



HAL
open science

Tokamak free-boundary plasma equilibrium computations in presence of non-linear materials

Cedric Boulbe, Blaise Faugeras, Francesca Rapetti

► **To cite this version:**

Cedric Boulbe, Blaise Faugeras, Francesca Rapetti. Tokamak free-boundary plasma equilibrium computations in presence of non-linear materials. 2022. hal-03423469v2

HAL Id: hal-03423469

<https://hal.science/hal-03423469v2>

Preprint submitted on 5 Dec 2022 (v2), last revised 6 Oct 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Tokamak free-boundary plasma equilibrium computations in presence of non-linear materials

Cedric Boulbe · Blaise Faugeras ·
Francesca Rapetti

Received: date / Accepted: date

Abstract We consider the axisymmetric formulation of the equilibrium problem for a hot plasma in a tokamak. We adopt a non-overlapping mortar element approach, that couples \mathcal{C}^0 piece-wise linear Lagrange finite elements in a region that does not contain the plasma and \mathcal{C}^1 piece-wise cubic reduced Hsieh-Clough-Tocher finite elements elsewhere, to approximate the magnetic flux field on a triangular mesh of the poloidal tokamak section. The inclusion of ferromagnetic parts is simplified by assuming that they fit within the axisymmetric modeling and a new formulation of the Newton algorithm for the problem solution is stated.

Keywords Tokamak · equilibrium · non-linear materials · reduced Hsieh-Clough-Tocher finite element · mortar element method · Newton method

Mathematics Subject Classification (2020) 65N30 · 65D05

1 Introduction

Theoretical and computational plasma physics is a wide subject with applications ranging from low temperature plasmas for lighting, thrusters and materials processing to hot plasmas for fusion; from ultra-cold plasmas to particle accelerators; from beams to pulsed power; and from intense kinetic non-equilibrium plasmas to high power microwaves. Each application is characterized by a proper space-time scaling, mathematical model and computational approach. In this work, we are interested in simulating the equilibrium of a plasma for fusion reaction in a tokamak [4]. We push forward the method proposed in [9] to compute a plasma equilibrium in tokamak devices that include ferromagnetic parts. The choice of an iron-transformer tokamak is due to Paul-Henri Rebut, a French physicist, working on nuclear fusion. From 1970 to 1973, Rebut contributed to the creation of TFR (Tokamak of Fontenay-aux-Roses), then of JET (Joint European Torus) and of Tore Supra (after the discontinuation of TFR). Tore Supra later became WEST

Cedric Boulbe · Blaise Faugeras · Francesca Rapetti
Dep. de Mathématiques, Univ. Côte d'Azur, CNRS, Castor INRIA Team, Nice (FR). E-mail: cedric.boulbe, blaise.faugeras, francesca.rapetti at univ-cotedazur.fr

(Tungsten (W) Environment in Steady-state Tokamak). The tokamak is a sort of huge transformer where the plasma current is the secondary circuit coupled to the primary one represented by the current in the coils that generate the poloidal field. In a tokamak with iron, the magnetic field lines are better conveyed (than by the air) leading to an increase in the poloidal flux thus generating a longer fusion reaction (at that time, the technology of supra-conducting coils to generate high intensity fields was not so well developed yet). However, the presence of the iron makes numerical computations more involved. Indeed, the magnetic permeability μ depends non linearly on the magnetic induction and the Green function, that relates directly the magnetic flux to the generating currents in an iron-free tokamak, cannot be used anymore. Moreover, the presence of iron parts (an internal kernel with an external arm) breaks the toroidal symmetry of the physical parameter distribution despite the plasma equilibrium is an axisymmetric phenomenon. For the TFR, that worked from 1973 to 1986, the presence of iron caused an instability on the horizontal displacement of the plasma, as described in [22, 5]. Tokamaks of new generation, such as ITER (under construction in Cadarache, France), are iron-free: thanks to modern technologies, the magnetic induction in the plasma can easily reach 10 teslas (and this would have not been possible with iron parts saturating at lower intensities). Iron-tokamaks such as WEST and JET are still used by scientists to make experiments.

We start in Section 2 by recalling some mathematical results about plasma modeling and by deriving the famous Grad-Shafranov-Schlüter equation [13], [23], [16], to solve for the numerical simulation of the axisymmetric equilibrium of the plasma. Then, in Section 3, on a triangular mesh of the tokamak poloidal cross-section, we propose a finite element approach involving highly regular approximations of the poloidal magnetic flux field, denoted by ψ . Finite elements providing piece-wise polynomial approximations of ψ that are only \mathcal{C}^0 have two main drawbacks: 1.) The definition of the plasma boundary hinges on the critical points of the unknown flux ψ . If the derivatives of ψ are not continuous, these points will neither be correctly calculated nor move in a continuous way during the plasma evolution. Indeed, with classical piece-wise finite elements, critical points of ψ are necessarily located at mesh nodes. 2.) The resistive diffusion and transport of the heat in plasma are described by one-dimensional equations containing metric coefficients that depend on the gradient of the solution ψ of the equilibrium problem. Many plasma characteristics (e.g. the so-called safety factor or the average current density profile), important to quantify stability or for monitoring during the experiment, are defined as integrals involving the gradient $\nabla\psi$ of the poloidal flux ψ (see [4]). These coefficients are not well-defined if the gradients are not continuous. Differently to other approaches in the recent literature (see for example [15, 18, 21]), to solve the axisymmetric formulation of the free-boundary plasma equilibrium in a tokamak, we rely on a non-conforming domain decomposition approach that couples \mathcal{C}^0 piece-wise linear Lagrange finite elements in a region that does not contain the plasma and \mathcal{C}^1 piece-wise cubic reduced Hsieh-Clough-Tocher [8] finite elements elsewhere. This approach gives the flexibility to achieve easily and at low cost higher order regularity for the approximation of the flux function ψ in the domain covered by the plasma, thus resolving the cited drawbacks, while preserving accurate meshing of the geometric details in the rest of the computational domain. The continuity of the numerical solution at the coupling interface is weakly enforced by mortar projection [2]. In Section 4 we write the

matrix problem and the modified Newton method to solve it. We conclude Section 4 with some numerical results and remarks, in order to compare the computed equilibrium (namely, the poloidal magnetic induction distribution and the number of Newton iterations before convergence at a fixed threshold) in the present case with that for the iron-free case, Numerical simulations are here performed with the software NICE (see [10]).

2 The direct static equilibrium problem

In a *plasma* for nuclear fusion, the charged particles (essentially, tritium and deuterium) at an extremely high temperature (ten times larger than that in the Sun) endure a fusion reaction, that is they stitch together, against the Coulomb repulsion, yielding production of energy, helium and neutrons. No material on Earth can support the temperature of such a hot mixture but due to the fact that the involved particles are charged, they can be confined in a toroidal chamber with magnetic field, *tokamak* in Russian. An additional iron structure can be installed in a tokamak to increase the poloidal flux thus generating a longer reaction. To keep up a fusion reaction we have, among many other tasks, to control the plasma in order to maintain it in equilibrium. A comprehensive survey of the (direct and inverse) mathematical problems associated with this equilibrium and of their low-order C^0 finite element modeling is described in [14] and the therein references. Here, we focus on the direct problem of computing a static equilibrium of a plasma in a tokamak by a mortar element approach coupling C^0 piece-wise linear with C^1 piece-wise cubic finite elements.

2.1 Mathematical properties for modeling the plasma at the equilibrium

The description of the plasma as a fluid that carries electrical currents and magnetic fields is surely simplified (*e.g.*, kinetic effects are ignored) but it enables the derivation and understanding of some of its most basic properties. In particular, the equations of magneto-hydro-dynamics (MHD) may be used to describe how the magnetic configuration of a tokamak holds the plasma in equilibrium. These are the continuity, momentum and energy equations in the plasma domain for the volume charge density ρ (with dimensions in the SI system¹ as $[L]^{-3}[T][I]$), the fluid velocity \mathbf{v} (as $[L][T]^{-1}$) and the pressure p (as $[M][L]^{-1}[T]^{-2}$), respectively,

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) &= 0 && \text{(continuity equation),} \\ \rho \frac{d\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} - \operatorname{grad} p && \text{(momentum equation),} \\ \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) &= S && \text{(energy equation),} \end{aligned} \tag{1}$$

being S a source collecting several terms, $\gamma > 1$ and $\frac{d}{dt}$ denotes the material time derivative, together with the magneto-quasi-static Maxwell equations in the whole

¹ In the Standard International (SI) unit system, mass M (kg), length L (m), time T (s) and current intensity I (A) are base dimensions (resp., units).

