedge v_2 \wedge v_3 \wedge v_4, \end{aligned} \tag{9.7}$$

$$\begin{aligned} \mathcal{T}_2\big((v_1 \wedge v_2) \circ (v_3 \wedge v_4)\big) &= \frac{1}{3}\big[(v_1 \wedge v_2) \circ (v_3 \wedge v_4) + (v_1 \wedge v_3) \circ (v_2 \wedge v_4)\big] \\ &\quad +\frac{1}{3}\big[(v_1 \wedge v_2) \circ (v_3 \wedge v_4) + (v_1 \wedge v_4) \circ (v_3 \wedge v_2)\big]. \end{aligned} \tag{9.8}$$

We verify that

$$\mathcal{T}_1 + \mathcal{T}_2 = \mathrm{Id}_{\mathrm{Sym}^2(\Lambda^2(V))}, \tag{9.9}$$

and that

$$\mathrm{Im}\, \mathcal{T}_1 = \Lambda^4(V), \quad \mathrm{Im}\, \mathcal{T}_2 = \mathbb{S}_{(2,2)}(V). \tag{9.10}$$

Indeed, this is clear for $\mathrm{Im}\, \mathcal{T}_1$. By expanding $((v_2+v_3) \wedge v_1) \circ ((v_2+v_3) \wedge v_4) \in \mathbb{S}_{(2,2)}(V)$, we get that the first bracket at the right-hand side of (9.8) is in $\mathbb{S}_{(2,2)}(V)$. We perform similarly for the second bracket with $((v_2 + v_4) \wedge v_1) \circ ((v_2 + v_4) \wedge v_3)$, so that $\mathrm{Im}\, \mathcal{T}_2 \subset \mathbb{S}_{(2,2)}(V)$. Because of (9.9) and (9.4), this inclusion is an equality. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ are the projections of $\mathrm{Sym}^2(\Lambda^2(V))$ on $\Lambda^4(V)$ and $\mathbb{S}_{(2,2)}(V)$ respectively.

Now, $\Lambda^4(V)$ is an irreducible representation of $\mathfrak{so}_n\mathbb{C}$ for $n \geq 9$ [42, Theorems 19.2 and 19.14]. From now on, we assume $n \geq 9$ and defer the examination of the special cases $n \in \{3, \ldots, 8\}$ to the end of the proof.

(i) Case $n \geq 9$. By contrast to $\Lambda^4(V)$, $\mathbb{S}_{(2,2)}(V)$ is not an irreducible representation of $\mathfrak{so}_n\mathbb{C}$ and we apply Weyl's construction using the contractions (see Section 6 or [42, §19.5]) to decompose it in irreducible representations. There are six pairs of indices for a rank-4 tensor. For an element of $\mathbb{S}_{(2,2)}(V) \subset \mathrm{Sym}^2(\Lambda^2(V))$, contractions with respect to

pairs $(1,2)$ and $(3,4)$ obviously lead to 0. The remaining four contractions all give rise, up to a sign, to the same linear map G of $\mathfrak{so}_n\mathbb{C}$-representations:

$$G:\left\{\begin{array}{lcl}\mathbb{S}_{(2,2)}(V) & \to & \mathrm{Sym}^2(V)\\ (z\wedge v)\circ(z\wedge w) & \mapsto & (z\cdot z)\,v\circ w+(v\cdot w)\,z\circ z-(z\cdot w)\,z\circ v-(z\cdot v)\,z\circ w.\end{array}\right.$$

By Weyl's construction, $\mathbb{S}_{[2,2]}(V)=\mathrm{Ker}\,G$ is an irreducible representation of $\mathfrak{so}_n\mathbb{C}$. The map G is surjective. Indeed, we check that

$$G\big((e_i\wedge e_j)\circ(e_i\wedge e_k)\big)=e_j\circ e_k+\delta_{jk}\,e_i\circ e_i,$$

for all i, j, k generate $\mathrm{Sym}^2(V)$. With (9.2), this allows us to write

$$\mathbb{S}_{(2,2)}(V)\approx\mathbb{S}_{[2,2]}(V)\oplus\mathrm{Sym}^2(V)\approx\mathbb{S}_{[2,2]}(V)\oplus\mathbb{S}_{[2]}(V)\oplus\mathbb{C}. \tag{9.11}$$

We now need to identify the sub-representations Σ_0 and Σ_1 of $\mathbb{S}_{(2,2)}(V)$ which are isomorphic to $\mathbb{C}$ and $\mathbb{S}_{[2]}(V)$ respectively and write how a generator of $\mathbb{S}_{(2,2)}(V)$ decomposes along (9.11). We define two linear endomorphisms $\mathcal{W}_0$ and $\mathcal{W}_1$ of $\mathbb{S}_{(2,2)}(V)$ by

$$\mathcal{W}_0\big((z\wedge v)\circ(z\wedge w)\big)=\frac{2}{n(n-1)}\big[(z\cdot z)(v\cdot w)-(z\cdot v)(z\cdot w)\big]\,\Xi, \tag{9.12}$$

with

$$\Xi=\sum_{i<j}(e_i\wedge e_j)\circ(e_i\wedge e_j)=\frac{1}{2}\sum_{i,\,j}(e_i\wedge e_j)\circ(e_i\wedge e_j),$$

and

$$\begin{aligned}\mathcal{W}_1\big((z\wedge v)\circ(z\wedge w)\big)=\frac{1}{n-2}\Big\{&(z\cdot z)\Big(\sum_{i=1}^n(e_i\wedge v)\circ(e_i\wedge w)-\frac{2}{n}(v\cdot w)\,\Xi\Big)\\ &+(v\cdot w)\Big(\sum_{i=1}^n(e_i\wedge z)\circ(e_i\wedge z)-\frac{2}{n}(z\cdot z)\,\Xi\Big)\\ &-(z\cdot v)\Big(\sum_{i=1}^n(e_i\wedge z)\circ(e_i\wedge w)-\frac{2}{n}(z\cdot w)\,\Xi\Big)\\ &-(z\cdot w)\Big(\sum_{i=1}^n(e_i\wedge z)\circ(e_i\wedge v)-\frac{2}{n}(z\cdot v)\,\Xi\Big)\Big\}.\end{aligned} \tag{9.13}$$