domain

$$\begin{aligned}
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \mathbf{0} && (\text{ideal Ohm law}), \\
\text{curl} \mathbf{E} &= -\partial_t \mathbf{B} && (\text{Faraday law}), \\
\text{curl} \left(\frac{1}{\mu} \mathbf{B} \right) &= \mathbf{J} && (\text{Ampere theorem}), \\
\text{div}(\mathbf{B}) &= 0 && (\text{solinoïdality condition}),
\end{aligned} \tag{2}$$

for the electric field \mathbf{E} (as $[M][L][T]^{-3}[I]^{-1}$), the magnetic induction field \mathbf{B} (as $[M][T]^{-2}[I]^{-1}$), the current density \mathbf{J} (as $[I][L]^{-2}$), with μ (as $[M][L][T]^{-2}[I]^{-2}$) the magnetic permeability and $\partial_t \mathbf{B}$ the time derivative of \mathbf{B} . Non-ferromagnetic parts of the tokamak have $\mu = \mu_0$, being μ_0 the magnetic permeability of the vacuum. Suitable boundary conditions close the MHD system. These conditions translate in mathematical terms the following facts: (1) the plasma is confined inside a perfectly conducting wall, (2) the wall separates the plasma from a vacuum region, and (3) the plasma is surrounded by external coils. We will detail these conditions when stating the final form of the problem to solve. The MHD system of equations (1), (2) is labeled as *ideal* since all resistive, viscous, conductive and diffusive terms have been neglected. Taking into account all these effects is mathematically and physically far from trivial and goes beyond the purpose of the present analysis.

To fulfill the solinoïdality condition on \mathbf{B} , a magnetic vector potential \mathbf{A} such that $\mathbf{B} = \text{curl} \mathbf{A}$ is introduced. The Coulomb gauge condition $\text{div} \mathbf{A} = 0$, which is typically used when the propagation velocity of the perturbations of the magnetic field lines is smaller than the speed of light, is imposed on \mathbf{A} to ensure uniqueness. We will see that, in an axisymmetric formulation as the one we will consider here, the Coulomb gauge on \mathbf{A} is automatically satisfied. In the following, when writing integrals, we will omit the integration element if this is not misleading. Let us introduce the magnetic helicity H_m defined by $H_m = \int_V \mathbf{A} \cdot \mathbf{B}$ where V denotes the volume of the domain of definition of the magnetic induction \mathbf{B} with $\mathbf{B} \cdot \mathbf{n}_{\partial V} = 0$ on the surface ∂V . Magnetic helicity H_m is gauge invariant since $\int_V \mathbf{A} \cdot \text{curl}(\mathbf{A} + \text{grad} \phi) = \int_V \mathbf{A} \cdot \text{curl} \mathbf{A}$, for any scalar field ϕ .

Property 1. *In the ideal MHD, magnetic helicity is conserved in the volume V , with boundary ∂V of outer normal vector $\mathbf{n}_{\partial V}$, when $\mathbf{B} \cdot \mathbf{n}_{\partial V} = 0$ on ∂V .*

Proof From the first two equations in (2), we have that the magnetic induction \mathbf{B} satisfies the induction equation $\partial_t \mathbf{B} = \text{curl}(\mathbf{v} \times \mathbf{B})$. In a fluid analogy, \mathbf{B} acts as the vorticity $\boldsymbol{\omega} = \text{curl} \mathbf{v}$ and \mathbf{A} as the velocity \mathbf{v} . By using $\mathbf{B} = \text{curl} \mathbf{A}$ and the fact that ∂_t commutes with curl , we get

$$\text{curl}(\partial_t \mathbf{A}) = \text{curl}(\mathbf{v} \times \text{curl} \mathbf{A}) \implies \partial_t \mathbf{A} = \mathbf{v} \times \text{curl} \mathbf{A} + \text{grad} \phi$$

where ϕ is any scalar field and thus

$$D_t \mathbf{A} = \partial_t \mathbf{A} - \mathbf{v} \times \text{curl} \mathbf{A} + \text{grad}(\mathbf{v} \cdot \mathbf{A}) \implies D_t \mathbf{A} = \text{grad}(\mathbf{v} \cdot \mathbf{A} + \phi)$$

where $D_t \mathbf{A}$ denotes the material derivative of \mathbf{A} . By using the vector identity

$$\text{curl}(\mathbf{z} \times \mathbf{w}) = (\text{div} \mathbf{w}) \mathbf{z} - (\text{div} \mathbf{z}) \mathbf{w} + (\mathbf{w} \cdot \text{grad}) \mathbf{z} - (\mathbf{z} \cdot \text{grad}) \mathbf{w},$$

with $\mathbf{z}, \mathbf{w} \in \mathbb{R}^3$, the material derivative $D_t \mathbf{B}$ of \mathbf{B} becomes

$$D_t \mathbf{B} = \partial_t \mathbf{B} + (\mathbf{v} \cdot \text{grad}) \mathbf{B} \implies D_t \mathbf{B} = (\mathbf{B} \cdot \text{grad}) \mathbf{v} - \mathbf{B} (\text{div} \mathbf{v})$$

since $\partial_t \mathbf{B}$ is equal to $\text{curl}(\mathbf{v} \times \mathbf{B})$. The continuity equation $\partial_t \rho + \text{div}(\rho \mathbf{v}) = 0$ can be written as follows

$$\partial_t \rho + (\mathbf{v} \cdot \text{grad}) \rho + \rho (\text{div} \mathbf{v}) = 0 \quad \Longrightarrow \quad D_t \rho = -\rho \text{div} \mathbf{v}$$

where $-\text{div} \mathbf{v}$ is the rate of compression of the volume element. We thus have

$$\begin{aligned} D_t \left(\frac{\mathbf{B}}{\rho} \right) &= \frac{1}{\rho} D_t \mathbf{B} - \frac{\mathbf{B}}{\rho^2} D_t \rho \\ &= \frac{1}{\rho} [(\mathbf{B} \cdot \text{grad}) \mathbf{v} - \mathbf{B} \text{div} \mathbf{v}] - \frac{\mathbf{B}}{\rho^2} (-\rho \text{div} \mathbf{v}) \\ &= \frac{1}{\rho} (\mathbf{B} \cdot \text{grad}) \mathbf{v}. \end{aligned}$$

We apply the Reynolds kinematic transport theorem to the volume V bounded by the closed surface ∂V (with normal vector $\mathbf{n}_{\partial V}$ of modulus equal to the measure $|\partial V|$) to compute $D_t H_m$. By using the results above, we get

$$\begin{aligned} D_t H_m &= D_t \int_V \frac{(\mathbf{A} \cdot \mathbf{B})}{\rho} \rho \\ &= \int_V D_t \left(\frac{(\mathbf{A} \cdot \mathbf{B})}{\rho} \rho \right) + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) \\ &= \int_V \left[(D_t \mathbf{A}) \cdot \frac{\mathbf{B}}{\rho} \rho + \rho \mathbf{A} \cdot D_t \left(\frac{\mathbf{B}}{\rho} \right) + \frac{(\mathbf{A} \cdot \mathbf{B})}{\rho} D_t \rho \right] + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) \\ &= \int_V \left[\text{grad}(\mathbf{v} \cdot \mathbf{A} + \phi) \cdot \mathbf{B} + \mathbf{A} \cdot (\mathbf{B} \cdot \text{grad}) \mathbf{v} + \frac{(\mathbf{A} \cdot \mathbf{B})}{\rho} (-\rho \text{div} \mathbf{v}) \right] \\ &\quad + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) \\ &= \int_V (\mathbf{B} \cdot \text{grad}) (2 \mathbf{v} \cdot \mathbf{A} + \phi) - \int_V (\mathbf{A} \cdot \mathbf{B}) \text{div} \mathbf{v} + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) \\ &= \int_V (\mathbf{B} \cdot \text{grad}) (2 \mathbf{v} \cdot \mathbf{A} + \phi) - \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) (\mathbf{v} \cdot \mathbf{n}_{\partial V}) \\ &= \int_{\partial V} (2 \mathbf{v} \cdot \mathbf{A} + \phi) \mathbf{B} \cdot \mathbf{n}_{\partial V} \\ &= 0 \end{aligned}$$

with the imposition of $\mathbf{B} \cdot \mathbf{n}_{\partial V} = 0$. \square

The term ‘‘helicity’’ recalls that the trajectory of a fluid particle in a fluid with velocity \mathbf{v} and vorticity $\boldsymbol{\omega} = \text{curl} \mathbf{v}$ forms a helix in regions where the fluid helicity $H_f = \int_V \mathbf{v} \cdot \boldsymbol{\omega} \neq 0$. This behavior is very similar for magnetic field lines. The helicity of a divergence-free vector field is a standard measure for the extent to which the field lines wrap and coil around one another (see, *e.g.*, [6]). The connection between twists and knots is analyzed in [20] and recently in [19]. The topological interpretation of helicity in terms of linking numbers is given, *e.g.*, in [3], [17] and references therein.