These endomorphisms commute with the action of $\mathfrak{so}_n\mathbb{C}$. Indeed, it is at the heart of Weyl's construction to remark that the operation consisting of the insertion of Ψ (see (9.3)) at any pair of positions inside a tensor of rank $d-2$ leading to a tensor of rank d commutes with the action of $\mathfrak{so}_n\mathbb{C}$. The tensors $\sum_i(e_i\wedge v)\circ(e_i\wedge w)$ or Ψ are obtained in this way.

Now, We check that $G(\Xi)=(n-1)\Psi$ and

$$G\Big(\sum_{i=1}^n(e_i\wedge v)\circ(e_i\wedge w)\Big)=(n-2)v\circ w+(v\cdot w)\Psi,\quad\forall v,\,w\in V,$$

so that:

$$(G\circ\mathcal{W}_0)\big((z\wedge v)\circ(z\wedge w)\big)=\frac{2}{n}\big[(z\cdot z)(v\cdot w)-(z\cdot v)(z\cdot w)\big]\,\Psi, \tag{9.14}$$

and

$$G\circ(\mathcal{W}_0+\mathcal{W}_1)=G. \tag{9.15}$$

It follows that

$$\mathrm{Im}\Big(\mathrm{Id}_{\mathbb{S}_{(2,2)}(V)}-\mathcal{W}_0-\mathcal{W}_1\Big)\subset\mathbb{S}_{[2,2]}(V). \tag{9.16}$$

Now, $\mathrm{Im}\,G\circ\mathcal{W}_0$ and $\mathrm{Im}\,G\circ\mathcal{W}_1$ are sub-representations of $\mathrm{Sym}^2(V)$ and, owing to the fact that the bracket at the right-hand side of (9.14) can be non-zero, $\mathrm{Im}\,G\circ\mathcal{W}_0=\mathbb{C}\Psi$. Now, $\mathrm{Im}\,G\circ\mathcal{W}_1\neq\{0\}$. Otherwise, from (9.15), $\mathrm{Im}\,G=\mathrm{Im}\,G\circ\mathcal{W}_0=\mathbb{C}\Psi$ which contradicts the fact that G is surjective on $\mathrm{Sym}^2(V)$. Let Φ be the contraction (7.4). We have $\Phi(\Psi)=2n$, from which we deduce that

$$(\Phi\circ G\circ\mathcal{W}_0)\big((z\wedge v)\circ(z\wedge w)\big)=4\big[(z\cdot z)(v\cdot w)-(z\cdot v)(z\cdot w)\big],$$

and thus

$$\Phi\circ G\circ\mathcal{W}_0=\Phi\circ G,\qquad \Phi\circ G\circ\mathcal{W}_1=0. \tag{9.17}$$

So, $\mathrm{Im}\,G\circ\mathcal{W}_1$ is a sub-representation of $\mathbb{S}_{[2]}(V)$. But $\mathbb{S}_{[2]}(V)$ is irreducible and $\mathrm{Im}\,G\circ\mathcal{W}_1\neq\{0\}$, thus $\mathrm{Im}\,G\circ\mathcal{W}_1=\mathbb{S}_{[2]}(V)$.

We define the following sub-representations of $\mathbb{S}_{(2,2)}(V)$:

$$\Sigma_0=\mathrm{Im}\,\mathcal{W}_0,\qquad \Sigma_1=\mathrm{Im}\,\mathcal{W}_1.$$

From (9.12), (9.13), we have:

$$\begin{aligned}\Sigma_0 &= \mathbb{C}\,\Xi,\\ \Sigma_1 &= \mathrm{Span}\Big\{\sum_{i=1}^{n}(e_i\wedge v)\circ(e_i\wedge w)-\frac{2}{n}(v\cdot w)\Xi \quad\Big|\quad v,\,w\in V\Big\}.\end{aligned} \tag{9.18}$$

From (9.16), we have

$$\mathbb{S}_{(2,2)}(V)=\Sigma_0+\Sigma_1+\mathbb{S}_{[2,2]}(V). \tag{9.19}$$

All the spaces at the right-hand side of (9.19) are sub-representations of $\mathbb{S}_{(2,2)}(V)$ and we know that Σ_0 and $\mathbb{S}_{[2,2]}(V)$ are irreducible. Furthermore, according to [42, Formulas 24.29 and 24.41], we have

$$\dim\mathbb{S}_{[2,2]}(V)=\frac{1}{12}(n^4-7n^2-6n). \tag{9.20}$$

We deduce that Σ_0 and $\mathbb{S}_{[2,2]}(V)$ are not isomorphic, since $\dim\Sigma_0=1$ is different from $\dim\mathbb{S}_{[2,2]}(V)$. This implies that $\Sigma_0\cap\mathbb{S}_{[2,2]}(V)=\{0\}$. Suppose now $\Sigma_1\cap\mathbb{S}_{[2,2]}(V)\neq\{0\}$. Then, $\Sigma_1\cap\mathbb{S}_{[2,2]}(V)=\mathbb{S}_{[2,2]}(V)$, i.e. $\mathbb{S}_{[2,2]}(V)\subset\Sigma_1$. But $\dim\Sigma_1\leq\frac{n(n+1)}{2}$, while, $\dim\mathbb{S}_{[2,2]}(V)$ is given by (9.20). We easily check that for $n\geq 5$, we cannot have $\dim\mathbb{S}_{[2,2]}(V)\leq\frac{n(n+1)}{2}$ (the special cases of small dimension will be examined later). So, we must have $\Sigma_1\cap\mathbb{S}_{[2,2]}(V)=\{0\}$. Finally, assume that $\lambda\Xi\in\Sigma_1$. Thanks to (9.17), we have $(\Phi\circ G)(\lambda\Xi)=0$ on the one hand and $(\Phi\circ G)(\lambda\Xi)=2n(n-1)\lambda$. This implies $\lambda=0$

and consequently, $\Sigma_0 \cap \Sigma_1 = \{0\}$. It follows that the sum (9.19) is a direct sum and that we have

$$\mathbb{S}_{(2,2)}(V) = \Sigma_0 \oplus \Sigma_1 \oplus \mathbb{S}_{[2,2]}(V). \tag{9.21}$$

Additionally, in view of (9.11), $\Sigma_1 \approx \mathbb{S}_{[2]}(V)$. In view of (9.16), $\mathcal{W}_0$, $\mathcal{W}_1$ and $1-(\mathcal{W}_0+\mathcal{W}_1)$ are the projections associated to this direct sum, respectively on the spaces Σ_0, Σ_1 and $\mathbb{S}_{[2,2]}(V)$. One last remark is that the description of Σ_1 can be slightly simplified from (9.18). We have

$$\Sigma_1 = \mathrm{Span}\Big\{ \sum_{i=1}^{n} (e_i \wedge v) \circ (e_i \wedge w) \quad \Big| \quad v,\, w \in V \text{ such that } v \cdot w = 0 \Big\}. \tag{9.22}$$

Indeed, the right-hand side of (9.22) is a sub-representation of Σ_1 which is obviously not $\{0\}$ and since Σ_1 is irreducible, the two spaces must be equal. With (9.22), the isomorphism $\mathcal{J}_1$ between $\mathbb{S}_{[2]}(V)$ and Σ_1 is simply given by

$$\mathcal{J}_1(v \circ w) = \sum_{i=1}^{n} (e_i \wedge v) \circ (e_i \wedge w), \quad \forall v,\, w \in V \text{ such that } v \cdot w = 0. \tag{9.23}$$

Obviously, the isomorphism $\mathcal{J}_0$ between $\mathbb{C}\Psi$ and Σ_0 is given by

$$\mathcal{J}_0(\lambda\Psi) = \lambda\Xi, \quad \forall \lambda \in \mathbb{C}. \tag{9.24}$$

Finally, with $\dim \Sigma_0 = 1$ and $\dim \Sigma_1 = \frac{n(n+1)}{2} - 1$, (9.20) and (9.5), we check the consistency of the dimensions with (9.21).

Collecting (9.4) and (9.21), we get the decomposition of $\mathrm{Sym}^2(\Lambda^2(V))$:

$$\mathrm{Sym}^2(\Lambda^2(V)) = \Sigma_0 \oplus \Sigma_1 \oplus \mathbb{S}_{[2,2]}(V) \oplus \Lambda^4(V), \tag{9.25}$$

and we denote by $\mathcal{Q}_0 \ldots, \mathcal{Q}_3$, the projections of $\mathrm{Sym}^2(\Lambda^2(V))$ on Σ_0, Σ_1, $\mathbb{S}_{[2,2]}(V)$ and $\Lambda^4(V)$ respectively. In particular, we have

$$\mathcal{Q}_0 = \mathcal{W}_0 \circ \mathcal{T}_2, \quad \mathcal{Q}_1 = \mathcal{W}_1 \circ \mathcal{T}_2, \tag{9.26}$$

where we have restricted the codomain of $\mathcal{T}_2$ to $\mathbb{S}_{(2,2)}(V)$. In the case $n \geq 9$, (9.25) is the decomposition of $\mathrm{Sym}^2(\Lambda^2(V))$ into irreducible representations of $\mathfrak{so}_n\mathbb{C}$. Since all these representations are of the form $\mathbb{S}_{[\lambda]}(V)$ for an appropriate partition λ of 4, they all lift into complex irreducible representations of $\mathrm{SO}_n\mathbb{C}$ and thus, of $\mathrm{SO}_n\mathbb{R}$.