Property 2. *In the ideal MHD, the magnetic flux $\Phi = \int_S \mathbf{B} \cdot \mathbf{n}_S$ through a surface S immersed in the magnetic field is conserved in time.*

Proof We apply a variant of the Reynolds kinematic transport theorem for a surface S (with normal vector \mathbf{n}_S of modulus equal to the measure $|S|$), bounded by the closed curved ∂S (with tangent vector $\mathbf{t}_{\partial S}$ of modulus equal to the length $|\partial S|$). It holds

$$D_t \Phi = \int_S \partial_t \mathbf{B} \cdot \mathbf{n}_S + \int_{\partial S} \mathbf{B} \cdot \mathbf{v} \times \mathbf{t}_{\partial S}$$

with \mathbf{v} the velocity field through S . We get

$$\begin{aligned} D_t \Phi &= \int_S \text{curl}(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{n}_S + \int_{\partial S} \mathbf{B} \cdot \mathbf{v} \times \mathbf{t}_{\partial S} \\ &= \int_{\partial S} \mathbf{v} \times \mathbf{B} \cdot \mathbf{t}_{\partial S} + \int_{\partial S} \mathbf{t}_{\partial S} \cdot \mathbf{B} \times \mathbf{v} \\ &= \int_{\partial S} \mathbf{v} \times \mathbf{B} \cdot \mathbf{t}_{\partial S} - \int_{\partial S} \mathbf{t}_{\partial S} \cdot \mathbf{v} \times \mathbf{B} \\ &= 0 \end{aligned}$$

thank to $\partial_t \mathbf{B} = \text{curl}(\mathbf{v} \times \mathbf{B})$. \square

In this work, we consider the equilibrium of the plasma, we remain at the diffusion time scale (the slowest one) in a device with characteristic length of meters. The equations describing an ideal MHD equilibrium are respectively, force balance (between the kinetic force and the magnetic force), Ampere theorem and the solinoidality condition, that are

$$\text{grad } p = \mathbf{J} \times \mathbf{B}, \quad \text{curl}\left(\frac{1}{\mu}\mathbf{B}\right) = \mathbf{J}, \quad \text{div}(\mathbf{B}) = 0. \quad (3)$$

We know that the distribution of iron structures in these tokamaks is not at all axisymmetric. Therefore, to fit within the axisymmetric modeling we make the following assumption: *all cross-sections of the considered iron tokamak are identical*. Thank to their twisting (rotation around its own axis), magnetic field line bundles in a close neighborhood remain localized in narrow flux tubes. Under the assumption of perfect axial symmetry of the device geometry and physical parameters' distribution, let \mathbf{e}_φ is the unit vector for the toroidal coordinate φ in the coordinate system (r, φ, z) . In these coordinates, r measures the distance from the tokamak axis, φ is the toroidal angle and z is the height along the tokamak axis. The magnetic induction field \mathbf{B} can be decomposed into the sum of $\mathbf{B}_t \parallel \mathbf{e}_\varphi$ and $\mathbf{B}_p \perp \mathbf{e}_\varphi$, with both \mathbf{B}_t and \mathbf{B}_p independent of φ . We express \mathbf{B} in terms of the poloidal flux function ψ and of another function f as given in [4], that is

$$\mathbf{B} = \frac{1}{r} \text{grad } \psi \times \mathbf{e}_\varphi + \frac{f}{r} \mathbf{e}_\varphi = \mathbf{B}_p + \mathbf{B}_t, \quad (4)$$

where $\psi|_{\partial V} = \text{constant}$. The first term, \mathbf{B}_p , in (4) is the poloidal component of \mathbf{B} that lies in the cross-section plane (r, z) also called poloidal section ($\varphi = C$) of the tokamak. The second term, \mathbf{B}_t , in (4) is the toroidal component of \mathbf{B} and $f \mathbf{e}_\varphi = r \mathbf{B}_t$.

In particular, $\mathbf{B}_p = \text{curl } \mathbf{A}_t$ and the poloidal magnetic flux ψ thus represents the scaled toroidal component of the vector potential \mathbf{A} , namely $\psi \mathbf{e}_\varphi = r \mathbf{A}_t$. We recall two important properties for the mathematical modeling of the plasma at the ideal MHD equilibrium. Property 3 states that the poloidal magnetic flux ψ is a key quantity in modeling plasma in tokamaks.

Property 3. *The lines of both the current density \mathbf{J} and magnetic induction \mathbf{B} are on surfaces of constant value for ψ (and p). They are called magnetic surfaces.*

Proof In cylindrical coordinates we have $\text{grad } p = (\partial_r p, \frac{1}{r} \partial_\varphi p, \partial_z p)^\top$. As we assume μ constant in the plasma domain, we obtain

$$\text{curl}\left(\frac{1}{\mu}\mathbf{B}_t\right) = \text{curl}\left(\frac{f}{\mu r}\mathbf{e}_\varphi\right) = \frac{1}{\mu r}(-\partial_z f)\mathbf{e}_r + \frac{1}{\mu r}(\partial_r f)\mathbf{e}_z \quad (5)$$

being \mathbf{e}_r and \mathbf{e}_z the unit vectors for the poloidal coordinates r, z , respectively. Let us consider the force balance identity in (3), then

$$0 = \text{grad } p \cdot \mathbf{B}_p = \text{grad } p \cdot \text{curl } \mathbf{A}_t = \frac{1}{r} \partial_r p (-\partial_z \psi) + \frac{1}{r} \partial_z p (\partial_r \psi)$$

hence we see that p is constant over surfaces where ψ is constant, so $p = p(\psi)$. Moreover, for the axisymmetry assumption,

$$0 = \text{grad } p \cdot \mathbf{J}_p = \text{grad } p \cdot \text{curl}\left(\frac{1}{\mu}\mathbf{B}_t\right) = \frac{1}{\mu r} \partial_r p (-\partial_z f) + \frac{1}{\mu r} \partial_z p (\partial_r f)$$

and thus f is constant over surfaces where p and ψ are constant, so $f = f(\psi)$. \square

The magnetic flux surfaces are hence strictly connected with the value of ψ , they are closed nested surfaces, that do not intersect with any material of the tokamak, and ensure the confinement of charged particles, namely the confinement of plasma inside a tokamak. The center of the plasma, where pressure has a maximum, is called the magnetic axis. The last closed surface defines the domain containing the plasma, say $\Omega_p(\psi)$. Property 4 relates the poloidal magnetic flux ψ to the current in the plasma domain $\Omega_p(\psi)$.

Property 4. *The flux ψ verifies the Grad-Shafranov-Schlüter equation in $\Omega_p(\psi)$.*

Proof We have

$$\begin{aligned} (-\partial_z f \mathbf{e}_r + \partial_r f \mathbf{e}_z) &= f'(\psi) (-\partial_z \psi \mathbf{e}_r + \partial_r \psi \mathbf{e}_z) && \text{(as } f = f(\psi)) \\ &= f'(\psi) \text{grad } \psi \times \mathbf{e}_\varphi && \text{(eq. (4))} \\ &= f'(\psi) r \mathbf{B}_p \\ &= f'(\psi) r \text{curl } \mathbf{A}_t. \end{aligned}$$

We consider the force balance equation in (3). We obtain

$$\text{grad } p = (\mathbf{J}_p + \mathbf{J}_t) \times (\mathbf{B}_t + \mathbf{B}_p) \quad \implies \quad \text{grad } p = \mathbf{J}_p \times \mathbf{B}_t + \mathbf{J}_t \times \mathbf{B}_p$$

as

$$\mathbf{J}_p \times \mathbf{B}_p = \text{curl} \left(\frac{1}{\mu} \mathbf{B}_t \right) \times \mathbf{B}_p = \frac{1}{\mu} f'(\psi) \text{curl } \mathbf{A}_t \times \text{curl } \mathbf{A}_t = \mathbf{0}$$

and $\mathbf{J}_t \times \mathbf{B}_t = \mathbf{0}$ by using vector identities. Let us now consider the identity

$$\text{grad } p \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) = (\mathbf{J}_b \times \mathbf{B}_t + \mathbf{J}_t \times \text{curl } \mathbf{A}_t) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t). \quad (6)$$

The magnetic surfaces are defined by a constant value of p and $p = p(\psi)$, as we have seen with Property 1, hence $\text{grad } p = p'(\psi) \text{grad } \psi$. The left-hand side gives

$$p'(\psi) \text{grad } \psi \cdot (\mathbf{e}_\varphi \times \frac{1}{r} \text{grad } \psi \times \mathbf{e}_\varphi) = \frac{1}{r} p'(\psi) |\text{grad } \psi|^2.$$

For the terms in the right-hand side of (6) we have

$$\begin{aligned} &(\mathbf{J}_b \times \mathbf{B}_t) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) \\ &= (\text{curl} \left(\frac{1}{\mu} \mathbf{B}_t \right) \times \mathbf{B}_t) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) \\ &= \left(\frac{1}{\mu} f'(\psi) \text{curl } \mathbf{A}_t \times \frac{f}{r} \mathbf{e}_\varphi \right) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) \\ &= - \frac{1}{\mu r} f f'(\psi) \left| \frac{1}{r} \text{grad } \psi \right|^2 = - \frac{1}{\mu r^3} f f'(\psi) |\text{grad } \psi|^2. \end{aligned}$$

The vector identity $(a \times b) \cdot (\alpha \times \beta) = (a \cdot \alpha)(b \cdot \beta) - (a \cdot \beta)(b \cdot \alpha)$ for generic vectors a, b, α, β , yields

$$\begin{aligned} &(\mathbf{J}_t \times \mathbf{B}_p) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) \\ &= (\mathbf{J}_t \times \text{curl } \mathbf{A}_t) \cdot (\mathbf{e}_\varphi \times \text{curl } \mathbf{A}_t) \\ &= (\mathbf{J}_t \cdot \mathbf{e}_\varphi) |\text{curl } \mathbf{A}_t|^2 - (\mathbf{J}_t \cdot \text{curl } \mathbf{A}_t) (\text{curl } \mathbf{A}_t \cdot \mathbf{e}_\varphi) \\ &= (\mathbf{J}_t \cdot \mathbf{e}_\varphi) \frac{1}{r^2} |\text{grad } \psi|^2 \end{aligned}$$

since $\text{curl } \mathbf{A}_t \cdot \mathbf{e}_\varphi = 0$. We have thus obtained

$$\frac{1}{r} p'(\psi) |\text{grad } \psi|^2 = \left[\frac{1}{r^2} \mathbf{J}_t \cdot \mathbf{e}_\varphi - \frac{1}{\mu r^3} f f'(\psi) \right] |\text{grad } \psi|^2.$$

We can simplify by $|\frac{1}{r} \text{grad } \psi|^2$ in the three terms of equation above and

$$\mathbf{J}_t \cdot \mathbf{e}_\varphi = \frac{1}{\mu r} f f'(\psi) + r p'(\psi) \quad (7)$$

is the current in the plasma domain $\Omega_p(\psi)$. In cylindrical coordinates, the left-hand side of (7) in terms of ψ becomes $-\text{div}(1/(\mu r) \text{grad } \psi) := -\Delta^* \psi$ and thus it holds

$$-\Delta^* \psi = \frac{1}{\mu r} f f'(\psi) + r p'(\psi) \quad (8)$$

that is the Grad-Shafranov-Schlüter equation for ψ in $\Omega_p(\psi)$. \square

For an air-transformer tokamak, $\mu = \mu_0$ everywhere and $-\Delta^*$ in equation (8) is a linear second-order elliptic operator. For an iron-transformer tokamak, μ is a given function of $|\mathbf{B}_p|^2$, thus of $|\frac{1}{r} \text{grad } \psi|^2$, in the ferromagnetic region and $-\Delta^*$ in equation (8) becomes a non-linear second-order elliptic operator. The main challenges for solving equation (8) numerically are its formulation on an infinite domain, the non-linear right-hand side, the non-linear permeability in iron and the non-linearity due to the free plasma boundary. In the following, we will go through these challenges up to a resolution algorithm and few numerical results.

2.2 The continuous problem in the poloidal section

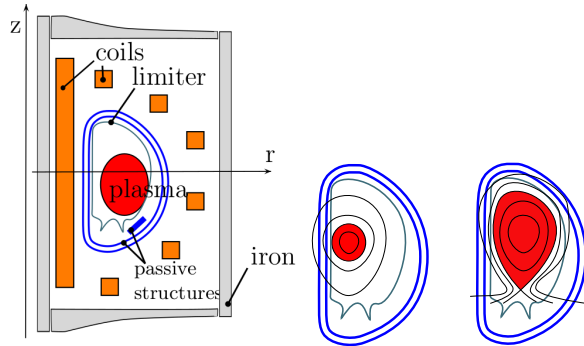


Fig. 1 Left: Geometric description of the tokamak in the poloidal plane. Middle and right, sketch for characteristic plasma shapes. The plasma boundary touches the limiter or the plasma is enclosed by a flux line that goes through an X-point.

To correctly state the equations and boundary conditions of the continuous problem to solve, we introduce $\mathcal{D} = [0, \infty] \times [-\infty, \infty]$, the positive half plane that contains the poloidal section. The geometry of the tokamak determines various

sub-domains (see Fig. 1, left) that are used to set the source current accordingly. In these pages, passive structures are not modeled (they are hence supposed to be characterized by an electric conductivity $\sigma = 0$). We have:

- $\Omega_{Fe} \subset \mathcal{D}$ denotes those parts of \mathcal{D} made of iron where the magnetic permeability μ is not constant and given as a non-linear function of ψ , namely $\mu(\psi) = \mu_{FE}(|\text{grad } \psi|^2 r^{-2})$. If $\Omega_{Fe} = \emptyset$, then $\mu = \mu_0$ everywhere;
- $\Omega_{c_i} \subset \mathcal{D}$, $1 \leq i \leq N_c$, denotes the intersection of the i th coil with the poloidal plane. Here, the i th coil cross section area is $|\Omega_{c_i}|$, with total current I_i ;
- $\Omega_L \subset \mathcal{D}$, denotes the domain bounded by the limiter, thus the domain accessible by the plasma;
- $\Omega_p \subset \Omega_L$, denotes the domain covered by the plasma and the boundary $\partial\Omega_p$ is the outermost closed ψ -isocontour contained within Ω_L .

The static direct equilibrium problem thus reads: find ψ such that

$$-\Delta^* \psi = \begin{cases} rp'(\psi) + \frac{1}{\mu_0 r} f f'(\psi) & \text{in } \Omega_p(\psi), \\ I_i(\psi)/|\Omega_{c_i}| & \text{in } \Omega_{c_i}, \quad i = 1, N_c, \\ 0 & \text{elsewhere,} \end{cases} \quad (9)$$

$$\psi(0, z) = 0 \quad \forall z \in \mathbb{R}, \quad \lim_{\|(r,z)\|_2 \rightarrow +\infty} \psi(r, z) = 0.$$

The plasma domain $\Omega_p(\psi)$ is unknown and depends non-linearly on the poloidal flux ψ (we have a free-boundary problem). The boundary of $\Omega_p(\psi)$ either touches that of Ω_L (limiter configuration, as in Fig. 1 middle) or contains one or more saddle points of ψ (divertor configuration, as in Fig. 1 right). The saddle points of ψ , denoted by $(r_X, z_X) = (r_X(\psi), z_X(\psi))$, are called X-points of ψ . The plasma domain $\Omega_p(\psi)$ is the largest sub-domain of Ω_L bounded by a closed ψ -isoline in Ω_L and containing the magnetic axis (r_a, z_a) . The magnetic axis is the point $(r_a, z_a) = (r_a(\psi), z_a(\psi))$, where ψ has its global maximum (or minimum, depending on axis positive direction) in Ω_L . Let $(r_b, z_b) = (r_b(\psi), z_b(\psi))$ be the point that determines the plasma boundary. Note that (r_b, z_b) is either an X-point of ψ or the contact point with $\partial\Omega_L$. The domain of p' and $f f'$ is the interval $[\psi_a, \psi_b]$ (supposing $\psi_a < \psi_b$) with the scalar values ψ_a and ψ_b being the flux values at the *magnetic axis* and at the boundary of the plasma. Since the domain of p' and $f f'$ depends on the poloidal flux itself, it is more practical to supply these profiles as functions of the normalized poloidal flux $\psi_N(r, z) = (\psi(r, z) - \psi_a(\psi))/(\psi_b(\psi) - \psi_a(\psi))$ and

$$p'(\psi) \approx S_{p'}(\psi_N), \quad f f'(\psi) \approx S_{f f'}(\psi_N).$$

These two functions, $S_{p'}$ and $S_{f f'}$, have, independently of ψ , a fixed domain $[0, 1]$ and their expression depends on parameters that are determined experimentally (see [4] for more details on these functions). To solve numerically problem (9) we need to work in a domain $\Omega \subset \mathcal{D}$, known as the ABB domain, named after Albanese-Blum-Barbieri who first introduced it [1], associated with \mathcal{D} (see Fig. 2, left), delimited by a *half-circle* γ of radius $\rho_\gamma > 0$ including $\Omega_L \cup \Omega_{Fe} \cup_i \Omega_{c_i}$ and the vertical *segment* $\Gamma_0 = \{0\}_r \times [-\rho_\gamma, \rho_\gamma]_z$.