Now, we express how $\mathcal{Q}_0$ and $\mathcal{Q}_1$ act on a generator of $\mathrm{Sym}^2(\Lambda^2(V))$. Inserting (9.8), (9.12) and (9.13) into (9.26), we find:

$$\mathcal{Q}_0\big((v_1 \wedge v_2) \circ (v_3 \wedge v_4)\big) = \frac{2}{n(n-1)} \big[(v_1 \cdot v_3)(v_2 \cdot v_4) - (v_1 \cdot v_4)(v_2 \cdot v_3)\big]\, \Xi, \tag{9.27}$$

and

$$\mathcal{Q}_1\big((v_1\wedge v_2)\circ(v_3\wedge v_4)\big)=\frac{1}{n-2}\Big\{(v_1\cdot v_3)\Big(\sum_{i=1}^n(e_i\wedge v_2)\circ(e_i\wedge v_4)-\frac{2}{n}(v_2\cdot v_4)\,\Xi\Big)$$
$$+(v_2\cdot v_4)\Big(\sum_{i=1}^n(e_i\wedge v_1)\circ(e_i\wedge v_3)-\frac{2}{n}(v_1\cdot v_3)\,\Xi\Big)$$
$$-(v_1\cdot v_4)\Big(\sum_{i=1}^n(e_i\wedge v_2)\circ(e_i\wedge v_3)-\frac{2}{n}(v_2\cdot v_3)\,\Xi\Big)$$
$$-(v_2\cdot v_3)\Big(\sum_{i=1}^n(e_i\wedge v_1)\circ(e_i\wedge v_4)-\frac{2}{n}(v_1\cdot v_4)\,\Xi\Big)\Big\}. \tag{9.28}$$

Then, let P, $Q\in\Lambda^2(V)$ and write their decomposition in the basis $\{e_i\wedge e_j\}_{i<j}$ as follows:

$$P=\sum_{1\le i<j\le n}P_{ij}\,e_i\wedge e_j=\frac{1}{2}\sum_{i,j=1}^n P_{ij}\,e_i\wedge e_j,$$

where $P_{ji}=-P_{ij}$ for $i>j$ and $P_{ii}=0$, and similarly for Q. We identify P and Q with the matrices $(P_{ij})_{i,j=1}^n$ and $(Q_{ij})_{i,j=1}^n$. Application of (9.27) and (9.28) lead to

$$\mathcal{Q}_0\big(P\circ Q\big)=-\frac{1}{n(n-1)}\,\mathrm{Tr}(PQ)\,\Xi, \tag{9.29}$$

and

$$\mathcal{Q}_1\big(P\circ Q\big)=-\frac{1}{n-2}\Big\{\sum_{i,j=1}^n(PQ)_{ij}\sum_{m=1}^n(e_m\wedge e_i)\circ(e_m\wedge e_j)-\frac{2}{n}\,\mathrm{Tr}(PQ)\,\Xi\Big\}$$
$$=-\frac{1}{n-2}\sum_{i,j=1}^n\Big(\frac{PQ+QP}{2}-\frac{1}{n}\mathrm{Tr}(PQ)\,\mathrm{Id}\Big)_{ij}\sum_{m=1}^n(e_m\wedge e_i)\circ(e_m\wedge e_j), \tag{9.30}$$

with PQ is the matrix product of P and Q, $\mathrm{Tr}(PQ)$, its trace and $(PQ)_{ij}$, its (i,j)-th entry.

Now, the map $\tilde{B}$ can be decomposed by blocks using (9.25) on the domain and (9.2) on the codomain. Applying the same arguments as in Section 8, Case (ii), there exist two constants C_3', $C_4'\in\mathbb{C}$ such that

$$\tilde{B}=C_3'\,\mathcal{J}_0^{-1}\circ\mathcal{Q}_0+C_4'\,\mathcal{J}_1^{-1}\circ\mathcal{Q}_1, \tag{9.31}$$

i.e., inserting (9.23), (9.24), (9.29), (9.30) into (9.31), there exist two constants C_3, $C_4\in\mathbb{C}$ such that (4.16) holds with B replaced by $\breve{B}$ and any $P,Q\in\Lambda^2(V)$. Now, if we take P and Q real, i.e. belonging to $\mathcal{A}_n$, then, $\breve{B}(P,Q)=B(P,Q)$ is real which implies that C_3, $C_4\in\mathbb{R}$ and finishes the proof of (4.16).

We now show (4.17) and (4.18). We have

$$\mathcal{P}_0\circ B(P,Q)\;=\;\int_{\mathrm{SO}_n\mathbb{R}}(A\cdot P)\,(A\cdot Q)\,\frac{1}{n}\,\mathrm{Tr}A\,g(A)\,dA\,\mathrm{Id}, \tag{9.32}$$

$$\mathcal{P}_1\circ B(P,Q)\;=\;\int_{\mathrm{SO}_n\mathbb{R}}(A\cdot P)\,(A\cdot Q)\,\Big(\frac{A+A^T}{2}-\frac{1}{n}\,\mathrm{Tr}A\,\mathrm{Id}\Big)\,g(A)\,dA. \tag{9.33}$$

Comparing (9.32) with (4.16), we get

$$C_3 \operatorname{Tr}(PQ) = \frac{1}{n} \int_{\mathrm{SO}_n\mathbb{R}} (A \cdot P)\,(A \cdot Q) \operatorname{Tr}A\, g(A)\, dA, \quad \forall P,\, Q \in \mathcal{A}_n.$$

Taking

$$P = e_i \wedge e_j,\ Q = e_k \wedge e_\ell, \quad \forall i,\, j,\, k,\, \ell \in \{1, \ldots, n\}, \quad i \neq j \text{ and } k \neq \ell, \tag{9.34}$$

we get

$$2C_3\,(\delta_{jk}\,\delta_{i\ell} - \delta_{j\ell}\delta_{ik}) = \frac{1}{n} \int_{\mathrm{SO}_n\mathbb{R}} (A_{ij} - A_{ji})(A_{k\ell} - A_{\ell k}) \operatorname{Tr}A\, g(A)\, dA.$$

we note that both sides are zero if $i = j$ or $k = \ell$ so the equality is valid for any $i,\, j,\, k,\, \ell \in \{1, \ldots, n\}$. Taking $k = j$ and $\ell = i$, and summing over all $i,\, j \in \{1, \ldots, n\}$, we get

$$C_3 = -\frac{1}{2n(n^2 - n)} \int_{\mathrm{SO}_n\mathbb{R}} \sum_{i,j=1}^{n} (A_{ij} - A_{ji})^2 \operatorname{Tr}A\, g(A)\, dA. \tag{9.35}$$

inserting (8.2) into (9.35) and making $g = M_{\mathrm{Id}}$ leads to (4.17).