2.3 The weak problem

Here comes the non-overlapping domain decomposition framework. We set $\Omega = \Omega^{\text{in}} \cup \Omega^{\text{ex}}$ where Ω^{in} is a bounded domain containing Ω_L (see Fig. 2, right) and

$\Omega^{\text{ex}} = \Omega \setminus \Omega^{\text{in}}$. The boundary of Ω^{in} is denoted \mathcal{I} , to recall that it is an interface between the two sub-domains Ω^{in} , Ω^{ex} , on which we will impose the continuity of ψ , in a weak sense, through a mortar-like L^2 projection [2].

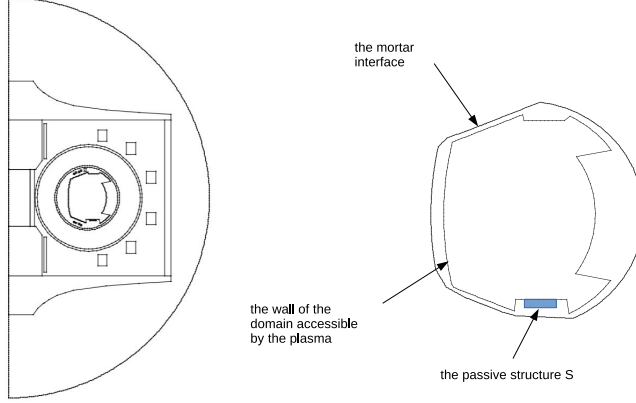


Fig. 2 The ABB domain (left) associated with \mathcal{D} and a zoom (right) on the sub-domain Ω^{in} involved in the domain decomposition formulation. The mortar interface \mathcal{I} in this case is closed. The passive structure S is drawn for completeness but it is not modeled here.

The weak formulation of (9) is: Find $\psi = (\psi_{\text{ex}}, \psi_{\text{in}}) \in \mathcal{V}$ such that

$$a(\psi, s) := a^{\text{ex}}(\psi_{\text{ex}}, v) + a^{\text{in}}(\psi_{\text{in}}, w) = \ell(I, s) \quad \forall s = (v, w) \in \mathcal{V}_{0, \mathcal{I}} \quad (10)$$

where $\mathcal{V} = \{(v, w) \in \mathcal{H}^1(\Omega^{\text{ex}}) \times \mathcal{H}^1(\Omega^{\text{in}}), v|_{\gamma_0} = 0, v|_{\mathcal{I}} = w|_{\mathcal{I}}\}$, being $\mathcal{H}^1(\Omega)$ the functional space defined as $\mathcal{H}^1(\Omega) = \{u \in L_*^2(\Omega), \nabla u \in L_*^2(\Omega)^2\}$ with $L_*^2(\Omega) = \{g : \Omega \rightarrow \mathbb{R}, \|g\|_{*, \Omega}^2 := \int_{\Omega} g^2 \frac{1}{r} dr dz < \infty\}$ and ∇ denoting the gradient operator in the poloidal variables. We have also set $\mathcal{V}_{0, \mathcal{I}} = \{(v, w) \in \mathcal{V}, v|_{\mathcal{I}} = w|_{\mathcal{I}} = 0\}$ and

$$\begin{aligned} a^{\text{ex}}(\psi, v) &:= \int_{\Omega^{\text{ex}}} \frac{1}{\mu r} \nabla \psi \cdot \nabla v dr dz + c(\psi, v), \\ a^{\text{in}}(\psi, w) &:= \int_{\Omega^{\text{in}}} \frac{1}{\mu_0 r} \nabla \psi \cdot \nabla w dr dz - J_{\text{p}}(\psi, w), \\ J_{\text{p}}(\psi, w) &:= \int_{\Omega_{\text{p}}(\psi)} \left(\frac{r}{r_0} \mathcal{A}(\psi_{\text{N}}) + \frac{r_0}{r} \mathcal{B}(\psi_{\text{N}}) \right) w dr dz, \\ \ell(I, s) &:= \sum_{i=1}^{N_c} \frac{I_i}{|\Omega_{c_i}|} \int_{\Omega_{c_i}} (\chi_{\Omega^{\text{ex}}} v + \chi_{\Omega^{\text{in}}} w) dr dz \end{aligned} \quad (11)$$

with, respectively, r_0 the characteristic radius (in meters) of Ω_L , λ a scaling coefficient such that the total plasma current is $I_{\text{p}} = \lambda |\mathbf{J}_{\text{p}}|$, the functions \mathcal{A} , \mathcal{B} parametric representations of $S_{\text{p}'}$, $S_{\text{ff}'}$. We recall that, in the expression of $a^{\text{ex}}(\psi, v)$, the magnetic permeability can depend on ψ as follows

$$\mu = \mu_{FE} \left(\frac{|\nabla \psi|^2}{r^2} \right) \chi_{\Omega_{FE}} + \mu_0 \chi_{\Omega^{\text{ex}} \setminus \Omega_{FE}},$$

where χ_V is the characteristic function of a set V and μ_{FE} is a given function. Note that $\ell(I, s)$ contains the expression $\chi_{\Omega^{\text{ex}}} v + \chi_{\Omega^{\text{in}}} w$ to deal with the presence of coils in Ω^{in} and Ω^{ex} . Moreover, $c(\psi, v) \approx \int_{\partial \Omega} v \partial_n \psi d\Gamma$ to take into account

the condition at infinity on γ . Under suitable assumptions, such as for example $\Omega_{Fe} = \emptyset$ or I_p assigned, it can be proved that problem (10) has a unique solution [12], in the general case the question is theoretically open². In the next section, we propose a Newton method to solve the discrete problem associated with (10) when $\Omega_{Fe} \neq \emptyset$ and μ needs to be estimated from experimental data. The reconstruction of the function μ for a given ferromagnetic material is performed on the tokamak in absence of plasma. The function $\mu_r(H_p)$ representing the relative magnetic permeability ($\mu = \mu_0 \mu_r$) is experimentally determined as a function of the modulus H_p of the poloidal magnetic field \mathbf{H}_p . We anticipate that, as remarked in [11], if μ was directly linked to B_p , the Newton algorithm generally used to solve the final discrete problem could be divergent, as B_p^2 and thus μ vary significantly from one iteration to another. Therefore, the function $\mu_r(H_p)$ is first reconstructed by relying on the Ampère theorem and then, at each iteration n , we use

$$\mu^n = g\left(\frac{|\nabla \psi_h^n|^2}{r^2 (\mu^{n-1})^2}\right).$$

To define the corresponding Jacobian matrix, we compute the derivatives w.r.t. the unknown field ψ of the non-linear operators in (10), and then we evaluate them on discrete fields with special care. By involving directional derivatives, we can define $D_\psi a^{\text{ex}}(\cdot, \cdot)$ as follows

$$D_\psi a^{\text{ex}}(\psi, s)(\tilde{\psi}) = a^{\text{ex}}(\tilde{\psi}, s) - 2 \int_{\Omega_{Fe}} \frac{g'(\cdot)}{g^2(\cdot)} \frac{1}{r^3} (\nabla \tilde{\psi} \cdot \nabla \psi) (\nabla \psi \cdot \nabla s) \quad (12)$$

where (\cdot) stands for $(|\nabla \psi|^2 / r^2 / (\mu^{n-1})^2)$. For $D_\psi a^{\text{in}}(\cdot, \cdot)$, the derivative w.r.t. ψ of $J_p(\cdot, \cdot)$ is computed analytically on an approximation of this functional by a quadrature formula³.

3 The discrete formulation of the equilibrium problem

A mortar finite element approach is applied to (10) to get the discrete problem. Let τ^{ex} (resp. τ^{in}) be a mesh of triangles that covers Ω^{ex} (resp. Ω^{in}). The two meshes τ^{ex} , τ^{in} are shape regular and quasi-uniform, with maximal diameters h_{ex} , h_{in} , respectively. We assume that \mathcal{I} is a polygonal with nodes and edges in τ^{ex} . We wish to use in $\Omega_L \subset \Omega^{\text{in}}$, a finite element approximation ψ_h for the poloidal flux ψ that is not only continuous but has also component-wise continuous gradient $\nabla \psi_h$. This is possible if we use the piece-wise cubic reduced or minimal Hsieh-Clough-Tocher (rHCT) finite element space, say \mathcal{V}^{in} , on τ^{in} (see [8]). This regularity is not necessary in Ω^{ex} therefore we couple rHCT finite elements in Ω^{in} with continuous piece-wise linear finite elements, say \mathcal{V}^{ex} , on τ^{ex} . The finite element space over the mesh τ^{ex} is $\mathcal{V}^{\text{ex}} = \{v \in C^0(\Omega^{\text{ex}}), v|_{\Gamma_0} = 0, v|_T \in \mathbb{P}_1(T), \forall T \in \tau^{\text{ex}}\}$, whereas over τ^{in} is

$$\mathcal{V}^{\text{in}} = \{w \in C^1(\Omega^{\text{in}}), w|_T \in P_{loc}(T), \forall T \in \tau^{\text{in}}\}.$$

² We wish to recall the fundamental contribution of professor Glowinski to the analysis, the finite element approximation and numerical resolution by Newton-like methods of such non-linear problems.

³ To approximate $J_p(\cdot, \cdot)$ in (11) by a quadrature formula we need to know the domain $\Omega_p(\psi)$ occupied by the plasma. This domain is an unknown of the equilibrium problem, as it depends on ψ . An efficient technique to determine it is stated in [9].

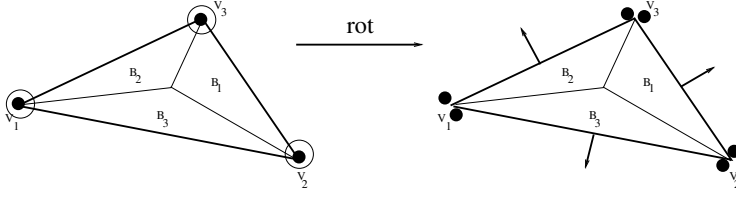


Fig. 3 Barycentric subdivision of $T = [V_1, V_2, V_3] \in \tau^{\text{in}}$ into three sub-triangles B_i . Locally on T , at the three vertices V_i , we reconstruct the height $\psi_h(V_i)$ (black filled thick points) of ψ and the tangent plane to the surface ψ as generated by $\partial_r \psi_h(V_i), \partial_z \psi_h(V_i)$ (empty circles around the vertices). Here $\mathbf{rot} \psi_h = (\partial_r \psi_h, -\partial_z \psi_h)^t$. If we compute $\mathbf{rot} \psi_h$, we get the restriction of $r \mathbf{B}_p$ to T with continuous components (the two black filled thick points) at the V_i and continuous normal component (the small arrows) at the $b_i = \partial B_i \cap \partial T$ (see details in [7]).

The space $P_{loc}(T)$ reads

$$P_{loc}(T) = \{w \in C^1(T), w|_{B_i} \in \mathbb{P}_3(B_i), (\partial_n w)|_{b_i} \in \mathbb{P}_1(b_i), \forall b_i \in \partial B_i \cap \partial T\},$$

with the triangle $T = [V_1, V_2, V_3]$ cut into three triangles $B_i = [G, V_m, V_\ell]$, having vertices in V_m, V_ℓ with $m, \ell \in \{1, 2, 3\} \setminus \{i\}$ and at the barycenter G , for each $i = 1, 2, 3$ as shown in Fig. 3. In the $P_{loc}(T)$ space definition, n is the outward normal vector to ∂T , b_i the edge $\partial B_i \cap \partial T$ and $(\partial_n w)|_{b_i}$ the normal derivative of w along the edge b_i .

Note that $\Omega_{Fe} \subset \Omega^{\text{ex}}$. Let us also write $\mathcal{V}^{\text{ex}} = \mathcal{V}_\circ^{\text{ex}} \oplus \mathcal{E} \mathcal{V}_\partial^{\text{ex}}$ and $\mathcal{V}^{\text{in}} = \mathcal{V}_\circ^{\text{in}} \oplus \mathcal{E} \mathcal{V}_\partial^{\text{in}}$, where, for example, $\mathcal{V}_\circ^{\text{ex}}$ (resp. $\mathcal{V}_\partial^{\text{ex}}$) is the subspace of \mathcal{V}^{ex} described by basis functions associated with dofs at nodes in $\bar{\Omega}^{\text{ex}} \setminus \mathcal{I}$ (resp., $\bar{\Omega}^{\text{ex}} \cap \mathcal{I}$) and \mathcal{E} denotes the corresponding trivial extension operator. The functions in $\mathcal{V}_\circ^{\text{ex}}$ and $\mathcal{V}_\circ^{\text{in}}$ have vanishing Dirichlet trace on \mathcal{I} . The discrete problem to solve reads: Find $\psi_h \in \mathcal{V}_h$ such that

$$a(\psi_h, s_h) = \ell(I, s_h) \quad \forall s_h = (v_h, w_h) \in \mathcal{V}_\circ^{\text{ex}} \times \mathcal{V}_\circ^{\text{in}} \quad (13)$$

where

$$\mathcal{V}_h = \{(u_h^{\text{in}}, u_h^{\text{ex}}) \in \mathcal{V}^{\text{in}} \times \mathcal{V}^{\text{ex}}, u_h^{\text{ex}}|_{\gamma_0} = 0, \int_{\mathcal{I}} (u_h^{\text{in}} - u_h^{\text{ex}}) z_h d\mathcal{I} = 0, \forall z_h \in \mathcal{M}_h\},$$

with $\mathcal{M}_h = \{\xi_h \in C^0(\mathcal{I}) : \xi_h|_e \in \mathbb{P}_1(e), \forall e \in (\tau^{\text{ex}})|_{\mathcal{I}}\}$ the mortar multiplier space. The bilinear and linear forms $a(\cdot, \cdot), \ell(I, \cdot)$ are defined as for the problem (10) and evaluated in (13) for functions in the discrete space \mathcal{V}_h .

4 The matrix problem and the Newton algorithm

Let us denote by $\{v_i^{\text{ex}}\}_{i=1, N^{\text{ex}}}$ the dual basis of \mathcal{V}^{ex} for the P1 dofs associated with the N^{ex} nodes $V_i \in \tau^{\text{ex}}$ and $\{w_j^{\text{in}}\}_{j=1, 3N^{\text{in}}}$ that of \mathcal{V}^{in} for the rHCT dofs at the N^{in} nodes $V_j \in \tau^{\text{in}}$. Let \mathbf{A} (resp. $\mathbf{C}, \mathbf{L}^{\text{in}}, \mathbf{L}^{\text{ex}}$) be the matrix associated with the integral expressions in (11) contained in $a(\cdot, \cdot)$ (resp., in $c(\cdot, \cdot)$, in $\ell(\cdot, \cdot)$ for the coil Ω_{c_i} if this coil is in Ω^{in} or in Ω^{ex}) and $\mathbf{J}(\cdot)$ (resp., $\mathbf{U}_I^{\text{in}}, \mathbf{U}_I^{\text{ex}}$) the vector with components resulting from $\mathbf{J}_p(\cdot)$ (resp., holding the currents I_i for the coil Ω_{c_i} if it is in Ω^{in} or in Ω^{ex}). To take into account iron parts in Ω^{ex} , we separate the

elliptic operator into the linear part and a nonlinear part, say $\mathbf{A}_0\boldsymbol{\psi} + \mathbf{A}_\mu(\boldsymbol{\psi})$, where the vector $\boldsymbol{\psi}$ gathers all dofs of $\psi_h \in \mathcal{V}_h$, the matrix \mathbf{A}_0 has entries

$$(\mathbf{A}_0)_{ij} = \int_{\Omega^{\text{ex}} \setminus \Omega_{F_e}} \frac{1}{\mu_0 r} \nabla v_i^{\text{ex}} \cdot \nabla v_j^{\text{ex}} dr dz, \quad i, j = 1, N^{\text{ex}},$$

and the vector $\mathbf{A}_\mu(\boldsymbol{\psi})$ has components

$$(\mathbf{A}_\mu(\boldsymbol{\psi}))_i = \int_{\Omega_{F_e}} \frac{1}{\mu(\psi_h)r} \nabla v_i^{\text{ex}} \cdot \nabla \psi_h dr dz, \quad i = 1, N^{\text{ex}}.$$