Now, comparing (9.33) with (4.16), we get

$$\begin{aligned} C_4 \Big(\frac{PQ + QP}{2} - \frac{1}{n}\operatorname{Tr}(PQ)\,\mathrm{Id}\Big) &= \\ = \int_{\mathrm{SO}_n\mathbb{R}} (A \cdot P)\,(A \cdot Q) \Big(\frac{A + A^T}{2} - \frac{1}{n} \operatorname{Tr}A\,\mathrm{Id}\Big)\, g(A)\, dA, &\quad \forall P,\, Q \in \mathcal{A}_n. \end{aligned} \tag{9.36}$$

Again, taking P and Q as in (9.34) and looking at the (p, q)-th entry of the matrix equation (9.36), we get

$$\begin{aligned} \Big\{ \frac{1}{2}\Big(&\delta_{jk}\,(\delta_{ip}\,\delta_{\ell q} + \delta_{iq}\,\delta_{\ell p}) + \delta_{i\ell}\,(\delta_{jp}\,\delta_{kq} + \delta_{jq}\,\delta_{kp}) - \delta_{j\ell}\,(\delta_{ip}\,\delta_{kq} + \delta_{iq}\,\delta_{kp}) \\ &-\delta_{ik}\,(\delta_{jp}\,\delta_{\ell q} + \delta_{jq}\,\delta_{\ell p})\Big) - \frac{2}{n}\,(\delta_{jk}\,\delta_{i\ell} - \delta_{j\ell}\,\delta_{ij})\,\delta_{pq} \Big\}\, C_4 \\ &= \int_{\mathrm{SO}_n\mathbb{R}} (A_{ij} - A_{ji})(A_{k\ell} - A_{\ell k}) \Big(\frac{A_{pq} + A_{qp}}{2} - \frac{1}{n} \operatorname{Tr}A\,\delta_{pq}\Big)\, g(A)\, dA. \end{aligned}$$

We note that both sides of this equation are equal to 0 if $i = j$ or $k = \ell$, so that it is valid for any integers $i,\, j,\, k,\, \ell,\, p,\, q \in \{1, \ldots, n\}$. Then, making $k = j$, $p = \ell$ and $q = i$ and summing over $i,\, j,\, \ell$, we get

$$\frac{(n-1)(n^2-4)}{2} C_4 = \int_{\mathrm{SO}_n\mathbb{R}} \sum_{i,\,j,\,\ell} (A_{ij} - A_{ji})\,(A_{j\ell} - A_{\ell j}) \Big(\frac{A_{\ell i} + A_{i\ell}}{2} - \frac{1}{n} \operatorname{Tr}A\,\delta_{\ell i}\Big)\, g(A)\, dA.$$

Then, the sum inside the integral is equal to $\operatorname{Tr}A^3 - \operatorname{Tr}A\left(\frac{2}{n}\operatorname{Tr}A^2 - 1\right)$, which leads to (4.18) by making $g = M_{\mathrm{Id}}$.

(ii) cases $n \in \{3, \ldots, 8\}$.

(ii)-a Case $n = 3$. This is the situation studied in [24, 26, 29, 31, 30]. With $V = \mathbb{C}^3$, it is classical that $\Lambda^4(V) = \{0\}$ and we also have $\mathbb{S}_{[2,2]}(V) = \{0\}$ [42, Exercise 19.20]. Furthermore, we have $\Lambda^2(V) \approx V$ through the isomorphism α: $\Lambda^2(V) \to V$, $v \wedge w \mapsto \alpha(v \wedge w)$ such that $(\alpha(v \wedge w) \cdot z) = \det(u, v, z)$, $\forall v,\, w,\, z \in V$. In other words, $\alpha(v \wedge w) = v \times w$ is the vector product of v and w. Through this isomorphism, (9.26) becomes equivalent to (9.2). Apart from simplifications to the computations, the stream of the proof and the final result remain identical.

(ii)-b Case $n = 4$. With $V = \mathbb{C}^4$, we have $\Lambda^4(V) = \mathbb{C}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \approx \mathbb{C}$. Therefore, in the decomposition (9.25), there are now two factors which are isomorphic to $\mathbb{C}$: $\Lambda^4(\mathbb{C}^4)$ and Σ_0. So, when applying Schur's Lemma, we have a third term in (9.31) and $\tilde{B}$ is now written:

$$\tilde{B} = C_3' \, \mathcal{J}_0^{-1} \circ \mathcal{Q}_0 + C_4' \, \mathcal{J}_1^{-1} \circ \mathcal{Q}_1 + C_5' \, \mathcal{J}_3^{-1} \circ \mathcal{T}_1,$$

where the projection $\mathcal{T}_1$: $\mathrm{Sym}^2(\Lambda^2(V)) \to \Lambda^4(V)$ is defined by (9.7) and $\mathcal{J}_3$: $\mathbb{C}\,\Psi \to \mathbb{C}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$, $\lambda\,\Psi \mapsto \lambda\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is a mapping between one-dimensional trivial representations. Introducing the automorphism α of $\Lambda^2(V)$ defined in Section 8, Case (ii), we note that $\mathcal{T}_1$ takes the form

$$\begin{aligned} \mathcal{T}_1\big((v_1 \wedge v_2) \circ (v_3 \wedge v_4)\big) &= \frac{1}{3} \det(v_1, v_2, v_3, v_4)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\ &= \frac{1}{3} \big(\alpha(v_1 \wedge v_2) \cdot (v_3 \wedge v_4)\big)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4. \end{aligned}$$

Thus, for two generic elements P, Q of $\Lambda^2(V)$, we have

$$\mathcal{T}_1(P \circ Q) = \frac{1}{3} \big(\alpha(P) \cdot Q\big)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

Using the same methodology as in the generic case, it follows that there exist constants C_3, C_4, C_5 in $\mathbb{C}$ such that

$$\tilde{B}(P \circ Q) = C_3 \,\mathrm{Tr}\,(PQ)\,\mathrm{Id} + C_4 \Big(\frac{PQ + QP}{2} - \frac{1}{4}\mathrm{Tr}\,(PQ)\,\mathrm{Id}\Big) + C_5 \big(\alpha(P) \cdot Q\big)\,\mathrm{Id}. \quad (9.37)$$

Now, we use the same methodology as in Section 8, Case (ii). We assume that g is invariant by all outer automorphisms defined by the conjugation with an element $U \in \mathrm{O}_4\mathbb{C} \setminus \mathrm{SO}_4\mathbb{C}$. It follows that $\tilde{B}$ is invariant by these outer automorphisms. On the other hand, α is alternating by these automorphisms. Applying such an outer automorphism to (9.37), we conclude that $C_5 = 0$ and the proof can be completed like in the generic case.

(ii)-c Cases $n = 5$, 6, 7. We recall that for $p < n$ and $V = \mathbb{C}^n$, we have $\Lambda^p(V) \approx \Lambda^{n-p}(V)$ thanks to the isomorphism

$$\alpha : \left\{ \begin{array}{l} \Lambda^p(V) \to \Lambda^{n-p}(V), \\ v_1 \wedge \ldots \wedge v_p \to \alpha(v_1 \wedge \ldots \wedge v_p), \end{array} \right.$$

such that

$$(\alpha(v_1 \wedge \ldots \wedge v_p) \cdot v_{p+1} \wedge \ldots \wedge v_n) = \det(v_1, \ldots, v_p, v_{p+1}, \ldots, v_n),\ \forall (v_1, \ldots, v_n) \in V^n.$$

The inner product at the left hand side is induced by that of V onto $V^{\otimes p}$ and its subspaces (such as $\Lambda^p(V)$) by $(v_1\otimes\ldots\otimes v_p\cdot w_1\otimes\ldots\otimes w_p)=(v_1\cdot w_1)\ldots(v_p\cdot w_p)$. This isomorphism is an isomorphism of representations of $\mathfrak{so}_n\mathbb{C}$. Consequently, we have $\Lambda^4(V)\approx\Lambda^1(V)=V$ if $n=5$, $\Lambda^4(V)\approx\Lambda^2(V)$ if $n=6$ and $\Lambda^4(V)\approx\Lambda^3(V)$ if $n=7$. All these spaces are irreducible representations of $\mathfrak{so}_n\mathbb{C}$ for $n=5$, 6, 7 respectively [42, Theorem 19.2 (i) and 19.14], and none of them is isomorphic to either $\mathbb{C}$ or $\mathbb{S}_{[2]}$. Thus the proof and conclusion of the generic case $n\geq 9$ apply to these three cases as well and the final result is identical.

(ii)-d Case $n=8$. The space $\Lambda^4(V)$ with $V=\mathbb{C}^8$ is not an irreducible representation of $\mathfrak{so}_8\mathbb{C}$ but it decomposes into two non-isomorphic sub-representations $\Pi_\pm$ of equal dimensions equal to 35 (with $\dim\Lambda^4(V)=\binom{8}{4}=70$). The representation $\Sigma_1\approx\mathbb{S}_{[2]}$ has also dimension 35. However, neither Π_+ nor Π_- is isomorphic to $\mathbb{S}_{[2]}$ (they have different heighest weight see [42]) so, it does not alter the fact that $\tilde{B}$ is given by (9.31) and the proof can be concluded like in the generic case. This ends the proof of Lemma 4.9 (ii). ∎

Remark 9.1. *Following Remark 4.2, we sketch a proof of Lemma 4.9 (ii) relying on elementary algebra only. The proof has three steps:*

1. *The goal is to compute $B(P,Q)\cdot S$ for $P,Q\in\mathcal{A}_n$ and $S\in\mathcal{S}_n$. By the invariance by conjugation* (9.1) *and the spectral theorem, one can take for S a diagonal matrix.*

2. *By linearity, it remains to compute the quantities $B(\alpha_{ij},\alpha_{k\ell})\cdot\sigma_m$ for $i,j,k,\ell,m\in\{1,\ldots,n\}$, $i<j$, $k<\ell$ and where $\alpha_{ij}:=e_i\wedge e_j$, $\sigma_m:=e_m\otimes e_m$.*