Equation (13) in its fully discretized form reads $\mathbf{e}(\boldsymbol{\psi}) = \mathbf{0}$ with

$$\mathbf{e}(\boldsymbol{\psi}) := (\mathbf{A}_0 + \mathbf{C})\boldsymbol{\psi} + \mathbf{A}_\mu(\boldsymbol{\psi}) - \mathbf{J}(\boldsymbol{\psi}) - \mathbf{L}^{\text{in}} \mathbf{U}_I^{\text{in}} - \mathbf{L}^{\text{ex}} \mathbf{U}_I^{\text{ex}} \quad (14)$$

where, for $k = 1, 3N^{\text{in}}$, we have

$$(\mathbf{J}(\boldsymbol{\psi}))_k = \int_{\Omega_p(\psi_h)} \mathbf{J}_p(\psi_{N,h}, r) w_k dr dz, \quad \text{and} \quad \mathbf{L}_{i,k}^{\text{in}} = \frac{1}{|\Omega_{c_i}|} \int_{\Omega_{c_i}} w_k^{\text{in}} dr dz,$$

for those indices $i = 1, N_c$ such that $\Omega_{c_i} \subset \Omega^{\text{in}}$. For the indices i such that $\Omega_{c_i} \subset \Omega^{\text{ex}}$, the definition of $\mathbf{L}_{i,j}^{\text{ex}}$, with $j = 1, N^{\text{ex}}$, is similar to that of $\mathbf{L}_{i,k}^{\text{in}}$, just replacing w_k^{in} by v_j^{ex} . Newton iterations for problem (14) are

$$\boldsymbol{\psi}^{n+1} = \boldsymbol{\psi}^n - [\mathbf{e}_\psi(\boldsymbol{\psi}^n)]^{-1} \mathbf{e}(\boldsymbol{\psi}^n), \quad (15)$$

with

$$[\mathbf{e}_\psi(\boldsymbol{\psi})] = D_\psi[(\mathbf{A}_0 + \mathbf{C})\boldsymbol{\psi} + \mathbf{A}_\mu(\boldsymbol{\psi})] - D_\psi \mathbf{J}(\boldsymbol{\psi}).$$

Let \mathbf{u}^{ex} and \mathbf{u}^{in} gather the values of dofs for $\psi_h^{\text{ex}} \in \mathcal{V}^{\text{ex}}$ and $\psi_h^{\text{in}} \in \mathcal{V}^{\text{in}}$, respectively. We have $\mathbf{u}^{\text{ex}} = (\mathbf{u}_o^{\text{ex}}, \mathbf{u}_\partial^{\text{ex}})$ and $\mathbf{u}^{\text{in}} = (\mathbf{u}_o^{\text{in}}, \mathbf{u}_\partial^{\text{in}})$ where \mathbf{u}_o^{ex} (resp. \mathbf{u}_o^{in}) and $\mathbf{u}_\partial^{\text{ex}}$ (resp. $\mathbf{u}_\partial^{\text{in}}$) are for dofs in V_o^{ex} (resp. V_o^{in}) and V_∂^{ex} (resp. V_∂^{in}). The mortar coupling condition in \mathcal{V}_h links the block $\mathbf{u}_\partial^{\text{ex}}$ to the block $\mathbf{u}_\partial^{\text{in}}$ by the matrix relation $\mathbf{P} \mathbf{u}_\partial^{\text{ex}} = \mathbf{D} \mathbf{u}_\partial^{\text{in}}$ with $(\mathbf{P})_{i,j} = \int_{\mathcal{I}} v_{\partial,i}^{\text{ex}} v_{\partial,j}^{\text{ex}} d\mathcal{I}$, for all $i, j = 1, N_\partial^{\text{ex}}$, and $(\mathbf{D})_{i,k} = \int_{\mathcal{I}} v_{\partial,i}^{\text{ex}} w_{\partial,k}^{\text{in}} d\mathcal{I}$, for all $i = 1, N_\partial^{\text{ex}}$ and $k = 1, N_\partial^{\text{in}}$. The inclusion of the coupling condition matrix form into the algebraic system associated with the discrete problem (13) is done by introducing the reduced variable, \mathbf{X} , such that

$$\boldsymbol{\psi} = \begin{pmatrix} \mathbf{u}_o^{\text{ex}} \\ \mathbf{u}_\partial^{\text{ex}} \\ \mathbf{u}_o^{\text{in}} \\ \mathbf{u}_\partial^{\text{in}} \end{pmatrix} = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{P}^{-1} \mathbf{D} \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{u}_o^{\text{ex}} \\ \mathbf{u}_o^{\text{in}} \\ \mathbf{u}_\partial^{\text{in}} \end{pmatrix} = \mathbf{Q} \mathbf{X}.$$

Equation (14) rewritten in terms of \mathbf{X} becomes

$$\mathbf{e}(\mathbf{X}) := \mathbf{Q}^\top [(\mathbf{A}_0 + \mathbf{C})\mathbf{Q} \mathbf{X} + \mathbf{A}_\mu(\boldsymbol{\psi}) - \mathbf{J}(\boldsymbol{\psi}) - \mathbf{L}^{\text{in}} \mathbf{U}_I^{\text{in}} - \mathbf{L}^{\text{ex}} \mathbf{U}_I^{\text{ex}}]. \quad (16)$$

For $\mathbf{J}(\boldsymbol{\psi}) = \mathbf{J}(\mathbf{Q} \mathbf{X}) = \mathbf{H}(\mathbf{X})$ we get

$$D_{\mathbf{X}} \mathbf{H}(\mathbf{X}) d\mathbf{X} = D_\psi \mathbf{J}(\boldsymbol{\psi}) \mathbf{Q} d\mathbf{X} = \text{Jac}_\psi(\boldsymbol{\psi}) \mathbf{Q} d\mathbf{X},$$

with $\text{Jac}_\psi(\boldsymbol{\psi})$ the matrix representing the derivative of $\mathbf{J}(\boldsymbol{\psi})$ w.r.t. $\boldsymbol{\psi}$. For the vector $\mathbf{A}_\mu(\boldsymbol{\psi}) = \mathbf{A}_\mu(\mathbf{Q} \mathbf{X}) = \mathbf{G}(\mathbf{X})$, we obtain $D_{\mathbf{X}} \mathbf{G}(\mathbf{X}) d\mathbf{X} = \mathbf{A}_{\mu,\psi}(\boldsymbol{\psi}) \mathbf{Q} d\mathbf{X}$ with

$[\mathbf{A}_{\mu,\psi}]_{i,j}$ given in (12) by setting $\psi_h, v_i^{\text{ex}}, v_j^{\text{ex}}$ at the place of $\psi, \tilde{\psi}$ and s , respectively. Newton iterations for problem (16) read

$$\mathbf{X}^{n+1} = \mathbf{X}^n - [\mathbf{e}_\mathbf{X}(\mathbf{X}^n)]^{-1} \mathbf{e}(\mathbf{X}^n) \quad (17)$$

where $[\mathbf{e}_\mathbf{X}(\mathbf{X}^n)] = \mathbf{Q}^\top [(\mathbf{A}_0 + \mathbf{C}) + \mathbf{A}_{\mu,\psi}(\psi^n) - \text{Jac}_\psi(\psi^n)] \mathbf{Q}$.

It does not exist, to our knowledge, analytical solutions for the free-boundary equilibrium problem considered in this paper. We provide nevertheless some numerical evidence of convergence for the proposed method in the next section.

5 Numerical results

The initial guess of the plasma domain $\Omega_p(\psi)$ for given currents in the poloidal field coils plays a crucial role in free-boundary equilibrium problems. Here, we find such initial guesses by solving inverse problems or optimal control problems, where a desired shape and position of the plasma domain is the objective and the precise values of the currents is unknown. In the present case we do not focus on this technical issue, but assume we have a good initial guess for the poloidal flux (e.g., \mathbf{X}^0 could be a non-mortar formulation of the free-boundary equilibrium problem involving piece-wise linear FEs everywhere, as explained in [10]). Then, the good convergence of the Newton iterations applied to (16) is shown in Table 1. We remark that the presence of iron parts slows down considerably the convergence speed. Fig. 4 shows a typical WEST poloidal flux map calculated by NICE.

Table 1 Convergence history of Newton iterations for WEST: iteration number n and residual relative error $\|\mathbf{X}^n - \mathbf{X}^{n-1}\|/\|\mathbf{X}^{n-1}\|$, with either $\Omega_{Fe} = \emptyset$ or $\Omega_{Fe} \neq \emptyset$.

n	if $\Omega_{Fe} = \emptyset$	if $\Omega_{Fe} \neq \emptyset$
1	1.77919×10^{-2}	9.78267×10^{-3}
2	4.35470×10^{-4}	8.56241×10^{-4}
3	4.05152×10^{-7}	6.17721×10^{-4}
4	2.80505×10^{-11}	3.46301×10^{-4}
5		9.38432×10^{-5}
6		3.09165×10^{-5}
7		8.76154×10^{-6}
8		3.11734×10^{-6}
9		5.16426×10^{-7}
10		2.58976×10^{-10}
11		6.29602×10^{-14}

A zoom on the distribution of ψ_h in Ω^{in} is proposed in Fig. 5, for the case $\Omega_{Fe} \neq \emptyset$. The X-point and plasma axis are enlightened in the small pictures. With the adopted piece-wise cubic FEs in Ω^{in} , these points do not coincide with nodes of the computational mesh, thus assuming a more physically meaningful position than the one that could be computed with low-order piece-wise linear FEs.