3. *Using the invariance by conjugation with the changes of variable described in [24, Definition 3.1], it can be proved that, at least when n is large enough, $B(\alpha_{ij},\alpha_{k\ell})\cdot\sigma_m=0$ if $(i,j)\neq(k,\ell)$ and there exist $\lambda,\mu\in\mathbb{R}$ such that $B(\alpha_{ij},\alpha_{ij})\cdot\sigma_i=B(\alpha_{ij},\alpha_{ij})\cdot\sigma_j=\lambda$ for all $i<j$ and $B(\alpha_{ij},\alpha_{ij})\cdot\sigma_m=\mu$ for all $i<j$ and $m\neq i,j$. The result then follows by a direct computation.*

The third step is quite tedious, especially for small dimensions, and does not explicitly use the underlying algebraic structure of the problem as in the main proof presented here. Moreover, representation theory and specifically Weyl's integration formula give explicit expressions for the coefficients (see Lemma 4.11). Finally, unlike the approach of [24], the proof presented in this article does not crucially use the fact that $\mathrm{SO}_n\mathbb{R}$ is a matrix group and may therefore be more easily generalized to other Lie groups.

10 Proof of Proposition 3.2 and dimension $n=3$ case

Proof of Proposition 3.2 From (3.7), we deduce (3.20) with $\mathbb{A}$ replaced by $\tilde{\mathbb{A}}$ defined by

$$\tilde{\mathbb{A}}\Theta=-\big((\nabla_x\wedge\Omega_1)\Theta+(\Omega_1\cdot\nabla_x)\Theta\big).$$

We now show that $\tilde{\mathbb{A}} = \mathbb{A}$ with $\mathbb{A}$ given by (3.17). On the one hand, we have:

$$\begin{aligned}(\tilde{\mathbb{A}}\Theta)_{ij} &= -\sum_{k=1}^{n}\left(\partial_{x_i}(\Omega_1)_k - \partial_{x_k}(\Omega_1)_i\right)\Theta_{kj} - (\Omega_1\cdot\nabla_x)\Theta_{ij}\\ &= -\sum_{k=1}^{n}\left(\partial_{x_i}(\Omega_1)_k - \partial_{x_k}(\Omega_1)_i\right)(\Omega_j)_k - (\Omega_1\cdot\nabla_x)(\Omega_j)_i\\ &= -\sum_{k=1}^{n}(\Omega_j)_k\partial_{x_i}(\Omega_1)_k + (\Omega_j\cdot\nabla_x)(\Omega_1)_i - (\Omega_1\cdot\nabla_x)(\Omega_j)_i.\end{aligned}$$

On the other hand, from (3.17), we have

$$\mathbb{A}_{ik} = \sum_{p,q=1}^{n}\Delta_{1pq}\Theta_{ip}\Theta_{kq},$$

so that, using (3.15) and (3.16), we have

$$\begin{aligned}(\mathbb{A}\Theta)_{ij} &= \sum_{k=1}^{n}\mathbb{A}_{ik}\Theta_{kj} = \sum_{k=1}^{n}\Delta_{1kj}\Theta_{ik} = \sum_{k=1}^{n}\Delta_{1kj}(\Omega_k)_i\\ &= \sum_{k=1}^{n}\Big(\big((\Omega_1\cdot\nabla_x)\Omega_k\big)\cdot\Omega_j + \big((\Omega_k\cdot\nabla_x)\Omega_j\big)\cdot\Omega_1 + \big((\Omega_j\cdot\nabla_x)\Omega_1\big)\cdot\Omega_k\Big)(\Omega_k)_i.\end{aligned}$$

The last term is equal to $(\Omega_j\cdot\nabla_x)(\Omega_1)_i$. For the first term, since $\Omega_k\cdot\Omega_j = \delta_{ij}$ is independent of x, we have

$$\sum_{k=1}^{n}(\Omega_k)_i\left((\Omega_1\cdot\nabla_x)\Omega_k\right)\cdot\Omega_j = -\sum_{k=1}^{n}(\Omega_k)_i\big((\Omega_1\cdot\nabla_x)\Omega_j\big)\cdot\Omega_k = -(\Omega_1\cdot\nabla_x)(\Omega_j)_i.$$

Similarly, for the second term, we have

$$\sum_{k=1}^{n}(\Omega_k)_i\left((\Omega_k\cdot\nabla_x)\Omega_j\right)\cdot\Omega_1 = -\sum_{k=1}^{n}(\Omega_k)_i\big((\Omega_k\cdot\nabla_x)\Omega_1\big)\cdot\Omega_j = -\sum_{k=1}^{n}(\Omega_j)_k\partial_{x_i}(\Omega_1)_k.$$

Thus, we get $\mathbb{A}\Theta = \tilde{\mathbb{A}}\Theta$ and since Θ is invertible, $\mathbb{A} = \tilde{\mathbb{A}}$, which ends the proof. ∎

Dimension $n = 3$: In [26], the constants (which we will temporarily call $\tilde{c}_1, \ldots, \tilde{c}_4$) were

given by

$$\tilde{c}_1 = \frac{1}{3}\,\frac{\displaystyle\int_0^{2\pi} (1+2\cos\theta)\,\exp\big(\kappa(1+2\cos\theta)\big)\,\sin^2\big(\frac{\theta}{2}\big)\,d\theta}{\displaystyle\int_0^{2\pi} \exp\big(\kappa\,(1+2\cos\theta)\,\big)\,\sin^2\big(\frac{\theta}{2}\big)\,d\theta}, \tag{10.1}$$

$$\tilde{c}_2 - \tilde{c}_4 = \frac{1}{5}\,\frac{\displaystyle\int_0^{2\pi} (2+3\cos\theta)\,\exp\big(\kappa(1+2\cos\theta)\big)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big)\,d\theta}{\displaystyle\int_0^{2\pi} \exp\big(\kappa(1+2\cos\theta)\big)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big)\,d\theta}, \tag{10.2}$$

$$\tilde{c}_3 = \frac{1}{2\kappa}, \tag{10.3}$$

$$\tilde{c}_4 = \frac{1}{5}\,\frac{\displaystyle\int_0^{2\pi} (1-\cos\theta)\,\exp\big(\kappa(1+2\cos\theta)\big)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big)\,d\theta}{\displaystyle\int_0^{2\pi} \exp\big(\kappa(1+2\cos\theta)\big)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big)\,d\theta}, \tag{10.4}$$

where we have modified the expressions from [26] to take into account that in [26] the matrix inner product involved a factor $1/2$ inside the trace. This modification multiplies by a factor 2 the expressions inside the exponentials compared with [26]. We now check that $\tilde{c}_i = c_i$ where c_i are the constants (3.11)-(3.14) in dimension $n = 3$

First, by comparing (3.13) and (10.3), we see that $\tilde{c}_3 = c_3$. We readily notice that Eq. (3.11) for c_1 with $n = 3$ is the same as Eq. (10.1) for $\tilde{c}_1$, upon changing $\sin^2(\theta/2)$ into $(1-\cos\theta)/2$ (note that the factors $1/2$ in the numerator and denominator cancel each other). So, we have $\tilde{c}_1 = c_1$. In passing, we realize that the exponential factors in all the integrals for the c's and $\tilde{c}$'s are the same, so we will only focus on the prefactors. The integrals at the denominator of Eqs. (3.12) and (3.14) for c_2 and c_4 are the same and, with $n = 3$, involve the factor

$$\big(1 - \frac{1}{3}C_3^{(2)}\big)\,u_3 = \big(1 - \frac{1}{3}(1+2\cos 2\theta)\big)\,(1-\cos\theta) = \frac{32}{3}\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big). \tag{10.5}$$

So, up to a numerical factor which we ignore for the time being, the denominators of the formulas for c_2 and c_4 match those of Eqs. (10.2) and (10.4) for $\tilde{c}_2$ and $\tilde{c}_4$. At the numerator of Eq. (3.14) for c_4 in the case $n = 3$, we find the factor

$$\begin{aligned}
&\big(C_3^{(3)} - \frac{2}{3}C_3^{(1)}\,C_3^{(2)} + C_3^{(1)}\big)\,u_3 = \\
&\quad = \big(1 + 2\cos 3\theta - \frac{2}{3}(1+2\cos\theta)(1+2\cos 2\theta) + (1+2\cos\theta)\big)\,(1-\cos\theta) \\
&\quad = \frac{8}{3}(1-\cos\theta)^2\,(1-\cos^2\theta) = \frac{64}{3}(1-\cos\theta)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big),
\end{aligned} \tag{10.6}$$

so that, still up to a numerical factor, it is equal to the numerator of Eq. (10.4) for $\tilde{c}_4$. Let's now check the numerical factor for c_4. From (3.14) and (10.5), there is a factor $10 \times \frac{32}{3}$ at the denominator and from (10.6), a factor $\frac{64}{3}$ at the numerator which results in a factor 5

at the denominator, matching that in Eq. (10.4). So, we have shown that $\tilde{c}_4 = c_4$. Finally, we need to compare c_2 with $\tilde{c}_2$ and we already know that up to a numerical factor, the denominators match. The prefactor of the exponential in the numerator of the formula for $\tilde{c}_2$ is $(3 + 2\cos\theta)\,\sin^4(\theta/2)\,\cos^2(\theta/2)$ and there is still a factor 5 at the denominator. Now, at the numerator of Eq. (3.12) for c_2 in the case $n = 3$, we find the factor

$$\begin{aligned}
&\big(2C_3^{(3)} - 3C_3^{(1)}\,C_3^{(2)} + 7C_3^{(1)}\big)\,u_3 = \\
&\quad = \big(2(1 + 2\cos 3\theta) - 3(1 + 2\cos\theta)(1 + 2\cos 2\theta) + 7(1 + 2\cos\theta)\big)\,(1 - \cos\theta) \\
&\quad = 4(3 + 2\cos\theta)\,(1 - \cos^2\theta)\,(1 - \cos\theta) = 32(3 + 2\cos\theta)\,\sin^4\big(\frac{\theta}{2}\big)\,\cos^2\big(\frac{\theta}{2}\big), \quad (10.7)
\end{aligned}$$

which also coincides with the prefactor of the exponential in the numerator of the formula for $\tilde{c}_2$, up to a numerical constant. Concerning the numerical prefactor, there is a factor $15 \times \frac{32}{3}$ at the denominator coming from (3.12) and (10.5) and a factor 32 at the numerator coming from (10.7). So, the numerical prefactor is $\frac{1}{5}$ which corresponds to that of $\tilde{c}_2$. So, we have finally shown that $c_2 = \tilde{c}_2$, confirming that the model of [26] and the model found here for $n = 3$ coincide, as they should.

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