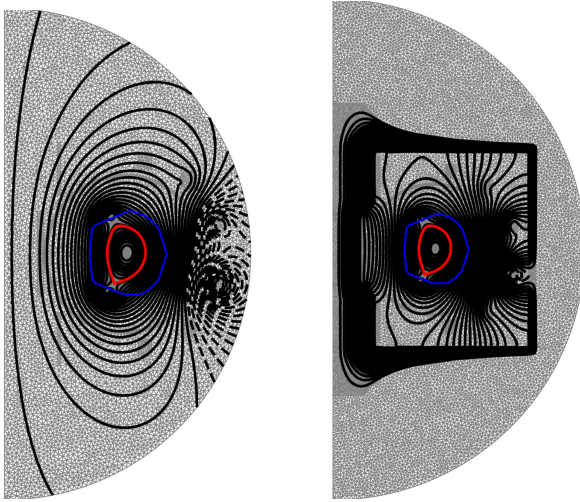


Fig. 4 Magnetic flux isolines in the poloidal section of WEST, supposing either $\Omega_{Fe} = 0$ (left) or $\Omega_{Fe} \neq 0$ (right).

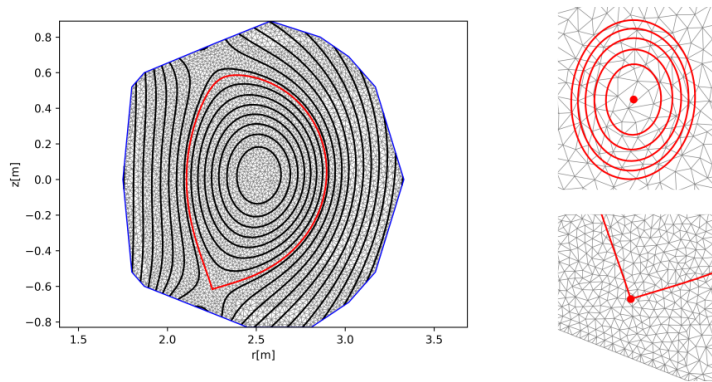


Fig. 5 Magnetic flux isolines in Ω^{in} (left), together with a zoom around the plasma axis (right, top) and X-point (right, bottom).

6 Conclusions

We have focused on the numerical computation of a MHD equilibrium for a hot plasma in iron tokamaks, such as WEST or JET, still in activity nowadays. A short overview on the mathematical complexity to treat the magnetic induction \mathbf{B} in tokamak plasmas has anticipated the equations. We have underlined that axisymmetric plasma equilibrium simulations need to rely on accurate reconstructions of the poloidal magnetic flux ψ and of its gradient $\nabla\psi$, at least in the part of the tokamak cross section that is accessible to the plasma. Therefore we have consid-

ered the C^1 , piece-wise cubic, reduced Hsieh-Clough-Tocher (rHCT) finite elements with C^0 piece-wise linear Lagrange ones by relying on a non-overlapping mortar element method and to discretize, in such a cross section, the Grad-Shafranov-Schlüter equation. At the discrete level a Newton method is proposed to solve the coupled nonlinear problem and it is given numerical evidence of its convergence. The presence of material non-linearities has been taken into account in the Newton algorithm by suitably modifying the computation of the Jacobian matrix.

Acknowledgements

This work is supported by the French National Research Agency grant SISTEM (ANR-19-CE46-0005-03). The first and second authors have worked within the framework of the EURO Fusion Consortium and have received funding from Euratom research and training programs 2014-2018, 2019-2020, under the grant agreement N.633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

References

1. Albanese, R., Blum, J., Barbieri, O.: On the solution of the magnetic flux equation in an infinite domain. In: EPS. 8th Europhysics Conference on Computing in Plasma Physics (1986), pp. 41–44 (1986)
2. Bernardi, C., Maday, Y., Patera, A.: A new nonconforming approach to domain decomposition: The mortar element method. In: Nonlinear Partial Differential Equations and Their Applications. in H. Brézis and J.-L. Lions (eds.), Collège de France Seminar XI. (1992)
3. Blackman, E.G.: Magnetic helicity and large scale magnetic fields: a primer. *J. Fluid Mechanics* **188**, 59–91 (2015)
4. Blum, J.: Numerical Simulation and Optimal Control in Plasma Physics with Applications to Tokamaks. Series in Modern Applied Mathematics. Wiley Gauthier-Villars, Paris (1989)
5. Blum, J., Le Foll, J., Thooris, B.: The self-consistent equilibrium and diffusion code SCED **24**, 235 – 254 (1981). DOI 10.1016/0010-4655(81)90149-1. URL <http://www.sciencedirect.com/science/article/pii/0010465581901491>
6. Cantarella, J., DeTurck, D., Gluck, H., Teytel, M.: Influence of geometry and topology on helicity. *Geophysical Monograph-American Geophysical Union* **111**, 17–24 (1999)
7. Christiansen, S.H., Hu, K.: Generalized finite element systems for smooth differential forms and Stokes’ problem. *Numer. Math.* **140**, 327–371 (2018)
8. Clough, R., Tocher, J.: Finite element stiffness matrices for analysis of plates in bending. In: Proc. Conf. Matrix Methods in Struct. Mech. Air Force Inst of Tech., Wright Patterson A.F Base, Ohio (1965)
9. Elarif, A., Faugeras, B., Rapetti, F.: Tokamak free-boundary plasma equilibrium computation using finite elements of class C^0 and C^1 within a mortar element approach. *J. Comput. Phys.* **439**, 110388 (2021)
10. Faugeras, B.: An overview of the numerical methods for tokamak plasma equilibrium computation implemented in the NICE code. *Fusion Eng. Design* **160**, 112020 (2020)
11. Glowinski, R., Marrocco, A.: Analyse numérique du champ magnétique d’un alternateur par éléments finis et sur-relaxation ponctuelle non linéaire. *Computer Meth. Appl. Mech. Engng.* **3**, 55–85 (1974)
12. Glowinski, R., Marrocco, A.: Sur l’approximation par éléments finis d’ordre 1, et la résolution par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires. *C.R.A.S. Serie A, Paris* **278**, 1649–1652 (1974)
13. Grad, H., Rubín, H.: Hydromagnetic equilibria and force-free fields. In: 2nd UN Conf. on the Peaceful Uses of Atomic Energy, vol. 31, p. 190 (1958)

14. Heumann, H., Blum, J., Boulbe, C., Faugas, B., Selig, G., Ané, J.M., Brémond, S., Grandgirard, V., Hertout, P., Nardon, E.: Quasi-static free-boundary equilibrium of toroidal plasma with CEDRES++: Computational methods and applications. *J. Plasma Physics* **81**, 905810301 (2015)
15. Heumann, H., Rapetti, F.: A finite element method with overlapping meshes for free-boundary axisymmetric plasma equilibria in realistic geometries. *J. Comput. Phys.* **334**, 522–540 (2017)
16. Lüst, R., Schlüter, A.: Axialsymmetrische magnetohydrodynamische gleichgewichtskonfigurationen. *Z. Naturforsch A* **12**, 850–854 (1957)
17. MacTaggart, D., Valli, A.: Magnetic helicity in multiply connected domains. *J. Plasma Physics* **85**, 775850501 (2019)
18. Minjeaud, S., Pasquetti, R.: Fourier-spectral elements approximation of the two fluid ion-electron braginskii system with application to tokamak edge plasma in divertor configuration. *J. Comput. Phys.* **321**, 492–511 (2016)
19. Moffatt, H., Ricca, R.: Helicity and the calugareanu invariant. pp. 411–429 (1992)
20. Moffatt, H.K.: The degree of knottedness of tangled vortex lines. *J. Fluid Mechanics* **35**, 117–129 (1969)
21. Ratnani, A., Crouseilles, N., Sonnendrücker, E.: An isogeometric analysis approach for the study of the gyrokinetic quasi-neutrality equation. *J. Comput. Phys.* **231**, 373–393 (2012)
22. Rebut, P.: Instabilités non magnétohydrodynamiques dans les plasmas à densités de courant élevé. *J. Nuclear Energy Part C* **4**, 159 (1963)
23. Shafranov, V.: On magnetohydrodynamical equilibrium configurations. *Soviet Journal of Experimental and Theoretical Physics* **6**, 545 (1958)